A LATTICE SCHEME FOR AN ELLIPTIC SPDE

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DESCRIPTION OF THE MODEL

We consider the elliptic SPDE on $D = (0,1)^d$, $d \geq 4$,

$$\begin{aligned} \Delta u(x) - f(u(x)) &= g(x) + \dot{F}(x), \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D. \end{aligned} \tag{1}$$

- $\dot{F}(x)$ is a colored noise,

- $f(x) = f_1(x) + f_2(x), \qquad f, f_1, f_2 : \mathbb{R} \to \mathbb{R},$

$(f1)$ f_1 continuous non-decreasing, bounded, $|f_1(x)| \leq M$
$(f2)$ f_2 Lipschitz, Lipschitz constant L (*small*)
- $g \in L^p(D)$

The noise

$F = \big(F(\psi),\ \psi \in \mathcal{D}(\mathbb{R}^d)\big)$ Gaussian, mean zero and covariance:

$$E\big(F(\psi_1)F(\psi_2)\big) = \int_{\mathbb{R}^d} \Gamma(dx)(\psi_1 * \tilde{\psi}_2)(x)$$

Γ non-negative, tempered, non-negative definite,
$\Gamma(dx) = \varphi(x)dx$.

Example: $\Gamma(dx) = |x|^{-\beta}\, dx$

References

Buckdahn, Pardoux (1990); Donati-Martin (1993); Nualart-Tindel (1994); Tindel (1998), ...

$d = 1, 2, 3$, white noise

Approximations

(1) Allen, Novosel, Zhang, *Stoch. Stoch. Reports*, 1998

(2) Gyöngy, Martínez, *Stoch. Stoch. Reports*, to appear ($d = 1, 2, 3$)

(3) Babuska, Chatzipantelidis, *Comp. Methods Appl. Mech. Engrg*, 2002 (random coefficients)

Our Aim: Extension of (2) to higher dimensions

Method: Finite difference scheme (Gyöngy, Krylov, ...)

I. Gyöngy: Potential Analysis **9**, (1998), **11** (1999)

FORMULATION OF THE EQUATION

Classical results on elliptic boundary value problems

$$\begin{aligned}
\Delta v(x) &= P(x),\ x \in D,\ v(x) = 0,\ x \in \partial D. \\
v(x) &= \textstyle\int_D G_D(x,y) P(y)\, dy, \\
G_D(x,y) &= G(x,y) + E_x(G(B_\tau, y)), \\
G(x,y) &= C_d\, |x-y|^{2-d},
\end{aligned}$$

$P \in L^2(D)$, C_d constant, B is a d-dimensional Brownian motion starting at x, stopped at the first exit time of D, τ.

Solution to the equation

$$\begin{aligned}
u(x) = &\textstyle\int_D G_D(x,y) f(u(y))\, dy + \int_D G_D(x,y) g(y)\, dy \\
&+ \textstyle\int_D G_D(x,y) dF(y),
\end{aligned} \tag{2}$$

The stochastic integral

Does $G_D(x,y)$ belong to the RKHS of F?

Lemma Given $g \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, $p \in (1,\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x)\varphi(x-y)g(y)\,dx\,dy \right|$$

$$\leq \|g\|_{L^p(\mathbb{R}^d)} \min\{\|g\|_{L^p(\mathbb{R}^d)}\|\varphi\|_{L^{p'/2}(\mathbb{R}^d)}, \|g\|_{L^1(\mathbb{R}^d)}\|\varphi\|_{L^{p'}(\mathbb{R}^d)}\}.$$

Young's inequalities

Fact: For any $\alpha \in [1, \frac{d}{d-2})$, $\sup_{x\in D} \|G_D(x,\cdot)\|_{L^\alpha(D)} < C$.

Hence, if $\varphi \in \mathcal{L}^\alpha = L^{\alpha'}(\mathbb{R}^d) \cup L^{\alpha'/2}(\mathbb{R}^d)$, $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, the stochastic integral $\int_D G_D(x,y)dF(y)$ is well defined.

For $\alpha \in [1, \frac{d}{d-2})$,

$$E\left|\int_D G_D(x,y)\, dF(y)\right|^2 \leq \sup_{x\in D} \|G_D(x,\cdot)\|^2_{L^\alpha(D)},$$

Notice: For $d \geq 4$, $\frac{d}{d-2} \leq 2$.

