Chow–Künneth decomposition for 3- and 4-folds fibred by varieties with trivial Chow group of zero-cycles

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Abstract

Let k be a field and let Ω be a universal domain over k. Let $f : X \to S$ be a dominant morphism defined over k from a smooth projective variety X to a smooth projective variety S of dimension ≤ 2 such that the general fibre of f_Ω has trivial Chow group of zero-cycles. For example, X could be the total space of a two-dimensional family of varieties whose general member is rationally connected. Suppose that X has dimension ≤ 4. Then we prove that X has a self-dual Murre decomposition, i.e. that X has a self-dual Chow–Künneth decomposition which satisfies Murre's conjectures (B) and (D). Moreover we prove that the motivic Lefschetz conjecture holds for X and hence so does the Lefschetz standard conjecture. We also give new examples of threefolds of general type which are Kimura finite-dimensional, new examples of fourfolds of general type having a self-dual Murre decomposition, as well as new examples of varieties with finite degree three unramified cohomology.

Introduction

Throughout this paper, algebraic cycles and Chow groups are with rational coefficients. Let X be a smooth projective complex variety. The Hodge conjecture predicts that every Hodge class in $H_{2i}(X) := H_{2i}(X(\mathbf{C}), \mathbf{Q})$ is the class of an algebraic cycle. In particular, given any two smooth projective complex varieties X and Y, the Hodge conjecture predicts that any morphism f of Hodge structures between $H_*(X)$ and $H_*(Y)$ comes from geometry. By this we mean that f is induced by a correspondence between X and Y, that is, by an algebraic cycle on $X \times Y$. Whether the Hodge conjecture happens to be true or not, Grothendieck pointed out that certain morphisms of Hodge structures play a more important role in the theory of algebraic cycles. If X has pure dimension d, he suggested that for all integer i the Künneth component in $H_{2d}(X \times X)$ inducing the projector on $H_i(X)$ should be induced by a correspondence. He also suggested that, for all $i \leq d$, the inverse to the Lefschetz isomorphism $H_{2d-i}(X) \to H_i(X)$ given by intersecting $d-i$ times with the class of a smooth hyperplane section should be induced by an algebraic cycle. The first conjecture is usually referred to as the Künneth standard conjecture and the second one to the Lefschetz standard conjecture. Classically [14, 4.1], it is known that the Lefschetz standard conjecture for X implies the Künneth standard conjecture for X.

If the Hodge conjecture gives a simple description of the image of the cycle class map $\mathrm{CH}_i(X) \to H_{2i}(X)$ in terms of the Hodge structure of $H_{2i}(X)$, it is a much more difficult problem to unravel the nature of its kernel. Beilinson and Bloch, inspired by Grothendieck's philosophy of motives, first proposed a description of such a kernel in terms of a descending filtration on Chow groups that would behave functorially with respect to the action of correspondences and would be such that its graded parts would depend solely on the homological motive of X.

More generally, the conjectures of Bloch and Beilinson can be formulated for smooth projective varieties defined over any field k if one uses ℓ-adic cohomology in place of Betti cohomology. In that setting, for X smooth projective over k, the Künneth and Lefschetz standard conjectures stipulate the existence of algebraic cycles in $X \times X$ inducing the right action on cohomology. This is consistent with the previous formulations. Indeed, let X be a smooth projective variety defined over a subfield $k \subseteq \mathbf{C}$ and assume that there is a cycle $\Gamma \in \mathrm{CH}^i(X_{\mathbf{C}} \times_{\mathbf{C}} X_{\mathbf{C}})$ inducing the inverse to the Lefschetz isomorphism $H_{2d-i}(X) \to H_i(X)$. Here, d is the dimension of X and i is a non-negative integer $\leq d$. It is a fact that, for a smooth projective variety Y defined over k, the base change map $\mathrm{CH}_*(Y_K) \to \mathrm{CH}_*(Y_L)$ is an isomorphism modulo homological equivalence for all extensions of algebraically closed fields L/K over k. Therefore, up to replacing Γ by a cycle homologically equivalent to it, we can assume that Γ is defined over the algebraic closure of k inside $\mathbf{C}$. But then, Γ is defined over a finite extension of k which can be chosen to be Galois, say of degree n, and with Galois group G. It is then straightforward to check that the cycle $\gamma := 1/n \cdot \sum_{\sigma \in G} \sigma_* \Gamma$ is defined over k and induces the inverse to the Lefschetz isomorphism $H_{2d-i}(X) \to H_i(X)$. The same arguments apply to the Künneth standard conjecture.

Twenty years ago, Murre proposed that not only should the Künneth projectors in cohomology be induced by correspondences, but also that they should be induced by correspondences that are idempotents modulo rational equivalence. Given X a smooth projective variety of dimension d over a field k, Murre [19] conjectured the following.

(A) X has a Chow–Künneth decomposition $\{\pi_0, \ldots, \pi_{2d}\}$: There exist mutually orthogonal idempotents $\pi_0, \ldots, \pi_{2d} \in \mathrm{CH}_d(X \times X)$ adding to the identity such that $(\pi_i)_* H_*(X) = H_i(X)$ for all i.

(B) $\pi_0, \ldots, \pi_{2l-1}, \pi_{d+l+1}, \ldots, \pi_{2d}$ act trivially on $\mathrm{CH}_l(X)$ for all l.

(C) $F^i \mathrm{CH}_l(X) := \mathrm{Ker}\,(\pi_{2l}) \cap \ldots \cap \mathrm{Ker}\,(\pi_{2l+i-1})$ does not depend on the choice of the π_j's. Here the π_j's are acting on $\mathrm{CH}_l(X)$.

(D) $F^1 \mathrm{CH}_l(X) = \mathrm{CH}_l(X)_{\mathrm{hom}}$, where the subscript 'hom' refers to homologically trivial cycles.

A variety X which has a Chow–Künneth decomposition that satisfies Murre's conjectures (B) and (D) is said to have a *Murre decomposition*. If moreover the Chow–Künneth decomposition of conjecture (A) can be chosen so that $\pi_i = {}^t\pi_{2d-i} \in \mathrm{CH}_d(X \times X)$, then X is said to have a *self-dual Murre decomposition*. Here, we understand Murre's conjecture (C) as saying that any two filtrations induced by two distinct Chow–Künneth decompositions for X coincide, not that they are merely isomorphic.

The relevance of Murre's conjectures was demonstrated by Jannsen [10] who showed that these hold for all smooth projective varieties if and only if Bloch's and Beilinson's conjectures hold for all smooth projective varieties. Murre's formulation of a conjectural descending filtration on Chow groups has the advantage over Beilinson's and Bloch's that it does not involve any functoriality properties and that it can therefore be proven on a case-by-case basis. Since Murre's paper [19] appeared, many authors have tried to prove those conjectures for certain classes of varieties. In this paper, we extend the list of cases for which these can be proven.

Let X be a smooth projective variety defined over a field k and let Ω be a universal domain over k, i.e. Ω is an algebraically closed field of infinite transcendence degree over k. A smooth projective variety X will be said to have trivial Chow group of zero-cycles if $\mathrm{CH}_0(X_\Omega) = \mathbf{Q}$. Our main result is the following

Theorem 1. *Let $f : X \to S$ be a dominant morphism between smooth projective varieties defined over a field k such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Suppose that S has dimension ≤ 2 and that X has dimension ≤ 4. Then X has a self-dual Murre decomposition. Moreover, the motivic Lefschetz conjecture, as stated in §5, holds for X and hence so does the Lefschetz standard conjecture.*

Let's stress that the theorem gives a self-dual Murre decomposition of X which is defined over a field of definition of f. Together with standard results on rationally connected varieties [15, IV.3.11] (see also the proof of Corollary 1.4), we deduce

Theorem 2. *Let $f : X \to S$ be an equidimensional dominant morphism between smooth projective varieties defined over a field k whose general fibre is separably rationally connected. Suppose that S has dimension ≤ 2 and that X has dimension ≤ 4. Then X has a self-dual Murre decomposition which satisfies the motivic Lefschetz conjecture.*

Theorem 1 contrasts with the approach of Gordon–Hanamura–Murre [8] where Chow–Künneth decompositions are constructed for varieties X that come with a fibration $f : X \to S$ which is "nice" enough: it is assumed, among other things, in *loc. cit.* that f should be smooth away from a finite number of points on S and that f should have a relative Chow–Künneth decomposition. Here, we do not even require f to be flat.

Theorem 1 was already proved in the case $\dim S \leq 1$: a self-dual Chow–Künneth decomposition for which the motivic Lefschetz conjecture holds was constructed in [22, 4.6] and Murre's conjectures were checked to hold in [21, 4.21]. The results in [21, 22] just mentioned are more generally valid for fourfolds with Chow group of zero-cycles supported on a curve. By Theorem 1.3, it is the case that, if X is as in Theorem 1 with $\dim S \leq 1$, then $\mathrm{CH}_0(X_\Omega)$ is supported on a curve. From now on, we will therefore focus on the case $\dim S = 2$.

Here, Murre's conjecture (C) is proved only on the grounds that the idempotents of a Chow–Künneth decomposition for X are supported in a specific dimension, cf. section 6. It is however fully proved under some extra assumption on X in Theorem 6.5 and Theorem 8.3. Let's mention that del Angel and Müller-Stach [4] proved the existence of a Murre decomposition for threefolds fibred by conics over a surface (see also the recent paper [17] where Chow–Künneth decompositions are constructed for some threefolds including conic fibrations). In addition to treating the four-dimensional case, our theorem makes more precise the result of [4] by showing that the Murre decomposition can be chosen to be self-dual and by showing the motivic Lefschetz conjecture for X. Also our approach is different from [4]. Del Angel and Müller-Stach assume that all the fibres of f are rationally connected, this allows them to compute the cohomology of X via the Leray spectral sequence. They then construct idempotents modulo rational equivalence and check that they act as the Künneth projectors on cohomology. Here, we do not make any assumptions on the bad fibres of f and we first compute the Chow group of zero-cycles of X to only then deduce that the idempotents we construct act as the Künneth projectors on cohomology.

A word about the proof of the theorem and about the organisation of the paper. In section 1, we make the simple observation that, for f as in Theorem 1, $f_* : \mathrm{CH}_0(X_\Omega) \to \mathrm{CH}_0(S_\Omega)$ is bijective with inverse induced by an algebraic correspondence which is defined over a field of definition of f. Together with Theorem 2.1, the proof of the existence of a Murre decomposition for X essentially reduces to the case of motives of surfaces. The validity of Murre's conjectures (A), (B) and (D) for surfaces goes back to Murre himself [18]. However, by the very nature of Theorem 2.1, we need Murre's conjectures not only for surfaces but for motives of surfaces (i.e. we need to deal with idempotents). This is the object of section 3. The construction of idempotents inducing the right Künneth projectors for X in homology is carried out in section 4. Constructing such idempotents is easy from the case of surfaces. However, these are not necessarily mutually orthogonal. The non-commutative Gram–Schmidt process which already appears in [22] and which is run on this set of idempotents is described in Lemma 4.11. This way, we obtain a *self-dual* Chow–Künneth decomposition for X. The motivic Lefschetz conjecture is formulated in section 5, its relevance is discussed and it is proved for X there. Murre's conjectures (B) and (D) are then proved for X in section 6 by using results of [21] which are recalled in Proposition 6.1 and 6.2. If C is a smooth projective curve, then the results of *loc. cit.* actually make it possible to prove Murre's conjectures (B) and (D) for $X \times C$. This is the object of section 7.

Murre's observation that the Künneth projectors should lift to idempotents modulo rational equivalence is crucial in the sense that a combination of Beilinson's and Bloch's conjectures with Grothendieck's standard conjectures imply that any projector in homology should be liftable to an idempotent modulo rational equivalence. Shun-Ichi Kimura [13] introduced a notion of finite-dimensionality for Chow motives which implies such a lifting property for projectors. Kimura's notion of finite-dimensionality has become widely popular for this reason and more importantly because of its relationship to Murre's conjectures. The simple observation of Theorem 1.3 is used in section 8 to give new examples of threefolds of general type which are Kimura finite-dimensional, namely threefolds fibred by Godeaux surfaces. There, using the result of section 4, we also show in Theorem 8.2 that, if X is a conic fibration over a surface which is Kimura finite-dimensional, then X is Kimura finite-dimensional. In section 9, we produce examples of fourfolds of general type with Chow group of zero-cycles not supported on a curve but which admit a self-dual Murre decomposition. Such examples will be given by fourfolds fibred by surfaces birational to Godeaux surfaces. Theorem 1.3 is slightly generalised in Theorem 1.7 ; this is used in section 10 to prove finiteness of unramified cohomology in some new cases.

