Bundles of semichained sets and their representations

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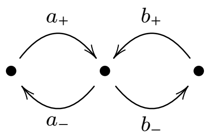
Remark of the translators

This translation was composed for pragmatic use. It is likely that mistakes were made during the translation. When in doubt, the reader should consult the original.

We are grateful to any mistakes pointed out to us and would be thankful if you could email those to uhansper@math.uni-bielefeld.de such that corrections can be made.

Also, we would like to mention that this translation intentionally is made close to the original such that the structure is not very neatly arranged, but it is easier to compare to the original this way. The translation uses in high means terms of the English version of a following paper also by V. M. Bondarenko, called "Representations of bundles of semichained sets, and their applications", published 1992 in St. Petersburg J., Vol. 3, No.5. We recommend reading the introduction and first chapter of this second paper before studying this one.

In [1], the problem of describing the representations of the quiver



with relations $a_+a_- = b_+b_-$ is considered, which was posed by I. M. Gelfand on the International Congress in Nice [2] in connection with the classification of Harish-Chandra-modules in a given special point for $\mathrm{SL}(2, R)$. In the solution of this problem there arose a certain class of matrix problems (representations of sets X of special structures, [1], §1) which are interesting by themselves. Many problems in representation theory reduce to such problems (cf. for example [3] - [7]).

With the help of the self-reproducing matrix problem method in §2 [1] it was proved that the indicated problems have tame type, and in §3 and §4 an algorithm of the construction of the indecomposable representations of the set X is considered.
In the present work we explicitly look at the indecomposable representations of a set X and remove certain inexactnesses and assumptions in [1]. Additionally in §1 we consider a wider class of matrix problems than in [1].

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1 Definition of bundles and their representations

A *semichained set* or simply a *semichain* is an arbitrary (finite) partially ordered set which does not contain subsets of the form $(1,1,1)$ and $(1,2)$ [8] [1]. It is evident that a semichain Π is uniquely represented in the form $\bigcup_{i=1}^{n} \Pi_i$, where each Π_i is of the form (1) or $(1,1)$ and $\Pi_i < \Pi_j$ for $i < j$ (that is, $x < y$ for all $x \in \Pi_i$, $y \in \Pi_j$). The set Π_i is called a *link* of the semichain Π. The set of points of the semichain Π that are comparable with all points $x \in \Pi$, will be denoted by Π^0.
Let $S = \{A_1, \dots, A_n, B_1, \dots, B_n\}$, $n \geq 1$, be some family of (pairwise disjoint) semichains, where $A_i \neq \varnothing$ or $B_i \neq \varnothing$ for each $1 \leq i \leq n$, and let α_0 be an involution on $S^0 = \left(\bigcup_{i=1}^{n} A_i^0\right) \cup \left(\bigcup_{i=1}^{n} B_i^0\right)$. The pair (S, α_0) is called a *bundle of semichains* $A_1, \dots, A_n, B_1, \dots, B_n$. The collection of all bundles $\overline{S} = (S, \alpha_0)$ is denoted by $\mathfrak{X}_0$.
We introduce for a bundle (S, α_0) the following sets: $A = \bigcup_{i=1}^{n} A_i$, $B = \bigcup_{i=1}^{n} B_i$, $S_i = A_i \cup B_i$. When we consider block matrices, $\dim P$ denotes the

[1] $(n_1, \dots, n_k)$ denotes the union of incomparable chains (i.e. linearly ordered sets) $Z_1, \dots, Z_k$ which contain $n_1, \dots, n_k$ elements.

number of rows (columns) of a horizontal (vertical) band P.
A *representation* of the bundle $\overline{S} = (S, \alpha_0)$ over a field k is a collection $\mathcal{U} = \{U_1, \dots, U_n\}$ of block matrices with coefficients in k, satisfying the following conditions:

(1) for each $1 \leq i \leq n$ there is a 1-1-correspondence between the points of the semichain A_i (B_i) and the horizontal (vertical) bands of the matrix U_i. We denote by $P(x)$ the band with number $x \in A \cup B$ (which belongs to U_i if $x \in S_i$);

(2) if $y = \alpha_0(x)$, then $\dim P(x) = \dim P(y)$;

(3) if $x < y$, where $x, y \in A_i$ (B_i), then the band $P(x)$ lies in the matrix U_i above (left of) the band $P(y)$.

Note that certain of the bands $P(x)$ can be empty. A representation $\mathcal{U}$ is called *exact* if $\dim P(x) \neq 0$ for all $x \in A \cup B$, and *inexact* otherwise. The *dimension of a representation* $\mathcal{U}$ is given by the sums of the numbers of rows and columns of all matrices U_i $(1 \leq i \leq n)$.
We will call the following sets of transformations of the matrices $U_1, \dots, U_n$ *admissible*:

(1) we can do arbitrary elementary transformations on rows (columns) within the band $P(x)$ where $x \in A$ (B); but in case that $y = \alpha_0(x)$, $y \neq x$, it is necessary to do the same transformation within the rows (columns) of the band $P(y)$, if $y \in A$ (B), and the inverse transformation within the columns (rows) of the band $P(y)$ if $y \in B$ (A);

(2) if $x < y$, where $x, y \in A_i$ (B_i), then one can add any multiplicative of a row (column) of $P(x)$ to a row (column) of $P(y)$ in the matrix U_i (here, *multiplicative* means multiplied by an element of the field k).

Two representations are called *equivalent*, if one can be obtained from the other by admissible transformations. Equivalence is denoted, as always, by $\simeq$.
Indecomposable and direct sums of representations are defined naturally. [2]
We remark that the theorem of Krull-Schmidt holds for representations of bundles (cf. for example [9]).

[2] Note that there are indecomposable representations which are empty representations (see [1]). More precisely, if $x \in A \cup B$ and either $x \notin S^0$ or $x \in S^0, \alpha_0(x) = x$, then there is an indecomposable representation I_x of dimension 1, for which $\dim P(x) = 1$ and $\dim P(y) = 0$ for $y \neq x$. If $\alpha_0(x) = y$ where $x \neq y, x \in A_i$ (B_i), $y \notin B_i$ (A_i), then the empty indecomposable representation $I_{x,y} = I_{y,x}$ has dimension 2 and is given by the equality: $\dim P(x) = \dim P(y) = 1$ and $\dim P(z) = 0$ for the other bands. The "empty" representation I_0 for which $\dim P(x) = 0$ for all $x \in A \cup B$, is not considered indecomposable.

2 X-chains and X-cycles

We denote by $X(\Pi)$ linear ordered sets consisting of links of the semichain Π (cf. §1). A link $\Pi_i = \{x\}$ will be identified with the point x. Links consisting of two points will also be denoted by one small letter x and the points of the link by x^+ and x^-. The number of points in the link x is denoted by $r(x)$. Let $\overline{S} = (S, \alpha_0)$ be a bundle of semichains $A_1, \dots, A_n, B_1, \dots, B_n$. Let $E_i = X(A_i)$, $F_i = X(B_i)$, $X_i = E_i \cup F_i$ $(1 \leq i \leq n)$, $E = \bigcup_{i=1}^n E_i$, $F = \bigcup_{i=1}^n F_i$. The union of the sets E and F is denoted by $X(S)$, or simply by X.
We introduce on the set X two binary operations α and β. Namely, two elements a and b from X are in relation α if and only if either $a \neq b$, $r(a) = r(b) = 1$ and $\alpha_0(a) = b$, or $a = b$ and $r(a) = 2$. The relation β is defined as $\beta(a, b)$ if and only if $a \in E_i$, $b \in F_i$ or $a \in F_i$, $b \in E_i$, $1 \leq i \leq n$. [3] We remark that if $r(a) = 1$ and $\alpha_0(a) = a$, then $\overline{\alpha}(a, x)$ for all $x \in X$. The relation β will also be represented by a straight mark (–) and the relation α by a wavy (~). The collection of sets X with the indicated structure is denoted by $\mathfrak{X}$. Clearly, there exists a natural 1-1-correspondence between the elements of the sets $\mathfrak{X}_0$ and $\mathfrak{X}$.
We will now introduce the notion of an X-graph [1].
Let Γ be the set of finite connected graphs C consisting of chains

$$c_1 \text{ —— } c_2 \text{ —— } \dots \text{ —— } c_m, \quad (m \geq 1)$$

and cycles

$$c_1 \text{ —— } c_2 \text{ —— } \dots \text{ —— } c_m \quad (m \geq 2).$$

An X-*graph* is given by a function g defined on an arbitrary $C \in \Gamma$ which puts each point $c_i \in C$ in correspondence with an element $g(c_i) \in X$ and each edge $\rho \in C$ in correspondence with a relation $g(\rho) \in \{\alpha, \beta\}$. Moreover it satisfies the following conditions:

(a) if ρ connects the points c_i and c_{i+1} in C, then $g(c_i)$ and $g(c_{i+1})$ satisfy the relation $g(\rho)$;[4]

(b) if ρ and δ are neighbouring edges in C, then $g(\rho) \neq g(\delta)$.

An X-graph corresponding to a chain (cycle) C is called an X-*chain* (X-*cycle*). An X-chain defined on $C \in \Gamma$ is called *admissible* if $\alpha(a, b)$ with $a \neq b$ and $g(c_i) = a$ implies the existence of an edge $\gamma \in C$ connecting the points c_i and c_j ($j = i - 1$ or $j = i + 1$) such that $g(c_j) = b$ and $g(\gamma) = \alpha$ (for an X-cycle, this condition is always satisfied). The set of admissible X-chains is denoted by $L(X)$ and the set of X-cycles by $Z(X)$. We define $\Gamma(X) = L(X) \cup Z(X)$. Thus, an X-chain (X-cycle) g gives sequences $g_0 = \{a_1, \dots, a_m\}$ of elements of

[3] We write $\gamma(a, b)$ ($\overline{\gamma}(a, b)$) for $\gamma = \alpha$ or $\gamma = \beta$ if the relation γ is satisfied (not satisfied) for the elements a and b.

[4] For a cycle, the indices $i < 1$ and $i > m$ are always considered modulo m.

