A MOTIVIC GLOBAL TORELLI THEOREM FOR ISOGENOUS K3 SURFACES

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Abstract. We prove that the Chow motives of twisted derived equivalent K3 surfaces are isomorphic, not only as Chow motives (due to Huybrechts), but also as Frobenius algebra objects. Combined with a recent result of Huybrechts, we conclude that two complex projective K3 surfaces are isogenous (i.e. their second rational cohomology groups are Hodge isometric) if and only if their Chow motives are isomorphic as Frobenius algebra objects ; this can be regarded as a motivic Torelli-type theorem. We ask whether, more generally, twisted derived equivalent hyper-Kähler varieties have isomorphic Chow motives as (Frobenius) algebra objects and in particular isomorphic graded rational cohomology algebras. In the appendix, we justify introducing the notion of "Frobenius algebra object" by showing the existence of an infinite family of K3 surfaces whose Chow motives are pairwise non-isomorphic as Frobenius algebra objects but isomorphic as algebra objects. In particular, K3 surfaces in that family are pairwise non-isogenous but have isomorphic rational Hodge algebras.

Introduction

Torelli theorems for K3 surfaces. A compact complex surface is called a K3 surface if it is simply connected and has trivial canonical bundle. The Hodge structure carried by the second singular cohomology contains essential information of the surface. Indeed, the global Torelli theorem says that the isomorphism class of a K3 surface S is determined by the Hodge structure $\mathrm{H}^2(S,\mathbb{Z})$ together with the intersection pairing on it [48, 11].

The following more flexible notion due to Mukai [40] turns out to be crucial in the study of their derived categories : two complex projective K3 surfaces S and S' are called *isogenous* if there exists a *Hodge isometry* $\varphi\colon \mathrm{H}^2(S,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(S',\mathbb{Q})$, *i.e.* an isomorphism of rational Hodge structures compatible with the intersection pairing on both sides. Recently, Buskin [12] proved that such an isometry φ is induced by an algebraic correspondence, as had previously been conjectured by Shafarevich [53] as a particularly interesting case of the Hodge conjecture. Let us call any such representative a *Shafarevich cycle* for this isogeny. Shortly afterwards, Huybrechts [29] gave another proof and showed that in fact φ is induced by a chain of exact linear equivalences between derived categories of twisted K3 surfaces ; thereby establishing the *twisted derived global Torelli theorem* [29, Corollary 1.4] :

$$S \text{ and } S' \text{ are isogenous} \Longleftrightarrow S \text{ and } S' \text{ are twisted derived equivalent.} \tag{1}$$

Here, following Huybrechts [29], we say that two K3 surfaces S and S' are *twisted derived equivalent* if there exist K3 surfaces $S = S_0, S_1, \ldots, S_n = S'$ and Brauer classes $\alpha = \beta_0 \in \mathrm{Br}(S), \alpha_1, \beta_1 \in \mathrm{Br}(S_1), \ldots, \alpha_{n-1}, \beta_{n-1} \in \mathrm{Br}(S_{n-1})$ and $\alpha' = \alpha_n \in \mathrm{Br}(S')$ and exact linear

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equivalences between bounded derived categories of twisted coherent sheaves

$$
\begin{aligned}
&\mathrm{D}^b(S,\alpha) \simeq \mathrm{D}^b(S_1,\alpha_1),\\
&\mathrm{D}^b(S_1,\beta_1) \simeq \mathrm{D}^b(S_2,\alpha_2),\\
&\vdots\\
&\mathrm{D}^b(S_{n-2},\beta_{n-2}) \simeq \mathrm{D}^b(S_{n-1},\alpha_{n-1}),\\
&\mathrm{D}^b(S_{n-1},\beta_{n-1}) \simeq \mathrm{D}^b(S',\alpha').
\end{aligned} \tag{2}
$$

Note that by [13] any exact linear equivalence between bounded derived categories of twisted coherent sheaves on smooth projective varieties is of Fourier–Mukai type, so that in (2), each equivalence $\mathrm{D}^b(S_i,\beta_i) \xrightarrow{\sim} \mathrm{D}^b(S_{i+1},\alpha_{i+1})$ is induced by a Fourier–Mukai kernel $\mathcal{E}_i \in \mathrm{D}^b(S_i \times S_{i+1}, \beta_i^{-1} \boxtimes \alpha_{i+1})$ (unique up to isomorphism).

Combined with his previous work [28] generalized to the twisted case, Huybrechts deduced that isogenous complex projective K3 surfaces have isomorphic Chow motives[1]. However, the converse does not hold in general : there are K3 surfaces having isomorphic Chow motives (hence isomorphic rational Hodge structures) without being isogenous. Examples of such K3 surfaces were constructed geometrically in [7] ; see Remark A.3 of Appendix A. We provide in Theorem A.13 two further constructions of infinite families of pairwise non-isogenous K3 surfaces with isomorphic Chow motives. The motivation of the paper is to complete the picture by giving a motivic characterization of isogenous K3 surfaces (Corollary 2). Our main result is the following :

Theorem 1. *Let S and S' be two twisted derived equivalent K3 surfaces over a field k. Then the Chow motives of S and S' are isomorphic as algebra objects, in fact even as Frobenius algebra objects (Definition 2.1), in the category of rational Chow motives over k.*

In concrete terms, there exists a correspondence $\Gamma \in \mathrm{CH}^2(S \times S')$ with $\Gamma \circ \delta_S = \delta_{S'} \circ (\Gamma \otimes \Gamma)$ ("algebra homomorphism") and such that Γ is invertible as a morphism between the Chow motives of S and S' with $\Gamma^{-1} = {}^t\Gamma$ ("orthogonality"), where ${}^t\Gamma$ denotes the transpose of Γ and where δ_S is the small diagonal in $S \times S \times S$ viewed as a correspondence between $S \times S$ and S. Equivalently, Γ is an isomorphism such that $(\Gamma \otimes \Gamma)_*\Delta_S = \Delta_{S'}$ and $(\Gamma \otimes \Gamma \otimes \Gamma)_*\delta_S = \delta_{S'}$. The notion of Frobenius algebra object provides a conceptual way to pack these conditions. Roughly speaking, a Frobenius algebra object in a rigid tensor category is an algebra object together with an isomorphism to its dual object[2] with some compatibility conditions. The Chow motive of a smooth projective variety carries a natural structure of Frobenius algebra object in the rigid tensor category of Chow motives. We refer to §2 for more details on Frobenius algebra objects.

Note that a particular consequence of Theorem 1 is that the induced action $\Gamma_* : \mathrm{CH}^*(S) \to \mathrm{CH}^*(S')$ on the Chow rings is an isomorphism of graded $\mathbb{Q}$-algebras – this can in fact be deduced from the previous work of Huybrechts [28] ; see Remark 3.2. However, having an isomorphism of Frobenius algebra objects allows us to derive the following much stronger result :

Corollary 1 (Powers and Hilbert schemes)**.** *Let S and S' be two twisted derived equivalent K3 surfaces defined over a field k. Then for any positive integers $n_1, \ldots, n_r$, there is an isomorphism of Frobenius algebra objects*

$$\mathfrak{h}\left(\mathrm{Hilb}^{n_1}(S) \times \cdots \times \mathrm{Hilb}^{n_r}(S)\right) \simeq \mathfrak{h}\left(\mathrm{Hilb}^{n_1}(S') \times \cdots \times \mathrm{Hilb}^{n_r}(S')\right).$$

[1] In this paper, Chow groups and motives are always with rational coefficients, except in Theorem 3.3.

[2] Technically, one needs to further tensor the dual object by certain power of some tensor-invertible objects ; this power is called the *degree* of this Frobenius structure.

As a consequence, there is an algebraic correspondence inducing an isomorphism of graded $\mathbb{Q}$*-algebras :*

$$\mathrm{CH}^*\left(\mathrm{Hilb}^{n_1}(S)\times\cdots\times\mathrm{Hilb}^{n_r}(S)\right)\simeq\mathrm{CH}^*\left(\mathrm{Hilb}^{n_1}(S')\times\cdots\times\mathrm{Hilb}^{n_r}(S')\right).$$

Here $\mathrm{Hilb}^n(S)$ *denotes the Hilbert scheme of length-n subschemes on S.*

Now let the base field be the field of complex numbers $\mathbb{C}$. Combining Theorem 1 with Huybrechts' twisted derived global Torelli theorem (1) mentioned above, we can establish :

Corollary 2 (Motivic global Torelli theorem for isogenous K3 surfaces)**.** *Let S and S' be two complex projective K3 surfaces. The following statements are equivalent :*

(i) S and S' are isogenous ;
(ii) S and S' are twisted derived equivalent ;
(iii) $\mathfrak{h}(S)$ and $\mathfrak{h}(S')$ are isomorphic as Frobenius algebra objects.

In the appendix, we construct in Theorem A.13 an infinite family of pairwise non-isogenous K3 surfaces whose motives are all isomorphic as algebra objects. This justifies introducing the Frobenius structure. In addition, Proposition A.14 gives some evidence that the isogeny class of a K3 surface cannot be determined by its Chow ring.

Finally, with integral coefficients, an algebra isomorphism between the motives of two K3 surfaces must respect the Frobenius structure. Therefore, the classical global Torelli theorem [48] can be upgraded to a *motivic global Torelli theorem* :

S and S' are isomorphic $\Longleftrightarrow$ their integral Chow motives are isomorphic as algebra objects.

We refer to Theorem 3.3 for a proof.

Orlov conjecture and multiplicative structure. The proof of Theorem 1 relies on the Beauville–Voisin decomposition of the small diagonal of a K3 surface (see Theorem 3.1) : given an exact linear equivalence $\mathrm{D}^b(S,\alpha)\xrightarrow{\sim}\mathrm{D}^b(S',\alpha')$ between twisted K3 surfaces, we are reduced to exhibiting a correspondence $\Gamma\in\mathrm{CH}^2(S\times S')$ such that $\Gamma\circ{}^t\Gamma=\Delta_{S'}$, ${}^t\Gamma\circ\Gamma=\Delta_S$ and $\Gamma_*o_S=o_{S'}$, where $o_S=\frac{1}{24}c_2(S)$ is the Beauville–Voisin 0-cycle [4]. The key point then consists in showing that if $v_2(\mathcal{E})$ denotes the dimension-2 component of the Mukai vector of the Fourier–Mukai kernel of an exact linear equivalence $\Phi_{\mathcal{E}}:\mathrm{D}^b(S,\alpha)\xrightarrow{\sim}\mathrm{D}^b(S',\alpha')$ between twisted smooth projective surfaces, then $v_2(\mathcal{E})$ induces an isomorphism $\mathfrak{h}^2_{\mathrm{tr}}(S)\xrightarrow{\sim}\mathfrak{h}^2_{\mathrm{tr}}(S')$ of the transcendental motives of S and S' with inverse given by $v_2(\mathcal{E}^\vee\otimes p^*\omega_S)$, where $\mathcal{E}^\vee$ denotes the derived dual of $\mathcal{E}$ and $p:S\times S'\to S$ is the natural projection. (In the case of K3 surfaces, we have $v_2(\mathcal{E}^\vee\otimes p^*\omega_S)={}^tv_2(\mathcal{E})$.) This is achieved by exploiting known cases of Murre's Conjecture 1.2(B), and we thereby give an alternative proof of Huybrechts' [28, Theorem 0.1], generalized to all surfaces : *two twisted derived equivalent smooth projective surfaces have isomorphic Chow motives* ; see Theorem 1.1. This confirms the two-dimensional case of the following conjecture due to Orlov :

Conjecture 1 (Orlov [46])**.** *Let X and Y be two derived equivalent smooth projective varieties. Then their Chow motives are isomorphic.*

We illustrate also in §1.3 how the same techniques can be used to establish Orlov's Conjecture 1 in some new cases in dimension 3 and 4 ; see Proposition 1.6.

In view of Theorem 1, we naturally ask that under what circumstances one could expect a "multiplicative Orlov conjecture", namely whether two derived equivalent smooth projective varieties have isomorphic Chow motives as algebra objects, or even as Frobenius algebra objects. According to the celebrated theorem of Bondal–Orlov [8], this holds true for varieties with ample or anti-ample canonical bundle, since any two such derived equivalent varieties must be isomorphic. The situation gets more intriguing for varieties with trivial canonical bundle and we cannot expect in general that derived equivalent varieties have isomorphic Chow motives as Frobenius algebra objects : there exists counter-examples for Calabi–Yau threefolds and abelian varieties, where even the graded cohomology algebras of the two derived equivalent varieties are not isomorphic as Frobenius algebras (see Example 4.3 and Proposition 4.5 (*ii*)). We notice that, on the other hand, two derived equivalent abelian varieties are isogenous and have isomorphic Chow motives as algebra objects (see Proposition 4.5 (*i*)).

Although we do not provide much evidence beyond the case of K3 surfaces, we are tempted to ask, because of the (expected) similarities of the intersection product on hyper-Kähler varieties with that on abelian varieties (*cf.* Beauville's seminal [5], and also [54, 20]), the following

Question 1. *Let X and Y be two twisted derived equivalent projective hyper-Kähler varieties. Are their Chow motives isomorphic as algebra objects or even as Frobenius algebra objects ? In particular, are their cohomology $\mathrm{H}^*(-,\mathbb{Q})$ isomorphic as graded $\mathbb{Q}$-algebras or even as Frobenius algebras ?*

Corollary 1 gives an example in higher dimensions. See §4 for other examples, conjectures and rudiment discussions on this subject.

Canonicity of the Shafarevich cycle. In [29], Huybrechts shows that the restriction to the transcendental cohomology of an isogeny $\varphi\colon \mathrm{H}^2(S,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(S',\mathbb{Q})$ is induced by the cycle $v_2(\mathcal{E}_{n-1}) \circ \cdots \circ v_2(\mathcal{E}_0) \in \mathrm{CH}^2(S\times S')$, where $\mathcal{E}_0,\ldots,\mathcal{E}_{n-1}$ are the Fourier–Mukai kernels in (2). This provides a Shafarevich cycle for the isogeny φ. In §5, we give some evidence for the above cycle to be canonical, that is, independent of the choice of a chain of twisted derived equivalence inducing the isogeny. This depends on extending a result of Huybrechts and Voisin (Theorem 5.1) to twisted equivalences. We do however prove unconditionally in Theorem 5.4 that the intersection of the second Chern classes of two objects $\mathcal{E}_1$ and $\mathcal{E}_2$ in $\mathrm{D}^b(S\times S')$ inducing an equivalence $\mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ is proportional to $c_2(S)\times c_2(S')$ in $\mathrm{CH}^2(S\times S')$. This suggests that the Mukai vectors of twisted derived equivalences between K3 surfaces can be added to the Beauville–Voisin ring ; see §5.

Notation and Conventions. We fix a base field k. By a derived equivalence between smooth projective k-varieties, we mean a k-linear exact equivalence of triangulated categories between their bounded derived categories of coherent sheaves. Chow groups will always be considered with rational coefficients. Concerning the category of Chow motives over k, we follow the notation and conventions of [3]. This category is a pseudo-abelian rigid tensor category, whose objects consist of triples (X,p,n), where X is a smooth projective variety of dimension d_X over k, $p\in \mathrm{CH}^{d_X}(X\times_k X)$ with $p\circ p = p$, and $n\in\mathbb{Z}$. Morphisms $f: M=(X,p,n)\to N=(Y,q,m)$ are elements $\gamma\in\mathrm{CH}^{d_X+m-n}(X\times_k Y)$ such that $\gamma\circ p = q\circ\gamma=\gamma$. The tensor product of two motives is defined in the obvious way, while the dual of $M=(X,p,n)$ is $M^\vee = (X, {}^tp, -n+d_X)$, where ${}^tp$ denotes the transpose of p. The Chow motive of a smooth projective variety X is defined as $\mathfrak{h}(X) := (X,\Delta_X,0)$, where Δ_X denotes the class of the diagonal inside $X\times X$, and the *unit motive* is denoted $\mathbb{1} := \mathfrak{h}(\mathrm{Spec}(k))$. In particular, we have $\mathrm{CH}^l(X) = \mathrm{Hom}(\mathbb{1}(-l),\mathfrak{h}(X))$. The *Tate motive* of weight $-2i$ is the motive $\mathbb{1}(i) := (\mathrm{Spec}(k),\Delta_{\mathrm{Spec}(k)},i)$. A motive is said to be of *Tate type* if it is isomorphic to a direct sum of Tate motives.

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1. Derived equivalent surfaces

The aim of this section is to provide an alternative proof to the following result of Huybrechts [28, 29] :

Theorem 1.1 (Huybrechts)**.** *Let S and S' be two (twisted) derived equivalent smooth projective surfaces defined over a field k. Then S and S' have isomorphic Chow motives.*

The reason for including such a proof is threefold : first it provides anyway all the prerequisites and notation for the proof of Theorem 1 which will be given in §3.1 ; second by avoiding Manin's identity principle[3] (as is employed in [28]) we obtain an explicit inverse to the isomorphism $v_2(\mathcal{E}) : \mathfrak{h}^2_{\mathrm{tr}}(S) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(S')$ which will be essential to the proof of Theorem 1 ; and third it provides somehow a link between Orlov's Conjecture 1 and Murre's Conjecture 1.2 which itself is intricately linked to the conjectures of Bloch and Beilinson (see [33]).

1.1. **Murre's conjectures.** We fix a base field k and a Weil cohomology theory $\mathrm{H}^*(-)$ for smooth projective varieties over k. Concretely, we think of $\mathrm{H}^*(-)$ as Betti cohomology in case $k \subseteq \mathbb{C}$, or as ℓ-adic cohomology when $\mathrm{char}(k) \neq \ell$.

Conjecture 1.2 (Murre [42])**.** *Let X be a smooth projective variety of dimension d over k.*

(A) The Chow motive $\mathfrak{h}(X)$ has a Chow–Künneth decomposition (also called weight decomposition*) $\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \cdots \oplus \mathfrak{h}^{2d}(X)$, meaning that $\mathrm{H}^*(\mathfrak{h}^i(X)) = \mathrm{H}^i(X)$ for all i.*

(B) $\mathrm{CH}^l(\mathfrak{h}^i(X)) := \mathrm{Hom}(\mathbb{1}(-l), \mathfrak{h}^i(X)) = 0$ for all $i > 2l$ and for all $i < l$.

(C) The filtration $\mathrm{F}^k\mathrm{CH}^l(X) := \mathrm{CH}^l\big(\bigoplus_{i \leq 2l-k} \mathfrak{h}^i(X)\big)$ does not depend on the choice of a Chow–Künneth decomposition.

(D) $\mathrm{F}^1\mathrm{CH}^l(X) = \mathrm{CH}^l(X)_{\mathrm{hom}} := \ker(\mathrm{CH}^l(X) \xrightarrow{\mathrm{cl}} \mathrm{H}^{2l}(X))$.

The filtration defined in (C) is conjecturally the Bloch–Beilinson filtration [6] (see also [58, Chapter 11]). In fact, as shown by Jannsen [33], the conjecture of Murre holds for all smooth projective varieties if and only if the conjectures of Bloch–Beilinson hold. We refer to [33] for precise statements.

Proposition 1.3. *Let X and Y be smooth projective varieties over k. Assume that $\mathfrak{h}(X)$ and $\mathfrak{h}(Y)$ admit a Chow–Künneth decomposition as in Conjecture 1.2(A). Then, with respect to the Chow–Künneth decomposition $\mathfrak{h}^n(X \times Y) = \bigoplus_{i+j=n} \mathfrak{h}^{2d_X-i}(X)^\vee(-d_X) \otimes \mathfrak{h}^j(Y)$, $X \times Y$ satisfies Conjecture 1.2(B) if and only if*

$$\mathrm{Hom}(\mathfrak{h}^i(X), \mathfrak{h}^j(Y)(k)) = 0 \quad \textit{for all } i < j - 2k \textit{ and for all } i > j + d_X - k. \tag{3}$$

Proof. This is formal : we have

$$\begin{aligned} \mathrm{Hom}(\mathbb{1}(-k-d_X), \mathfrak{h}^n(X \times Y)) &= \bigoplus_{i-j=2d_X-n} \mathrm{Hom}(\mathbb{1}(-k), \mathfrak{h}^j(Y) \otimes \mathfrak{h}^i(X)^\vee) \\ &= \bigoplus_{i-j=2d_X-n} \mathrm{Hom}(\mathfrak{h}^i(X), \mathfrak{h}^j(Y)(k)). \end{aligned}$$

□

In other words, Murre's conjecture (B) implies that a motive of pure weight does not admit any non-trivial morphism to a motive of pure larger weight.

