

*Branching Processes in Random Environment –
A Phase Transition in the subcritical regime*

Götz Kersting

University of Frankfurt/Main

joint work with V. Avanashev, J. Geiger, V. Vatutin

The random environment

Let

$$Q_0, Q_1, \dots$$

be a sequence of i.i.d. copies of the random probability measure Q on $\mathbb{N}_0 = \{0, 1, \dots\}$ (i.e. a random variable with values in the set Δ of all probability measures on $\mathbb{N}_0$). Then

$$\mathcal{E} = (Q_0, Q_1, \dots)$$

is called a *random environment*.

Q_n is the *offspring distribution* of an individual in generation n .

The embedded branching process

Denote

$$\begin{aligned} Z_n &= \text{number of individuals in generation } n \\ Y_{i,n} &= \text{number of children of the} \\ &\quad i\text{th individual in generation } n , \end{aligned}$$

thus

$$Z_{n+1} = \sum_{i=1}^{Z_n} Y_{i,n} .$$

Then (Z_n) is called a *branching process in random environment (BPRE)*, if given the environment the numbers of children are independent with offspring distributions as in the environment, or

$$\mathcal{L}\left((Y_{1,n}, \dots, Y_{Z_n,n}) \mid Z_0, \dots, Z_n, \mathcal{E}\right) = Q_n^{\otimes Z_n} .$$

The associated random walk

Define the random variables

$$X_n := \log m(Q_{n-1}) , \quad n \geq 1$$

with the offspring means

$$m(Q_{n-1}) := \int x Q_{n-1}(dx) ,$$

being independent copies of $X := \log m(Q)$.

The characteristics of the BPRE (Z_n) is first of all determined by the random walk

$$S_n := X_1 + \cdots + X_n , \quad S_0 := 0 .$$

Survival probabilities 1

By means of the conditional moment

$$\mathbf{E}[Z_n | Z_0, \mathcal{E}] = Z_0 \cdot m(Q_0) \cdots m(Q_{n-1}) = Z_0 e^{S_n}$$

and $\mathbf{P}\{Z_n > 0 | \dots\} \leq \mathbf{P}\{Z_k > 0 | \dots\} \leq \mathbf{E}[Z_k | \dots]$ for $k \leq n$, we obtain for the probabilities of survival the exponential estimate

$$\mathbf{P}\{Z_n > 0 | Z_0, \mathcal{E}\} \leq Z_0 \exp \left(\min(S_0, \dots, S_n) \right)$$

and

$$\mathbf{P}\{Z_n > 0\} \leq \mathbf{E} \left[Z_0 \exp \left(\min(S_0, \dots, S_n) \right) \right]$$