X and $g_1 = \{\gamma_{1,2}, \gamma_{2,3}, \dots, \gamma_{m-1,m}\}$ ($g_1 = \{\gamma_{1,2}, \gamma_{2,3}, \dots, \gamma_{m-1,m}, \gamma_{m,1}\}$) where $\gamma_{i,i+1} \in \{\alpha, \beta\}$, $\gamma_{i-1,i} \neq \gamma_{i,i+1}$ and $\gamma_{i,i+1}(a_i, a_{i+1}) \in X$. The number m will be called *length* of the X-chain (X-cycle) and be denoted by $|g|$. It is evident that the length of an arbitrary X-cycle is even.
The left (right) end a_1 (a_m) of an admissible chain is called *double* if $\gamma_{1,2} \neq \alpha$ ($\gamma_{m-1,m} \neq \alpha$) [5] and $\alpha(a_1, a_1)$ ($\alpha(a_m, a_m)$) holds in X. The number of double ends of an X-chain $g \in L(X)$ is denoted by $\mathrm{d}(g)$ [6].
Let G be the automorphism group of the graph $C \in \Gamma$. Then the action of G defined on C naturally carries over to the set of X-graphs on C. Two X-graphs are called *equivalent* if they are defined on one and the same C and they can be transformed into each other by some element of the group G. Each $s \in G$ such that $s(g) = g$ is called *automorphism* on the X-graph g, i.e. $s(a_i) = a_i$ for every element of the set X and $s(\gamma_{i,i+1}) = \gamma_{i,i+1}$ for an element of the set $\{\alpha, \beta\}$ (for any $a_i \in g_0, \gamma_{i,i+1} \in g_1$). An automorphism s of the X-cycle g is called *rotation* if s translates a_i into a_{i+k} ($1 \le i \le m$), where k is an integer not depending on i. The group of automorphisms of the X-graph g is denoted by $\mathrm{Aut}(g)$. An X-chain (X-cycle) g is called *symmetric* (*symmetric*) if the group $\mathrm{Aut}(g)$ (factor group $\mathrm{Aut}(g)$ by subgroup of rotations) is non-trivial.
The natural form of an X-*subchain* (or simply called *subchain*) of an X-graph g is given by the restriction of the function g to the connected subchain $C' = \{\, c_i \text{ --- } c_{i+1} \text{ --- } \dots \text{ --- } c_{i+k} \,\}$ of the graph C.
Let $g^{(1)}$ and $g^{(2)}$ be two admissible X-chains such that the right point $a_m \in g_0^{(1)}$ and the left point $b_1 \in g_0^{(2)}$ are double and such that $b_1 = a_m$. If the two points a_m and b_1 are connected by relation α, then we obtain a new X-chain which will be denoted by $g^{(1)} \sim g^{(2)}$. Analogously we define X-chains $g^{(1)} \sim g^{(2)} \sim \dots \sim g^{(k)}$ for any $k \ge 2$.
Let h be an X-chain, $h_0 = \{b_1, \dots, b_s\}$, $h_1 = \{\gamma_{1,2}, \dots, \gamma_{s-1,s}\}$. We denote by h^* the following X-chain: $h_0^* = \{b_1^*, \dots, b_s^*\}$, $h_1^* = \{\gamma_{1,2}^*, \dots, \gamma_{s-1,s}^*\}$, where $b_1^* = b_s$, $b_2^* = b_{s-1}$, $\dots$, $b_{s-1}^* = b_2$, $b_s^* = b_1$ and $\gamma_{1,2}^* = \gamma_{s-1,s}$, $\dots$, $\gamma_{s-1,s}^* = \gamma_{1,2}$. If both ends of h are double, then there exists an X-chain of form $h^{(1)} \sim h^{(2)} \sim \dots \sim h^{(k)}$ ($k \ge 1$), where $h^{(i)} = h$ for odd i and $h^{(i)} = h^*$ for even i. An X-chain of this form will be denoted by $h^{[k]}$. If h only has a double point at the right end of h, then one can construct $h^{[k]}$ only for $k \le 2$. We remark that any X-chain $g \in L(X)$ is represented in the form $g = h^{[k]}$, where h is a simple X-chain and $k \ge 1$, clearly. This fact follows from the following lemma which is also necessary in the following paragraph.

Lemma 1. If u is a simple X-chain of which both ends are double, and $u \sim u^* \sim u = v \sim v^* \sim w$, then $|v| \le |u|$.

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[5] Writing $\gamma_{1,2} \neq \alpha$ ($\gamma_{m-1,m} \neq \alpha$) means either $\gamma_{1,2} = \beta$ ($\gamma_{m-1,m} = \beta$) or length g is equal to 1.

[6] For an X-chain $g = \{a_1\}$ with $\alpha(a_1, a_1)$ it is natural to assume $\mathrm{d}(g) = 1$.

Proof. We suppose otherwise and choose u and v such that the difference $|v| - |u|$ is minimal. Let $v = u \sim v'$. Then by the assumption of the lemma $u^* = (v')^{[k]} \sim u'$ where $k > 1$ and $0 < |u'| < |v'|$ and also, if k is odd (even), then $(v')^* = u' \sim v''$ ($v' = u' \sim v''$) and for this $v'' \sim (v'')^* \sim (u')^* \sim u' = (u')^* \sim u' \sim v'' \sim (v'')^*$. From this equality we easily obtain that $u^* \sim u \sim u^* = \overline{v} \sim \overline{v^*} \sim \overline{w}$ where $\overline{v} = u^* \sim (u')^*$, $\overline{w} = (v')^{[k-1]} \sim (v'')^*$. This contradicts the choice of u and v, since $|u'| < |v'|$. □

We denote by $L_0(X)$ the set of all simple admissible X-chains, and by $Z_0(X)$ the set of X-cycles with trivial group of rotations. We set $\Gamma_0(X) = L_0(X) \cup Z_0(X)$. It is clear that $|\operatorname{Aut}(g)| = 1$ for all $g \in L_0(X)$ and $|\operatorname{Aut}(g)| \leq 2$ for all $g \in Z_0(X)$. If g is a symmetric X-cycle in $Z_0(X)$ and $g_0 = \{a_1, \dots, a_m\}$, we set $\sigma_0(g) = \frac{1}{2}\sigma(g)$ where $\sigma(g)$ is the number of pairs (a_i, a_{i+1}) such that $a_i, a_{i+1} \in E$ or $a_i, a_{i+1} \in F$ and additionally also $a_i \neq a_{i+1}$. Let $P_i(X) = \{g \in L_0(X) \mid \mathrm{d}(g) = i\}$, $i \in \{0, 1, 2\}$; N the set of natural numbers; $N_j = \{1, \dots, j\}$. We put $P(X) = P_0(X) \cup [P_1(X) \times N_2] \cup [P_2(X) \times N_4 \times N]$.
We denote by $k_0[t]$ the set of all irreducible polynomials over the field k (with highest coefficient 1). For an element $a \in k$ we set $K_a = \{\varphi_0^K \mid \varphi_0 \in k_0[t], \varphi_0 \neq t, t + a, K \in N\}$. We further take

$$Q_1(X) = \{g \in Z_0(X) \mid |\operatorname{Aut}(g)| = 1\},$$
$$Q_2^1(X) = \{g \in Z_0(X) \mid |\operatorname{Aut}(g)| = 2, \sigma_0(g) \text{ odd}\},$$
$$Q_2^2(X) = \{g \in Z_0(X) \mid |\operatorname{Aut}(g)| = 2, \sigma_0(g) \text{ even}\},$$

and

$$Q(X, k) = [Q_1(X) \times K_0] \cup [Q_2^1(X) \times K_{-1}] \cup [Q_2^2 \times K_1].$$

Let finally $\mathcal{I}(X, k) = P(X) \cup Q(X, k)$.
It will follow from the main theorem (of the following paragraph) that there exists a 1-1-correspondence between the elements of $\mathcal{I}(X, k)$ (considered up to the equivalence of X-graphs) and equivalence classes of indecomposable representations of the bundle $\overline{S}$.

3 Main results

3.1 Orientation of X-graphs. Elementary subchains.

Let g be an X-graph and $g_0 = \{a_1, \dots, a_m\}$. Denote by $\mathcal{D}(g_0)$ the set of pairs (a_i, a_{i+1}) such that $a_i = a_{i+1}$ (then evidently: $\gamma_{i,i+1} = \alpha$). A mapping ε from the set $\mathcal{D}(g_0)$ into the set $\{1, -1\}$ will be called *orientation* of the X-graph g.
We define for each $g \in \Gamma(X)$ a certain orientation ε_0 which will play a fundamental role for the construction of the canonical representations of the bundle $\overline{S}$.

We consider first the case where g is a simple X-chain. We will introduce for each $1 \le i \le m$ elements $x_i, y_i \in X \cup \{0\}$ such that $(a_i, a_{i+1}) \in \mathcal{D}(g_0)$. We insert g in the following X-chain:

$$\tilde{g} = \begin{cases} g & \text{if } \mathrm{d}(g) = 0, \\ g^* \sim g \ (g \sim g^*) & \text{if } \mathrm{d}(g) = 1 \text{ and } a_1 \ (a_m) \text{ is double end}, \\ g^* \sim g \sim g^* & \text{if } \mathrm{d}(g) = 2. \end{cases}$$

Let $g(i)$ be the maximal subchain of $\tilde{g}$ of the form $w \sim w^*$ where the right end of w coincides with the element a_i of the chain g. We remark that $s = |w| = \frac{1}{2}|g(i)|$ is an odd number (in case of $\mathrm{d}(g) = 2$ this follows from Lemma 1). If $g(i)$ does not contain the left (right) end of the X-chain $\tilde{g}$, then we denote by x_i (y_i) the element of $\tilde{g}_0$ which is connected in $\tilde{g}$ by the relation β with the left (right) end of the subchain $g(i)$. Otherwise we set $x_i = \infty$ $(y_i = \infty)$ [7]. We remark that in all cases $x_i \ne y_i$, moreover, if $x_i \in E_j \cup \{\infty\}$ $(F_j \cup \{\infty\})$ then y_i also belongs to this set. Therefore the elements x_i and y_i are always comparable (we consider $\infty > x$ for all $x \in X$). We define now for an X-chain $g \in L_0(X)$ the orientation ε_0, deeming that $\varepsilon_0(a_i, a_{i+1}) = 1$ $(\varepsilon_0(a_i, a_{i+1}) = -1)$ in the following cases:

a) $x_i < y_i$ $(x_i > y_i)$ and either $a_i \in E$, $x_i \in E \cup \{\infty\}$ or $a_i \in F$, $x_i \in F \cup \{\infty\}$,

b) $x_i > y_i$ $(x_i < y_i)$ and either $a_i \in E$, $x_i \in F \cup \{\infty\}$ or $a_i \in F$, $x_i \in E \cup \{\infty\}$.

Let now g be a composite X-chain. We represent it in the form $g = h^{[k]} = h^{(1)} \sim \dots \sim h^{(k)}$, where h is a simple X-chain and $k > 1$. Let $|h| = p$ and $h_0^{(i)} = \{a_1^{(i)}, \dots, a_p^{(i)}\}$. The orientation ε_0 is already defined for each simple subchain $h^{(i)}$, and at the "joints" ε_0 is defined by $\varepsilon_0(a_p^{(i)}, a_1^{(i+1)}) = 1$ if $a_i \in E$ and $\varepsilon_0(a_p^{(i)}, a_1^{(i+1)}) = -1$ if $a_i \in F$ $(1 \le i \le k)$.
We consider now the case where g is an X-cycle. We denote by $\overline{\mathcal{D}}(g_0)$ the set of pairs $(a_i, a_{i+1}) \in \mathcal{D}(g_0)$ for which there exists an automorphism $z \in \mathrm{Aut}(g)$ which transfers a_i into a_{i+1} (and thus a_{i+1} into a_i). Clearly, $\overline{\mathcal{D}}(g_0) \ne \varnothing$ only for symmetric X-cycles. If $(a_i, a_{i+1}) \notin \overline{\mathcal{D}}(g_0)$, then $\varepsilon_0(a_i, a_{i+1})$ is defined in the same way as for a simple X-chain g with $\mathrm{d}(g) = 0$ (in this case affine subchains $g(i)$ of the X-cycle g have length $2s < m$ where s is odd), and if $(a_i, a_{i+1}) \in \overline{\mathcal{D}}(g_0)$, then in the same way as "at the joints" of composite X-chains, i.e. $\varepsilon_0(a_i, a_{i+1}) = 1$ (-1), if $a_i \in E$ (F).
In the case where $(a_i, a_{i+1}) \in \mathcal{D}(g_0)$ and $\varepsilon_0(a_i, a_{i+1}) = 1$ $(\varepsilon_0(a_i, a_{i+1}) = -1)$, we will write $\overrightarrow{a_i \sim a_{i+1}}$ $(\overleftarrow{a_i \sim a_{i+1}})$.
Any subchain of the following form is called *elementary subchain* of the X-graph $g \in \Gamma(X)$:

[7] In other words, if $\gamma_{1,2} = \alpha$, set $a_0 = \infty$, and if $\gamma_{m-1,m} = \alpha$, set $a_{m+1} = \infty$ $(\gamma_{1,2}, \gamma_{m-1,m} \in g_1)$. Then, in all cases we have: $x_i = a_{i-s}$ for $s \le i$, $x_i = a^*_{m-s+i} = a_{s-i+1}$ for $s > i$, $y_i = a_{i+1+s}$ for $s \le m-i$ and $y_i = a^*_{s-m+i+1} = a_{2m-s-i}$ for $s > m-i$.

1) $a_{i-1} - a_i$;

2) $\overrightarrow{a_{i-1} \sim a_i} - a_{i+1}$;

2') $a_{i-1} - \overleftarrow{a_i \sim a_{i+1}}$;

3) $\overrightarrow{a_{i-1} \sim a_i} - \overleftarrow{a_{i+1} \sim a_{i+1}}$.

