H-Spaces I

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Abstract

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1 H-Spaces

The concept of an *H-space* is what results when that of a topological magma is generalised to a homotopy-invariant notion. Essentially the H-spaces are the unital magmas in the homotopy category of based spaces. The term H-space was coined by J.P. Serre in his 1951 paper [11], with the 'H' being a tribute to the German Mathematician H. Hopf for his influential work [4]. A consequence of the main result of [4] is essentially that the cohomology ring of a compact, connected Lie group G is an exterior algebra[1] on a collection of odd-dimensional classes

$$H^*(G;\mathbb{Q}) \cong \Lambda(x_{2n_1+1},\dots,x_{2n_k+1}). \tag{1.1}$$

As it turned out, the only two necessary features of G used to prove this isomorphism were the basic finiteness assumptions and the presence of the multiplication. Thus were born H-spaces.

[1]The **exterior algebra** over a field $\mathbb{K}$ generated by odd-dimensional elements $u_1,\dots,u_n$ is the free $\mathbb{K}$-vector space with basis consisting of all products $u_{i_1}\dots u_{i_k}$, $i_1 < \cdots < i_k$, equipped with an associative, distributive multiplication subject to the relations $u_i\cdot u_j = -u_j\cdot u_i$ for $i\neq j$ and $u_i^2 = 0$. The empty product is understood to correspond to an algebra identity in degree 0.

Definition 1 *An* ***H-space*** *is a pair* (X, μ) *consisting of a pointed space* X *and a map* $\mu : X \times X \to X$ *which makes the next diagram commute up to homotopy*

$$\begin{tikzcd} X \vee X \arrow[d, hook] \arrow[dr, "\nabla"] & \\ X \times X \arrow[r, "\mu"'] & X. \end{tikzcd} \tag{1.2}$$

□

Thus we are asking for the existence of homotopies $\mu(-, *) \simeq id_X \simeq \mu(*, -)$, although we do not fix these as part of the structure. The map μ is called a **multiplication** on X, and although we will normally leave it implicit from notation, you should understand that it is an integral part of the structure: the same underlying space may admit many different multiplications. Another thing to bear in mind is that although the homotopy commutativity of (1.2) is a requirement only on the homotopy class of μ, we have asked that a particular representative be fixed.

Lemma 1.1 *Let* (X, μ) *be an H-space. If* X *is well-pointed, then* μ *is homotopic to a map* μ' *satisfying* $\mu'(x, *) = \mu'(*, x) = x$ *for all* $x \in X$.

Proof Since X is well-pointed the inclusion

$$j : X \vee X \cong (X \times *) \cup (* \times X) \subseteq X \times X \tag{1.3}$$

is a closed cofibration. Choosing a homotopy $F : \mu j \simeq \nabla$ and applying the HEP we get a homotopy $\widetilde{F} : X \times F \to X$ starting at μ and ending at the required map μ'. ∎

Generally we shall be working with well-pointed spaces like CW complexes, and in this case the lemma shows that there is little loss of generality in assuming that any H-space multiplication is strict.

Example 1.1

1. A topological group G is an H-space. The group multiplication makes (1.2) commute strictly. Of course the multiplication on G is associative, and strictly so, and G has inverses. None of this additional structure is assumed in Definition 1.

2. More generally any topological magma with unit is an H-space for which (1.2) commutes strictly. Lemma 1.1 shows that Definition 1 isn't really that far away from this. Contrast this to the situation for co-H-spaces, where you showed that the only 'strict' co-H-space is the one-point space! □

Given the first example here it is natural to ask when the multiplication on an H-space satisfies extra properties.

Definition 2 *An H-space (X, μ) is said to be* ***homotopy associative*** *if the diagram*

$$\begin{tikzcd} X \times X \times X \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd} \tag{1.4}$$

commutes up to homotopy. □

Definition 3 *An H-space (X, μ) is said to be* ***homotopy commutative*** *if the diagram*

$$\begin{tikzcd} X \times X \arrow[rr, "T"] \arrow[dr, "\mu"'] & & X \times X \arrow[dl, "\mu"] \\ & X & \end{tikzcd} \tag{1.5}$$

commutes up to homotopy, where $T : X \times X \to X \times X$, $(x, y) \mapsto (y, x)$, is the twist map. □

Definition 4 *A* ***homotopy inverse*** *for a multiplication $\mu : X \times X \to X$ is a map $\kappa : X \to X$ which makes both the following diagrams commute up to homotopy*

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "*"'] & X \times X \arrow[d, "1\times\kappa"] \\ X & X \times X \arrow[l, "\mu"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "*"'] & X \times X \arrow[d, "\kappa\times 1"] \\ X & X \times X \arrow[l, "\mu"'] \end{tikzcd} \tag{1.6}$$

□

A homotopy associative H-space with homotopy inverse is said to be an **H-group**. Loop spaces are particularly important examples of H-spaces which are homotopy associative but not strictly associative. Similarly there areplentiful examples of non-commutative topological monoids which are homotopy commutative. It is interesting in such cases to form to form homotopy-theoretic measures of their failure to be commutative.

Example 1.2 If $\mathbb{K}$ is a real normed division algebra, then its unit sphere $S(\mathbb{K}) = \{x \in \mathbb{K} \mid \|x\|^2 = 1\}$ inherits a multiplication which turns it into an H-space. It is a classical result of Hurewitz [6] that the there are exactly four normed division algebras and that these are exactly the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. These algebras may be obtained successively by Cayley-Dickson doubling [6]. Repeating the process yield the sedenions, which fail to be a division algebra and have zero-divisors, the presence of which obstruct its unit sphere S^{15} from inheriting a product in this manner.

1. The unit sphere in $\mathbb{R}$ is $\mathbb{Z}_2$ as a discrete topological group.

2. The unit sphere in $\mathbb{C}$ is S^1, which is both associative and commutative.

3. The quaternions $\mathbb{H}$ form a 4-dimensional real associative division algebra. The unit sphere in $\mathbb{H} \cong \mathbb{R}^4$ is S^3 and the multiplication induced on it is associative with inverses. The multiplication is smooth and with it S^3 becomes a Lie group. Other multiplications on S^3 are discussed in §4 below. The Lie multiplication is neither commutative, nor commutative up to homotopy. We discuss a homotopy-theoretic measure of the failure of the commutivity in future.

4. The octonions $\mathbb{O}$ form a 8-dimensional nonassociative real algebra. The unit sphere in the algebra is S^7 and it inherits a non-associative multiplication. Thus unlike the previous examples S^7 is not a topological group. In fact it is known that the multiplication is not even associative up to homotopy [3].

