DISTINGUISHED CYCLES ON VARIETIES WITH MOTIVE OF ABELIAN TYPE AND THE SECTION PROPERTY

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Abstract. A remarkable result of Peter O'Sullivan asserts that the algebra epimorphism from the rational Chow ring of an abelian variety to its rational Chow ring modulo numerical equivalence admits a (canonical) section. Motivated by Beauville's splitting principle, we formulate a conjectural Section Property which predicts that for projective holomorphic symplectic varieties there exists such a section of algebra whose image contains all the Chern classes of the variety. In this paper, we investigate this property for (not necessarily symplectic) varieties with motive of abelian type. We provide a sufficient condition for a smooth projective variety to admit such a section, and give series of examples of varieties for which our theory works. For instance, we prove the existence of such a section for arbitrary products of varieties with Chow groups of finite rank, abelian varieties, hyperelliptic curves, Fermat cubic hypersurfaces, Hilbert schemes of points on an abelian surface or a Kummer surface or a K3 surface with Picard number at least 19, and generalized Kummer varieties. The latter cases provide evidence for the conjectural Section Property and exemplify the mantra that the motives of holomorphic symplectic varieties should behave as the motives of abelian varieties as algebra objects.

Contents

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Introduction

Let X be a smooth projective variety. We denote by $\mathrm{CH}(X)$ its Chow ring with rational coefficients, and by $\overline{\mathrm{CH}}(X)$ the quotient of $\mathrm{CH}(X)$ by numerical equivalence of cycles. The aim of this work is to build upon a recent result of O'Sullivan [35] and give sufficient conditions on a smooth projective variety X for the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ to admit a section that contains the Chern classes of X. This amounts to *lift* numerical cycle classes to classes in Chow groups such that the lifted cycles form a subalgebra and the lifting of the Chern classes are the corresponding Chow-theoretic Chern classes.

0.1. Motivation : the motives of holomorphic symplectic varieties.

It is an insight of Beauville that the motives of smooth projective holomorphic symplectic varieties should behave in a similar way to the motives of abelian varieties as algebra objects. Following the seminal work [9], Beauville [8] (meta-)conjectured that the conjectural Bloch–Beilinson filtration on the Chow ring of holomorphic symplectic varieties should split. This will subsequently be referred to as the *splitting principle*. That the conjectural Bloch–Beilinson filtration on the Chow ring of abelian varieties should split was established by Beauville in [7].

0.1.1. *The conjecture of Beauville.* A first verifiable consequence of this splitting principle for holomorphic symplectic varieties is the following concrete conjecture, called *weak splitting property* ; see [8] for details.

Conjecture (Beauville [8])**.** *Let X be a holomorphic symplectic variety, and denote by $R(X)$ the subalgebra of $\mathrm{CH}(X)$ generated by divisors. Then the composition*

$$R(X) \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$$

is injective.

This conjecture was checked for K3 surfaces in the seminal work of Beauville and Voisin [9], and in [8] Beauville checked it for Hilbert schemes of length-2 and length-3 subschemes on a K3 surface. The conjecture was later strengthened by Voisin [45] who added the Chern classes of X to the set of generators of $R(X)$ (see also [48]). Since then, the strengthened conjecture has been checked to hold in a number of cases; see [45], [17], [49], [38], [18, §10] and [20].

0.1.2. *Multiplicative Chow–Künneth decompositions.* Another verifiable consequence of this splitting principle can be formulated directly on the level of Chow motives. Deninger and Murre [15] constructed a canonical Chow–Künneth decomposition of the motive of an abelian variety, lifting to the motivic level the decomposition of Beauville on the level of the Chow ring [7]. It can be checked that the decomposition of Deninger–Murre is compatible with the algebra structure on the Chow motives of abelian varieties; we say that abelian varieties admit a *multiplicative Chow–Künneth decomposition.* We refer to Section 5 for definitions and properties of (multiplicative) Chow–Künneth decompositions.

Conjecture (Multiplicative Chow–Künneth decomposition [39])**.** *A holomorphic symplectic variety X admits a multiplicative self-dual Chow–Künneth decomposition with the additional property that the Chern classes $c_i(X)$ belong to* $\mathrm{CH}(X)_{(0)}$.[1]

The decomposition of the small diagonal for K3 surfaces of Beauville–Voisin [9] in fact establishes the existence of a multiplicative Chow–Künneth decomposition for K3 surfaces; see [39, Proposition 8.14]. The existence of a multiplicative Chow–Künneth decomposition was established for the Hilbert scheme of length-2 subschemes on a K3 surface in [39], more generally for the Hilbert scheme of length-n subschemes on a K3 surface in [44], and for generalized Kummer varieties in [19].

0.1.3. *O'Sullivan's theorem.* There is yet another verifiable consequence of this splitting principle, which will be our main focus here. The Bloch–Beilinson conjectures (or Murre's conjecture (D) [33]) predict that for any smooth projective variety, the composite $\mathrm{CH}^i(X)_{(0)} \hookrightarrow \mathrm{CH}^i(X) \twoheadrightarrow \overline{\mathrm{CH}}^i(X)$ is an isomorphism of $\mathbb{Q}$-vector spaces for all i. In the case where the conjectural Bloch–Beilinson conjecture splits, $\mathrm{CH}(X)_{(0)}$ is a $\mathbb{Q}$-subalgebra of $\mathrm{CH}(X)$ and we would therefore expect that $\mathrm{CH}(X)_{(0)}$ provides a section to the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$. In the case of abelian varieties, this was conjectured by Beauville [7]. A breakthrough in that direction is the following result due to O'Sullivan, based on the work of Kimura [24] and André–Kahn [4].

Theorem (O'Sullivan [35])**.** *Let A be an abelian variety. Then the $\mathbb{Q}$-algebra epimorphism*

$$\mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$$

admits a section (as $\mathbb{Q}$-algebras), whose image consists of symmetrically distinguished cycles *in the sense of Definition 1.5.*

See Theorems 1.2 and 1.6 for a more precise version of O'Sullivan's theorem. In particular, O'Sullivan's theorem establishes the following version[2] of Beauville's

[1] See (10) for the definition of the grading $\mathrm{CH}(X)_{(*)}$.

[2] This question was asked by Voisin as a more accessible consequence of Beauville's more general conjecture in [7].

conjecture for abelian varieties (see [1] and [31] for alternative proofs): *if A is an abelian variety, then the subalgebra of* $\mathrm{CH}(A)$ *generated by symmetric divisors injects into cohomology via the cycle class map.* In this paper, inspired by the work of O'Sullivan [35] on the Chow rings of abelian varieties, we would like to address the following consequence of Beauville's splitting principle.

Conjecture 1 (Section Property)**.** *Let X be a smooth projective holomorphic symplectic variety. Then the $\mathbb{Q}$-algebra epimorphism*

$$\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$$

admits a section (as $\mathbb{Q}$-algebras) whose image contains the Chern classes of X.

Conjecture 1 implies the Conjecture of Beauville [8], as well as its refinement due to Voisin [45], because $\mathrm{CH}^1(X) \twoheadrightarrow \overline{\mathrm{CH}}^1(X)$ is an isomorphism for smooth projective varieties X with vanishing first Betti number. We prove the following result (Propositions 4.10, 4.11, 4.12 and 4.13) in support of Conjecture 1.

Theorem 1. *Let X be either the Hilbert scheme of length-n subschemes on an abelian surface or a Kummer surface or a K3 surface with Picard number ≥ 19, or a generalized Kummer variety. Then Conjecture 1 holds for X.*

0.2. **Distinguished cycles on varieties with motive of abelian type.** Although our primary motivation for this work was to establish Theorem 1, we were led to consider the following broader question (see Question 2.7): Suppose X is a smooth projective variety whose Chow motive is isomorphic to a direct summand of the motive of an abelian variety (such varieties are said to have *motive of abelian type*). Are there sufficient conditions on X that ensure that the epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ admits a section that is compatible with the intersection product? For that purpose we introduce the notion of *distinguished cycles* on a certain class of varieties with motive of abelian type; see Definition 2.4. Precisely, assuming the existence of a *marking*, *i.e.* an isomorphism $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ of Chow motives, where M is of the form $\oplus_i(A_i, p_i, n_i)$ with A_i an abelian variety, $p_i \in \mathrm{CH}(A_i \times A_i)$ a *symmetrically distinguished* projector and $n_i \in \mathbb{Z}$, the group of distinguished cycles $\mathrm{DCH}_\phi(X)$ consists of the symmetrically distinguished cycles on each A_i, in the sense of O'Sullivan (see Definition 1.5), transported via the induced isomorphism $\phi_* : \mathrm{CH}(X) \xrightarrow{\simeq} \mathrm{CH}(M)$. The question becomes: What are sufficient conditions on the marking ϕ for $\mathrm{DCH}_\phi(X)$ to be a subalgebra of $\mathrm{CH}(X)$? In Proposition 2.11, we show that it suffices that

- ($\star$1) the diagonal Δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 2}}(X^2)$, that is, under the induced isomorphism $\phi_*^{\otimes 2} : \mathrm{CH}(X^2) \xrightarrow{\simeq} \mathrm{CH}(M^{\otimes 2})$, the image of Δ_X is symmetrically distinguished;
- ($\star$2) the small diagonal δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 3}}(X^3)$.

Since it is natural to expect that the Chern classes are distinguished, we will also require that the Chern classes of X are transported to symmetrically distinguished cycles via ϕ, *i.e.* that the marking ϕ also satisfies

- ($\star$3) all Chern classes $c_i(X)$ belong to $\mathrm{DCH}_\phi(X)$.

These conditions are gathered to Condition ($\star$) in Definition 2.8, which includes the more general situation where X is endowed with the action of a finite group G.

Therefore in order to prove Theorem 1 it is enough to exhibit a suitable marking for X such that the Chern classes, the diagonal and the small diagonal are distinguished with respect to the (product) markings. If such a suitable marking for X exists, we will say that X satisfies $(\star)$; see Definition 2.8. This condition is strictly stronger than the condition of having motive of abelian type ; see Section 6 for examples of varieties with motive of abelian type that do not satisfy $(\star)$ and/or are such that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ does not admit a section. Thus that smooth projective holomorphic symplectic varieties should satisfy the Section Property 1 is remarkable. In view of Proposition 2.11, one could also be optimistic and go as far as asking whether smooth projective holomorphic symplectic varieties admit a marking that satisfy $(\star)$; in particular, whether they have motive of abelian type. Some evidence towards the latter is provided by recent work of Kurnosov–Soldatenkov–Verbitsky [28].

Although smooth projective holomorphic symplectic varieties seem to play a central role, we provide many other examples of smooth projective varieties X that satisfy $(\star)$ and hence are such that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ admits a section whose image contains the Chern classes of X. The building blocks (see Section 4) are given by abelian varieties (O'Sullivan's theorem), varieties with Chow groups of finite rank (Proposition 4.2), hyperelliptic curves (Corollary 4.4), cubic Fermat hypersurfaces (Proposition 4.6), K3 surfaces with Picard rank ≥ 19 (Proposition 4.11), and generalized Kummer varieties (Proposition 4.13). One can then construct new examples (see Section 3) of varieties satisfying $(\star)$ by taking products (Proposition 3.1), certain projective bundles and blow-ups (Propositions 3.6 and 3.10, here that the Chern classes are distinguished plays a central role), Hilbert squares and some nested Hilbert schemes (Propositions 3.12 and 3.13), and Hilbert schemes of length-n subschemes of curves or surfaces satisfying $(\star)$ (Remark 4.5 and Proposition 4.12). Combining the above-mentioned results, we obtain

Theorem 2. *Let E be the smallest collection of smooth projective varieties that contains varieties with Chow groups of finite rank (as $\mathbb{Q}$-vector spaces), abelian varieties, hyperelliptic curves, cubic Fermat hypersurfaces, K3 surfaces with Picard rank ≥ 19, and generalized Kummer varieties, and that is stable under the following operations :*

- *(i) if X and Y belong to E, then so is $X \times Y$;*
- *(ii) if X belongs to E, then $\mathbb{P}(\oplus_i \mathbb{S}_{\lambda_i} T_X)$ belongs to E, where T_X is the tangent bundle of X, the λ_i's are non-increasing sequences of integers and $\mathbb{S}_{\lambda_i}$ is the Schur functor associated to λ_i ;*
- *(iii) if X belongs to E, then the Hilbert scheme of length-2 subschemes $X^{[2]}$, as well as the nested Hilbert schemes $X^{[1,2]}$ and $X^{[2,3]}$ belong to E ;*
- *(iv) if X is a curve or a surface that belongs to E, then the Hilbert scheme of length-n subschemes $X^{[n]}$ belongs to E.*

If X is a smooth projective variety that is isomorphic to a variety in E, then the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ admits a section (as $\mathbb{Q}$-algebras) whose image contains the Chern classes of X.

Note that all smooth projective varieties which we can show satisfy $(\star)$ were already shown to admit a self-dual multiplicative Chow–Künneth decomposition ; see [40, Theorem 2], and [19] for the case of generalized Kummer varieties. In fact, condition $(\star)$ implies the existence of a multiplicative Chow–Künneth decomposition

(Proposition 5.1). Note also that the structure of Section 3 is similar to the structure of [40, Section 3]. We refer to Section 5 for more on multiplicative Chow–Künneth decompositions and links to this work. Finally, we note that while the result of Beauville–Voisin [9] shows that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(S) \twoheadrightarrow \overline{\mathrm{CH}}(S)$ admits a section whose image contains the Chern classes of S, for a K3 surface S, and while it can be shown [44] that the Hilbert scheme of length-n subschemes on a K3 surface has a self-dual multiplicative Chow–Künneth decomposition, we do not know how to show in general that a K3 surface satisfies the condition $(\star)$, nor do we know how to show that the Hilbert scheme of length-n subschemes on a K3 surface satisfies the Section Property 1. In fact it is even an open problem to show in general that K3 surfaces have motive of abelian type.

Conventions and Notations. We work throughout the paper over an algebraically closed base field k. Chow groups CH^i are always understood to be with rational coefficients. For a smooth projective variety X, we will write $\mathrm{CH}(X)$ for the (graded) Chow ring $\bigoplus_i \mathrm{CH}^i(X)$. We will denote $\overline{\mathrm{CH}}^i$ the Chow groups modulo numerical equivalence and $\overline{\mathrm{CH}}(X)$ the Chow ring modulo numerical equivalence. An *abelian variety* is always assumed to be connected and with a fixed origin.

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1. Symmetrically distinguished cycles

In this section, we review the theory of symmetrically distinguished cycles developed by O'Sullivan in [35] and, with a view towards applications, extend it slightly following the authors' previous work [19] joint with Zhiyu Tian.

1.1. O'Sullivan's theorem : categorical version.

Let $\mathrm{CHM} := \mathrm{CHM}(k)_{\mathbb{Q}}$ and $\mathrm{NumM} := \mathrm{NumM}(k)_{\mathbb{Q}}$ be respectively the category of rational Chow motives and that of rational numerical motives over the base field k. By definition, there is a natural (full) projection functor :

$$\mathrm{CHM} \to \mathrm{NumM},$$

which sends a Chow motive to the corresponding numerical motive and sends any cycle/correspondence modulo rational equivalence to its class modulo numerical equivalence. See [2] for the basic notions.

Let us introduce some subcategories of CHM and NumM that will be our main interest of study. Let CHM^{ab} (*resp.* NumM^{ab}) be the full, $\mathbb{Q}$-linear, additive, pseudo-abelian, tensor subcategory of CHM (*resp.* NumM) generated by the Tate objects and the motives of abelian varieties. We have the restriction of the projection functor :

$$P : \mathrm{CHM}^{ab} \to \mathrm{NumM}^{ab}.$$

One important feature of these subcategories of abelian motives, proved by Shun-Ichi Kimura [24], is that every object is *finite dimensional* in the following sense.

Definition 1.1. Let $\mathcal{C}$ be a full, pseudo-abelian tensor subcategory of the category of rational motives modulo certain adequate equivalence relation.

(1) An object M of $\mathcal{C}$ is called *even*, if its sufficiently high exterior power vanishes : $\bigwedge^i M = 0$ when $i \gg 0$;

(2) An object M of $\mathcal{C}$ is called *odd*, if its sufficiently high symmetric power vanishes: $\mathrm{Sym}^i M = 0$ when $i \gg 0$;
(3) An object is called *finite dimensional*, if it can be written as the direct sum of an even object and an odd object.

The above conditions make sense for any $\mathbb{Q}$-linear, additive, rigid symmetric monoïdal, pseudo-abelian category: categories whose objects are finite dimensional are called *Kimura categories* and are further studied in [4]. We refer to André's Bourbaki talk [3] for more details.