Lemma 2. An arbitrary X-cycle g contains a maximal elementary subchain of length 2 (i.e. not belonging to an elementary subchain of length 3).

Proof. Let $a_i = a_{i+1}$ each time when $\gamma_{i,i+1} = \alpha$ (in opposite case the assertion is clear). We will suppose that $\gamma_{m,1} = \beta$. We have fixed an element $a_s \in g_0$ such that $a_s \in E_k$ for some $1 \leq k \leq n$ and $a_i \geq a_s$ if $a_i \in E_k$. We set $C' = \{a_i \mid a_{i-1} = a_s, i \text{ odd}\}$, $C'' = \{a_i \mid a_{i+1} = a_s, i \text{ even}\}$ and $C = C' \cup C''$. We remark that each element of C belongs to F_k. If a_j is some minimal element of C (with respect to the ordering on F_k) and $a_j \in C'$ ($a_j \in C''$) then it is easily seen that $\varepsilon_0(a_{j-2}, a_{j-1}) = -\varepsilon_0(a_j, a_{j+1})$ ($\varepsilon_0(a_{j-1}, a_j) = -\varepsilon_0(a_{j+1}, a_{j+2})$), whence the assertion of the lemma follows. □

3.2 Canonical representations. Main theorem.

Let $g \in \Gamma_0(X)$, $g_0 = \{a_1, \dots, a_m\}$ and $g_{0,1} = \{a_i \in g_0 \mid \alpha(a_i, a_i)\}$. Denote by $\Psi(g)$ the set of mappings $\psi : g_{0,1} \to \{1, -1\}$ such that $\psi(a_i) = 1$ ($\psi(a_i) = -1$) each time that $a_i = a_{i+1}$ ($a_i = a_{i-1}$). If $a_i \neq a_{i+1}$ and $a_i \neq a_{i-1}$ ($a_i \in g_{0,1}$) then $\psi(a_i)$ can be equal to 1 or -1 (in this case, clearly, $g \in L_0(X)$ and a_i is a double end of g). Thus, if g is an X-chain without double ends or an X-cycle, then $\Psi(g)$ consists of one mapping which is denoted ψ_1. If g is a X-chain with one double end a_1 (a_m) then $\Psi(g)$ consists of two mappings ψ_1 and ψ_2; for definiteness we will suppose that $\psi_1(a_1) = -1$ and $\psi_2(a_1) = 1$ ($\psi_1(a_m) = 1$, $\psi_2(a_m) = -1$) [8]. Finally, if g is an X-chain for which $\mathrm{d}(g) = 2$, then $\Psi(g)$ consists of four maps ψ_s, $1 \leq s \leq 4$; we will suppose that $\psi_1(a_1) = -1$, $\psi_1(a_m) = 1$, $\psi_2(a_1) = 1$, $\psi_2(a_m) = 1$, $\psi_3(a_1) = -1$, $\psi_3(a_m) = -1$, $\psi_4(a_1) = 1$, $\psi_4(a_m) = -1$. Each map $\psi_s \in \Psi(g)$ induces a map $\psi_s^* \in \Psi(g^*)$ which acts on each $a_i \in g_{0,1}$ with opposite sign, i.e. $\psi_s^*(a_j^*) = -\psi_s(a_{m+1-j})$ for each $a_j^* \in g_{0,1}^*$ (cf. page 5).
We denote by $\delta(a_i)$, where $a_i \in g_0$, the number of those $a_j \in g_0$, $0 < j \leq i$, for which $a_j = a_i$ and by $\delta(a, g)$, for $a \in X$, the number of elements $a_j \in g_0$ equal to a. If $a_i \in g_{0,1}$ and $0 < s \leq |\Psi(g)|$, then we denote by $\delta_s(a_i)$ the number of those $a_j \in g_0$, $0 < j \leq i$, for which $a_j = a_i$ and $\psi_s(a_j) = \psi_s(a_i)$. By $\delta_s^+(a, g)$ ($\delta_s^-(a, g)$) where $a \in X$, $\alpha(a, a)$, we denote the number of elements $a_j \in g_{0,1}$ such that $a_j = a$ and $\psi(a_j) = 1$ ($\psi(a_j) = -1$).

[8] The X-chain $g = \{a_1\}$, where $\alpha(a_1, a_1)$, has one double end. In this case we will suppose that $\psi_1(a_1) = -1$, $\psi_2(a_1) = 1$.

We associate now with each X-graph $g \in \Gamma_0(X)$, where $X = X(S)$, a representation of special form for the bundle $\overline{S} = (S, \alpha_0)$. Namely, we associate an X-chain $g \in L_0(X)$ with representations $\mathcal{U}_s(g)$ if $\mathrm{d}(g) \leq 1$ and with representations $\mathcal{U}_s(g,p)$ if $\mathrm{d}(g) = 2$, where $1 \leq s \leq |\Psi(g)|$ and p is any natural number. We associate an X-cycle $g \in Z_0(X)$ with representations $\mathcal{U}(g,\varphi)$, where $\varphi = \varphi(t)$ is a polynomial equal to a power of an irreducible polynomial φ_0 over the field k (with highest coefficient 1), moreover $\varphi_0 \neq t$ for asymmetric g and $\varphi_0 \neq t, t+1$ ($\varphi_0 \neq t, t-1$) for symmetric g for even (odd) $\sigma_0(g)$.
We construct first for X-chains $g \in L_0(X)$ representations of the form $\mathcal{U} = \mathcal{U}_s(g)$ and $\mathcal{U} = \mathcal{U}_s(g,1)$, $(1 \leq s \leq |\Psi(g)|)$.
We establish, first of all, in a 1-1-manner a correspondence between rows and columns of the "future" matrices $U_1, \dots, U_n$ of the representation $\mathcal{U}$ and the elements from g_0. If $a \in E_k$ ($a \in F_k$) and $\overline{\alpha}(a,a)$, then the band $P(a)$ of the matrix U_k consists of $\delta(a,g)$ rows (columns), moreover to the j-th row (column) of this band corresponds an element $a_i \in g_0$, equal to a and such that $\delta(a_i) = j$. If however $a \in E_k$ ($a \in F_k$) and $\alpha(a,a)$, then the band $P(a^+)$ of the matrix U_k consists of $\delta_s^+(a,g)$ rows (columns) and $P(a^-)$ of $\delta_s^-(a,g)$ rows (columns). To this j-th row (column) of the band $P(a^\pm)$ corresponds an element $a_i \in g_0$ equal to a and such that $\psi_s(a_i) = \pm 1$ and $\delta_s(a_i) = j$. We will always suppose that the band $P(a^+)$ stands above (left of) the band $P(a^-)$.
In the matrices U_k $(1 \leq k \leq n)$ stands at the intersection of a row corresponding to an element $a_i \in g_0$ and a column corresponding to an element $a_j \in g_0$, the identity element if there is an elementary subchain of length $2, 3$ or 4 in g, having at its ends the elements a_i and a_j, and the zero element otherwise. Thus there exists a 1-1-correspondence between nonzero (identity) elements of the representations $\mathcal{U}$ ($\mathcal{U} = \mathcal{U}_s(g)$ or $\mathcal{U} = \mathcal{U}_s(g,1)$) and elementary subchains in g.
In case when g is an X-chain, it remains to construct representations of the form $\mathcal{U}_s(g,p)$ for $p > 1$.
We set $h = g^{[p]} = g^{(1)} \sim \dots \sim g^{(p)}$, where $g^{(i)} = g$ ($g^{(i)} = g^*$) for odd (even) i. For a composite X-chain h, we define the set $\Psi(h)$ in the following way: $\Psi(h) = \{\overline{\psi}_s \mid \psi_s \in \Psi(g)\}$, where $\overline{\psi}_s$ is a mapping from the set $h_{0,1} = \bigcup_{k=1}^{p} g_{0,1}^{(k)}$ into the set $\{1, -1\}$ which induces the mapping $\psi_s \in \Psi(g)$, i.e. which coincides on each subchain $g^{(i)} = g$ with ψ_s, and on each subchain $g^{(i)} = g^*$ with ψ_s^*. The representation $\mathcal{U}_s(g,p)$ for $p > 1$ is built in the same way as the representation $\mathcal{U}_s(g,1)$ but now g has to be replaced by $h = g^{[p]}$ and ψ_s by $\overline{\psi}_s$ (in particular, $\delta_s(b_i)$, where $b_i \in h_0$, and $\delta_s^\pm(a,h)$ are already defined with respect to $\overline{\psi}_s$).
Let now g be an X-cycle from $Z_0(X)$, $g_0 = \{a_1, \dots, a_m\}$. We denote by h the maximal elementary subchain of the form $a_{i-1} - a_i$ (cf. Lemma 2) with minimal i $(1 \leq i \leq m)$.
The representation $\mathcal{U}(g,\varphi) = \{U_1, \dots, U_n\}$ is built analogously to the repres-

entation $\mathcal{U}_1(g)$ for an X-chain g without double ends. The only difference is that, firstly, in the matrix U_k, $(1 \le k \le n)$, the bands of the form $P(a)$ and $P(a^{\pm})$, where $a \in E_k$ $(a \in F_k)$, consist of, respectively, the subbands $\delta(a,g)$ and $\delta_1^{\pm}(a,g)$, each of which contains $j = \deg(\varphi)$ rows (columns), and, additionally, the elements $a_i \in g_0$ are no longer associated with rows and columns, but rather with the mentioned horizontal and vertical subchain matrices $U_1, \dots, U_n$, and, secondly, to each elementary subchain, except for h, corresponds a unity cell E of the size $j \times j$, and to the subchain h corresponds a Frobenius cell ϕ with characteristic polynomial φ [9].
We remark that if $g = \{\, b \sim\sim b - c \sim\sim c \,\}$, where $b \in E_k$ (F_k), $c \in F_k$ (E_k), then at the intersection of the bands $P(b^+)$ and $P(c^-)$ ($P(b^-)$ and $P(c^+)$) in U_k stands the matrix $E + \phi$, in so far as first and last (second and third) elements of g appear as double ends at the elementary subchains of length 2 and 4.
The representations of form $\mathcal{U}_s(g)$, $\mathcal{U}_s(g,p)$ and $\mathcal{U}(g,\varphi)$ will be called the *canonical representations of the bundle* $\overline{S} = (S, \alpha_0)$.
The class of all canonical representations which correspond to an X-graph g, is denoted by $K(g)$. It is evident that $|K(g)| = |\Psi(g)|$, if $X \in L_0(X)$, $\mathrm{d}(g) < 2$, and $|K(g)| = \infty$ in all remaining cases. Two classes of canonical representations $K(g)$ and $K(h)$ are called *equivalent* if for each representation $\mathcal{U} \in K(g)$ there exists an equivalence with a representation $\mathcal{V} \in K(h)$, and conversely.
The main results of this work are the following assertions:

Main Theorem.

1) An arbitrary indecomposable representation of the bundle $\overline{S} = (S, \alpha_0)$ is equivalent to some canonical representation.

2) All canonical representations are indecomposable.

3) If two X-graphs g and h are equivalent, then the classes $K(g)$ and $K(h)$ are also equivalent. Otherwise $K(g)$ and $K(h)$ do not contain equivalent representations.

This theorem gives a complete classification of the indecomposable representations of the bundle $\overline{S} = (S, \alpha_0)$ (in order to obtain a full list of indecomposable pairs of inequivalent representations of the bundle $\overline{S}$, it is necessary to construct in each equivalence class of the X-graph the canonical representations [10].

[9] If the field k is algebraically closed, then the subchain h corresponds to a Jordan cell with characteristic polynomial φ.