[3] Manin's identity principle only establishes that $v_2(\mathcal{E}^\vee \otimes p^*\omega_S) \circ v_2(\mathcal{E})$ acts as the identity on $\mathrm{CH}^2(\mathfrak{h}^2_{\mathrm{tr}}(S))$ which implies that it is unipotent as an endomorphism of $\mathfrak{h}^2_{\mathrm{tr}}(S)$.

Theorem 1.4 (Murre [41])**.** *Let X be a smooth projective irreducible variety of dimension d_X over a field k.*

(i) The Chow motive of X admits a decomposition

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus M \oplus \mathfrak{h}^{2d_X-1}(X) \oplus \mathfrak{h}^{2d_X}(X)$$

such that $\mathrm{H}^*(\mathfrak{h}^i(X)) = \mathrm{H}^i(X)$; in particular, Conjecture 1.2(A) holds for curves and surfaces. Moreover, such a decomposition can be chosen such that

- $\mathfrak{h}^{2d_X}(X)(d_X) = \mathfrak{h}^0(X)^\vee \simeq \mathfrak{h}^0(X)$ *and* $\mathfrak{h}^{2d_X-1}(X)(d_X) = \mathfrak{h}^1(X)^\vee \simeq \mathfrak{h}^1(X)(1)$ *;*
- $\mathfrak{h}^0(X)$ *is the unit motive* $\mathbb{1}$ *and* $\mathfrak{h}^1(X) \simeq \mathfrak{h}^1(\mathrm{Pic}^0(X)_{\mathrm{red}})$ *;*
- $\mathrm{Hom}(\mathbb{1}(-i), \mathfrak{h}^1(X)) = 0$ *for* $i \neq 1$*, and* $\mathrm{Hom}(\mathbb{1}(-1), \mathfrak{h}^1(X)) = \mathrm{Pic}^0(X)_{\mathrm{red}}(k) \otimes \mathbb{Q}$.

(ii) Equation (3) holds in case X and Y are varieties of dimension ≤ 2 endowed with a Chow–Künneth decomposition as in (i).

Proof. Item (i) in the case of surfaces is the main result of [41]. In fact, for any smooth projective variety X of any dimension, $\mathfrak{h}^1(X)$ can be constructed as a direct summand of the motive of a smooth projective curve. As for (ii), this was checked by Murre [43] in the case one of X and Y has dimension ≤ 1. Thanks to item (i) and Proposition 1.3, it only remains to check that $\mathrm{CH}^l(\mathfrak{h}^2(X) \otimes \mathfrak{h}^2(Y)) = 0$ for $l = 0, 1$ for smooth projective surfaces X and Y. For that purpose, we simply observe that for any choice of a Chow–Künneth decomposition (if it exists) on the motive of a smooth projective variety Z we have $\mathrm{CH}^0(Z) = \mathrm{CH}^0(\mathfrak{h}^0(Z))$ and $\mathrm{CH}^1(Z) = \mathrm{CH}^1(\mathfrak{h}^2(Z) \oplus \mathfrak{h}^1(Z))$. (Indeed, denote π^i_Z the projectors corresponding to the Chow–Künneth decomposition of Z, then by definition π^{2i}_Z acts as the identity on $\mathrm{H}^{2i}(Z)$ and hence on $\mathrm{im}(\mathrm{CH}^i(Z) \to \mathrm{H}^{2i}(Z))$, and by Murre [41] π^1_Z acts as the identity on $\ker(\mathrm{CH}^1(Z) \to \mathrm{H}^2(Z))$. Therefore $\pi^2_Z + \pi^1_Z$, which is a projector, acts as the identity on $\mathrm{CH}^1(Z)$.) □

The following terminology will be convenient for our purpose. We say that a Chow motive M is of *curve type* (or of *pure weight 1*) if it is isomorphic to a direct summand of a direct sum of motives of the form $\mathfrak{h}^1(C)$, where C is a smooth projective curve defined over k. Motives of curve type form a thick additive subcategory[4] and enjoy the following property, which is also shared by Tate motives :

Proposition 1.5. *The full subcategory of motives whose objects are of curve type is abelian semi-simple. Moreover, the realization functor $M \mapsto \mathrm{H}^*(M)$ is conservative.*

Proof. The first statement follows from the fact that this full subcategory of motives of curve type is equivalent to the category of abelian varieties up to isogeny, via the Jacobian construction ; see [3, Proposition 4.3.4.1]. The second statement follows from the first one together with the fact that $\mathrm{H}^*(\mathfrak{h}^1(A))$ is a $2g$-dimensional vector space for an abelian variety A of dimension g. □

1.2. **Proof of Theorem 1.1.** First, we observe as in [29, Section 2] that for twisted equivalences the yoga of Fourier–Mukai kernels, their action on Chow groups induced by Mukai vectors, and how they behave under convolutions works as in the untwisted case. Therefore, for ease of notation, we will only give a proof of Theorem 1.1 in the untwisted case.

1.2.1. *Derived equivalences and motives, following Orlov.* In general, let $\Phi_\mathcal{E} : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ be an exact equivalence with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^b(X \times Y)$ between the derived categories of two smooth projective k-varieties of dimension d. Its inverse can be described as $\Phi_\mathcal{E}^{-1} \simeq$

[4] It is however not stable under tensor products and it does not contain any Tate motives. The tensor category generated by the Tate motives and the motives of curve type consists of the so-called motives of *abelian type*.

$\Phi_{\mathcal{E}^\vee \otimes p^*\omega_X[d]} \simeq \Phi_{\mathcal{E}^\vee \otimes q^*\omega_Y[d]}$, where $\mathcal{E}^\vee$ is the derived dual of $\mathcal{E}$ and p, q are the projections from $X \times Y$ to X and Y respectively. As observed by Orlov [46], the Mukai vector

$$v(\mathcal{E}) := \mathrm{ch}(\mathcal{E}) \cdot \sqrt{\mathrm{td}(X \times Y)} \in \mathrm{CH}^*(X \times Y)$$

induces a split injective morphism of motives $\mathfrak{h}(X) \longrightarrow \bigoplus_{i=-d}^{d} \mathfrak{h}(Y)(i)$ with left inverse given by $v(\mathcal{E}^\vee \otimes p^*\omega_X[d])$, *i.e.*

$$\mathrm{id}: \ \mathfrak{h}(X) \xrightarrow{\quad v(\mathcal{E}) \quad} \bigoplus_{i=-d}^{d} \mathfrak{h}(Y)(i) \xrightarrow{v(\mathcal{E}^* \otimes p^*\omega_X[d])} \mathfrak{h}(X).$$

In particular, $v(\mathcal{E}^\vee \otimes p^*\omega_X[d]) \circ v(\mathcal{E}) = \Delta_X$. In fact, as observed by Orlov [46], the latter identity shows that $v(\mathcal{E})$ seen as a morphism of ind-motives $\bigoplus_{i\in\mathbb{Z}} \mathfrak{h}(X)(i) \to \bigoplus_{i\in\mathbb{Z}} \mathfrak{h}(Y)(i)$ is an isomorphism with inverse given by $v(\mathcal{E}^\vee \otimes p^*\omega_X[d])$.

1.2.2. *The refined decomposition of the motive of surfaces, following Kahn–Murre–Pedrini.* Let S be a smooth projective surface over k. The motive $\mathfrak{h}(S)$ admits a Chow–Künneth decomposition as in Murre's Theorem 1.4(i) ; in particular $\mathfrak{h}^0(X) = \mathfrak{h}^4(X)(2)^\vee$ is the unit motive $\mathbb{1}$ and $\mathfrak{h}^1(X) = \mathfrak{h}^3(X)(2)^\vee$ is of curve type. Following Kahn–Murre–Pedrini [34], the summand $\mathfrak{h}^2(S)$ admits a further decomposition

$$\mathfrak{h}^2(S) = \mathfrak{h}^2_{\mathrm{alg}}(S) \oplus \mathfrak{h}^2_{\mathrm{tr}}(S)$$

defined as follows. Let k^s be a separable closure of k and let $E_1, \ldots, E_\rho$ be non-isotropic divisors in $\mathrm{CH}^1(S_{k^s})$ whose images in $\mathrm{NS}(X_{k^s})_\mathbb{Q}$ form an orthogonal basis. Up to replacing each E_i by $E_i - (\pi^1_S)_*E_i$, we can assume that $(\pi^1_S)_*E_i = 0$ for all i. Consider then the idempotent correspondence

$$\pi^2_{\mathrm{alg},S} := \sum_{i=1}^{\rho} \frac{1}{\deg(E_i \cdot E_i)} E_i \times E_i.$$

Since $\pi^2_{\mathrm{alg},S}$ is the intersection form on $\mathrm{NS}(X_{k^s})_\mathbb{Q}$, it is Galois-invariant, and hence does define an idempotent in $\mathrm{CH}^2(S\times S)$. The motive $(S, \pi^2_{\mathrm{alg},S}, 0)$ is clearly isomorphic, after base-change to k^s, to the direct sum of ρ copies of the Tate motive $\mathbb{1}(-1)$. Moreover, it is easy to check that $\pi^2_{\mathrm{alg},S}$ is orthogonal to π^i_S for $i \neq 2$ (use $(\pi^1_S)_*E_i = 0$). Equivalently, we have $\pi^2_{\mathrm{alg},S} \circ \pi^2_S = \pi^2_S \circ \pi^2_{\mathrm{alg},S}$, so that $(S, \pi^2_{\mathrm{alg},S}, 0)$ does define a direct summand of $\mathfrak{h}^2(S)$, denoted by $\mathfrak{h}^2_{\mathrm{alg}}(S)$. We then define $\pi^2_{\mathrm{tr},S} := \pi^2_S - \pi^2_{\mathrm{alg},S}$ and $\mathfrak{h}^2_{\mathrm{tr}}(S) := (S, \pi^2_{\mathrm{tr},S}, 0)$. It is then straightforward to check that such a decomposition satisfies $\mathrm{Hom}(\mathbb{1}(-i), \mathfrak{h}^2_{\mathrm{tr}}(S)) = 0$ for all $i \neq 2$. We note that ${}^t\pi^2_{\mathrm{alg},S} = \pi^2_{\mathrm{alg},S}$, and since ${}^t\pi^0_S = \pi^4_S$ and ${}^t\pi^1_S = \pi^3_S$ we also have ${}^t\pi^2_{\mathrm{tr},S} = \pi^2_{\mathrm{tr},S}$. Moreover, although this won't be of any use to us, we mention for comparison to [28] that $\mathrm{Hom}(\mathbb{1}(-2), \mathfrak{h}^2_{\mathrm{tr}}(S))$ coincides with the Albanese kernel.

More generally, the above refined decomposition can be performed for direct summand of motives of surfaces, *i.e.* for motives of the form $(S, p, 0)$, where S is a smooth projective k-surface and $p \in \mathrm{CH}^2(S \times S)$ is an idempotent. This will be used in the proof of Proposition 1.6. Indeed, by the above together with Theorem 1.4(ii), we have a decomposition

$$\mathfrak{h}(S) = \mathfrak{h}^0(S) \oplus \mathfrak{h}^1(S) \oplus \mathfrak{h}^2_{\mathrm{alg}}(S) \oplus \mathfrak{h}^2_{\mathrm{tr}}(S) \oplus \mathfrak{h}^3(S) \oplus \mathfrak{h}^4(S), \tag{4}$$

where none of the direct summands admit a non-trivial morphism to another direct summand placed on its right. It follows that the morphism p, expressed with respect to the decomposition (4) is upper-triangular. By [55, Lemma 3.1], the motive $M = (S, p, 0)$ admits a weight decomposition $M = M^0 \oplus M^1 \oplus M^2_{\mathrm{alg}} \oplus M^2_{\mathrm{tr}} \oplus M^3 \oplus M^4$, where each factor is isomorphic to a direct summand of the corresponding factor in the decomposition (4). In particular, this decomposition of M inherits the properties of the decomposition (4), *e.g.* M^0 and M^4 are of Tate type, M^2_{alg} becomes of Tate type after base-change to k^s and M^1 and $M^3(1)$ are of curve type.

1.2.3. *A weight argument.* Thanks to Theorem 1.4(ii), $v(\mathcal{E})$ maps $\mathfrak{h}^2_{\mathrm{tr}}(S)$ possibly non-trivially only in summands of

$$\bigoplus_{i=-2}^{2} \left(\mathfrak{h}^0(S')(i) \oplus \mathfrak{h}^1(S')(i) \oplus \mathfrak{h}^2_{\mathrm{alg}}(S')(i) \oplus \mathfrak{h}^2_{\mathrm{tr}}(S')(i) \oplus \mathfrak{h}^3(S')(i) \oplus \mathfrak{h}^4(S')(i)\right) \tag{5}$$

of weight ≤ 2. Since $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathbb{1}(-1)) = \mathrm{Hom}(\mathbb{1}(1), \mathfrak{h}^2_{\mathrm{tr}}(S)^\vee) = \mathrm{Hom}(\mathbb{1}(-1), \mathfrak{h}^2_{\mathrm{tr}}(S)) = 0$, we see that $\mathfrak{h}^2_{\mathrm{tr}}(S')$ is the only direct summand of weight 2 in (5) that admits a possibly non-trivial morphism from $\mathfrak{h}^2_{\mathrm{tr}}(S)$. Likewise, the only direct summand of (5) of weight ≤ 2 that maps possibly non-trivially in $\mathfrak{h}^2_{\mathrm{tr}}(S)$ via $v(\mathcal{E}^\vee \otimes p^*\omega_S)$ is $\mathfrak{h}^2_{\mathrm{tr}}(S')$. It follows that the restriction of $v_2(\mathcal{E})$ induces an isomorphism

$$\pi^2_{\mathrm{tr},S'} \circ v_2(\mathcal{E}) \circ \pi^2_{\mathrm{tr},S} : \mathfrak{h}^2_{\mathrm{tr}}(S) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(S')$$

with inverse $\pi^2_{\mathrm{tr},S} \circ v_2(\mathcal{E}^\vee \otimes p^*\omega_S) \circ \pi^2_{\mathrm{tr},S'}$; this fact will be used in the proof of Theorem 1.

In a similar fashion, thanks to Theorem 1.4(ii) and having in mind that $\mathfrak{h}^1(S) \simeq \mathfrak{h}^3(S)(1)$ is the direct summand of the motive of a curve (and similarly for S'), $v(\mathcal{E})$ induces isomorphisms

$$\mathfrak{h}^0(S) \oplus \mathfrak{h}^2_{\mathrm{alg}}(S)(1) \oplus \mathfrak{h}^4(S)(2) \xrightarrow{\sim} \mathfrak{h}^0(S') \oplus \mathfrak{h}^2_{\mathrm{alg}}(S')(1) \oplus \mathfrak{h}^4(S')(2)$$

and

$$\mathfrak{h}^1(S) \oplus \mathfrak{h}^3(S)(1) \xrightarrow{\sim} \mathfrak{h}^1(S') \oplus \mathfrak{h}^3(S')(1).$$

The first isomorphism yields an isomorphism $\mathfrak{h}^2_{\mathrm{alg}}(S) \simeq \mathfrak{h}^2_{\mathrm{alg}}(S')$, while the second one yields, thanks to Theorem 1.4(i), together with the semi-simplicity statement of Proposition 1.5, isomorphisms $\mathfrak{h}^1(S) \simeq \mathfrak{h}^1(S')$ and $\mathfrak{h}^3(S) \simeq \mathfrak{h}^3(S')$. This finishes the proof of Theorem 1.1. □

1.3. **A slight generalization to Theorem 1.1.** The content of this paragraph won't be used in the proof of Theorem 1. Recall that Theorem 1.1 fits more generally into the Orlov Conjecture 1.

The method of proof of Theorem 1.1 can be pushed through to establish the following :

Proposition 1.6. *Let X and Y be two smooth projective varieties of dimension 3 or 4 over a field k. Assume either of the following :*

- $\dim X = 3$ *and* $\mathrm{CH}_0(X)$ *is representable ;*
- $\dim X = 4$, $\mathrm{CH}_0(X)$ *and* $\mathrm{CH}_0(Y)$ *are both representable, and X and Y have same Picard rank.*

Then $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ *implies that* $\mathfrak{h}(X) \simeq \mathfrak{h}(Y)$.

Here, we say that a smooth projective k-variety X of dimension d has *representable* CH_0 if for a choice of universal domain (*i.e.*, algebraically closed field of infinite transcendence degree over its prime subfield) Ω containing k, there exists a smooth projective Ω-curve C and a correspondence $\gamma \in \mathrm{Hom}(\mathfrak{h}(X_\Omega), \mathfrak{h}(C))$ such that $\gamma^*\mathrm{CH}_0(C) = \mathrm{CH}_0(X_\Omega)$. Examples of such varieties include varieties with maximally rationally connected quotient of dimension ≤ 1, and in particular rationally connected varieties.

Proof. We start with the case of threefolds. By [21], X admits a Chow–Künneth decomposition, where the even-degree summands are of Tate type, while the odd-degree summands are Tate twists of motives of curve type. The arguments of §1.2.1 show that $\mathfrak{h}(Y)$ is a direct summand of $\bigoplus_{i=-3}^{3} \mathfrak{h}(X)(i)$; in particular, by Kimura finite-dimensionality (or by Theorem 1.4(ii) together with [55, Lemma 3.1] as used in §1.2.2), $\mathfrak{h}(Y)$ has a Chow–Künneth decomposition with a similar property to that of X (and hence has representable CH_0). The arguments of §1.2.3 provide isomorphisms

$$\mathfrak{h}^0(X)\oplus\mathfrak{h}^2(X)(1)\oplus\mathfrak{h}^4(X)(2)\oplus\mathfrak{h}^6(X)(3)\simeq\mathfrak{h}^0(Y)\oplus\mathfrak{h}^2(Y)(1)\oplus\mathfrak{h}^4(Y)(2)\oplus\mathfrak{h}^6(Y)(3)$$

and

$$\mathfrak{h}^1(X)\oplus\mathfrak{h}^3(X)(1)\oplus\mathfrak{h}^5(X)(2)\simeq\mathfrak{h}^1(Y)\oplus\mathfrak{h}^3(Y)(1)\oplus\mathfrak{h}^5(Y)(2), \tag{6}$$

Since the even-degree summands are of Tate type, and since $\dim \mathrm{H}^{2i}(X)=\dim \mathrm{H}^{6-2i}(X)$ by Poincaré duality (and similarly for Y), we conclude that $\mathfrak{h}^{2i}(X)\simeq\mathfrak{h}^{2i}(Y)$ for all i. Now, a theorem of Popa–Schnell [49] says that two derived equivalent complex varieties have isogenous reduced Picard scheme (see [24, Appendix] for the case of k-varieties). It follows from Theorem 1.4(i) that $\mathfrak{h}^1(X)\simeq\mathfrak{h}^1(Y)$, and then by duality that $\mathfrak{h}^5(X)\simeq\mathfrak{h}^5(Y)$. Since all terms of (6) are of curve type, we deduce from the semi-simplicity statement of Proposition 1.5 that $\mathfrak{h}^3(X)\simeq\mathfrak{h}^3(Y)$. Alternately, from [2], two derived equivalent threefolds have degree-wise isomorphic cohomology groups (the isomorphisms being induced by some algebraic correspondences) ; it then follows from the description of the motives of X and Y together with Proposition 1.5 that X and Y have isomorphic motives.