The main result of [35] can be stated as follows:

Theorem 1.2 (O'Sullivan [35])**.** *The projection functor $P : \mathrm{CHM}^{ab} \to \mathrm{NumM}^{ab}$ has a unique section (*i.e. *a right inverse functor) T satisfying:*

(i) $P \circ T = \mathrm{id}_{\mathrm{NumM}^{ab}}$;
(ii) T is a tensor functor;
(iii) T preserves Tate objects strictly;
(iv) In the following diagram we have $\mathfrak{h} = T \circ \overline{\mathfrak{h}}$:

$$\begin{tikzcd} \mathscr{A}\mathscr{V}^{op} \arrow[r, "\mathfrak{h}"] \arrow[dr, "\overline{\mathfrak{h}}"'] & \mathrm{CHM}^{ab} \arrow[d, "P"', shift right] \\ & \mathrm{NumM}^{ab} \arrow[u, bend right, "T"'] \end{tikzcd}$$

where $\mathscr{A}\mathscr{V}$ is the category whose objects are abelian varieties and morphisms are homomorphisms between abelian varieties, $\mathfrak{h}$ and $\overline{\mathfrak{h}}$ are the contra-variant functors which associate to an abelian variety its motive.

Remark 1.3. If one only requires $(i) - (iii)$, the existence of a tensor section is ensured by a general result of André–Kahn [4] concerning the so-called Wedderburn categories, and such section is unique only up to a tensor conjugacy. The condition (iv) together with the Hopf algebra structure on the motive of an abelian variety allows O'Sullivan to single out the canonical section T. Let us also mention that the image of the functor T consists of as objects as in the subcategory CHM^{ab}_{sd} of Definition 2.1 and as morphisms only *symmetrically distinguished morphisms* between them.

Definition 1.4 (Symmetrically distinguished morphisms)**.** For any two objects $M, N \in \mathrm{CHM}^{ab}$, the functor T in Theorem 1.2 gives a splitting of the surjective morphism

$$P_{M,N} : \mathrm{Hom}_{\mathrm{CHM}^{ab}}(M, N) \twoheadrightarrow \mathrm{Hom}_{\mathrm{NumM}^{ab}}(M, N).$$

We call the image of $T_{M,N}$, which is a sub-vector space of $\mathrm{Hom}_{\mathrm{CHM}^{ab}}(M, N)$, the space of *symmetrically distinguished* morphisms from M to N. Note that by definition, this subspace is mapped isomorphically to $\mathrm{Hom}_{\mathrm{NumM}^{ab}}(M, N)$.

In particular, for any object $M \in \mathrm{CHM}^{ab}$ and any integer i, we have a canonical subgroup of $\mathrm{CH}^i(M) = \mathrm{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), M)$, denoted by $\mathrm{DCH}^i(M)$, consisting of *symmetrically distinguished cycles*, such that $\mathrm{DCH}^i(M) \to \overline{\mathrm{CH}}^i(M)$ is an isomorphism. (Beware that our notation conflicts with the notation of [8], where $\mathrm{DCH}^*(X)$ stands for the sub-algebra generated by divisors.)

1.2. **On abelian varieties.** O'Sullivan defines the following, more concrete, notion of symmetrically distinguished cycles on an abelian variety A, and shows (Theorem 1.6) not only that it agrees with the notion of symmetrically distinguished cycles of Definition 1.4, but also that these provide a section to $\mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$ that is compatible with the intersection product.

Definition 1.5 (Symmetrically distinguished cycles on abelian varieties [35])**.** Let A be an abelian variety and $\alpha \in \mathrm{CH}(A)$. For each integer $m \geq 0$, denote by $V_m(\alpha)$ the $\mathbb{Q}$-vector subspace of $\mathrm{CH}(A^m)$ generated by elements of the form

$$p_*(\alpha^{r_1} \times \alpha^{r_2} \times \cdots \times \alpha^{r_n}),$$

where $n \leq m$, $r_j \geq 0$ are integers, and $p : A^n \to A^m$ is a closed immersion with each component $A^n \to A$ being either a projection or the composite of a projection with $[-1] : A \to A$. Then α is *symmetrically distinguished* if for every m the restriction of the projection $\mathrm{CH}(A^m) \to \overline{\mathrm{CH}}(A^m)$ to $V_m(\alpha)$ is injective.

Theorem 1.6 (O'Sullivan [35])**.** *Let A be an abelian variety. Then* $\mathrm{DCH}(A)$*, the symmetrically distinguished cycles in* $\mathrm{CH}(A)$*, form a graded sub-*$\mathbb{Q}$*-algebra that contains symmetric divisors and that is stable under pull-backs and push-forwards along homomorphisms of abelian varieties. Moreover the composition*

$$\mathrm{DCH}(A) \hookrightarrow \mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$$

is an isomorphism of $\mathbb{Q}$*-algebras.*

Remark 1.7. For an abelian variety A, O'Sullivan also shows that $\mathrm{DCH}(A)$ is a subalgebra of $\mathrm{CH}(A)_{(0)}$, where $\mathrm{CH}(A)_{(*)}$ refers to Beauville's decomposition [7][3]. Moreover, the inclusion $\mathrm{DCH}^i(A) \subseteq \mathrm{CH}^i(A)_{(0)}$ is an equality for $i \leq 1$ as well as for $i \geq \dim A - 1$ by the Fourier transform [5]. Beauville's conjectures on abelian varieties in [7] would imply that the subalgebra $\mathrm{DCH}(A)$ is equal to the direct summand $\mathrm{CH}(A)_{(0)}$. In this sense, O'Sullivan's work [35] can be viewed as a step towards Beauville's conjectures.

Warning 1.8. Unlike the usual convention, the subcategories CHM^{ab} and NumM^{ab} in this paper are not *strictly* full subcategories of CHM and NumM, *i.e.* they are not closed under isomorphisms. The point is that Theorem 1.2 cannot be true once one includes in the category for example the abelian torsors. Indeed, let B be a torsor under an abelian variety A of dimension g. Obviously A and B have isomorphic Chow motives. If a canonical section T were constructed for morphisms between $\mathbb{1}(-g)$ and $\mathfrak{h}(B)$, then we would have a canonical 1-dimensional subspace $\mathrm{DCH}_0(B)$ inside $\mathrm{CH}_0(B)$, hence a canonical degree-one 0-cycle of B. However, as the origin of B is not fixed, there is no privileged point or non-trivial 0-cycle. We will study the closures by isomorphisms of CHM^{ab} and NumM^{ab} in §2, see Definition 2.1.

1.3. **On abelian torsors with torsion structures.** For later use, we give here a minor extension of O'Sullivan's theory. The main idea appeared in our previous work [19] : to treat the Chow motives of some algebraic varieties like Hilbert schemes of abelian surfaces and generalized Kummer varieties, it is inevitable to deal with 'disconnected abelian varieties' where there is no natural choice for the origins on the components, whence the notion of symmetrically distinguished cycles *a priori*

[3]Beauville's decomposition coincides with the decomposition induced, as in (10), by the Chow–Künneth decomposition of Deninger–Murre [15].

fails. However, a simple but crucial observation made in [19] is that we have a canonical notion of torsion points on these components.

Definition 1.9 (Abelian torsors with torsion structure [19])**.** An *abelian torsor with torsion structure*, or an *a.t.t.s* for short, is a pair (X, Q_X) where X is a connected smooth projective variety and Q_X is a subset of X such that there exists an isomorphism, as algebraic varieties, $f : X \xrightarrow{\simeq} A$ from X to an abelian variety A which induces a bijection between Q_X and $\mathrm{Tor}(A)$, the set of all torsion points of A. A choice of such isomorphism f is called a *marking*. A morphism of a.t.t.s's $(X, Q_X) \to (Y, Q_Y)$ consists of a morphism of algebraic varieties $f : X \to Y$ such that $f(Q_X) \subseteq Q_Y$.

This notion of *a.t.t.s* is in between of the notion of abelian variety (with fixed origin) and that of abelian torsor (without origin).

Definition 1.10 (Symmetrically distinguished cycles on a.t.t.s's)**.** Given an a.t.t.s (X, Q_X), an algebraic cycle $\gamma \in \mathrm{CH}(X)$ is called *symmetrically distinguished*, if for a marking $f : X \xrightarrow{\simeq} A$, the cycle $f_*(\gamma) \in \mathrm{CH}(A)$ is symmetrically distinguished in the sense of O'Sullivan (Definition 1.5). By [19, Lemma 6.7], this definition is independent of the choice of marking. An algebraic cycle on a disjoint union of a.t.t.s's is symmetrically distinguished if it is so on each component. We denote by $\mathrm{DCH}(X)$ the subspace of symmetrically distinguished cycles.

We have the following generalization of Theorem 1.6 ; see [19, Proposition 6.9]. Its proof uses the fact that torsion points on an abelian variety are all rationally equivalent (with $\mathbb{Q}$-coefficients).

Theorem 1.11. *Let (X, Q_X) be an a.t.t.s. Then the symmetrically distinguished cycles in* $\mathrm{CH}(X)$ *form a graded sub-*$\mathbb{Q}$*-algebra that is stable under pull-backs and push-forwards along morphisms of a.t.t.s's. Moreover the composition* $\mathrm{DCH}(X) \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ *is an isomorphism of* $\mathbb{Q}$*-algebras.*

We refer to [19, §6.2] for more properties of symmetrically distinguished cycles on a.t.t.s's. Here we want to extend, in a canonical way, O'Sullivan's section T to the larger category CHM^{atts}, which is by definition the full, $\mathbb{Q}$-linear, additive, pseudo-abelian, tensor subcategory of CHM generated by the Tate objects and the motives of abelian torsors with torsion structures. Similarly for NumM^{atts}. Obviously, the inclusion functor $\mathrm{CHM}^{ab} \subset \mathrm{CHM}^{atts}$ is an equivalence of categories, but it does not imply that the following result is trivial (*cf.* Warning 1.8).

Proposition 1.12. *The projection functor $P : \mathrm{CHM}^{atts} \to \mathrm{NumM}^{atts}$ has a unique section (*i.e. *a right inverse functor) T, extending O'Sullivan's section from* NumM^{ab} *to* CHM^{ab} *in Theorem 1.2, satisfying :*

(i) $P \circ T = \mathrm{id}$ *;*
(ii) T is a tensor functor ;
(iii) T preserves Tate objects strictly ;
(iv) In the following diagram we have $\mathfrak{h} = T \circ \bar{\mathfrak{h}}$:

$$\begin{tikzcd} \mathscr{A}tts^{op} \arrow[r, "\mathfrak{h}"] \arrow[dr, "\bar{\mathfrak{h}}"'] & \mathrm{CHM}^{atts} \arrow[d, shift right, "P"'] \\ & \mathrm{NumM}^{atts} \arrow[u, shift right, bend right, "T"'] \end{tikzcd}$$

where $\mathscr{A}tts$ is the category of a.t.t.s's as in Definition 1.9.

Moreover, for cycles on a.t.t.s's, T gives back the notion of symmetrically distinguished cycles in Definition 1.10.

Proof. Using additivity, we only need to define the subspace of symmetrically distinguished morphisms from (X, p, m) to (Y, q, n), where X, Y are a.t.t.s's, $p \in \mathrm{CH}^{\dim X}(X^2)$, $q \in \mathrm{CH}^{\dim Y}(Y^2)$ are projectors and $m, n \in \mathbb{Z}$. Then we take $q \circ \mathrm{DCH}^{\dim X}(X \times Y) \circ p$ as the subspace of symmetrically distinguished morphisms in $\mathrm{Hom}_{\mathrm{CHM}}((X, p, m), (Y, q, n)) := q \circ \mathrm{CH}^{\dim X}(X \times Y) \circ p$. It is straightforward to check that this defines a section functor T satisfying all the desired properties. □

2. Distinguished cycles

In this section, we use the theory of symmetric distinguishedness recalled in §1 to study the cycles of algebraic varieties whose Chow motive are of abelian type (but not necessarily in CHM^{ab} or CHM^{atts}, see Warning 1.8).

2.1. **Some categories of motives of abelian type.** To make the words 'of abelian type' precise, let us introduce several subcategories of CHM closely related to CHM^{ab}:

(1)
$$\begin{array}{ccc} \mathrm{CHM}^{ab}_{sd} & \subset & \mathrm{CHM}^{ab} \\ \bigcap & & \bigcap \\ \mathrm{CHM}^{atts}_{sd} & \subset & \mathrm{CHM}^{atts} \\ \bigcap & & \bigcap \\ \mathscr{M} & \subset & \mathscr{M}' \end{array}$$

Definition 2.1. In the above,

(i) CHM^{ab}_{sd} is the full subcategory of CHM^{ab} consisting of objects of the form $\oplus_i(A_i, p_i, n_i)$ with A_i an abelian variety, $p_i \in \mathrm{DCH}(A_i \times A_i)$ a *symmetrically distinguished* projector and $n_i \in \mathbb{Z}$.

(ii) In the same fashion, let CHM^{atts}_{sd} be the full subcategory of CHM^{atts} (see §1.3) consisting of objects of the form $\oplus_i(X_i, p_i, n_i)$ with X_i an abelian torsor with torsion structure, $p_i \in \mathrm{DCH}(X_i \times X_i)$ a *symmetrically distinguished* projector and $n_i \in \mathbb{Z}$.

(iii) Let $\mathscr{M}$ (*resp.* $\mathscr{M}'$) be the strictly full subcategory of CHM consisting of objects isomorphic to some object in CHM^{ab}_{sd} (*resp.* in CHM^{ab}).

The relation between these subcategories is summarized in the Diagram (1), where all vertical inclusions are equivalences of categories. Note that when we apply the natural functor $\mathrm{CHM} \to \mathrm{NumM}$ on the subcategories in (1), all horizontal inclusions become equalities and we get inclusions of categories $\mathrm{NumM}^{ab} \subset \mathrm{NumM}^{atts} \subset \overline{\mathscr{M}'}$, where $\overline{\mathscr{M}'}$ is the strictly full subcategory of NumM consisting of objects isomorphic to some object in NumM^{ab}.

We will call objects in $\mathscr{M}'$ and $\overline{\mathscr{M}'}$ *motives of abelian type* (while objects in CHM^{ab} and NumM^{ab} will be called *abelian motives*).

Remark 2.2. As is pointed out in Warning 1.8, $\mathscr{M}'$ and CHM^{ab} are usually confused in the literature, but it is crucial in this paper to separate them in order to have the section functor T. Although objects in CHM^{ab} have rather rigid shapes (they are of the form $\oplus(A_i, p_i, n_i)$ with A_i an abelian variety, p_i a projector and n_i an integer), the larger category $\mathscr{M}'$ contains a lot more interesting motives arising from geometry. Here are some examples :

Example 2.3. The Chow (*resp.* numerical) motives of the following algebraic varieties belong to the category $\mathscr{M}$ (*resp.* $\overline{\mathscr{M}}$), hence to $\mathscr{M}'$ (*resp.* $\overline{\mathscr{M}'}$), but not necessarily to CHM^{ab} (*resp.* NumM^{ab}) :

(i) projective spaces, Grassmannian varieties and more generally homogeneous varieties and toric varieties ;
(ii) smooth projective curves ;
(iii) Kummer K3 surfaces ; K3 surfaces with Picard numbers at least 19 as well as their Hilbert schemes ;
(iv) abelian torsors ;
(v) Hilbert schemes of abelian surfaces ;
(vi) generalized Kummer varieties ;
(vii) Fermat hypersurfaces ;
(viii) projective bundles over and products of the examples above.

As far as the authors know, all examples of motives that are proved to be (Kimura) finite dimensional for now, belong[4] to the category $\mathscr{M}'$.

Now we can say precisely that the philosophy of the paper is to study the Chow rings of algebraic varieties with motive lying in $\mathscr{M}'$, via the section $T : \mathrm{NumM}^{ab} \to \mathrm{CHM}^{ab}$ of O'Sullivan in Theorem 1.2 as well as its extension in Proposition 1.12.

2.2. **Distinguished cycles : the basics.** Here comes the key notion of the paper :

Definition 2.4 (Distinguished cycles)**.** Let X be smooth projective variety such that its Chow motive $\mathfrak{h}(X) \in \mathscr{M}$.

(1) A *marking* for X is an isomorphism $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$, in CHM, with target $M \in \mathrm{CHM}^{ab}_{sd}$, *i.e.* M is of the form $\oplus_i(A_i, p_i, n_i)$ with A_i being an abelian variety, $p_i \in \mathrm{DCH}(A_i \times A_i)$ a projector and $n_i \in \mathbb{Z}$.
(2) Slightly more generally, A *marking* for X is an isomorphism $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$, in CHM, with target $M \in \mathrm{CHM}^{atts}_{sd}$, *i.e.* M is of the form $\oplus_i(A_i, p_i, n_i)$ with A_i being an abelian torsor with torsion structures, $p_i \in \mathrm{DCH}(A_i \times A_i)$ a projector and $n_i \in \mathbb{Z}$.
(3) Given a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ (thus $M \in \mathrm{CHM}^{ab}_{sd}$ or even CHM^{atts}_{sd}), we define the subgroup of *distinguished cycles* of X, denoted by $\mathrm{DCH}_\phi(X)$, or sometimes $\mathrm{DCH}(X)$ if ϕ is clear from the context, to be the pre-image of $\mathrm{DCH}(M)$ (see Definition 1.4) via the induced isomorphism $\phi_* : \mathrm{CH}(X) \xrightarrow{\simeq} \mathrm{CH}(M)$.

Almost by construction, we have :

Lemma 2.5. *For any smooth projective variety X such that $\mathfrak{h}(X) \in \mathscr{M}$ and any marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ (with $M \in \mathrm{CHM}^{ab}_{sd}$), the composition*

$$\mathrm{DCH}_\phi(X) \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$$

is an isomorphism. In other words, ϕ provides a section (as graded vector spaces) of the natural projection $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$.