It is a celebrated result of Adams [1] that no other sphere outside of the list above admits an H-space multiplication. $\blacksquare$

Example 1.3 By mapping

$$(a_0, \dots, a_n) \mapsto a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{1.7}$$

we can view $\mathbb{C}_*^{n+1} = \mathbb{C}^{n+1} \setminus 0$ as the space of non-zero complex polynomials of degree $\leq n$. Consider then the map defined by multiplication of polynomials

$$\begin{gathered} \mathbb{C}_*^{m+1} \times \mathbb{C}_*^{n+1} \to \mathbb{C}_*^{m+n+1} \\ (\sum_{i=0}^{m} a_i z^i, \sum_{j=0}^{n} b_j z^j) \mapsto \sum_{i,j \geq 0} a_i b_j z^{i+j} = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i b_j \right) z^k \end{gathered} \tag{1.8}$$

Projecting this to $\mathbb{C}P^{n+m}$ there is an induced map

$$\mathbb{C}P^m \times \mathbb{C}P^n \to \mathbb{C}P^{m+n}. \tag{1.9}$$

We check easily that if $m \leq m'$ and $n \leq n'$, then the following diagram commutes

$$\begin{tikzcd} \mathbb{C}P^m \times \mathbb{C}P^n \arrow[r] \arrow[d] & \mathbb{C}P^{m+n} \arrow[d] \\ \mathbb{C}P^{m'} \times \mathbb{C}P^{n'} \arrow[r] & \mathbb{C}P^{m'+n'} \end{tikzcd} \tag{1.10}$$

where the vertical arrows are the inclusions. This implies that the collection of all such maps is coherent enough to induce a multiplication

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \tag{1.11}$$

which is associative, commutative, and with unit $[1, 0, \dots, 0, \dots]$. The fact that this product has inverses up to homotopy will follow from later results.

On the other hand, $\mathbb{C}P^n$ admits no H-space structure for any finite n. The reasons for this is essentially because its rational cohomology is not an exterior algebra. $\square$

Example 1.4 If X is an H-group, then all path components of X are of the same homotopy type. For let X_x denote the path component of a point $x \in X$ and let X_0 denote the path component of X containing the basepoint. Then the maps

$$\mu(\kappa(x), -) : X_x \to X_0, \qquad \mu(x, -) : X_0 \to X_x \tag{1.12}$$

are inverse homotopy equivalences. $\square$

Let (X, μ) be an H-space and Y a space. Given maps $f, g : Y \to X$ we define $f + g : Y \to X$ to be the composite

$$f + g : Y \xrightarrow{\Delta} Y \times Y \xrightarrow{f \times g} X \times X \xrightarrow{\mu} X. \tag{1.13}$$

Homotopies $f \simeq f'$ and $g \simeq g'$ induce a homotopy

$$f + g \simeq f' + g' \tag{1.14}$$

so we see that the operation $(f, g) \mapsto f + g$ descends to the homotopy category and defines a product on the homotopy set $[Y, X]$. The homotopy commutativity of the diagrams

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\Delta"'] & X \arrow[d, "in_1"] \arrow[dr, equal] & \\ Y \times Y \arrow[r, "f \times *"] & X \times X \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\Delta"'] & X \arrow[d, "in_2"] \arrow[dr, equal] & \\ Y \times Y \arrow[r, "* \times f"] & X \times X \arrow[r, "\mu"] & X \end{tikzcd} \tag{1.15}$$

encodes the equations

$$* + f \simeq f \simeq f + * \tag{1.16}$$

and shows that the constant map is a two-sided unit for the product on $[Y, X]$. In particular $[Y, X]$ is a unital magma. It need not be associative, and despite our additive notation is not commutative in general.

If $\alpha : Y' \to Y$ is a map, then for $f, g : Y \to X$ we have a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "\Delta"] \arrow[d, "\alpha"'] & Y' \times Y' \arrow[r, "f\alpha \times f\alpha"] \arrow[d, "\alpha \times \alpha"'] & X \times X \arrow[d, equal] \arrow[dr, "\mu"] & \\ Y \arrow[r, "\Delta"] & Y \times Y \arrow[r, "f \times g"] & X \times X \arrow[r, "\mu"] & X \end{tikzcd} \tag{1.17}$$

which gives us

$$\alpha^*(f + g) = \alpha^* f + \alpha^* g. \tag{1.18}$$

Since $\alpha^*(*) = *$, the function

$$\alpha^* : [Y, X] \to [Y', X] \tag{1.19}$$

is a homomorphism of unital magmas depending only on the homotopy class of α.

Now assume that (X, μ) is homotopy associative. Then for maps $f, g, h : Y \to X$ we have a homotopy commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\Delta"] & Y \times Y \times Y \arrow[r, "f \times g \times h"] & X \times X \times X \arrow[r, "1 \times \mu"] \arrow[d, "\mu \times 1"] & X \times X \arrow[d, "\mu"] \\ & & X \times X \arrow[r, "\mu"] & X \end{tikzcd} \tag{1.20}$$

which displays a homotopy

$$f + (g + h) \simeq (f + g) + h. \tag{1.21}$$

The conclusion is that in this case the product on $[Y, X]$ is associative.

On the other hand, if (X, μ) is homotopy commutative, then

$$\begin{tikzcd} Y \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & Y \times Y \arrow[r, "f\times g"] \arrow[d, "T"] & X \times X \arrow[d, "T"] \arrow[dr, "\mu"] & \\ & Y \times Y \arrow[r, "g\times f"'] & X \times X \arrow[r, "\mu"'] & X \end{tikzcd} \tag{1.22}$$

homotopy commutes and gives a homotopy $f + g \simeq g + f$. Thus in this case the product on $[Y, X]$ is abelian. Note that commutativity does not imply associativity.

Finally let us consider the presence of an inverse $\kappa : X \to X$ for (X, μ). Given $f : Y \to X$ we write

$$-f = \kappa f : Y \xrightarrow{f} X \xrightarrow{\kappa} X. \tag{1.23}$$

Then the following diagram commutes up to homotopy

$$\begin{tikzcd} Y \times Y \arrow[r, "f\times f"] & X \times X \arrow[r, "1\times\kappa"] & X \times X \arrow[d, "\mu"] \\ Y \arrow[u, "\Delta"] \arrow[r, "f"] & X \arrow[u, "\Delta"] \arrow[r, "*"] & X. \end{tikzcd} \tag{1.24}$$

Note that the clockwise composite around this diagram is $f + (-f)$, while the anticlockwise composite is $*$. Thus in $[Y, X]$ we have

$$f + (-f) = *. \tag{1.25}$$

Similarly we show that $(-f) + f = *$ in $[Y, X]$. Thus if (X, μ) has a homotopy inverse, then the elements in $[Y, X]$ are invertible.