[10] From formal considerations, we take as invariant several other X-graphs, than in [1]. In reality, we get, from our considerations, the enumeration of indecomposable representations, given in Theorem 7 of [1] (for some modification of the definition of a special

Remark. In certain cases, the identity elements (blocks) of canonical representations corresponding to elementary subchains of length 4, can be "removed" with the help of admissible transformations (for example if $g = \{b - \overrightarrow{c \sim c} - \overleftarrow{b \sim b}\}$). However, the definition of canonical representation is easily modified thus, the "new" representations which we will denote by $\mathcal{U}_s^0(g)$, $\mathcal{U}_s^0(g,p)$ and $\mathcal{U}^0(g,\varphi)$ do not already contain "superfluous" identity elements (cells). Let $g \in \Gamma(X)$, $g_0 = \{a_1, \dots, a_m\}$.
If $\gamma_{i,i+1} = \beta$, we define by $g(a_i, a_{i+1})$ the maximal subchain of g which contains the elements a_i, a_{i+1} and any element a_j which is equal to a_i or a_{i+1} (if g is an X-cycle and any element $a_j \in g_0$ is equal to a_i or a_{i+1}, then we take $g(a_i, a_{i+1}) = \{a_{i-1} \sim a_i - \cdots - a_{i-3} \sim a_{i-2}\}$). An elementary subchain $\overrightarrow{a_{i-1} \sim a_i} - \overleftarrow{a_{i+1} \sim a_{i+2}}$ of the X-graph g will be called *important* if $g(a_i, a_{i+1})$ has no double ends and the direction of the arrows of $g(a_i, a_{i+1})$ (which give the direction ε_0 of the X-graph g) alternate. If g is an X-chain and $\gamma_{1,2} = \beta$ ($\gamma_{m-1,m} = \beta$), we denote by $g_0(a_1)$ ($g_0(a_m)$) the set of elements $a_k \in g_0$ that are equal to a_1 (a_m) and belong to some subchain $\overleftarrow{a_i \sim a_{i+1}}$ or $\overrightarrow{a_j \sim a_{j+1}}$, where $a_{i+1} \in g(a_1, a_2)$, $a_{j+2} \in g(a_1, a_2)$ ($a_{i-1} \in g(a_{m-1}, a_m)$, $a_j \in g(a_{m-1}, a_m)$). In all other cases, in particular for an X-cycle g, we take $g_0(a_1) = \varnothing$ ($g_0(a_m) = \varnothing$). We now associate to each mapping $\psi_s \in \Psi(g)$, where g is a simple X-chain, a mapping $\psi_s^0 : g_{0,1} \to \{1, -1\}$, where we take $\psi_s^0(a_i) = \psi_s(a_i)$ if $a_i \notin g_0(a_1) \cup g_0(a_m)$; $\psi_s^0(a_i) = \mp\psi_s(a_1)$ if $a_i \in g_0(a_1)$, $a_i = a_{i\pm1}$; $\psi_s^0(a_i) = \pm\psi_s(a_m)$ if $a_i \in g_0(a_m)$, $a_i = a_{i\pm1}$. The representations $\mathcal{U}_s^0(g)$, $\mathcal{U}_s^0(g,p)$ and $\mathcal{U}^0(g,\varphi)$ are defined in the same way as the canonical representations, but to the elementary subchains of length 4, which are not important, now correspond zero elements (blocks) and, additionally, instead of mappings ψ_s the mappings ψ_s^0 are considered. It is not hard to prove that $\mathcal{U}^0 \cong \mathcal{U}$ for any canonical representation $\mathcal{U}$.

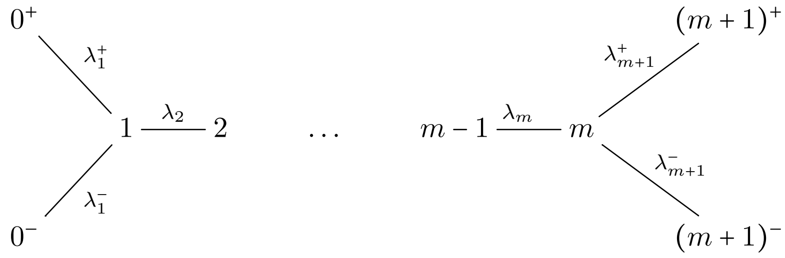
4 Examples

4.1 Quivers

It is clear that the problem of representing a quiver Λ, which has the shape of a cycle (chain), is posed in the shape of a bundle (S, α_0) of singleton sets $A_1 = \{a_1\}, \dots, A_n = \{a_n\}$, $B_1 = \{b_1\}, \dots, B_n = \{b_n\}$, where n is the number of arrows of the quiver. If $\lambda_1, \lambda_2, \dots, \lambda_n$ is some numbering of the arrows of Λ and $\varepsilon_1(\lambda)$ ($\varepsilon_2(\lambda)$) denotes the beginning point (end point) of the arrow λ, then the involution α_0 is given the following way: $\alpha_0(a_i) = a_j$ ($i \neq j$), $\alpha_0(b_i) = b_j$ ($i \neq j$), $\alpha_0(a_i) = b_j$ if $\varepsilon_1(\lambda_i) = \varepsilon_1(\lambda_j)$, $\varepsilon_2(\lambda_i) = \varepsilon_2(\lambda_j)$, $\varepsilon_1(\lambda_i) = \varepsilon_2(\lambda_j)$; additionally, $\alpha_0(a_i) = a_i$ ($\alpha_0(b_i) = b_i$) if Λ is a chain and

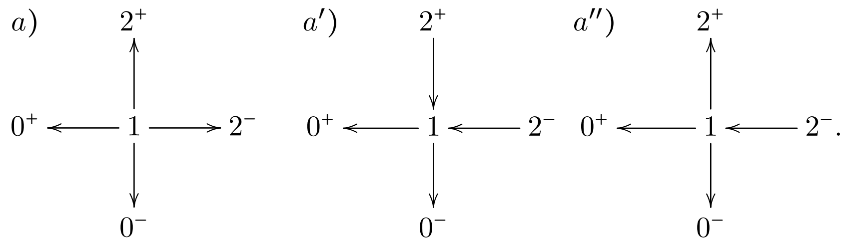
chain, see [1], p. 61). However, representations corresponding to one invariant work [1] do not always belong to one weak equivalence class (in the case when both ends of the X-chain appear as double ends and are equal among themselves, the four corresponding indecomposable representations turn into not one, but two classes).

$\varepsilon_1(\lambda_i)$ $(\varepsilon_2(\lambda_i))$ is one of its links. In particular, for the classical problem of matrix similarity and of a matrix bundle, we get the following: $n = 1$, $\alpha_0(a_1) = b_1$ and $n = 2$, $\alpha_0(a_1) = a_2$, $\alpha_0(b_1) = b_2$. Let Λ now be a quiver of the following type:



$(m \geq 1)$

with some direction on the edges.
Let us assume that $m = 1$. Now it it sufficient to look at (up to duality of quivers) the following cases:



The problem of giving the representations of the quiver a) [10]-[12] is the problem of giving the representations of the bundle (S, α_0) of semichains A_1, A_2, B_1, B_2, where $A_1 = \{p_1^1\}$, $A_2 = \{p_1^2\}$, $B_1 = \{(p_{0^+}, p_{0^-})\}$, $B_2 = \{(p_{2^+}, p_{2^-})\}$ and $\alpha_0(p_1^1) = p_1^2$ [11]. In case a'), the bundles are defined analogously (replacing A_2 by B_2 and vice-versa). The problem of giving the representation of the quiver of a"), is easily reduced to a bundle of semichain sets, however, unlike in cases a) and a'), it cannot be directly represented in this way.
Let T be a (matrix) representation of the quiver a"). We will bring the matrices $T_{\lambda_2^+}$ and $T_{\lambda_2^-}$ to the following structural shape:

$$T_{\lambda_2^+} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline E & 0 & 0 \\ \hline 0 & E & 0 \end{array}\right), \qquad T_{\lambda_2^-} = \left(\begin{array}{c|c|c|c} E & 0 & 0 & 0 \\ \hline 0 & 0 & E & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

(the horizontal bandstructure of $T_{\lambda_2^+}$ corresponds to the vertical bandstucture of $T_{\lambda_2^-}$) and we make the corresponding subdivision in the matrices $T_{\lambda_1^+}$

[11] By p_1^k and (p_{i^+}, p_{i^-}) we denote the one-point link and the two-point links, generated by the point 1 and the points i^+, i^-, respectively.

and $T_{\lambda_1^-}$:

$$T_{\lambda_1^+} = \begin{pmatrix} C^{(1)} \\ \hline C^+ \\ \hline C^- \\ \hline C^{(2)} \end{pmatrix}, \qquad T_{\lambda_1^-} = \begin{pmatrix} D^{(1)} \\ \hline D^+ \\ \hline D^- \\ \hline D^{(2)} \end{pmatrix}.$$

Now we will perform on T only transformations that do not disturb the shape of the matrices $T_{\lambda_2^+}$, $T_{\lambda_2^-}$. Now it is easy to check that for the matrix

$$\mathcal{U} = (T_{\lambda_1^+} | T_{\lambda_1^-})$$

only transformations that are defined by the bundle of semichains $A_1 = \{p_1^1 < (p_1^+, p_1^-) < p_1^2\}$ and $B_1 = \{(p_{0^+}, p_{0^-})\}$ (with the involution being trivial) are admissible. Here, if T does not contain direct summands that are "empty" representations which are exact in exactly one point 2^+ or 2^- , then $\mathcal{U}$ is indecomposable (as a representation of a bundle) if and only if T is indecomposable. [12]
In the general case we consider the following cases:

b) $\varepsilon_1(\lambda_1^-) = \varepsilon_1(\lambda_1^+)$, $\varepsilon_1(\lambda_{m+1}^-) = \varepsilon_1(\lambda_{m+1}^+)$;

b') $\varepsilon_1(\lambda_1^-) = \varepsilon_1(\lambda_1^+)$, $\varepsilon_2(\lambda_{m+1}^-) = \varepsilon_1(\lambda_{m+1}^+) = \varepsilon_1(\lambda_m)$;

b") $\varepsilon_2(\lambda_1^-) = \varepsilon_1(\lambda_1^+) = \varepsilon_1(\lambda_2)$, $\varepsilon_2(\lambda_{m+1}^-) = \varepsilon_1(\lambda_{m+1}^+) = \varepsilon_1(\lambda_m)$

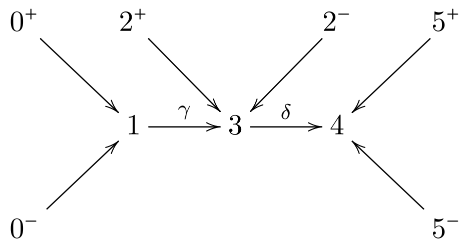
(the other cases can be considered in an analogous way).
We denote by $\overline{S} = (S, \alpha_0)$ the bundle of singleton sets $A_1, \ldots, A_{m+1}, B_1, \ldots, B_{m+1}$ that correspond to the chain

$$0^+ \xrightarrow{\lambda_1^+} 1 \xrightarrow{\lambda_2} 2 \longrightarrow \ldots \longrightarrow m \xrightarrow{\lambda_{m+1}^+} (m+1)^+ \qquad \text{(cf. above)} .$$

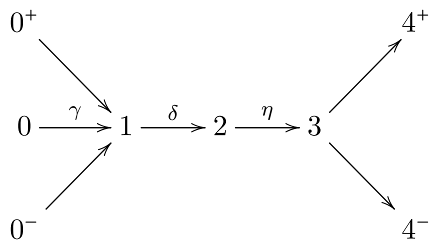
In case b) the problem of giving the representation of the quiver Λ is given in the shape of the bundle $(\hat{S}, \hat{\alpha}_0)$ of semichains $A_1, \ldots, A_{m+1}, B_1^2, B_2, \ldots, B_m, B_{m+1}^2$ where B_1^2 (B_{m+1}^2) are obtained from B_1 (B_{m+1}) by "fractoring" its one-point link. In the case b'), after reducing the matrix with respect to the arrows λ_{m+1}^+ and λ_{m+1}^- (see case a")), we get the bundle $(\hat{S}, \hat{\alpha}_0)$ of semichains $A_1, \ldots, A_{m-1}, \hat{A}_m, B_1^2, B_2, \ldots, B_m$, where $\hat{A}_m = \{p_m^1 < (p_m^+, p_m^-) < p_m^2\}$, $\hat{\alpha}_0(p_m^1) = p_m^1$, $\hat{\alpha}_0(p_m^2) = p_m^2$ (in the other cases we have $\hat{\alpha}_0(x) = \alpha_0(x)$). In the case b") we get the bundle of semichains $\hat{A}_2, A_3, \ldots, A_{m-1}, \hat{A}_m, B_2, \ldots, B_m$ in an analogous way.
Note that if one puts $0^+ = 0^-$ and $\lambda_1^+ = \lambda_1^-$ in Λ then we get a quiver of finite

[12] The transition from the quiver a") to the bundle of semichains A_1 and B_1 can be given in the language of bigraphs [9, 13] if first one induces the arrow γ_2^- and after that the new arrow $\overline{\gamma}_2^+$ (the reincarnation of γ_2^+ after reducing γ_2^-) and, finally, the arrow γ_2^+ itself.