Survival probabilities 2

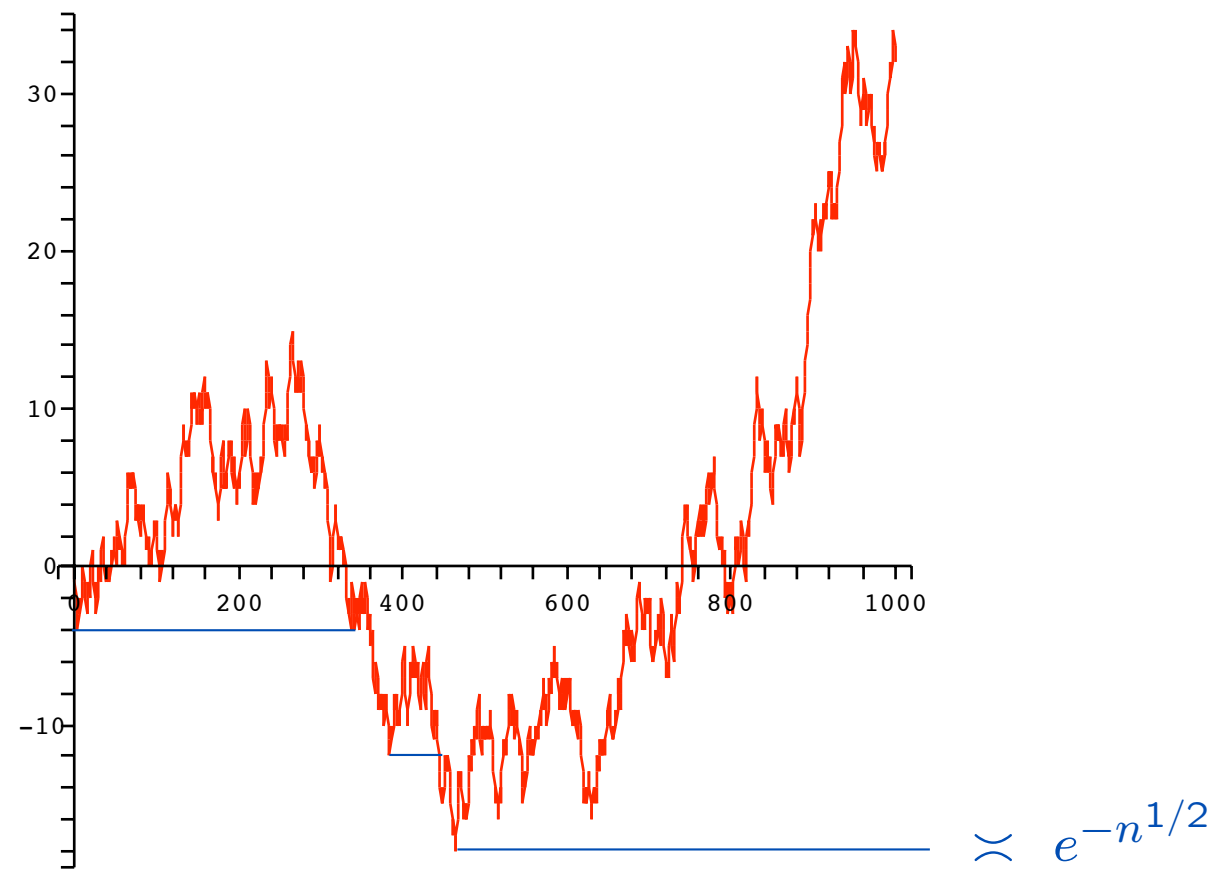
In fact (we assume $Z_0 = 1$ from now on)

$$\mathbf{P}\{Z_n > 0 \mid \mathcal{E}\} \approx \exp\left(\min(S_0, \dots, S_n)\right) .$$

(Kozlov 1975, Geiger, Kersting 2001)

That is: For survival the population first of all has to overcome the moment, when the conditional mean population size is particularly small. After this hurdle the population experiences a kind of supercritical growth.

The effect of the environment



Criticality

The BPRE is called

<i>critical,</i>	if $\mathbf{E}[X] = 0$,
<i>supercritical,</i>	if $\mathbf{E}[X] > 0$,
<i>subcritical,</i>	if $\mathbf{E}[X] < 0$.

(Smith, Wilkinson *AMS* 1969, Athreya, Karlin *AMS* 1971) or more generally, if (S_n) is an oscillating random walk, or if it has drift to ∞ or to $-\infty$ (Avanasyev, Geiger, Kersting, Vatutin *AP*, 2005).

In the critical and subcritical case

$$\mathbf{P}\{Z_n > 0 \mid \mathcal{E}\} \rightarrow 0 \text{ a.s.} \quad \text{and} \quad \mathbf{P}\{Z_n > 0\} \rightarrow 0 ,$$

as follows from the above exponential estimate.

