Principal Bundles

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Abstract

For these notes we will follow chapter 9 from [Die00] combined with chapter 14 from [Die08].

1 Definition of principal bundles

A principal G-bundle is a local trivial fibration where the charts are compatible with a group action. We will specify this by definition 1.1. Some basic definitions about group actions can be found in the appendix.

Definition 1.1. *Let G be a topological group. A* ***principal*** G***-bundle*** *is a map $p\colon E \to B$ together with a right action $r\colon E \times G \to E$ satisfying the following conditions.*

1. *For all $x \in E$ and $g \in G$ we have $p(xg) = p(x)$.*
2. *For every $b \in B$ there is an open neighbourhood U and a G-homeomorphism $\varphi\colon p^{-1}(U) \to U \times G$ (where G acts on $p^{-1}(U)$ by restriction of r and on $U \times G$ by $((u, x), g) \mapsto (u, xg)$) such that the following diagram commutes.*

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[d] & U \times G \arrow[dl, "\mathrm{pr}_U"] \\ U & \end{tikzcd} \tag{1}$$

First we notice that the action of G on $p^{-1}(U)$ is well defined by the first condition. Also because of the first condition, the map p induces a map $\overline{p}\colon E/G \to B$. From the second condition we get that G acts freely on E and that the map $\overline{p}\colon E/G \to B$ is a homeomorphism (we can verify that $\overline{p}$ is bijective but the map p is, as a local trivial fibration, a quotient map (Fibrations III Proposition 1.3) hence the set inverse of $\overline{p}$ is continuous).

Theorem 1.2. *Let $p\colon X \to B$ and $q\colon Y \to B$ be principal G-bundles and $F\colon X \to Y$ a G-map such that $qF = p$. Then F is a homeomorphism.*

Proof. We will show that F is locally a homeomorphism. We therefore assume that p and q are trivial. Then F is of the form:

$$F\colon B \times G \to B \times G, \qquad F(b,g) = (b, \alpha(b)g) \tag{2}$$

with a map $\alpha\colon B \to G$. Then an inverse of F is given by $(b,g) \mapsto (b, \alpha^{-1}(b)g)$. The general case follows since we can show that F is bijective. □

Definition 1.3. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{F} & X \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ C & \xrightarrow{f} & B \end{array} \tag{3}$$

*be a commutative square with principal G-bundles q and p and a G-map $F\colon Y \to X$. Then the map F or (F, f) is called **bundle map**.*
*If f is the identity then F is a homeomorphism by Theorem 1.2 and is called a **bundle isomorphism**.*

Let $p\colon E \to B$ be a principal G-bundle and $f\colon X \to B$ be a map. Then the map $X \times_B E \to X$ from the pullback

$$\begin{array}{ccc} X \times_B E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array} \tag{4}$$

is also a principal G-bundle. Here G acts on $X \times_B E$ by $(x,e)g = (x, eg)$. We call the map $X \times_B E \to X$ the bundle **induced** from p by f.

Definition 1.4. *A right G-space U is called **trivial** if there exists a continuous G-map $f\colon U \to G$ into the G-space G with right translation action. A right G-space is called **locally trivial** if it has an open covering by trivial G-subspaces.*

Lemma 1.5. *A G-space U is trivial if and only if $U \to U/G$ is isomorphic to the trivial principal G-bundle* $\mathrm{pr}\colon U/G \times G \to U/G$.

Proof. Let U be a trivial G-space. Hence there is a G-map $f\colon U \to G$. Let $p\colon U \to U/G$ be the projection map. Then we get a continuous G-map $(p, f)\colon U \to U/G \times G$ over U/G. Because the map $U \to U, u \mapsto u \cdot f^{-1}(u)$ factors over p we get a map $s\colon U/G \to U$. Now we can verify that $U/G \times G \to U, (x,g) \mapsto s(x)g$ is an inverse of (p,f). □

With this Lemma get the following theorem:

Theorem 1.6. *The total space E of a principal G-bundle is locally trivial. If E is locally trivial, then $E \to E/G$ is a principal G-bundle.*