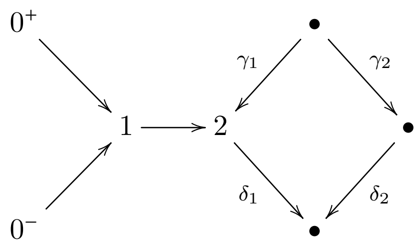
type. Additionally, here case b") is impossible and in cases b) and b') the link B_1 does not split into two.
Multiple problems of giving the representations of quivers with relations can be reduced to bundles of semichains. To exemplify this we will look at several such problems (over an arbitrary field), that arise in the study of different kinds of classes of algebras [7, 14, 15][13]:

c) 

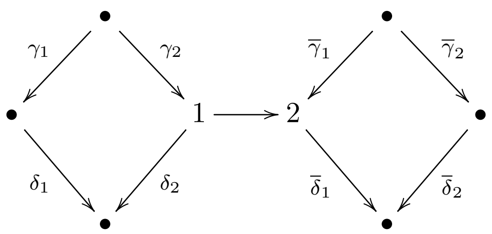
$\gamma\delta = 0;$

d) 

$\gamma\delta\eta = 0;$

e) 

$\gamma_1\delta_1 = \gamma_2\delta_2;$

f) 

$\gamma_1\delta_1 = \gamma_2\delta_2,$

$\overline{\gamma}_1\overline{\delta}_1 = \overline{\gamma}_2\overline{\delta}_2;$

(we take the convention of right hand side writing of morphisms).
In each of the cases $c) - f)$ the problem of representing the quiver can be reduced to some bundle (S, α_0) analogously to what was done in case $a'')$. Here we consider respectively the following matrices: T_γ; T_γ and T_δ; T_{γ_i} and T_{δ_i} $(i = 1, 2)$; T_{γ_i}, T_{δ_i}, $T_{\overline{\gamma}_i}$ and $T_{\overline{\delta}_i}$ $(i = 1, 2)$.
It is easy to check that the bundle (S, α_0) has the following form:

[13] For more complicated examples see paragraph 4–6

c) $A_1 = \{(p_{0^+}, p_{0^-})\}$, $A_2 = \{(p_{2^+}, p_{2^-})\}$, $A_3 = \{p_3^1\}$, $A_4 = \{p_{5^+}, p_{5^-}\}$, $B_1 = \{p_1^1 < p_1^2\}$, $B_2 = \{p_3^2 < p_3^3\}$, $B_3 = \{p_4^1\}$, $B_4 = \{p_4^2\}$, $\alpha_0(p_1^1) = p_3^3$, $\alpha_0(p_1^2) = p_1^2$, $\alpha_0(p_3^1) = p_3^2$, $\alpha_0(p_4^1) = p_4^2$;

d) $A_1 = \{(p_{0^+}, p_{0^-})\}$, $A_2 = \{p_2^1 < p_2^2\}$, $A_3 = \{p_3^1\}$, $B_1 = \{p_1^1 < (p_1^+, p_1^-) < p_1^2\}$, $B_2 = \{p_3^2\}$, $B_3 = \{(p_{4^+}, p_{4^-})\}$, $\alpha_0(p_1^1) = p_2^1$, $\alpha_0(p_1^2) = p_1^2$, $\alpha_0(p_2^2) = p_2^2$, $\alpha_0(p_3^1) = p_3^2$;

e) $A_1 = \{(p_{0^+}, p_{0^-})\}$, $A_2 = \{p_1^1\}$, $B_1 = \{p_1^2\}$, $B_2 = \{p_2^1 < p_2^2 < (p_2^+, p_2^-) < p_2^3 < p_2^4\}$, $\alpha_0(p_1^1) = p_1^2$, $\alpha_0(p_2^j) = p_2^j$ $(1 \le j \le 4)$;

f) $A_1 = \{p_1^1 < p_1^2 < (p_1^+, p_1^-) < p_1^3 < p_1^4\}$, $B_1 = \{p_2^1 < p_2^2 < (p_2^+, p_2^-) < p_2^3 < p_2^4\}$, $\alpha_0(p_i^j) = p_i^j$ $(i = 1, 2,\ 1 \le j \le 4)$

(see footnote on page 13).

4.2 Partially ordered sets

Let $S = \{A_1, A_2, B_1, B_2\}$ where $A_i = \{a_i\}$ and B_i is any semichain $(i = 1, 2)$, $\alpha_0(a_1) = a_1$ and $\alpha_0(x) = x$ for any $x \in B_1^0 \cup B_2^0$. The problem of finding representations of the bundle (S, α_0) is the problem of finding representations of partially ordered sets of the form $\mathcal{H}(B_1, B_2) = B_1 \cup B_2$ (i.e. the points of different semichains are incomparable).
The set $\mathcal{H}(B_1, B_2)$ plays the main role in the study of partially ordered sets of infinite growth. In [16] it has been proven that a partially ordered set has infinite growth if and only if it does not contain a subset of the form $\mathcal{H}(C, D)$ where $C = \{c^+, c^-\}$, $D = \{d_1^\pm, d_2^\pm \mid d_1^\gamma < d_2^\delta, \gamma, \delta \in \{+, -\}\}$. Additionally any exact set of infinite growth has the form $\bar{\mathcal{H}}(B_1, B_2)$, where the bar over $\mathcal{H}$ expresses the presence of some number $k > 0$ of additional comparisons between points of the semichains B_1 and B_2 [17] [14].

4.3 Π-matrices

Let us consider the class of matrix problems which arises in the study of the representations of some algebras (see in particular paragraph 4 and 5).
Let Π be a semichain with the involution γ_0 on the subset Π^0. A Π-*matrix* over the field k is a block-square matrix U (with coefficients in k), which satisfies the following conditions:

a) there exists a 1-1-correspondence between points of the semichain Π and the horizontal bands of U, additionally a band with index x stands above the band with index y if $x < y$;

b) horizontal bands with index x and $\gamma_0(x)$ have the same number of rows;

[14] Exact partially ordered sets of infinite growth are described in [18].

c) the matrix U has the same number of horizontal and vertical bands and additionally all diagonal blocks are square.

On the lines of the Π-matrix U one can perform any transformation which is given by the semichain Π and the involution γ_0 (see transformations 1) and 2), §1), but here one has to perform the inverse transformation on the columns of U. Two Π-matrices which can be transformed into one another by using these transformations will be called *equivalent.*
It is easy to prove that the problem of describing (up to equivalence) a Π-matrix over a field k is wild if $|\Pi| > 1$.
In the work [5] one considers the problem of describing Π-matrices (over a field k), of which the square is zero [15] (in particular the case where Π a chain is also studied in work [19] in connection with the description of the representations of the algebra $\Lambda = \langle a, b \mid a^2 = 0, b^2 = 0 \rangle$).
Let us denote by Π' the semichain which one obtains from Π by adding single point links $\bar{x}$, where x runs through the set $X(\Pi)$, where $\bar{x} = x$, if $\gamma_0(x) = x$ and $\bar{x} < x$ in all other cases, and the comparison $\bar{x} < y$, $x < \bar{y}$, $\bar{x} < \bar{y}$ for links $x, y, \bar{x}, \bar{y} \in X(\Pi')$ $(x \neq y)$ is performed if and only if $x < y$. The dual semichain of Π will be denoted by Π^* (i.e. $x < y$ in Π^* if $x > y$ in Π). We will take the following bundle corresponding to the semichain Π: $T(\Pi) = (T, \alpha_0)$ of semichains $A_1 = \Pi'$ and $B_1 = (\Pi^*)'$, where $\alpha_0(x) = y$ in the following cases: a) $x, y \in \Pi \subset A_1 (x, y \in \Pi^* \subset B_1)$ and $\gamma_0(x) = y$; b) $x = \bar{a} \in A_1 \setminus \Pi$, $y = \bar{b} \in B_1 \setminus \Pi^*$ and $\gamma(a, b)$ (γ is binary relationship on $X(\Pi)$ corresponding to the involution γ_0; see §2).
In §2 [5] it is proven that the problem of describing Π-matrices U of which the square is zero is equivalent to the problem of describing the row-wise non-degenerate representations of the bundle $T(\Pi)$ [16].

4.4 Representation of the algebra $\Lambda = \langle a, b | (a - a_1)(a - a_2) = 0, (b - b_1)(b - b_2) = 0 \rangle$

Let us consider the problem of finding the representations of the algebra Λ over any field k $(a_i, b_i \in k)$. Here we consider the following cases:
a) $a_1 \neq a_2, b_1 \neq b_2$, b) $a_1 \neq a_2, b_1 = b_2$, c) $a_1 = a_2, b_1 = b_2$.
In case a) the problem of finding representations of the algebra Λ is clearly equivalent to the problem of describing non-degenerate representations of the bundle $\bar{S}$ of the semichains A_1 and B_1 which consist respectively of links $x = (x^+, x^-)$ and $y = (y^+, y^-)$ (see [10]).
In case b) after diagonalising the matrix which corresponds to the elements

[15] In [5] for such Π-matrices the term "S-representation" is used.
[16] Let $U = \{U_1, \ldots, U_n\}$ be the canonical representation of the bundle $\bar{S} = (S, \alpha_0)$ corresponding to the X-graph g. From the results §5 and §6 it follows that the matrix U_k is row-wise (column-wise) non-degenerate if and only if any element a_i of g_0 which is an element of the set E_k (F_k) is connected in g with some neighbouring elements by the relation β. In particular if g is an X-cycle, then all matrices $U_1, \ldots, U_n$ are non-degenerate.

a, one obtains a problem about Π-matrices of which the square is zero, where Π consists of one link $x = (x^+, x^-)$ (see example 3).
Finally the problem of describing representations of the algebra Λ in case c) was solved in work [19] (see also [20]). Note that it is equivalent to the problem about Π-matrices of which the square is zero where $\Pi = \{x < y < z\}$, $\alpha_0(x) = z$, $\alpha_0(y) = y$.

