Systems of quadratic diophantine inequalities

by

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Let $Q_1, \ldots, Q_r$ be quadratic forms in s variables with real coefficients. We ask whether the system of quadratic inequalities

$$|Q_1(x)| < \epsilon, \ldots, |Q_r(x)| < \epsilon \tag{1}$$

has a nonzero integer solution for every $\epsilon > 0$. In the case $r = 1$ the *Oppenheim conjecture* says that this is true if Q_1 is a non degenerate indefinite quadratic form in $s \geq 5$ variables. In 1987 G. Margulis proved this conjecture under the less restrictive assumption $s \geq 3$ using ergodic theory. Margulis's method does not apply to general systems of quadratic forms. In 1999 V. Bentkus and F. Götze (Ann. Math. **150**, (1999), 977-1027) developed a completely different method to prove the Oppenheim conjecture for $s \geq 9$. Using a multidimensional variant of their method we prove the following theorem. Recall that the real pencil generated by the quadratic forms $Q_1, \ldots, Q_r$ is defined as the set of all forms

$$Q_\alpha = \sum_{i=1}^{r} \alpha_i Q_i$$

where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$, $\alpha \neq 0$.

Theorem. *Let $Q_1, \ldots, Q_r$ be quadratic forms with real coefficients. Then for every $\epsilon > 0$ the system* (1) *has a nonzero integer solution provided that*

- *(i) the system $Q_1(x) = 0, \ldots, Q_r(x) = 0$ has a nonsingular real solution,*
- *(ii) each form in the real pencil is irrational and has rank $> 8r$.*

This generalizes a result of D. Freeman (Journal of Number Theory **89**, 268-307 (2001)) who considered diagonal forms.