[4]Of course, there are certainly many varieties whose motive is not in $\mathscr{M}'$, while conjecturally all varieties have finite dimensional motive.

Proof. In the commutative diagram

$$\begin{tikzcd} \mathrm{DCH}_\phi(X) \arrow[r, hook] \arrow[d, "\simeq"', "\phi_*"] & \mathrm{CH}(X) \arrow[r, two heads] \arrow[d, "\simeq"', "\phi_*"] & \overline{\mathrm{CH}}(X) \arrow[d, "\simeq"', "\overline{\phi}_*"] \\ \mathrm{DCH}_\phi(M) \arrow[r, hook] & \mathrm{CH}(M) \arrow[r, two heads] & \overline{\mathrm{CH}}(M) \end{tikzcd}$$

the composition of the bottom line is an isomorphism since $P \circ T = \mathrm{id}$. Therefore the composition of the top line is also an isomorphism, hence $\mathrm{DCH}_\phi(X)$ gives a section as graded vector spaces. □

Distinguished cycles behave well with respect to tensor/exterior products :

Proposition 2.6 (Tensor products)**.** *Let X, Y be two smooth projective varieties.*

(a) If $\mathfrak{h}(X) \in \mathscr{M}$ and $\mathfrak{h}(Y) \in \mathscr{M}$, then $\mathfrak{h}(X \times Y) \in \mathscr{M}$.
Assuming this,

(b) Given markings $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} N$ with $M, N \in \mathrm{CHM}^{ab}_{sd}$, the exterior product $\mathrm{CH}(X) \times \mathrm{CH}(Y) \xrightarrow{\otimes} \mathrm{CH}(X \times Y)$ respects distinguished cycles :

$$\mathrm{DCH}_\phi(X) \times \mathrm{DCH}_\psi(Y) \xrightarrow{\otimes} \mathrm{DCH}_{\phi\otimes\psi}(X \times Y).$$

Proof. For *(a)*, it is enough to observe that CHM^{ab}_{sd}, hence also $\mathscr{M}$, is closed under tensor products. Indeed, let A, B be two abelian varieties, $p \in \mathrm{DCH}_{\dim A}(A^2)$, $q \in \mathrm{DCH}_{\dim B}(B^2)$ be symmetrically distinguished projectors and $m, n \in \mathbb{Z}$. Then $(A, p, m) \otimes (B, q, n) = (A \times B, p \otimes q, m + n)$, where $p \otimes q$ is the exterior product, which is symmetrically distinguished by Theorem 1.6.
For *(b)*, let $\alpha \in \mathrm{DCH}^i_\phi(X)$ and $\beta \in \mathrm{DCH}^j_\psi(Y)$. We consider them as morphisms $\alpha : \mathbb{1}(-i) \to \mathfrak{h}(X)$ and $\beta : \mathbb{1}(-j) \to \mathfrak{h}(Y)$, then $\phi \circ \alpha : \mathbb{1}(-i) \to M$ and $\psi \circ \beta : \mathbb{1}(-j) \to N$ are symmetrically distinguished morphisms in CHM^{ab}. Since T is a tensor functor, $(\phi \otimes \psi) \circ (\alpha \otimes \beta) = (\phi \circ \alpha) \otimes (\psi \circ \beta) : \mathbb{1}(-i - j) \to M \otimes N$ is also symmetrically distinguished, which means exactly that $\alpha \otimes \beta$ is in $\mathrm{DCH}_{\phi\otimes\psi}(X \times Y)$. □

2.3. **The main questions and the key condition** ($\star$)**.**

Question 2.7. Here are the most important properties of the distinguished cycles that we are going to investigate :

- When does $\mathrm{DCH}_\phi(X)$ form a $\mathbb{Q}$-sub-algebra of $\mathrm{CH}(X)$?
- When do the Chern classes of X belong to $\mathrm{DCH}_\phi(X)$?

To this end, let us introduce the following condition for smooth projective varieties whose Chow motive is of abelian type :

Definition 2.8. We say that a smooth projective variety X with $\mathfrak{h}(X) \in \mathscr{M}$ satisfies the condition ($\star$) if :

($\star$) There exists a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ with $M \in \mathrm{CHM}^{ab}_{sd}$ such that

- ($\star$1) **(Auto-duality)** the diagonal Δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 2}}(X^2)$, that is, under the induced isomorphism $\phi_*^{\otimes 2} : \mathrm{CH}(X^2) \xrightarrow{\simeq} \mathrm{CH}(M^{\otimes 2})$, the image of Δ_X is symmetrically distinguished, *i.e.* in $\mathrm{DCH}(M^{\otimes 2})$.
- ($\star$2) **(Multiplicativity)** the small diagonal δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 3}}(X^3)$;
- ($\star$3) **(Chern classes)** all Chern classes $c_i(X)$ belong to $\mathrm{DCH}_\phi(X)$;

More generally, if X is a smooth projective variety equipped with the action of a finite group G, we say that (X,G) satisfies $(\star)$ if there exists a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ with $M \in \mathrm{CHM}^{ab}_{sd}$ that satisfies, in addition to $(\star 1)$, $(\star 2)$ and $(\star 3)$ above:

- $(\star_G)$ **(G-invariance)** the graph g_X of $g : X \to X$ belongs to $\mathrm{DCH}_{\phi^{\otimes 2}}(X^2)$ for any $g \in G$.

Remark 2.9 (Fundamental class)**.** The condition $(\star 3)$ above says in particular that the fundamental class $\mathbb{1}_X$, which is $c_0(X)$, is distinguished. However, we want to point out that $\mathbb{1}_X$ is distinguished for any choice of marking. Indeed, we can assume that X is connected, thus $\mathrm{CH}^0(X) = \mathbb{Q} \cdot \mathbb{1}_X$, and Lemma 2.5 ensures that $\mathbb{1}_X$ is distinguished.

The following interpretations of the conditions in Definition 2.8 will be useful:

Lemma 2.10 (Equivalent formulations of $(\star 1)$ and $(\star 2)$)**.** *Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ be as above and d_X be the dimension of X.*

(a) The condition $(\star 1)$ is equivalent to say that the isomorphism σ, given by the commutativity of the following diagram, is symmetrically distinguished in the sense of Definition 1.4, where the top morphism is the Poincaré duality in CHM *(induced by Δ_X).*

$$\begin{tikzcd} \mathfrak{h}(X)(d_X) \arrow[r, "PD_X", "\simeq"'] \arrow[d, "\phi(d_X)", "\simeq"'] & \mathfrak{h}(X)^\vee \\ M(d_X) \arrow[r, "\simeq", "\sigma"'] & M^\vee \arrow[u, "\phi^\vee"', "\simeq"] \end{tikzcd}$$

(b) In the presence of condition $(\star 1)$, the condition $(\star 2)$ is equivalent to say that the morphism μ, determined by the commutativity of the following diagram, is a symmetrically distinguished morphism, where the top morphism is the intersection product in CHM *induced by the small diagonal.*

$$\begin{tikzcd} \mathfrak{h}(X)^{\otimes 2} \arrow[r, "\delta_X"] \arrow[d, "\phi^{\otimes 2}"', "\simeq"] & \mathfrak{h}(X) \arrow[d, "\simeq"', "\phi"] \\ M^{\otimes 2} \arrow[r, "\mu"'] & M \end{tikzcd}$$

Proof. *(a)* is tautological.
(b) According to *(a)*, the condition $(\star 1)$ implies that the following isomorphism, induced by composing with $\sigma^{\otimes 2} \otimes \mathrm{id}_M$, preserves the symmetrically distinguished elements:

$$\mathrm{CH}^{2d_X}(M^{\otimes 3}) = \mathrm{Hom}(\mathbb{1}, M(d_X)^{\otimes 2} \otimes M) \xrightarrow{\simeq} \mathrm{Hom}(\mathbb{1}, (M^\vee)^{\otimes 2} \otimes M) = \mathrm{Hom}(M^{\otimes 2}, M).$$

One can conclude by observing that this isomorphism sends $\phi_*^{\otimes 3}(\delta_X)$ to μ. $\square$

The motivation to study the condition $(\star)$ is the following:

Proposition 2.11 (Sub-algebra)**.** *Let X be a smooth projective variety with $\mathfrak{h}(X) \in \mathscr{M}$. If X satisfies the condition $(\star)$, then there is a section, as* graded algebras, *for the natural surjective morphism $\mathrm{CH}(X) \to \overline{\mathrm{CH}}(X)$ such that all Chern classes of X are in the image of this section.*

In other words, under ($\star$), we have a graded $\mathbb{Q}$*-sub-algebra* $\mathrm{DCH}(X)$ *of the Chow ring, which contains all the Chern classes of* X *and is mapped isomorphically to* $\overline{\mathrm{CH}}(X)$. *We call elements of* $\mathrm{DCH}(X)$ distinguished cycles *of* X.

Proof. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ be a marking, thus $M \in \mathrm{CHM}^{ab}_{sd}$.
If ϕ satisfies ($\star$), then we define $\mathrm{DCH}(X) := \mathrm{DCH}_\phi(X)$ as in Definition 2.4 (3), which is a section of graded vector spaces by Lemma 2.5. To show that it is a section of algebras, one has to show that $\mathrm{DCH}_\phi(X)$ is closed under the intersection product of X (the unit $\mathbb{1}_X$ is automatically distinguished by Remark 2.9). Let $\alpha \in \mathrm{DCH}^i_\phi(X)$ and $\beta \in \mathrm{DCH}^j_\phi(X)$. Then by definition the morphisms $\phi\circ\alpha : \mathbb{1}(-i) \to M$ and $\phi\circ\beta : \mathbb{1}(-j) \to M$ are symmetrically distinguished morphisms in CHM^{ab}. Since T is a tensor functor, $(\phi^{\otimes 2}) \circ (\alpha \otimes \beta) = (\phi \circ \alpha) \otimes (\phi \circ \beta) : \mathbb{1}(-i-j) \to M^{\otimes 2}$ is also symmetrically distinguished.

$$\begin{tikzcd}
\mathbb{1}(-i-j) \arrow[r, "\alpha\otimes\beta"] \arrow[dr] & \mathfrak{h}(X)^{\otimes 2} \arrow[r, "\delta_X"] \arrow[d, "\phi^{\otimes 2}", "\simeq"'] & \mathfrak{h}(X) \arrow[d, "\phi", "\simeq"'] \\
 & M^{\otimes 2} \arrow[r, "\mu"] & M
\end{tikzcd}$$

Condition ($\star$) implies that μ, which is determined by the above commutative diagram, is a symmetrically distinguished morphism. Therefore, the total composition $\phi \circ \delta_X \circ (\alpha \otimes \beta)$ in the above diagram is symmetrically distinguished, which means that $\alpha \cdot \beta = \delta_{X,*}(\alpha \otimes \beta)$ is in $\mathrm{DCH}_\phi(X)$. The assertion concerning Chern classes is tautological. $\square$

We deduce that the conditions ($\star 1$) and ($\star 2$) together actually already imply all the analogous statements for all sorts of diagonals on higher powers (note the analogy with [39, Prop. 8.7.(iii)] in the context of self-dual multiplicative Chow–Künneth decompositions):

Corollary 2.12. *Let* X *be a smooth projective variety with* $\mathfrak{h}(X) \in \mathscr{M}$. *If* X *satisfies the condition ($\star$), then all partial diagonals*[5] *in a self-product of* X *are distinguished.*

Proof. Let us fix a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ (with $M \in \mathrm{CHM}^{ab}_{sd}$) satisfying the conditions ($\star 1$) and ($\star 2$) and write DCH for $\mathrm{DCH}_{\phi^{\otimes ?}}$. Observe that any partial diagonal can be written as the intersection product of several big diagonals[6]. By Proposition 2.11, we only have to show that any big diagonal of a self-product is distinguished. However, a big diagonal is a exterior product of distinguished classes $\Delta_X \in \mathrm{DCH}(X \times X)$ (by ($\star 1$)), $\mathbb{1}_X \in \mathrm{DCH}(X)$ (see Remark 2.9). Therefore itself must be so, thanks to Proposition 2.6. $\square$

3. Operations preserving the condition ($\star$)

In this section, we provide some standard operations on varieties that preserve ($\star$).

[5] A *partial diagonal* of a self-product X^n is a subvariety of the form $\{(x_1, \cdots, x_n) \in X^n \mid x_i = x_j \text{ for all } i \sim j\}$ for an equivalence relation $\sim$ on $\{1, \cdots, n\}$.

[6] A *big diagonal* of a self-product X^n is a subvariety of the form $\{(x_1, \cdots, x_n) \in X^n \mid x_i = x_j\}$ for some $1 \leq i \neq j \leq n$.

3.1. **Product varieties.** Given two smooth projective varieties X and Y with markings $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} N$ with $M, N \in \mathrm{CHM}^{ab}_{sd}$, their product will always be understood to be endowed with the marking

$$\phi \otimes \psi : \mathfrak{h}(X \times_k Y) \cong \mathfrak{h}(X) \otimes \mathfrak{h}(Y) \xrightarrow{\simeq} M \otimes N,$$

which we will refer to as the *product marking.* If X and Y are endowed with the action of a finite group G, then $X \times Y$ is endowed with the natural diagonal action of G. Our condition $(\star)$ (see Definition 2.8) behaves well with respect to products:

Proposition 3.1 (Products)**.** *Assume X and Y are two smooth projective varieties satisfying the condition $(\star)$. Then the natural marking on the product $X \times Y$ satisfies $(\star)$ and has the additional property that the graphs of the two natural projections are distinguished.*

If in addition X and Y are equipped with the action of a finite group G and the respective markings satisfy $(\star_G)$, then the product marking on $X \times Y$ satisfies $(\star_G)$.

Proof. By assumption, there are markings $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} N$ with $M, N \in \mathrm{CHM}^{ab}_{sd}$. The assertion $(\star 1)$ (*resp.* $(\star_G)$, *resp.* $(\star 2)$) follows from Proposition 2.6 applied to X and Y replaced by X^2 and Y^2 (*resp.* X^2 and Y^2, *resp.* X^3 and Y^3). Indeed, $\Delta_{X\times Y} = \Delta_X \otimes \Delta_Y$ (*resp.* $g_{X\times Y} = g_X \otimes g_Y$, *resp.* $\delta_{X\times Y} = \delta_X \otimes \delta_Y$).
The assertion $(\star 3)$ concerning the Chern classes follows directly from the formula

$$c_i(X \times Y) = \sum_{j=0}^{i} c_j(X) \otimes c_{i-j}(Y)$$

and Proposition 2.6.
Finally, as the diagonal $\Delta_X \in \mathrm{CH}(X \times X)$ and fundamental class $\mathbb{1}_Y$ of Y are by assumption distinguished (Remark 2.9), Proposition 2.6 tells us that the graph of the projection $X \times Y \to X$, which is equal to $\Delta_X \otimes \mathbb{1}_Y \in \mathrm{CH}(X \times X \times Y)$, is distinguished. The proof is similar for the other projection $X \times Y \to Y$. $\square$

Remark 3.2. Suppose X has a marking that satisfies $(\star)$. Then any permutation of the factors of X^n defines a distinguished correspondence in $\mathrm{DCH}(X^{2n})$ for the product marking by Corollary 2.12.

Remark 3.3. Assume X and Y are two smooth projective varieties endowed with the action of the finite groups G and H, respectively. The product $G \times H$ acts naturally on the product $X \times Y$. Suppose X and Y satisfy $(\star_G)$ and $(\star_H)$, respectively. Then the same arguments as above show that product marking on $X \times Y$ satisfies $(\star_{G\times H})$.

3.2. **Projective bundles.** We show in this subsection that the condition $(\star)$ is stable by forming projective bundles as long as the Chern classes of the vector bundle are distinguished.

Let X be a smooth projective variety of dimension d and E be a vector bundle over X of rank $r+1$. Let $\pi : \mathbb{P}(E) \to X$ be the associated projective bundle[7]. Let ξ be the first Chern class of $\mathcal{O}_\pi(1)$.

[7] The $\mathbb{P}$ we are using here is the space of 1-dimensional subspaces, thus different from Grothendieck's convention.

Recall the *projective bundle formula* (see [2, §4.3.2]) :

$$b : \bigoplus_{k=0}^{r} \mathfrak{h}(X)(-k) \xrightarrow{\simeq} \mathfrak{h}(\mathbb{P}E), \tag{2}$$

which is given by $\xi^k \cdot \pi^* : \mathfrak{h}(X)(-k) \to \mathfrak{h}(\mathbb{P}E)$ for any $0 \le k \le r$.

The following two lemmas[8] compute the diagonal and the small diagonal for $\mathbb{P}E$. A piece of notation is convenient : for an element $\omega \in \mathrm{CH}^k(X)$, viewed as a morphism $\mathbb{1} \to \mathfrak{h}(X)(k)$, we will talk about the morphism *multiplication by* ω, denoted by $\cdot\omega : \mathfrak{h}(X) \to \mathfrak{h}(X)(k)$, which is by definition the following composition :

$$\mathfrak{h}(X) \xrightarrow{\mathrm{id} \otimes \omega} \mathfrak{h}(X) \otimes \mathfrak{h}(X)(k) \xrightarrow{\delta_X(k)} \mathfrak{h}(X)(k).$$

Obviously, with a marking fixed, if $\omega \in \mathrm{DCH}(X)$ and X satisfies $(\star)$, then multiplication by ω is a distinguished morphism too.