4.5 Representation of the algebra $\Lambda_n = \langle a, b | a^3 = 0, b^2 = 0, a^2 = (ba)^n b\rangle$

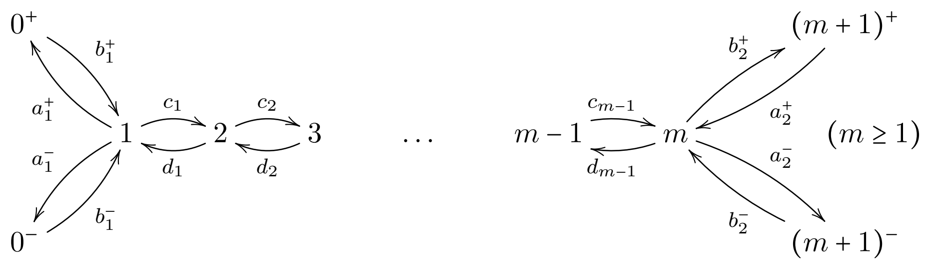
The classical representation of the algebra Λ_n was given in [5][17].
Let Π_{n+1} denote the semichain composed by the links

$$C_{-(n+1)} < C_{-n} < \cdots < C_{-1} < C_0 < C_1 < C_2 < \cdots < C_{n+1},$$

where all links but C_0 are trivial (contain only one point). We define an involution on Π^0_{n+1} in the following way: $\gamma(c_i) = c_{-i}$ for all $i \neq 0$. In §3 [5] it was proven that representing the algebra Λ_n is equivalent to describing the Π_{n+1}-matrices whose squares are zero. And thus this problem is reduced to describing the non-singular row-wise permutations of the bundles $T(\Pi_{n+1})$ (see section 3). Additionally the Π_{n+1}-matrices, and thus the representations of $T(\Pi_{n+1})$ satisfy additional conditions ([5], p. 39,40). These conditions mean that, essentially, the representations of the bundle of semichains $\hat{T}(\Pi_{n+1})$ with $\hat{A}_1 = A_1 \smallsetminus \{C_{n+1}, C_{-(n+1)}, \overline{C}_{-(n+1)}\}$, $\hat{B}_1 = B_1 \smallsetminus \{\overline{C}_n, C_{n+1}, \overline{C}_{n+1}\}$ are considered (the involution γ_0 is induced in a natural way). In this setting the canonical representations are only constructed for X-graphs $g(X = X(S), S = \{\hat{A}_1, \hat{B}_1\})$, that do not contain neighbors a_i and $a_{i\pm1}$ connected in g by β and belonging to the set $\{C_i, \overline{C}_i | i < 0\} \cap \hat{A}_1$ and $\{C_i, \overline{C}_i | i < 0\} \cap \hat{B}_1$, respectively. Additionally, since only row-wise non-singular permutations of bundles $\hat{T}(\Pi_{n+1})$ are considered, the X-graph g satisfies the condition mentioned on the previous page (relating the sets $\hat{E}_1 = X(\hat{A}_1)$).

4.6 Generalization of the I.M. Gelfand Problem

Consider the problem of describing (over an arbitrary field) the quiver



[17] A special case of this problem is the problem of describing representations of the Quasi-dieder-group $\mathcal{O}_m = \langle x, y \mid x^2 = y^{2^m} = 1, yx = xy^{2^{m-1}-1}\rangle$ $(m \geq 3)$ over a field with characteristic 2 (see §1 [5]).

with the relations $a_i^+ b_i^+ = a_i^- b_i^-$ $(i = 1, 2)$ and $d_1 a_1^\pm = 0, b_1^\pm c_1 = 0,\ c_j c_{j+1} = 0,$ $d_{j+1} d_j = 0$ $(1 \le j < m-1)$, $c_{m-1} a_2^\pm = 0,\ b_2^\pm d_{m-1} = 0$ (for $m = 1 :\ b_k^\sigma a_s^\tau = 0$ for any $\sigma, \tau \in \{+, -\}$ and $\{k, s\} = \{1, 2\},\ k \ne s$) [18].
Using easy arguments (see, in particular, lemma I in [5]) one can prove that any matrix representation T has the following shape

$$T_{a_1^\pm} = \left(\begin{array}{c|c|c} 0 & \pm A_{11} & A_{11}^\pm \\ \hline 0 & 0 & 0 \\ \hline 0 & \pm A_{21} & A_{21}^\pm \\ \hline 0 & 0 & 0 \end{array}\right), \quad T_{b_1^\pm} = \left(\begin{array}{c|c|c|c} 0 & 0 & B_{11}^\pm & B_{21}^\pm \\ \hline 0 & 0 & B_{11} & B_{21} \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

$$T_{c_i} = \left(\begin{array}{c|c|c|c} 0 & 0 & C_{1i} & C_{2i} \\ \hline 0 & 0 & C_{3i} & C_{4i} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad T_{d_i} = \left(\begin{array}{c|c|c|c} 0 & D_{1i} & 0 & D_{2i} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & D_{3i} & 0 & D_{4i} \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad (1 \le i \le m-1),$$

$$T_{a_2^\pm} = \left(\begin{array}{c|c|c} 0 & \pm A_{12} & A_{12}^\pm \\ \hline 0 & \pm A_{22} & A_{22}^\pm \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad T_{b_2^\pm} = \left(\begin{array}{c|c|c|c} 0 & B_{12}^\pm & 0 & B_{22}^\pm \\ \hline 0 & B_{12} & 0 & B_{22} \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

where the matrices

$$V_1 = \left(\begin{array}{c|c|c} A_{11} & A_{11}^+ & A_{11}^- \\ \hline A_{21} & A_{21}^+ & A_{21}^- \end{array}\right), \qquad V_2 = \left(\begin{array}{c|c} B_{11}^- & B_{21}^- \\ \hline B_{11}^+ & B_{21}^+ \\ \hline B_{11} & B_{21} \end{array}\right), \qquad V_{2i+1} = \left(\begin{array}{c|c} C_{1i} & C_{2i} \\ \hline C_{3i} & C_{4i} \end{array}\right),$$

$$V_{2i+2} = \left(\begin{array}{c|c} D_{1i} & D_{2i} \\ \hline D_{3i} & D_{4i} \end{array}\right), \quad V_{2m} = \left(\begin{array}{c|c|c} A_{12} & A_{12}^+ & A_{12}^- \\ \hline A_{22} & A_{22}^+ & A_{22}^- \end{array}\right), \quad V_{2m+2} = \left(\begin{array}{c|c} B_{12}^- & B_{22}^- \\ \hline B_{12}^+ & B_{22}^+ \\ \hline B_{12} & B_{22} \end{array}\right),$$

$(1 \le i \le m-1)$ are row-wise non degenerate. [19]
We consider in the matrix V_i $(1 \le i \le 2m+2)$ the horizontal block-rows and write them in the opposite order. The resulting matrix we denote by U_i. To the first and the second block-row (block-column) of the matrix U_i, for $i \ne 2, 2m+2 (i \ne 1, 2m+1)$, we assign the number a_i^1 and a_i^2 (b_i^1 and b_i^2). For the other matrices U_i we number the block-rows (block-columns) as follows: a_i^1, a_i^+, a_i^- (b_i^1, b_i^+, b_i^-). We denote by $\overline{S} = (S, \alpha_0)$ the bundles of

[18]Some particular cases of this problem were considered in work [21] in the study of indecomposable representations of the group $SO(1,n)$ and its connecting group $SO_0(1,n)$.
[19]If $\varepsilon_1(x) = \varepsilon_1(y)$ $(\varepsilon_2(x) = \varepsilon_2(y))$, then the horizontal (vertical) subbands of the matrices T_x T_y correspond. If $\varepsilon_1(x) = \varepsilon_2(y)$, then the horizontal subband structure of T_x corresponds to the vertical subband structure of T_y.

semichains $A_1,...,A_{2m+2},B_1,...,B_{2m+2}$ where $A_i=\{a_i^1<a_i^2\}$ for $i\neq 2,2m+2$; $A_i=\{a_i^1<(a_i^+,a_i^-)\}$ for $i=2,2m+2$, $B_i=\{b_i^1<b_i^2\}$ for $i\neq 1,2m+1$; $B_i=\{b_i^1<(b_i^+,b_i^-)\}$ for $i=1,2m+1$ and $\alpha_0(a_1^2)=a_3^2$, $\alpha_0(b_2^2)=b_4^2$, $\alpha_0(a_{2i+1}^i)=b_{2i+2}^1$, $\alpha_0(a_{2j}^1)=b_{2j-1}^1$, $\alpha_0(a_{2k+1}^2)=a_{2k}^2$, $\alpha_0(b_{2s-1}^2)=b_{2s+2}^2$ $(0\leq i\leq m,\ 1\leq j\leq m+1,\ 1<k,\ s\leq m)$. Let us consider on T only the operations that conserve the given shape. In this case it is easy to see that for the matrices U_i only the operations are allowed that are defined by the bundle (S,α_0).
In this way, our problem is reduced to the question of describing the representation $\mathcal{U}=\{U_1...U_{2m+2}\}$ of the bundle (S,α_0) for which each matrix U_i is row-wise non-degenerate (see footnote on page 17).
Remark that for each $1\leq i\leq 2m+2$ the set A_i or B_i is a chain. In this case the canonical representation is easier than in the general case, because in the X-graph there are no elementary semichains of length 4.

5 Selfreproduction

In the work [1] (see §2) it is shown that any non-zero representation of a bundle $\overline{S}$ of two semichain sets can be reduced to the representation of a lower dimesional bundle $\overline{S'}$ by some block operations. This result easily translates to general bundles $\overline{S}=(S,\alpha_0)$ of semichains $A_1,...,A_n,B_1,...B_n$.
Let $\mathcal{U}=\{U_1,\dots,U_n\}$ be a non-zero representation of a bundle $\overline{S}$. Without loss of generality, we can say that $U_1\neq 0$. If e is a link of the semichain A_1 and f is a link of the semichain B_1, then we denote by $U_1(e,f)$ the part of the matrix U_1 on the intersection of the bands corresponding to the elements of these links. It is clear that $U_1(e,f)$ consists of one entry of the matrix U_1 if $r(e)=r(f)=1$, of two entries if $r(e)=1$, $r(f)=2$ $(r(e)=2,\ r(f)=1)$, and of four entries if $r(e)=r(f)=2$.
Let us fix links e_0 and f_0 such that $U_1(e_0,f_0)\neq 0$ and $U_1(e,f)=0$ when $e<e_0$ or $e=e_0$, $f<f_0$. The following cases are possible: 1) $r(e_0)=r(f_0)=1$ and $\alpha_0(e_0)\neq f_0$; 2) $r(e_0)=2$, $r(f_0)=1$; 2') $r(e_0)=1$, $r(f_0)=2$; 3) $r(e_0)=r(f_0)=1$ and $\alpha_0(e_0)=f_0$; 4) $r(e_0)=r(f_0)=2$.
The matrix $U_1(e_0,f_0)$ can be seen as a representation of the bundle $\overline{S}_0=(S_0,\eta_0)$ of the semichains A_0 and B_0 where A_0 (B_0) consists out of one link e_0 (f_0) and η_0 is the restricted involution α_0 on $S_0^\circ=A_0^\circ\cup B_0^\circ$ (that is $\eta_0(e_0)=f_0$ in case 3) and η_0 is trivial in the other cases). Let us assume that $X_0=X(S_0)$ (see §2) and let η denote the relation on X_0 corresponding to η_0. It is clear that $X_0=E_0\cup F_0$, where $E_0=\{e_0\}$, $F_0=\{f_0\}$ and we have $\overline{\eta}(e_0,e_0),\overline{\eta}(f_0,f_0)$ in case 1), $\eta(e_0,e_0),\overline{\eta}(f_0,f_0)$ in case 2), $\overline{\eta}(e_0,e_0),\eta(f_0,f_0)$ in case 2'), $\eta(e_0,f_0)$ in case 3) and $\eta(e_0,e_0),\eta(f_0,f_0)$ in case 4). Let us state explicitly the X_0-graph $g\in\Gamma_0(X_0)$ in all the cases 1)-4) (up to equivalence).
In the cases 1),2) and 2') we have X_0-chains:
a) e_0; b) f_0; c) e_0-f_0 and additionally in case 2) - d) $f_0-\overleftarrow{e_0\sim e_0}$ and in case 2') - e) $e_0-\overleftarrow{f_0\sim f_0}$. In case 3) we have the X_0 -chain $e_0\sim f_0-e_0\sim f_0-\cdots-e_0\sim f_0$

and X_0-cycle $e_0 \simeq f_0$. In case 4) we have the X_0-chains: a) $\overrightarrow{e_0 \sim e_0} - \overleftarrow{f_0 \sim f_0} - \cdots - \overrightarrow{e_0 \sim e_0} - \overleftarrow{f_0 \sim f_0}$, b) $e_0 - \overrightarrow{f_0 \sim f_0} - \overleftarrow{e_0 \sim e_0} - \cdots - \overrightarrow{f_0 \sim f_0} - \overleftarrow{e_0 \sim e_0}$, b') $f_0 - \overrightarrow{e_0 \sim e_0} - \overleftarrow{f_0 \sim f_0} - \cdots - \overrightarrow{e_0 \sim e_0} - \overleftarrow{f_0 \sim f_0}$, c) $e_0 - \overleftarrow{f_0 \sim f_0} - \overrightarrow{e_0 \sim e_0} - \cdots - \overrightarrow{e_0 \sim e_0} - \overleftarrow{f_0 \sim f_0}$, c') $f_0 - \overleftarrow{e_0 \sim e_0} - \overrightarrow{f_0 \sim f_0} - \cdots - \overrightarrow{f_0 \sim f_0} - \overleftarrow{e_0 \sim e_0}$, d) $e_0 - f_0$ and the X_0-cycle $e_0 \overset{\longrightarrow}{\approx\approx} e_0 \text{———} f_0 \overset{\longleftarrow}{\approx\approx} f_0$. It is easy to see that, in the cases 1)-3)for the bundle S_0 the main theorem, formulated in §3, holds, in the case 4) this follows from [10].