A phase transition in the domain of subcriticality

The precise asymptotics for the survival probabilities are as follows (Kozlov 1975, Avanashev 1980, Dekking 1988, ...):

case:	condition:	$\mathbf{P}\{Z_n > 0\} \propto$
<i>critical</i>	$\mathbf{E}[X] = 0$	$n^{-1/2}$
<i>weakly subcritical</i>	$\mathbf{E}[X] < 0 < \mathbf{E}[Xe^X]$	$n^{-3/2}\gamma^n$, with $0 < \gamma < \mathbf{E}[m(Q)]$
<i>moderate subcritical</i>	$\mathbf{E}[Xe^X] = 0$	$n^{-1/2}\mathbf{E}[m(Q)]^n$
<i>strongly subcritical</i>	$\mathbf{E}[Xe^X] < 0$	$\mathbf{E}[m(Q)]^n$

The Method: Change of measure

We introduce the family $\hat{\mathbf{P}}_\beta$, $\beta \geq 0$ of probability measures, given by

$$\begin{aligned}\hat{\mathbf{E}}_\beta[\phi(Q_1, \dots, Q_n, Z_1, \dots, Z_n)] \\ = \gamma^{-n} \mathbf{E}[\phi(Q_1, \dots, Q_n, Z_1, \dots, Z_n) e^{\beta S_n}]\end{aligned}$$

with

$$\gamma = \gamma_\beta := \mathbf{E}[e^{\beta X}] .$$

Or conversely:

$$\mathbf{E}[\psi(Q_1, \dots, Z_n)] = \gamma^n \hat{\mathbf{E}}_\beta[\psi(Q_1, \dots, Z_n) e^{-\beta S_n}] .$$

Survival probabilities 3

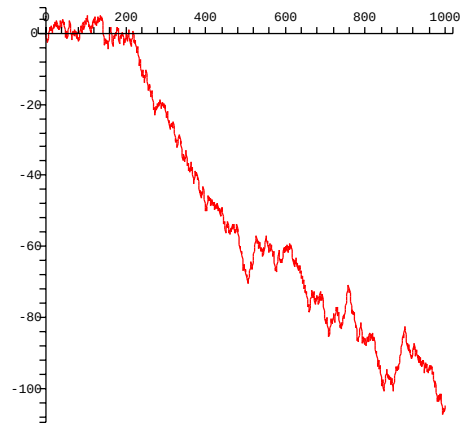
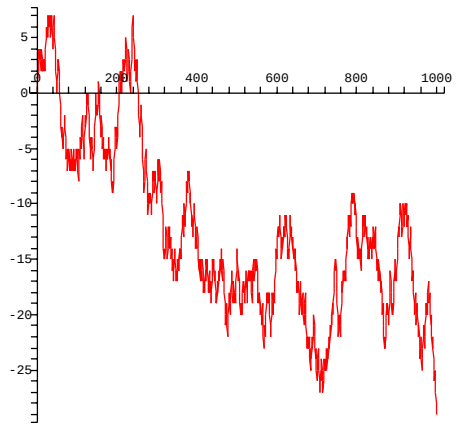
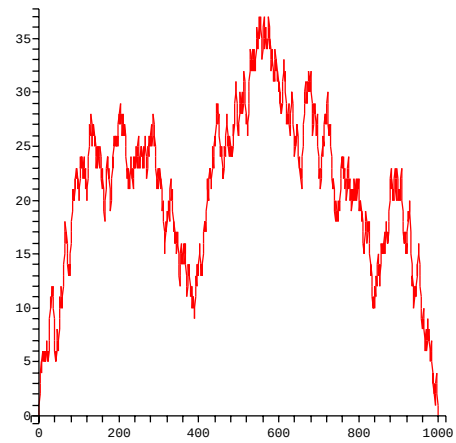
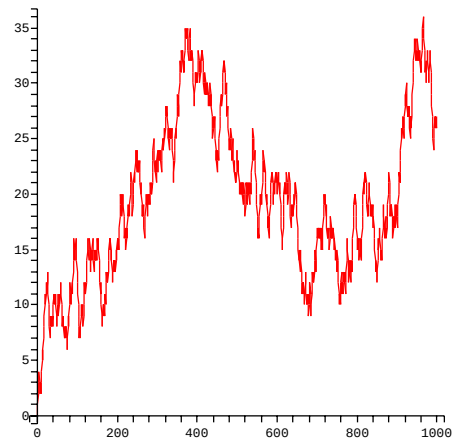
In particular