2 Universal Bundle

A principal G-bundle[1] $p\colon EG \to BG$ is called **universal** if it is numerable[2] trivial and if for every numerable trivial principal G-bundle $q\colon E \to B$ there exist up to homotopy a unique bundle map from q to p.
Two bundle maps $\alpha_0, \alpha_1\colon E \to EG$ are homotopic as bundle maps if there is a homotopy $H_t\colon \alpha_0 \simeq \alpha_1$ such that H_t is a G-equivariant map for all $t \in I$.
Now assume that $p'\colon E'G \to B'G$ is another universal bundle. Then by definition there are up to homotopy unique bundle maps $\beta\colon EG \to E'G$ and $\gamma\colon E'G \to EG$ where the compositions $\beta\gamma$ and $\gamma\beta$ are bundle maps and hence are homotopic to the identity. Also BG and $B'G$ are homotopy equivalent. The map $p\colon EG \to EB$ is called **universal bundle** and the space BG is called **classifying space** of G.
Let $q\colon E \to B$ be a numerable principal G-bundle. Then there is up to homotopy a unique map $k\colon B \to BG$ induced from the bundle map from q to $EG \to BG$. This map is called **classifying map**. We denote by $\mathcal{B}(G,B)$ the set of isomorphism classes of numerable principal G-bundles over B. By assigning each isomorphism class of a bundle the corresponding homotopy class of the classifying map, we get a well defined map $\kappa\colon \mathcal{B}(G,B) \to [B, BG]$. An inverse is given by assigning each $k\colon E \to BG$ the induced principal G-bundle $k^*\colon B \times_{BG} EG \to B$. This inverse is well defined because of the homotopy theorem for principal G-bundles (see Theorem 3.7 Fibrations III).
Hence the classification of bundles is reduced to a problem in homotopy theory:

Theorem 2.1 (Classification Theorem)**.** *We assign to each isomorphism class of numerable principal G-bundles the homotopy class of a classifying map and obtain a well-defined bijection $\mathcal{B}(G,B) \cong [B, BG]$. The inverse assigns to $k\colon B \to BG$ the bundle induced by k from the universal bundle.*

2.1 Construction of a universal bundle

In this section we want to actually construct a universal bundle for a given topological group G.

Theorem 2.2. *For every topological group G there exists a universal bundle $p\colon EG \to BG$.*

Therefore we will use the infinite join of topological spaces. Let $(X_j \mid j \in J)$ be a family of topological spaces. The join

$$X = \star_{j\in J} X_j \tag{5}$$

can be defined as the following space. The elements of X are are represented by families $(t_j, x_j)_{j\in J}$ with $t_j \in [0,1]$ and $x_j \in X_j$ such that only finitely many t_j are not zero and $\sum_{j\in J} t_j = 1$. The families (t_j, x_j) and (u_j, y_j) represent the same element if and only if

1. $t_j = u_j$ for each $j \in J$
2. $x_j = y_j$ whenever $t_j \neq 0$.

[1] Note that the spaces EG and BG are related to G but not to E and B.
[2] i.e. there is a numerable covering of BG such that p is trivial over each member of this open covering.

We will use the notation t_jx_j for (t_j, x_j). This is suggestive since we can replace $0x_j$ by $0y_j$ for any x_j and y_j in X_j. We have coordinate functions:

$$t_j\colon X \to [0,1], \quad (t_ix_i) \mapsto t_j \qquad p_j\colon t_j^{-1}(0,1] \to X_j, \quad (t_ix_i) \mapsto x_j \tag{6}$$

The topology on X will be the coarsest topology such that all maps t_j and p_j are continuous. This topology is characterized by the following universal property: A map $f\colon Y \to X$ for any space Y is continuous if and only if the maps $t_jf\colon Y \to [0,1]$ and $p_jf\colon f^{-1}t_j^{-1}(0,1] \to X_j$ are continuous.
If all spaces X_j are G spaces, then $((t_jx_j), g) \mapsto (t_jx_jg)$ defines a right G action on X. The continuity can be verified with the universal property from above.
Now we can set

$$EG = G \star G \star G \star \dots \tag{7}$$

as the join of countably many copies of G. We write $BG = EG/G$ and therefore get the orbit map $p\colon EG \to BG$.
We want to show that this map $p\colon EG \to BG$ is a numerable principal G-bundle. The coordinate functions t_j are G-invariant and induce therefore functions τ_j on BG. The τ_j are a point-finite partition of unity subordinate to the open covering by the V_j/G, $V_j = t_j^{-1}(0,1]$. The bundle is trivial over V_j/G, since we have, by construction, G-maps $p_j\colon V_j \to G$.