Summarising the previous paragraphs we have the following.

Proposition 1.2 *If (X, μ) is an H-space, then for each space Y, the homotopy sets $[Y, X]$ is a unital monoid. If X is an H-group, then $[Y, X]$ is a group which is abelian if in addition (X, μ) is homotopy commutative. Moreover all this structure is natural with respect to maps $Y \to Y'$.* ∎

The proposition has a sort of converse. To formalise it we will introduce a further definition.

Definition 5 *Let X be a space. A* ***contravariant binary operation*** *on the functor $Top_*^{op} \xrightarrow{[-,X]} Set_*$ is a lifting of it into the category Mag_* of unital magmas as indicated in the left-hand diagram below*

$$\begin{tikzcd} & Mag_* \arrow[d, "U"] \\ hTop_*^{op} \arrow[ur, dashed] \arrow[r, "{[-,X]}"'] & Set_* \end{tikzcd} \qquad \begin{tikzcd} & Gr \arrow[d, "U"] \\ hTop_*^{op} \arrow[ur, dashed] \arrow[r, "{[-,X]}"'] & Set_* \end{tikzcd} \qquad \begin{tikzcd} & Ab \arrow[d, "U"] \\ hTop_*^{op} \arrow[ur, dashed] \arrow[r, "{[-,X]}"'] & Set_* \end{tikzcd} \tag{1.26}$$

*Similarly we define a **contravariant group operation** on $[-, X]$ to be a lifting into the category Gr of groups as indicated in the middle diagram, and a **contravariant abelian operation** to be a lifting into the category Ab of abelian groups as indicated in the right-hand diagram. In each diagram U denotes the obvious forgetful functor.* $\square$

Thus $[-, X]$ has a contravariant binary operation if for each space Y, there is a unital binary operation on the homotopy set $[Y, X]$, and a (homotopy class of) map $\alpha : Y' \to Y$ induces a unit-preserving homomorphism $\alpha^* : [Y, X] \to [Y', X]$. We have seen above how the presence of an H-space multiplication on X gives rise to exactly such a structure.

Proposition 1.3 *Let X be a space. Then the homotopy classes of multiplications on X are in one-to-one correspondence with contravariant binary operations on $[-, X]$. Moreover, homotopy classes of homotopy associative multiplications with inverses on X are in correspondenec with contravariant group operations, and amongst these, the homotopy commutative multiplications correspond to contravariant abelian operations.*

Proof We have already seen that every homotopy class of multiplication gives rise to a contravariant binary operation, so we only show here the converse. Thus assume that $[-, X]$ has been equipped with the structure of a contravariant binary operation. The trick to producing a multiplication on X is to study $[X \times X, X]$ and use its abstract product to define one. We'll first need to notice that for each space Y the constant map is the unit in $[Y, X]$. Indeed, $[*, X]$ has a single point so must be the trivial magma with only an identity element. Then the unique map $Y \to *$ gives rise to a homomorphism $[*, X] \to [Y, X]$ whose image must be the unit in its codomain.

Returning now to the problem let $pr_1, pr_2 : X \times X \to X$ be the projections onto the first and second factors and define

$$\mu = pr_1 + pr_2 \in [X \times X, X] \tag{1.27}$$

where $+$ denotes the abstract product granted by the binary operation. We have to check that μ satisfies the requirements for it to be a multiplication. This is equivalent to showing that $\mu \circ j_1 = id_X = \mu \circ j_2 \in [X, X]$, where $j_a : X \hookrightarrow X \times X$ is the inclusion into the a^{th} factor. To get this we can use the contravariant functorality. For instance

$$j_1^* \mu = j_1^*(pr_1 + pr_2) = j_1^* pr_1 + j_1^* pr_2 = (pr_1 \circ j_1) + (pr_2 \circ j_1) = id_X + * = id_X \tag{1.28}$$

and similarly $j_2^* \mu = * + id_X = id_X$. This proves the first statement.

Now assume that $[-, X]$ is a group operation. Then in $[X \times X \times X, X]$ we have

$$\mu(\mu \times 1) = (\mu \times 1)^*(pr_1 + pr_2) = pr_1(\mu \times 1) + pr_2(\mu \times 1) = (pr_1 + pr_2) + pr_3. \tag{1.29}$$

and similarly $\mu(1 \times \mu) = pr_1 + (pr_2 + pr_3)$. Since these two classes are equal we conclude that the product (1.27) is homotopy associative. Next we get a homotopy inverse for μ by setting

$$\kappa = -id_X \in [X, X] \tag{1.30}$$

and checking that

$$\mu(1 \times \kappa)\Delta = \Delta^*(1 \times \kappa)^* pr_1 + \Delta^*(1 \times \kappa)^* pr_2 = \Delta^*(pr_1 - pr_2) = id_X - id_X = 0 \tag{1.31}$$

and that $\mu(\kappa \times 1)\Delta = 0$. Thus we have the second part of the statement.

In the case that $[-, X]$ is contravariantly commutative we have

$$\mu T = T^* pr_1 + T^* pr_2 = pr_2 + pr_1 = pr_1 + pr_2 = \mu \tag{1.32}$$

and so can complete the proof. ∎

Example 1.5 Let X, Y be H-spaces. Then $X \times Y$ is an H-space. Simply observe that $[-, X \times Y] \cong [-, X] \times [-, Y]$ naturally. □

Example 1.6 If X is an H-space then there is an induced operation on its set of path components $\pi_0 X$. If X is homotopy-associative with inverse, then $\pi_0 X$ is a group. ∎

Example 1.7 If X is an H-space and K is any locally compact space, then $C_*(K, X)$ is an H-space. The bijections

$$[Y, C_*(K, X)] \cong [Y \wedge K, X] \tag{1.33}$$

are natural in Y and so it is possible to apply 1.3. If μ is the multiplication on X, then

$$\widehat{\mu} : C_*(K, X) \times C_*(K, X) \cong C_*(K, X \times X) \xrightarrow{\mu_*} C_*(K, X) \tag{1.34}$$

is the multiplication on $C_*(K, X)$. If X admits an H-inverse, then $C_*(K, X)$ admits an H-inverse. If X is homotopy associative, then $C_*(K, X)$ is homotopy associative. In particular if X is homotopy associative and has an inverse, then all components of $C_*(K, X)$ are of the same homotopy type and $\pi_0(X^K) = [K, X]$ is a group. Notice that these conditions are fulfilled when $X = \Omega Y$ is a loop space (see §2). □

To complete the analogue with topological picture we need a notion of a homomorphism up to homotopy.