Finally, although we don't state it here, the methods of this paper actually show that Murre's conjectures hold for smooth projective fourfolds X for which there exist a smooth projective surface S and correspondences $\alpha \in \mathrm{CH}_2(S \times X)$ and $\beta \in \mathrm{CH}^2(X \times S)$ such that $\beta \circ \alpha = \Delta_S \in \mathrm{CH}_2(S \times S)$ and such that $\alpha \circ \beta$ acts as the identity on $\mathrm{CH}_0(X_\Omega)$. Consequently, if $f : X \to S$ is a dominant morphism to a surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles, then Murre's conjectures hold for any smooth projective variety X' which is birational to X.

Notations. We refer to [20] for the notion of Chow motive and to [12] for the covariant notations we use here. Briefly, a Chow motive M is a triple (X, p, n) where X is a smooth projective variety over k of pure dimension d, $p \in \mathrm{CH}_d(X \times X)$ is an idempotent ($p \circ p = p$) and n is an integer. The motive M is said to be effective if $n \geq 0$. The dual of M is the motive $M^\vee := (X, {}^tp, -d-n)$, where ${}^tp$ denotes the transpose of p. A morphism between two motives (X, p, n) and (Y, q, m) is a correspondence in $q \circ \mathrm{CH}_{d+n-m}(X \times Y) \circ p$. We write $\mathfrak{h}(X)$ for the motive of X, i.e. for the motive $(X, \Delta_X, 0)$ where Δ_X is the class of the diagonal inside $\mathrm{CH}_d(X \times X)$. We have $\mathrm{CH}_i(X, p, n) = p_* \mathrm{CH}_{i-n}(X)$ and $H_i(X, p, n) = p_* H_{i-2n}(X)$, where we write $H_i(X) := H^{2d-i}(X(\mathbf{C}), \mathbf{Q})$ for Betti homology when $k \subseteq \mathbf{C}$. The results of this paper are valid more generally for any field k if one considers ℓ-adic homology $H_i(X, \mathbf{Q}_\ell) := H^{2d-i}(X_{\bar{k}}, \mathbf{Q}_\ell)$ with $\ell \neq \mathrm{char}\, k$ in place of Betti homology. In that case the action of a correspondence $\Gamma \in \mathrm{CH}_i(X \times Y)$ on $H_i(X, \mathbf{Q}_\ell)$ is given by the action of $\Gamma \otimes_{\mathbf{Q}} 1 \in \mathrm{CH}_i(X \times Y) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$.

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1 A geometric result on zero-cycles

Let X and S be smooth projective varieties over k and let $f : X \to S$ be a dominant morphism. Then any general linear section $\sigma : H \to X$ of dimension $\dim S$ is smooth over k and is such that the morphism $\pi := f|_H : H \to S$ is dominant. In particular, for any general H, π is generically finite and its degree is written n.

Proposition 1.1. *Let $f : X \to S$ be a dominant morphism. Then, for any general H as above, $\Gamma_f \circ \Gamma_\sigma \circ {}^t\Gamma_\pi = n \cdot \Delta_S \in \mathrm{CH}_{\dim S}(S \times S)$.*

Proof. This follows from the projection formula applied to $\pi = f \circ \sigma$ and Manin's identity principle. See [16, Example 1 p. 450]. $\square$

Definition 1.2. Let $f : X \to S$ be a dominant morphism between smooth projective varieties defined over a field k. A general point of S is a closed point sitting outside a given proper closed subset of S. By *general fibre* of f, we mean the fibre of f over a general point of S.

Theorem 1.3. *Let $f : X \to S$ be a dominant morphism between smooth projective varieties defined over a field k. Assume that a general fibre Y of f satisfies $\mathrm{CH}_0(Y) = \mathbf{Q}$. Then the induced map $f_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(S)$ is bijective and its inverse is induced by a correspondence $\Gamma \in \mathrm{CH}^{\dim X}(S \times X)$. Moreover Γ can be chosen to be defined over a field of definition of f.*

Proof. Let's show that the correspondence Γ can be chosen to be $\frac{1}{n}\Gamma_\sigma \circ {}^t\Gamma_\pi$. According to Proposition 1.1, it suffices to prove that the correspondence $\Gamma_\sigma \circ {}^t\Gamma_\pi$ induces a surjective map $(\Gamma_\sigma \circ {}^t\Gamma_\pi)_* : \mathrm{CH}_0(S) \to \mathrm{CH}_0(X)$.

Let's fix an open subset U of S such that $\pi : H_U \to U$ is finite and such that the fibres of f_U satisfy $\mathrm{CH}_0(X_u) = \mathbf{Q}$ for all closed points u in U. Let p be a closed point of X and let $[p]$ denote the class of p in $\mathrm{CH}_0(X)$. By Chow's moving lemma, the zero-cycle $[p] \in Z_0(X)$ is rationally equivalent to a zero-cycle $\alpha = \sum a_i \cdot [p_i]$ supported on X_U. This means that each p_i is a closed point of X that belongs to the open subset X_U of X. Let $u_i := f(p_i)$. Since $\mathrm{CH}_0(X_{u_i}) = \mathbf{Q}$ and $\deg(\sigma_*\pi^*[u_i]) \neq 0$, we see that p_i is rationally equivalent to $\sigma_*\pi^*[u_i]$ taken with an appropriate rational coefficient. Thus, $\sigma_*\pi^*$ is surjective on zero-cycles. $\square$

For example, we get as a corollary the following which is used in [21, Cor. 4.23] and [22, Cor. 4.7] in the cases when S is a curve.

Corollary 1.4. *Let $f : X \to S$ be an equidimensional dominant morphism between smooth projective varieties defined over a field k. Assume that the general fibre of f is separably rationally connected (e.g. X could be a Fano fibration). Then $f_* : \mathrm{CH}_0(X_\Omega) \to \mathrm{CH}_0(S_\Omega)$ is bijective and there is a correspondence $\Gamma \in \mathrm{CH}^{\dim X}(S \times X)$ such that $\Gamma_* : \mathrm{CH}_0(S_\Omega) \to \mathrm{CH}_0(X_\Omega)$ is the inverse of f_*.* $\square$

Proof. According to Theorem 1.3, it suffices to prove that a general fibre of f_Ω has trivial Chow group of zero-cycles. By considering a smooth closed fibre of f which is separably rationally connected, we see after pulling back to Ω that f_Ω has a smooth closed fibre which is separably rationally connected. It follows from [15, IV.3.11] that a general fibre of f_Ω is separably rationally connected and hence that a general fibre of f_Ω has trivial Chow group of zero-cycles. $\square$

Theorem 1.7 below, which generalises Theorem 1.3 is irrelevant to the proof of Theorem 1. However, we include it here because of the interesting consequences it has for unramified cohomology, see section 10.

Definition 1.5. A smooth projective variety X over k is said to have *representable* Chow group of algebraically trivial i-cycles if there exists a curve C over Ω and a correspondence $\gamma \in \mathrm{CH}_{i+1}(C \times X_\Omega)$ such that $\gamma_* \mathrm{CH}_0(C)_{alg} = \mathrm{CH}_i(X_\Omega)_{alg}$.

Lemma 1.6. *Let X be a smooth projective variety over k. Then the following statements are equivalent.*

1. $\mathrm{CH}_0(X)_{\mathrm{alg}}$ *is representable.*
2. *The Albanese map* $\mathrm{alb}_{X_\Omega} : \mathrm{CH}_0(X_\Omega)_{alg} \to \mathrm{Alb}_{X_\Omega}(\Omega)$ *is an isomorphism (this map is always surjective).*
3. *If $\iota : C \to X$ is any smooth linear section of X of dimension* 1*, the induced map* $\iota_* : \mathrm{CH}_0(C_\Omega) \to \mathrm{CH}_0(X_\Omega)$ *is surjective.*

Proof. This was proved by Jannsen [10, 1.6]. □

Theorem 1.7. *Let $f : X \to S$ be a generically smooth and dominant morphism between smooth projective varieties defined over a field k. Assume that the general fibre Y of f_Ω is such that* $\mathrm{CH}_0(Y)_{\mathrm{alg}}$ *is representable. Then* $\mathrm{CH}_0(X)$ *is supported in dimension* $\dim S+1$. *This means that there exists a smooth projective variety T over k of dimension* $\dim S+1$ *and a correspondence* $\Gamma \in \mathrm{CH}^{\dim X}(T \times X)$ *such that* $(\Gamma_\Omega)_* : \mathrm{CH}_0(T_\Omega) \to \mathrm{CH}_0(X_\Omega)$ *is surjective.*

Proof. Let $\iota : H \to X$ be a smooth linear section of X of dimension $\dim S + 1$ such that f restricted to H is dominant and generically smooth. Let U be an open subset of S_Ω such that $f_U : X_U \to U$ is smooth, $f_U|_{H_U} : H_U \to U$ is smooth and such that the fibres of f_U have representable Chow group of zero-cycles.

Let's prove that $(\iota_\Omega)_* : \mathrm{CH}_0(H_\Omega) \to \mathrm{CH}_0(X_\Omega)$ is surjective. Let p be a closed point of X_Ω. By Chow's moving lemma, the zero-cycle $[p]$ is rationally equivalent to a cycle $\alpha = \sum a_i \cdot [p_i]$ supported on X_U. Let $s_i := f(p_i) \in U$. Then, by choice of U, each cycle $[p_i]$ is rationally equivalent on X_{s_i} to a cycle β_i supported on H_{s_i}. Now clearly $\sum a_i \cdot \beta_i$ is in the image of $(\iota_\Omega)_* : \mathrm{CH}_0(H_\Omega) \to \mathrm{CH}_0(X_\Omega)$ and hence so is $[p]$. □

Remark 1.8. The descent properties of Theorems 1.3 & 1.7, i.e. the fact that the correspondence Γ in those theorems can be chosen to be defined over a field of definition of f, are essential to proving that X has a Chow–Künneth decomposition defined over the field of definition of f (Theorem 4.1) and to proving Proposition 10.1.

Remark 1.9. Under the assumptions of the above theorem, it is not true that if $\mathrm{CH}_0(S)$ is supported in dimension, say n, then $\mathrm{CH}_0(X)$ is supported in dimension $n+1$. Consider for example a Lefschetz fibration $S \to \mathbf{P}^1$ associated to a smooth projective surface S with non-representable Chow group of zero-cycles.

Remark 1.10. Let Y be a smooth projective variety over k and let $H \to Y$ be a smooth linear section of dimension n. A consequence of the conjectures of Bloch and Beilinson is that, if $\mathrm{CH}_0(Y_\Omega)$ is supported in dimension n, then $\mathrm{CH}_0(H_\Omega) \to \mathrm{CH}_0(Y_\Omega)$ is surjective. Therefore, if one believes in the conjectures of Bloch and Beilinson, then the above theorem can be extended to the following. Let $f : X \to S$ be a generically smooth and dominant morphism between smooth projective varieties defined over a field k. Let n be a positive integer and assume that the general fibre Y of f is such that $\mathrm{CH}_0(Y)$ is supported in dimension n. Then $\mathrm{CH}_0(X)$ is supported in dimension $\dim S + n$.

2 Effective motives with trivial Chow group of zero-cycles

The following theorem was mentioned to me by Bruno Kahn. Its proof uses, among other things, the technique of Bloch and Srinivas [1] together with Theorem 2.4.1 of Kahn–Sujatha [11] where it is shown that a correspondence $\Gamma \in \mathrm{CH}_d(X \times Y) = \mathrm{Hom}(\mathfrak{h}(X), \mathfrak{h}(Y))$ which vanishes in $\mathrm{CH}_d(U \times Y)$ for some open subset $U \subset X$ factors through some motive $\mathfrak{h}(Z)(1)$ with $\dim Z = d - 1$.