In the case of fourfolds, we first note by [56, Theorem 3.11] that the motive of a fourfold X with representable CH_0 admits a decomposition of the form

$$\mathfrak{h}(X)\simeq(C,p,0)\oplus(S,q,1)\oplus(C,{}^tp,3),$$

for some curve C and some surface S. It follows from the arguments of §1.2.2 that $\mathfrak{h}(X)$ admits a Chow–Künneth decomposition such that $\mathfrak{h}^{2i+1}(X)(i)$ is of curve type for all i, $\mathfrak{h}^{2i}(X)$ is of Tate type for all $i\neq 2$, and $\mathfrak{h}^4(X)$ further decomposes as $\mathfrak{h}^4_{\mathrm{alg}}(X)\oplus\mathfrak{h}^4_{\mathrm{tr}}(X)$, with the property that $\mathfrak{h}^4_{\mathrm{alg}}(X)$ becomes, after base-change to k^s, a direct sum of Tate motives $\mathbb{1}(-2)$ and $\mathfrak{h}^4_{\mathrm{tr}}(X)(1)$ is a direct summand of the motive of a surface with $\mathrm{Hom}(\mathfrak{h}^4_{\mathrm{tr}}(X),\mathbb{1}(-i))=0$ for $i\neq 3$. The arguments of §1.2.3 then provides isomorphisms

$$\mathfrak{h}^0(X)\oplus\mathfrak{h}^2(X)(1)\oplus\mathfrak{h}^4_{\mathrm{alg}}(X)(2)\oplus\mathfrak{h}^6(X)(3)\oplus\mathfrak{h}^8(X)(4)\simeq\mathfrak{h}^0(Y)\oplus\mathfrak{h}^2(Y)(1)\oplus\mathfrak{h}^4_{\mathrm{alg}}(Y)(2)\oplus\mathfrak{h}^6(Y)(3)\oplus\mathfrak{h}^8(Y)(4),$$

$$\mathfrak{h}^1(X)\oplus\mathfrak{h}^3(X)(1)\oplus\mathfrak{h}^5(X)(2)\oplus\mathfrak{h}^7(X)(3)\simeq\mathfrak{h}^1(Y)\oplus\mathfrak{h}^3(Y)(1)\oplus\mathfrak{h}^5(Y)(2)\oplus\mathfrak{h}^7(Y)(3)$$
$$\text{and}\quad \mathfrak{h}^4_{\mathrm{tr}}(X)\simeq\mathfrak{h}^4_{\mathrm{tr}}(Y).$$

As in the case of threefolds, since the Picard numbers of X and Y agree, we conclude that $\mathfrak{h}^{2i}(X)\simeq\mathfrak{h}^{2i}(Y)$ for all $i\neq 2$ and that $\mathfrak{h}^4_{\mathrm{alg}}(X)\simeq\mathfrak{h}^4_{\mathrm{alg}}(Y)$, while by utilizing the Theorem of Popa–Schnell [49], we conclude from Theorem 1.4(i) that $\mathfrak{h}^1(X)\simeq\mathfrak{h}^1(Y)$ and then by duality that $\mathfrak{h}^7(X)\simeq\mathfrak{h}^7(Y)$. It follows by cancellation (Proposition 1.5) that $\mathfrak{h}^3(X)\oplus\mathfrak{h}^5(X)(1)\simeq\mathfrak{h}^3(Y)\oplus\mathfrak{h}^5(Y)(1)$. Since there is a Lefschetz isomorphism $\mathrm{H}^3(X)\simeq\mathrm{H}^5(X)(1)$ and similarly for Y, we conclude (again from Proposition 1.5) that $\mathfrak{h}^3(X)\simeq\mathfrak{h}^3(Y)$ and $\mathfrak{h}^5(X)\simeq\mathfrak{h}^5(Y)$. $\square$

2. Motives of varieties as Frobenius algebra objects

2.1. **Algebras and Frobenius algebras.** Let $\mathcal{C}$ be a symmetric monoidal category with tensor unit $\mathbb{1}$. An *algebra object* in $\mathcal{C}$ is an object M together with a unit morphism $\eta:\mathbb{1}\to M$ and a multiplication morphism $\mu: M\otimes M\to M$ satisfying the unit axiom $\mu\circ(\mathrm{id}\otimes\eta)=\mathrm{id}=\mu\circ(\eta\otimes\mathrm{id})$ and the associativity axiom $\mu\circ(\mu\otimes\mathrm{id})=\mu\circ(\mathrm{id}\otimes\mu)$. It is called *commutative* if moreover $\mu=\mu\circ c_{M,M}$ is satisfied, where $c_{M,M}$ is the commutativity constraint of the category $\mathcal{C}$. A *morphism of algebra objects* between two algebra objects M and N is a morphism $\phi: M\to N$ in $\mathcal{C}$ that preserves the multiplication μ and the unit η. We note that an algebra structure on an object M of $\mathcal{C}$ induces naturally an algebra structure on the n-th tensor powers $M^{\otimes n}$ of M, and that a morphism $\phi: M\to N$ of algebra objects induces naturally a morphism of algebra objects $\phi^{\otimes n}: M^{\otimes n}\to N^{\otimes n}$ which is an isomorphism if ϕ is.

If $\mathcal{C}$ is moreover rigid and possesses a $\otimes$-invertible object then we can speak of Frobenius algebra objects in $\mathcal{C}$:

Definition 2.1 (Frobenius algebra objects)**.** Let $(\mathcal{C}, \otimes, \vee, \mathbb{1})$ be a rigid symmetric monoidal category admitting a $\otimes$-invertible object denoted by $\mathbb{1}(1)$. Let d be an integer. A *degree-d Frobenius algebra object* in $\mathcal{C}$ is the data of an object $M \in \mathcal{C}$ endowed with

- $\eta : \mathbb{1} \to M$, a unit morphism ;
- $\mu : M \otimes M \to M$, a multiplication morphism ;
- $\lambda : M^\vee \xrightarrow{\sim} M(d)$, an isomorphism, called the *Frobenius structure* ;

satisfying the following axioms:

(i) (Unit) $\mu \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = \mu \circ (\eta \otimes \mathrm{id})$;
(ii) (Associativity) $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$;
(iii) (Frobenius condition) $(\mathrm{id} \otimes \mu) \circ (\delta \otimes \mathrm{id}) = \delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$,

where the *comultiplication* morphism $\delta : M \to M \otimes M(d)$ is defined by dualizing μ via the following commutative diagram :

$$\begin{tikzcd} M^\vee \arrow[r, "{}^t\mu"] \arrow[d, "\lambda"', "\simeq"] & M^\vee \otimes M^\vee \arrow[d, "\lambda\otimes\lambda"', "\simeq"] \\ M(d) \arrow[r, "\delta(d)"] & M(d) \otimes M(d) \end{tikzcd}$$

We define also the *counit morphism* $\epsilon : M \to \mathbb{1}(-d)$ by dualizing η via the following diagram:

$$\begin{tikzcd} M^\vee \arrow[r, "{}^t\eta"] \arrow[d, "\simeq"', "\lambda"] & \mathbb{1} \\ M(d) \arrow[ur, "\epsilon(d)"'] & \end{tikzcd}$$

We remark that ϵ and δ satisfy automatically the counit and coassociativity axioms.

A Frobenius algebra object M is called *commutative* if the underlying algebra object is commutative : $\mu \circ c_{M,M} = \mu$. Commutativity is equivalent to the cocommutativity of δ. The morphism $\beta = \epsilon \circ \mu : M \otimes M \to \mathbb{1}(-d)$, called the *Frobenius pairing*, is also sometimes used. It is a symmetric pairing if M is commutative.

Remark 2.2. In the case of Frobenius algebra objects of degree 0, the $\otimes$-invertible object $\mathbb{1}(1)$ is not needed in the definition, and it is reduced to the usual notion of Frobenius algebra object in the literature (see for example [1], [36]). In this sense, Definition 2.1 generalizes the existing definition of Frobenius structure by allowing non-zero twists. We believe that our more flexible notion is necessary and adequate for more sophisticated tensor categories than that of vector spaces, such as the categories of Hodge structures, Galois representations, motives, *etc.*

Remark 2.3 (Morphisms)**.** Morphisms of Frobenius algebra objects are defined in the natural way, that is, as morphisms $\phi : M \to N$ such that all the natural diagrams involving the structural morphisms are commutative. In particular, in order to admit non-trivial morphisms, the degrees of the Frobenius algebra objects M and N must coincide and the following diagram is then commutative :

$$\begin{tikzcd} N^\vee \arrow[r, "{}^t\phi"] \arrow[d, "\lambda_N"', "\simeq"] & M^\vee \arrow[d, "\simeq"', "\lambda_M"] \\ N(d) & M(d) \arrow[l, "\phi(d)"] \end{tikzcd}$$

As a result, all morphisms between Frobenius algebra objects are in fact invertible. It is an exercise to show that an isomorphism $\phi : M \to N$ between two Frobenius algebra objects respects the Frobenius algebra structures if and only if it is compatible with the algebra structure (*i.e.* with μ) and the Frobenius structure (*i.e.* with λ). This is proved in Proposition 2.11 in the case of Chow motives of smooth projective varieties. In addition, $\phi^{\otimes n} : M^{\otimes n} \to N^{\otimes n}$ is naturally an isomorphism of Frobenius algebra objects, as is the dual ${}^t\phi : N^\vee \to M^\vee$.

We summarize the above discussion in the following

Lemma 2.4. *Let M, N be two Frobenius algebra objects of degree d. A morphism of algebra objects $\phi : M \to N$ is a morphism of Frobenius algebra objects if and only if it is an isomorphism and it is* orthogonal *in the sense that $\phi(d)^{-1} = \lambda_M \circ {}^t\phi \circ \lambda_N^{-1}$, or more succinctly, $\phi^{-1} = {}^t\phi$.*

Proof. The "if" part follows from the definition. The "only if" part is explained in Remark 2.3. □

Now let us turn to important examples of Frobenius algebra objects.

Example 2.5 (Cohomology as a graded vector space)**.** Let X be a connected compact orientable (real) manifold of dimension d. Then its cohomology group $\mathrm{H}^*(X, \mathbb{Q})$ is naturally a Frobenius algebra object of degree d in the category of $\mathbb{Z}$-graded $\mathbb{Q}$-vector spaces (where morphisms are *degree-preserving* linear maps and the $\otimes$-invertible object is chosen to be $\mathbb{Q}[1]$, the 1-dimensional vector space sitting in degree -1). The unit morphism $\eta : \mathbb{Q} \to \mathrm{H}^*(X, \mathbb{Q})$ is given by the fundamental class $[X]$; the multiplication morphism $\mu : \mathrm{H}^*(X, \mathbb{Q}) \otimes \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(X, \mathbb{Q})$ is the cup-product ; the Frobenius structure comes from the Poincaré duality

$$\lambda : \mathrm{H}^*(X, \mathbb{Q})^\vee \xrightarrow{\sim} \mathrm{H}^*(X, \mathbb{Q})[d] = \mathrm{H}^*(X, \mathbb{Q}) \otimes \mathbb{Q}[d].$$

The induced comultiplication morphism $\delta : \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(X, \mathbb{Q}) \otimes \mathrm{H}^*(X, \mathbb{Q})[d]$ is the Gysin map for the diagonal embedding $X \hookrightarrow X \times X$; the counit morphism $\epsilon : \mathrm{H}^*(X, \mathbb{Q}) \to \mathbb{Q}[-d]$ is the integration $\int_X$. The Frobenius condition is a classical exercise. Note that $\mathrm{H}^*(X, \mathbb{Q})$ is commutative, because the commutativity constraint in the category of graded vector spaces is the super one.

If instead we consider the cohomology group as merely an ungraded vector space, then it becomes a Frobenius algebra object of degree 0 (*i.e.* in the usual sense) ; this is one of the main examples in the literature.

Example 2.6 (Hodge structures)**.** A pure (rational) Hodge structure is a finite-dimensional $\mathbb{Z}$-graded $\mathbb{Q}$-vector space $H = \oplus_{n \in Z} H^{(n)}$ such that each $H^{(n)}$ is given a Hodge structure of weight n. A morphism between two Hodge structures is required to preserve the weights. The category of pure Hodge structures is naturally a rigid symmetric monoidal category. The $\otimes$-invertible object is chosen to be $\mathbb{Q}(1)$, which is the 1-dimensional vector space $(2\pi i) \cdot \mathbb{Q}$ with Hodge structure purely of type $(-1, -1)$.

Let X be a compact Kähler manifold of (complex) dimension d. Then $\mathrm{H}^*(X, \mathbb{Q})$ is naturally a commutative Frobenius algebra object of degree d in the category of pure $\mathbb{Q}$-Hodge structures. The structural morphisms are the same as in Example 2.5 up to replacing $[d]$ by (d). For instance, $\lambda : \mathrm{H}^*(X, \mathbb{Q})^\vee \xrightarrow{\sim} \mathrm{H}^*(X, \mathbb{Q})(d)$.

Our main examples of Frobenius algebra objects are the Chow motives of smooth projective varieties.

2.2. **Frobenius algebra structure on the motives of varieties.** The category of rational Chow motives over a field k is rigid and symmetric monoidal. We choose the $\otimes$-invertible object to be the Tate motive $\mathbb{1}(1)$. Then for any smooth projective k-variety X of dimension d, its Chow motive $\mathfrak{h}(X)$ is naturally a commutative Frobenius algebra object of degree d in the category of Chow motives. Let us explain the structural morphisms in detail.

Let δ_X denote the class of the small diagonal $\{(x,x,x) : x \in X\}$ in $\mathrm{CH}_d(X \times X \times X)$. Note that for α and β in $\mathrm{CH}^*(X)$, we have $(\delta_X)_*(\alpha \times \beta) = \alpha \cdot \beta$, so that δ_X seen as an element of $\mathrm{Hom}(\mathfrak{h}(X) \otimes \mathfrak{h}(X), \mathfrak{h}(X))$ describes the intersection theory on the Chow ring of X, as well as the cup product of its cohomology ring. So it is natural to define the multiplication morphism

$$\mu : \mathfrak{h}(X) \otimes \mathfrak{h}(X) \longrightarrow \mathfrak{h}(X)$$

to be the one given by the small diagonal $\delta_X \in \mathrm{CH}^{2d}(X \times X \times X) = \mathrm{Hom}(\mathfrak{h}(X) \otimes \mathfrak{h}(X), \mathfrak{h}(X))$; it can be checked to be commutative and associative. The unit morphism $\eta : \mathbb{1} \to \mathfrak{h}(X)$ is again given by the fundamental class of X. The unit axiom is very easy to check.

The Frobenius structure is defined as the following canonical isomorphism, called the motivic Poincaré duality, given by the class of diagonal $\Delta_X \in \mathrm{CH}^d(X \times X) = \mathrm{Hom}(\mathfrak{h}(X)^\vee, \mathfrak{h}(X)(d))$:

$$\lambda : \mathfrak{h}(X)^\vee \xrightarrow{\sim} \mathfrak{h}(X)(d).$$

One readily checks that the induced comultiplication morphism

$$\delta : \mathfrak{h}(X) \to \mathfrak{h}(X) \otimes \mathfrak{h}(X)(d)$$

is given by the small diagonal $\delta_X \in \mathrm{CH}^{2d}(X \times X \times X) = \mathrm{Hom}(\mathfrak{h}(X), \mathfrak{h}(X) \otimes \mathfrak{h}(X)(d))$, while the counit morphism

$$\epsilon : \mathfrak{h}(X) \to \mathbb{1}(-d)$$

is given by the fundamental class.

The following lemma proves that, endowed with these structural morphisms, $\mathfrak{h}(X)$ is indeed a Frobenius algebra object.

Lemma 2.7 (Frobenius condition)**.** *Notation is as above. We have an equality of endomorphisms of* $\mathfrak{h}(X) \otimes \mathfrak{h}(X)$ *:*

$$(\mathrm{id} \otimes \mu) \circ (\delta \otimes \mathrm{id}) = \delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta).$$

Proof. We only show $\delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$, the other equality being similar. We have a commutative cartesian diagram without excess intersection :

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "\Delta\times\mathrm{id}"] \\ X \times X \arrow[r, "\mathrm{id}\times\Delta"'] & X \times X \times X, \end{tikzcd}$$

where $\Delta : X \to X \times X$ denotes the diagonal embedding. The base-change formula yields

$$(\Delta \times \mathrm{id})^* \circ (\mathrm{id} \times \Delta)_* = \Delta_* \circ \Delta^*$$

on Chow groups, hence also for Chow motives by Manin's identity principle [3, §4.3.1]. Now it suffices to notice that Δ_* is the comultiplication δ and Δ^* is the multiplication μ. ☐

Remark 2.8. In general, a tensor functor $F : \mathcal{C} \to \mathcal{C}'$ between two rigid symmetric monoidal categories sends a Frobenius algebra object in $\mathcal{C}$ to such an object in $\mathcal{C}'$. Example 2.6 is obtained by applying the Betti realization functor from the category of Chow motives to that of pure Hodge structures ; Example 2.5 (for Kähler manifolds) is obtained by further applying the forgetful functor ($\mathbb{Q}(1)$ is sent to $\mathbb{Q}[2]$).

2.3. **(Iso)morphisms of Chow motives as Frobenius algebra objects.** The notion of morphisms between two algebra objects is the natural one. Let us spell it out for motives of varieties. A non-zero morphism $\Gamma : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ between the motives of two smooth projective varieties over a field k is said to *preserve the algebra structures* if the following diagram commutes[5]

$$\begin{tikzcd}
\mathfrak{h}(X) \otimes \mathfrak{h}(X) \arrow[r, "\delta_X"] \arrow[d, "\Gamma\otimes\Gamma"] & \mathfrak{h}(X) \arrow[d, "\Gamma"] \\
\mathfrak{h}(Y) \otimes \mathfrak{h}(Y) \arrow[r, "\delta_Y"] & \mathfrak{h}(Y).
\end{tikzcd} \tag{7}$$

For example, if $f : Y \to X$ is a k-morphism, then $f^* : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ is a morphism of algebra objects. Note that if $\Gamma : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ preserves the algebra structures, then $\Gamma_* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ is a $\mathbb{Q}$-algebra homomorphism. In fact, since in that case $\Gamma^{\otimes n} : \mathfrak{h}(X^n) \to \mathfrak{h}(Y^n)$ also preserves the algebra structures for all $n > 0$, $(\Gamma^{\otimes n})_* : \mathrm{CH}^*(X^n) \to \mathrm{CH}^*(Y^n)$ is also a $\mathbb{Q}$-algebra homomorphism. We say that the Chow motives of X and Y are *isomorphic as algebra objects* if there exists an isomorphism $\Gamma : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ that preserve the algebra structures. The following lemma is a formal consequence of the definition.

Lemma 2.9 (Algebra morphisms)**.** *Let X and Y be connected smooth projective varieties and let $\Gamma : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ be a non-zero morphism that preserves the algebra structures.*

(i) *Γ preserves the units : if $[X]$ is the fundamental class of X in $\mathrm{CH}^0(X)$ and similarly for Y, then*
$$\Gamma_*[X] = [Y].$$
(ii) *Suppose X and Y have same dimension and define c to be the rational number such that $\Gamma^*[Y] = c\,[X]$; then*
$$(\Gamma \otimes \Gamma)^*\Delta_Y = c\,\Delta_X.$$
In particular, Γ is an isomorphism if and only if $c \neq 0$, and in this case, due to Lieberman's formula[6]*, the inverse of Γ is equal to $\frac{1}{c}{}^t\Gamma$.*

Proof. (i) This is the analogue of the basic fact that a non-trivial homomorphism of unital algebras preserves the units. Concretely, the fundamental class of X provides a morphism $1_X : \mathbb{1} \to \mathfrak{h}(X)$, and similarly for Y, and we need to show that $\Gamma \circ 1_X = 1_Y$. First, for dimension reasons we have $\Gamma \circ 1_X = \lambda \cdot 1_Y$ for some $\lambda \in \mathbb{Q}$. Compose then the diagram (7) with the morphism $1_X \otimes 1_X$; one obtains $\lambda^2 = \lambda$. If $\lambda = 0$, then by composing diagram (7) with the morphism $1_X \otimes \mathrm{id}_X$, we find that $\Gamma = 0$. Hence $\lambda = 1$ and we are done.

(ii) The commutativity of (7) provides the identity $\Gamma \circ \delta_X = \delta_Y \circ (\Gamma \otimes \Gamma)$. Letting the latter act contravariantly on $[Y]$ yields
$$c\,\Delta_X = (\Gamma \otimes \Gamma)^*\Delta_Y = {}^t\Gamma \circ \Gamma,$$
where c is the rational number such that $\Gamma^*[Y] = c\,[X]$ and where the second equality is Lieberman's formula. Since we assume that Γ is invertible, we get that $\Gamma^{-1} = \frac{1}{c}\,{}^t\Gamma$. ☐

As is alluded to in Lemma 2.4, the notion of orthogonality is highly relevant when considering morphisms between Frobenius algebras. Let us recast it in the context of motives :

[5]As explained in Lemma 2.9, a non-zero morphism between algebra objects that preserves the multiplication morphisms must also preserve the unit morphisms, and hence is a morphism of algebra objects in the sense of §2.1.