Definition 6 *Let (X, m), (Y, n) be H-spaces. A map $f : X \to Y$ is said to be an* ***H-map*** *if the following diagram commutes up to homotopy*

$$\begin{tikzcd} X \times X \arrow[r, "f \times f"] \arrow[d, "m"'] & Y \times Y \arrow[d, "n"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{1.35}$$

We say that f is a ***H-equivalence*** *if it is both an H-map and a homotopy equivalence.* □

Clearly the property of a map being an H-map (H-equivalence) depends only on the homotopy class of that map. It is also easy to see that a composite of H-maps (H-equivalences) is an H-map (H-equivalence).

Lemma 1.4 *If $f : (X, m) \xrightarrow{\simeq} (Y, n)$ is an H-equivalence, then so is any homotopy inverse $g : Y \to X$.*

Proof We have $gn \simeq gn(fg \times fg) \simeq gfm(g \times g) \simeq m(g \times g)$. ∎

Corollary 1.5 *If two H-spaces are H-equivalent and one is homotopy associative (commutative), then so is the other. If one admits a homotopy inverse, then so does the other.*

Proof Using Lemma 1.4 reduces the proof to studying a few simple diagrams, a task we leave to the reader. ∎

Proposition 1.6 *Let $f : (X, m) \to (Y, n)$ be an H-map. Then for any space K, the induced map $f_* : [K, X] \to [K, Y]$ is a homomorphism. If f is a H-equivalence, then f_* is an isomorphism.*

Proof The first statement is clear from the definition of the binary operations in $[K, X], [K, Y]$. The second statement follows with the help of 1.5. ∎

Example 1.8 Let $f : X \to Y$ be a map where (X, μ_X) is a well-pointed H-space and (Y, μ_Y) is an H-group. When is f an H-map? Consider the difference

$$\widetilde{\mathcal{O}}(f) = f\mu_X - \mu_Y(f \times f) \in [X \times X, Y]. \tag{1.36}$$

The cofiber sequence

$$X \vee X \xrightarrow{j} X \times Y \xrightarrow{q} X \wedge X \xrightarrow{\delta} \Sigma X \vee \Sigma X \dots \tag{1.37}$$

gives an exact sequence of groups

$$0 \leftarrow [X \vee X, Y] \leftarrow [X \times X, Y] \leftarrow [X \wedge X, Y] \leftarrow 0. \tag{1.38}$$

The left-hand arrow is surjective, since the multiplication μ_Y can be used to cosntruct a (non-multiplicative) splitting, and the right hand arrow is injective, since the connecting map δ is null homotopic (this was shown in the examples of *Cofiber Sequences*). The element $\widetilde{\mathcal{O}}(f)$ restricts to 0 in $[X \vee X, Y]$ and so by exactness defines a unique obstruction class

$$\mathcal{O}(f) \in [X \wedge X, Y] \tag{1.39}$$

which we call the *H-deviation* of f. The map f is an H-map if and only if $\mathcal{O}(f) = 0$. □

Example 1.9 If $f : X \to Y$ is an H-map between H-spaces X, Y, then a choice of homotopy $\psi : f\mu_X \simeq \mu_Y(f \times f)$ give rise to a multiplication $\mu_{f,\psi}$ on the homotopy fibre $F_f = \{(x, l) \in X \times Y^I \mid f(x) = l(0), l(1) = *\}$. For simplicity let us assume that the multiplications μ_X and μ_Y are strict. Then $\mu_{f,\psi}$ is given by

$$\mu_{f,\psi}\left((x, l), (y, m)\right) = (\mu_X(x, y), \psi(x, y) + \mu_Y(l(-), m(-))). \tag{1.40}$$

Different choices for the homotopy ψ will generally give rise to different multiplication. □

Finally we address the problem of transferring H-structures across a homotopy equivalence. Ideally the condition for a space to admit a multiplication would depend only on its homotopy type, and indeed this is true.

Proposition 1.7 *Let (X, m) be an H-space and Y a space. Assume that $f : Y \to X$ has a left homotopy inverse $g : X \to Y$. Then the map*

$$\mu : Y \times Y \xrightarrow{f \times f} X \times X \xrightarrow{m} X \xrightarrow{g} Y \tag{1.41}$$

is an H-space multiplication on Y.

If f is a homotopy equivalence then the maps f, g are H-equivalences between (X, m) and (Y, μ). It follows in this case if (X, m) is homotopy associative (commutative), then so is (Y, μ), and if X admits a homotopy inverse, then so does Y.

Proof The first statement follows by inspecting the homotopy commutative diagram

$$\begin{tikzcd}
Y \vee Y \arrow[r, "f \vee f"'] \arrow[d] \arrow[rr, bend left, equal] & X \vee X \arrow[r, "g \vee g"'] \arrow[d] \arrow[dr, "\nabla"] & Y \vee Y \arrow[dr, "\nabla"] & \\
Y \times Y \arrow[r, "f \times f"'] & X \times X \arrow[r, "m"'] & X \arrow[r, "g"'] & Y.
\end{tikzcd} \tag{1.42}$$

If f is a homotopy equivalence, then g is its unique inverse and we have

$$f\mu = gfm(f \times f) \simeq m(f \times f), \qquad gm \simeq gm(fg \times fg) = \mu(g \times g) \tag{1.43}$$

which show that f, g are H-maps. The remaining statements follow from Corollary (1.5). ∎

There is another proof of the proposition which uses Proposition 1.3. Note that if f is not a homotopy equivalence, then no structure need be transferred along the retraction g. For instance, it does not follow that Y is homomotopy associative or commutative when X is, nor that Y has an inverse when X does. Another subtle point is that even when f, g are inverse homotopy equivalences, if one of X, Y is strictly associative (commutative), then the other is only guaranteed to be associative (commutative) up to homotopy.

2 Spaces of Loops

The loop space $\Omega X = C_*(S^1, X)$ of any pointed space X becomes an H-space when given the **loop multiplication**

$$\mu(\alpha, \beta) = \alpha + \beta : t \mapsto \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t - 1) & \frac{1}{2} \le t \le 1. \end{cases} \tag{2.1}$$

The unit is the loop which is constant at the basepoint of X, and the required homotopies for (1.2) are

$$R_s(\alpha)(t) = \begin{cases} \alpha\left(\frac{2}{1+s}t\right) & 0 \le t \le \frac{1+s}{2} \\ * & \frac{1+s}{2} \le t \le 1, \end{cases} \qquad L_s(\alpha)(t) = \begin{cases} * & 0 \le t \le \frac{1-s}{2} \\ \alpha\left(\frac{2}{1+s}t + \frac{s-1}{s+1}\right) & \frac{1-s}{2} \le t \le 1. \end{cases} \tag{2.2}$$

In particular ΩX is not strictly unital. If we wished to apply Lemma 1.1, then a sufficient condition for ΩX to be well-pointed is that X is. Below we will focus on developing the loop multiplication.