Hölder continuity

Fix $\lambda \in (0,1)$. If $\varphi \in \mathcal{L}^\alpha(D)$ with $\alpha \in [1, \frac{d}{(d-2)(2-\lambda)\vee(d-1)\lambda})$,

$$\|G_D(x,\cdot) - G_D(z,\cdot)\|_{L^\alpha(D)} \le C|x-z|^\lambda,$$

and from here

$$E\left|\int_D G_D(x,y)\,dF(y) - \int_D G_D(z,y)\,dF(y)\right|^2 \le C|x-z|^{2\lambda}.$$

Thus, for $\varphi \in \mathcal{L}^\alpha$, the Gaussian process

$$v(x) = \int_D G_D(x,y)\,dF(y), \quad x \in D,$$

is γ-Hölder continuous with $\gamma < \lambda$.

EXISTENCE AND UNIQUENESS OF SOLUTION

Hypotheses

(i) $\varphi \in \mathcal{L}^{\alpha}$, for some $\alpha \in [1, \frac{d}{(d-2)(2-\lambda)\vee(d-1)\lambda})$,

(ii) $g \in L^{\alpha'}(D)$,

(iii) $f(x) = f_1(x) + f_2(x), \qquad f, f_1, f_2 : \mathbb{R} \to \mathbb{R},$

$(f1)$ f_1 continuous non-decreasing, bounded, $|f_1(x)| \leq M$

$(f2)$ f_2 Lipschitz, Lipschitz constant L small

Results

1 There exist a unique solution of the SPDE with paths in

$$\mathcal{B} = \{w : w \in \mathcal{C}(D),\ w|_{\partial D} = 0\},$$

a.s.,

2 For any $p \in [1, \alpha']$,

$$\|u\|_{L^p(\Omega; L^{\alpha'}(D))} \leq C.$$

3 The solution has a.s. Hölder continuous sample paths of order $\gamma < \lambda - \frac{d}{\alpha'}$.

Method based on classical results on nonlinear monotone operators (see e.g. Zeidler)

$$T(w)(x) = w(x) - \int_D G_D(x,y) f(w(y))\, dy,$$
$$b(x) = \int_D G_D(x,y) g(y)\, dy + \int_D G_D(x,y)\, dF(y) \in \mathcal{B},$$

Prove that the operator equation $Tw = b$ has a unique solution in $\mathcal{B}$.

Uniqueness: Poincaré's inequality implies that T is one to one

Existence: The operator

$$A : H_0^1(D) \longrightarrow (H_0^1(D))^*, \quad Au = -\Delta u + f(u)$$

is monotone, coercive, hemicontinuous

(Buckdahn,Pardoux; Donati-Martin)

APPROXIMATION SCHEME BY FINITE DIFERENCES

Method applied in previous articles

1. Fix a grid on $D = (0,1)^d$

2. Replace the differential operator governing the SPDE, $\mathcal{L}$, by the one obtained by finite differences on the grid, $\mathcal{L}_n$

3. Write a finite dimensional equation giving approximations of the solution on the grid, by means of the equation driven by $\mathcal{L}_n$

4. Compare the kernels of $\mathcal{L}^{-1}$ and $\mathcal{L}_n^{-1}$: Green functions, G, G_n, by using their Fourier series expansion.

ALL RIGHT IF $G \in L^2(D \times D)$. But, In our problem $G_D \in L^\alpha(D \times D)$, with $\alpha < 2$!

Our strategy

1. Regularisation of G_D by means of a partition of the identity (parameter ϵ): $G_D^\varepsilon, u^\varepsilon$

2. Discretisation of the Laplacian with appropriate weights coming from step one (parameters ϵ, n): $G_{D,n}^\varepsilon, u_n^\varepsilon$

3. Comparison of G_D^ε and $G_{D,n}^\varepsilon$ by means of their Fourier expansion is now possible

4. Choose a ε cleverly related to n

Smoothing G_D

- $\Psi(x) = \Pi_{i=1}^{d} \psi(x_i)$, $\psi(x) \in \mathcal{C}_c^\infty((-1,1))$ even function, $0 \leq \psi \leq 1$, $\int_{-1}^{1} \psi = 1$

- $G_D^\varepsilon(x,y) = G_D(x,\cdot) * (\varepsilon^{-d})\Psi(\frac{\cdot}{\varepsilon})$, $\varepsilon > 0$