Lemma 3.4 (Diagonal of projective bundles). *Notation is as above. Via the projective bundle formula (2), the Poincaré duality $PD_{\mathbb{P}E} : \mathfrak{h}(\mathbb{P}E)(d+r) \xrightarrow{\simeq} \mathfrak{h}(\mathbb{P}E)^\vee$ induces an isomorphism*

$$b^\vee \circ PD_{\mathbb{P}E} \circ b(d+r) : \bigoplus_{k=0}^{r} \mathfrak{h}(X)(d+r-k) \xrightarrow{\simeq} \bigoplus_{l=0}^{r} \mathfrak{h}(X)^\vee(l)$$

where for any $0 \le k, l \le r$, the morphism $\mathfrak{h}(X)(d+r-k) \to \mathfrak{h}(X)^\vee(l)$ is described as :

- *if $l + k \ge r$, it is the composition :*

  $$\mathfrak{h}(X)(d+r-k) \xrightarrow{\cdot s_{k+l-r}(E)} \mathfrak{h}(X)(d+l) \xrightarrow{PD_X(l)} \mathfrak{h}(X)^\vee(l),$$

  where the first morphism is multiplication by the Segre class[9].
- *If $k + l < r$, it is the zero map.*

Proof. By Manin's identity principle ([2, §4.3.1]), it suffices to prove the lemma for Chow groups. Note that the Poincaré dualities induce simply the identity morphisms on Chow groups. Thus we have to compute the composition ${}^t b \circ b$ whose (k, l)-component for any $0 \le k, l \le r$ is

$$\mathrm{CH}^{*+d+r-k}(X) \xrightarrow{\xi^k \cdot \pi^*(-)} \mathrm{CH}^{*+d+r}(\mathbb{P}E) \xrightarrow{\pi_*(\xi^l \cdot -)} \mathrm{CH}^{*+d+l}(X).$$

Now for any $z \in \mathrm{CH}(X)$, $\pi_*(\xi^l \cdot \xi^k \cdot \pi^*(z)) = z \cdot \pi_*(\xi^{k+l}) = z \cdot s_{k+l-r}(E)$ by the definition of Segre class. We conclude by remarking that all negative Segre classes vanish. □

Lemma 3.5 (Small diagonal of projective bundles). *Notation is as above. The intersection product*

$$\delta_{\mathbb{P}E} : \mathfrak{h}(\mathbb{P}E) \otimes \mathfrak{h}(\mathbb{P}E) \to \mathfrak{h}(\mathbb{P}E)$$

induces, via (2), a morphism $\left(\bigoplus_{k=0}^{r} \mathfrak{h}(X)(-k)\right)^{\otimes 2} \to \bigoplus_{m=0}^{r} \mathfrak{h}(X)(-m)$, such that for any $0 \le k, l, m \le r$, the morphism

$$\mathfrak{h}(X)(-k) \otimes \mathfrak{h}(X)(-l) \to \mathfrak{h}(X)(-m)$$

is described as :

[8] They should be well-known but the authors cannot find a proper reference.

[9] The total Segre class is by definition the inverse of the total Chern class, *cf.* [21, Chapter 3].

- *If $m > k+l$ or $m > r$, it is the zero map.*
- *If $m = k+l \leq r$, it is induced by the intersection product product of X, namely, δ_X.*
- *If $k+l \leq r$ and $m \neq k+l$, it is the zero map.*
- *If $m \leq r < k+l$, then it is the composition*

$$\mathfrak{h}(X)(-k) \otimes \mathfrak{h}(X)(-l) \xrightarrow{\delta_X(-k-l)} \mathfrak{h}(X)(-k-l) \xrightarrow{\cdot\omega} \mathfrak{h}(X)(-m),$$

where the second morphism is the multiplication by the following characteristic class

$$\omega := \sum_{t=0}^{r-m} c_t(E) s_{k+l-m-t}(E) \in \mathrm{CH}^{k+l-m}(X).$$

Proof. By Manin's identity principle ([2, §4.3.1]), we only have to prove the lemma for Chow groups. Let us first compute the inverse b^{-1} of the isomorphism in the projective bundle formula

$$b : \bigoplus_{k=0}^{r} \mathrm{CH}^{*-k}(X) \xrightarrow{\simeq} \mathrm{CH}^*(\mathbb{P}E).$$

Assume $\gamma \in \mathrm{CH}^*(\mathbb{P}E)$ is the image of $(z_0, z_1, \cdots, z_r) \in \oplus_{k=0}^{r} \mathrm{CH}^{*-k}(X)$, *i.e.*

$$\gamma = \sum_{k=0}^{r} \pi^*(z_k) \cdot \xi^k.$$

For any $t \geq 0$, $\pi_*(\gamma \cdot \xi^t) = \sum_{k=0}^{r} \pi_*(\pi^*(z_k) \cdot \xi^{k+t}) = \sum_{k=0}^{r} z_k \cdot s_{k+t-r}(E)$. Since the total Segre class is the inverse of the total Chern class, we have for any $0 \leq k \leq r$,

$$z_k = \sum_{t=0}^{r-k} c_t(E) \cdot \pi_*(\gamma \cdot \xi^{r-k-t}).$$

This gives b^{-1}. Now let us go back to the product formula. We have to compute the composition $b^{-1} \circ (b \otimes b)$, whose (k,l,m)-th component for any $0 \leq k, l, m \leq r$ is the composition :

$$\mathrm{CH}(X) \otimes \mathrm{CH}(X) \xrightarrow{(\xi^k \cdot \pi^*, \xi^l \cdot \pi^*)} \mathrm{CH}(\mathbb{P}E) \otimes \mathrm{CH}(\mathbb{P}E) \xrightarrow{\cdot} \mathrm{CH}(\mathbb{P}E) \xrightarrow{b_m^{-1}} \mathrm{CH}(X),$$

where the last morphism is $\sum_{t=0}^{r-m} c_t(E) \cdot \pi_*(\bullet \cdot \xi^{r-m-t})$ by the formula for b^{-1}. Now for any $z, z' \in \mathrm{CH}(X)$, the m-th component of $\pi^*(z) \cdot \xi^k \cdot \pi^*(z') \cdot \xi^l = \pi^*(z \cdot z') \cdot \xi^{k+l}$ is $\sum_{t=0}^{r-m} c_t(E) \cdot \pi_*(\pi^*(z \cdot z') \cdot \xi^{k+l} \cdot \xi^{r-m-t}) = z \cdot z' \cdot (\sum_{t=0}^{r-m} c_t(E) s_{k+l-m-t}(E))$. We can conclude in all cases easily. □

Proposition 3.6 (($\star$) and projective bundles)**.** *Let X be a smooth projective variety and let E be a vector bundle over X of rank $r+1$. Let $\pi : \mathbb{P}(E) \to X$ be the associated projective bundle. If we have a marking for X satisfying ($\star$) such that all Chern classes of E are distinguished, then $\mathbb{P}E$ has a natural marking such that $\mathbb{P}E$ satisfies ($\star$) and such that the graph of the projection $\pi : \mathbb{P}E \to X$ is distinguished.*

If in addition X is equipped with the action of a finite group G such that E is G-equivariant and such that the marking of X satisfies ($\star_G$), then the natural marking of $\mathbb{P}E$ satisfies ($\star_G$).

Proof. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ be a marking that satisfies $(\star)$ and is such that $c_k(E) \in \mathrm{DCH}(X)$. Using the projective bundle formula (2), we obtain a marking for $\mathbb{P}E$:

$$\lambda : \mathfrak{h}(\mathbb{P}E) \xrightarrow{\simeq} \bigoplus_{k=0}^{r} M(-k).$$

Let us show that λ satisfies $(\star)$.
For $(\star 1)$, it is equivalent, by Lemma 2.10, to say that the bottom isomorphism in the following commutative diagram is symmetrically distinguished :

$$\begin{tikzcd}
\mathfrak{h}(\mathbb{P}E)(d+r) \arrow[r, "PD_{\mathbb{P}E}"] \arrow[dd, bend right=60, "\lambda(d+r)"'] & \mathfrak{h}(\mathbb{P}E)^\vee \arrow[d, "b^\vee", "\simeq"'] \\
\bigoplus_{k=0}^r \mathfrak{h}(X)(d+r-k) \arrow[r] \arrow[u, "b(d+r)", "\simeq"'] \arrow[d, "\oplus\phi(\cdot)"', "\simeq"] & \bigoplus_{l=0}^r \mathfrak{h}(X)^\vee(l) \\
\bigoplus_{k=0}^r M(d+r-k) \arrow[r] & \bigoplus_{l=0}^r M^\vee(l) \arrow[u, "\simeq", "\oplus\phi^\vee(\cdot)"'] \arrow[uu, bend right=60, "\lambda^\vee"']
\end{tikzcd}$$

However, this is a consequence of the description of the middle morphism in Lemma 3.4, since all the Chern classes (and hence Segre classes) of E as well as PD_X is distinguished by assumption.
For $(\star 2)$: with $(\star 1)$ being prove, one uses the interpretation of $(\star 2)$ in Lemma 2.10. $(\star 2)$ then follows from Lemma 3.5 by the distinguishedness of δ_X as well as the Chern classes and Segre classes of E.
For $(\star 3)$, we first claim that for any k, the cycle $\pi^*(\alpha) \cdot \xi^k$ is distinguished if $\alpha \in \mathrm{CH}(X)$ is so. Indeed, for $k \leq r$, it is by definition ; for $k > r$, we use the equality $\xi^{r+1} + \pi^*(c_1(E))\xi^r + \cdots + \pi^*(c_{r+1}(E)) = 0$ and the distinguishedness of Chern classes of E to reduce to the treated cases. Now from the short exact sequences :

$$0 \to \mathcal{O}_{\mathbb{P}E} \to \pi^*(E) \otimes \mathcal{O}_\pi(1) \to T_{\mathbb{P}E/X} \to 0\,;$$
$$0 \to T_{\mathbb{P}E/X} \to T_{\mathbb{P}E} \to \pi^* T_X \to 0,$$

we see that all Chern characters of $\mathbb{P}E$ are linear combinations of terms of the form $\pi^*(\alpha)\cdot\xi^k$ with α being a polynomial of Chern classes of X and of E. By assumption α is distinguished hence so is the Chern characters of $\mathbb{P}E$. With $(\star 1)$ and $(\star 2)$ being proved for $\mathbb{P}E$, we know that $\mathrm{DCH}(\mathbb{P}E)$ is a sub-algebra by Proposition[10] 2.11. We are done as Chern classes are polynomials of Chern characters.
The distinguishedness of (the graph of) the projection $\pi : \mathbb{P}(E) \to X$ is obvious : via the markings ϕ and λ, it is equivalent to say that the inclusion of the first summand

$$M \hookrightarrow M \oplus M(-1) \oplus \cdots \oplus M(-r)$$

is a symmetrically distinguished morphism in CHM^{ab}_{sd}.
Finally, assume that X is equipped with the action of a finite group G such that E is G-equivariant. Note that with the induced action of G on $\mathbb{P}E$, we have that π is G-equivariant and we have that $(g_{\mathbb{P}E})_*\xi = \xi$ (since G preserves $\mathcal{O}_\pi(1)$). Thus the action of G commutes with b and $b^\vee$. Since we are assuming that the marking ϕ of X satisfies $(\star_G)$, we find that the marking λ satisfies $(\star_G)$. □

[10]In Proposition 2.11, the assumption on Chern classes is not used for the proof that DCH forms a sub-algebra.

Example 3.7. To apply Proposition 3.6, as examples of vector bundles with distinguished Chern classes, we have the tangent bundle T_X as well as other vector bundles obtained from it by performing duals, tensor products, direct sums and direct summands *etc.* Concretely, these are direct sums of vector bundles of the form $\mathbb{S}_\lambda T_X$, where λ is a non-increasing sequence of integers and $\mathbb{S}_\lambda$ is the associated Schur functor.

3.3. **Blow-ups.** We will show in this subsection that the condition $(\star)$ in Definition 2.8 passes to a blow-up in the expected way.

We fix the following notation throughout this subsection. Let X be a smooth projective variety of dimension d, $i : Y \hookrightarrow X$ be a closed smooth subvariety of codimension c and $\mathscr{N} := \mathscr{N}_{Y/X}$ be the normal bundle. Let $\widetilde{X}$ be the blow-up of X along Y and E the exceptional divisor, which is identified with $\mathbb{P}(\mathscr{N})$. Denote by ξ the first Chern class of $\mathscr{O}_p(1) = \mathscr{N}^\vee_{E/\widetilde{X}}$. The names of some relevant morphisms are in the following cartesian diagram :

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "\tau"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd} \tag{3}$$

Recall the blow-up formula (see [2, §4.3.2]) :

$$b : \mathfrak{h}(X) \oplus \bigoplus_{k=1}^{c-1} \mathfrak{h}(Y)(-k) \xrightarrow{\simeq} \mathfrak{h}(\widetilde{X}), \tag{4}$$

which is given by :

- $\tau^* : \mathfrak{h}(X) \to \mathfrak{h}(\widetilde{X})$;
- for any $1 \leq k \leq c-1$, $j_*(\xi^{k-1} \cdot p^*(-)) : \mathfrak{h}(Y)(-k) \to \mathfrak{h}(\widetilde{X})$.

The following two lemmas[11] compute the diagonal and the small diagonal for $\widetilde{X}$.

Lemma 3.8 (Diagonal of blow-ups)**.** *Notation is as above. Then the isomorphism*

$$b^\vee \circ PD_{\widetilde{X}} \circ b(d) : \mathfrak{h}(X)(d) \oplus \bigoplus_{k=1}^{c-1} \mathfrak{h}(Y)(d-k) \xrightarrow{\simeq} \mathfrak{h}(X)^\vee \oplus \bigoplus_{l=1}^{c-1} \mathfrak{h}(Y)^\vee(l)$$

induced by the Poincaré duality $PD_{\widetilde{X}} : \mathfrak{h}(\widetilde{X})(d) \xrightarrow{\simeq} \mathfrak{h}(\widetilde{X})^\vee$ via (4), is described as follows :

- *$\mathfrak{h}(X)(d) \to \mathfrak{h}(X)^\vee$ is the Poincaré duality isomorphism PD_X ;*
- *For any $1 \leq k, l \leq c-1$ such that $l + k \geq c$, then*
  $$\mathfrak{h}(Y)(d-k) \to \mathfrak{h}(Y)^\vee(l)$$
  is the composition :
  $$\mathfrak{h}(Y)(d-k) \xrightarrow{-\cdot s_{k+l-c}(\mathscr{N})} \mathfrak{h}(Y)(d-c+l) \xrightarrow{PD_Y(l)} \mathfrak{h}(Y)^\vee(l),$$
  where the first morphism is the multiplication by the opposite of some Segre class.
- *Zero everywhere else.*

[11]These are certainly well-known but the authors could not find a proper reference.