Proposition 2.3. *Let E be a G space. Any two G-maps $f, g\colon E \to EG$ are G-homotopic.*

Proof. We consider the coordinate form of $f(x)$ and $g(x)$

$$(t_1(x)f_1(x), t_2(x)f_2(x), \dots) \quad \text{and} \quad (u_1(x)g_1(x), u_2(x)g_2(x), \dots), \tag{8}$$

and show that f and g are G-homotopic to maps with coordinate form

$$(t_1(x)f_1(x), 0, t_2(x)f_2(x), 0, \dots) \quad \text{and} \quad (0, u_1(x)g_1(x), 0, u_2(x)g_2(x), \dots) \tag{9}$$

where 0 denotes an element of the form $0 \cdot y$. In order to achieve this, for f say, we construct a homotopy in an infinite number of steps. The first step has in the homotopy parameter t the form

$$(t_1f_1, tt_2f_2, (1-t)t_2f_2, tt_3f_3, (1-t)t_3f_3, \dots) \tag{10}$$

It removes the first zero in the final result (9). We now iterate this process appropriately. We obtain the desired homotopy by using the first step on the interval $[0, \frac{1}{2}]$, the second step on the interval $[\frac{1}{2}, \frac{3}{4}]$, and so on. The total homotopy is continuous since in each coordinate place only a finite number of homotopies are relevant.
Having arrived at the two forms (9), they are now connected by the homotopy

$$((1-t)t_1f_1, tu_1g_1, (1-t)t_2f_2, tu_2g_2, \dots) \tag{11}$$

in the parameter t. $\square$

Proposition 2.4. *Let E be a G space. Let $(U_n \mid n \in \mathbb{N})$ be an open covering by G trivial sets. Suppose there exists a point-finite partition of unity $(v_n \mid n \in \mathbb{N})$ by G-invariant functions subordinate to the covering (U_n). Then there exists a G-map $\varphi\colon E \to EG$.*
A numerable free G-space E admits a G-map $E \to EG$.

Proof. By definition of a G-trivial space, there exist G-maps $\varphi_j \colon U_j \to G$. The desired map φ is now given by $\varphi(z) = (v_1(z)\varphi_1(z), v_2(z)\varphi_2(z), \dots)$. It is continuous, by the universal property of the topology of EG. In order to apply the last result to the general case, we reduce arbitrary partitions of unity to countable ones (see Partitions of Unity notes). $\square$

A Appendix: Group actions

In this section we want to recall some basic definitions.

Definition A.1. *A (right)* ***group action*** *of a topological group G on a space E is a continous map $r\colon E \times G \to E$ such that*

1. $r(x, e) = x$ *for all* $x \in E$
2. $r(x, g_1 g_2) = r(r(x, g_1), g_2)$ *for all* $x \in E$ *and* $g_1, g_2 \in G$

We will write xg for $r(x, g)$. Similarly we can define a left action as a map $l\colon G \times E \to E$. Note that a left action is in principal something different than a right action since the second condition will be $(g_1 g_2)x = g_1(g_2 x)$. But with the map $(g, x) \mapsto xg^{-1}$ we can turn a right action into a left one.

Definition A.2. *Here is some more terminology.*

1. *The* ***orbit*** *of an element $x \in E$ is the set $x \cdot G = \{xg \mid g \in G\}$.*
2. *For every $x \in E$ the* ***stabilizer subgroup*** *of G with respect to x is the set of all elements in G which fix x:*
$$\mathrm{Stab}_G(x) = \{g \in G \mid xg = x\}$$
3. *A group action is called* ***free*** *if all stabilizer subgroups are trivial.*
4. *A group action is called* ***transitive*** *if $x \cdot G = X$ for some $x \in E$ (equivalent for all $x \in E$).*
5. *The* ***orbit space*** *E/G is the topological space $E/\sim$ where $x_1 \sim x_2$ if and only if there exist a $g \in G$ such that $x_1 g = x_2$.*

Definition A.3. *A G-**equivariant map** $f\colon A \to B$ is a map of G-spaces (spaces which are equipped with a right G-action) such that $f(ag) = f(a)g$ for all $a \in A$ and $g \in G$.*
A G-homeomorphism is a G-map which is also a homeomorphism. The topological inverse is then also a G-map.

References

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[Die08] Tammo tom Dieck. *Algebraic Topology*. Zürich: European Mathematical Society, 2008. ISBN: 978-3-037-19048-7.