Odd extension to $[-1,1]^d$, identification of real functions f defined on $[-1,1)^d$ with $\tilde{f}$ defined on the torus $\mathbb{T}^d$ endowed with the Haar measure dx: $\tilde{f}(e^{i\pi x}) = \tilde{f}(e^{i\pi x_1}, \ldots, e^{i\pi x_d}) := f(x)$, yielding

$$\int_{(-1,1)^d} f(t)\, dt = \int_{\mathbb{T}^d} \tilde{f}(e^{i\pi t})\, dt$$

Facts:

(1) $\alpha \in [1, \frac{d}{(d-2)(2-\lambda)\vee(d-1)\lambda})$, $\varepsilon > 0$,

$$\|G_D - G_D^\varepsilon\|_{L^\alpha(D\times D)} \le C\varepsilon^\lambda.$$

(2) For any $\varepsilon > 0$,

$$G_D^\varepsilon(x,y) = \sum_{\beta\in I^d} \frac{-\hat\Psi(\varepsilon\beta)2^d}{\pi^2|\beta|^2} v_\beta(x)v_\beta(y),$$

in $L^2(D\times D)$ and a.e., where $\hat\Psi(\xi)$ is a rapidly decreasing function

• $v_\beta(x) = \sin(\beta_1\pi x_1)\dots\sin(\beta_d\pi x_d)$, $\beta \in I^d = \{1,2,\cdots\}^d$ orthogonal complete system in $L^2(D)$, with $\|v_\beta\|_{L^2(D)} = 2^{-d/2}$

Discrete approximations

The grid

$$\mathcal{G}_n = \left\{ \frac{j}{n} = \left(\frac{j_1}{n}, \ldots, \frac{j_d}{n} \right) : \ j_k = 0, 1 \ldots, n, \ k = 1, \ldots, d \right\} \subset [0,1]^d$$

$$D_j = \left[\frac{j_1}{n}, \frac{j_1 + 1}{n} \right) \times \cdots \times \left[\frac{j_d}{n}, \frac{j_d + 1}{n} \right)$$

Finite dimensional approximations based on the grid

$$\sum_{j=1}^{d} n^2 \left[u_n\left(\frac{i - e_j}{n} \right) - 2u_n\left(\frac{i}{n} \right) + u_n\left(\frac{i + e_j}{n} \right) \right]$$
$$= f\left(u_n\left(\frac{i}{n} \right) \right) + g_n\left(\frac{i}{n} \right) + n^d F(D_i),$$

$\|g - g_n\|_{L^{\alpha'}(D)} \leq C/n, \ i \in I_n^d = \{1, \cdots, n\}^d$

The previous equation is of the form $Au = w$, with

$$(Au)_i = \sum_{j=1}^{d} n^2 \big[u_{i-e_j} - 2u_i + u_{i+e_j}\big], u \in \mathbb{R}^{(n-1)^d}.$$

The set of vectors of $\mathbb{R}^{(n-1)^d}$

$$\left\{\left(\frac{2}{n}\right)^{d/2} U_\beta\right\}_{\beta \in I_n^d}, \quad (U_\beta)_i = v_\beta\left(\frac{i}{n}\right), \quad i \in I_n^d,$$

form an orthonormal basis of $\mathbb{R}^{(n-1)^d}$ of eigenvectors of A with eigenvalues

$$\lambda_\beta = -\pi^2((\beta^1)^2 c_{\beta^1} + \cdots + (\beta^d)^2 c_{\beta^d}).$$

Thus,

$$Au = U^t \Lambda U u,$$

Λ diagonal matrix, $\Lambda_{i,i} = \lambda_{\beta_i}$. (lexicographic order in I_n^d)

Smoothing the discrete scheme: $u_n^\varepsilon(\frac{j}{n}), \frac{j}{n} \in \mathcal{G}_n$

Notation: For $i \in I_n^d$,

$\mathbf{F} = \{F(D_i)\}$, $\tilde{\Lambda} = \text{diag}\left(\frac{\lambda_{\beta_i}}{\widehat{\Psi}(\varepsilon\beta_i)}\right)$,

$\kappa_n(x) = \frac{j}{n}$, for $x \in D_j$.