Proof. Using Manin's identity principle ([2, §4.3.1]), we only need to show the lemma for Chow groups. Note that Poincaré duality induces identity on Chow groups. We have to compute $b^\vee \circ b$ on the level of Chow groups : one checks easily that the composition

$$\mathrm{CH}(X) \xrightarrow{\tau^*} \mathrm{CH}(\widetilde{X}) \xrightarrow{\tau_*} \mathrm{CH}(X)$$

is the identity ; for any $1 \leq k \leq c-1$, the compositions

$$\mathrm{CH}(X) \xrightarrow{\tau^*} \mathrm{CH}(\widetilde{X}) \xrightarrow{p_*(j^*(-)\cdot \xi^{k-1})} \mathrm{CH}(Y)$$

$$\mathrm{CH}(Y) \xrightarrow{j_*(p^*(-)\cdot \xi^{k-1})} \mathrm{CH}(\widetilde{X}) \xrightarrow{\tau_*} \mathrm{CH}(X)$$

are zero ; and finally for any $1 \leq k, l \leq c-1$, the composition

$$\mathrm{CH}(Y) \xrightarrow{j_*(p^*(-)\cdot \xi^{k-1})} \mathrm{CH}(\widetilde{X}) \xrightarrow{p_*(j^*(-)\cdot \xi^{l-1})} \mathrm{CH}(Y)$$

sends a cycle $z \in \mathrm{CH}(Y)$ to

$$p_*\left(j^*\left(j_*(p^*(z)\cdot \xi^{k-1})\right)\cdot \xi^{l-1}\right) = -p_*\left(p^*(z)\cdot \xi^{k+l-1}\right) = -z\cdot p_*(\xi^{k+l-1}) = -z\cdot s_{k+l-c}(\mathscr{N}).$$

□

Lemma 3.9 (Small diagonal of blow-ups)**.** *The intersection product*

$$\delta_{\widetilde{X}} : \mathfrak{h}(\widetilde{X}) \otimes \mathfrak{h}(\widetilde{X}) \to \mathfrak{h}(\widetilde{X})$$

is described via the isomorphism (4) as follows :

- $\mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ *is the intersection product (induced by* δ_X*) ;*
- *For any* $1 \leq k \leq c-1$, $\mathfrak{h}(X) \otimes \mathfrak{h}(Y)(-k) \to \mathfrak{h}(Y)(-k)$ *is the composition*

$$\mathfrak{h}(X) \otimes \mathfrak{h}(Y)(-k) \xrightarrow{i^*\otimes \mathrm{id}} \mathfrak{h}(Y) \otimes \mathfrak{h}(Y)(-k) \xrightarrow{\delta_Y(-k)} \mathfrak{h}(Y)(-k) ;$$

- *For any* $1 \leq k, l \leq c-1$,

$$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \to \mathfrak{h}(X)$$

  is the composition :

$$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \xrightarrow{\delta_Y(-k-l)} \mathfrak{h}(Y)(-k-l) \xrightarrow{-\cdot s_{k+l-c}(\mathscr{N})} \mathfrak{h}(Y)(-c) \xrightarrow{i_*} \mathfrak{h}(X)$$

- *For any* $1 \leq k, l, m \leq c-1$,

$$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \to \mathfrak{h}(Y)(-m)$$

  is as follows :
  - *if* $m \geq c$ *or* $m > k+l$*, it is the zero map.*
  - *if* $m = k+l \leq c-1$*, then it is induced by* $-\delta_Y$*.*
  - *if* $m \neq k+l \leq c-1$*, then it is the zero map.*
  - *if* $m \leq c-1 < k+l$*, it is the composition*

$$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \xrightarrow{\delta_Y(-k-l)} \mathfrak{h}(Y)(-k-l) \xrightarrow{\cdot\omega} \mathfrak{h}(Y)(-m),$$

    where the second morphism is the multiplication by the following characteristic class

$$\omega := -\sum_{t=1}^{c-m} s_{k+l-m-t+1}(\mathscr{N}) \cdot c_{t-1}(\mathscr{N}) \in \mathrm{CH}^{k+l-m}(Y).$$

Proof. We only have to prove for Chow groups thanks to Manin's identity principle ([2, §4.3.1]). As in Lemma 3.5, we compute the inverse of

$$b : \mathrm{CH}^*(X) \oplus \bigoplus_{k=1}^{c-1} \mathrm{CH}^{*-k}(Y) \to \mathrm{CH}^*(\widetilde{X}).$$

Assume $\gamma = \tau^*(z_0) + \sum_{k=1}^{c-1} j_*(p^*(z_k) \cdot \xi^{k-1})$ where $z_0 \in \mathrm{CH}(X)$ and $z_k \in \mathrm{CH}(Y)$ for all $1 \le k \le c-1$. Then b^{-1} is given by $z_0 = \tau_*(\gamma)$; and for all $1 \le k \le c-1$,

$$z_k = -\sum_{t=1}^{c-k} p_*(j^*(\gamma) \cdot \xi^{c-k-t}) \cdot c_{t-1}(\mathscr{N}).$$

Now for intersection products, we have to compute $b^{-1} \circ (b \otimes b)$, which we only give the computation of the (k,l,m)-th component when $1 \le k,l,m \le c-1$ and leave the other cases to the reader. Let $z, z' \in \mathrm{CH}(Y)$, then the m-th component of the product $j_*(p^*(z) \cdot \xi^{k-1}) \cdot j_*(p^*(z') \cdot \xi^{l-1}) = j_*\left(p^*(z) \cdot \xi^{k-1} \cdot j^*(j_*(p^*(z') \cdot \xi^{l-1}))\right) = -j_*\left(p^*(z \cdot z') \cdot \xi^{k+l-1}\right)$ is

$$\begin{aligned} & \sum_{t=1}^{c-m} p_*\left(j^* j_*\left(p^*(z \cdot z') \cdot \xi^{k+l-1}\right) \cdot \xi^{c-m-t}\right) \cdot c_{t-1}(\mathscr{N}) \\ = \quad & -\sum_{t=1}^{c-m} p_*\left(p^*(z \cdot z') \cdot \xi^{k+l+c-m-t}\right) \cdot c_{t-1}(\mathscr{N}) \\ = \quad & -\sum_{t=1}^{c-m} z \cdot z' \cdot s_{k+l-m-t+1}(\mathscr{N}) \cdot c_{t-1}(\mathscr{N}). \end{aligned}$$

Then all cases follow easily. □

Proposition 3.10 (($\star$) and blow-ups)**.** *Let X be a smooth projective variety and let $i : Y \hookrightarrow X$ be a closed smooth subvariety. If we have markings satisfying the condition ($\star$) for X and Y such that $i^* : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ is distinguished, then $\widetilde{X}$, the blow-up of X along Y, has a natural marking that satisfies ($\star$) and is such that the graphs of the morphisms in Diagram* (3) *are all distinguished* [12].

If in addition X is equipped with the action of a finite group G such that $G \cdot Y = Y$ and such that the markings of X and Y satisfy ($\star_G$), then the natural marking of $\widetilde{X}$ also satisfies ($\star_G$).

Proof. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} N$ be markings satisfying ($\star$). The distinguishedness of i^* means that the induced morphism $\psi \circ i^* \circ \phi^{-1} : M \to N$ is symmetrically distinguished, or equivalently, the graph of the inclusion $\Gamma_i \in \mathrm{DCH}(X \times Y)$ is distinguished. Using the blow-up formula (4), ϕ and ψ induce a marking for $\widetilde{X}$:

$$\lambda : \mathfrak{h}(\widetilde{X}) \xrightarrow{\simeq} M \oplus \bigoplus_{k=1}^{c-1} N(-k), \tag{5}$$

which we will show to satisfy ($\star$).

We first point out that the distinguishedness of i^* implies $i^*(\mathrm{DCH}(X)) \subset \mathrm{DCH}(Y)$. By Poincaré duality and condition ($\star$1) (interpreted in Lemma 2.10) for X and Y,

[12] The exceptional divisor E is endowed with the natural marking of Proposition 3.6 by its projective bundle structure over Y.

$i_* : \mathfrak{h}(Y) \to \mathfrak{h}(X)(-c)$ is also distinguished. In particular, $i_*(\mathrm{DCH}(Y)) \subset \mathrm{DCH}(X)$. Using the short exact sequence $0 \to T_Y \to T_X|_Y \to \mathscr{N} \to 0$, we see that the Chern classes of $\mathscr{N}$ is a polynomial of Chern classes of Y and Chern classes of X restricted to Y, which are all in $\mathrm{DCH}(Y)$ by hypothesis ($\star 3$) for X and Y. Since $\mathrm{DCH}(Y)$ is a sub-algebra (Proposition 2.11), all Chern classes of $\mathscr{N}$ are distinguished on Y.
For ($\star 1$), it amounts to show that the bottom isomorphism in the following diagram determined by the commutativity is symmetrically distinguished :

$$\begin{tikzcd}
\mathfrak{h}(\widetilde{X})(d) \arrow[r, "PD_{\widetilde{X}}"] \arrow[d, bend right=60, "\lambda(d)"'] & \mathfrak{h}(\widetilde{X})^{\vee} \arrow[d, "\simeq"', "b^{\vee}"] \\
\mathfrak{h}(X)(d) \oplus \bigoplus_{k=1}^{c-1} \mathfrak{h}(Y)(d-k) \arrow[r] \arrow[u, "b(d)", "\simeq"'] \arrow[d, "\phi{,}\psi"', "\simeq"] & \mathfrak{h}(X)^{\vee} \oplus \bigoplus_{l=1}^{c-1} \mathfrak{h}(Y)^{\vee}(l) \\
M(d) \oplus \bigoplus_{k=1}^{c-1} N(d-k) \arrow[r] & M^{\vee} \oplus \bigoplus_{l=1}^{c-1} N^{\vee}(l) \arrow[u, "\simeq", "\phi^{\vee}{,}\psi^{\vee}"'] \arrow[u, uu, bend right=60, "\lambda^{\vee}"']
\end{tikzcd}$$

However, this is a consequence of the description of the middle morphism given in Lemma 3.8, since PD_X, PD_Y and the Segre classes of $\mathscr{N}$ are all distinguished by assumption.
With ($\star 1$) being proved, the condition ($\star 2$) follows similarly from Lemma 3.9, since all Segre and Chern classes as well as the morphisms $i^* : \mathfrak{h}(X) \to \mathfrak{h}(Y)$, $i_* : \mathfrak{h}(Y) \to \mathfrak{h}(X)(c)$, the intersection products $\delta_X : \mathfrak{h}(X)^{\otimes 2} \to \mathfrak{h}(X)$ and $\delta_Y : \mathfrak{h}(Y)^{\otimes 2} \to \mathfrak{h}(Y)$ are all distinguished by assumption. Note that the 'quick' proof here does not mean ($\star 2$) is tautological : all the difficulties are put into the proof of ($\star 1$) and two preparation Lemmas 3.8 and 3.9, without which the equivalent interpretation of ($\star 2$) by the intersection product in Lemma 2.10 is not available.
Now for ($\star 3$), we use the formula for Chern classes of the blow-up in [21, Theorem 15.4]. Given the distinguishedness of Chern classes of T_X, T_Y and $\mathscr{N}$, we only have to show that for any $\alpha \in \mathrm{DCH}(Y)$ and $k \in \mathbb{N}$, the class $j_*(p^*(\alpha)\cdot \xi^k) \in \mathrm{CH}(\widetilde{X})$ is distinguished. For $k < c-1$, it is by definition. For $k = c-1$, by the excess intersection formula[13] ([21, §6.3]), $j_*(p^*(\alpha)\cdot\xi^k)$ is a linear combination of $\tau^*(i_*(\alpha))$ with some $j_*(p^*(\alpha)\cdot \xi^l)$ for $l < k$. By the remark in the beginning of the proof, $i_*(\alpha) \in \mathrm{DCH}(X)$ and hence $\tau^*(i_*(\alpha))$ is distinguished by definition. Finally, for $k > c-1$, one uses the equality $\xi^c + p^*(c_1(\mathscr{N}))\xi^{c-1} + \cdots + p^*(c_c(\mathscr{N})) = 0$ to reduce to the treated cases.
The graph of $i : Y \hookrightarrow X$ is distinguished by assumption ; the graph of $p : E \to Y$ is distinguished thanks to Proposition 3.6 ; the distinguishedness of the graph of τ is equivalent to say that (via the markings ϕ and λ) the inclusion of the first summand $M \hookrightarrow M \oplus \bigoplus_{k=1}^{c-1} N(-k)$ is symmetrically distinguished, which is obvious ; finally, one checks easily that via the natural markings, the morphism $j^* : \mathfrak{h}(\widetilde{X}) \to \mathfrak{h}(E)$ corresponds to the morphism

$$(i^*, -\mathrm{id}, \cdots, -\mathrm{id}) : M \oplus N(-1) \oplus \cdots \oplus N(-c+1) \to N \oplus N(-1) \oplus \cdots \oplus N(-c+1),$$

which is obviously symmetrically distinguished.
Finally, assume that X is equipped with the action of a finite group G such that $G \cdot Y = Y$. Note that with the induced action of G on E and $\widetilde{X}$, we have that the

[13] In this case, it is called the 'key formula' in [21, Proposition 6.7(a)].

morphisms in diagram (3) are G-equivariant. Thus the action of G commutes with b and $b^\vee$. Since we are assuming that the markings of X and Y satisfy $(\star_G)$, we find that the marking λ satisfies $(\star_G)$. $\square$

3.4. **Finite group quotients.** In this subsection, let us fix a smooth projective variety X of dimension d upon which a finite group G acts and denote by $\pi : X \to Y := X/G$ the quotient. We assume that Y is also smooth.

Proposition 3.11 (($\star$) and quotient)**.** *Notation and assumptions are as above. If there is a marking for (X, G) satisfying $(\star 1)$, $(\star 2)$ and $(\star_G)$, then Y has a natural marking that satisfies $(\star 1)$ and $(\star 2)$ and is such that the graph of the quotient morphism $\pi : X \to Y$ is distinguished.*

Moreover, if $\pi : X \to Y$ is a cyclic covering along a divisor D such that $D \in \mathrm{DCH}(X)$ and if the marking for X satisfies $(\star 3)$, then the natural marking for Y also satisfies $(\star 3)$.

Proof. Let $\phi : X \xrightarrow{\simeq} M$ (with $M \in \mathrm{CHM}^{ab}_{sd}$) be the marking in question. By assumption, the induced endomorphisms $g \in \mathrm{End}(M)$ are all symmetrically distinguished. In particular

$$M^G := \mathrm{Im}\left(\frac{1}{|G|}\sum_{g\in G} g : M \to M\right)$$

belongs to CHM^{ab}_{sd} and the following composition of isomorphisms

$$\lambda : \mathfrak{h}(Y) \xrightarrow{\pi^*} \mathfrak{h}(X)^G = \left(X, \frac{1}{|G|}\sum_{g\in G}\Gamma_g, 0\right) \xrightarrow{\phi^G} M^G$$

provides a marking for Y, for which we will prove to satisfy $(\star)$.

For $(\star 1)$, by the definition of λ, we have the following diagram:

$$\begin{tikzcd} M(d) \arrow[r, "\sigma_X", "\simeq"'] & M^\vee \arrow[d, two heads] \\ M^G(d) \arrow[u, hook] \arrow[r, "\sigma_Y", "\simeq"'] & (M^G)^\vee = (M^\vee)^G \end{tikzcd}$$

where σ_X and σ_Y are the isomorphisms determined by the Poincaré duality for X and Y respectively (see Lemma 2.10), and the vertical arrows are the natural inclusion and projection, which are symmetrically distinguished morphisms due to the symmetric distinguishedness of the induced action of G on M. As σ_X is symmetrically distinguished by assumption, so is σ_Y, hence $(\star 1)$ is true by Lemma 2.10.

For $(\star 2)$, we use the interpretation in Lemma 2.10 by intersection product and it amounts to show that the morphism μ_Y in the diagram below is symmetrically distinguished:

$$\begin{tikzcd} M \otimes M \arrow[r, "\mu_X"] & M \arrow[d, two heads] \\ M^G \otimes M^G \arrow[u, hook] \arrow[r, "\mu_Y"] & M^G \end{tikzcd}$$

which follows from the symmetric distinguishedness of μ_X, the inclusion and the projection.

The distinguishedness of the graph of the quotient map $\pi : X \to Y$ is equivalent to the condition that the natural inclusion $M^G \hookrightarrow M$ is a symmetrically distinguished morphism in CHM^{ab}_{sd}, which follows directly from the assumption that the action of G on M is symmetrically distinguished.

Finally, suppose that the marking for X satisfies ($\star 3$) and that $\pi : X \to Y$ is a degree n cyclic covering along a divisor D such that $D \in \mathrm{DCH}(X)$. In order to show that the natural marking on Y satisfies ($\star 3$), it suffices to show by the projection formula that $\pi^* \operatorname{ch}(T_Y)$ is distinguished. We have a short exact sequence

$$0 \longrightarrow T_X \longrightarrow \pi^* T_Y \longrightarrow O_D(nD) \longrightarrow 0.$$

Since X satisfies ($\star 3$), it is enough to show that $\operatorname{ch}(O_D(nD))$ belongs to $\mathrm{DCH}(X)$. Now $O_D(nD)$ fits into the short exact sequence

$$0 \longrightarrow O_X((n-1)D) \longrightarrow O_X(nD) \longrightarrow O_D(nD) \longrightarrow 0.$$

Since the class of the divisor D is assumed to belong to the $\mathbb{Q}$-algebra $\mathrm{DCH}(X)$, we find that indeed $\operatorname{ch}(O_D(nD)) = \operatorname{ch}(O_X(nD)) - \operatorname{ch}(O_X((n-1)D))$ belongs to $\mathrm{DCH}(X)$, which concludes the proof. □

3.5. **Hilbert squares and nested Hilbert schemes.**

Proposition 3.12 (Hilbert squares)**.** *Assume X is a smooth projective variety with a marking that satisfies ($\star$). Then $X^{[2]}$ has a natural marking that satisfies ($\star$) and is such that the universal family $\{(x, z) : x \in \operatorname{Supp}(z)\} \subseteq X \times X^{[2]}$ is distinguished (with respect to the product marking).*

Proof. The product $X \times X$ is naturally endowed with the action of $G := \mathbb{Z}/2$ that switches the factors, and the locus of fixed points is the diagonal, which is isomorphic to X. By Remark 3.2, the product marking on $X \times X$ satisfies ($\star_G$). Therefore, we may apply Proposition 3.10 to obtain a marking on the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal that satisfies ($\star$) and ($\star_G$). Now $X^{[2]}$ is the quotient of the latter blow-up by the cyclic action of $\mathbb{Z}/2$. Thus Proposition 3.11 provides a marking for $X^{[2]}$ that satisfies ($\star$).