$$\mathbf{P}\{Z_n > 0\} = \gamma^n \hat{\mathbf{E}}_\beta \left[\mathbf{P}\{Z_n > 0 \mid \mathcal{E}\} e^{-\beta S_n} \right]$$

and

$$\mathbf{P}\{Z_n > 0\} \approx \gamma^n \hat{\mathbf{E}}_\beta \left[\exp \left(\min(S_0, \dots, S_n) - \beta S_n \right) \right] .$$

What is the right choice for β ?

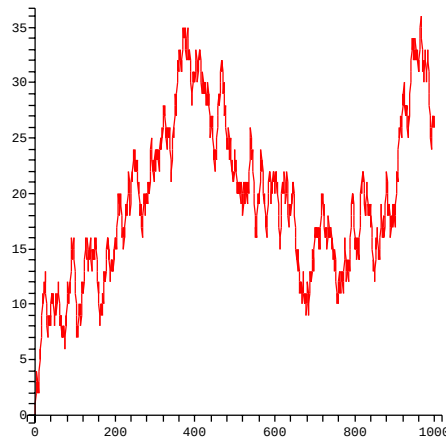


Criticality

When $\mathbf{E}[X] = 0$, it is appropriate to choose $\beta = 0$ and

$$\mathbf{E}[e^{\min(S_0, \dots, S_n)}] \approx \mathbf{P}\{\min(S_0, \dots, S_n) \geq -c\} \approx n^{-1/2}.$$

Given $Z_n > 0$ the random walk $S_0, \dots, S_n$ behaves like a *Random Walk Meander*,



Criticality 2

Consequently: Given $Z_n > 0$ the branching process exhibits supercritical behaviour. Let for $0 \leq k \leq n$

$$X_t^{k,n} := Z_{k+[t(n-k)]} / \exp \left(S_{k+[t(n-k)]} \right) , \quad 0 \leq t \leq 1 .$$

Theorem. (Conditional supercriticality of the critical process) *There is a process $Y = (Y_t)_{0 \leq t \leq 1}$ with strictly positive, a.s. constant paths, i.e. $Y_t = Y_0$ a.s., such that for $0 \leq k_n \leq n$ and $k_n \rightarrow \infty$*

$$\mathcal{L}(X^{k_n,n} \mid Z_n > 0) \Rightarrow \mathcal{L}(Y) .$$

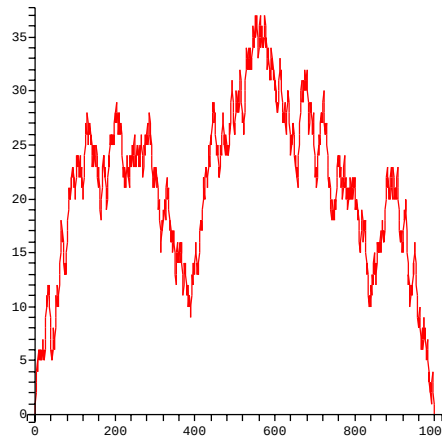
Remark. This is analogue to the fact that for an ordinary supercritical Galton-Watson process Z_n/m^n is a.s. convergent.