Theorem 2.1. *Let $M = (X, p)$ be an effective Chow motive such that $\mathrm{CH}_0(M_\Omega) = 0$. Then there exist a smooth projective variety Y of dimension $\dim X - 1$ and an idempotent $q \in \mathrm{CH}_{\dim Y}(Y \times Y)$ such that $(X, p, 0) \simeq (Y, q, 1)$.*

Proof. Without loss of generality, we can assume that X is connected. Since we are working with rational coefficients, the assumption $\mathrm{CH}_0(M_\Omega) = 0$ implies $\mathrm{CH}_0(M_{k(X)}) = 0$ which means $p_* \mathrm{CH}_0(X_{k(X)}) = 0$. In particular, if η_X denotes the generic point of X, then we have $p_*\eta_X = 0$. But $p_*\eta_X$ is the restriction of $p \in \mathrm{CH}_d(X \times X)$ to $\varinjlim \mathrm{CH}_d(U \times X) = \mathrm{CH}_0(X_{k(X)})$, where the limit is taken over all open subsets U of X. Therefore, by the localization exact sequence for Chow groups, there exist a proper closed subset $D \subset X$ and a correspondence $\gamma \in \mathrm{CH}_d(D \times X)$ such that γ maps to p via the inclusion $D \times X \to X \times X$. Up to shrinking the open U for which $p|_{U \times X}$ vanishes, we can assume that D has pure dimension $d - 1$. Let $Y \to D$ be an alteration of D and let $\sigma : Y \to D \hookrightarrow X$ be the composite morphism. The induced map $\mathrm{CH}_d(Y \times X) \to \mathrm{CH}_d(D \times X)$ is surjective and we have $p = (\sigma \times \mathrm{id}_X)_* f$, where $f \in \mathrm{CH}_d(Y \times X)$ is a lift of γ. Then, by [5, 16.1.1], we have $(\sigma \times \mathrm{id}_X)_* f = f \circ {}^t\Gamma_\sigma$. This yields a factorisation $p = f \circ g$, where $f \in \mathrm{CH}_d(Y \times X)$ and $g = {}^t\Gamma_\sigma \in \mathrm{CH}^d(X \times Y)$. Let's consider the correspondence $q := g \circ f \circ g \circ f = g \circ p \circ f \in \mathrm{CH}_{d-1}(Y \times Y)$. It is straightforward to check that q is an idempotent, and that $p \circ f \circ q \circ g \circ p = p$ as well as $q \circ g \circ p \circ f \circ q = q$. These last two equalities exactly mean that $p \circ f \circ q$ seen as a morphism of Chow motives from $(Y, q, 1)$ to $(X, p, 0)$ is an isomorphism with inverse $q \circ g \circ p$. □

As noted by Sergey Gorchinskiy [7], this theorem admits the following corollary

Corollary 2.2. *Let m and n be positive integers. Let $M = (X, p)$ be an effective Chow motive such that $\mathrm{CH}_i(M_\Omega) = 0$ for $i \leq n-1$ and $\mathrm{CH}_j(M_\Omega^\vee(\dim X)) = \mathrm{CH}_{j-\dim X}(M_\Omega^\vee) = 0$ for $j \leq m-1$. Then there exists a smooth projective variety Y of dimension $\dim X - m - n$ such that M is isomorphic to a direct summand of $\mathfrak{h}(Z)(n)$.*

Proof. By Theorem 2.1 applied n times, there is a smooth projective variety Y of dimension $\dim X - n$ and an idempotent $q \in \mathrm{CH}_{\dim X - n}(Y \times Y)$ such that $M \simeq (Y, q, n)$. We then have by duality $M^\vee(\dim X) \simeq (Y, {}^tq)$. Applying m times Theorem 2.1 gives a smooth projective variety Z of dimension $\dim Y - m = \dim X - n - m$ such that

$M^\vee(\dim X)$ is isomorphic to a direct summand of $\mathfrak{h}(Z)(m)$. Therefore, after dualizing, we see that M is isomorphic to a direct summand of $\mathfrak{h}(Z)(n)$. $\square$

3 The Albanese motive and the Picard motive

The results of the previous section show that it is convenient not only to deal with smooth projective varieties but also with idempotents. It may, however, be difficult to deal with idempotents because these are usually not central. Here, we extend the construction of Murre's Albanese projector to the case of Chow motives.

I thank Sergey Gorchinskiy [7] for mentioning the following basic lemma and the construction that ensues. The difference between Lemma 3.1 and Lemma 4.11 is that Lemma 4.11 makes it possible to preserve self-duality when orthonormalising a family of idempotents.

Lemma 3.1. *Let p be an idempotent endomorphism of an object $A \oplus B$ in a Karoubi closed additive category. Let p_A denote the composition*

$$A \hookrightarrow A \oplus B \xrightarrow{p} A \oplus B \to A$$

and similarly for p_B. Assume that p is upper-triangular, that is, the composition

$$A \hookrightarrow A \oplus B \xrightarrow{p} A \oplus B \to B$$

vanishes. Then p_A and p_B are idempotents and there is a canonical isomorphism

$$\mathrm{Im}\,(p) \simeq \mathrm{Im}\,(p_A) \oplus \mathrm{Im}\,(p_B).$$

Proof. It is immediate that p_A and p_B are projectors. The required isomorphism is given in the opposite direction by p. $\square$

Given a smooth projective variety X of dimension $d \geq 2$, consider the decomposition constructed by Murre [18]:

$$\mathfrak{h}(X) = \mathbb{1} \oplus \mathfrak{h}_1(X) \oplus M \oplus \mathfrak{h}_{2d-1}(X) \oplus \mathbb{1}(d). \qquad (*)$$

Since $\mathrm{Hom}(\mathbb{1}(d), \mathfrak{h}(X)) = \mathrm{Hom}(\mathbb{1}(d), \mathbb{1}(d))$, we obtain

$$\mathrm{Hom}(\mathbb{1}(d), \mathbb{1}) = \mathrm{Hom}(\mathbb{1}(d), \mathfrak{h}_1(X)) = \mathrm{Hom}(\mathbb{1}(d), M) = \mathrm{Hom}(\mathbb{1}(d), \mathfrak{h}_{2d-1}(X)) = 0.$$

For any curve C, we have

$$\mathrm{Hom}(\mathfrak{h}(C)(d-1), \mathfrak{h}(X)) = \mathrm{Pic}(C \times X).$$

This implies that

$$\mathrm{Hom}(\mathfrak{h}_{2d-1}(X), M) = 0.$$

By duality, we conclude that there are no morphisms going from the right to the left in the decomposition of $\mathfrak{h}(X)$ as above. By Lemma 3.1 applied several times, for any idempotent endomorphism p of $\mathfrak{h}(X)$, we have a decomposition

$$\mathrm{Im}\,(p) \simeq \mathrm{Im}\,(p_0) \oplus \mathrm{Im}\,(p_1) \oplus \mathrm{Im}\,(p_M) \oplus \mathrm{Im}\,(p_{2d-1}) \oplus \mathrm{Im}\,(p_{2d}),$$

where $\mathrm{Im}\,(p)$ is a direct summand in $\mathfrak{h}(X)$ and where each direct summand appearing in the decomposition of $\mathrm{Im}\,(p)$ above is a direct summand of the corresponding direct summand appearing in the decomposition $(*)$ of $\mathfrak{h}(X)$. The following proposition is then straightforward.

Proposition 3.2. *Let (X,p) be a motive. The idempotents p_0, p_1, p_{2d-1} and p_{2d} constructed above enjoy the following properties :*

- *(X,p_0) is isomorphic to $\mathbb{1}^{\oplus n}$ for some n and $H_*(X,p_0) = H_0(X,p)$.*
- *(X,p_1) is isomorphic to a direct summand of the $\mathfrak{h}_1(C)$ for some curve C and $H_*(X,p_1) = H_1(X,p)$.*
- *(X,p_{2d-1}) is isomorphic to a direct summand of the $\mathfrak{h}_1(C)(d-1)$ for some curve C and $H_*(X,p_{2d-1}) = H_{2d-1}(X,p)$.*
- *(X,p_{2d}) is isomorphic to $\mathbb{1}(d)^{\oplus n}$ for some n and $H_*(X,p_{2d}) = H_{2d}(X,p)$.*

Definition 3.3. The idempotent p_1 is called the *Albanese projector* and the idempotent p_{2d-1} is called the *Picard projector*.

Remark 3.4. The Albanese and Picard projectors are not unique.

As an immediate corollary, we can extend Murre's theorem on surfaces [18] to direct summands of Chow motives of surfaces.

Theorem 3.5. *Let $M = (S,p)$ be a Chow motive where S is a smooth projective surface. Then M has a Murre decomposition. If, moreover, M is Kimura finite-dimensional [13], then M satisfies Murre's conjecture (C).*

Proof. The correspondences p_0, p_1, p_3 and p_4 of Proposition 3.2 together with $p_2 := p - \sum_{i\neq 2} p_i$ give a Chow–Künneth decomposition for M. That such a decomposition satisfies Murre's conjecture (B) is obvious. Murre's conjecture (D) is possibly unclear only for one-cycles. Given a motive N, a correspondence $\gamma \in \mathrm{Hom}(\mathfrak{h}(S), N)$ that acts trivially on $H^1(S)$ acts trivially on $\mathrm{Pic}^0_S = \mathrm{CH}^1(S)_{\mathrm{hom}}$. Since p_2 and p_3 are the only projectors that act possibly non-trivially on $\mathrm{CH}_1(S)$ and since $(p_3)_* \mathrm{CH}_1(M) \subseteq \mathrm{CH}_1(M)_{\mathrm{hom}}$, we get conjecture (D) for one-cycles on M, see also Proposition 6.2. In the case when M is Kimura finite-dimensional, Murre's conjecture (C) for M can be obtained by applying [21, Proposition 3.1]. □

4 Self-dual Chow–Künneth decompositions

Let X be a smooth projective variety of dimension d over k. It is proved in [22, Theorem 4.2] that if the cohomology of X in degree $\neq d$ is generated by the cohomology of curves, then X admits a self-dual Chow–Künneth decomposition. Precisely, if $H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$ for all $i \neq d$, where N is the coniveau filtration, then X has a Chow–Künneth decomposition. It follows from Theorem 1.3, together with a decomposition of the diagonal argument à la Bloch–Srinivas, that a fourfold which is fibred by rationally connected threefolds over a curve has a self-dual Chow–Künneth decomposition [22, Corollary 4.7]. Del Angel and Müller-Stach [4] proved that unirational threefolds have a Chow–Künneth decomposition. To do so, they use Mori theory to reduce to the case of a conic fibration. In this section, we generalise the aforementioned results by proving the following:

Theorem 4.1. *Let $f : X \to S$ be a dominant morphism defined over a field k from a smooth projective variety X to a smooth projective surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Suppose that X has dimension $d \leq 4$. Then X has a self-dual Chow–Künneth decomposition $\{p_i\}_{0 \leq i \leq 2d}$.*
Moreover, this decomposition can be chosen so as to satisfy the following properties:

- *p_0 factors through a point P_0, i.e. (X, p_0) is isomorphic to $\mathfrak{h}(P_0)$.*
- *p_1 and p_3 factor through a curve, i.e. there is a curve C_0 (resp. C_1) such that (X, p_1) (resp. (X, p_3)) is a direct summand of $\mathfrak{h}_1(C_0)$ (resp. $\mathfrak{h}_1(C_1)(1)$).*
- *p_2 factors through a surface, i.e. there is a surface S_0 such that (X, p_2) is isomorphic to a direct summand of $\mathfrak{h}(S_0)$.*
- *If $d \leq 4$, p_4 factors through a surface, i.e. there is a surface S_1 such that (X, p_4) is isomorphic to a direct summand of $\mathfrak{h}(S_1)(1)$.*

In particular, this will give an alternate proof to del Angel and Müller-Stach's result for conic fibrations over a surface.
We divide the proof into several steps.

4.1 The projectors π_0, π_1 and π_2^{tr}

The surface S has a Chow–Künneth decomposition [18, 20] $\{\pi_0^S, \pi_1^S, \pi_2^S, {}^t\pi_1^S, {}^t\pi_0^S\}$. The motive (S, π_2^S) admits a direct summand $(S, \pi_2^{tr,S})$ called its transcendental part, cf [12]. The action of the idempotent $\pi_2^{tr,S}$ on the homology of S is the orthogonal projector on the orthogonal complement for cup-product of the span of the classes of algebraic one-cycles inside $H_2(S)$. In characteristic zero, for Betti cohomology, $(\pi_2^{tr,S})_* H_*(S)$ is the sub-Hodge structure of $H_2(S)$ generated by $H^{2,0}(S) = H^2(S, O_S)$ thanks to the Lefschetz $(1,1)$-theorem. The idempotent $\pi_2^{tr,S}$ acts trivially on $\mathrm{CH}_1(S_\Omega)$ and on $\mathrm{CH}_2(S_\Omega)$ so that $\mathrm{CH}_*(S, \pi_2^{tr,S}) = \mathrm{CH}_0(S, \pi_2^{tr,S})$.