Finally, we show that the universal family $Y := \{(x, z) : x \in \operatorname{Supp}(z)\}$ is distinguished. First note that Y is isomorphic to $\widetilde{X \times X}$, so that Y is endowed by the natural marking coming from that of X. In order to conclude, we only need to show that the graph Γ of the inclusion morphism $Y \hookrightarrow X \times X^{[2]}$ is distinguished. Since the quotient morphism $\widetilde{X \times X} \to X^{[2]}$ is distinguished (Proposition 3.11), it is enough to show that the pull-back Γ' of Γ to $Y \times X \times Y$ is distinguished. But then, this is clear since Γ' consists of the two irreducible components $\{((x, y), x, (x, y) : x, y \in X\}$ and $\{(y, x), x, (x, y) : x, y \in X\}$, and the cycle classes of both components are distinguished by Corollary 2.12 and Proposition 3.10. □

Recall that by a result of Cheah [13], for a smooth projective variety X of dimension ≥ 3, the only smooth nested Hilbert schemes of finite length subschemes on X are $X^{[2]}$, $X^{[3]}$, $X^{[1,2]}$ and $X^{[2,3]}$. By the same method, we have :

Proposition 3.13 (Nested Hilbert schemes)**.** *Assumption is as in Proposition 3.12. Then $X^{[1,2]}$ and $X^{[2,3]}$ have natural markings satisfying ($\star$) and are such that the classes of the universal subschemes are distinguished.*

Proof. It is clear that $X^{[1,2]}$ is isomorphic to $\widetilde{X \times X}$, the blow-up of $X \times X$ along the diagonal, hence satisfies $(\star)$ by Proposition 3.10. Similarly, $X^{[2,3]}$ is isomorphic to the blow-up of $X \times X^{[2]}$ along the universal subscheme Y. As is mentioned in the proof of the previous proposition, Y is isomorphic to $X^{[1,2]}$ hence to $\widetilde{X \times X}$, thus it has a marking satisfying $(\star)$. As $X^{[2]}$ is endowed with the marking in Proposition 3.12, $X \times X^{[2]}$ is endowed with the product marking satisfying $(\star)$ by Proposition 3.1. Moreover, the Chern classes of the normal bundle of Y in $X \times X^{[2]}$ is distinguished since it is a polynomial of the Chern classes of T_Y, of T_X pulled-back to $Y = \widetilde{X \times X}$ via the first projection and of $T_{X^{[2]}}$ pulled-back to Y via the $\mathbb{Z}/2$ quotient map (*cf.* the computation in [40, Theorem 6.1]), which are all distinguished by Propositions 3.10 and 3.11. Again by Proposition 3.10, $X^{[2,3]}$ has a marking satisfying $(\star)$. The assertions about the universal subschemes follow from Corollary 2.12. □

Remark 3.14 (Hilbert cubes)**.** The similar argument as above combined with the explicit description of the Hilbert cube $X^{[3]}$ in [40] shows that $X^{[3]}$ satisfies $(\star)$ once X does. Indeed, $X^{[3]}$ is constructed from X^3 in five steps (*cf.* [40]) : the first three are successive blow-ups of X^3, each time along a center satisfying $(\star)$ with normal bundle having distinguished Chern classes ; the fourth step is a quotient map by a distinguished cyclic $\mathbb{Z}/3$-action ; the final step is a blow-down of divisor with distinguished normal bundle to a center satisfying $(\star)$. Thus using Propositions 3.1, 3.6, 3.10, 3.11 and Corollary 2.12 repeatedly in the first four steps, and using in the final step the analogue of the technical [40, Lemma 6.4] (with $\mathrm{CH}(-)_{(0)}$ replaced by $\mathrm{DCH}(-)$), one can obtain a marking of $X^{[3]}$ satisfying $(\star)$. The details are left to the interested reader.

4. Examples of varieties satisfying the condition $(\star)$

We provide in this section some examples of varieties satisfying the condition $(\star)$. Together with the operations in §3, we obtain even more examples. Thanks to Proposition 2.11, the rational Chow ring of each of them possesses a sub-algebra consisting of distinguished cycles, which is mapped isomorphically to the numerical Chow ring and contains all Chern classes of the variety.

4.1. **Easy examples.** First of all, as $(\star)$ is certainly a property preserved by isomorphisms of algebraic varieties, we have by definition :

Lemma 4.1. *Any abelian torsor, that is, a variety isomorphic to an abelian variety, satisfies* $(\star)$*.*

Another set of examples generalizes the projective spaces :

Proposition 4.2. *Let X be a smooth projective variety satisfying at least one of the following conditions :*

1. *$X \simeq G/P$ is a homogeneous variety, where G is a linear algebraic group and P is a parabolic subgroup.*
2. *X is a toric variety.*
3. *The derived category $D^b_{coh}(X)$ has a full exceptional collection.*
4. *The cycle class map $\mathrm{CH}^*(X_{\mathbb{C}}) \to H^*(X_{\mathbb{C}}, \mathbb{Q})$ is injective.*
5. *The Chow group $\mathrm{CH}^*(X_{\mathbb{C}})$ is a finite-dimensional $\mathbb{Q}$-vector space.*

Then X satisfies $(\star)$*.*

Proof. Actually any of these conditions ensures that the Chow motive of X is of Lefschetz-Tate type:

$$\mathfrak{h}(X) \simeq \bigoplus_i \mathbb{1}(a_i),$$

with $a_i \in \mathbb{Z}$. It is well-known for (1) and (2); while for (3) it is established in [10] and [30]. For (4), it is the main result of [29], see also [46, §2.2] for a recent account, and for (5), it is proved in [25], [43]. $\square$

4.2. **Curves.** Recall that the smooth projective curves of genus 0 and 1 are covered in §4.1. We consider in this subsection higher genus cases.

Let C be a smooth projective curve with genus $g \geq 2$. Its Jacobian variety JC is a principally polarized abelian variety of dimension g with origin denoted by O and theta divisor denoted by $\Theta \in \mathrm{CH}^1(JC)$, which is always assumed to be symmetric. By choosing a base point $z \in C$, we have the Abel-Jacobi embedding:

$$\begin{array}{rcl} \iota_z : C & \hookrightarrow & JC \\ x & \mapsto & \mathcal{O}_C(x-z). \end{array}$$

Associated to z, there is also the motivic decomposition of C:

$$\mathfrak{h}(C) = \mathfrak{h}^0(C) \oplus \mathfrak{h}^1(C) \oplus \mathfrak{h}^2(C),$$

where $\mathfrak{h}^0(C) := (C, z \times C, 0) \simeq \mathbb{1}$, $\mathfrak{h}^2(C) := (C, C \times z, 0) \simeq \mathbb{1}(-1)$ and $\mathfrak{h}^1(C) := (C, \Delta_C - z \times C - C \times z, 0)$.

Proposition 4.3. *Let C be a smooth projective curve of genus $g \geq 2$. If there exists a point $z \in C$ such that $\iota_z(C) \in \mathrm{CH}_1(JC)$ is symmetrically distinguished*[14]*, then C satisfies the condition $(\star)$ in Definition 2.8.*

Proof. Let us fix z and simply write $\iota := \iota_z$ and $C := \iota_z(C)$. Assume that $C \in \mathrm{CH}_1(JC)$ is symmetrically distinguished. Since the 1-cycles C and $\frac{1}{(g-1)!}\Theta^{g-1}$ are numerically equivalent and symmetrically distinguished, they are actually equal (*i.e.* rationally equivalent), thanks to Theorem 1.6.

Deninger and Murre construct in [16] a canonical motivic decomposition

$$\mathfrak{h}(JC) = \oplus_{i=0}^{2g} \mathfrak{h}^i(JC).$$

Let $\pi^i \in \mathrm{CH}^g(JC \times JC)$ be the projector corresponding to $\mathfrak{h}^i(JC)$. For example, $\pi^0 = [O] \times JC$ and $\pi^{2g} = JC \times [O]$. See [27] for the explicit formulae of the other projectors π^i. One important feature, easily seen from Theorem 1.6, is that they are all symmetrically distinguished.

We claim that $\Gamma_\iota =: \iota_* : \mathfrak{h}(C) \to \mathfrak{h}(JC)(g-1)$ induces isomorphisms:

- $\mathfrak{h}^2(C) \xrightarrow{\simeq} \mathfrak{h}^{2g}(JC)(g-1) := (JC, JC \times [O], g-1)$;
- $\mathfrak{h}^1(C) \xrightarrow{\simeq} \mathfrak{h}^{2g-1}(JC)(g-1) := (JC, \pi^{2g-1}, g-1)$;
- $\mathfrak{h}^0(C) \xrightarrow{\simeq} L^{g-1}\mathfrak{h}^0(JC)(g-1) := (JC, \frac{1}{g!}\Theta \times \Theta^{g-1}, g-1)$; the latter is a direct summand of $\mathfrak{h}^{2g-2}(JC)(g-1)$ in the Lefschetz decomposition constructed by Künnemann in [26],

where L is the Lefschetz operator (see [26]). Indeed, all these morphisms are in the Kimura category $\mathscr{M}'$ (see Definition 1.1 and Definition 2.1). The functor $\mathscr{M}' \to \overline{\mathscr{M}'}$ is therefore conservative (*c.f.* [3, Corollary 3.16]). One checks easily that

[14] equivalently by Remark 1.7, $\iota_z(C) \in \mathrm{CH}_1(JC)_{(0)}$.

these morphisms are isomorphisms modulo homological, thus *a fortiori* numerical, equivalence.

Putting them together, we have a marking for C:

$$\phi := \iota_* : \mathfrak{h}(C) \xrightarrow{\simeq} M := (JC, JC \times [O] + \pi^{2g-1} + \frac{1}{g!}\Theta \times \Theta^{g-1}, g-1),$$

where M is obviously in CHM^{ab}_{sd}. The proofs for ($\star$1) and ($\star$2) are similar, let us just give the latter: since the inclusion of the direct summand M into $\mathfrak{h}(JC)$ is clearly symmetrically distinguished, to show that $\phi_*^{\otimes 3}(\delta_C)$ is symmetrically distinguished, it suffices to show that $\iota^3_* : \mathrm{CH}_1(C^3) \to \mathrm{CH}_1(JC^3)$ sends the small diagonal δ_C to a symmetrically distinguished cycle of $JC \times JC \times JC$. However, by the following commutative diagram

$$\begin{tikzcd} C \arrow[r, hook, "\delta_C"] \arrow[d, hook, "\iota"'] & C \times C \times C \arrow[d, hook, "\iota^3"] \\ JC \arrow[r, hook, "\delta_{JC}"] & JC \times JC \times JC \end{tikzcd}$$

we have $\iota^3_*(\delta_C) = \delta_{JC,*}(\iota(C))$ is symmetrically distinguished by the assumption and Theorem 1.6.

It remains to show the condition ($\star$3) on Chern classes. The fundamental class $\mathbb{1}_C$ being clearly distinguished (*c.f.* Remark 2.9), it is enough to show that in $\mathrm{CH}_0(JC)$ we have

$$\iota_*(K_C) = (2g-2)[O]. \tag{6}$$

It is well-known that

$$\iota^*(\Theta) = \frac{1}{2}K_C + z \in \mathrm{CH}_0(C).$$

Apply ι_* to this equality, by projection formula, we obtain

$$\frac{1}{2}\iota_*(K_C) + \iota_*(z) = \Theta \cdot C = \Theta \cdot \frac{\Theta^{g-1}}{(g-1)!} = g[O].$$

The desired equality (6) follows by observing that $\iota(z) = O$. □

Corollary 4.4. *All hyperelliptic curves satisfy the condition* ($\star$).

Proof. For a hyperelliptic curve C, choose any Weierstrass point to define the Abel-Jacobi embedding, then the involution $[-1]$ on JC preserves C and acts on C by the hyperelliptic involution. By [42, Proposition 2.1], in the Beauville decomposition of $\mathrm{CH}^{g-1}(JC)$, the class of C belongs to $\mathrm{CH}^{g-1}(JC)_{(0)}$. On the other hand, $\mathrm{CH}^{g-1}(JC)_{(0)}$ is the Fourier transform [7] of $\mathrm{CH}^1(JC)_{(0)}$ which maps isomorphically to $\overline{\mathrm{CH}}^1(JC)$. Therefore, the natural cycle class map $\mathrm{CH}^{g-1}(JC)_{(0)} \to \overline{\mathrm{CH}}^{g-1}(JC)$ is also an isomorphism. Consequently, all cycles in $\mathrm{CH}^{g-1}(JC)_{(0)}$, in particular the class of C, are symmetrically distinguished. One can now conclude by invoking Proposition 4.3. □

Remark 4.5 (Hilbert schemes of a hyperelliptic curve)**.** Recall that the Hilbert scheme of length-n subschemes on a smooth curve C is nothing but the n-th symmetric power $C^{(n)}$ of C. Now if C satisfies ($\star$1) and ($\star$2), then by Proposition 3.1 C^n satisfies ($\star$1) and ($\star$2) and by Proposition 3.11, $C^{(n)}$ satisfies ($\star$1) and ($\star$2). If in addition C satisfies ($\star$3), then the same computation as in [39, p. 95] shows that $C^{(n)}$ satisfies ($\star 3$). Therefore, it follows from Corollary 4.4 that the Hilbert schemes of a hyperelliptic curve satisfy ($\star$).

4.3. **Fermat hypersurfaces.** An important class of (higher dimensional) varieties whose motive is known to be of abelian type is provided by the Fermat hypersurfaces, by using the inductive structure discovered by Shioda–Katsura [41]. Note that Proposition 4.2 implies that smooth quadric hypersurfaces satisfy ($\star$) since they have finite dimensional Chow groups.

In the sequel of this subsection, we fix a degree $d \geq 3$ and, for any $r \in \mathbb{N}$, we let X_r denote the Fermat hypersurface of degree d in $\mathbb{P}^{r+1}$:

$$X_r := \{x_0^d + \cdots + x_{r+1}^d = 0\} \subset \mathbb{P}^{r+1}.$$

Recall the inductive structure (*cf.* [41, Theorem 1]) : let ϵ be a (fixed) d-th root of -1 and ζ be a (fixed) d-th root of unity. For any $r, s \in \mathbb{N}$, we have the following commutative diagram :

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[r] \arrow[d, "\beta"] \arrow[dr, "\psi"] & Z/\mu_d \arrow[d, "\tau"] \arrow[dr, phantom, "\square"] & X_{r-1} \times \mathbb{P}^s \coprod \mathbb{P}^r \times X_{s-1} \arrow[l, hook'] \arrow[d] \\ X_{r-1} \times X_{s-1} \arrow[r, hook, "i_r \times i_s"] & X_r \times X_s \arrow[r, dashed, "\varphi"] & X_{r+s} & X_{r-1} \coprod X_{s-1} \arrow[l, hook'] \end{tikzcd} \tag{7}$$

where $i_r : X_{r-1} \hookrightarrow X_r$ is the embedding given by $(x_0, \ldots, x_r) \mapsto (x_0, \ldots, x_r, 0)$; $\varphi : ((x_0, \ldots, x_{r+1}), (y_0, \ldots, y_{s+1})) \mapsto (y_{s+1}x_0, \ldots, y_{s+1}x_r, \epsilon x_{r+1}y_0, \ldots, \epsilon x_{r+1}y_s)$; β and τ are blow-ups ; the action of μ_d on the blow-up Z is induced by its action on X_r and X_s given by $(x_0, \ldots, x_{r+1}) \mapsto (x_0, \ldots, x_r, \zeta x_{r+1})$ and $(y_0, \ldots, y_{s+1}) \mapsto (y_0, \ldots, y_s, \zeta y_{s+1})$, respectively.

The main result of this subsection is the following.

Proposition 4.6 (Fermat cubics). *If $d = 3$, then there exist, for all $r \in \mathbb{N}$, a marking $\phi_r : \mathfrak{h}(X_r) \xrightarrow{\simeq} M_r$, with $M_r \in \mathrm{CHM}^{ab}_{sd}$, for the cubic Fermat hypersurface X_r, such that*

(*i*) *The embedding $i_r : X_{r-1} \hookrightarrow X_r$ is distinguished* [15] *;*

(*ii*) *The action of μ_d on X_r is distinguished, i.e. satisfies ($\star_G$) ;*

(*iii*) *ϕ_r satisfies the condition ($\star$) in Definition 2.8.*

In particular, all Fermat cubic hypersurfaces satisfy the condition ($\star$).

Proof. We proceed by induction on r. For $r = 1$, $X_1 = \{x_0^3 + x_1^3 + x_2^3 = 0\}$ is a cubic curve in $\mathbb{P}^2$; by fixing an origin, it becomes an elliptic curve. We fix $(-1, 1, 0)$ as its origin. Trivially, X_1 satisfies ($\star$) (§4.1). The embedding $X_0 \hookrightarrow X_1$ is given by three points $(-1, 1, 0), (-\zeta, 1, 0), (-\zeta^2, 1, 0)$, which are of 3-torsion [16], therefore distinguished. As for the action of μ_d, which is given by $(x_0, x_1, x_2) \mapsto (x_0, x_1, \zeta x_2)$, it is clearly an automorphism of abelian variety hence also distinguished.

[15] that is, its graph is distinguished, or equivalently, the induced morphism $\phi_{r-1} \circ i_r^* \circ \phi_r^{-1} : M_r \to M_{r-1}$ is symmetrically distinguished, *cf.* the beginning of the proof of Proposition 3.10.

[16] In fact, the nine 3-torsion points of the Fermat elliptic curve are exactly its intersection with the coordinate axes $(x_0 = 0), (x_1 = 0)$ and $(x_2 = 0)$. Indeed, these nine points lie on 12 lines. Each line contains three of these points and each point lies on four lines. Now use the fact that the sum of the three points in the intersection of any line with the elliptic curve is the hyperplane section class, we easily deduce that 3 times any of the nine points is the hyperplane section class. Hence they are all 3-torsion points if any one of them is fixed as the origin.