We will call the subset $X_0 = \{e_0, f_0\}$ of the set $X = X(S)$ *closed* if $\alpha_0(x_0) \in X_0$ for each $x_0 \in S_0^\circ$. In the case that X_0 is closed, denote by R_0 the set of permutations $W = \{W_1\}$ of the bundle $\overline{S}_0$, consisting of square non-singular matrices W_1. If X_0 is not closed, we define $R_0 = \varnothing$.

Assume that $\mathcal{U} \notin R_0$. Let us decompose the matrix $U_1(e_0, f_0)$ into a direct sum of canonical representations of the bundle $\overline{S}_0$ (with respect to the X_0-graphs that we mentioned earlier). Let us denote the representations by V^i $(1 \le i \le l)$. The X_0-graph corresponding to the representation V^i will be denoted by h^i (i.e. $V^i \in K(h^i)$) and the dimension of the representation V^i by s_i. If h^i is an X_0-cycle or an X_0-chain, for which $d(h^i) = 2$, then V^i can be separated in the direct sum of $\mathcal{U}$, because in this case $V^i \in R_0$. If h^i is an X_0-chain for which $d(h^i) < 2$ and $h_0^i = \{a_1, ...a_m\}, h_1^i = \{\gamma_{12}, ..., \gamma_{m-1,m}\}$ $(m \ge 1)$, then in U_1 all entries on the columns/rows of V^i are zero, except those that correspond to the end a_1 if $\gamma_{12} \ne \beta$ and the end a_m if $\gamma_{m-1,m} \ne \beta$ (see [1]). In this case, if $V^i = U_s(h^i)$ where $s \in \{1, 2\}$, then each such end $a_j \in h_0^i$ "creates" a subband of the band $P(a_j)$, if $r(a_j) = 1$ and $P(a_j^\pm)$, if $r(a_j) = 2$ and $\psi_s(a_j) = \pm 1$ (the number of rows or columns of this band is equal to the number of representations V^k equal to V^i). Let us give this new band the number $(a_j, (-1)^{s_i - 1} s_i - 1)$, if $|\Psi(h^i)| = 1$ and $(a_j^\pm, (-1)^{s_i - 1} s_i - 1)$, if $|\Psi(h^i)| = 2$ and $\psi_s(a_k) = \pm 1$ where a_k is the double end of h^i. Additionally, in the case where a_j is the end of the X_0-chain h^i and $\alpha(a_j) = b$ where $b \notin \{e_0, f_0\}$, the element a_j "creates" some subband of the band $P(b)$. Let $\sigma_x = 1$ $(\sigma_x = -1)$ where $x \in A$ $(x \in B)$ and let us give this band the number $(b, \sigma_b \sigma_{a_j}(s_i - 1))$ if $|\Psi(h^i)| = 1$, and $(b^\pm, \sigma_b \sigma_{a_j}(s_i - 1))$ if $|\Psi(h^i)| = 2$ and $\psi_s(a_k) = \pm 1$ (where a_k is the double end of h^i). Note that if the X_0-chain h^i does not create any new bands, then V^i can be separated as a summand from the direct sum of U (it is clear that in this case $V^i \in R_0$).

In this way the representations V^i $(1 \le i \le l)$ create a family of bands in U, numbered by the pairs (c, p) and $(c^\pm, p)$. For $s_i = 1$ let us say that $(c, p) = (c, +0)$, $(c^\pm, p) = (c^\pm, +0)$ if $c \in \{E_0, F_0\}$; and $(c, p) = (c, \pm 0)$ if $c \notin \{E_0, F_0\}$ and $\sigma_c \sigma_{\alpha_0(c)} = \pm 1$. Additionally, the elements $(x, \pm 0)$ and x are always identified with one another.

Let us now perform on the matrices $U_1, \ldots, U_n$ transformations that conserve the shape of the matrix U_1. Then, after removing all rows and columns, in which we got zero entries by the representations V^i, from the bands corresponding to the links e_0, f_0 of the matrix U_1, we get a collection of matrices

$U' = (U'_1, \dots U'_n)$, which is a representation of some bundle $\overline{S'} = (S', \alpha'_0)$ (see §2 [1]).
Let us give the explicit form of $\overline{S'}$. Let us denote by A'_k (B'_k) the set that we get from A_k (B_k) by adding all elements (c,p) and $(c^{\pm},p)$ where $c \in E_k$ (F_k). Let $S' = \{A'_1, \dots, A'_n, B'_1, \dots, B'_n\}$, $A' = \bigcup_{k=1}^{n} A'_k, B' = \bigcup_{k=1}^{n} B'_k$ and thus depending on the earlier mentioned cases $A' \cup B'$ is obtained by adding the following elements:

1) (a, σ_a) if $\alpha_0(e_0) = a$, $a \neq e_0$; and $(b, -\sigma_b)$ if $\alpha_0(f_0) = b$, $b \neq f_0$;

2) $(e_0, 2)$ and additionally $(b^{\pm}, -\sigma_b)$ and $(b, -2\sigma_b)$, if $\alpha_0(f_0) = b$, $b \neq f_0$;

2') $(f_0, 2)$ and additionally $(a^{\pm}, \sigma_a)$ and $(a, 2\sigma_a)$ if $\alpha_0(e_0) = a$, $a \neq e_0$;

3) $(e_0, -2s-1)$, $(f_0, -2s-1)$, $s \geq 0$;

4) $(e_0, -4s-1)$, $(f_0, -4s-1)$, $(e_0^{\pm}, 4s-2)$,$(f_0^{\pm}, 4s-2)$, $(e_0^{\pm}, 4s)$, $(f_0^{\pm}, 4s)$, $s \geq 1$. [20]

The sets A'_k and B'_k $(1 \leq k \leq n)$ are semichains. The links of A'_k (B'_k) will be, additional to the links of A_k (B_k), single point links (c,p) and two point links consisting of the elements (c^+, p) and (c^-, p), which we denote by (c,p) ($c \in E_k(F_k)$). It is clear that for $k > 1$ all new links of A'_k (B'_k) are single point links. The linear order on the links of A'_k (B'_k) is the previous one and

$$(a,p) \lessgtr b(a \neq b) \leftrightarrow a \lessgtr b, (a,p) < (a,q) \leftrightarrow p^{-1} > q^{-1}$$

(where $(+0)^{-1} = +\infty, (-0)^{-1} = -\infty$).
To finish the construction of the bundle $\overline{S'} = (S', \alpha'_0)$, it remains to define α'_0. On the old elements the involution remains the previous, on the new elements we have $\alpha'_0(x) = y$ for the following elements x and y (depending on the case):

1) $x = (a, \sigma_a)$, $y = (b, -\sigma_b)$;

2) $x = (e_0, 2)$, $y = (b, -2\sigma_b)$;

2') $x = (f_0, 2)$, $y = (a, 2\sigma_a)$;

3) $x = (e_0, -2s-1)$, $y = (f_0, -2s-1)$, $s \geq 0$;

4) $x = (e_0, p)$, $y = (f_0, p)$, where $p < 0$.

Note that if the representation U is a zero-representation (in particular "empty"), then it is natural to say that $\overline{S'} = \overline{S}$ and $U' = U$. However, if $U \in R_0$, then $\overline{S'} = \overline{S}$ and $U' = I_0$.

[20] In the cases 3) and 4) A'_1 (B'_1) are infinite, but for every finite U the new bands of U'_1 correspond to elements of some finite subset $A''_1 \subset A'_1$ ($B''_1 \subset B'_1$).

From the explicit construction of the transition from the representation U to the representation U' we can conclude the following statements:

Statement 1.
If U is indecomposable and $U \notin R_0$ then U' is indecomposable. If U' is indecomposable then $U \cong V \oplus W$, where V is indecomposable and $W \in R_0$.

Statement 2.
Representations U and V of the bundle $\overline{S}$ that do not contain representations from R_0[21], as direct summands, are equivalent if and only if U' and V' are equivalent.

Let X', a set from $\mathfrak{X}$, correspond to the bundle $\overline{S'}$. (The binary relations on X' will be denoted by α' and β'.) For each $x \in X$ with $\alpha(x,x)$ we have the corresponding subset $[x]$ of X' given by:

- $[c] = \{c\}$ if $c \notin \{e_0, f_0\}$;
- $[e_0] = \{e_0, (b, -\sigma_b)\}$ in case 2);
- $[f_0] = \{f_0, (a, \sigma_a)\}$ in case 2');
- $[e_0] = \{e_0, (e_0, 4q), (f_0, 4q-2) \mid q \geq 1\}$ and $[f_0] = \{f_0, (f_0, 4q), (e_0, 4q-2) \mid q \geq 1\}$ in case 4).

Let U be a representation of the bundle $\overline{S}$, then we will denote by U^x the representation that we obtain from U after interchanging the bands $P(x^+)$ and $P(x^-)$. Analogously we define the representation $V^{[x]}$ where V is the representation of the bundle $\overline{S'}$ (by interchanging the bands $P(y^+)$ and $P(y^-)$ for all $y \in [x]$).
From the definition of U' the following lemma follows.

Lemma 3. $(U^x)' \cong (U')^{[x]}$.

For each X-graph $g \in \Gamma(X)$ we have (for fixed e_0 and f_0) a corresponding X'-graph $\psi(g)$ (see [1]). The mapping ψ is consistent with the transition from the canonical representation of the bundle $\overline{S}$ to the representation of the bundle $\overline{S'}$ (see next paragraph).
Let $g \in \Gamma(X)$. We denote by $M_0(g)$ the set of the maximal subchains of g, consisting of e_0 and f_0 (if g is an X-cycle consisting of the elements e_0 and f_0, then we take $M_0(g) = \{g\}$). The elements of the X-graph $h \in M_0(g)$ that are connected inside of h by the relationship β will be called the *main ones*, the others will be called *additional ones*.
We construct the X'-graph $\psi(g)$ in the following way:

[21] it is assumed that for the representations U and V the elements e_0 and f_0 are one and the same.

1) we throw out from g the main elements a_i in all $h \in M_0(g)$ (together with the relations $\gamma_{i-1,i}$ and $\gamma_{i,i+1}$);

2) instead of the additional elements e_0 and f_0 of the X-chain $h \in M_0(g)$, we put the corresponding elements $(e_0, (-1)^{s-1}s-1)$ and $(f_0, (-1)^{s-1}s-1)$ where $s = \frac{|h|}{|Aut(h)|}$;

3) an element b that is connected in g by α with an element of an X-chain $h \in M_0(g)$ but which is not an element of h is transformed into $(b, \sigma_b\sigma_a(s-1))$ where $s = \frac{|h|}{|Aut(h)|}$;

4) in all places where an X-chain (X-cycle) is broken we introduce the relation α';

5) we replace the relation β by the relation β'.