[6]see e.g. [3, §3.1.4], and [56, Lemma 3.3] for a proof.

Definition 2.10 (Orthogonal isomorphisms)**.** Let X and Y be two smooth projective varieties of the same dimension and $\Gamma : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ be an isomorphism between their Chow motives. Then by Lieberman's formula we see that $\Gamma^{-1} = {}^t\Gamma$ if and only if $(\Gamma \otimes \Gamma)_* \Delta_X = \Delta_Y$. In this case, Γ is called an *orthogonal* isomorphism.

Finally, we can unravel the meaning of being isomorphic as Frobenius algebra objects for the motives of two varieties (and the same holds for Hodge morphisms between the cohomology algebras of smooth projective varieties of same dimension).

Proposition 2.11 (Frobenius algebra isomorphisms)**.** *Let X and Y be two smooth projective varieties of the same dimension and $\Gamma : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ be a morphism between their motives. Then the following are equivalent:*

(*i*) Γ *is an isomorphism of Frobenius algebra objects.*
(*ii*) Γ *is an algebra isomorphism and* Γ *is* orthogonal *: that is,* $\Gamma^{-1} = {}^t\Gamma$ *or equivalently,* $(\Gamma \otimes \Gamma)_* \Delta_X = \Delta_Y$.
(*iii*) Γ *is an algebra isomorphism and* $\deg(\Gamma) = 1$ *: that is,* $\deg(\Gamma_*[pt]) = 1$ *or equivalently,* $\Gamma^*[Y] = [X]$.
(*iv*) Γ *is an isomorphism and* $(\Gamma \otimes \Gamma)_* \Delta_X = \Delta_Y$ *and* $(\Gamma \otimes \Gamma \otimes \Gamma)_* \delta_X = \delta_Y$.

Proof. The equivalence between (*i*) and (*ii*) is a special case of Lemma 2.4. The equivalence between (*ii*) and (*iii*) can be read off Lemma 2.9(*ii*). For the equivalence between (*ii*) and (*iv*), one only needs to see that an orthogonal isomorphism ($\Gamma^{-1} = {}^t\Gamma$) is an algebra morphism ($\Gamma \circ \delta_X = \delta_Y \circ (\Gamma \otimes \Gamma)$) if and only if $(\Gamma \otimes \Gamma \otimes \Gamma)_* \delta_X = \delta_Y$. But this again follows from Lieberman's formula. ☐

3. Derived equivalent K3 surfaces

The aim of this section is to prove Theorem 1, Corollaries 1 and 2.

3.1. Proof of Theorem 1.

The proof relies crucially on the Beauville–Voisin description of the algebra structure on the motive of K3 surfaces :

Theorem 3.1 (Beauville–Voisin [4])**.** *Let S be a K3 surface and let o_S be the class of any point lying on a rational curve on S. Then, as cycle classes in* $\mathrm{CH}_2(S \times S \times S)$*, we have*

$$\delta_S = p_{12}^*\Delta_S \cdot p_3^* o_S + p_{13}^*\Delta_S \cdot p_2^* o_S + p_{23}^*\Delta_S \cdot p_1^* o_S - p_1^* o_S \cdot p_2^* o_S - p_1^* o_S \cdot p_3^* o_S - p_2^* o_S \cdot p_3^* o_S, \tag{8}$$

where $p_k : S \times S \times S \to S$ and $p_{ij} : S \times S \times S \to S \times S$ denote the various projections.

Note that, for a K3 surface S, Theorem 3.1 implies that $\alpha \cdot \beta = \deg(\alpha \cdot \beta)\, o_S$ for all divisors $\alpha, \beta \in \mathrm{CH}^1(S)$, and that $c_2(S) = (\delta_S)_* \Delta_S = \chi(S)\, o_S = 24\, o_S \in \mathrm{CH}^2(S)$. (Of course, this is due originally to Beauville–Voisin [4].)

According to Proposition 2.11, in order to establish Theorem 1, it is necessary and sufficient to produce a correspondence $\Gamma : \mathfrak{h}(S) \to \mathfrak{h}(S')$ which is invertible and such that

(*i*) $(\Gamma \otimes \Gamma)_* \Delta_S = \Delta_{S'}$, or equivalently $\Gamma^{-1} = {}^t\Gamma$;
(*ii*) $(\Gamma \otimes \Gamma \otimes \Gamma)_* \delta_S = \delta_{S'}$.

By the Beauville–Voisin Theorem 3.1, it is sufficient (in fact also necessary by looking at the contravariant action of (*ii*) on $\Delta_{S'}$) to produce a correspondence $\Gamma : \mathfrak{h}(S) \to \mathfrak{h}(S')$ which is invertible and such that

(*i*) $(\Gamma \otimes \Gamma)_* \Delta_S = \Delta_{S'}$, or equivalently $\Gamma^{-1} = {}^t\Gamma$;
(*ii'*) $\Gamma_* o_S = o_{S'}$.

We now proceed to the proof of Theorem 1, *i.e.* to constructing an invertible correspondence satisfying (i) and (ii') above. Given a K3 surface S, we consider the refined Chow–Künneth decomposition of Kahn–Murre–Pedrini as described in §1.2.2 given by

$$\mathfrak{h}(S) = \mathfrak{h}^0(S) \oplus \mathfrak{h}^2_{\mathrm{alg}}(S) \oplus \mathfrak{h}^2_{\mathrm{tr}}(S) \oplus \mathfrak{h}^4(S),$$

with $\pi^0_S = o_S \times S$ and $\pi^4_S = S \times o_S$, where π^i_S denote the projectors on the corresponding direct summands and where o_S denotes the Beauville–Voisin zero-cycle as in Theorem 3.1. Moreover, the decomposition is such that ${}^t\pi^2_{\mathrm{alg},S} = \pi^2_{\mathrm{alg},S}$ and ${}^t\pi^2_{\mathrm{tr},S} = \pi^2_{\mathrm{tr},S}$.

Consider now two twisted derived equivalent K3 surfaces S and S'. As in the proof of Theorem 1.1, we only give a proof in the untwisted case, the twisted case being similar. We fix an exact linear equivalence $\Phi_{\mathcal{E}} : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^b(S \times S')$. The proof will proceed in two steps. First, we will construct an invertible correspondence

$$\Gamma_{\mathrm{alg}} : \mathfrak{h}_{\mathrm{alg}}(S) \to \mathfrak{h}_{\mathrm{alg}}(S') \quad \text{with } (\Gamma_{\mathrm{alg}})^{-1} = {}^t\Gamma_{\mathrm{alg}} \text{ and } (\Gamma_{\mathrm{alg}})_* o_S = o_{S'},$$

where $\mathfrak{h}_{\mathrm{alg}}(S) = \mathfrak{h}^0(S) \oplus \mathfrak{h}^2_{\mathrm{alg}}(S) \oplus \mathfrak{h}^4(S)$ (and similarly for S') is the algebraic summand of the motive of S; second, we will construct an invertible correspondence on the transcendental summands of the motives of S and S':

$$\Gamma_{\mathrm{tr}} : \mathfrak{h}^2_{\mathrm{tr}}(S) \to \mathfrak{h}^2_{\mathrm{tr}}(S') \quad \text{with } (\Gamma_{\mathrm{tr}})^{-1} = {}^t\Gamma_{\mathrm{tr}} \text{ and } (\Gamma_{\mathrm{tr}})_* o_S = 0.$$

The correspondence

$$\Gamma := \Gamma_{\mathrm{alg}} + \Gamma_{\mathrm{tr}} : \mathfrak{h}(S) \to \mathfrak{h}(S')$$

will then provide the desired isomorphism of Frobenius algebra objects.

First, the numerical Grothendieck group K_0^{num} equipped with the Euler pairing is clearly a derived invariant. Using the Chern character isomorphism, we obtain an isometry between the quadratic spaces $\widetilde{\mathrm{NS}}(S_{k^s})_{\mathbb{Q}}$ and $\widetilde{\mathrm{NS}}(S'_{k^s})_{\mathbb{Q}}$, where $\widetilde{\mathrm{NS}}$ is the extended Néron–Severi group equipped with the Mukai pairing, hence is isometric to the (orthogonal) direct sum of the Néron–Severi lattice (endowed with the intersection pairing) and a copy of the hyperbolic plane. By Witt's cancellation theorem, the Néron–Severi groups $\mathrm{NS}(S_{k^s})_{\mathbb{Q}}$ and $\mathrm{NS}(S'_{k^s})_{\mathbb{Q}}$ of two derived equivalent surfaces are isomorphic both as $\mathrm{Gal}(k)$-representations and as quadratic spaces; there exists therefore a correspondence $M = \pi^2_{\mathrm{alg},S'} \circ M \circ \pi^2_{\mathrm{alg},S}$ in $\mathrm{CH}^2(S \times_k S')$ inducing an isometry $\mathrm{NS}(S_{k^s})_{\mathbb{Q}} \simeq \mathrm{NS}(S'_{k^s})_{\mathbb{Q}}$. This means that M induces an isomorphism $\mathfrak{h}^2_{\mathrm{alg}}(S) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{alg}}(S')$ with inverse given by its transpose. It follows that $\Gamma_{\mathrm{alg}} := o_S \times S' + M + S \times o_{S'}$ induces an isomorphism $\mathfrak{h}_{\mathrm{alg}}(S) \xrightarrow{\sim} \mathfrak{h}_{\mathrm{alg}}(S')$ with inverse ${}^t\Gamma_{\mathrm{alg}}$. In addition, we have $(\Gamma_{\mathrm{alg}})_* o_S = o_{S'}$.

Second, recall from §1.2.3 that $v_2(\mathcal{E})$ induces an isomorphism $\mathfrak{h}^2_{\mathrm{tr}}(S) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(S')$ with inverse induced by $v_2(\mathcal{E}^\vee \otimes p^*\omega_S)$. Since K3 surfaces have trivial first Chern class and trivial canonical bundle, it follows that the inverse of $v_2(\mathcal{E})$ is in fact its transpose. In other words, $\Gamma_{\mathrm{tr}} := \pi^2_{\mathrm{tr},S'} \circ v_2(\mathcal{E}) \circ \pi^2_{\mathrm{tr},S}$ induces an isomorphism of Chow motives $\mathfrak{h}^2_{\mathrm{tr}}(S) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(S')$ with inverse its transpose. Finally, we do have $(\pi^2_{\mathrm{tr}})_* o_S = 0$ because of the orthogonality of $\pi^2_{\mathrm{tr},S}$ with π^4_S.

The required correspondences Γ_{alg} and Γ_{tr} have thus been constructed and this concludes the proof of Theorem 1. □

3.2. **Proof of Corollary 1.** Let S and S' be two twisted derived equivalent K3 surfaces. Then due to Theorem 1 their motives are isomorphic as Frobenius algebra objects. As is explained in §2.1, isomorphisms of Frobenius algebra objects behave well with respect to (tensor) products, hence it suffices to see that for any $n \in \mathbb{Z}_{>0}$ the Hilbert schemes of n points $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(S')$ have isomorphic Chow motives as Frobenius algebra objects. To this end, we use the result of Fu–Tian [18] that describes the algebra object $\mathfrak{h}(\mathrm{Hilb}^n(S))$ in terms of the algebra objects $\mathfrak{h}(S^m)$ for $m \leq n$, together with some explicit combinatorial rules. More precisely, by

[18, Theorem 1.6 and Remark 1.7], for a K3 surface S, we have an isomorphism of algebra objects :

$$\phi : \mathfrak{h}(\mathrm{Hilb}^n(S)) \simeq \left(\bigoplus_{g\in\mathfrak{S}_n} \mathfrak{h}\left(S^{O(g)}\right), \star_{\mathrm{orb,dt}}\right)^{\mathfrak{S}_n}, \tag{9}$$

where $\mathfrak{S}_n$ is the symmetric group acting naturally on S^n ; for a permutation g, $O(g)$ is its set of orbits in $\{1, \cdots, n\}$, $S^{O(g)}$ is canonically identified with the fixed locus $(S^n)^g$, and finally $\star_{\mathrm{orb,dt}}$ is the orbifold product with discrete torsion (see [17, 18]) defined as follows (let us omit the Tate twists for ease of notation) : it is compatible with the $\mathfrak{S}_n$-grading, and for any $g, h \in \mathfrak{S}_n$, $\mathfrak{h}\left(S^{O(g)}\right) \otimes \mathfrak{h}\left(S^{O(h)}\right) \to \mathfrak{h}\left(S^{O(g,h)}\right)$ is given by the pushforward via the diagonal inclusion $S^{O(g,h)} \hookrightarrow S^{O(g)} \times S^{O(h)} \times S^{O(gh)}$ of the cycle $\epsilon(g,h)c_{g,h} \in \mathrm{CH}(S^{O(g,h)})$, by [18, Lemma 9.3] :

$$c_{g,h} := \begin{cases} 0, & \text{if } \exists\, t \in O(g,h) \text{ with } d_{g,h}(t) \geq 2\,; \\ \prod_{t\in I} \left(24\,\mathrm{pr}_t^*(o_S)\right), & \text{if } \forall t \in O(g,h) \text{ has } d_{g,h}(t) = 0 \text{ or } 1, \end{cases} \tag{10}$$

where $\epsilon(g,h) := (-1)^{\frac{n-|O(g)|-|O(h)|+|O(gh)|}{2}}$, $O(g,h)$ is the set of orbits in $\{1, \cdots, n\}$ under the subgroup generated by g and h ; for any orbit $t \in O(g,h)$,

$$d(g,h)(t) := \frac{2 + |t| - |t/g| - |t/h| - |t/gh|}{2}$$

is the *graph defect* function [18, Lemma 9.1] and $I := \{t \in O(g,h) \mid d_{g,h}(t) = 1\}$ is the subset of orbits with graph defect 1.

As our isomorphism of algebra objects $\Gamma : \mathfrak{h}(S) \to \mathfrak{h}(S')$ satisfies $\Gamma_*(o_S) = o_{S'}$, it is now clear from the above precise description that the right-hand side of (9) for S and for S' are isomorphic algebra objects, and the isomorphism can be chosen orthogonal. As the morphism ϕ in (9) satisfies $\phi^{-1} = {}^t\phi$, we have $\mathfrak{h}(\mathrm{Hilb}^n(S))$ and $\mathfrak{h}(\mathrm{Hilb}^n(S'))$ are isomorphic Frobenius algebra objects. This completes the proof. ☐

Remark 3.2 (Chow rings *vs.* algebra objects)**.** It turns out that we do not need Theorem 1 to show that two twisted derived equivalent K3 surfaces have isomorphic Chow rings. Indeed, Huybrechts' result [28] (generalized to the twisted case in [29]) provides a correspondence $\Gamma \in \mathrm{CH}^2(S \times S')$ that induces an isomorphism of graded $\mathbb{Q}$-vector spaces $\Gamma_* : \mathrm{CH}^*(S) \xrightarrow{\sim} \mathrm{CH}^*(S')$ with the extra property of being isometric on the Néron–Severi spaces $\mathrm{CH}^1(S) \xrightarrow{\sim} \mathrm{CH}^1(S')$. Now thanks to the theorem of Beauville–Voisin [4] saying that the image of the intersection product of two divisors on a K3 surface is of dimension 1, this already implies that Γ_* is actually an isomorphism of graded $\mathbb{Q}$-algebras.

In contrast, in the situation of Corollary 1, a derived equivalence between $D^b(S)$ and $D^b(S')$ does give rise to a derived equivalence between their powers and Hilbert schemes, thanks to Bridgeland–King–Reid [10] and Haiman [22]. However, it is not at all clear for the authors how to produce an isomorphism of the Chow rings (or even the rational cohomology rings) of two derived equivalent holomorphic symplectic varieties starting from the Fourier–Mukai kernel ; see Conjecture 4.6.

3.3. **Proof of Corollary 2.** The equivalence of (i) and (ii) is due to Huybrechts [29, Corollary 1.4], while the implication $(ii) \Rightarrow (iii)$ is Theorem 1. We now prove the implication $(iii) \Rightarrow (i)$. Suppose that $\Gamma : \mathfrak{h}(S) \to \mathfrak{h}(S')$ is an isomorphism that preserves the algebra structures. Let c be the rational number such that $\Gamma^*[S'] = c\,[S]$, or equivalently such that $\Gamma^{-1} = \frac{1}{c}{}^t\Gamma$ by Lemma 2.9 ; then the following diagram is commutative :

$$\begin{tikzcd} \mathrm{H}^2(S)\otimes \mathrm{H}^2(S) \arrow[r, "\cup"] \arrow[d, "(\Gamma\otimes\Gamma)_*"] & \mathrm{H}^4(S) \arrow[r, "\deg"] \arrow[d, "\Gamma_*"] & \mathbb{Q} \arrow[d, "\cdot c"] \\ \mathrm{H}^2(S')\otimes \mathrm{H}^2(S') \arrow[r, "\cup"] & \mathrm{H}^4(S') \arrow[r, "\deg"] & \mathbb{Q}. \end{tikzcd} \tag{11}$$

The commutativity of the left-hand square of (11) is implied directly by the assumption that Γ preserves the algebra structures, while the commutativity of the right-hand square follows from the Poincaré dual of the identity $\Gamma^*[S'] = c\,[S]$. If in addition Γ preserves the Frobenius algebra structure, then $c = 1$ by Proposition 2.11. This means that S and S' are isogenous. □

3.4. **A motivic global Torelli theorem.** The aim of this section is to show that Lemma 2.9 directly allows to upgrade motivically the global Torelli theorem, without utilizing the decomposition of the diagonal of Beauville–Voisin (Theorem 3.1). We denote $\mathfrak{h}(X)_{\mathbb{Z}}$ the Chow motive of X with integral coefficients.

Theorem 3.3 (Motivic global Torelli theorem for K3 surfaces)**.** *Let S and S' be two complex projective K3 surfaces. The following statements are equivalent :*

(i) *S and S' are isomorphic ;*
(ii) *$\mathrm{H}^2(S,\mathbb{Z})$ and $\mathrm{H}^2(S',\mathbb{Z})$ are Hodge isometric ;*
(iii) *$\mathfrak{h}(S)_{\mathbb{Z}}$ and $\mathfrak{h}(S')_{\mathbb{Z}}$ are isomorphic as algebra objects.*

Proof. The equivalence of items (i) and (ii) is the global Torelli theorem. The implication $(i) \Rightarrow (iii)$ is obvious. It remains to check that $(iii) \Rightarrow (ii)$. Once it is observed that Lemma 2.9 holds with integral coefficients, we obtain the following commutative diagram (with $c \in \mathbb{Z}$), which is similar to (11) in the proof of Corollary 2

$$\begin{tikzcd} \mathrm{H}^2(S,\mathbb{Z})\otimes \mathrm{H}^2(S,\mathbb{Z}) \arrow[r, "\cup"] \arrow[d, "(\Gamma\otimes\Gamma)_*", "\simeq"'] & \mathrm{H}^4(S,\mathbb{Z}) \arrow[r, "\deg", "\simeq"'] \arrow[d, "\Gamma_*", "\simeq"'] & \mathbb{Z} \arrow[d, "\cdot c", "\simeq"'] \\ \mathrm{H}^2(S',\mathbb{Z})\otimes \mathrm{H}^2(S',\mathbb{Z}) \arrow[r, "\cup"] & \mathrm{H}^4(S',\mathbb{Z}) \arrow[r, "\deg", "\simeq"'] & \mathbb{Z}. \end{tikzcd}$$

Therefore, there is an isometry of lattices between $\mathrm{H}^2(S,\mathbb{Z}) \otimes \langle c\rangle$ and $\mathrm{H}^2(S',\mathbb{Z})$, which implies that $c = 1$. □

4. Beyond K3 surfaces

Orlov's conjecture 1 predicts that the Chow motives of two derived equivalent smooth projective varieties are isomorphic. Motivated by Theorem 1, we raised the following question in the introduction :

Question 4.1. When can we expect more strongly that a derived equivalence between two smooth projective varieties implies an isomorphism between their rational Chow motives as *Frobenius algebra objects* ?

We make some remarks and speculations on this question in this section.