Weak Subcriticality

If there is a $0 < \beta < 1$ such that $\mathbf{E}[Xe^{\beta X}] = 0$, then $\hat{\mathbf{E}}_\beta[X] = 0$ and the random walk is recurrent under $\hat{\mathbf{P}}_\beta$. From

$$\hat{\mathbf{E}}_\beta \left[e^{\min(S_0, \dots, S_n) - \beta S_n} \right] \approx \hat{\mathbf{P}}_\beta \left\{ \min(S_0, \dots, S_n) \approx 0, S_n \approx 0 \right\} \approx n^{-3/2}$$

it becomes plausible that given $Z_n > 0$ the path $S_0, \dots, S_n$ behaves like a *Random Walk Excursion*



Weak subcriticality 2

Again: Given $Z_n > 0$ the branching process exhibits supercritical behaviour till almost time n . Now let for $0 \leq k \leq n$

$$\widetilde{X}_t^{k,n} := Z_{k+[t(n-2k)]} / \exp \left(S_{k+[t(n-2k)]} \right), \quad 0 \leq t \leq 1 .$$

Theorem. (Conditional supercriticality of the weakly subcritical process) There is a process $Y = (Y_t)_{0 \leq t \leq 1}$ with strictly positive, a.s. constant paths, i.e. $Y_t = Y_0$ a.s., such that for $0 \leq k_n \leq n/2$ and $k_n \rightarrow \infty$

$$\mathcal{L}(\widetilde{X}^{k_n,n} \mid Z_n > 0) \Rightarrow \mathcal{L}(Y) .$$

However

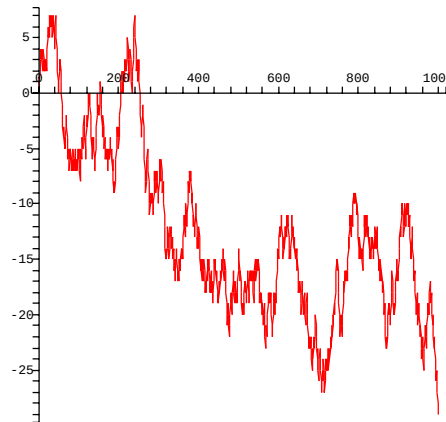
Theorem. (Conditional subcriticality of the weakly subcritical process) $\mathcal{L}(Z_n \mid Z_n > 0)$ is weakly convergent.

Moderate subcriticality

If $\mathbf{E}[Xe^X] = 0$, then $\hat{\mathbf{E}}_1[X] = 0$ and the random walk is recurrent under $\hat{\mathbf{P}}_1$. Now

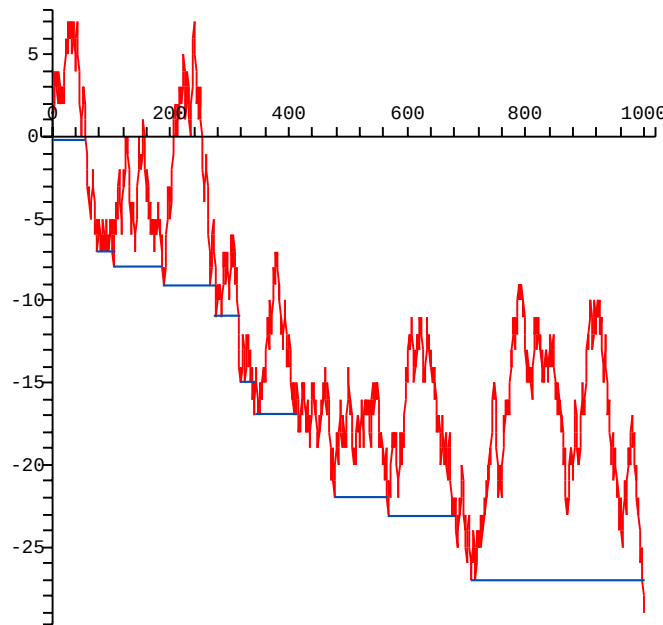
$$\hat{\mathbf{E}}_1[e^{\min(S_0, \dots, S_n) - S_n}] \approx \hat{\mathbf{P}}_1\{\min(S_0, \dots, S_n) \approx S_n\} \approx n^{-1/2}$$

and given $Z_n > 0$ the path $S_0, \dots, S_n$ behaves like a *conditioned random walk*



Moderate subcriticality 2

Supercriticality is only present during random time intervals and alternates with subcritical phases (similar to the unconditioned critical behavior!).