Note that despite our additive notation the loop multiplication is not commutative nor even homotopy commutative in general. We check also that neither is it strictly associative

$$((\alpha+\beta)+\gamma)(t) = \begin{cases} \alpha(4t) & 0 \leq t \leq \frac{1}{4} \\ \beta(4t-1) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \leq t \leq 1, \end{cases} \qquad (\alpha+(\beta+\gamma))(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(4t-2) & \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma(4t-3) & \frac{3}{4} \leq t \leq 1. \end{cases}$$

However it's easy to see that both of these sums are homotopic to

$$(\alpha+\beta+\gamma)(t) = \begin{cases} \alpha(3t) & 0 \leq t \leq \frac{1}{3} \\ \beta(3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \gamma(3t-2) & \frac{2}{3} \leq t \leq 1 \end{cases} \tag{2.3}$$

by homotopies which are linear on each subinterval (and in particular independent of the particular loops α, β, γ). See *Elementary Homotopy Theory IV* pg. 5 for a similar homotopy. The conclusion is that the loop multiplication is homotopy associative.

Now consider the map $\kappa : \Omega X \to \Omega X$, $\alpha \mapsto -\alpha$, given by

$$(-\alpha)(t) = \alpha(1-t). \tag{2.4}$$

This provides a homotopy inverse for the loop multiplication. Indeed, we get a homotopy $J : \Omega X \times I \to \Omega X$ for the left-hand diagram in (1.6) by setting

$$J_s(\alpha)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1-s}{2} \\ \alpha(1-s) & \frac{1-s}{2} \leq t \leq \frac{1+s}{2} \\ \alpha(2-2t) & \frac{1+2}{2} \leq t \leq 1. \end{cases} \tag{2.5}$$

Similarly we find a homotopy for the right-hand diagram.

In summary we have:

Proposition 2.1 *For any space X, the loop space ΩX is an H-group. A pointed map $f : X \to Y$ induces an H-map $\Omega f : \Omega X \to \Omega Y$.*

Proof Only the last statement needs proof and in fact it is easy to see that Ωf is a strict H-map. ∎

The presence of all this is understandable in another way. Recall the co-H-structure on S^1. We check that the map

$$c^* : \Omega X \times \Omega X \cong C_*(S^1 \vee S^1, X) \to C_*(S^1, X) = \Omega X \tag{2.6}$$

induced by the suspension comultiplication $c : S^1 \to S^1 \vee S^1$ is exactly the loop multiplication (2.1). The same goes for the loop inverse (2.4), which is exactly the map induced by the suspension coinverse $\iota : S^1 \to S^1$. If you wondered why I did not define π_1 in terms of loops, it is because the approach through co-H-structure is more fundamental. Generalising all of the above we have the following.

Proposition 2.2 *Let (X, c) be a co-H-space and Y a space. Then $C_*(X, Y)$ is an H-space with multiplication*

$$c^* : C_*(X, Y) \times C_*(X, Y) \cong C_*(X \vee X, Y) \xrightarrow{c^*} C_*(X, Y). \tag{2.7}$$

If X is coassociative (cocommutative), then $C_*(X, Y)$ is homotopy associative (commutative). If X has a coinverse, then $C_*(X, Y)$ has an inverse. In particular $C_*(X, Y)$ is an H-group when X is a cogroup.

Proof We need to prove that the multiplication satisfies condition (1.2). We get this by studying the diagram

$$\begin{tikzcd} C_*(X,Y) \vee C_*(X,Y) \arrow[r, "(pr_1^*, pr_2^*)"] \arrow[d] & C_*(X \times X, Y) \arrow[d] \arrow[dr, "\Delta^*"] & \\ C_*(X,Y) \times C_*(X,Y) \arrow[r, "\cong"] & C_*(X \vee X, Y) \arrow[r, "c^*"] & C_*(X,Y) \end{tikzcd} \tag{2.8}$$

The right-hand vertical arrow is induced by the inclusion $X \vee X \hookrightarrow X \times X$ and so the triangle homotopy commutes by virtue of c being a comultiplication. We get by inspection that the square in the diagram commutes strictly. Since $\Delta^*(pr_1^*, pr_2^*) = (\Delta^* pr_1, \Delta^* pr_2^*) = (id_X^*, id_X^*)$ we see that (1.2) is satisfied. This shows that $C_*(X, Y)$ is an H-space. The other statements follow similarly. ∎

2.1 The Moore Loop Space

Let X be a based space. We show in this section we give an alternative construction of a space of loops on X which is a strictly associative topological monoid. This construction is called the *Moore Loop Space*. In the case that X is well-pointed, the Moore loop space and ordinary loop space of X are H-equivalent.

Define the **Moore loop space** of X by putting

$$\Omega_M X = \{(\alpha, v) \in C([0, \infty), X) \times [0, \infty) \mid \alpha(0) = * \text{ and } \alpha(t) = * \text{ for } t \geq v\} \tag{2.9}$$

and topologising it as a subspace of $C([0, \infty), X) \times [0, \infty)$. We define a product $\Omega_M X \times \Omega_M X \to \Omega_M X$ by letting

$$(\alpha_1, v_1) + (\alpha_2, v_2) = (\alpha_1 \star \alpha_2, v_1 + v_2) \tag{2.10}$$

where

$$(\alpha_1 \star \alpha_2)(t) = \begin{cases} \alpha_1(t) & 0 \leq t \leq v_1 \\ \alpha_2(t - v_1) & v_1 \leq t \leq v_1 + v_2 \\ * & v_1 + v_2 \leq t. \end{cases} \tag{2.11}$$

The constant loop of 0 length $(*, 0)$ furnishes the product with a strict unit. Notice that the associativity relation

$$\big((\alpha_1, v_1) + (\alpha_2, v_2)\big) + (\alpha_3, v_3) = (\alpha_1, v_1) + \big((\alpha_2, v_2) + (\alpha_3, v_3)\big) \tag{2.12}$$

holds strictly. That is, $\Omega_M X$ is a topological monoid. On the other hand, $\Omega_M X$ has inverses only up to homotopy, in this case provided by the map

$$\kappa : \Omega_M X \to \Omega_M X, \qquad (\alpha, v) \mapsto (-\alpha, v) \tag{2.13}$$

where

$$(-\alpha)(t) = \begin{cases} \alpha(v-t) & 0 \le t \le v \\ * & v \le t \end{cases} \qquad t \in [0, \infty). \tag{2.14}$$

Proposition 2.3 *If X is a based space, then $\Omega_M X$ is a topological monoid and a H-group.* ∎

A basepoint preserving map $f : X \to Y$ induces a continuous monoid homomorphism

$$\Omega_M f : \Omega_M X \to \Omega_M Y, \qquad (\alpha, v) \mapsto (f\alpha, v). \tag{2.15}$$

Thus we can view $\Omega_M(-)$ as a functor from Top_* into the category of topological monoids. In particular, if f is a homeomorphism, then $\Omega_M f$ is a monoid isomorphism. On the other hand, as an H-map, clearly $\Omega_M f$ depends only on the homotopy class of f, and in particular is an H-equivalence whenever f is a homotopy equivalence.