Note that if the X-graph g consists of elements equal to e_0 and f_0 and does not contain additional elements, then $\psi(g)$ is an empty X'-graph (to which corresponds a representation I_0 which, as mentioned in §1, is not considered indecomposable).
It is clear that if g is an X-chain (X-cycle) and $\psi(g) \neq \varnothing$, then $\psi(g)$ is an X'-chain (X'-cycle), where $\psi(g) \in \Gamma(X')$. Additionally $\psi(g) \in \Gamma_0(X')$ if $g \in \Gamma_0(X)$.
The following lemma follows directly from the definition of the X'-graph $\psi(g)$:

Lemma 4. If $g, h \in \Gamma(X)$ and $\psi(g) \neq \varnothing$, $\psi(h) \neq \varnothing$, then $\psi(g)$ and $\psi(h)$ are equivalent if and only if g and h are equivalent.

6 Proof of the main theorem

Recall (see §5), that U' denotes a representation of a bundle $\overline{S'} = (S', \alpha_0')$ constructed from the representation U of the bundle $\overline{S} = (S, \alpha_0)$ with respect to e_0 and f_0; there we also introduced the representation U^x, the set of representations R_0 and the X'-graph $\psi(g)$ which we will denote by g' for simplicity.

Statement 3. Let $g \in L_0(X)$, $d(g) < 2$, $c \in X$, $\alpha(c,c)$. We have for the canonical representation $U_s(g)$:

1. If $g' \neq \varnothing$ then $[U_s(g)]' \cong U_s(g)'$. Conversely, if $U' \cong U_s(g')$ and U does not contain representations from R_0, as direct summands, then $U \cong U_s(g)$.

2. $[U_s(g)]^c \cong U_{\hat{s}}(g)$ where $\hat{1} = 2$, $\hat{2} = 1$ if g has a double end equal to c and $\hat{s} = s$ in all other cases.

First we prove part 1. An admissible transformation which adds to a row (column) with number $a_i \in g_0$ the row (column) with the number $a_j \in g_0$ multiplied by $x \in k$ will be denoted by $P_x(a_i, a_j)$. The multiplication of the row (column) with number a_i by an element $x \neq 0$ will be denoted by $Q_x(a_i)$. Let us first assume that $s = 1$ and $U = U_1(g) = \{U_1, \dots, U_n\}$. Let us choose e_0 and f_0 as in §5 (we can assume that $U_1 \neq 0$, otherwise we perform a renumbering of the semichains of the bundle $\overline{S}$). In the case that e_0 and f_0 satisfy condition 1) or 3) the matrix $U_1(e_0, f_0)$ is automatically a direct sum of canonical representations of the bundle $\overline{S}_0 = (S_0, \eta_0)$. By removing the rows and columns of U_1 that correspond to the main elements of all subchains $h \in M_0(g)$ (see §5) it is easy to see that we get the representation $U_1(g')$. Analogously in the case 2) (2')) if $\alpha_0(f_0) = f_0$ ($\alpha_0(e_0) = e_1$). In the other cases, that is in case 2) with $\alpha_0(f_0) = b$, $b \neq f_0$, in case 2') with $\alpha_0(e_0) = a, a \neq e_0$ and in the case 4), among the subchains $h \in M_0(g)$ that contain at least one additional element, there could be symmetrical subchains. In this case their direct summands can always be decomposed (by using admissible transformations on U) into a direct sum of two canonical representations of the bundle $\overline{S}_0$ corresponding to the left and right "half" of the subchain h. We give these transformations. Let us assume, to concretize, that a_j and a_{j+1} are the middle elements of h, then $\overrightarrow{a_j \sim a_{j+1}}$ ($\overleftarrow{a_j \sim a_{j+1}}$) for $a_j = e_0$ ($a_j = f_0$).

In case 2) (2')) we do for the subchain $h = \{a_i - a_{i+1} \sim a_{i+2} - a_{i+3}\}$ the transformations $P_{-1}(a_i, a_{i+3})$, $P_1(a_{i+4}, a_{i-1})$ ($P_{-1}(a_{i+3}, a_i)$, $P_{-1}(a_{i+4}, a_{i-1})$) if $b \in E$ ($a \in E$) and $P_{-1}(a_i, a_{i+3})$, $P_{-1}(a_{i-1}, a_{i+4})$ ($P_{-1}(a_{i+3}, a_i)$, $P_1(a_{i-1}, a_{i+4})$) if $b \in F$ ($a \in F$). Additionally, if $b \in F$ ($a \in E$), "new" elements of the representation U that are equal to -1 we will make into 1 by transformations $Q_{-1}(a_j), a_j \in g_0$.

In case 4) we will assume that the first element a_i of the subchain h is equal to e_0 (the second case can be understood in a dual way); additionally, the elements of the subchain $h = \{a_i \sim a_{i+1} - a_{i+2} \sim a_{i+3} - \dots - a_{i+2k-2} \sim a_{i+2k-1}\}$ where k is an odd number (for a symmetric h) will be numbered, for convenience, in the following way: $a_i = e_0^1$, $a_{i+1} = e_0^2$, $a_{i+2} = f_0^1$, $a_{i+3} = f_0^2$, $a_{i+4} = e_0^3$, $a_{i+5} = e_0^4$ etc.. In this case we do the transformation $P_{-1}(e_0^{2p+1}, e_0^{2q+1})$ for $p < q$, $2p + 2q = k \pm 1$, $P_{-1}(e_0^{2p}, e_0^{2q})$ for $p < q$, $2p + 2q = k + 3$, $P_1(f_0^{2p+1}, f_0^{2q+1})$ for $p > q$, $2p+2q$ equal to $k-1$ or $k-3$, $P_1(f_0^{2p}, f_0^{2q})$ for $p > q$, $2p+2q = k+1$. Additionally, to obtain zero elements in the matrix U_1 in all rows and columns going through $U_1(e_0, f_0)$ which correspond to main elements of the subchain h, we have to perform the transformation $P_1(f_0^1, a_{i+2k})$; and if $\overleftarrow{a_{i+2k} \sim a_{i+2k+1}}$, then also the transformation $P_1(f_0^1, a_{i+2k+1})$.

After performing these transformations for all symmetric subchains $h \in M_0(g)$ and going to the representation U' of the bundle $\overline{S'} = (S', \alpha_0')$, we get

that $U' \cong U_1(g')$ [22]!
In an analogous way one can study the case where $U = U_2(g)$.
Conversely, if U satisfies the condition in the statement and $U' \cong U_s(g')$ then by applying the inverse to the previously mentioned transformations we get $U \cong U_s(g)$.
Part 2 is now easy to prove by induction over the length of the X-chain by using part 1, Lemma 3 and Statement 2.
Let $g \in \Gamma(X)$ and $g_0 = \{a_1, \dots, a_n\}$. Let us define the operation ε_0^* of the X-chain (X-cycle) g in the following way: $\varepsilon_0^*(a_i, a_{i+1}) = -\varepsilon_0(a_i, a_{i+1})$ on the "junction" of the compound X-chain (for a symmetric X-cycle with $(a_i, a_{i+1}) \in \overline{D}(g_0)$) and $\varepsilon_0^*(a_i, a_{i+1}) = \varepsilon_0(a_i, a_{i+1})$ in all other cases. Note that $\varepsilon_0^* = \varepsilon_0$ for every simple X-chain (for every non-symmetric X-cycle). If $g \in L_0(X)$ and $d(g) = 2$ ($g \in Z_0(X)$), then we denote by $U_s^*(g,p)$ ($U^*(g,\varphi)$) the representation of the bundle $\overline{S}$ which is constructed analogously to the canonical representation $U_s(g,p)$ ($U(g,\varphi)$) but with respect to the operation ε_0^*.

Statement 4.
Let $g \in L_0(X)$, $d(g) = 2$ and $c \in X$, $\alpha(c,c)$. Then we have for a canonical representations $U_s(g,p)$:

1. If $g' \neq \varnothing$, then $[U_s(g,p)]' \cong U_s(g',p)$. Conversely, if $U' \cong U_s(g',p)$ and U does not contain representations from R_0, as direct summands, then $U \cong U_s(g,p)$.

2. $[U_s(g,p)]^c \cong U_{\hat{s}}(g',p)$ where $\hat{s} \neq \hat{k}$ for $s \neq k$. Additionally, $\hat{s} = s$ if $a_1 \neq c, a_m \neq c$;

$$\hat{1} = 2, \hat{2} = 1, \hat{3} = 4, \hat{4} = 3 \quad \text{if } a_1 = c, a_m \neq c;$$
$$\hat{1} = 3, \hat{2} = 4, \hat{3} = 1, \hat{4} = 2 \quad \text{if } a_1 \neq c, a_m = c;$$
$$\hat{1} = 4, \hat{2} = 3, \hat{3} = 2, \hat{4} = 1 \quad \text{if } a_1 = c, a_m = c.$$

3. $U_s^*(g,p) \cong U_s(g,p)$.

The proof is done by induction over the length of g analogously to the proof of Statement 3. Note that in the proof of part 1. instead of the canonical representation $U_s(g',p)$ the representation $U_s^*(g',p)$ can appear. In this case

[22] In §5 we agreed on decomposing the matrix $U_1(e_0, f_0)$ into a direct sum of canonical representations of the bundle $\overline{S}_0$ that correspond to some fixed (pairwise non-equivalent) X_0 - chains (see page 20-21). Because of this, formally, by the transition to U' in case 4 we need to obtain (by using admissible transformations) that all canonical representations that are direct summands of $U_1(e_0, f_0)$ correspond only to fixed X_0-chains (in particular, we need to "replace" canonical representations that correspond to the left part of symmetric subchains $h \in M_0(g)$ by equivalent canonical representations that correspond to the right part of the respective subchain).

one needs to use the induction assumption of part 3. (more precisely, part 3. for the X'-chain g'). Analogously, one can prove part 1. for the representations $U_s^*(g,p)$. Let us additionally note that for the proof of part 2. the equality $(U^c)^* = (U^*)^c$, where $U = U_s(g,p)$ is used and in the proof of part 3., part 1. for the representations $U_s(g,p)$ and $U_s^*(g,p)$ is used.
In an analogous way one can look at the cases where g is an X-cycle. More precisely, we have the following statement:

Statement 5.
Let $g \in Z_0(X)$ and $c \in X, \alpha(c,c)$. Then it holds for the canonical representations $U(g,\varphi)$:

1. If $g' \neq \varnothing$, then $[U(g,\varphi)]' \cong U(g',\varphi')$, where $\varphi_1' \neq \varphi_2'$ for $\varphi_1 \neq \varphi_2$. Conversely, if $U' \cong U(g',\varphi)$ and U does not contain representations from R_0, as direct summands, then $U \simeq U(g,\varphi')$.
2. $[U(g,\varphi)]^c \simeq U(g,\varphi')$, where $\varphi_1' \neq \varphi_2'$ for $\varphi_1 \neq \varphi_2$.
3. $U^*(g,\varphi) \simeq U(g,\varphi')$, where $\varphi_1' \neq \varphi_2'$ for $\varphi_1 \neq \varphi_2$.

Note that a Frobenious block ϕ of the canonical representation $U(g,\varphi)$ corresponding to the subchain $h = \{a_{i-1}{-}a_i\}$ can be "moved" by using admissible transformations of kind 1) to any other place that corresponds to a maximal elementary subchain $\overline{h} = \{a_{j-1} - a_j\}$. Hereby, the new block is equal to ϕ or ϕ^{-1}. Because of this in the condition of part 1. we can assume that φ' is the characteristic polynomial of the matrix $\sigma_1\phi^{\sigma_2}$, where $\sigma_1, \sigma_2 \in \{1,-1\}$ (σ_1 and σ_2 are uniquely defined by the X-chain g). In the condition of part 2. (3.) φ' is the characteristic polynomial of the matrix ϕ^{σ_2} if g is not a symmetric X-chain of length 4, and of the matrix $(E+\phi)^{-1} - E$ (ϕ) otherwise.
The main theorem is now easily to prove by induction over the dimension of representations by using statements 1-5.

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