Moderate subcriticality 3

Theorem. Let $B^l = (B_t^l)_{0 \leq t \leq 1}$ be a Brownian motion conditioned to take its minimum at time $t = 1$ and let $e_1, e_2, \dots$ its excursion intervals (in some order). Write $j(t) = j$, if $t \in e_j$. Then for $0 < t_1 < \dots < t_k < 1$, as $n \rightarrow \infty$

$$\mathcal{L}\left(\left(\frac{Z_{nt_1}}{\exp(S_{nt_1})}, \dots, \frac{Z_{nt_k}}{\exp(S_{nt_k})}\right) \middle| Z_n > 0\right) \xrightarrow{d} (W_{j(t_1)}, \dots, W_{j(t_k)}) ,$$

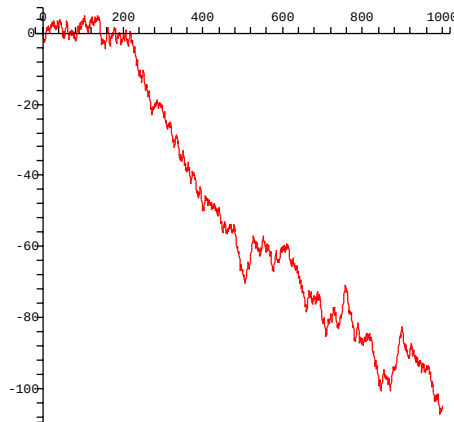
where $W_1, W_2, \dots$ are independent of B^l and i.i.d. copies of some strictly positive random variable W .

Strong subcriticality

If $\mathbf{E}[Xe^X] < 0$, then $\beta = 1$ still is the right choice, $\hat{\mathbf{E}}_1[X] < 0$ and the random walk is transient under $\hat{\mathbf{P}}_1$. Now

$$\hat{\mathbf{E}}_1[e^{\min(S_0, \dots, S_n) - S_n}] \approx \hat{\mathbf{P}}_1\{\min(S_0, \dots, S_n) \approx S_n\} \approx c$$

and given $Z_n > 0$ the path $S_0, \dots, S_n$ behaves like a *transient random walk*



Strong supercriticality 2

Now given $Z_n > 0$ no supercritical behavior remains.

Theorem. *Let $0 < t_1 < \dots < t_k < 1$, then*

$$\left((Z_{nt_1}, \dots, Z_{nt_k}) \mid Z_n > 0 \right) \xrightarrow{d} (W_1, \dots, W_k) ,$$

where $W_1, W_2, \dots$ are i.i.d. copies of some strictly positive random variable W .

Conclusion

For the conditioned process the borderline from supercritical to subcritical behavior shifts:

case	unconditioned behavior	conditioned behavior
<i>critical</i>	alternating	supercritical
<i>weakly subcritical</i>	subcritical	supercritical
<i>moderate subcritical</i>	subcritical	alternating
<i>strongly subcritical</i>	subcritical	subcritical

