

Mass Transportation, Equilibration for Nonlinear Diffusions, and Ricci Curvature

We will give a survey on recent developments in optimal mass transportation. Among others,

- We explain how nonlinear diffusions (e.g. porous medium equ., fast diffusion, McKean-Vlasov equ.) on $\mathbb{R}^n$ or on a Riemannian manifold M are related to gradient flows of certain functionals S (e.g. Rényi entropy functionals) on $\mathcal{P}_2(M)$, the L_2 -Wasserstein space of probability measures on M .
- We show how convexity properties of S on $\mathcal{P}_2(M)$ imply functional inequalities (e.g. Talagrand inqu., logarithmic Sobolev inequality) and equilibration of the underlying diffusion on M . In particular, we deduce the analogue to the Bakry-Émery condition for nonlinear diffusions.
- We introduce and analyze generalized Ricci curvature bounds for metric measure spaces (M, d, m) , based on convexity properties of the relative entropy $Ent(\cdot|m)$. For Riemannian manifolds, $Curv(M, d, m) \geq K$ if and only if $Ric_M \geq K$ on M . For the Wiener space, $Curv(M, d, m) = 1$.

One of the main results is that these lower curvature bounds are stable under convergence. This solves one of the basic problems in this field, open for many years.

The notion of convergence comes from a complete separable metric D on the space of normalized metric measure spaces, again defined in terms of mass transportation.

- We construct a canonical stochastic process on the the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ associated with any given probability measure m on $\mathbb{R}$. This process has an invariant measure which may be characterized as the 'uniform distribution' on $\mathcal{P}_2(\mathbb{R})$ with weight function $\frac{1}{Z} \exp(-\beta \cdot Ent(\cdot|m))$.