Assuming the assertions $(i)-(iii)$ for $r \leq n$, let us establish them for $r = n+1$. We set in the sequel $s = 1$ in the diagram (7). By the induction hypothesis and the fact that distinguished morphisms are stable under products, the embedding $X_{n-1} \times X_0 \hookrightarrow X_n \times X_1$ is distinguished. Therefore Z satisfies $(\star)$ by Proposition 3.10. Again by the induction hypothesis, the action of μ_d on $X_n \times X_1$ is distinguished with distinguished ramification locus, which implies by Proposition 3.11 that Z/μ_d satisfies $(\star)$. Via τ, this determines a marking for X_{n+1}. As $\mathfrak{h}(X_{n+1})$ is a subalgebra object of $\mathfrak{h}(Z/\mu_d)$ (*cf.* Lemma 3.9), we see that this marking of X_{n+1} satisfies $(\star)$ and the morphism ψ is distinguished. In particular, (iii) for $r = n+1$ is proven.
For (i), we have the following commutative diagram, where i is the embedding determined by the point $(1, 0, -\zeta) \in X_1$.

$$\begin{tikzcd}
Z \arrow[d, "\beta"] \arrow[dr, "\psi"] & \\
X_n \times X_1 \arrow[r, dashed, "\varphi"] & X_{n+1} \\
X_n \arrow[u, hook, "i"] \arrow[ur, hook, "i_{n+1}"'] &
\end{tikzcd}$$

Since $(1, 0, -\zeta)$ is a torsion point of X_1, i^* is distinguished. Therefore, with ψ and β being distinguished by construction, $i_{n+1}^* = i_{n+1}^* \circ \psi_* \circ \psi^* = i^* \circ \beta_* \circ \psi^*$ is also distinguished.
Finally for (ii), the action of μ_d on X_{n+1} comes, via the diagram (7), from the action of μ_d on X_1 which is given by $(y_0, y_1, y_2) \mapsto (y_0, \zeta y_1, y_2)$. It is clearly an automorphism of abelian variety hence is distinguished. $\square$

We are not able to determine whether other Fermat hypersurfaces satisfy $(\star)$ but we would like to make the following conjecture:

Conjecture 4.7. *The Fermat hypersurfaces which are Calabi–Yau or Fano,* i.e. $d \leq r+2$, *satisfy the condition* $(\star)$.

Remark 4.8. Conjecture 4.7 does not hold in general for Fermat hypersurfaces of general type; *cf.* Proposition 6.3 (together with Proposition 5.1) below for counter-examples in the curve case starting from degree 4.

Remark 4.9. It is interesting to notice that for $d = 4$, we know that the quartic Fermat surface satisfies $(\star)$ for different reasons: it is a Kummer surface (*cf.* [23, Chapter 14, Example 3.18]) and Proposition 4.10 applies. One can show Conjecture 4.7 for $d = 4$ by a similar induction argument as in Proposition 4.6 once we know the case of Fermat quartic threefold (and some natural compatibilities with the Fermat quartic surface).

4.4. **K3 surfaces with large Picard number.** While K3 surfaces are expected to have motive of abelian type via the Kuga–Satake construction, this has only been established in scattered cases. This includes Kummer surfaces, and [37, Theorem 2] K3 surfaces with Picard number ≥ 19.

4.4.1. *Kummer surfaces.* By definition the Kummer surface $K_1(A)$ attached to the abelian surface A is the fiber over 0 of the morphism $A^{[2]} \to A^{(2)} \to A$, which is the composition of the sum morphism $A^{(2)} \to A$ with the Hilbert–Chow morphism $A^{[2]} \to A^{(2)}$.

Proposition 4.10. *A Kummer surface admits a marking that satisfies* ($\star$).

Proof. The Kummer surface $K_1(A)$ has the following alternative description : the $[-1]$-involution on A induces an involution, denoted ι, on the blow-up $\widetilde{A}$ of A along its subgroup of 2-torsion points, and $K_1(A)$ is the $\mathbb{Z}/2$-quotient of $\widetilde{A}$ for that action. By Proposition 3.10, $(\widetilde{A}, \mathbb{Z}/2)$ has a marking that satisfies ($\star$). We can then conclude from Proposition 3.11 that $K_1(A)$ has a marking that satisfies ($\star$). $\square$

Later on (*cf.* Proposition 4.13), we will generalize Proposition 4.10 by establishing that generalized Kummer varieties admit a marking that satisfies ($\star$).

4.4.2. *K3 surfaces with Picard number* ≥ 19. Such K3 surfaces admit [32] a Nikulin involution (that is, a symplectic involution) with quotient birationally isomorphic to a Kummer surface.

Proposition 4.11. *A K3 surface with Picard number* ≥ 19 *admits a marking that satisfies* ($\star$).

Proof. Let X be a K3 surface with a Nikulin involution ; by [34, §5] X has eight isolated fixed points, which we denote $Q_1, \ldots, Q_8$. Let $\pi : X \to X/\iota$ be the quotient morphism ; X/ι has ordinary double points at the points $P_i := \pi(Q_i)$, so that if $f : Y \to X/\iota$ denotes the minimal resolution, then the exceptional divisors of f are smooth rational (-2)-curves $C_i := f^{-1}(P_i)$.

Let X now be a K3 surface with Picard number ≥ 19. According to [32, Corollary 6.4], X admits a Shioda–Inose structure, meaning that X admits a Nikulin involution ι such that Y is a Kummer surface and such that $f^*\pi_*$ induces a Hodge isometry $T_X(2) \simeq T_Y$, where T_X refers to the transcendental lattice of X. The latter was upgraded to an isomorphism of Chow motives by Pedrini [37, Theorem 2]. Precisely, given S a K3 surface, let us denote o_S the Beauville–Voisin zero-cycle ; *cf.* [9]. We fix a basis D_j of $\mathrm{CH}^1(S)$, and denote $D_j^\vee$ the dual basis with respect to the intersection product. We then define the idempotent correspondences $\pi_S^0 := o_S \times S$, $\pi_S^4 := S \times o_S$, $\pi_S^{2,alg} := \sum_j D_j^\vee \times D_j$, and $\pi_S^{2,tr} := \Delta_S - \pi_S^0 - \pi_S^4 - \pi_S^{2,alg}$. The motive $\mathfrak{h}^{alg}(S) := (S, \pi_S^0 + \pi_S^{2,alg} + \pi_S^4)$ is the algebraic motive of S (it is isomorphic to a direct sum of Lefschetz–Tate motives), and the motive $\mathfrak{t}^2(S) := (S, \pi_S^{2,tr})$ is the transcendental motive of S. Pedrini showed that $f^*\pi_*$ induces an isomorphism of motives $\mathfrak{t}^2(X) \simeq \mathfrak{t}^2(Y)$ (with inverse $\frac{1}{2}\pi^* f_*$).

We fix a marking for the Kummer surface Y that satisfies ($\star$) ; such a marking does exist by Proposition 4.13. Since $\mathrm{DCH}^1(Y) = \mathrm{CH}^1(Y)$, we have that the classes of the smooth rational curves C_i are distinguished, and we also have that the idempotents $\pi_Y^0, \pi_Y^4, \pi_Y^{2,alg}$, and $\pi_Y^{2,tr}$ are distinguished. Then we claim that the marking given by the decomposition $\mathfrak{h}(X) \simeq \pi^* f_* \mathfrak{t}^2(Y) \oplus \mathfrak{h}^{alg}(X)$ satisfies ($\star$). That it satisfies ($\star$3) is obvious since $c_1(X) = 0$ and since by [9] $c_2(X)$ is a multiple of o_X and hence is mapped to zero in $\mathrm{CH}^2(\mathfrak{t}^2(Y))$. By refined intersection [21], the cycle $(f,f)^*(\pi,\pi)_*\Delta_X$ is supported on $(f,f)^{-1}(\pi,\pi)(\Delta_X) = \Delta_Y \cup \bigcup_i C_i \times C_i$. Therefore the cycle $(f,f)^*(\pi,\pi)_*\Delta_X$ is a linear combination of Δ_Y and the $C_i \times C_i$. These cycles belong to $\mathrm{DCH}(Y \times Y)$, thereby establishing ($\star$1), *i.e.* that Δ_X is distinguished. Again, by refined intersection [21], the cycle $(f,f,f)^*(\pi,\pi,\pi)_*\delta_X$ is supported on $(f,f,f)^{-1}(\pi,\pi,\pi)(\delta_X) = \delta_Y \cup \bigcup_i C_i \times C_i \times C_i$. Since C_i is a smooth rational curve, we have that $\mathrm{CH}_2(C_i \times C_i \times C_i)$ admits $c_i \times C_i \times C_i$, $C_i \times c_i \times C_i$ and $C_i \times C_i \times c_i$ as a basis, where c_i is any point on C_i. The cycle $(f,f,f)^*(\pi,\pi,\pi)_*\delta_X$ is therefore a linear combination of δ_Y and, for $1 \leq i \leq 8$, of $c_i \times C_i \times C_i$, $C_i \times c_i \times C_i$ and $C_i \times C_i \times c_i$. By [9], the class of c_i in $\mathrm{CH}^2(Y)$ is the Beauville–Voisin zero-cycle o_Y ; thus $c_i \in \mathrm{DCH}^2(Y)$. The cycles $c_i \times C_i \times C_i$, $C_i \times c_i \times C_i$ and $C_i \times C_i \times c_i$ therefore belong to $\mathrm{DCH}(Y \times Y \times Y)$ by Proposition 2.6. Since δ_Y is distinguished, this establishes $(\star 2)$, *i.e.* that δ_X is distinguished. □

4.5. **Hilbert schemes of surfaces, and generalized Kummer varieties.** In this subsection, we produce series of varieties satisfying $(\star)$.

The first series of examples is given by the Hilbert schemes of points on a surface that satisfies $(\star)$, *e.g.* an abelian surface, a Kummer surface (Proposition 4.10), a K3 surface with Picard rank ≥ 19 (Proposition 4.11) or the product of two hyperelliptic curves (Corollary 4.4).

Proposition 4.12. *Let S be a smooth projective surface that satisfies $(\star)$. Then, for any $n \in \mathbb{N}$, the Hilbert scheme of length-n subschemes on S, denoted $S^{[n]}$, satisfies the condition $(\star)$.*

The second series of example is built from an abelian surface A : the associated Kummer K3 surface as well as its higher dimensional generalizations. Recall that the n-th *generalized Kummer* variety (see [6]) is the symplectic resolution of the quotient $A_0^{n+1}/\mathfrak{S}_{n+1}$, where A_0^{n+1} is the abelian variety $\ker\big(+ : A^{n+1} \to A\big)$, upon which the symmetric group acts naturally by permutations.

Proposition 4.13. *For any $n \in \mathbb{N}$, the generalized Kummer variety $K_n(A)$ satisfies the condition $(\star)$.*

The proofs of Propositions 4.12 and 4.13 will be given concomitantly in full in §4.5.2. Note that the case of Kummer surfaces (which are the generalized Kummer varieties of dimension 2) was already treated in Proposition 4.10. We start by recalling some results of de Cataldo and Migliorini [14] concerning the motives of Hilbert schemes of surfaces, or more generally that of a semi-small resolution.

4.5.1. *The motive of semi-small resolutions.* Recall that a morphism $f : Y \to X$ is called *semi-small* if for all integer $k \geq 0$, the codimension of the locus $\big\{x \in X : \dim f^{-1}(x) \geq k\big\}$ is at least $2k$. In particular, f is generically finite. In [14], assuming $f : Y \to X$ is a semi-small resolution with Y smooth and projective, de Cataldo and Migliorini computed the Chow motive of Y in terms of the Chow motives of projective compactifications of *relevant strata* of f provided these are finite group quotients of smooth varieties ; we refer to [14] for a precise statement. In our case of interest, this has the following consequence. Suppose S is a smooth projective surface and suppose A is an abelian surface ; then the strata associated to the semi-small resolutions

$$S^{[n+1]} \to S^{(n+1)} \quad \text{and} \quad K_n(A) \to A_0^{(n+1)}$$

are indexed by the set of partitions $\mathscr{P}(n+1)$ of $n+1$, and we have morphisms (in fact, isomorphisms by Theorem 4.14 below) of Chow motives

$$\Gamma := \bigoplus_{\lambda \in \mathscr{P}(n+1)} \Gamma^{(\lambda)} \ : \ \mathfrak{h}(S^{[n+1]}) \longrightarrow \bigoplus_{\lambda \in \mathscr{P}(n+1)} \mathfrak{h}(S^{(\lambda)})(|\lambda| - n - 1) \tag{8}$$

and

$$\Gamma_0 := \bigoplus_{\lambda\in\mathscr{P}(n+1)} \Gamma_0^{(\lambda)} \ : \ \mathfrak{h}(K_n(A)) \longrightarrow \bigoplus_{\lambda\in\mathscr{P}(n+1)} \mathfrak{h}(A_0^{(\lambda)})(|\lambda|-n-1). \tag{9}$$

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{l(\lambda)}) = (1^{a_1}\cdots(n+1)^{a_{n+1}})$ of $n+1$ where $a_i = \#\{j : 1 \leq j \leq n+1 ; \lambda_j = i\}$ and where $l(\lambda) := a_1 + \cdots + a_{n+1}$ denotes the length of λ, we define $\mathfrak{S}_\lambda := \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_{n+1}}$. We define S^λ to be $S^{l(\lambda)}$, equipped with the natural action of $\mathfrak{S}_\lambda$ and with the natural morphism to $A^{(n+1)}$ by sending $(x_1, \cdots, x_{l(\lambda)})$ to $\sum_{j=1}^{l(\lambda)} \lambda_j [x_j]$. We define A^λ similarly; in addition A^λ comes equipped with the natural morphism $A^\lambda \to A^{(n+1)} := A^{n+1}/\mathfrak{S}_{n+1}, (x_1, \ldots, x_{l(\lambda)}) \mapsto \sum_i a_i x_i$. We denote the quotient $S^{(\lambda)} := S^\lambda/\mathfrak{S}_\lambda$ and we define the incidence correspondence $\Gamma^\lambda := (S^{[n+1]} \times_{S^{n+1}} S^\lambda)_{\mathrm{red}} \subset S^{[n+1]} \times S^\lambda$. The correspondence $\Gamma^{(\lambda)} \subset S^{[n+1]} \times S^{(\lambda)}$ is then the quotient $\Gamma^\lambda/\mathfrak{S}_\lambda$. The motive of the quotient $S^{(\lambda)}$ is thought of as the direct summand of the motive of S^λ with respect to the idempotent $\frac{1}{\ell(\lambda)!}\sum_{\sigma\in\mathfrak{S}_\lambda} \sigma$. When S is an abelian surface, this idempotent is symmetrically distinguished, while in the case when S is a smooth projective surface satisfying $(\star)$ it is distinguished (see Remark 3.2). In the case $S = A$, taking the fiber over 0 of the sum map $A^{n+1} \to A$ and of the sum map composed with the Hilbert–Chow morphism $A^{[n+1]} \to A^{(n+1)} \to A$, we define likewise $A_0^\lambda, A_0^{(\lambda)}, \Gamma_0^\lambda$, and $\Gamma_0^{(\lambda)}$.

Theorem 4.14 (de Cataldo and Migliorini)**.** *The morphisms of Chow motives Γ and Γ_0 are isomorphisms with inverses given respectively by*

$$\Gamma' := \sum_{\lambda\in\mathscr{P}(n+1)} \frac{1}{m_\lambda} {}^t\Gamma^\lambda \quad \textit{and} \quad \Gamma_0' := \sum_{\lambda\in\mathscr{P}(n+1)} \frac{1}{m_\lambda} {}^t\Gamma_0^\lambda,$$

where the superscript 't' indicates transposition, and where $m_\lambda := (-1)^{n+1-l(\lambda)} \prod_{i=1}^{l(\lambda)} \lambda_i$ is a non-zero constant.

Proof. The proof that the morphism (8) is bijective with inverse given by Γ' can be found in [14], and the proof that the morphism (9) is bijective with inverse given by Γ_0' can be found in [19, Corollary 6.3]. □

4.5.2. *Proof of Propositions 4.12 and 4.13.* The argument is based on Voisin's universally defined cycle theorem on self-products of surfaces [47, Theorem 5.12]. Let us write X for either (i) the Hilbert scheme of length-n subschemes on a surface S satisfying $(\star)$ (Proposition 4.12), or (ii) a generalized Kummer variety $K_n(A)$ (Proposition 4.13). We are going to show that the markings given by (8) and (9) satisfy $(\star)$. For that purpose, we have to show that the class of the diagonal Δ_X (resp. the class of the small diagonal δ_X, resp. the Chern classes of X) are mapped in case (i) to a distinguished cycle on S under the correspondence $\Gamma\otimes\Gamma$ (resp. $\Gamma\otimes\Gamma\otimes\Gamma$, resp. Γ), where Γ is the isomorphism (8), and in case (ii) to a symmetrically distinguished cycle on an a.t.t.s. under the correspondence $\Gamma_0\otimes\Gamma_0$ (resp. $\Gamma_0\otimes\Gamma_0\otimes\Gamma_0$, resp. Γ_0), where Γ_0 is the isomorphism (9).