In the next paragraph we will compare the Moore loop space with the standard loop space. Unfortunately this will take us outside of the pointed category. There is an *unpointed* inclusion

$$j : \Omega X \to \Omega_M X, \qquad l \mapsto (l, 1) \tag{2.16}$$

where on the right-hand side we extend the domain of l to $[0, \infty)$ by letting it be constant at the basepoint on $[1, \infty)$. There is also a pointed retraction

$$r : \Omega_M X \to \Omega X, \qquad (\alpha, v) \mapsto [\alpha_v : t \mapsto \alpha(tv)] \tag{2.17}$$

where on the right-hand side we understand α_v to have its domain restricted to $[0, 1]$.

Lemma 2.4 *ΩX is an unpointed deformation retract of $\Omega_M X$.*

Proof Clearly $rj = id_{\Omega X}$. A homotopy $H : id_{\Omega_M X} \simeq jr$ is defined at $(\alpha, v) \in \Omega_M X$ with $v > 0$ by

$$H_s(\alpha, t_0) = (h_s(\alpha, v), (1-s)v + s), \qquad s \in I, \tag{2.18}$$

where

$$h_s(\alpha, v)(t) = \alpha\left(\frac{tv}{(1-s)v + s}\right), \qquad t \in [0, \infty). \tag{2.19}$$

For the constant loop we set $H_s(*, 0) = (*, 0)$ for all $s \in I$. ∎

Notice that while the inclusion $j : \Omega X \xrightarrow{\subseteq} \Omega_M X$ does not respect basepoints, the retraction $r : \Omega_M X \to \Omega X$ is a pointed map, and the homotopy H defined in the lemma is one of pointed maps.

Lemma 2.5 *For any pointed space X the map $r : \Omega_M X \to \Omega X$ is an H-map and an unpointed homotopy equivalence. If X is well-pointed, then r is an H-equivalence.*

Proof That r is a free homotopy equivalence was shown in 2.4. To show that it is an H-map consider the homotopy $G : \Omega_M X \times \Omega_M X \times I \to \Omega X$ given by

$$G_s((\alpha,u),(\beta,v))(t) = \begin{cases} \alpha\left(t((1+s)u+(1-s)v)\right) & 0 \le t \le \frac{u}{(1+s)u+(1-s)v} \\ \beta\left(\frac{(1+s)uv+(1-s)v^2}{su+(1-s)v}t - \frac{uv}{su+(1-s)v})\right) & \frac{u}{(1+s)u+(1-s)v} \le t \le 1. \end{cases} \tag{2.20}$$

This has $G_0((\alpha,u),(\beta,v)) = r((\alpha,u)+(\beta,v))$ and $G_1((\alpha,u),(\beta,v)) = r(\alpha,u)+r(\beta,v)$, and moreover is a based map. Thus we have the first part of the claim.

To see the final part notice that if X is well-pointed, then so are ΩX and $\Omega_M X$. We can use a Strøm structure of $* \hookrightarrow X$ to define ones for the loop spaces. In any case the claim follows because the well-pointedness of the loop spaces implies that r is a pointed homotopy equivalence. To see now that it is an H-equivalence we simply apply Lemma 1.4. ∎

Remark The inclusion $j : \Omega X \to \Omega_M X$ is an unbased H-map. A free homotopy $H : \Omega X \times \Omega X \times I \to \Omega_M X$ intertwining the two multiplications is given by

$$F_s(\alpha,\beta)(t) = (f_s(\alpha,\beta), 1+s), \qquad \alpha,\beta \in \Omega X,\ s \in I, \tag{2.21}$$

where

$$f_s(\alpha,\beta)(t) = \begin{cases} \alpha((2-s)t) & 0 \le t \le \frac{1}{2-s} \\ \beta\left(\frac{(2-s)t-1}{1+s-s^2}\right) & \frac{1}{2-s} \le t \le 1+s \\ * & t \ge 1+s \end{cases} \qquad t \in [0,\infty). \tag{2.22}$$

□

Corollary 2.6 *Let X be a space. Then for any space K, the map*

$$r_* : [K, \Omega_M X] \to [K, \Omega X] \tag{2.23}$$

is a homomorphism. If X is well-pointed, then it is an isomorphism of groups. ∎

3 H-Structure vs. Co-H-Structure

Consider the following scenario: (X,c) is a co-H-space and (Y,m) is an H-space and we are presented with the homotopy set $[X,Y]$. The set has two different multiplicative structures. Which will be the more fruitful one to try to understand? As it turns out, the question is unnecessary: they are the same. One proof of this hinges upon the following more general statement.

Theorem 3.1 (The Eckmann-Hilton Argument) *Let M be a set and $* : M \times M \to M$, $(x,y) \mapsto x*y$, and $\cdot : M \times M \to M$, $(x,y) \mapsto x\cdot y$, a pair of unital binary operations. Assume they satisfy the interchange law*

$$(w*x)\cdot(y*z) = (w\cdot y)*(x\cdot z), \qquad \forall w,x,y,z \in M. \tag{3.1}$$

Then the operations $*$ and $\cdot$ coincide, and moreover are both commutative and associative.