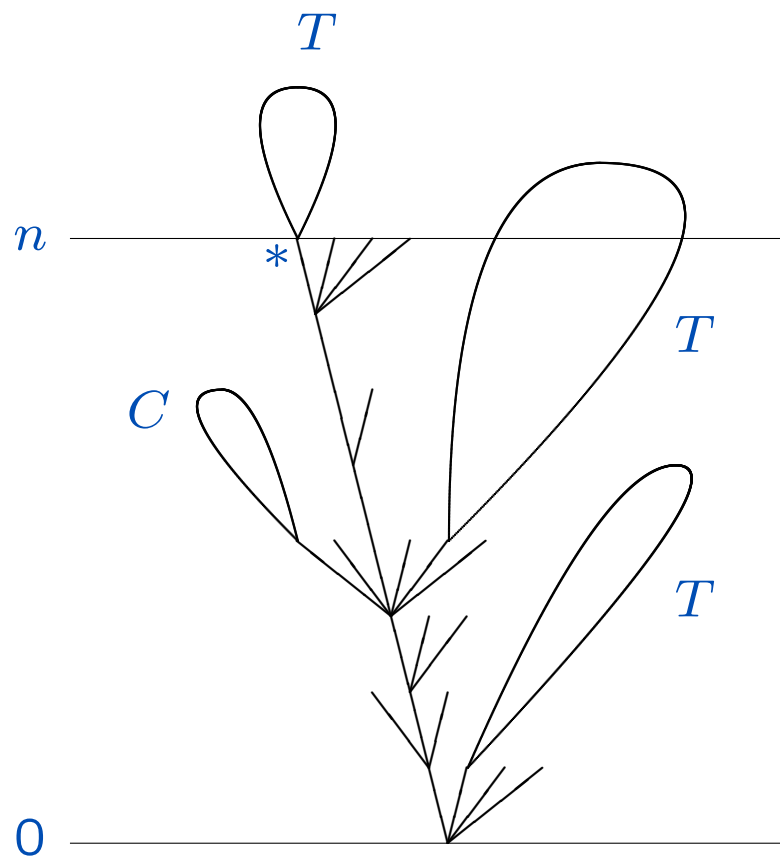
A formula for survival probabilities

From definition

$$\begin{aligned}\mathbf{P}\{Z_n > 0 \mid \mathcal{E}\} &= \mathbf{E}[Z_n \mid \mathcal{E}] / \mathbf{E}[Z_n \mid Z_n > 0, \mathcal{E}] \\ &= e^{S_n} / \mathbf{E}[Z_n \mid Z_n > 0, \mathcal{E}] .\end{aligned}$$

Thus it is sufficient to consider $\mathbf{E}[Z_n \mid Z_n > 0, \mathcal{E}]$.

Geiger's tree



$*$ = leftmost individual
in generation n

T = independent GW-trees

C = independent
conditioned GW-trees

Thus

$$\mathbf{E}[Z_n \mid Z_n > 0, \mathcal{E}] = 1 + \mathbf{E}\left[\sum_T Z_{n,T} \mid Z_n > 0, \mathcal{E}\right]$$

(where $Z_{n,T}$ is the number of individuals in generation in subtree T)

$$= 1 + \sum_{k=1}^n \mathbf{E}[R_{k,n} \mid Z_n > 0, \mathcal{E}] \exp(S_n - S_k) ,$$

where $R_{k,n}$ is the number of siblings to the right of $*$'s ancestor in generation $k - 1$. Altogether

$$\mathbf{P}\{Z_n > 0 \mid \mathcal{E}\}^{-1} = \sum_{k=0}^n \eta_{k,n} e^{-S_k} .$$

References

- V. Afanasyev, J. Geiger, G. Kersting, V. Vatutin (2005) *Criticality for branching processes in random environment*. Ann. Probab. **33**, 645-673.
- V. Afanasyev, J. Geiger, G. Kersting, V. Vatutin (2005) *Functional limit theorems for strongly subcritical branching processes in random environment*. to appear in Stoch. Proc. Appl.
- M. Birkner, J. Geiger, G. Kersting (2005) *Branching processes in random environment - a view on critical and subcritical cases*. Interacting stochastic systems (Eds. J. Deuschel, A. Greven). Springer, 269-291
- J. Geiger (1999) *Elementary new proofs of classical limit theorems for Galton-Watson processes*. J. Appl. Probab. **36**, 301-309.
- J. Geiger, G. Kersting (2000) *The survival probability of a critical branching process in random environment*. Teor. Veroyatnost. i Primenen **45**, 607-615.
- J. Geiger, G. Kersting, V. Vatutin (2003) *Limit theorems for subcritical branching processes in random environment*. AIHP **39**, 593-620.