- Boundary conditions: $u_n^\varepsilon(\frac{j}{n}) = 0, j_k = 0, n.$

- $i \in I_n^d$: $u_n^\varepsilon(i/n)$ solution to

$$U^t \tilde{\Lambda} U \, u_n^\varepsilon = f(u_n^\varepsilon) + g_n + n^d \mathbf{F},$$

- $u_n^\varepsilon(x) = u_n^\varepsilon(\kappa_n(x))$,

u_n^ε, $x \in D$, satisfies

$$u_n^\varepsilon(x) = \int_D G_{D,n}^\varepsilon(x,y) f(u_n^\varepsilon(y))\, dy + \int_D G_{D,n}^\varepsilon(x,y) g_n(y)\, dy + \int_D G_{D,n}^\varepsilon(x,y) dF(y),$$

where

$$G_{D,n}^\varepsilon(x,y) = \sum_{\beta \in I_n^d} \frac{\hat{\Psi}(\varepsilon\beta) 2^d}{\lambda_\beta} v_\beta(\kappa_n(x)) v_\beta(\kappa_n(y)).$$

- A solution to this equation exists

Lemma Set $\varepsilon := \varepsilon_n = n^{(2d-4-\theta)/\theta}$ with $\theta > 2d-4$. For any $p \in [1, \alpha']$,

$$\sup_{n \geq 1} \left(\|u_n^\varepsilon\|_{L^p(\Omega; L^{\alpha'}(D))} \right) \leq C(\theta).$$

Lemma Set $\varepsilon := \varepsilon_n = n^{(2d-4-\theta)/\theta}$ with $\theta > 2d-4$. For any $\gamma > 0$, there exists a constant $C(\gamma)$ such that

$$\sup_{x \in D} \|G_D^{\varepsilon}(x) - G_{D,n}^{\varepsilon}(x)\|_{L^2(D)} \leq C(\gamma) n^{-\gamma}.$$

Comparison of Green kernels

Lemma Fix $\lambda \in (0,1)$. Assume $\varphi \in \mathcal{L}^\alpha$, $\alpha \in [1, \frac{d}{(d-2)(2-\lambda)\vee(d-1)\lambda})$. Set $\varepsilon := \varepsilon_n = n^{(2d-4-\theta)/\theta}$ with $\theta > 2d-4$.

There exists a constant $C := C(\lambda, \theta, d)$ not depending on n, such that

$$\|G_D - G^{\varepsilon}_{D,n}\|_{L^\alpha(D\times D)} \leq Cn^{-\gamma},$$

with $\gamma = \frac{\lambda}{\theta}(\theta + 4 - 2d)$.

Remarks

- The constant θ is related to the Fourier transform of Ψ: $\forall \theta \in (0,\infty)^d$, $\exists C(\theta)$ such that $|\xi|^\theta |\hat{\Psi}(\xi)| \leq C(\theta)$.

- For $\theta \sim \infty$, $\gamma \sim \lambda$

Convergence result

Theorem Let $\varepsilon := \varepsilon_n = n^{\frac{2d-4-\theta}{\theta}}$ with $\theta > 2d-4$.

Assume:
(a) $\varphi \in \mathcal{L}^\alpha$, with $\alpha \in (1, \frac{d}{(d-2)(2-\lambda)\vee(d-1)\lambda})$

(b) the Lipschitz constant L small

Set $\gamma = \frac{\lambda}{\theta}(\theta + 4 - 2d)$. Then, there exists $C(\gamma)$ such that for each $p \in [1, \alpha']$,

$$\|u - u_n^{\varepsilon_n}\|_{L^p(\Omega; L^2(D))} \leq C(\gamma) n^{-\gamma \frac{\alpha}{2\alpha'}}.$$

Open problem

What about $\|u - u_n^{\varepsilon_n}\|_{L^p(\Omega;L^{\alpha'}(D))}$?

Poincaré's inequality: For any $u \in W_0^{1,p}(D)$, $1 \leq p < \infty$,

$$\|u\|_p \leq C_{p,d} \|\nabla u\|_p.$$

When applied with $p = 2$, to the solution of the elliptic boundary value problem, we obtain for $\Phi \in L^2(D)$,

$$\int_D \left(\int_D G_D(x,y)\Phi(y)dy \right) \Phi(x)dx \leq -a \int_D \left(\int_D G_D(x,y)\Phi(y)dy \right)^2 dx.$$

Requirement on the Lipschitz constant L

$L < \min(a, 4d, C^{-1}, C(\theta)^{-1})$,
where

$$\sup_{x \in D} \|G_D(x, \cdot)\|_{L^\alpha(D)} \leq C$$
$$\sup_{n \geq 1} \sup_{x \in D} \|G^{\varepsilon}_{D,n}(x, \cdot)\|_{L^2(D)} \leq C(\theta)$$

The End