Remark 4.2. By Bondal–Orlov [8], two derived equivalent smooth projective varieties that are either Fano or with ample canonical bundle are isomorphic ; in particular, their motives are isomorphic as Frobenius algebra objects. Similarly, Question 4.1 also has a positive answer for curves, as they do not have non-isomorphic Fourier–Mukai partners [25, Corollary 5.46].

In general, one cannot expect in general a positive answer to Question 4.1. In fact, if $\mathfrak{h}(X)$ and $\mathfrak{h}(Y)$ are isomorphic as Frobenius algebra objects then by applying the Betti realization functor, their cohomology are isomorphic as Frobenius algebras, that is, due to Proposition 2.11, there is a (graded) isomorphism of $\mathbb{Q}$-algebras $\mathrm{H}^*(X,\mathbb{Q}) \to \mathrm{H}^*(Y,\mathbb{Q})$ sending the class of a point on X to the class of a point on Y. However, as we will see below, this is not the case in general for derived equivalent varieties.

4.1. **Calabi–Yau varieties.**

Example 4.3. Borisov and Căldăraru [9] constructed derived equivalent (but non-birational) Calabi–Yau threefolds X and Y with the following properties : $\mathrm{Pic}(X) = \mathbb{Z}H_X$ with $\deg(H_X^3) = 14$ and $\mathrm{Pic}(Y) = \mathbb{Z}H_Y$ with $\deg(H_Y^3) = 42$; hence there is no graded $\mathbb{Q}$-algebra isomorphism between $\mathrm{H}^*(X,\mathbb{Q})$ and $\mathrm{H}^*(Y,\mathbb{Q})$ that respects the point class. Therefore, $\mathfrak{h}(X)$ and $\mathfrak{h}(Y)$ are not isomorphic as Frobenius algebra objects. Nevertheless, thanks to the following proposition, $\mathrm{H}^*(X,\mathbb{Q})$ and $\mathrm{H}^*(Y,\mathbb{Q})$ are Hodge isomorphic as graded $\mathbb{Q}$-algebras and also as graded Frobenius algebras after extending the coefficients to $\mathbb{R}$ [7].

Proposition 4.4. *Let X and Y be two derived equivalent Calabi–Yau varieties of dimension $d \geq 3$. Suppose their Hodge numbers satisfy*

- $h^{p,q} = 0$ *for all* $p \neq q$ *and* $p + q \neq d$*;*
- $h^{p,p} = 1$ *for all* $2p \neq d$ *and* $0 \leq p \leq d$*.*

Then

(*i*) *There is a (graded) real Frobenius algebras isomorphism between* $\mathrm{H}^*(X,\mathbb{R})$ *and* $\mathrm{H}^*(Y,\mathbb{R})$ *preserving the real Hodge structures.*

(*ii*) *If d is odd or d is even and $s := \frac{\deg(Y)}{\deg(X)}$ is a square in $\mathbb{Q}$, then $\mathrm{H}^*(X,\mathbb{Q})$ and $\mathrm{H}^*(Y,\mathbb{Q})$ are isomorphic as graded $\mathbb{Q}$-Hodge algebras. Here the degree is the top self-intersection number of the ample generator of the Picard group.*

Proof. We first prove (*ii*). Let $\mathcal{E}$ be the Fourier–Mukai kernel of the equivalence from $\mathrm{D}^b(X)$ to $\mathrm{D}^b(Y)$. By [27, Proposition 5.44], the correspondence given by the Mukai vector $v(\mathcal{E}) \in \mathrm{CH}^*(X \times Y)$ induces a $\mathbb{Z}/2\mathbb{Z}$-graded Hodge isometry

$$\Phi_{\mathcal{E}}^{\mathrm{H}} : \mathrm{H}^*(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q}),$$

where both sides are equipped with the Mukai pairing. Note that as the varieties are Calabi–Yau, the Mukai pairing is simply given by the intersection pairing with some extra sign changes ([27, Definition 5.42]). The *transcendental cohomology* denoted by $\mathrm{H}^*_{\mathrm{tr}}(-,\mathbb{Q})$ is defined to be the orthogonal of all the Hodge classes ; it is obviously preserved by $\Phi_{\mathcal{E}}^{\mathrm{H}}$. Thanks to our assumption on the Hodge numbers, the transcendental cohomology is concentrated in degree d. Therefore by restricting $\Phi_{\mathcal{E}}^{\mathrm{H}}$, we get a Hodge isometry

$$\phi_{\mathrm{tr}} : \mathrm{H}^d_{\mathrm{tr}}(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^d_{\mathrm{tr}}(Y,\mathbb{Q}).$$

On the other hand, if d is even, $\Phi_{\mathcal{E}}^{\mathrm{H}}$ also provides an isometry between the subalgebras of Hodge classes $\mathrm{Hdg}^*_{\mathbb{Q}}(X)$ and $\mathrm{Hdg}^*_{\mathbb{Q}}(Y)$. Since the quadratic space $\mathrm{H}^0 \oplus \cdots \oplus \mathrm{H}^{d-2} \oplus \mathrm{H}^{d+2} \oplus \cdots \oplus \mathrm{H}^{2d}$ equipped with the restriction of the Mukai pairing is isometric to $U^{\frac{d}{2}} \otimes \mathbb{Q}$ for both X and Y, the quadratic spaces $\mathrm{Hdg}^d_{\mathbb{Q}}(X)$ and $\mathrm{Hdg}^d_{\mathbb{Q}}(Y)$ are isometric by Witt cancellation theorem. Due to the assumption that $s := \frac{\deg(Y)}{\deg(X)}$ is a square and to Witt's theorem, we have an isometry

$$\phi_{\mathrm{Hdg}} : \mathrm{Hdg}^d_{\mathbb{Q}}(X)\,(s) \longrightarrow \mathrm{Hdg}^d_{\mathbb{Q}}(Y)$$

[7] The authors do not have examples of derived equivalent smooth projective varieties with non-isomorphic cohomology as $\mathbb{Q}$-algebras or as $\mathbb{R}$-Frobenius algebras.

that sends $H_X^{\frac{d}{2}}$ to $H_Y^{\frac{d}{2}}$, where H_X and H_Y denote the ample generators of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$, respectively.

Let us know try to define a graded Hodge algebra isomorphism $\psi : \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(Y, \mathbb{Q})$. Consider the following formulas with the numbers a, b to be determined later :

- $H_X^i \mapsto a^i \cdot H_Y^i$ for all $0 \leq i \leq d$ and consequently $[\mathrm{pt}_X] \mapsto a^d\, s \cdot [\mathrm{pt}_Y]$, where $[\mathrm{pt}]$ is the class of a point ;
- $a^{\frac{d}{2}} \cdot \phi_{\mathrm{Hdg}} : \mathrm{Hdg}^d_{\mathbb{Q}}(X) \to \mathrm{Hdg}^d_{\mathbb{Q}}(Y)$;
- $b \cdot \phi_{\mathrm{tr}} : \mathrm{H}^d_{\mathrm{tr}}(X, \mathbb{Q}) \to \mathrm{H}^d_{\mathrm{tr}}(Y, \mathbb{Q})$.

These formulas define an algebra isomorphism if and only if $b^2 = a^d\, s$. This equation has non-zero rational solutions when d is odd or d is even and s is a square in $\mathbb{Q}$. Item (ii) is therefore proved.

The proof of (i) goes similarly as for (ii) by replacing $\mathbb{Q}$ by $\mathbb{R}$. Notice that the analogous assumption that s is a square in $\mathbb{R}$ is automatically satisfied. So it is enough to see that there are always non-zero real solutions to the equation $b^2 = a^d\, s = 1$, where the last equality reflects the Frobenius condition. □

4.2. **Abelian varieties.**

Proposition 4.5 (Isogenous abelian varieties)**.** *Let A and B be two isogenous abelian varieties of dimension g. Then*

(i) *$\mathfrak{h}(A)$ and $\mathfrak{h}(B)$ are isomorphic as algebra objects.*

(ii) *The following conditions are equivalent:*

  (a) *There is an isomorphism of Frobenius algebra objects between $\mathfrak{h}(A)$ and $\mathfrak{h}(B)$.*

  (b) *There is a graded Hodge isomorphism of Frobenius algebras between $\mathrm{H}^*(A, \mathbb{Q})$ and $\mathrm{H}^*(B, \mathbb{Q})$.*

  (c) *There exists an isogeny of degree m^{2g} between A and B for some $m \in \mathbb{Z}_{>0}$.*

  In the case that these equivalent conditions hold, the isomorphism in (a), denoted by $\Gamma : \mathfrak{h}(A) \to \mathfrak{h}(B)$, can be chosen to respect moreover the motivic decomposition of Deninger–Murre [16] in the sense that $\Gamma \circ \pi_A^i = \pi_B^i \circ \Gamma$ for any i, where the π^i's are the projectors corresponding to the decomposition.

(iii) *$\mathfrak{h}(A)_{\mathbb{R}}$ and $\mathfrak{h}(B)_{\mathbb{R}}$ are isomorphic as Frobenius algebra objects in the category of Chow motives with real coefficients.*

Proof. (i) Consider any isogeny $f : B \to A$. Then $f^* : \mathfrak{h}(A) \to \mathfrak{h}(B)$ is an isomorphism of algebra objects with inverse given by $\frac{1}{\deg(f)} f_*$.

(ii) The implication $(a) \Longrightarrow (b)$ is obtained by applying the realization functor.

$(b) \Longrightarrow (c)$. Let $\gamma : \mathrm{H}^*(A, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(B, \mathbb{Q})$ be a Frobenius algebra isomorphism preserving the Hodge structures, and let $\gamma_i : \mathrm{H}^i(A, \mathbb{Q}) \to \mathrm{H}^i(B, \mathbb{Q})$ be its i-th component, for all $0 \leq i \leq 2g$. There exist a rational number λ and an isogeny $f : B \to A$, such that $\gamma_1 : \mathrm{H}^1(A, \mathbb{Q}) \to \mathrm{H}^1(B, \mathbb{Q})$ is equal to $\frac{1}{\lambda} f^*|_{\mathrm{H}^1}$. As $\mathrm{H}^*(A, \mathbb{Q}) \cong \bigwedge^\bullet \mathrm{H}^1(A, \mathbb{Q})$ as algebras and similarly for B, γ is in fact determined by γ_1 in the following way : for any i, $\gamma_i = \wedge^i \gamma_1 = \frac{1}{\lambda^i} f^*|_{\mathrm{H}^i}$. We compute that

$$\mathrm{id} = {}^t\gamma \circ \gamma = \left(\sum_i \frac{1}{\lambda^i} f_*|_{\mathrm{H}^{2g-i}} \right) \circ \left(\sum_i \frac{1}{\lambda^i} f^*|_{\mathrm{H}^i} \right) = \frac{1}{\lambda^{2g}} \deg(f) \cdot \mathrm{id}.$$

This yields that the isogeny f is of degree λ^{2g}.

$(c) \Longrightarrow (a)$ If there is an isogeny $f : B \to A$ of degree m^{2g}, then for any $0 \leq i \leq 2g$ consider the morphism $\Gamma_i := \frac{1}{m^i} \pi_B^i \circ f^* \circ \pi_A^i = \frac{1}{m^i} f^* \circ \pi_A^i$ from $\mathfrak{h}^i(A)$ to $\mathfrak{h}^i(B)$, which is an isomorphism with inverse $\Gamma_i^{-1} = \frac{1}{m^{2g-i}} \pi_A^i \circ f_*$. Here we use the motivic decomposition of Deninger–Murre

[16] for abelian varieties $\mathfrak{h}(A) = \oplus_{i=0}^{2g}\mathfrak{h}^i(A)$, and π^i is the projector corresponding to $\mathfrak{h}^i$. One readily checks that $\Gamma := \sum_i \Gamma_i : \mathfrak{h}(A) \to \mathfrak{h}(B)$ is an isomorphism of algebra objects. Moreover, as $\pi^i = {}^t\pi^{2g-i}$ for all i, we have that $\Gamma_i^{-1} = {}^t\Gamma_{2g-i}$, hence $\Gamma^{-1} = {}^t\Gamma$, that is, Γ respects the Frobenius structures. Notice that by construction, Γ respects the decomposition of Deninger–Murre.
The proof of *(iii)* is similar to the last part of the proof of *(ii)*. One only needs to notice that there is no obstruction to taking the $2g$-th root of a positive number in $\mathbb{R}$. □

As a consequence, given two derived equivalent abelian varieties, in general there is no isomorphism of Frobenius algebra objects between their Chow motives (or their cohomology). Indeed, by Proposition 4.5*(ii)*, the motives of two derived equivalent abelian varieties that cannot be related by an isogeny of degree the $2g$-th power of some positive integer are not isomorphic as Frobenius algebra objects. For instance, if one considers an abelian variety A with Néron–Severi group generated by one ample line bundle L, then any isogeny between A and $A^\vee$ is of degree $\chi(L)^2m^{4g}$ for some $m \in \mathbb{Z}_{>0}$. But in general, $\chi(L)$ is not a g-th power in $\mathbb{Z}$. On the other hand, A and $A^\vee$ are always derived equivalent by Mukai's classical result [39].

4.3. **Hyper-Kähler varieties.** One particularly interesting class of varieties for which we expect a positive answer consists of (projective) hyper-Kähler varieties ; these constitute higher-dimensional generalizations of K3 surfaces. Note that by Huybrechts–Nieper-Wißkirchen [30], any Fourier–Mukai partner of a hyper-Kähler variety remains hyper-Kähler.

Conjecture 4.6. *Let X and Y be two projective hyper-Kähler varieties. If there is an exact equivalence between triangulated categories $\mathrm{D}^b(X)$ and $\mathrm{D}^b(Y)$, then there exists an isomorphism of Chow motives $\mathfrak{h}(X)$ and $\mathfrak{h}(Y)$, as Frobenius algebra objects in the categories of Chow motives. In particular, their Chow rings as well as cohomology rings are isomorphic.*

The following result is known to the experts ; it answers the last part of Conjecture 4.6 for cohomology with complex coefficients.

Proposition 4.7. *Let X and Y be two derived equivalent projective hyper-Kähler varieties. Then their cohomology rings with complex coefficients are isomorphic as $\mathbb{C}$-algebras.*

Proof. [8] As any exact equivalence $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ is given by Fourier–Mukai kernel, it lifts naturally to an equivalence of differential graded categories. Therefore we have an isomorphism of graded $\mathbb{C}$-algebras between their Hochschild cohomology:

$$\mathrm{HH}^*(X) \simeq \mathrm{HH}^*(Y).$$

By a result of Calaque–Van den Bergh [14], which was also previously announced by Kontsevich, the Hochschild–Kostant–Rosenberg isomorphism twisted by the square root of the Todd genus gives rise to an isomorphism of $\mathbb{C}$-algebras

$$\mathrm{HH}^*(X) \simeq \bigoplus_{i+j=*} \mathrm{H}^i(X, \bigwedge^j T_X).$$

Now the symplectic forms on X induce an isomorphism between T_X and Ω_X, which yields isomorphisms of $\mathbb{C}$-algebras:

$$\bigoplus_{i+j=*} \mathrm{H}^i(X, \bigwedge^j T_X) \simeq \bigoplus_{i+j=*} \mathrm{H}^i(X, \Omega_X^j) \simeq \mathrm{H}^*(X, \mathbb{C}).$$

We can conclude by combining these isomorphisms. □

[8] It is not clear to the authors whether the isomorphism constructed in the proof preserves the Frobenius structure.

There are not so many known examples of derived equivalent hyper-Kähler varieties. Let us test Conjecture 4.6 for the available ones.

Example 4.8. Let S and S' be two derived equivalent K3 surfaces. Then for any $n \in \mathbb{N}^*$, the n-th Hilbert schemes $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(S')$ are derived equivalent. Indeed, by combining the results of Bridgeland–King–Reid [10] and Haiman [22], we have exact linear equivalences of triangulated categories:

$$\mathrm{D}^b(\mathrm{Hilb}^n(S)) \simeq \mathrm{D}^b(\mathfrak{S}_n - \mathrm{Hilb}(S^n)) \simeq \mathrm{D}^b_{\mathfrak{S}_n}(S^n),$$

and similarly for S'; the Fourier–Mukai kernel $\mathcal{E} \boxtimes \cdots \boxtimes \mathcal{E}$ induces an equivalence

$$\mathrm{D}^b_{\mathfrak{S}_n}(S^n) \simeq \mathrm{D}^b_{\mathfrak{S}_n}(S'^n),$$

where $\mathcal{E} \in \mathrm{D}^b(S \times S')$ is the original Fourier–Mukai kernel inducing the equivalence between $\mathrm{D}^b(S)$ and $\mathrm{D}^b(S')$. We showed in Corollary 1 that $\mathfrak{h}(\mathrm{Hilb}^n(S))$ and $\mathfrak{h}(\mathrm{Hilb}^n(S'))$ are isomorphic as Frobenius algebra objects.

Example 4.9. Conjecturally two birationally equivalent hyper-Kähler varieties are derived equivalent [25, Conjecture 6.24]. Thanks to the result of Rieß [51], or rather its proof, we know that birational hyper-Kähler varieties have isomorphic Chow motives as Frobenius algebra objects, hence compatible with Conjecture 4.6. There are by now some cases where the derived equivalence is known. The easiest example might be the so-called Mukai flop. Another instance of interest is as follows: given a projective K3 surface S and a Mukai vector v, when the stability condition σ varies in the chambers of the distinguished component $\mathrm{Stab}^*(S)$ of the manifold of stability conditions on $\mathrm{D}^b(S)$, the moduli spaces $M_\sigma(v)$ of σ-stable objects are all birational to each other, and their derived equivalence has been announced by Halpern-Leistner in [23].

Example 4.10. If one is willing to enlarge a bit the category of hyper-Kähler varieties to that of hyper-Kähler *orbifolds*[9], Conjecture 4.6 is closely related to the so-called *motivic hyper-Kähler resolution conjecture* investigated in [17] and [18]. Indeed, let M be a projective holomorphic symplectic variety endowed with a faithful action of a finite group G by symplectic automorphisms. The quotient stack $[M/G]$ is a hyper-Kähler (or rather symplectic) orbifold. If the main component of the G-invariant Hilbert scheme $X := G{-}\mathrm{Hilb}(M)$ is a symplectic (or equivalently crepant) resolution of the singular variety M/G, then by Bridgeland–King–Reid [10, Corollary 1.3] there is an equivalence of derived categories $\mathrm{D}^b(X) \simeq \mathrm{D}^b([M/G])$. On the other hand, the motivic hyper-Kähler resolution conjecture [17] predicts that the orbifold motive of $[M/G]$ endowed with the *orbifold product* is isomorphic to the motive of X as algebra objects. In this sense, forgetting the Frobenius structure, we can obtain some evidences for the orbifold analogue of Conjecture 4.6: for example between a K3 orbifold and its minimal resolution by [19], between $[\ker(A^{n+1} \xrightarrow{+} A)/\mathfrak{S}_n]$ and the n-th generalized Kummer variety associated to an abelian surface A by [17], and between $[S^n/\mathfrak{S}_n]$ and the n-th Hilbert scheme of a K3 surface S by [18]. In fact, the authors suspect that the motivic hyper-Kähler resolution conjecture can be stated more strongly as an isomorphism of Frobenius algebra objects with *complex* coefficients, and the proofs of our aforementioned results do confirm this stronger version.

5. Chern classes of Fourier–Mukai equivalences between K3 surfaces

The aim of this final section is to provide evidence for the fact that the Chern classes of Fourier–Mukai equivalences between two K3 surfaces S and S' define "distinguished" classes in the Chow ring of $S \times S'$, in the sense that they can be added to the Beauville–Voisin ring of $S \times S'$ and the resulting ring would still inject into cohomology *via* the cycle class map.

[9] Here by an *orbifold* we mean a smooth proper Deligne–Mumford stack with trivial generic stabilizer.

5.1. **The Beauville–Voisin ring, and generalizations.** Let S be a K3 surface and define its Beauville–Voisin ring $R^*(S)$ to be the subring of $\mathrm{CH}^*(S)$ generated by divisors and Chern classes of the tangent bundle. By Beauville–Voisin's Theorem 3.1, this ring has the property that it injects into cohomology via the cycle class map.