Proof Included among the assumptions is the existence of units $1_*, 1_\bullet \in M$ satisfying

$$x * 1_* = 1_* * x = x \tag{3.2}$$
$$x \cdot 1_\bullet = 1_\bullet \cdot x = x \tag{3.3}$$

for each $x \in M$. Applying the interchange law we see that the two units are actually equal

$$1_\bullet = 1_\bullet \cdot 1_\bullet = (1_* * 1_\bullet) \cdot (1_\bullet * 1_*) = (1_* \cdot 1_\bullet) * (1_\bullet \cdot 1_*) = 1_* * 1_* = 1_*. \tag{3.4}$$

Now with this in hand write $1 = 1_* = 1_\bullet$ and take $x, y \in M$ to get

$$x * y = (1 \cdot x) * (y \cdot 1) = (1 * y) \cdot (x * 1) = y \cdot x \tag{3.5}$$

as well as

$$y \cdot x = (y * 1) \cdot (1 * x) = (y \cdot 1) * (1 \cdot x) = y * x. \tag{3.6}$$

Switching x and y gives $y * x = x \cdot y$, and putting everything together gives

$$x * y = y \cdot x = y * x = x \cdot y \tag{3.7}$$

which prove that the two operations coincide and are commutative. To show that they are associative we take $x, y, z \in M$ and get

$$(x * y) * z = (x \cdot y) * (1 \cdot z) = (x * 1) \cdot (y * z) = x \cdot (y * z) = x * (y * z). \tag{3.8}$$

Since the operations coincide this proves their associativity. ∎

To relate this theorem to $[X, Y]$ we have to show that the interchange law is satisfied. Write $+_c$ for the product formed using the comultiplication on X and $+_m$ for the product formed using the multiplication on Y. Then if $f, g, h, k : X \to Y$ are given, the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[dr, "\Delta"] & X \vee X \arrow[r, "\Delta\vee\Delta"] \arrow[dr, "\Delta"] & X^2 \vee X^2 \arrow[r, "(f\times g)\vee(h\times k)"] & Y^2 \vee Y^2 \arrow[r, "m\vee m"] \arrow[dr, "\nabla"] & Y \vee Y \arrow[dr, "\nabla"] & \\
& X \times X \arrow[r, "c\vee c"] & (X \vee X)^2 \arrow[r, "(f\vee h)\times(g\vee k)"] & (Y \vee Y)^2 \arrow[r, "\nabla"] & Y \times Y \arrow[r, "m"] & Y
\end{tikzcd} \tag{3.9}$$

encodes the equation

$$(f +_m g) +_c (h +_m k) = (f +_c h) +_m (g +_c k). \tag{3.10}$$

In the diagram we write a superscript 2 to denote a two-fold cartesian product. Note the switching of g, h on the bottom line that arises at the interchange of the diagonals on X and on $X \vee X$. In any case we are free to apply 3.1.

Corollary 3.2 *If (X, c) is a co-H-space and (Y, m) is an H-space, then binary operations on $[X, Y]$ which are induced respectively by the co-H-structure on X and the H-structure on Y coincide, and moreover are associative and commutative.* ∎

This observation is useful for several reasons. Firstly because it aids greatly with the computability of $[X, Y]$. More subtly is to do with functorality. For instance if Y is an H-space, then any map $\alpha : X \to X$ between co-H-spaces X, X' will induce a homomorphism $\alpha^* : [X, Y] \to [X', Y]$, whereas normally we would need to assume that α were a co-H-map to guarantee this. This observation has some application when studying compositions in $\pi_* X$.

Example 3.1 Let (X, μ) be an H-space. Then then the algebraic structure on the homotopy groups of X is induced by its multiplication. We see this explicitly in the following diagram in which the vertical isomorphism is the map $(\alpha, \beta) \mapsto (\alpha \times \beta)\Delta$

$$\begin{tikzcd}
\pi_n X \times \pi_n X \arrow[d, "\cong"'] \arrow[dr, "+"] & \\
\pi_n(X \times X) \arrow[r, "\mu_*"'] & \pi_n X.
\end{tikzcd} \tag{3.11}$$

Moreover, if $\alpha, \beta \in \pi_k X$ and $f \in \pi_n S^k$, then

$$f^*(\alpha + \beta) = f^*\alpha + f^*\beta \in \pi_n X \tag{3.12}$$

□

There are also other consequences of 3.2.

Corollary 3.3 *Let X be a space. Assume that X admits an H-space multiplication. Then $\pi_1 X$ is abelian.* ▮

Thus we have immediate criteria to discount such spaces as $S^1 \vee S^1$ from being H-spaces. There are also other applications of this corollary.

Proposition 3.4 *Let (X, m) be a connected H-space and Y any a well-pointed space. Then there is a bijection $[Y, X]_0 \cong [Y, X]$ between the free and pointed homotopy classes of maps $Y \to X$.*

Proof The special case that X was a connected topological group was Exercise 1.5 in *Pointed and Unpointed Homotopy Sets*. The general proof goes through almost unchanged. ▮

Example 3.2 Assume that S^n admits a multiplication $\mu : S^n \times S^n \to S^n$. Then the degree -1 map is a unique homotopy inverse for μ. This is true because any inverse $\kappa : S^n \to S^n$ must solve the equations

$$0 = 1 +_\mu \kappa = 1 +_c \kappa \tag{3.13}$$

in $\pi_n S^n$, where c is the suspension comultiplication. Clearly this equation has a unique solution.

More generally, if X is a space which possesses both a comultiplication $c : X \to X \vee X$ and a multiplication $\mu : X \times X \to X$, then a map $\kappa : X \to X$ is a coinverse for c if and only if it is an inverse for m. □

One upshot of the work above is a calculation of the fundamental group of the circle by purely abstract reasoning.

Proposition 3.5 *There is an isomorphism*

$$\pi_1 S^1 \cong \mathbb{Z} \tag{3.14}$$

induced by sending $n \in \mathbb{N}$ to the map which is n times the identity.

Proof We have shown in Corollary 3.3 above the fundamental group of an H-space is abelian. On the other hand it was an exercise to show that the fundamental group of a co-H-space is free. Thus if a space is both a co-H-space and an H-space, then its fundamental group is either trivial or $\mathbb{Z}$, since these are the only free groups which are abelian. In particular this applies to S^1. In *Elementary Homotopy Theory I* we showed that S^1 is not contractible, so in particular id_{S^1} is not null homotopic and represents a non-trivial element in $\pi_1 S^1 = [S^1, S^1]$. Clearly id_{S^1} cannot be divisible in this group, and so represents a generator. ∎

Example 3.3 Let (X, μ) be an H-group. For an integer $k \in \mathbb{Z}$ the k^{th} **power map** on X is the map $\overline{k} : X \to X$ which is k times the identity in $[X, X]$. In particular $\overline{0}$ is the constant map, and $\overline{k}$ is defined iteratively for $k > 0$ by

$$\overline{k} : X \xrightarrow{\Delta} X \times X \xrightarrow{id_X \times \overline{k-1}} X \times X \xrightarrow{\mu} X. \tag{3.15}$$

If $k < 0$, then $\overline{k}$ is iteratively defined by

$$\overline{k} : X \xrightarrow{\Delta} X \times X \xrightarrow{\kappa \times \overline{k+1}} X \times X \xrightarrow{\mu} X \tag{3.16}$$

where κ is the homotopy inverse of X. The map $\overline{k}$ induces multiplication by k on $\pi_* X$.