In case (i), one argues as in [44, §3.2] or as in [19, Proposition 5.7]. The main idea is that, thanks to Voisin's theorem [47, Theorem 5.12], $\Gamma_* c_i(X)$, $(\Gamma\otimes\Gamma)_*\Delta_X$ and $(\Gamma\otimes\Gamma\otimes\Gamma)_*\delta_X$ are cycles that are polynomials in pull-backs along projections of Chern classes of S and the diagonal Δ_S. Since S is assumed to satisfy $(\star)$, diagonals and Chern classes are distinguished, and hence the above cycles are all distinguished.

In case (ii), this is achieved for the diagonals by arguing as in the proof of [19, Proposition 6.12] and for the Chern classes as in the proof of [19, Proposition 7.13]. We only show ($\star 2$). A key point is that the small diagonal $\delta_{K_n(A)}$ is the restriction of the small diagonal $\delta_{A^{[n+1]}}$ under the 3-fold product of the inclusion $K_n(A) \to A^{[n+1]}$. The proofs of ($\star 1$) and ($\star 3$) are similar once one has observed that the diagonal $\Delta_{K_n(A)}$ is the restriction of the diagonal $\Delta_{A^{[n+1]}}$ and that the Chern classes $c_i(K_n(A))$ are the restrictions of the Chern classes $c_i(A^{[n+1]})$. One cannot invoke Voisin's theorem directly here, and one has to utilize the commutativity of the following diagram, whose squares are all cartesian and without excess intersections,

$$\begin{tikzcd}
(A^{[n+1]})^3 \arrow[dr, phantom, "\Box"] & \Gamma^\lambda \times \Gamma^\mu \times \Gamma^\nu \arrow[l, "p''"'] \arrow[r, "q''"] \arrow[dr, phantom, "\Box"] & A^\lambda \times A^\mu \times A^\nu \\
(A^{[n+1]})^{3/A} \arrow[u, hook] \arrow[dr, phantom, "\Box"] & \Gamma^\lambda \times_A \Gamma^\mu \times_A \Gamma^\nu \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[u, hook] \arrow[dr, phantom, "\Box"] & A^\lambda \times_A A^\mu \times_A A^\nu \arrow[u, hook, "j"'] \\
K_n(A)^3 \arrow[u, hook] & \Gamma_0^\lambda \times \Gamma_0^\mu \times \Gamma_0^\nu \arrow[l, "p"'] \arrow[r, "q"] \arrow[u, hook] & A_0^\lambda \times A_0^\mu \times A_0^\nu \arrow[u, hook, "i"']
\end{tikzcd}$$

Here λ, μ, ν are partitions of $n+1$; all fiber products in the second row are over A; the second row is the base change by the inclusion of small diagonal $A \hookrightarrow A^3$ of the first row; the third row is the base change by $O_A \hookrightarrow A$ of the second row.

We need to show that $(\Gamma_0^\lambda \times \Gamma_0^\mu \times \Gamma_0^\nu)_*(\delta_{K_n(A)}) = q_* p^*(\delta_{K_n(A)})$ is symmetrically distinguished on the a.t.t.s. $A_0^\lambda \times A_0^\mu \times A_0^\nu$ for all partitions λ, μ, ν of $n+1$.

As in the proof of [19, Proposition 6.12], we have thanks to [19, Lemma 6.6] that $A^\lambda \times_A A^\mu \times_A A^\nu$ and $A_0^\lambda \times A_0^\mu \times A_0^\nu$ are naturally disjoint unions of a.t.t.s. and the inclusions i and j are morphisms of a.t.t.s. on each component.

Denote $\delta_{A^{[n+1]}/A}$ the small diagonal inside the relative product $(A^{[n+1]})^{3/A}$. Now by functorialities and the base change formula (*cf.* [21, Theorem 6.2]), we have

$$j_* \circ q'_* \circ p'^*(\delta_{A^{[n+1]}/A}) = q''_* \circ p''^*(\delta_{A^{[n+1]}}),$$

which is a polynomial of big diagonals of $A^{l(\lambda)+l(\mu)+l(\nu)}$ by Voisin's result [47, Proposition 5.6], thus symmetrically distinguished in particular. By [19, Lemma 6.10], $q'_* \circ p'^*(\delta_{A^{[n+1]}/A})$ is symmetrically distinguished on each component of $A^\lambda \times_A A^\mu \times_A A^\nu$. Again by functorialities and the base change formula, we have

$$q_* \circ p^*(\delta_{K_n(A)}) = i^* \circ q'_* \circ p'^*(\delta_{A^{[n+1]}/A}).$$

Since i is a morphism of a.t.t.s on each component, one concludes that $q_* \circ p^*(\delta_{K_n(A)})$ is symmetrically distinguished on each component, which concludes the proof. $\Box$

5. Link with multiplicative Chow–Künneth decompositions

A *Chow–Künneth decomposition* on a smooth projective variety X of dimension d is a set $\{\pi_X^i : 0 \leq i \leq 2d\}$ of mutually orthogonal idempotent correspondences in $X \times X$ that add up to Δ_X and whose cohomology classes in $H^{2d}(X \times X)$ are the components of the diagonal in $H^{2d-i}(X) \otimes H^i(X)$ for the Künneth decomposition. The notion of Chow–Künneth decomposition was introduced by Murre, who conjectured that all smooth projective varieties should admit such a decomposition

[33]. Murre's conjecture is intimately linked to the conjectures of Beilinson and Bloch ; *cf.* [2].

The notion of *multiplicative* Chow–Künneth decomposition was introduced in [39] and further studied in [19], [40], [44] and [18]. A Chow–Künneth decomposition $\{\pi^i_X : 0 \leq i \leq 2d\}$ on a smooth projective variety X of dimension d induces a bigrading decomposition of the Chow groups of self-powers of X via the formula

$$\mathrm{CH}^i(X^n)_{(j)} := (\pi^{2i-j}_{X^n})_*\mathrm{CH}^i(X^n), \tag{10}$$

where by definition X^n is endowed with the product Chow–Künneth decomposition

$$\pi^k_{X^n} := \sum_{k_1+\cdots+k_n=k} \pi^{k_1}_X \otimes \cdots \otimes \pi^{k_n}_X.$$

A Chow–Künneth decomposition $\{\pi^i_X : 0 \leq i \leq 2d\}$ is *self-dual* if $\pi^i_X = {}^t\pi^{2d-i}_X$ for all i ; this ensures that Δ_X belongs to $\mathrm{CH}^d(X \times X)_{(0)}$. A self-dual Chow–Künneth decomposition $\{\pi^i_X : 0 \leq i \leq 2d\}$ is *multiplicative* if δ_X belongs to $\mathrm{CH}^{2d}(X \times X \times X)_{(0)}$. This ensures in particular that $\mathrm{CH}^*(X)_{(0)}$ is a graded sub-algebra of $\mathrm{CH}^*(X)$. Finally, a natural condition that appeared in [40] is that the Chern classes of X belongs to $\mathrm{CH}^*(X)_{(0)}$. As is apparent from the previous sections, the theory for DCH^* is in every way similar to that of $\mathrm{CH}^*(-)_{(0)}$ (compare with [40]).

According to Murre's conjecture (D), for any choice of a Chow–Künneth decomposition $\{\pi^i_X : 0 \leq i \leq 2d\}$, we should have that the restriction of the projection morphism $\mathrm{CH}^*(X) \to \overline{\mathrm{CH}}^*(X)$ to $\mathrm{CH}^*(X)_{(0)}$ is an isomorphism ; see [33]. Thus conjecturally the existence of a self-dual multiplicative Chow–Künneth decomposition for X provides a splitting to the algebra homomorphism $\mathrm{CH}^*(X) \to \overline{\mathrm{CH}}^*(X)$, in the same that a marking that satisfies $(\star)$ does.

Proposition 5.1. *Let X be a smooth projective variety with a marking ϕ that satisfies $(\star 1)$ and $(\star 2)$. Then X has a self-dual multiplicative Chow–Künneth decomposition with the property that $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) \subseteq \mathrm{CH}^*(X^n)_{(0)}$. Moreover, equality holds if Murre's conjecture (D) is true.*

Proof. The proof of Proposition 3.1 shows that if X and Y are two smooth projective varieties endowed each with markings satisfying $(\star 1)$ and $(\star 2)$, then the product marking on $X \times Y$ also satisfies $(\star 1)$ and $(\star 2)$. Moreover, the graphs of the projection morphisms are distinguished for the product markings. Therefore, composition of distinguished correspondences are distinguished.

Let A be an abelian variety, and let $p \in \mathrm{DCH}(A \times A)$ be a symmetrically distinguished projector. The Deninger–Murre Chow–Künneth projectors π^i_A in [15] of A are symmetrically distinguished. Since the Chow–Künneth projectors are central modulo homological equivalence, we see that $p \circ \pi^i_A = \pi^i_A \circ p \in \mathrm{CH}^*(A \times A)$ and in particular that these provide distinguished Chow–Künneth projectors for (A, p).

It follows that, assuming X has a marking ϕ that satisfies $(\star 1)$ and $(\star 2)$, X admits a distinguished Chow–Künneth decomposition. We conclude that X has a self-dual multiplicative Chow–Künneth decomposition by noting that since a Künneth decomposition is always self-dual and multiplicative, any distinguished Chow–Künneth decomposition is self-dual and multiplicative.

Finally, the inclusion $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) \subseteq \mathrm{CH}^*(X^n)_{(0)}$ is due to the following three facts : the product Chow–Künneth decomposition $\{\pi^i_{X^n}\}$ is distinguished, the cycle $(\pi^i_{X^n})_*\alpha$ is homologically trivial (and hence numerically trivial) for all $\alpha \in \mathrm{CH}^j(X^n)$

and all $i \neq 2j$, and $(\pi^i_{X^n})_*\alpha$ is distinguished if α is. Murre's conjecture (D) for X^n stipulates that $\mathrm{CH}^i(X^n)_{(0)}$ should inject in cohomology via the cycle class map, and in particular that the surjective quotient morphism $\mathrm{CH}^i(X^n) \to \overline{\mathrm{CH}}^*(X^n)$ is an isomorphism when restricted to $\mathrm{CH}^i(X^N)_{(0)}$. Since the quotient morphism is surjective when restricted to $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n)$, Murre's conjecture implies $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) = \mathrm{CH}^*(X^n)_{(0)}$. $\square$

6. Varieties with motive of abelian type that do not satisfy ($\star$)

6.1. **The Ceresa cycle and the condition** ($\star$)**.** Let C be a smooth projective curve. In this section we give a necessary condition on the Ceresa cycle of C for C to admit a marking that satisfies ($\star$). In fact, we give a necessary condition on the Ceresa cycle of C for C to admit a self-dual multiplicative Chow–Künneth decomposition; see Proposition 5.1.

Fix a zero-cycle α of degree 1 on C, and denote $\iota : C \to J(C)$ the Abel–Jacobi map which maps a point $c \in C$ to the divisor class $[c] - \alpha$. We denote $[C]$ the class of the image of C under ι. Denote $[k] : J(C) \to J(C)$ the multiplication by k homomorphism. The *Ceresa cycle* is then the one-cycle $[C] - [-1]_*[C]$; it is numerically trivial, and its class modulo algebraic equivalence does not depend on the choice of the degree 1 zero-cycle α.

Proposition 6.1. *Let C be a smooth projective curve. If C has a self-dual multiplicative Chow–Künneth decomposition, then the Ceresa cycle is algebraically trivial.*

Proof. Since a smooth projective curve has finite-dimensional motive in the sense of Kimura [24], any idempotent that is homologically equivalent to the Künneth projector on $H^0(C)$ is rationally equivalent to $\alpha \times C$ for some zero-cycle α of degree 1. Thus if C has a self-dual multiplicative Chow–Künneth decomposition, it must be of the form $\pi^0_C := \alpha \times C$, $\pi^2_C := C \times \alpha$, $\pi^1_C := \Delta_C - \pi^0_C - \pi^2_C$ for some zero-cycle α of degree 1. According to [39, Proposition 8.14] this decomposition is multiplicative if and only if the modified diagonal cycle

$$\mathfrak{z} := \delta_C - \{(x,x,\alpha)\} - \{(x,\alpha,x)\} - \{(\alpha,x,x)\} + \{(x,\alpha,\alpha)\} + \{(\alpha,x,\alpha)\} + \{(\alpha,\alpha,x)\}$$

is zero in $\mathrm{CH}_1(C \times C \times C)$. Now we argue as in the proof of [9, Proposition 3.2]. Let $\iota : C \to J(C)$ be the Abel–Jacobi map which maps a point $c \in C$ to the divisor class $[c] - \alpha$, and let $\iota^3 : C^3 \to J(C)$ be the map deduced from ι by summation. We have

$$(\iota^3)_*(\mathfrak{z}) = [3]_*[C] - 3[2]_*[C] + 3[C] = 0 \quad \text{in } \mathrm{CH}_1(J(C)).$$

According to the Beauville decomposition [7], we have

$$\mathrm{CH}_1(J(C)) = \mathrm{CH}_1(J(C))_{(0)} \oplus \cdots \oplus \mathrm{CH}_1(J(C))_{(g-1)},$$

where g is the dimension of $J(C)$, and where $[k]_*$ acts on $\mathrm{CH}_1(J(C))_{(s)}$ by multiplication by k^{2+s}. Since $3^{2+s} - 3 \cdot 2^{2+s} + 3 > 0$ for $s > 0$, we find that $[C]$ belongs to $\mathrm{CH}_1(J(C))_{(0)}$. In particular, taking $k = -1$, we see that $[C] - [-1]_*[C] = 0$ in $\mathrm{CH}_1(J(C))$, and hence that the Ceresa cycle is algebraically trivial. $\square$

6.2. **A very general curve of genus > 2 does not satisfy ($\star$).** Although motives of curves are of abelian type, they do not necessarily have a marking that satisfies ($\star$):

Proposition 6.2. *Let C be a curve, and let α be a degree 1 zero-cycle on C. If C is very general of genus > 2, then the self-dual Chow–Künneth decomposition $\pi^0_C := \alpha \times C$, $\pi^2_C := C \times \alpha$, $\pi^1_C := \Delta_C - \pi^0_C - \pi^2_C$ is not multiplicative, and C does not satisfy ($\star$).*

Proof. Ceresa [12] proved that the Ceresa cycle of a very general curve of genus > 2 is algebraically non-trivial. The proposition follows then from Proposition 6.1 (together with Proposition 5.1). □

6.3. **The Fermat quartic curve does not satisfy ($\star$).**

Proposition 6.3. *Let C be a Fermat curve of degree d with $d \geq 4$, and let α be a zero-cycle of degree one on C. If $d \leq 1000$, then the self-dual Chow–Künneth decomposition $\pi^0_C := \alpha \times C$, $\pi^2_C := C \times \alpha$, $\pi^1_C := \Delta_C - \pi^0_C - \pi^2_C$ is not multiplicative, and C does not satisfy ($\star$).*

Proof. B. Harris [22] and S. Bloch [11] proved that the Ceresa cycle of quartic Fermat curves is algebraically non-trivial, and Otsubo [36] proved that the Ceresa cycle of Fermat curves of degree $4 \leq d \leq 1000$ is algebraically non-trivial. We can now apply Proposition 6.1 (together with Proposition 5.1). □

6.4. **Varieties with motive of abelian type that do not admit a section.** By considering a K3 surface of Picard rank ≥ 19, the following proposition provides a simple example of a variety X whose motive is of abelian type but for which the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ does not admit a section. In particular, by Proposition 2.11, such a variety X does not satisfy ($\star$).

Proposition 6.4. *Let S be a complex K3 surface, and let P be a point in S that is not rationally equivalent to the Beauville–Voisin zero-cycle. Denote $\widetilde{S}$ the blow-up of S along P. Then the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(\widetilde{S}) \twoheadrightarrow \overline{\mathrm{CH}}(\widetilde{S})$ does not admit a section.*

Proof. The theorem of Beauville–Voisin [9] asserts that $\mathrm{Im}(\mathrm{CH}^1(S) \otimes \mathrm{CH}^1(S) \to \mathrm{CH}^2(S))$ has rank one and is spanned by the class of any point lying on a rational curve on S. Such a class is called the Beauville–Voisin zero-cycle. Since $\dim_{\mathbb{Q}} \mathrm{CH}^2(S) = \infty$, there exists a point P on S whose class is not rationally equivalent to the Beauville–Voisin zero-cycle. It is then straightforward to check that $\mathrm{Im}(\mathrm{CH}^1(\widetilde{S}) \otimes \mathrm{CH}^1(\widetilde{S}) \to \mathrm{CH}^2(\widetilde{S}))$ has rank 2 and is spanned by the class of P and the Beauville–Voisin zero-cycle. Since $\mathrm{CH}^1(\widetilde{S}) \twoheadrightarrow \overline{\mathrm{CH}}^1(\widetilde{S})$ is an isomorphism, if $\mathrm{CH}(\widetilde{S}) \twoheadrightarrow \overline{\mathrm{CH}}(\widetilde{S})$ had a section, then $\mathrm{Im}(\mathrm{CH}^1(\widetilde{S}) \otimes \mathrm{CH}^1(\widetilde{S}) \to \mathrm{CH}^2(\widetilde{S}))$ would have rank 1 (equal to $\mathrm{rk}\, \overline{\mathrm{CH}}^2(\widetilde{S})$). This is a contradiction. □

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