Let $\mathfrak{h}(S) = \mathfrak{h}^0(S) \oplus \mathfrak{h}^2(S) \oplus \mathfrak{h}^4(S)$ be the Chow–Künneth decomposition induced by $\pi^0_S = o_S \times S$, $\pi^4_S = S \times o_S$ and $\pi^2_S = \Delta_S - \pi^0_S - \pi^4_S$. In [54, Proposition 8.14] it was observed that the decomposition of the small diagonal (8) is equivalent to the above Chow–Künneth decomposition being *multiplicative*, meaning that the multiplication morphism $\mathfrak{h}(S)\otimes\mathfrak{h}(S) \to \mathfrak{h}(S)$ is compatible with the grading given by the Chow–Künneth decomposition.

The following (formal) facts about multiplicative Chow–Künneth decompositions will be used. Let X and Y be two smooth projective varieties, both having motive endowed with a multiplicative Chow–Künneth decomposition. Then [54, Theorem 8.6] the *product Chow–Künneth decomposition* $\mathfrak{h}^n(X\times Y) = \bigoplus_{i+j=n} \mathfrak{h}^i(X)\otimes\mathfrak{h}^j(Y)$ is multiplicative. Moreover, if $p : X\times Y \to X$ denotes the projection, then $p^* : \mathfrak{h}(X) \to \mathfrak{h}(X \times Y)$ is graded (*i.e.* compatible with the Chow–Künneth decompositions) and $p_* : \mathfrak{h}(X \times Y) \to \mathfrak{h}(X)$ shifts the gradings by $-2\dim Y$.

A Chow–Künneth decomposition on the motive of X induces a bigrading on the Chow groups of X given by

$$\mathrm{CH}^i(X)_{(j)} := \mathrm{CH}^i(\mathfrak{h}^{2i-j}(X)),$$

which in case the Chow–Künneth decomposition is multiplicative satisfies

$$\mathrm{CH}^i(X)_{(j)} \cdot \mathrm{CH}^{i'}(X)_{(j')} \subseteq \mathrm{CH}^{i+i'}(X)_{(j+j')}.$$

Given smooth projective varieties endowed with multiplicative Chow–Künneth decompositions, the products of which are endowed with the product Chow–Künneth decompositions, we therefore see that $\mathrm{CH}^*(-)_{(0)}$ defines a subalgebra of $\mathrm{CH}^*(-)$ that is stable under pushforwards and pullbacks along projections, and stable under composition of correspondences belonging to $\mathrm{CH}^*(- \times -)_{(0)}$.

Murre's conjecture 1.2(B) and (D) imply that $\mathrm{CH}^i(X)_{(0)} := \mathrm{CH}^i(\mathfrak{h}^{2i}(X))$ injects in cohomology with image the Hodge classes for any choice of Chow–Künneth decomposition. (This is known unconditionally in the case $i = 0$ and $i = \dim X$.) In particular, in the above situation of smooth projective varieties endowed with multiplicative Chow–Künneth decompositions, it is expected that the subalgebra $\mathrm{CH}^*(-)_{(0)}$ injects into cohomology with image the Hodge classes. In that sense, $\mathrm{CH}^*(-)_{(0)}$ is a maximal subalgebra of $\mathrm{CH}^*(-)$ with the property that it injects into cohomology *via* the cycle class map.

5.2. **Adding the second Chern class of Fourier–Mukai equivalences to the BV ring.** Recall the following theorem of Huybrechts [26, Theorem 2] and Voisin [57, Corollary 1.10].

Theorem 5.1 (Huybrechts, Voisin). *Let $\Phi_{\mathcal{E}} : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ be an exact linear equivalence between K3 surfaces with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^b(S \times S')$. Then $v(\mathcal{E})$ preserves the Beauville–Voisin ring.*

In light of the discussion in §5.1, it is natural to ask whether a more general statement could be true, namely :

Question 5.2. Let $\Phi_{\mathcal{E}} : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ be an exact linear equivalence between K3 surfaces with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^b(S \times S')$. Then does $v(\mathcal{E})$ belong to $\mathrm{CH}^*(S \times S')_{(0)}$?

For $i = 0$ or 1, the Mukai vectors $v_i(\mathcal{E})$ obviously belong to $\mathrm{CH}^i(S\times S')_{(0)}$, since in those cases $\mathrm{CH}^i(S \times S') = \mathrm{CH}^i(S \times S')_{(0)}$. In the case of $v_2(\mathcal{E})$, this can be deduced from Theorem 5.1 :

Proposition 5.3. *Let $\Phi_{\mathcal{E}} : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ be an exact equivalence with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^b(S \times S')$. Then $v_2(\mathcal{E})$ belongs to $\mathrm{CH}^2(S \times S')_{(0)}$.*

Proof. Since $\mathrm{CH}^2(S\times S') = \mathrm{CH}^2(S\times S')_{(0)} \oplus \mathrm{CH}^2(S\times S')_{(2)}$, it is enough to show that $v_2(\mathcal{E})_{(2)} = 0$. Let γ be any cycle in $\mathrm{CH}^2(S\times S')$. On the one hand, we have $\gamma_{(2)} = \pi^2_{\mathrm{tr},S'} \circ \gamma \circ \pi^4_S + \pi^0_{S'} \circ \gamma \circ \pi^2_{\mathrm{tr},S}$. On the other hand, we have $\gamma \circ \pi^4_S = (p')^*\gamma_* o_S$. Now setting $\gamma = v_2(\mathcal{E})$, Theorem 5.1 yields that $v_2(\mathcal{E}) \circ \pi^4_S$ is a multiple of $(p')^* o_{S'}$ and henceforth since $(\pi^2_{\mathrm{tr},S'})_* o_{S'} = 0$ that $\pi^2_{\mathrm{tr},S'} \circ \gamma \circ \pi^4_S = 0$. Likewise, we have $\pi^0_{S'} \circ v_2(\mathcal{E}) \circ \pi^2_{\mathrm{tr},S} = 0$, and the proposition is established. □

We deduce from §5.1 the following

Theorem 5.4. *Let $\tilde{R}^*(S \times S')$ be the subring of $\mathrm{CH}^*(S \times S')$ generated by divisors, $p^*c_2(S)$, $(p')^*c_2(S')$ and $c_2(\mathcal{E})$, where $\mathcal{E}$ runs through the objects in $\mathrm{D}^b(S \times S')$ inducing exact linear equivalences $\mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$. The cycle class map $\tilde{R}^n(S \times S') \to \mathrm{H}^{2n}(S \times S', \mathbb{Q})$ is injective for $n = 3, 4$. In particular, if $\Phi_{\mathcal{E}_1}, \Phi_{\mathcal{E}_2} : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S')$ are two exact linear equivalences with Fourier–Mukai kernels $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{D}^b(S \times S')$, then $c_2(\mathcal{E}_1) \cdot c_2(\mathcal{E}_2) \in \mathbb{Z}[o_S \times o_{S'}]$.* □

5.3. **Some speculations concerning Chern classes of twisted derived equivalent K3 surfaces.** It is natural to ask whether Theorem 5.1 extends to derived equivalences between twisted K3 surfaces :

Question 5.5. Let $\Phi_{\mathcal{E}} : \mathrm{D}^b(S, \alpha) \xrightarrow{\sim} \mathrm{D}^b(S', \alpha')$ be an exact equivalence between twisted K3 surfaces with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^b(S \times S', \alpha^{-1} \boxtimes \alpha')$. Then does $v(\mathcal{E})$ preserve the Beauville–Voisin ring ? More generally, does $v(\mathcal{E})$ belong to $\mathrm{CH}^*(S \times S')_{(0)}$?

We note that if $v(\mathcal{E})$ preserves the Beauville–Voisin ring, then the same argument as in the proof of Proposition 5.3 gives that $v_2(\mathcal{E})$ belongs to $\mathrm{CH}^2(S \times S')_{(0)}$.

Let us now define $E^*(S \times S')$ to be the subalgebra of $\mathrm{CH}^*(S \times S')$ generated by divisors, $p^*c_2(S)$, $(p')^*c_2(S')$, and the Chern classes of $\mathcal{E}$, where $\mathcal{E}$ runs through objects in $\mathrm{D}^b(S\times S', \alpha^{-1}\boxtimes \alpha')$ inducing exact equivalences $\Phi_{\mathcal{E}} : \mathrm{D}^b(S, \alpha) \xrightarrow{\sim} \mathrm{D}^b(S', \alpha')$ for some Brauer classes $\alpha \in \mathrm{Br}(S)$ and $\alpha' \in \mathrm{Br}(S')$. We then define $\tilde{E}^*(S \times S')$ to be the subalgebra of $\mathrm{CH}^*(S \times S')$ generated by cycles of the form

$$\gamma_{n-1} \circ \cdots \circ \gamma_0,$$

where $\gamma_i \in E^*(S_i \times S_{i+1})$ for all i for some K3 surfaces $S = S_0, S_1, \ldots, S_n = S'$. According to the discussion in §5.1, a positive answer to Question 5.5 would suggest that the following question should have a positive answer.

Question 5.6. Does $\tilde{E}^*(S \times S')$ inject into cohomology via the cycle class map ?

In particular, if $\mathrm{H}^2(S, \mathbb{Q}) \simeq \mathrm{H}^2(S', \mathbb{Q})$ is an isogeny, then the cycle class $v_2(\mathcal{E}_{n-1}) \circ \cdots \circ v_2(\mathcal{E}_0)$ inducing the isogeny between $T(S)_{\mathbb{Q}} := \mathrm{H}^*(\mathfrak{h}^2_{\mathrm{tr}}(S))$ and $T(S')_{\mathbb{Q}} := \mathrm{H}^*(\mathfrak{h}^2_{\mathrm{tr}}(S))$ (with $\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}$ as in (2)) should be canonically defined, *i.e.* should not depend on the choice of twisted derived equivalence between S and S' as in (2) inducing the isogeny.

Appendix A. Non-isogenous K3 surfaces with isomorphic Hodge structures

Recall that two K3 surfaces S and S' are said to be *isogenous* if their second rational cohomology groups are Hodge isometric, that is, if there exists an isomorphism of Hodge structures

$f : \mathrm{H}^2(S,\mathbb{Q}) \xrightarrow{\simeq} \mathrm{H}^2(S',\mathbb{Q})$ making the following diagram commute :

$$\begin{tikzcd} \mathrm{H}^2(S,\mathbb{Q})\otimes \mathrm{H}^2(S,\mathbb{Q}) \arrow[r, "\cup"] \arrow[d, "f\otimes f"] & \mathrm{H}^4(S,\mathbb{Q}) \arrow[r, "\deg"] & \mathbb{Q} \arrow[d, equal] \\ \mathrm{H}^2(S',\mathbb{Q})\otimes \mathrm{H}^2(S',\mathbb{Q}) \arrow[r, "\cup"] & \mathrm{H}^4(S',\mathbb{Q}) \arrow[r, "\deg"] & \mathbb{Q}. \end{tikzcd}$$

We provide in this appendix infinite families of pairwise non-isogenous K3 surfaces with isomorphic rational Hodge structures, that is, for any two K3 surfaces S and S' belonging to the same family, we have $\mathrm{H}^2(S,\mathbb{Q}) \simeq \mathrm{H}^2(S',\mathbb{Q})$ as $\mathbb{Q}$-Hodge structures but there does not exist any isomorphism of $\mathbb{Q}$-Hodge structures $\mathrm{H}^2(S,\mathbb{Q}) \to \mathrm{H}^2(S',\mathbb{Q})$ that is compatible with the intersection pairings given by $(\alpha,\beta)\mapsto \deg(\alpha\cup\beta)$. By Mukai [40], such K3 surfaces are not derived equivalent to each other, and actually not even through a chain of twisted derived equivalences (*cf.* Huybrechts [29, Theorem 0.1]).

In fact, our families have a stronger property : any two K3 surfaces S and S' in the same family have Hodge-isomorphic rational cohomology *algebras*, that is, for any two K3 surfaces S and S' of the family, we have a graded Hodge isomorphism $g : \mathrm{H}^*(S,\mathbb{Q}) \xrightarrow{\simeq} \mathrm{H}^*(S',\mathbb{Q})$ that respects the cup-product, *i.e.* such that the following diagram commutes :

$$\begin{tikzcd} \mathrm{H}^*(S,\mathbb{Q})\otimes \mathrm{H}^*(S,\mathbb{Q}) \arrow[r, "\cup"] \arrow[d, "g\otimes g"] & \mathrm{H}^*(S,\mathbb{Q}) \arrow[d, "g"] \\ \mathrm{H}^*(S',\mathbb{Q})\otimes \mathrm{H}^*(S',\mathbb{Q}) \arrow[r, "\cup"] & \mathrm{H}^*(S',\mathbb{Q}). \end{tikzcd}$$

We say that $\mathrm{H}^*(S,\mathbb{Q})$ and $\mathrm{H}^*(S',\mathbb{Q})$ are *Hodge algebra isomorphic*.

A.1. **Motivation and statements.** The aim of this appendix is to show that, for K3 surfaces, the notion of *isogeny* is strictly more restrictive than the notion of *Hodge algebra isomorphic*. In §A.5 this is upgraded motivically : we show that, for Chow motives of K3 surfaces, the notion of being *isomorphic as Frobenius algebra objects* is strictly more restrictive than the notion of being *isomorphic as algebra objects*, thereby justifying the somewhat technical condition (*iii*) involving Frobenius algebra objects in the motivic Torelli statement of Corollary 2.

First, we provide an infinite family of K3 surfaces whose rational cohomology rings are all Hodge algebra isomorphic (and, *a fortiori*, Hodge isomorphic), but they are pairwise non-isogenous. Precisely we have

Theorem A.1. *There exists an infinite family $\{S_i\}_{i\in\mathbb{Z}_{>0}}$ of pairwise non-isogenous K3 surfaces such that, for all $j,k\in\mathbb{Z}_{>0}$, $\mathrm{H}^*(S_j,\mathbb{Q})$ and $\mathrm{H}^*(S_k,\mathbb{Q})$ are Hodge algebra isomorphic. Moreover, such a family can be chosen to consist of K3 surfaces of maximal Picard rank 20.*

We will present two proofs of this theorem, one in §A.3 and the other in §A.4. Furthermore, we will show in Theorem A.13 that the Chow motives of K3 surfaces belonging to the family of K3 surfaces of Theorem A.1 are all isomorphic as algebra objects.

Second, if the Picard number is maximal, a family of K3 surfaces as in Theorem A.1 must have non-isometric Néron–Severi spaces (Lemma A.8). In this sense, we can improve Theorem A.1 in order to have in addition isometric $\mathbb{Q}$-quadratic forms on the Néron–Severi spaces :

Theorem A.2. *There exists an infinite family $\{S_i\}_{i\in\mathbb{Z}_{>0}}$ of pairwise non-isogenous K3 surfaces such that, for all $j\neq k\in\mathbb{Z}_{>0}$, we have*

- $\mathrm{H}^*(S_j,\mathbb{Q})$ *and* $\mathrm{H}^*(S_k,\mathbb{Q})$ *are Hodge algebra isomorphic.*
- $\mathrm{H}^2_{\mathrm{tr}}(S_j,\mathbb{Q})\simeq \mathrm{H}^2_{\mathrm{tr}}(S_k,\mathbb{Q})$, *hence* $\mathrm{NS}(S_j)_{\mathbb{Q}}\simeq \mathrm{NS}(S_k)_{\mathbb{Q}}$, *as* $\mathbb{Q}$*-quadratic spaces.*

Moreover, such a family can be chosen to consist of K3 surfaces with transcendental lattice being any prescribed even lattice with square discriminant and of signature $(2,2),(2,4),(2,6)$ *or* $(2,8)$.

Furthermore, we will show in Proposition A.14 that, assuming Conjecture A.9, the K3 surfaces in such a family all have isomorphic Chow motives and isomorphic Chow rings, but their Chow motives are pairwise non-isomorphic as Frobenius algebra objects. This gives evidence that one cannot characterize the isogeny class of a complex K3 surface with its Chow ring.

Remark A.3. As mentioned to us by Chiara Camere, examples of a pair of non-isogenous K3 surfaces with isomorphic rational Hodge structures were constructed geometrically, via the so-called *Inose isogenies*[10] [32], by Boissier–Sarti–Veniani [7]. Precisely, let f be a symplectic automorphism of prime order p of a K3 surface S and let S' be the minimal resolution of the quotient $S/\langle f\rangle$; the surface S' is a K3 surface and, by definition, the rational map $S \dashrightarrow S'$ is a degree-p Inose isogeny between these two K3 surfaces. On the one hand, we have isomorphisms of Hodge structures $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q}) = \mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q})^f \simeq \mathrm{H}^2_{\mathrm{tr}}(S/\langle f\rangle,\mathbb{Q}) \simeq \mathrm{H}^2_{\mathrm{tr}}(S',\mathbb{Q})$. On the other hand, [7, Theorem 1.1 and Corollary 1.2] provide many situations where $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q})$ and $\mathrm{H}^2_{\mathrm{tr}}(S',\mathbb{Q})$ are not isometric and consequently $\mathrm{H}^2(S,\mathbb{Q})$ and $\mathrm{H}^2(S',\mathbb{Q})$ are not Hodge isometric. Note that this approach only produces finitely many non-isogenous K3 surfaces with isomorphic Hodge structures. The aim of this appendix is to show, via the surjectivity of the period map, that in fact one can produce an *infinite* family of such pairwise non-isogenous K3 surfaces.

A.2. **Hodge isomorphic *vs.* Hodge algebra isomorphic.** All the examples of non-isogenous K3 surfaces that we will consider will consist of K3 surfaces S and S' whose respective transcendental lattices T and T' become Hodge isometric after some twist, *i.e.* such that $T \simeq T'(m)$ as Hodge lattices for some integer m. The aim of this paragraph is to show that the cohomology algebra of any two such K3 surfaces are Hodge algebra isomorphic ; *cf.* Lemma A.5 below. This will reduce the proofs of Theorems A.1 and A.2 to showing that the K3 surfaces in the families have Hodge isomorphic transcendental cohomology groups.

First we state a general lemma based on the classical classification of quadratic forms over $\mathbb{Q}$. This lemma will also be used in the proof of Theorem A.2.

Lemma A.4. *Let Q be a non-degenerate $\mathbb{Q}$-quadratic form of even rank.*

- *If* $\mathrm{disc}(Q) = 1$ *and* $\mathrm{rk}(Q) \equiv 0$ *mod* 4*, or if* $\mathrm{disc}(Q) = -1$ *and* $\mathrm{rk}(Q) \equiv 2$ *mod* 4*, then for any* $m \in \mathbb{Q}_{>0}$*,* Q *and* $Q(m)$ *are isometric* $\mathbb{Q}$*-quadratic forms.*
- *If* $\mathrm{disc}(Q) = 1$ *and* $\mathrm{rk}(Q) \equiv 2$ *mod* 4*, or if* $\mathrm{disc}(Q) = -1$ *and* $\mathrm{rk}(Q) \equiv 0$ *mod* 4*, then for any* $m \in N\mathbb{Q}(i)^\times$*,* Q *and* $Q(m)$ *are isometric* $\mathbb{Q}$*-quadratic forms, where* $N\mathbb{Q}(i)^\times = \{x^2+y^2 \mid (x,y) \neq (0,0) \in \mathbb{Q}^2\}$ *is the norm group of the field extension* $\mathbb{Q}(i)/\mathbb{Q}$.

Proof. Obviously Q and $Q(m)$ have the same rank and signature. As the rank of Q is even, their discriminants are also the same (that is, only differ by a square). By the classification theory of quadratic forms over $\mathbb{Q}$, we only need to check that for any prime number ℓ, their ε-invariants are equal at all places ℓ. Assume that Q is equivalent to the diagonal form $\langle a_1,\dots,a_r\rangle$ where r is the rank of Q and $a_i \in \mathbb{Q}$; its discriminant is thus given by $\mathrm{disc}(Q) = \prod_{i=1}^r a_i$ and we have

$$\begin{aligned}\varepsilon_\ell(Q(m)) &:= \prod_{i<j}(a_im,a_jm)_\ell = \prod_{i<j}(a_i,a_j)_\ell\left(\prod_{i=1}^r(a_i,m)_\ell\right)^{r-1}(m,m)_\ell^{r(r-1)/2}\\ &= \varepsilon_\ell(Q)\big((\mathrm{disc}(Q)^{r-1}(-1)^{r(r-1)/2},m\big)_\ell.\end{aligned}$$

where $(a,b)_\ell$ is the Hilbert symbol of a,b for the local field $\mathbb{Q}_\ell$ and the last equality uses the identity $(m,m)_\ell = (m,-1)_\ell$. One concludes by using the fact that $(m,-1)_\ell = 1$ for all prime numbers ℓ when $m \in N\mathbb{Q}(i)^\times$. $\square$

[10] Note that an Inose isogeny is not an isogeny in our sense.