By Proposition 1.1, there is a correspondence $\Gamma \in \mathrm{CH}^d(S \times X)$ such that $\Gamma_f \circ \Gamma = \Delta_S$, so that the Chow motive of S is a direct summand of the Chow motive of X. We thus define mutually orthogonal idempotents $\pi_0 := \Gamma \circ \pi_0^S \circ \Gamma_f$, $\pi_1 := \Gamma \circ \pi_1^S \circ \Gamma_f$ and $\pi_2^{tr} := \Gamma \circ \pi_2^{tr,S} \circ \Gamma_f$. Because the idempotents π_0^S, π_1^S and $\pi_2^{tr,S}$ in $\mathrm{CH}_2(S \times S)$ satisfy $(\pi_0^S)_* H_*(S) = H_0(S)$, $(\pi_1^S)_* H_*(S) = H_1(S)$ and $(\pi_2^{tr,S})_* H_*(S) \subset H_2(S)$, we see that $(\pi_0)_* H_*(X) \subset H_0(X)$, $(\pi_1)_* H_*(X) \subset H_1(X)$ and $(\pi_2^{tr})_* H_*(X) \subset H_2(X)$.

Since $\mathrm{CH}_0(S_\Omega) = (\pi_0^S + \pi_1^S + \pi_2^{tr,S})_* \mathrm{CH}_0(S_\Omega)$ and since both $(\Gamma_f)_* : \mathrm{CH}_0(X_\Omega) \to \mathrm{CH}_0(S_\Omega)$ and $\Gamma_* : \mathrm{CH}_0(S_\Omega) \to \mathrm{CH}_0(X_\Omega)$ are isomorphisms by Theorem 1.3, we get

Proposition 4.2. $(\pi_0 + \pi_1 + \pi_2^{tr})_* \mathrm{CH}_0(X_\Omega) = \mathrm{CH}_0(X_\Omega)$. $\square$

This yields that the decomposition $\mathfrak{h}(X) = (X, \pi_0) \oplus (X, \pi_1) \oplus (X, \pi_2^{tr}) \oplus M$ satisfies $\mathrm{CH}_0(M_\Omega) = 0$. Theorem 2.1 gives a smooth projective variety Y of dimension one less than the dimension d of X together with an idempotent $q \in \mathrm{CH}_{d-1}(Y \times Y)$ such that $M \simeq (Y, q, 1)$.

By definition $H_*(Y, q, 1) = H_{*-2}(Y, q) = q_* H_{*-2}(Y)$. Consequently, we see that $H_0(M) = 0$ and also that $H_1(M) = 0$. Therefore, $(\pi_0)_* H_*(X) = H_0(X)$ and $(\pi_1)_* H_*(X) = H_1(X)$.

It is interesting to note that we can show that the π_i's act as the Künneth projectors on homology only after having determined their action on Chow groups.

4.2 Chow–Künneth decomposition for $\dim X = 3$

When $\dim X = 3$, the Chow motive $\mathfrak{h}(X)$ decomposes as $(X, \pi_0) \oplus (X, \pi_1) \oplus (X, \pi_2^{tr}) \oplus M$ where M is isomorphic to a motive $(Y, q, 1)$ with $\dim Y = 2$. In other words, $\mathfrak{h}(X)$ is isomorphic to a direct sum of direct summands of twisted motives of surfaces. Theorem 3.5 then says that X has a Murre decomposition. This is made more precise in §4.7.

4.3 A first approach to splitting the motive of X when $\dim X \geq 3$

Ultimately, our goal is to define a self-dual Chow–Künneth decomposition for X with $\dim X \leq 4$. Let's thus study the orthogonality relations between the idempotents π_0, π_1, π_2^{tr} and their transpose ${}^t\pi_0$, ${}^t\pi_1$, ${}^t\pi_2^{tr}$. We already know that π_0, π_1 and π_2^{tr} are mutually orthogonal. For dimension reasons (see also §4.6), the only possible missing orthogonality relations concern $\pi_2^{tr} \circ {}^t\pi_2^{tr}$ and ${}^t\pi_2^{tr} \circ \pi_2^{tr}$. The crucial point here is that $\pi_2^{tr} \circ {}^t\pi_2^{tr} = 0$ and that ${}^t\pi_2^{tr} \circ \pi_2^{tr}$ acts trivially on $\mathrm{CH}_*(X_\Omega)$. Let's prove these facts.

Proposition 4.6 gives two proofs that $\pi_2^{tr} \circ {}^t\pi_2^{tr} = 0$. The first proof relies on Lemma 4.3 and is particular to our geometric situation. The second proof relies on Lemma 4.5; it is more general and could be useful in other situations.

On the one hand, we have

Lemma 4.3. *Let $f : X \to S$ be a dominant map between two smooth projective varieties with $\dim X > \dim S$. Then $\Gamma_f \circ {}^t\Gamma_f = 0$.*

Proof. By definition, we have $\Gamma_f \circ {}^t\Gamma_f = (p_{1,3})_*(p_{1,2}^* {}^t\Gamma_f \cap p_{2,3}^* \Gamma_f)$, where $p_{i,j}$ denotes projection from $S \times X \times S$ to the (i,j)-th factor. These projections are flat morphisms, therefore, by flat pullback, we have $p_{1,2}^* {}^t\Gamma_f = [{}^t\Gamma_f \times S]$ and $p_{2,3}^* \Gamma_f = [S \times \Gamma_f]$. It is easy to see that the closed subschemes ${}^t\Gamma_f \times S$ and $S \times \Gamma_f$ of $S \times X \times S$ intersect properly. Their intersection is given by $\{(f(x), x, f(x)) : x \in X\} \subset S \times X \times S$. This is a closed subset of dimension $\dim X$ and its image under the projection $p_{1,3}$ has dimension $\dim S$, which is strictly less than $\dim X$ by assumption. The projection $p_{1,3}$ is a proper map and hence, by proper pushforward, we get that $(p_{1,3})_*[\{(f(x), x, f(x)) \in S \times X \times S : x \in X\}] = 0$. $\square$

On the other hand, we have the following two lemmas, the first of which will be used later on.

Lemma 4.4. *Let $\gamma \in \mathrm{CH}^0(V \times W)$ be a correspondence such that γ_* acts trivially on zero-cycles. Then $\gamma = 0$.*

Proof. We can assume that V and W are both connected. The cycle γ is equal to $a \cdot [V \times W]$ for some $a \in \mathbf{Q}$. Let z be a zero-cycle on V. Then $\gamma_* z = a \cdot \deg z \cdot [W]$. This immediately implies $a = 0$. $\square$

Lemma 4.5. *Let $\gamma \in \mathrm{CH}^1(V \times W)$ be a correspondence such that both γ_* and γ^* act trivially on zero-cycles after base-change to an algebraically closed field over k. Then $\gamma = 0$.*

Proof. Since base-change to a field extension induces an injective map on Chow groups with rational coefficients, we may assume that the base field k is algebraically closed. We may also assume that V and W are connected. We have

$$\mathrm{Pic}(V \times W) = \mathrm{Pic}(V) \times [W] \oplus [V] \times \mathrm{Pic}(W) \oplus \mathrm{Hom}(\mathrm{Alb}_V, \mathrm{Pic}^0_W) \otimes \mathbf{Q}.$$

Let $\varphi \in \mathrm{Hom}(\mathrm{Alb}_V, \mathrm{Pic}^0_W) \otimes \mathbf{Q}$ be the component of γ under the above decomposition. By assumption, γ_* acts trivially on $\mathrm{CH}_0(V)$ and, hence, also on $\mathrm{CH}_0(V)_{\mathrm{hom}}$. The albanese map $\mathrm{CH}_0(V)_{\mathrm{hom}} \to \mathrm{Alb}_V(k)$ is surjective and it follows that $\varphi = 0$. The cycle γ is thus equal to $D_1 \times [W] + [V] \times D_2$ for some divisors $D_1 \in \mathrm{CH}^1(V)$ and $D_2 \in \mathrm{CH}^1(W)$. Let z be a zero-cycle on V. Then $\gamma_* z = \deg z \cdot D_2$. This immediately implies that $D_2 = 0$. Likewise, if $z \in \mathrm{CH}_0(W)$, then $\gamma^* z = 0$ implies $D_1 = 0$. We have thus proved that $\gamma = 0$. $\square$

Proposition 4.6. $\pi_2^{tr} \circ {}^t\pi_2^{tr} = 0$. *More generally,* $\mathrm{Hom}((X, {}^t\pi_2^{tr}), (X, \pi_2^{tr})) = 0$.

Proof. From Lemma 4.3 and from the very definition of π_2^{tr}, it is immediate that $\pi_2^{tr} \circ {}^t\pi_2^{tr} = 0$. Let now α be a correspondence in $\mathrm{CH}_d(X \times X)$. The correspondence $\pi_2 \circ \alpha \circ {}^t\pi_2^{tr}$ factors through a correspondence $\gamma \circ \pi_2^{tr,S} \in \mathrm{CH}_d(S \times S)$. If $d > 4$, then the statement is clear. If $d = 4$, we use the fact that $\pi_2^{tr,S}$ sends zero-cycles on S to zero-cycles in the Albanese kernel of S. Hence, $\gamma \circ \pi_2^{tr,S}$ sends zero-cycles on S to homologically trivial two-cycles on S. In particular, $\gamma \circ \pi_2^{tr,S}$ acts trivially on zero-cycles on S and we can therefore apply Lemma 4.4. Let's now assume that $d = 3$ and let's give a proof using Lemma 4.5 when the base field is a subfield of $\mathbf{C}$. The reason is that we use Abel–Jacobi maps (although it is almost certainly true that the Albanese variety and the Picard variety enjoy the required functoriality properties over any base field). The correspondence $\pi_2^{tr} \circ \alpha \circ {}^t\pi_2^{tr} \in \mathrm{CH}_3(X \times X)$ factors through a correspondence $\pi_2^{tr,S} \circ \gamma \circ {}^t\pi_2^{tr,S} \in \mathrm{CH}^1(S \times S)$. In particular, by functoriality of the Abel–Jacobi map, $\pi_2^{tr,S} \circ \gamma \circ {}^t\pi_2^{tr,S}$ sends 0-cycles on $S_{\mathbf{C}}$ to 1-cycles on $S_{\mathbf{C}}$ which lie in the kernel of the Abel–Jacobi map. This last kernel is trivial. Therefore, $\pi_2^{tr,S} \circ \gamma \circ {}^t\pi_2^{tr,S}$ acts trivially on zero-cycles on S_Ω. Clearly the same holds for its transpose. Therefore, $\pi_2^{tr,S} \circ \gamma \circ {}^t\pi_2^{tr,S} = 0$ and, hence, $\pi_2^{tr} \circ \alpha \circ {}^t\pi_2^{tr} = 0$. $\square$

Proposition 4.7. *${}^t\pi_2^{tr} \circ \pi_2^{tr}$ acts trivially on $\mathrm{CH}_*(X_\Omega)$.*

Proof. The correspondence ${}^t\pi_2^{tr} \circ \pi_2^{tr}$ factors through a correspondence $\gamma \circ \pi_2^{tr,S} \in \mathrm{CH}_{4-d}(S \times S)$. The proposition follows immediately since the idempotent $\pi_2^{tr,S}$ acts trivially on $\mathrm{CH}_1(X_\Omega)$ and on $\mathrm{CH}_2(X_\Omega)$. $\square$

Let's then define $p_0 := \pi_0$, $p_1 := \pi_1$ and $p_2^{tr} := (1 - \frac{1}{2}{}^t\pi_2^{tr}) \circ \pi_2^{tr}$. It is clear that these are idempotents and that $\{p_0, p_1, p_2^{tr}, {}^tp_2^{tr}, {}^tp_1, {}^tp_0\}$ is a set of mutually orthogonal idempotents in $\mathrm{CH}_d(X \times X)$. This yields a splitting

$$\mathfrak{h}(X) = (X, p_0) \oplus (X, p_1) \oplus (X, p_2^{tr}) \oplus (X, {}^tp_2^{tr}) \oplus (X, {}^tp_1) \oplus (X, {}^tp_0) \oplus M.$$

Proposition 4.8. $(p_0 + p_1 + p_2^{tr})_* \mathrm{CH}_0(X_\Omega) = \mathrm{CH}_0(X_\Omega)$.