In the case that X is also a cogroup, the k^{th} power map $\overline{k}$ coincides with the degree k map $\underline{k} : X \to X$, which is formed using the co-H-structure as k times the identity in $[X, X]$. Recall that the the map $\underline{k}$ induces multiplication by k on $H_* X$. In particular these observations apply to any sphere with a multiplication. □

4 Counting Multiplications

Here we consider problem of the uniqueness of a given H-space structure. Examples are given at the end of the section.

Proposition 4.1 *Let X be a space. Assume that X is well-pointed that that it admits a homotopy associative H-space multiplication $\mu : X \times X \to X$ with inverse. Then the set of all homotopy classes of multiplications on X is in in one-to-one correspondence with the set $[X \wedge X, X]$.*

Proof Because X is well-pointed there is a cofiber sequence

$$X \vee X \xrightarrow{j^*} X \times X \xrightarrow{q^*} X \wedge X \xrightarrow{\delta} \Sigma X \vee \Sigma X \to \dots \tag{4.1}$$

and applying $[-, X]$ to this leads to an exact sequence

$$[X \vee X, X] \leftarrow [X \times X, X] \leftarrow [X \wedge X, X] \xleftarrow{\delta^*} [\Sigma X \vee \Sigma X, X] \leftarrow \dots \tag{4.2}$$

Now the multiplication on X endows each of these sets with a natural group structure, so this is an exact sequence of groups. It was shown in the examples of the *Cofiber Sequences* exercise sheet that the connecting map δ is null homotopic, so δ^* is the zero homomorphism and q^* is an injection of groups. At the other end, the j^* is surjective, since we can use the multiplication μ to construct a (non-multiplicative) section. Hence we have a short exact sequence of groups

$$0 \leftarrow [X, X] \times [X, X] \xleftarrow{j^*} [X \times X, X] \xleftarrow{q^*} [X \wedge X, X] \leftarrow 0 \tag{4.3}$$

Now, if μ' is any H-space multiplication on X, then it must satisfy $j^*\mu' = \mu' j = (id_X, id_X) \in [X, X] \times [X, X]$. In particular the set of all homotopy classes of multiplications on X is exactly the inverse image of (id_X, id_X) under j^*. But by exactness we have

$$(j^*)^{-1}(id_X, id_X) = q_*([X \wedge X, X]) \tag{4.4}$$

and since q^* is injective we get the proposition. ■

Note that it is essential for the proposition that X have a good multiplication to begin with, and the multiplications it generates will not in general be homotopy associative or have inverses. It is not the most general statement possible, but will suffice for our examples. It yields the slightly surprising fact that although H-structures are fairly rare amongst topological spaces, when a space has one, it will generally have a large number of others. The proposition also tells us how to find these other multiplications. With its notation, if $\xi : X \wedge X \to X$ is a map, then

$$\mu_\xi = \mu + \xi q \tag{4.5}$$

is a multiplication, where μ is the fixed multiplication on X and the sum is formed using the group structure on $[X \times X, X]$ which is again induced by μ.

Example 4.1 Any contractible H-space has a unique multiplication which ithomotopy associative, commutative and admits an inverse. □

Example 4.2 Among the spheres which are H-spaces we have $[S^n \wedge S^n, S^n] = \pi_{2n}S^n$, and these sets are known for $n = 1, 3, 7$.

1. S^1 has a unique H-space structure, since $\pi_2 S^1 = 0$.

2. There are twelve different multiplications on S^3, since $\pi_6 S^3 \cong \mathbb{Z}_{12}$ and it is known that exactly 8 of them are homotopy associative [3]. Let $\mu : S^3 \times S^3 \to S^3$ be its Lie multiplication and fix a generator $\nu' \in \pi_6 S^3$. Then, for each mod 12 integer n we get a (homotopy class of) multiplication $\mu_n : S^3 \times S^3 \to S^3$ by setting

   $$\mu_n = \mu + n \cdot \nu' q \tag{4.6}$$

   where $q : S^3 \times S^3 \to S^3 \wedge S^3$ is the quotient map. These 12 maps realise all the multiplications on S^3, and $\mu_0 = \mu$ is the Lie multiplication.

Proposition 4.2 (James, Slifker) *The multiplication $\mu_n : S^3 \times S^3 \to S^3$ is homotopy associative if and only if $n \equiv 0, 1 \mod 3$. For each homotopy associative multiplication there is a topological group X_n and an H-homotopy equivalence $(S^3, \mu_n) \xrightarrow{\simeq} X_n$.* ∎

The first statement is James's and the second Slifker's.

Now, the standard Lie multiplication and its opposite are the only associative multiplications which come from any Lie group structure on a manifold homotopy equivalent to S^3. This follows from the classification theorem for compact Lie groups. Thus of the eight homotopy associative multiplications on S^3, six represent multiplications on non-smooth topological groups of the homotopy type of S^3.

It is a subtle point that none of these non-Lie homotopy associative multiplications is homotopic to a strictly associative multiplication on S^3 itself, which is the reason for needing to introduce the topological goups X_n in the above statement.

3. On S^7 there are $\pi_{14}S^7 \cong \mathbb{Z}_{120}$ distinct multiplications. None of them are homotopy associative. Note that S^7 does not satisfy the conditions of Proposition 4.1. Neverthless, we have seen in the previous section that the degree -1 map on S^7 is an H-inverse, and that fact that the standard multiplication on S^7 is Moufang gives us enough structure to be able to extend the statement to cover this case. □

Example 4.3 H-space structures on products of spheres has been an interesting question and can be a difficult problem. For instance Loibel [5] and Norman [10] have calculated that on $S^3 \times S^3$ there are $2^{20} \cdot 3^{16}$ distinct homotopy classes of multiplication, with $2^{16} \times 3^{16}$ of them being homotopy associative. □

Example 4.4 Nonstandard multiplications on compact Lie groups have been studied and considered.

1. SO_3 has 768 different multiplications [9].
2. SU_3 has $2^{15} \cdot 3^9 \cdot 5 \cdot 7$ multiplications and Sp_2 has $2^{20} \cdot 3 \cdot 5^5 \cdot 7$ multiplications. [7]
3. As far as I am aware, the number of multiplications on SU_4 and SU_5 is an open question, although it is known that these numbers are finite, and partial results for SU_4 are obtained by Murley in his thesis [8].
4. For $n \geq 6$, the number of multiplications on SU_n is known to be infinite [2]. □

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