Lemma A.5. *Let S and S' be two complex K3 surfaces. The following statements are equivalent:*

(i) There is a Hodge isometry $\mathrm{H}^2_{\mathrm{tr}}(S',\mathbb{Q}) \simeq \mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q})(m)$, for some $m \in \mathbb{Q}_{>0}$;
(ii) There is a Hodge isometry $\mathrm{H}^2(S',\mathbb{Q}) \simeq \mathrm{H}^2(S,\mathbb{Q})(m)$, for some $m \in \mathbb{Q}_{>0}$;
(iii) The cohomology rings $\mathrm{H}^*(S,\mathbb{Q})$ and $\mathrm{H}^*(S',\mathbb{Q})$ are Hodge algebra isomorphic.

Proof. $(i) \Rightarrow (ii)$. Clearly any choice of linear isomorphism between the orthogonal complements of $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q})$ and $\mathrm{H}^2_{\mathrm{tr}}(S',\mathbb{Q})$ provides a Hodge isomorphism $\mathrm{H}^2(S,\mathbb{Q}) \simeq \mathrm{H}^2(S',\mathbb{Q})$ that extends the Hodge isomorphism between $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q})$ and $\mathrm{H}^2_{\mathrm{tr}}(S',\mathbb{Q})$. Since the K3 lattice $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ and its twist $\Lambda(m)$ are $\mathbb{Q}$-isometric for all positive rational numbers m (*cf.* Lemma A.4), the Hodge isometry $\mathrm{H}^2_{\mathrm{tr}}(S',\mathbb{Q}) \simeq \mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q}))(m)$ extends to a Hodge isometry $\mathrm{H}^2(S',\mathbb{Q}) \simeq \mathrm{H}^2(S,\mathbb{Q})(m)$ by Witt's theorem, where the twist (m) refers to the quadratic form.

$(ii) \Rightarrow (iii)$. Item (ii) means that there is a Hodge class $\Gamma \in \mathrm{H}^4(S \times S',\mathbb{Q})$ making the diagram

$$\begin{tikzcd} \mathrm{H}^2(S,\mathbb{Q}) \otimes \mathrm{H}^2(S,\mathbb{Q}) \arrow[r, "\cup"] \arrow[d, "(\Gamma\otimes\Gamma)_*"] & \mathrm{H}^4(S,\mathbb{Q}) \arrow[r, "\deg"] & \mathbb{Q} \arrow[d, "\cdot m"] \\ \mathrm{H}^2(S',\mathbb{Q}) \otimes \mathrm{H}^2(S',\mathbb{Q}) \arrow[r, "\cup"] & \mathrm{H}^4(S',\mathbb{Q}) \arrow[r, "\deg"] & \mathbb{Q} \end{tikzcd} \tag{12}$$

commute. By imposing the $(4,0)$- and $(0,4)$-Künneth components of Γ to be $[\mathrm{pt}] \times [S']$ and $m\,[S] \times [\mathrm{pt}]$, respectively, we obtain that $\Gamma_* : \mathrm{H}^*(S,\mathbb{Q}) \to \mathrm{H}^*(S',\mathbb{Q})$ is an isomorphism of algebras.

$(iii) \Rightarrow (ii)$. Let $\Gamma : \mathrm{H}^*(S,\mathbb{Q}) \to \mathrm{H}^*(S',\mathbb{Q})$ be a Hodge algebra isomorphism. Then its restriction to $\mathrm{H}^2(S,\mathbb{Q})$ induces the commutative diagram (12), where m is the rational number such that $\Gamma_*[\mathrm{pt}] = m\,[\mathrm{pt}]$.

$(ii) \Rightarrow (i)$. As any Hodge isomorphism must preserve the transcendental part, we see that the restriction of a Hodge isometry as in (ii) gives a Hodge isometry $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q}) \to \mathrm{H}^2_{\mathrm{tr}}(S',\mathbb{Q})$. □

A.3. **First construction: an elementary approach.** Our first construction is for K3 surfaces of maximal Picard number $\rho = 20$, and provides a proof of Theorem A.1.

Given a K3 surface S of Picard rank 20, the subspace $\mathrm{H}^{2,0}(S) \oplus \mathrm{H}^{0,2}(S) \subset \mathrm{H}^2(S,\mathbb{Q}) \otimes \mathbb{C}$ is defined over $\mathbb{Q}$, and consequently, up to multiplying by a real scalar, we can fix a generator σ of the 1-dimensional space $\mathrm{H}^{2,0}(S)$ so that $\sigma + \bar{\sigma}$ is rational, i.e., lies in $\mathrm{H}^2(S,\mathbb{Q})$.

Let $(-,-)$ denote the intersection pairing and set $v := (\sigma,\bar{\sigma}) > 0$. Since $(\sigma+\bar{\sigma},\sigma+\bar{\sigma}) = 2(\sigma,\bar{\sigma}) = 2v$, we see that $v \in \mathbb{Q}_{>0}$. Since the real element $i(\sigma-\bar{\sigma})$ spans the orthogonal complement of the rational line spanned by $\sigma+\bar{\sigma}$, the real line spanned by $i(\sigma-\bar{\sigma})$ is also defined over $\mathbb{Q}$. Since $i(\sigma-\bar{\sigma})$ has norm $(i(\sigma-\bar{\sigma}), i(\sigma-\bar{\sigma})) = 2(\sigma,\bar{\sigma}) = 2v$, there exists a rational number $\alpha > 0$ such that $i\sqrt{\alpha}(\sigma-\bar{\sigma}) \in \mathrm{H}^2(S,\mathbb{Q})$.

Let $\{D_1,\ldots,D_\rho\}$ be an orthogonal basis of the Néron–Severi space $\mathrm{NS}(S)_{\mathbb{Q}}$. Write $(D_j,D_j) = 2d_j$ with $d_j \in \mathbb{Q}$. We then have the following $\mathbb{Q}$-basis of $\mathrm{H}^2(S,\mathbb{Q})$:

$$\{\sigma+\bar{\sigma}, i\sqrt{\alpha}(\sigma-\bar{\sigma}), D_1,\ldots,D_\rho\}.$$

Let V be the lattice $\mathrm{H}^2(S,\mathbb{Z})$ equipped with the intersection pairing. We aim, via the (surjective) period map, to construct new K3 surfaces by constructing new Hodge structures on the lattice V. To this end, define a $\mathbb{Q}$-linear map ϕ from the transcendental part $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q})$ to $V_{\mathbb{Q}}$ by the following formula

$$\begin{cases} \phi(\sigma) = a\sigma + \bar{b}\bar{\sigma} + \sum_{j=1}^{\rho} c_j D_j \\ \phi(\bar{\sigma}) = b\sigma + \bar{a}\bar{\sigma} + \sum_{j=1}^{\rho} \bar{c}_j D_j \end{cases} \tag{13}$$

with $a, b, c_j \in \mathbb{C}$. The condition that the image of ϕ lies in $V_{\mathbb{Q}}$ is equivalent to saying that

$$a, b, c_j \in \mathbb{Q}(\sqrt{-\alpha}), \tag{14}$$

where $\mathbb{Q}(\sqrt{-\alpha})$ is the imaginary quadratic extension of $\mathbb{Q}$ with basis 1 and $i\sqrt{\alpha}$.

Consider the following subspaces of $V_{\mathbb{C}}$:

$$V^{2,0} := \mathbb{C} \cdot \phi(\sigma), \quad V^{0,2} := \mathbb{C} \cdot \phi(\bar{\sigma}), \quad V^{1,1} := (V^{0,2} + V^{2,0})^{\perp}.$$

By the global Torelli theorem [48], they define a Hodge structure of a K3 surface S' if and only if

$$(\phi(\sigma), \phi(\sigma)) = 0 \text{ and } (\phi(\sigma), \phi(\bar{\sigma})) > 0,$$

or equivalently, the following numerical conditions are satisfied : [11]

$$\begin{cases} va\bar{b} + \sum_{j=1}^{\rho} c_j^2 d_j = 0 \\ v(|a|^2 + |b|^2) + 2\sum_{j=1}^{\rho} |c_j|^2 d_j > 0. \end{cases} \tag{15}$$

By construction, there is an isomorphism between the $\mathbb{Q}$-Hodge structures of the two K3 surfaces S and S' :

$$\widetilde{\phi} : \mathrm{H}^2(S, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(S', \mathbb{Q}),$$

where $\widetilde{\phi}$ is given on $\mathrm{H}^2_{\mathrm{tr}}(S, \mathbb{Q})$ by ϕ and on $\mathrm{NS}(S)_{\mathbb{Q}}$ by any isomorphism to $\mathrm{NS}(S')_{\mathbb{Q}}$.

Note that there is *a priori* no reason for $\widetilde{\phi}$ to be an isometry. Our goal is actually to provide examples where no Hodge isometry exists. If there exists a Hodge isometry

$$\psi : \mathrm{H}^2(S, \mathbb{Q}) \to \mathrm{H}^2(S', \mathbb{Q}),$$

then as $\mathrm{H}^{2,0}$ is 1-dimensional, there is a $\lambda \in \mathbb{C}^{\times}$, such that

$$\psi(\sigma) = \frac{1}{\lambda}\phi(\sigma) = \frac{1}{\lambda}\left(a\sigma + \bar{b}\bar{\sigma} + \sum_{j=1}^{\rho} c_j D_j\right).$$

Since ψ respects the $\mathbb{Q}$-structures, we find that $\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c_j}{\lambda} \in \mathbb{Q}(\sqrt{-\alpha})$, hence from (14) that $\lambda \in \mathbb{Q}(\sqrt{-\alpha})^{\times}$. The condition that ψ is an isometry implies in particular that

$$(\psi(\sigma), \psi(\bar{\sigma})) = (\sigma, \bar{\sigma}),$$

that is,

$$|\lambda|^2 = |a|^2 + |b|^2 + 2\sum_{j=1}^{\rho} |c_j|^2 \frac{d_j}{v}.$$

To summarize, any solution $a, b, c_j \in \mathbb{Q}(\sqrt{-\alpha})$ of (15) gives rise to a K3 surface S' with $\mathrm{H}^2(S, \mathbb{Q})$ isomorphic to $\mathrm{H}^2(S, \mathbb{Q})$ as $\mathbb{Q}$-Hodge structures and with $\mathrm{H}^2(S, \mathbb{Q}) \otimes \langle m \rangle$ isometric to $\mathrm{H}^2(S', \mathbb{Q})$ with $m := (\phi(\sigma), \phi(\bar{\sigma}))/v$, but it would be isogenous to S only if

$$|a|^2 + |b|^2 + 2\sum_{j=1}^{\rho} |c_j|^2 \frac{d_j}{v} \in N\mathbb{Q}(\sqrt{-\alpha})^{\times}, \tag{16}$$

[11] We observe that these conditions imply that a and b cannot be both zero. Indeed, by the Hodge index theorem, one of the d_i's is positive, say d_1, while all the others are negative. The triangular inequality applied to $c_1^2 d_1 = \sum_{j=2}^{\rho} c_j^2(-d_j)$ yields $|c_1|^2 d_1 \leq \sum_{j=2}^{\rho} |c_j|^2(-d_j)$, i.e., $\sum_{j=1}^{\rho} |c_j|^2 d_j \leq 0$.

where $N\mathbb{Q}(\sqrt{-\alpha})^\times = \left\{|z|^2 \mid z \in \mathbb{Q}(\sqrt{-\alpha})^\times\right\} = \left\{x^2+\alpha y^2 \mid (x,y) \neq (0,0) \in \mathbb{Q}^2\right\}$ is the norm group of the extension $\mathbb{Q}(\sqrt{-\alpha})/\mathbb{Q}$. By the same argument, for two such K3 surfaces S' and S'', corresponding to solutions (a,b,c_j) and (a',b',c'_j) of (15), if they are isogenous, then

$$\frac{|a|^2+|b|^2+2\sum_{j=1}^{\rho}|c_j|^2 d_j/v}{|a'|^2+|b'|^2+2\sum_{j=1}^{\rho}|c'_j|^2 d_j/v} \in N\mathbb{Q}(\sqrt{-\alpha})^\times. \tag{17}$$

First proof of Theorem A.1. By the previous discussion (together with Lemma A.5), it is enough to provide infinitely many solutions of (15) such that (16) does not hold for each of them and such that (17) does not hold for any two of them.

Let $S = \left(T_0^4+\cdots+T_3^4=0\right) \subset \mathbb{P}^3$ be the Fermat quartic surface. We know (see [31, Appendix A]) that S is the Kummer surface associated to an abelian surface A, which has a degree-two isogeny from $E\times E$, where $E \simeq \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\cdot i$ is the elliptic curve with j-invariant 1728.

$$\begin{tikzcd}
 & & S \arrow[d] \\
E\times E \arrow[rru, dashed, "\pi"] \arrow[r, "\text{degree-2 isogeny}"'] & A \arrow[r] & A/\{\pm 1\}
\end{tikzcd}$$

Consider the basis $\{e_1 := 1, e_2 := i\}$ of $\mathrm{H}_1(E,\mathbb{Z})$ and let $\{e_1^*, e_2^*\}$ be the dual basis of $\mathrm{H}^1(E,\mathbb{Z})$. Denote by z the holomorphic coordinate of $\mathbb{C}$, so that $\mathrm{H}^{1,0}(E)$ is generated by dz. We see that

$$\begin{cases} dz = e_1^* + ie_2^*, \\ d\bar{z} = e_1^* - ie_2^*, \end{cases} \tag{18}$$

and that $\int_E dz\wedge d\bar{z} = -2i$.

Let σ be the generator of $\mathrm{H}^{2,0}(S)$ such that $\pi^*(\sigma) = dz_1\wedge dz_2$ in $\mathrm{H}^{2,0}(E\times E)$. Using (18), one checks readily that $\sigma+\bar{\sigma}$ and $i\,(\sigma-\bar{\sigma})$ belong to the rational lattice $\mathrm{H}^2(S,\mathbb{Q})$ because

$$\begin{cases} \pi^*(\sigma+\bar{\sigma}) = dz_1\wedge dz_2 + d\bar{z}_1\wedge d\bar{z}_2 = 2e_{1,1}^*\wedge e_{2,1}^* - 2e_{1,2}^*\wedge e_{2,2}^*; \\ \pi^*\left(i\sigma - i\bar{\sigma}\right) = idz_1\wedge dz_2 - id\bar{z}_1\wedge d\bar{z}_2 = -2e_{1,1}^*\wedge e_{2,2}^* - 2e_{1,2}^*\wedge e_{2,1}^*. \end{cases}$$

Hence $\{\sigma+\bar{\sigma}, i\,(\sigma-\bar{\sigma})\}$ is a $\mathbb{Q}$-basis of $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q})$; one can therefore take $\alpha = 1$, and $\mathbb{Q}(\sqrt{-\alpha})$ is simply $\mathbb{Q}(i)$.

On the other hand,

$$v = (\sigma,\bar{\sigma}) := \int_S \sigma\wedge\bar{\sigma} = \frac{1}{\deg(\pi)}\int_{E\times E} dz_1\wedge dz_2\wedge d\bar{z}_1\wedge d\bar{z}_2 = -\frac{1}{\deg(\pi)}\left(\int_E dz\wedge d\bar{z}\right)^2 = 1.$$

Now take an orthogonal $\mathbb{Q}$-basis $\{D_1,\cdots,D_\rho\}$ of $\mathrm{NS}(S)_\mathbb{Q}$ such that $d_1 = 2$, $d_2 = -1$; this is possible since the Néron–Severi lattice of S is isomorphic to $E_8(-1)^{\oplus 2}\oplus U\oplus\mathbb{Z}(-8)^{\oplus 2}$ (see [52]). We will only use solutions of (15) with $c_3 = \cdots = c_\rho = 0$ (and remember that $\alpha = v = 1$, $d_1 = 2$, $d_2 = -1$):

$$\begin{cases} a\bar{b} + 2c_1^2 - c_2^2 = 0 \\ |a|^2+|b|^2+4|c_1|^2-2|c_2|^2 > 0. \end{cases}$$

For any integer $m>0$, we have a solution

$$a = 2m, \quad b = -\frac{1}{m}(1+i), \quad c_1 = 1, \quad c_2 = 1+i,$$

where

$|a|^2+|b|^2+4|c_1|^2-2|c_2|^2 = \frac{2}{m^2}(2m^4+1) = |\frac{1+i}{m}|^2(2m^4+1)$. By the previous discussion, for our purpose, it suffices to produce an infinite sequence of positive integers $\{m_j\}_{j=1}^\infty$ such that for any j, $2m_j^4+1 \notin N\mathbb{Q}(i)^\times$ and for any $j \neq k$, $\frac{2m_j^4+1}{2m_k^4+1} \notin N\mathbb{Q}(i)^\times$, where $N\mathbb{Q}(i) = \{x^2+y^2 \mid (x,y) \neq (0,0) \in \mathbb{Q}^2\}$.

We conclude the proof of Theorem A.1 by constructing such a sequence inductively by means of elementary arithmetics. This is the object of Lemma A.6 below. □

Lemma A.6. *Let $m_1 := 1$ and for any j, $m_{j+1} := \prod_{l=1}^{j}(2m_l^4+1)$. Then for any j, $2m_j^4+1 \notin N\mathbb{Q}(i)^\times$ and for any $j \neq k$, $\frac{2m_j^4+1}{2m_k^4+1} \notin N\mathbb{Q}(i)^\times$.*

Proof. A standard argument of infinite descent shows that a positive integer n is not the sum of squares of two rational numbers, *i.e.* not in $N\mathbb{Q}(i)^\times$, if and only if it is not the sum of squares of two integers. By the theorem of sum of two squares, the latter is equivalent to the condition that any prime divisor of n with odd adic valuation is not congruent to 3 modulo 4.
As the m_j's are all odd, $2m_j^4+1 \equiv 3 \mod 4$, hence is not the sum of two squares. On the other hand, by construction, for any $j \neq k$, $2m_j^4+1$ and $2m_k^4+1$ are coprime to each other, therefore their product admits a prime divisor p congruent to 3 modulo 4 with odd adic valuation. Hence $\frac{2m_j^4+1}{2m_k^4+1} \notin N\mathbb{Q}(i)^\times$. □

Remark A.7. The reason for requiring that c_1 and c_2 are not zero in the proof of Theorem A.1 is the following. Let $\alpha \in \mathbb{Q}_{>0}$, and consider a solution to (15) with $c_2 = \cdots = c_\rho = 0$, that is, a solution to

$$\begin{cases} va\bar{b} + c^2 d = 0 \\ v(|a|^2+|b|^2) + 2|c|^2 d > 0, \end{cases}$$

with $a, b, c \in \mathbb{Q}(\sqrt{-\alpha})$, $v \in \mathbb{Q}_{>0}$ and $d \in \mathbb{Z}$. We observe that $va\bar{b} + c^2 d = 0$ implies that $|a||b| = \mp |c|^2 \frac{d}{v}$ is a non-negative rational number, where the sign depends on the sign of d. Recall from footnote 11 that a and b cannot be both zero ; if one of them is zero, we may assume without loss of generality that it is b. Since $|a|^2$ is a rational number, this implies that $|b| = t|a|$ for some rational number $t \in \mathbb{Q}_{\geq 0}$. It is then immediate to check that

$$|a|^2 + |b|^2 + 2|c|^2 \frac{d}{v} = |a|^2 + |b|^2 \pm 2|a||b| = |a|^2(1 \pm t)^2.$$

In other words, the equation $va\bar{b} + c^2 d = 0$ (with a and b not both zero) forces the positivity of $|a|^2 + |b|^2 + 2|c|^2 d/v$, but also implies that it belongs to $N\mathbb{Q}(\sqrt{-\alpha})^\times$. Therefore, the new K3 surface S' obtained, via the global Torelli theorem, by imposing $c_2 = \cdots = c_\rho = 0$ in (13) must be isogenous to S.