Proof. By Proposition 4.7 we see that $(p_2^{tr})_* x = (\pi_2^{tr})_* x$ for all $X \in \mathrm{CH}_*(X_\Omega)$. We can therefore conclude with Proposition 4.2. $\square$

Theorem 4.9. *There exists a smooth projective variety Z of dimension $d-2$ and an idempotent $q \in \mathrm{CH}_{d-2}(Z \times Z)$ such that*

$$\mathfrak{h}(X) = (X, p_0) \oplus (X, p_1) \oplus (X, p_2^{tr}) \oplus (X, {}^tp_2^{tr}) \oplus (X, {}^tp_1) \oplus (X, {}^tp_0) \oplus (Z, q, 1).$$

Proof. The theorem is a combination of Proposition 4.8 and Corollary 2.2. $\square$

Theorem 4.10. *If $d \leq 4$, then X has a Murre decomposition.*

Proof. The theorem follows from Theorem 4.9 and Theorem 3.5. $\square$

Let's write $\mathfrak{h}(X) = (X, p) \oplus (Z, q, 1)$, where $p = p_0 + p_1 + p_2^{tr} + {}^tp_2^{tr} + {}^tp_1 + {}^tp_0$. Although $\mathfrak{h}(X) = \mathfrak{h}(X)^\vee(d) = (X, p) \oplus (Z, {}^tq, 1)$, it is not clear that $(Z, q, 1)$ is self-dual, i.e. isomorphic to $(Z, {}^tq, 1)$. Thus we need to refine the above construction.

4.4 The projectors π_2^{alg} and π_3

Until §4.7, the dimension d of X is supposed to be ≥ 4.

Let's go back to the situation and notations of §4.1. Let $p := \Delta_X - (\pi_0 + \pi_1 + \pi_2^{tr})$. We have the decomposition $\mathfrak{h}(X) = (X, \pi_0) \oplus (X, \pi_1) \oplus (X, \pi_2^{tr}) \oplus M$ with $M = (X, p)$ isomorphic to $(Y, q, 1)$. Choose an isomorphism $f : (Y, q, 1) \to M$ and let $g : M \to (Y, q, 1)$ be its inverse. Let q_0^Y and q_1^Y be respectively the point projector and the Albanese projector of Proposition 3.2 for $(Y, q, 0)$. We define idempotents $\pi_2^{alg} := f \circ q_0^Y \circ g$ and $\pi_3 := f \circ q_1^Y \circ g$.

These two idempotents are orthogonal and are obviously orthogonal to the idempotents π_0, π_1 and π_2^{tr} previously defined. Their action on cohomology is the expected one: we have $H_2(X) = H_2(X, \pi_2^{tr}) \oplus H_2(M)$ but $H_2(M) = H_0(Y, q) = H_0(Y, \pi_0^Y)$. Therefore $\pi_2 := \pi_2^{tr} + \pi_2^{alg}$ induces the Künneth projector $H_*(X) \to H_2(X) \to H_*(X)$. We also have $H_3(X) = H_3(M) = H_1(Y, q) = H_1(Y, \pi_1^Y)$ and hence $(\pi_3)_* H_*(X) = H_3(X)$.

4.5 The remaining projectors

We now define $\pi_{2d} := {}^t\pi_0$, $\pi_{2d-1} := {}^t\pi_1$, $\pi_{2d-2} := {}^t\pi_2$ and $\pi_{2d-3} := {}^t\pi_3$. By Poincaré duality, these idempotents satisfy $(\pi_i)_* H_*(X) = H_i(X)$.

4.6 Orthonormalising the projectors

We have the following non-commutative Gram–Schmidt process [22, Lemma 2.12]

Lemma 4.11. *Let V be a $\mathbf{Q}$-algebra and let k and n be positive integers. Let $\pi_0, \ldots, \pi_n$ be idempotents in V such that $\pi_i \circ \pi_j = 0$ whenever $i - j < k$ and $i \neq j$. Then the endomorphisms*

$$p_i := (1 - \frac{1}{2}\pi_n) \circ \cdots \circ (1 - \frac{1}{2}\pi_{i+1}) \circ \pi_i \circ (1 - \frac{1}{2}\pi_{i-1}) \circ \cdots \circ (1 - \frac{1}{2}\pi_0)$$

define idempotents such that $p_i \circ p_j = 0$ whenever $i - j < k + 1$ and $i \neq j$.

Let's state an orthonormalisation result which is valid in a general setting and that we can apply to our particular case of interest.

Theorem 4.12. *Let X be a smooth projective variety of dimension d. Let $\pi_0, \ldots, \pi_n \in \mathrm{CH}_d(X \times X)$ be idempotents such that $\pi_r \circ \pi_s = 0$ for all $r < s$. Then applying n times the Gram–Schmidt process of Lemma 4.11 gives mutually orthogonal idempotents $p_0, \ldots, p_n$ such that $(X, p_r) \simeq (X, \pi_r)$ for all r. Furthermore,*

- *if there exists r such that $(\pi_r)_* H_*(X) = H_r(X)$, then $(p_r)_* H_*(X) = H_r(X)$.*
- *if $\pi_r = {}^t\pi_{n-r}$ for all r, then $p_r = {}^t p_{n-r}$ for all r;*
- *if there exists r such that $\pi_s \circ \pi_r$ acts trivially on $\mathrm{CH}_*(X)$ for all $s > r$, then $(p_r)_* \mathrm{CH}_*(X) = (\pi_r)_* \mathrm{CH}_*(X)$ inside $\mathrm{CH}_*(X)$.*

Proof. The idempotents $\pi_0, \ldots, \pi_n$ satisfy the assumptions of Lemma 4.11 with $k = 1$. Therefore, after having run n times the orthonormalisation process of Lemma 4.11, we get mutually orthogonal idempotents. In order to prove the theorem, it suffices to prove each statement after each application of the orthonormalisation process. Given r, the isomorphism $(X, p_r) \simeq (X, \pi_r)$ is simply given by the correspondence $\pi_r \circ p_r$; its inverse is $p_r \circ \pi_r$ as can be readily checked.

If $(\pi_r)_* H_*(X) = H_r(X)$, then the image of π_r in $H_d(X \times X) \simeq \mathrm{Hom}(H_*(X), H_*(X))$ is central. Therefore, if $\pi_r \circ \pi_s = 0$ for all $r < s$, then the image of $\pi_s \circ \pi_r$ in $H_d(X \times X)$ is trivial for all $s > r$. It is then straightforward to conclude that $(p_r)_* H_*(X) = H_r(X)$.

If $\pi_r = {}^t\pi_{n-r}$ for all r, then it is straightforward to check from the formula of Lemma 4.11 that $p_r = {}^t p_{n-r}$ for all r

Let's fix r. Given the isomorphism $(X, p_r) \simeq (X, \pi_r)$, it is very tempting to conclude that $(p_r)_* \mathrm{CH}_*(X_\Omega) = (\pi_r)_* \mathrm{CH}_*(X_\Omega)$ in $\mathrm{CH}_*(X_\Omega)$. However, this appears not to be obvious at all and a careful analysis of the non-commutative Gram–Schmidt process needs to be carried on. By examining the formula defining the idempotent p_r, together with the assumption that $\pi_s \circ \pi_r$ acts trivially on $\mathrm{CH}_*(X)$ for all $s > r$, we see that, for $x \in \mathrm{CH}_*(X_\Omega)$, we have $(p_r)_* x = (\pi_r)_* x \in \mathrm{CH}_*(X_\Omega)$. □

We wish to apply Theorem 4.12 to the set of idempotents $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_{2d-3}, \pi_{2d-2}, \pi_{2d-1}, \pi_{2d}\}$. In order to do so, we have to show that $\pi_i \circ \pi_j = 0$ whenever $i < j$. We already know that π_0, π_1, π_2 and π_3 are mutually orthogonal. Let's prove the missing orthogonality relations. First we have :

- $\pi_0 \circ {}^t\pi_0 = \pi_0 \circ {}^t\pi_1 = \pi_0 \circ {}^t\pi_2 = \pi_0 \circ {}^t\pi_3 = 0$.
- $\pi_1 \circ {}^t\pi_1 = \pi_1 \circ {}^t\pi_2 = \pi_1 \circ {}^t\pi_3 = 0$.
- $\pi_2 \circ {}^t\pi_2^{alg} = 0$ and hence $\pi_2 \circ {}^t\pi_2 = 0$ thanks to Proposition 4.6.

These relations are obvious : one uses a dimension argument as well as the fact that π_0 (resp. π_1, π_2^{tr}, π_2^{alg}, π_3) factors through a variety P_0 (resp. C_0, S, P_1, C_1) of dimension 0 (resp. 1, 2, 0, 1). For instance, $\pi_1 \circ {}^t\pi_3$ factors through a correspondence in $\mathrm{CH}_{d-1}(C_1 \times C_0)$. If $d \geq 4$, then this last group is trivial.

Using the same arguments, the following orthogonality relations can be further proved. These relations are not necessary to run the non-commutative Gram–Schmidt process but are essential to the proof of Proposition 4.15.

- ${}^t\pi_0 \circ \pi_0 = {}^t\pi_0 \circ \pi_1 = {}^t\pi_0 \circ \pi_2 = {}^t\pi_0 \circ \pi_3 = 0$.
- ${}^t\pi_1 \circ \pi_1 = {}^t\pi_1 \circ \pi_2 = {}^t\pi_1 \circ \pi_3 = 0$.
- ${}^t\pi_2 \circ \pi_2^{alg} = 0$.
- ${}^t\pi_3 \circ \pi_2^{alg} = 0$.

Secondly, the remaining orthogonality relations needed to run the non-commutative Gram–Schmidt process follow from Lemma 4.4.

- $\pi_2 \circ {}^t\pi_3 = 0$. The correspondence $\pi_2 \circ {}^t\pi_3$ factors through a correspondence $\gamma \in \mathrm{CH}_{d-1}(C_1 \times S_0)$, where S_0 is a surface, that sends zero-cycles to homologically trivial cycles on S_0. Again, if $d > 4$, then the result is trivial. If $d = 4$, then we conclude by Lemma 4.4.
- $\pi_3 \circ {}^t\pi_3 = 0$. The correspondence $\pi_3 \circ {}^t\pi_3$ factors through a correspondence $\gamma \in \mathrm{CH}_{d-2}(C_1 \times C_1)$ that sends zero-cycles to homologically trivial cycles on C. Again, if $d > 4$, then the result is trivial. If $d = 4$, then we conclude by Lemma 4.4.

We are now in a position to apply Theorem 4.12 to obtain a set of mutually orthogonal idempotents $\{p_0, p_1, p_2, p_3, p_{2d-3}, p_{2d-2}, p_{2d-1}, p_{2d}\}$ such that $p_{2d-i} = {}^tp_i$ which induce the expected Künneth projectors modulo homological equivalence.

Remark 4.13. It follows from the above discussion that the only possible missing orthogonality relations among the idempotents $\pi_0, \pi_1, \pi_2^{alg}, \pi_2^{tr}, \pi_3$ and their transpose are the following.

- ${}^t\pi_3 \circ \pi_2^{tr}$.

- ${}^t\pi_2^{tr} \circ \pi_2^{tr}$.
- ${}^t\pi_3 \circ \pi_3$.

It can then be checked that it is actually enough to run the non-commutative Gram–Schmidt process only once on the set of idempotents $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_{2d-3}, \pi_{2d-2}, \pi_{2d-1}, \pi_{2d}\}$ to obtain a set of mutually orthogonal idempotents. We can therefore describe the p_i's in terms of the π_i's by not too complicated explicit formulas. Such formulas may then be used, for instance, to give a quicker proof of the motivic Lefschetz conjecture for X. However, we describe a method that might be useful in other situations where the Gram–Schmidt process needs to be run several times.

The following proposition is fundamental to proving Proposition 4.15 and, hence, to proving Murre's conjectures for X.