A.4. **Second construction : via Nikulin's embedding.** Our second construction is *via* Nikulin's embedding theorem [44] and provides proofs for Theorems A.1 and A.2.

A.4.1. *Hodge algebra isomorphic but non-isometric transcendental cohomology.* This lattice-theoretic approach to show the existence of non-isogenous K3 surfaces with Hodge isomorphic second cohomology group was communicated to us by Benjamin Bakker, who attributes it to Huybrechts. Let S be a projective K3 surface with Picard number ρ. Denote its transcendental lattice by $T := \mathrm{H}^2_{\mathrm{tr}}(S, \mathbb{Z})$; it is an even lattice of signature $(2, 20-\rho)$. By Nikulin's embedding theorem [44, Theorem 1.14.4] (see also [27, Theorem **14**.1.12, Corollary **14**.3.5]), when $12 \leq \rho \leq 20$, for any integer $m > 0$, the lattice $T(m)$ admits a primitive embedding into the K3 lattice $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$, unique up to $O(\Lambda)$. Now consider the (new) Hodge structure on Λ given by declaring that the Hodge structure on $T(m)$ is the same as the one on T and $T(m)^\perp$ is of type $(1,1)$. By the surjectivity of the period map, there exists a K3 surface S_m, such that there is a Hodge isometry $T(m) \simeq \mathrm{H}^2_{\mathrm{tr}}(S_m, \mathbb{Z})$. In particular, for all $m > 0$, the Hodge structures $\mathrm{H}^2(S_m, \mathbb{Q})$ are all isomorphic to $\mathrm{H}^2(S, \mathbb{Q})$ and in fact, for all $m > 0$, the cohomology algebras $\mathrm{H}^*(S_m, \mathbb{Q})$ are Hodge algebra isomorphic due to Lemma A.5.

Second proof of Theorem A.1. We now take $\rho = 20$ and let S be the Fermat quartic surface; its transcendental lattice T is isomorphic to $\mathbb{Z}(8) \oplus \mathbb{Z}(8)$. In order to prove Theorem A.1, it is enough to construct an infinite sequence of positive integers $\{m_j\}_{j=1}^{\infty}$, such that the lattices $T(m_j) \simeq \mathbb{Z}(8m_j) \oplus \mathbb{Z}(8m_j)$ are pairwise non-isometric over $\mathbb{Q}$. However, it is easy to see that for $m, m' \in \mathbb{Z}_{>0}$, the two $\mathbb{Q}$-quadratic forms $\mathbb{Q}(8m) \oplus \mathbb{Q}(8m)$ and $\mathbb{Q}(8m') \oplus \mathbb{Q}(8m')$ are isometric if and only if mm' belongs to $\{x^2 + y^2 \mid x, y \in \mathbb{Z}\}$, which in turn is equivalent to the condition that any prime factor of mm' congruent to 3 modulo 4 has even exponent, by the theorem of the sum of two squares. Therefore a desired sequence is easy to construct, for example, one can take m_j to be the j-th prime number congruent to 3 modulo 4. □

A.4.2. *Hodge algebra isomorphic and isometric but non-Hodge isometric transcendental cohomology.* In the previous example, the K3 surfaces S_{m_j} all have isomorphic $\mathrm{H}^2(-, \mathbb{Q})$ as rational Hodge structures and their isogeny classes are distinguished from each other by the $\mathbb{Q}$-quadratic forms on $\mathrm{H}^2_{\mathrm{tr}}(-, \mathbb{Q})$. We would like to go further and produce non-isogenous K3 surfaces with $\mathrm{H}^2_{\mathrm{tr}}(-, \mathbb{Q})$ both isomorphic as Hodge structures and isometric as $\mathbb{Q}$-quadratic forms. Let us first remark that no such examples of K3 surfaces exist in the case of maximal Picard number $\rho = 20$. Indeed, we have the following elementary result:

Lemma A.8. *Let S, S' be two K3 surfaces with maximal Picard number $\rho = 20$.*

- *If $\mathrm{H}^2_{\mathrm{tr}}(S, \mathbb{Z})$ and $\mathrm{H}^2_{\mathrm{tr}}(S', \mathbb{Z})$ are isometric lattices, then S and S' are isomorphic.*
- *If $\mathrm{H}^2_{\mathrm{tr}}(S, \mathbb{Q})$ and $\mathrm{H}^2_{\mathrm{tr}}(S', \mathbb{Q})$ are isometric $\mathbb{Q}$-quadratic forms, then S and S' are isogenous.*
- *If $\mathrm{H}^2_{\mathrm{tr}}(S, \mathbb{Q})$ and $\mathrm{H}^2_{\mathrm{tr}}(S', \mathbb{Q})$ are Hodge isomorphic, then $\mathrm{H}^*(S, \mathbb{Q})$ and $\mathrm{H}^*(S', \mathbb{Q})$ are Hodge algebra isomorphic.*

Proof. We only prove the first two points; the third is left to the reader. Let T be the quadratic space underlying their transcendental cohomologies. Due to the Hodge–Riemann bilinear relations, T is positive definite. Choose an orthogonal basis $\{e_1, e_2\}$ of T and let $d_i := (e_i, e_i) \in \mathbb{Q}_{>0}$. One observes that there are only two isotropic directions in $T \otimes \mathbb{C}$, namely $\sqrt{d_1}e_1 \pm i\sqrt{d_2}e_2$, hence only two possible Hodge structures of K3 type on T. However, these two Hodge structures are Hodge isometric via the $\mathbb{Q}$-linear transformation $e_1 \mapsto e_1$; $e_2 \mapsto -e_2$. □

For K3 surfaces with $\rho = 12, 14, 16, 18$, there are indeed examples of non-isogenous K3 surfaces with $\mathrm{H}^2_{\mathrm{tr}}(-, \mathbb{Q})$ both isometric and isomorphic as rational Hodge structures, as is stated in Theorem A.2.

Proof of Theorem A.2. Given any even lattice T of signature $(2, 2), (2, 4), (2, 6)$ or $(2, 8)$ whose discriminant is a square, by Lemma A.4, the $\mathbb{Q}$-quadratic forms $T(m) \otimes \mathbb{Q}$ and $T \otimes \mathbb{Q}$ are isometric for any integer $m \in \mathbb{Z}_{>0}$ which is the sum of two squares. On the other hand, a generic choice of an isotropic element $\sigma \in T \otimes \mathbb{C}$ gives rise to an irreducible $\mathbb{Q}$-Hodge structure on T, hence on all its twists $T(m)$, with minimal endomorphism algebra; i.e. $\mathrm{End}_{HS}(T_{\mathbb{Q}}) \simeq \mathbb{Q}$. So for all $m \in \mathbb{Z}_{>0}$ which is the sum of two squares, the twists $T(m) \otimes \mathbb{Q}$ are all Hodge isomorphic and isometric. However, since for any such integers m and m', $\mathrm{Hom}_{HS}(T(m) \otimes \mathbb{Q}, T(m') \otimes \mathbb{Q}) = \mathbb{Q}$, we see that $T(m) \otimes \mathbb{Q}$ and $T(m') \otimes \mathbb{Q}$ are Hodge isometric if and only if mm' is a square.

In order to realize the twists $T(m)$ as transcendental lattices of K3 surfaces, we use Nikulin's embedding theorem [44, Theorem 1.14.4] to get a primitive embedding of $T(m)$ into the K3 lattice Λ. For each $m \in \mathbb{Z}_{>0}$, we can therefore construct a Hodge structure on Λ by declaring that $T(m)$ carries the Hodge structure on T and $T(m)^{\perp}$ is of type $(1, 1)$. By the surjectivity of the period map, for any $m \in \mathbb{Z}_{>0}$, there exists a K3 surface S_m with $\mathrm{H}^2_{\mathrm{tr}}(S_m, \mathbb{Q})$ Hodge isometric to $T(m)$.

Now, thanks to Lemma A.5, it remains to construct an infinite sequence of positive integers $\{m_j\}_{j=1}^{\infty}$ which are sums of two squares, such that the product of any two different terms is not a square. This is easily achieved : for example, one can take m_j to be the j-th prime number congruent to 1 modulo 4. ☐

A.5. **Consequences on motives.**

A.5.1. *The general expectations.* The following conjecture is a combination of the Hodge conjecture and of the conservativity conjecture (which itself is a consequence of the Kimura–O'Sullivan finite-dimensionality conjecture or of the Bloch–Beilinson conjectures).

Conjecture A.9. *Two smooth projective varieties X and Y have isomorphic Chow motives if and only if their rational cohomologies are isomorphic as Hodge structures :*

$$\mathfrak{h}(X) \simeq \mathfrak{h}(Y) \text{ as Chow motives } \iff \mathrm{H}^*(X,\mathbb{Q}) \simeq \mathrm{H}^*(Y,\mathbb{Q}) \text{ as graded Hodge structures.}$$

The implication $\Rightarrow$ holds unconditionally and is simply attained by applying the Betti realization functor. Regarding the implication $\Leftarrow$, the Hodge conjecture predicts that an isomorphism $\mathrm{H}^*(X,\mathbb{Q}) \simeq \mathrm{H}^*(Y,\mathbb{Q})$ of graded Hodge structures and its inverse are induced by the action of a correspondence. Hence the homological motives of X and Y are isomorphic. By conservativity, such an isomorphism lifts to rational equivalence, *i.e.* lifts to an isomorphism between the Chow motives of X and Y.

Obviously, if $\mathfrak{h}(X) \simeq \mathfrak{h}(Y)$ as (Frobenius) algebra objects in the category of Chow motives, then by realization $\mathrm{H}^*(X,\mathbb{Q})$ and $\mathrm{H}^*(Y,\mathbb{Q})$ are Hodge (Frobenius) algebra isomorphic. We would like to discuss to which extent the converse statement could be true. In general, this is not the case : consider for instance a complex K3 surface S ; then the blow-up S_1 of S at a point lying on a rational curve and the blow-up S_2 of S at a very general point have Hodge isomorphic cohomology Frobenius algebras, but due to the Beauville–Voisin Theorem 3.1 we have

$$1 = \mathrm{rk}\left(\mathrm{CH}^1(S_1)\otimes \mathrm{CH}^1(S_1) \to \mathrm{CH}^2(S_1)\right) \neq \mathrm{rk}\left(\mathrm{CH}^1(S_2)\otimes \mathrm{CH}^1(S_2) \to \mathrm{CH}^2(S_2)\right) = 2,$$

in particular their Chow motives are not isomorphic as algebra objects.

However, in the case of hyper-Kähler varieties, one can expect :

Conjecture A.10. *Two smooth projective hyper-Kähler varieties X and Y have isomorphic Chow motives as (Frobenius) algebra objects if and only if their rational cohomology rings are Hodge (Frobenius) algebra isomorphic :*

$$\begin{aligned} &\mathfrak{h}(X) \simeq \mathfrak{h}(Y) \text{ as (Frobenius) algebra objects} \\ &\iff \mathrm{H}^*(X,\mathbb{Q}) \text{ and } \mathrm{H}^*(Y,\mathbb{Q}) \text{ are Hodge (Frobenius) algebra isomorphic.}\end{aligned}$$

We note that Corollary 2 establishes this conjecture in the case of K3 surfaces and Frobenius algebra structures. In general, Conjecture A.10 is implied by the combination of the Hodge conjecture and of the "distinguished marking conjecture" for hyper-Kähler varieties [20, Conjecture 2].

Proposition A.11. *Let X and Y be hyper-Kähler varieties of same dimension d. Assume :*

- *The Hodge conjecture in codimension d for $X \times Y$;*
- *X and Y satisfy the "distinguished marking conjecture"* [20, Conjecture 2].

Then Conjecture A.10 holds for X and Y.

Proof. By [20, §3], the distinguished marking conjecture for X and Y provides for all non-negative integers n and m a section to the graded algebra epimorphism $\mathrm{CH}^*(X^n \times Y^m) \to \overline{\mathrm{CH}}^*(X^n \times Y^m)$ in such a way that these are compatible with push-forwards and pull-backs along projections. In addition, the images of the sections corresponding to $\mathrm{CH}^*(X^2) \to \overline{\mathrm{CH}}^*(X^2)$ and $\mathrm{CH}^*(Y^2) \to \overline{\mathrm{CH}}^*(Y^2)$ contain the diagonals Δ_X and Δ_Y, respectively. Here, $\overline{\mathrm{CH}}^*(-)$ denotes the Chow ring modulo numerical equivalence. In fact, since numerical and homological equivalence agree for abelian varieties, the same holds for X and Y (*via* their markings).

As before, the Hodge conjecture predicts that a Hodge isomorphism $\mathrm{H}^*(X,\mathbb{Q}) \simeq \mathrm{H}^*(Y,\mathbb{Q})$ and its inverse are induced by the action of an algebraic correspondence. We fix the isomorphism $\phi : \mathfrak{h}(X) \xrightarrow{\sim} \mathfrak{h}(Y)$ to be the correspondence that is the image of the Hodge class under the section to $\mathrm{CH}^*(X \times Y) \to \overline{\mathrm{CH}}^*(X \times Y)$ inducing the Hodge isomorphism of (Frobenius) algebras $\mathrm{H}^*(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q})$. Since the (Frobenius) algebra structure on the motives of varieties is simply described in terms of the rational equivalence class of the diagonal and of the small diagonal, the isomorphism ϕ provides, thanks to the compatibilities of the sections on the product of various powers of X and Y, a morphism compatible with the (Frobenius) algebra structures. ☐

Although we do not know how to establish Conjecture A.10 in general for K3 surfaces and algebra structures, we can still say something for K3 surfaces with a *Shioda–Inose structure*. Recall that a Shioda–Inose structure on a K3 surface S consists of a Nikulin involution (that is, a symplectic involution) with rational quotient map $\pi : S \dashrightarrow Y$ such that Y is a Kummer surface and π_* induces a Hodge isometry $T_S(2) \simeq T_Y$, where T_S and T_Y denote the transcendental lattices of S and Y. If S admits a Shioda–Inose structure, let $f : A \to Y$ be the quotient morphism from the complex abelian surface whose Kummer surface is Y. By [38, §6], there is a Hodge isometry of transcendental lattices $T_S \simeq T_A$, and $f^*\pi_*$ induces an isomorphism $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{tr}}(A,\mathbb{Q})$ with inverse $\frac{1}{2}f_*\pi^*$.

Proposition A.12. *Let S and S' be two K3 surfaces with a Shioda–Inose structure (*e.g.* with Picard rank ≥ 19, [38, Corollary 6.4]). The following conditions are equivalent.*

(i) $\mathrm{H}^*(S,\mathbb{Q})$ *and* $\mathrm{H}^*(S',\mathbb{Q})$ *are Hodge algebra isomorphic.*
(ii) $\mathfrak{h}(S) \simeq \mathfrak{h}(S')$ *as algebra objects in the category of rational Chow motives.*

Proof. Let S and S' be two K3 surfaces with a Shioda–Inose structure. In a similar vein to Proposition A.11, the proposition is a combination of the validity of the Hodge conjecture for $S \times S'$, together with the fact [20, Proposition 5.12] that S and S' satisfy the distinguished marking conjecture of [20, Conjecture 2]. The fact that the Hodge conjecture holds for $S \times S'$ reduces, *via* the correspondence-induced isomorphism $\mathrm{H}^2_{\mathrm{tr}}(S,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{tr}}(A,\mathbb{Q})$ described above, to the fact that the Hodge conjecture holds for the product of any two abelian surfaces. The latter is proven in [50]. ☐

A.5.2. *Non-isogenous K3 surfaces with Chow motives isomorphic as algebra objects.* By combining Theorem A.1 with the fact [38, Corollary 6.4] that K3 surfaces of maximal Picard rank admit a Shioda–Inose structure, we can establish :

Theorem A.13. *There exists an infinite family of K3 surfaces such that*

- *they are pairwise non-isogenous ;*
- *their Chow motives are pairwise non-isomorphic as Frobenius algebra objects ;*
- *their Chow motives are all isomorphic as algebra objects.*

Proof. Let $\{S_i\}_{i\in\mathbb{Z}_{>0}}$ be a family of pairwise non-isogenous K3 surfaces of maximal Picard rank such that $\mathrm{H}^*(S_j,\mathbb{Q})$ and $\mathrm{H}^*(S_k,\mathbb{Q})$ are Hodge algebra isomorphic for all $j,k \in \mathbb{Z}$. Such a family of K3 surfaces exist thanks to Theorem A.1. By Corollary 2 the Chow motives of these K3 surfaces are pairwise non-isomorphic as Frobenius algebra objects. The fact that the Chow motives of any two surfaces in the family are isomorphic as algebra objects is Proposition A.12. ☐

Finally, the following proposition gives evidence that the notion of "isogeny" for K3 surfaces is strictly more restrictive than the notion of "isomorphic Chow rings" (see Remark 3.2):

Proposition A.14. *Assume that Conjecture A.9 holds for K3 surfaces. Then there exists an infinite family* $\{S_i\}_{i\in\mathbb{Z}_{>0}}$ *of pairwise non-isogenous K3 surfaces with the property that, for all* $j,k\in\mathbb{Z}_{>0}$*, there exists an isomorphism* $\mathfrak{h}(S_j)\xrightarrow{\sim}\mathfrak{h}(S_k)$ *of Chow motives inducing a ring isomorphism* $\mathrm{CH}^*(S_j)\xrightarrow{\sim}\mathrm{CH}^*(S_k)$ *such that the distinguished class* o_{S_j} *is mapped to the distinguished class* o_{S_k}*.*

Moreover, such a family can be chosen to consist of K3 surfaces with transcendental lattice being any prescribed even lattice with square discriminant and of signature $(2,2),(2,4),(2,6)$ *or* $(2,8)$*.*

Proof. We consider the infinite family constructed in Theorem A.2. For any $j\neq k$, S_j and S_k are not isogenous. On the other hand, Conjecture A.9 implies that the Chow motives of S_j and S_k are isomorphic; in particular, by the same weight argument as in §1.2.3, there exists an isomorphism between their transcendental motives:

$$\Gamma_{\mathrm{tr}}:\mathfrak{h}^2_{\mathrm{tr}}(S_j)\simeq\mathfrak{h}^2_{\mathrm{tr}}(S_k).$$

As $\mathrm{NS}(S_j)_{\mathbb{Q}}$ and $\mathrm{NS}(S_k)_{\mathbb{Q}}$ are isometric by construction, there is an isomorphism between the algebraic part of their weight-2 motives $\Gamma^2_{\mathrm{alg}}:\mathfrak{h}^2_{\mathrm{alg}}(S_j)\to\mathfrak{h}^2_{\mathrm{alg}}(S_k)$ which induces the isometry between the Néron–Severi spaces. Combining them together, $\Gamma:=o_{S_j}\times S_k+\Gamma^2_{\mathrm{alg}}+\Gamma_{\mathrm{tr}}+S_j\times o_{S_k}$ yields an isomorphism between their Chow motives:

$$\Gamma:\mathfrak{h}(S_j)=\mathfrak{h}^0(S_j)\oplus\mathfrak{h}^2_{\mathrm{alg}}(S_j)\oplus\mathfrak{h}^2_{\mathrm{tr}}(S_j)\oplus\mathfrak{h}^4(S_j)\xrightarrow{\sim}\mathfrak{h}(S_k)=\mathfrak{h}^0(S_k)\oplus\mathfrak{h}^2_{\mathrm{alg}}(S_k)\oplus\mathfrak{h}^2_{\mathrm{tr}}(S_k)\oplus\mathfrak{h}^4(S_k),$$

with the extra property that it induces an isometry between the $\mathbb{Q}$-quadratic spaces $\mathrm{CH}^1(S_j)$ and $\mathrm{CH}^1(S_k)$. Now as in Remark 3.2, according to the Beauville–Voisin theorem [4], the image of the intersection product $\mathrm{CH}^1(S_j)\otimes\mathrm{CH}^1(S_j)\to\mathrm{CH}^2(S_j)$ is 1-dimensional and similarly for S_k. This implies that Γ induces an isomorphism of Chow rings with $\Gamma_*o_{S_j}=o_{S_k}$. □

Remark A.15. In Proposition A.14, if one assumes Conjecture A.10 for K3 surfaces instead of Conjecture A.9, then the K3 surfaces of the family have isomorphic Chow motives as algebra objects.

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