Proposition 4.14. *Let p and q be any two distinct idempotents among the idempotents π_0, π_1, π_2, π_3, π_{2d-3}, π_{2d-2}, π_{2d-1} and π_{2d}. Then $p \circ q$ acts trivially on $\mathrm{CH}_l(X_\Omega)$ for all l.*

Proof. From remark 4.13 we only need to prove that ${}^t\pi_3 \circ \pi_2^{tr}$, ${}^t\pi_2^{tr} \circ \pi_2^{tr}$ and ${}^t\pi_3 \circ \pi_3$ act trivially on $\mathrm{CH}_*(X_\Omega)$. In the first case, ${}^t\pi_3 \circ \pi_2^{tr}$ factors through a correspondence $\gamma \circ \pi_2^{tr,S} \in \mathrm{CH}_0(S \times C_1)$ for some curve C_1 and it therefore acts trivially on $\mathrm{CH}_*(X_\Omega)$ because $\pi_2^{tr,S}$ only acts possibly non-trivially on $\mathrm{CH}_0(S_\Omega)$. In the second case, ${}^t\pi_2^{tr} \circ \pi_2^{tr}$ factors through a correspondence $\gamma \circ \pi_2^{tr,S} \in \mathrm{CH}_0(S \times S)$ and we conclude in the same way. In the last case, ${}^t\pi_3 \circ \pi_3$ factors through a correspondence $\gamma \circ \pi_1^{C_1} \in \mathrm{CH}_0(C_1 \times C_1)$ which also acts trivially on $\mathrm{CH}_*((C_1)_\Omega)$ because $\pi_1^{C_1}$ acts trivially on $\mathrm{CH}_1((C_1)_\Omega)$. ☐

Proposition 4.15. $(p_0 + p_1 + p_2)_* \mathrm{CH}_0(X_\Omega) = \mathrm{CH}_0(X_\Omega)$.

Proof. Proposition 4.14 and Theorem 4.12 imply that $(\pi_i)_* x = (p_i)_* x$ for $i = 0, 1$ or 2 and for all $x \in \mathrm{CH}_0(X_\Omega)$. By Proposition 4.2, this yields $(p_0+p_1+p_2)_* \mathrm{CH}_0(X_\Omega) = \mathrm{CH}_0(X_\Omega)$ as claimed. ☐

Finally, when $d = 4$, we define $p_4 := \Delta_X - \sum_{i \neq 4} p_i$. The set $\{p_i\}_{0 \leq i \leq 8}$ is then a self-dual Chow–Künneth decomposition for X. Moreover, p_4 has the following property.

Proposition 4.16. *(X, p_4) is isomorphic to a direct summand of $\mathfrak{h}(S_1)(1)$ for some smooth projective surface S_1.*

Proof. Proposition 4.15 implies that $(p_4)_* \mathrm{CH}_0(X_\Omega) = 0$. Also we know that $p_4 = {}^t p_4$. Therefore, the result follows immediately from Corollary 2.2. ☐

4.7 Back to the case $\dim X = 3$

Let's now consider the case of a conic fibration over a surface. In section 4.2, we already gave a quick argument showing that X has a Chow–Künneth decomposition. As in the case $\dim X = 4$, we want to show that a Chow–Künneth decomposition for X can be chosen to be self-dual, a result which is not shown in [4]. In order to prove Murre's conjectures for such a decomposition (which will be done in section 6), we also want to show that the middle idempotent factors through a curve.

For this purpose, we define π_0, π_1, π_2^{tr}, π_2^{alg} and $\pi_2 := \pi_2^{tr} + \pi_2^{alg}$ the same way we did in sections 4.1 and 4.4. The only difference with section 4.4 is that we don't define an idempotent π_3. We then define $\pi_6 = {}^t\pi_0$, $\pi_5 = {}^t\pi_1$ and $\pi_4 = {}^t\pi_2$. As in sections 4.1 and 4.4, it is easy to see that these do define the Künneth projectors in homology.

As before, we have $\pi_i \circ \pi_j = 0$ for all $i < j$ not equal to 3. These relations make it possible to run the non-commutative Gram–Schmidt process and to get mutually orthogonal idempotents $p_0, p_1, p_2, p_4, p_5, p_6$ such that $(p_i)_* H_*(X) = H_i(X)$ and $p_{6-i} = {}^t p_i$ for all $i \neq 3$. Setting $p_3 := \Delta_X - \sum_{i\neq 3} p_i$, we thus get a self-dual Chow–Künneth decomposition for X. Again, as before, we have that $\pi_j \circ \pi_i$ acts trivially on $\mathrm{CH}_*(X_\Omega)$ for all $j > i$ not equal to 3. The middle idempotent p_3 has thus the following property.

Proposition 4.17. *There exists a curve C_1 such that (X, p_3) is isomorphic to a direct summand of $\mathfrak{h}_1(C_1)(1)$.*

Proof. As in the proof of Proposition 4.15, we have $(p_0+p_1+p_2)_* \mathrm{CH}_0(X_\Omega) = \mathrm{CH}_0(X_\Omega)$. Therefore, $\mathrm{CH}_0(X_\Omega, p_3) = 0$. But $p_3 = {}^t p_3$, so that $\mathrm{CH}_0(X_\Omega, {}^t p_3) = 0$ too. It follows from Corollary 2.2 that there exists a curve C_1 such that (X, p_3) is isomorphic to a direct summand of $\mathfrak{h}(C_1)(1)$. The fact that the motive (X, p_3) is pure of weight 3 makes it possible to conclude. $\square$

5 The motivic Lefschetz conjecture for X

Let X be a smooth projective variety of dimension d over a field k. Let $i \leq d$ and let $\iota : H \to X$ be a smooth linear section of dimension i and let $L := (\iota, \mathrm{id}_X)_*\Gamma_\iota = \Gamma_\iota \circ {}^t\Gamma_\iota \in \mathrm{CH}_i(X \times X)$. The correspondence L acts on cohomology or Chow groups as intersecting $d - i$ times by a smooth hyperplane section of X. The variety X is said to satisfy the motivic Lefschetz conjecture in degree i if there exist mutually orthogonal idempotents π_i and π_{2d-i} such that $H_*(X, \pi_i) = H_i(X)$ and $H_*(X, \pi_{2d-i}) = H_{2d-i}(X)$ and such that the induced map

$$L : (X, \pi_{2d-i}) \to (X, \pi_i, d - i)$$

is an isomorphism of Chow motives. The variety X is said to satisfy the *motivic Lefschetz conjecture* if it satisfies the motivic Lefschetz conjecture in all degrees $< d$. Note that if X satisfies the motivic Lefschetz conjecture in degree i then X satisfies the Lefschetz standard conjecture in degree i, i.e. there exists a correspondence $\Gamma \in \mathrm{CH}^i(X \times X)$ such that $\Gamma_* : H_i(X) \to H_{2d-i}(X)$ is the inverse to $L : H_{2d-i}(X) \to H_i(X)$. The motivic Lefschetz conjecture for X follows from a combination of the Lefschetz standard conjecture for X and of Kimura's finite dimensionality conjecture for X ; it is thus expected to hold for all smooth projective varieties.

Proposition 5.1. *Let P be a zero-dimensional variety over k. Let $p \in \mathrm{CH}_d(X \times X)$ be an idempotent such that (X, p) is isomorphic to $\mathfrak{h}(P)(i)$ for some integer i satisfying*

$2i \leq d$. *If the induced map* $L : H_{2d-2i}(X, {}^tp) \to H_{2i}(X, p)$ *is an isomorphism, then* $L : (X, {}^tp) \to (X, p, d-2i)$ *is an isomorphism of Chow motives.*

Proof. There exist correspondences $f \in \mathrm{Hom}(\mathfrak{h}(P)(i), (X, p))$ and $g \in \mathrm{Hom}((X, p), \mathfrak{h}(P)(i))$ such that $g \circ f = \mathrm{id}_{\mathfrak{h}(P)(i)}$ and $f \circ g = p$. The correspondence $g \circ L \circ {}^tg \in \mathrm{End}(\mathfrak{h}(P))$ induces an automorphism of $H_0(P)$ and, hence, is itself an automorphism. Therefore, it admits an inverse $\alpha \in \mathrm{End}(\mathfrak{h}(P))$. It is now straightforward to check that ${}^tp \circ {}^tg \circ \alpha \circ g \circ p$ is the inverse of $p \circ L \circ {}^tp$. $\square$

Proposition 5.2. *Let J be an abelian variety over k. Let $p \in \mathrm{CH}_d(X \times X)$ be an idempotent such that (X, p) is isomorphic to $\mathfrak{h}_1(J)(i)$ for some integer i satisfying $2i+1 \leq d$ and such that p is orthogonal to ${}^tp$ (this last condition is automatically satisfied if $2i+1 < d-1$). If the induced map $L : H_{2d-2i-1}(X, {}^tp) \to H_{2i+1}(X, p)$ is an isomorphism, then $L : (X, {}^tp) \to (X, p, d-2i-1)$ is an isomorphism of Chow motives.*

Proof. There exist correspondences $f \in \mathrm{Hom}(\mathfrak{h}_1(J)(i), (X, p))$ and $g \in \mathrm{Hom}((X, p), \mathfrak{h}_1(J)(i))$ such that $g \circ f = \mathrm{id}_{\mathfrak{h}_1(J)(i)}$ and $f \circ g = p$. The correspondence $g \circ L \circ {}^tg \in \mathrm{End}(\mathfrak{h}_1(J))$ induces an automorphism of $H_1(J)$ and, hence, is itself an automorphism (indeed by [20, Prop. 4.5] we have $\mathrm{End}(\mathfrak{h}_1(J)) = \mathrm{End}_k(J) \otimes \mathbf{Q}$ and it is well-known that a map between abelian varieties which induces an isomorphism in degree one homology must be an isogeny). Therefore, it admits an inverse $\alpha \in \mathrm{End}(\mathfrak{h}_1(J))$. It is now straightforward to check that ${}^tp \circ {}^tg \circ \alpha \circ g \circ p$ is the inverse of $p \circ L \circ {}^tp$. $\square$

As already proved by Scholl [20], every smooth projective variety satisfies the motivic Lefschetz conjecture in degrees ≤ 1.

Theorem 5.3. *Let $f : X \to S$ be a dominant morphism defined over a field k from a smooth projective variety X to a smooth projective surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Then X satisfies the motivic Lefschetz conjecture in degrees ≤ 3. In particular, if X has dimension ≤ 4, then X satisfies the motivic Lefschetz conjecture and hence the Lefschetz standard conjecture.*

Proof. By Theorem 4.1, p_0 factors through a point, and p_1 and p_3 factor through the $\mathfrak{h}_1$ of a curve. The hard Lefschetz theorem says that the map $H_{2d-i}(X) \to H_i(X)$ induced by intersecting $d-i$ times with a smooth hyperplane section is an isomorphism. Therefore, the two propositions above give the motivic Lefschetz conjecture in degrees 0, 1 and 3 for X.

Let π_2^{tr} be the idempotent of section 4.1. Let's prove that $L : (X, {}^t\pi_2^{tr}, 0) \to (X, \pi_2^{tr}, d-2)$ is an isomorphism of Chow motives. Because $\iota : H \to X$ is a linear section of X of dimension 2, Proposition 1.1 gives a non-zero integer m such that $\Gamma_f \circ L \circ {}^t\Gamma_f = m \cdot \Delta_S$. It is then straightforward to check that $\frac{1}{m} \cdot {}^t\pi_2^{tr} \circ {}^t\Gamma_f \circ \Gamma_f \circ \pi_2^{tr}$ is the inverse of $\pi_2^{tr} \circ L \circ {}^t\pi_2^{tr}$.

Let π_2^{alg} be the idempotent of section 4.3. Because $L : (X, {}^t\pi_2^{tr}, 0) \to (X, \pi_2^{tr}, d-2)$ is an isomorphism and because $L_* : H_{2d-2}(X) \to H_2(X)$ is an isomorphism by the hard Lefschetz theorem, we see that L induces an isomorphism $L : H_{2d-2}(X, {}^t\pi_2^{alg}) \to$

$H_2(X, \pi_2^{alg})$. Proposition 5.1 then shows that $\pi_2^{alg} \circ L \circ {}^t\pi_2^{alg} \in \mathrm{Hom}((X, {}^t\pi_2^{alg}), (X, \pi_2^{alg}, d-2))$ is an isomorphism.

We have thus showed that $\pi_2 \circ L \circ {}^t\pi_2 \in \mathrm{Hom}((X, {}^t\pi_2), (X, \pi_2, d-2))$ is an isomorphism. Since, by Theorem 4.12, we know that $(X, p_2, d-2) \simeq (X, \pi_2, d-2)$ and $(X, {}^tp_2) \simeq (X, {}^t\pi_2)$, we get that $(X, p_2, d-2)$ is isomorphic to $(X, {}^tp_2)$. However, the isomorphism is induced by $p_2 \circ \pi_2 \circ L \circ {}^t\pi_2 \circ {}^tp_2$ which is not quite the isomorphism we were aiming at.

By Remark 4.13, it can be checked that in our particular setting we have $p_2 \circ \pi_2 = p_2$ so that $p_2 \circ L \circ {}^tp_2$ is an isomorphism with inverse $\frac{1}{m} \cdot {}^tp_2 \circ {}^t\pi_2 \circ {}^t\Gamma_f \circ \Gamma_f \circ \pi_2 \circ p_2$.

Let's however give another proof that $p_2 \circ L \circ {}^tp_2$ is an isomorphism that might be useful in other situations. We can conclude that $p_2 \circ L \circ {}^tp_2$ is an isomorphism if we can show that it is equal to $p_2 \circ \pi_2 \circ L \circ {}^t\pi_2 \circ {}^tp_2$. For this purpose, after examining the formula of Lemma 4.11 defining p_2, it is enough to check that, for all correspondences $\alpha \in \mathrm{CH}_2(X \times X)$, we have $\pi_r \circ \alpha \circ {}^t\pi_2 = 0$ for $r = 0, 1$. This is recorded in the lemma below. $\square$

Lemma 5.4. $\mathrm{Hom}((X, {}^t\pi_2), (X, \pi_r, d-2)) = 0$ *for* $r = 0$ *or* 1.

Proof. When $r = 0$, $\pi_0 \circ \alpha \circ {}^t\pi_2^{alg}$ factors through a correspondence $\gamma \in \mathrm{CH}_1(P_1 \times P_0)$ for some zero-dimensional P' and is thus zero for dimension reasons and $\pi_0 \circ \alpha \circ {}^t\pi_2^{tr}$ factors through a correspondence $\gamma \in \mathrm{CH}^0(S \times P_0)$ with $\gamma^* z = 0$ for any $z \in \mathrm{CH}_0(P)$. Lemma 4.4 then shows that $\gamma = 0$ and hence $\pi_0 \circ \alpha \circ {}^t\pi_2^{tr} = 0$. When $r = 1$, on the one hand, we have that $\pi_1 \circ \alpha \circ {}^t\pi_2^{alg}$ factors through a correspondence $\gamma \in \mathrm{CH}^0(P_1 \times C_0)$ with $\gamma^* z = 0$ for any zero-cycle z on C_0. Lemma 4.4 then shows that $\gamma = 0$ and hence $\pi_1 \circ \alpha \circ {}^t\pi_2^{alg} = 0$. On the other hand, $\pi_1 \circ \alpha \circ {}^t\pi_2^{tr}$ factors through a correspondence $\gamma \in \mathrm{CH}^1(S \times C_0)$ with $\gamma^* z = 0$ for any zero-cycle z on $(C_0)_\Omega$ and $\gamma_* z'$ for any zero-cycle z' on S_Ω by functoriality of the Abel–Jacobi map. Therefore thanks to Lemma 4.5, we get $\gamma = 0$ and hence $\pi_1 \circ \alpha \circ {}^t\pi_2^{tr} = 0$. $\square$

Remark 5.5. The results of this section actually show that, for X as in the theorem above and for the idempotents p_i constructed in §4, the map $L : (X, p_{2d-i}) \to (X, p_i, d-i)$ is an isomorphism for $i \leq 3$ for any choice of a smooth linear section $\iota : H \hookrightarrow X$ of dimension i.

6 Murre's conjectures for X

As shown by Jannsen [10], Murre's conjectures [19] are equivalent to Bloch and Beilinson's. Let's recall them :

(A) X has a Chow–Künneth decomposition $\{\pi_0, \ldots, \pi_{2d}\}$: There exist mutually orthogonal idempotents $\pi_0, \ldots, \pi_{2d} \in \mathrm{CH}_d(X \times X)$ adding to the identity such that $(\pi_i)_* H_*(X) = H_i(X)$ for all i.

(B) $\pi_0, \ldots, \pi_{2l-1}, \pi_{d+l+1}, \ldots, \pi_{2d}$ act trivially on $\mathrm{CH}_l(X)$ for all l.

(C) $F^i\,\mathrm{CH}_l(X) := \mathrm{Ker}\,(\pi_{2l}) \cap \ldots \cap \mathrm{Ker}\,(\pi_{2l+i-1})$ doesn't depend on the choice of the π_j's. Here the π_j's are acting on $\mathrm{CH}_l(X)$.

(D) $F^1\,\mathrm{CH}_l(X) = \mathrm{CH}_l(X)_{\mathrm{hom}}$.

Before we consider Murre's conjectures for X as in Theorem 1, let's consider the following situation. Let $\Pi \in \mathrm{CH}_d(X \times X) = \mathrm{End}(\mathfrak{h}(X))$ be an idempotent which factors as

$$\mathfrak{h}(X) \xrightarrow{g} \mathfrak{h}(Y) \xrightarrow{f} \mathfrak{h}(X)$$

where Y is a smooth projective variety of dimension $\leq l+1$. (Actually, up to replacing Y with $Y \times \mathbf{P}^{l+1-\dim Y}$, we can assume $\dim Y = l+1$.) The arguments in the proof of Propositions 6.1 and 6.2 below are essentially contained in [21] and [22].

Proposition 6.1. *Let Π be as above.*

- *If $\Pi_* H_{2l+1}(X) = 0$, then Π acts trivially on $\mathrm{CH}_l(X)_{\mathrm{hom}}$.*
- *If, moreover, $\Pi_* H_{2l}(X) = 0$, then Π acts trivially on $\mathrm{CH}_l(X)$.*

Proof. Because Π is an idempotent, we see that $g \circ f$ acts trivially on $H^1(Y)$. Therefore, $g \circ f$ acts trivially on $\mathrm{CH}^1(Y)_{\mathrm{hom}}$. Thus, Π acts trivially on $\mathrm{CH}_l(X)_{\mathrm{hom}}$. If, moreover, $\Pi_* H_{2i}(X) = 0$, then $g \circ f$ acts trivially on $\mathrm{CH}^1(Y)$. Thus, Π acts trivially on $\mathrm{CH}_l(X)$. $\square$

Proposition 6.2. *Let Π be as above.*

- *If $\Pi_* H_*(X) = H_{2l}(X)$, then $\mathrm{Ker}\,\big(\Pi_* : \mathrm{CH}_l(X) \to \mathrm{CH}_l(X)\big) = \mathrm{CH}_l(X)_{\mathrm{hom}}$.*
- *Assume that $k \subseteq \mathbf{C}$. If $\Pi_* H_*(X) = H_{2l+1}(X)$, then $\mathrm{Ker}\,\big(\Pi_* : \mathrm{CH}_l(X)_{\mathrm{hom}} \to \mathrm{CH}_l(X)_{\mathrm{hom}}\big) = \mathrm{Ker}\,\big(AJ_l : \mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X) \otimes \mathbf{Q}\big)$. Here AJ_l is the Abel–Jacobi map.*

Proof. In the first case, the inclusion $\subseteq$ follows immediately from the functoriality of the cycle class map with respect to the action of correspondences. The reverse inclusion $\supseteq$ follows from the first point of Proposition 6.1.

In the second case, we consider the Abel–Jacobi map $AJ_l : \mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X) \otimes \mathbf{Q}$ instead of the cycle class map $\mathrm{CH}_l(X) \to H_{2l}(X)$. The Abel–Jacobi map is functorial with respect to the action of correspondences and, if a correspondence $\alpha \in \mathrm{End}(\mathfrak{h}(X))$ induces the identity on $H_{2l+1}(X)$, then α_* induces the identity on $J_l(X) \otimes \mathbf{Q}$. This yields a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_l(X)_{\mathrm{hom}} \arrow[r, "g_*"] \arrow[d, "AJ_l"] & \mathrm{CH}^1(Y)_{\mathrm{hom}} \arrow[r, "f_*"] \arrow[d] & \mathrm{CH}_l(X)_{\mathrm{hom}} \arrow[d, "AJ_l"] \\
J_l(X)(\mathbf{C}) \otimes \mathbf{Q} \arrow[r] & \mathrm{Pic}^0_Y(\mathbf{C}) \otimes \mathbf{Q} \arrow[r] & J_l(X)(\mathbf{C}) \otimes \mathbf{Q}.
\end{tikzcd}$$

where the composite of the two bottom arrows is the identity and where the middle vertical arrow is injective. It is then straightforward to conclude by a simple diagram chase. $\square$

Definition 6.3. A smooth projective variety X of dimension d is said to have a *special* Chow–Künneth decomposition $\{\pi_i\}_{0 \leq i \leq 2d}$ if, for all i,

• π_{2i} factors through a surface, i.e. there is a surface S_i such that (X, π_{2i}) is a direct summand of $\mathfrak{h}(S_i)(i-1)$.

• π_{2i+1} factors through a curve, i.e. there is a curve C_i such that (X, π_{2i+1}) is a direct summand of $\mathfrak{h}_1(C_i)(i)$.

Proposition 6.4. *Let X be a smooth projective variety that has a special Chow–Künneth decomposition. Then homological and algebraic equivalence agree on X, and X satisfies Murre's conjectures (A), (B) and (D). Moreover, if $k \subseteq \mathbf{C}$, then the filtration F on $\mathrm{CH}_l(X)$ does not depend on the choice of a special Chow–Künneth decomposition for X.*

Proof. That homological and algebraic equivalence agree on X follows from the well-known fact that they agree on zero-cycles and on codimension-one cycles. That X satisfies Murre's conjectures (B) is obvious and that X satisfies (D) follows from the first point of Proposition 6.2. That the induced filtration on the Chow groups of X is independent of the choice of a special Chow–Künneth decomposition for X is contained in Proposition 6.2. $\square$

Since the Chow–Künneth decomposition of X as in Theorem 4.1 is a special Chow–Künneth decomposition, we can state the following.

Theorem 6.5. *Let $f : X \to S$ be a dominant morphism defined over a field k between a smooth projective variety X of dimension ≤ 4 and a smooth projective surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Then X has a special Chow–Künneth decomposition which is self-dual and which satisfies Murre's conjectures (B) and (D) and, if $k \subseteq \mathbf{C}$, then the induced filtration F on $\mathrm{CH}_l(X)$ does not depend on the choice of a special Chow–Künneth decomposition for X. Finally, whichever the characteristic of k is, if X is Kimura finite-dimensional, then F does not depend on the choice of a Chow–Künneth decomposition for X.*

Proof. Theorem 4.1 says that X has a self-dual Chow–Künneth decomposition $\{p_i : 0 \leq i \leq 2d\}$ which is special. We may then conclude with Proposition 6.4 that X satisfies Murre's conjectures (B) and (D) and that, if char $k = 0$, then the induced filtration F on $\mathrm{CH}_l(X)$ does not depend on the choice of a special Chow–Künneth decomposition for X. When X is Kimura finite-dimensional, Murre's conjecture (C) follows from applying [21, Proposition 3.1] to X endowed with the Chow–Künneth decomposition given by the p_i's. $\square$

7 Murre's conjectures for $X \times C$

Let $f : X \to S$ be a dominant morphism from a smooth projective fourfold to a smooth projective surface such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Consider the self-dual Chow–Künneth decomposition $\{p_i : 0 \leq i \leq 8\}$ of X given in Theorem 4.1 which, by Proposition 6.4, is a Murre decomposition. Let C be a smooth projective curve and let $\{p_0^C, p_1^C, p_2^C\}$ be a self-dual Chow–Künneth decomposition for C

as described in [20]. Then the variety $X \times C$ has a self-dual Chow–Künneth decomposition given by $q_l := \sum_{i+j=l} p_i \times p_j^C$. The results of [21] make it possible to prove the following.

Theorem 7.1. *The fivefold $X \times C$ endowed with the above self-dual Chow–Künneth decomposition satisfies Murre's conjectures (A), (B) and (D).*

Proof. The idempotents p_0, p_1, p_2, p_3, p_0^C and p_1^C factor through varieties of respective dimension $0, 1, 2, 1, 0$ and 1. Therefore, (X, q_i) is isomorphic to a direct summand of the twisted motive of a surface for i even. It is isomorphic to the direct summand of the twisted motive of a curve for $i = 1$ or 9, and it is isomorphic to the direct summand of the twisted motive of a threefold for i odd $\neq 1, 9$. Murre's conjectures (B) and (D) for X endowed with the Chow–Künneth decomposition given by the q_l's then follow immediately from Proposition 6.1 and from the first point of Proposition 6.2. $\square$

8 Application to the finite-dimensionality problem

Kimura [13] introduced the notion of finite-dimensionality for Chow motives. There he proved that any variety dominated by a product of curves is finite-dimensional. It was proved by Guletskii and Pedrini [9] that a surface with representable Chow group of zero-cycles is Kimura finite-dimensional. Gorchinskiy and Guletskii [6] then proved that a threefold with representable Chow group of zero-cycle is Kimura finite-dimensional. This was subsequently generalised to varieties of any dimension in [22] and to pure motives in [23]. In their paper, Gorchinskiy and Guletskii also prove [6, Theorem 15] that, when X is fibred over a curve by del Pezzo or Enriques surfaces over an algebraically closed field of characteristic zero, then X has representable Chow group of zero-cycles. Their method involves looking at the singular fibres of the family. Our Theorem 1.3 is more general and immediately gives

Theorem 8.1. *Let X be a smooth projective threefold over a field k and let $f : X \to C$ be a dominant morphism over a curve C such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Then X has representable Chow group of zero-cycles and is finite-dimensional in the sense of Kimura.*

Godeaux surfaces are examples of surfaces of general type with trivial Chow group of zero-cycles [25]. Therefore, new cases encompassed by the above theorem are given by threefolds fibred by Godeaux surfaces over a curve. Let's make this more precise.

Let ζ be a primitive fifth root of unity. The group $G = \mathbf{Z}/5\mathbf{Z}$ acts on the complex projective space $\mathbf{P}^3_{\mathbf{C}}$ in the following way : $\zeta \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : \zeta x_1 : \zeta^2 x_2 : \zeta^3 x_3]$. Let $\bar{V} := H^0(\mathbf{P}^3, O_{\mathbf{P}^3}(5))^G$ be the subspace of $H^0(\mathbf{P}^3, O_{\mathbf{P}^3}(5))$ consisting of elements invariant under the action of G and let $V \hookrightarrow \bar{V}$ be the Zariski open subset of $\bar{V}$ consisting of elements defining smooth quintic surfaces. The monomials X_i^5 belong to $\bar{V}$ so that the dimension of $\bar{V}$ is at least 4. If Y_v is a smooth quintic in $\mathbf{P}^3$ given by the equation $v \in V$, then a local computation shows that Y_v cannot contain the fixed points of the action of G on $\mathbf{P}^3$, so that the action of G restricts to a free action on Y_v. The quotient space

$X_v := Y_v/G$ is a smooth projective surface called a Godeaux surface. These Godeaux surfaces fit into a family $X \to \mathbf{P}(\bar{V})$.

Let's consider a smooth projective curve C in $\mathbf{P}(\bar{V})$ that meets $\mathbf{P}(\bar{V}-V)$ transversely. Then X restricted to C gives a smooth projective threefold $X|_C \to C$ with general fibre a Godeaux surface. If C is of general type ($g(C) \geq 2$), then, by a result of Viehweg [24] which is a special instance of the Iitaka conjecture, $X|_C$ is a threefold of general type. We have thus exhibited new examples of threefolds of general type with representable Chow group of zero-cycles (obvious examples are given by the product of a curve of general type with a Godeaux surface). Such threefolds are also Kimura finite-dimensional thanks to [6].

In the following theorem, by conic fibration, we mean a dominant morphism $X \to S$ whose general fibre is a conic.

Theorem 8.2. *Let X be a smooth projective threefold which is a conic fibration over a surface S which is Kimura finite-dimensional. Then X is finite-dimensional in the sense of Kimura.*

Proof. In section 4, we proved that there is an orthogonal decomposition of the diagonal $\Delta_X = p_0+p_1+p_2^{tr}+p_2^{alg}+p_3+{}^tp_2^{alg}+{}^tp_2^{tr}+{}^tp_1+{}^tp_0$ with (X,p_0) and (X,p_2^{alg}) isomorphic to twisted motives of points, (X,p_1) and (X,p_3) isomorphic to direct summands of twisted motives of curves; and with (X,p_2^{tr}) isomorphic to $(S,\pi_2^{tr,S})$. Motives of points and motives of curves are finite-dimensional [13]. Since S is Kimura finite-dimensional by assumption and since finite-dimensionality is stable under direct summand [13], we have that (X,p_2^{tr}) is finite-dimensional. Therefore X is Kimura finite-dimensional. □

Theorem 8.3. *Let X be as in Theorem 8.1 or as in Theorem 8.2. Then X satisfies Murre's conjectures (A), (B), (C) and (D).*

Proof. By Theorem 6.5, X is Kimura finite-dimensional and has a Chow–Künneth decomposition that satisfies Murre's conjectures (B) and (D). According to [21, Theorem 4.8], we can conclude that X satisfies Murre's conjecture (C) if the cohomology of X is generated via the action of correspondences by the cohomology of surfaces. By Theorem 6.5 again, X satisfies the Lefschetz standard conjecture. Therefore, it suffices to show that $H_3(X)$ is generated by the H_1 of a curve. But then, this follows, via Bloch–Srinivas [1], from the fact that $\mathrm{CH}_0(X_\Omega)$ is supported on a surface. □

9 A fourfold of general type satisfying Murre's conjectures

In this section we wish to give explicit examples of fourfolds satisfying the assumptions of Theorem 1. For this purpose, we consider two-dimensional families of surfaces having trivial Chow group of zero-cycles.

A first type of such families was already given in Theorem 2 and consisted in separably rationally connected fibrations over a surface (separably rationally connected is the same as rationally connected if the base field has characteristic zero). Precisely, Theorem 2 considered smooth projective varieties X over a field k with an equidimensional map $X \to S$ to a smooth projective surface with general fibre being separably rationally connected. However, such a fourfold is not of general type. A natural question is to ask whether it is possible to construct a fourfold of general type that has a self-dual Murre decomposition. In [21, §2.3 & Cor 4.12], we produced examples of such fourfolds. These fourfolds have the property of having trivial Chow group of zero-cycles. Obvious examples were given by the product of two surfaces of general type with trivial Chow group of zero-cycles (e.g. Godeaux surfaces). Another example, a fourfold of Godeaux type, was given. The strategy consisted in checking the validity of the generalised Hodge conjecture for this fourfold.

We are now going to give an example of fourfold of general type with non-trivial (and in fact non-representable) Chow group of zero-cycles that has a self-dual Murre decomposition. Let's take up the notations of the previous section and let's consider the family $X \to \mathbf{P}(\bar{V})$. Let then S be a high-degree (i.e. ≥ 5) complete intersection which is a smooth surface in $\mathbf{P}(\bar{V})$ meeting $\mathbf{P}(\bar{V} - V)$ transversely. Then $X|_S$ is a projective fourfold with $X|_S \to S$ having a smooth Godeaux surface as a general fibre. A desingularization $X' \to X|_S$ gives a morphism $X' \to S$ with general fibre being of general type and having trivial Chow group of zero-cycles. This is because these two conditions are birational invariants. The high-degree condition on S imposes that S is of general type and has non-representable $\mathrm{CH}_0(S)_{\mathrm{alg}}$. Therefore, by Viehweg's result [24], X' is of general type; and, by Theorem 1, X' has a self-dual Murre decomposition which satisfies the motivic Lefschetz conjecture.

10 Application to unramified cohomology

Following the fundamental result of Colliot-Thélène, Sansuc and Soulé [3] which asserts that the degree-three unramified cohomology groups $H^3_{nr}(S/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$ vanish for all prime numbers l for S a smooth projective surface defined over a field k which is either finite or separably closed, it is proved in [2, Proposition 3.2] that, if X is a smooth projective variety defined over a field k which is either finite or separably closed such that its Chow group of zero-cycles is supported on a surface, then the groups $H^3_{nr}(X/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$ are finite for all prime numbers l and vanish for almost all l. Therefore, any variety X defined over a finite field or a separably closed field such that its restriction X_Ω to a universal domain Ω satisfies the assumptions of Theorem 1.3 has finite degree-three unramified cohomology $\bigoplus_l H^3_{nr}(X/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$. In particular, the fourfold of general type of section 9, when defined over a finite field or a separably closed field, has finite degree-three unramified cohomology. Furthermore, as a straightforward application of Theorem 1.7, we get

Proposition 10.1. *Let $f : X \to C$ be a dominant and generically smooth morphism from a smooth projective variety X to a smooth projective curve C defined over a field k which is either finite or separably closed. Assume that the general fibre Y of f_Ω is such that $\mathrm{CH}_0(Y)_{\mathrm{alg}}$ is representable. Then $H^3_{nr}(X/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$ is finite for all prime numbers l and vanishes for almost all l.*

Since unramified cohomology is a birational invariant for smooth projective varieties, the conclusion of the above theorem still holds for a smooth projective variety X' which is birational to the variety X of the theorem. For instance, we get finiteness of degree three unramified cohomology for threefolds which are the smooth compactification of one-dimensional families of smooth projective surfaces defined over a finite field or a separably closed field whose generic member is a bielliptic surface.

References

[1] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. *Amer. J. Math.*, 105(5):1235–1253, 1983.

[2] J.-L. Colliot-Thélène and B. Kahn. Cycles de codimension 2 et H^3 non ramifié pour les variétés sur les corps finis. Preprint, arXiv:1104.3350v2.

[3] Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Christophe Soulé. Torsion dans le groupe de Chow de codimension deux. *Duke Math. J.*, 50(3):763–801, 1983.

[4] Pedro Luis del Angel and Stefan Müller-Stach. Motives of uniruled 3-folds. *Compositio Math.*, 112(1):1–16, 1998.

[5] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[6] S. Gorchinskiy and V. Guletskii. Motives and representability of algebraic cycles on threefolds over a field. JAG, to appear.

[7] Sergey Gorchinskiy. Letter to the author. Feb. 10, 2012.

[8] B. Brent Gordon, Masaki Hanamura, and Jacob P. Murre. Absolute Chow-Künneth projectors for modular varieties. *J. Reine Angew. Math.*, 580:139–155, 2005.

[9] V. Guletskiĭ and C. Pedrini. Finite-dimensional motives and the conjectures of Beilinson and Murre. *K-Theory*, 30(3):243–263, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part III.

[10] Uwe Jannsen. Motivic sheaves and filtrations on Chow groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 245–302. Amer. Math. Soc., Providence, RI, 1994.

[11] B. Kahn and R. Sujatha. Birational motives, I pure birational motives. Preprint, February 27, 2009. arXiv:0902.4902.

[12] Bruno Kahn, Jacob P. Murre, and Claudio Pedrini. On the transcendental part of the motive of a surface. In *Algebraic cycles and motives. Vol. 2*, volume 344 of *London Math. Soc. Lecture Note Ser.*, pages 143–202. Cambridge Univ. Press, Cambridge, 2007.

[13] Shun-Ichi Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.

[14] Steven L. Kleiman. The standard conjectures. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 3–20. Amer. Math. Soc., Providence, RI, 1994.

[15] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 1996.

[16] Ju. I. Manin. Correspondences, motifs and monoidal transformations. *Mat. USSR-Sb.*, 6:439–470, 1968.

[17] S. Müller-Stach and M. Saito. Relative Chow-Künneth decompositions for morphisms of threefolds. to appear in Crelle's journal.

[18] J. P. Murre. On the motive of an algebraic surface. *J. Reine Angew. Math.*, 409:190–204, 1990.

[19] J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples. *Indag. Math. (N.S.)*, 4(2):177–188, 1993.

[20] A. J. Scholl. Classical motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 163–187. Amer. Math. Soc., Providence, RI, 1994.

[21] Charles Vial. Niveau and coniveau filtrations on cohomology groups and Chow groups. *Proc. Lond. Math. Soc.* to appear.

[22] Charles Vial. Projectors on the intermediate algebraic Jacobians. Preprint.

[23] Charles Vial. Pure motives with representable Chow groups. *C. R. Math. Acad. Sci. Paris*, 348(21-22):1191–1195, 2010.

[24] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 329–353. North-Holland, Amsterdam, 1983.

[25] Claire Voisin. Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(4):473–492, 1992.

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