THE GENERALIZED FRANCHETTA CONJECTURE FOR SOME HYPER-KÄHLER VARIETIES

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Abstract. We study the generalized Franchetta conjecture for holomorphic symplectic varieties. The conjecture predicts that the restriction of an algebraic cycle on the universal family of certain polarized hyper-Kähler varieties to a fiber is rationally equivalent to zero if and only if its cohomology class vanishes. We provide the following evidence : (1) The Beauville–Donagi family of Fano varieties of lines on cubic fourfolds ; (2) The relative square and the relative Hilbert square of the families of K3 surfaces which are complete intersections ; (3) Zero-cycles and codimension 2 cycles for the Lehn–Lehn–Sorger–van Straten family of hyper-Kähler eightfolds. We also draw some consequences in the direction of the Beauville–Voisin conjecture as well as Voisin's refinement for coisotropic subvarieties.

1. Introduction

The original Franchetta conjecture [9] (proved in [13], see also [16] and [2]) states the following :

Theorem 1.1 ([9], [13], [16], [2]). *Let $g \geq 2$ be an integer and $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus g. Denote by $\mathcal{M}_g^\circ \subset \mathcal{M}_g$ the Zariski open subset parametrizing those curves without nontrivial automorphisms and by $\mathcal{C} \to \mathcal{M}_g^\circ$ the universal curve. Then for any line bundle L on $\mathcal{C}$ and any $b \in \mathcal{M}_g^\circ$, the restriction of L to the fiber C_b is a multiple of the canonical bundle of C_b.*

In the case of the universal family of K3 surfaces $\mathcal{S} \to \mathcal{F}_g^\circ$, where $\mathcal{F}_g^o$ is the moduli space of polarized K3 surfaces of genus g without nontrivial automorphisms, O'Grady proposed in [17] the following analogue of the Franchetta conjecture. Recall that the Beauville–Voisin class ([5]) of a projective K3 surface S is the 0-cycle class $\mathfrak{o}_S$ represented by any point on a rational curve of the K3 surface. It enjoys the property that the intersections of any two divisors, as well as the second (Chow-theoretic) Chern class of S, are multiples of $\mathfrak{o}_S$.

Conjecture 1.2 (O'Grady [17]). *Notation is as above. Then for any algebraic cycle $z \in \mathrm{CH}^2(\mathcal{S})$ and any point $b \in \mathcal{F}_g^\circ$, the restriction of z to the fiber K3 surface S_b is a multiple of the Beauville–Voisin class of S_b.*

Using Mukai models, Conjecture 1.2 is verified in [18] for K3 surfaces with genus $g \leq 10$ and $g = 12, 13, 16, 18, 20$. Otherwise, Conjecture 1.2 is still wide open.

The main goal of the paper is to investigate the following higher dimensional analogue of O'Grady's Conjecture 1.2 concerning self-products of projective hyper-Kähler varieties. Recall that a smooth projective variety is called *hyper-Kähler* or *irreducible holomorphic symplectic*, if it is simply connected and $H^{2,0}$ is generated by a nowhere degenerate holomorphic 2-form.

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Conjecture 1.3 (Generalized Franchetta Conjecture, *cf.* [6]). *Let $\mathcal{F}$ be the moduli space of polarized hyper-Kähler varieties, $\mathcal{F}^\circ$ be its open subset parametrizing hyper-Kähler varieties with trivial automorphism groups and $\mathcal{X}^\circ \to \mathcal{F}^\circ$ be the universal family. For any $z \in \mathrm{CH}(\mathcal{X}^\circ)_{\mathbf{Q}}$, if its restrictions to the very general fibers are homologically trivial then its restriction to any fiber is (rationally equivalent to) zero.*

One could also be optimistic and expect that Conjecture 1.3 holds more generally for self-products of hyper-Kähler varieties, *i.e.* , for any $z \in \mathrm{CH}(\mathcal{X}^\circ \times_{\mathcal{F}^\circ} \cdots \times_{\mathcal{F}^\circ} \mathcal{X}^\circ)_{\mathbf{Q}}$, if its restrictions to the very general fibers are homologically trivial then its restriction to any fiber is (rationally equivalent to) zero.

Recently, Bergeron and Li [6] proved the cohomological version of the generalized Franchetta Conjecture 1.3 for relative 0-cycles when the second Betti number is sufficiently large, which is an important support in favor of the conjecture.

Let us also mention that Conjecture 1.3 is closely related to the so-called Beauville–Voisin conjecture and its refinement (see Conjecture 2.3 and 2.4). On the one hand, the proof of some of our main results actually uses some known cases of Beauville–Voisin conjecture ; on the other hand, the Generalized Franchetta Conjecture implies the part of Beauville–Voisin conjecture involving only Chern classes and the polarization, see Proposition 2.5.

We outline the main results of the paper, which provide more evidences to the Generalized Franchetta Conjecture 1.3.

1.1. **Squares and Hilbert squares of some K3 surfaces.** We can establish Conjecture 1.3 for the relative squares, as well as the relative Hilbert squares, of the universal family of K3 surfaces which are complete intersections.

Theorem 1.4. *Let B° be the parameter space of smooth K3 surfaces which are complete intersections in projective spaces and $\mathcal{S}^\circ \to B^\circ$ be the universal family. Let $\mathcal{X}^\circ$ be either $\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ$ or $\mathrm{Hilb}^2_{B^\circ} \mathcal{S}^\circ$. For any cycle $z \in \mathrm{CH}(\mathcal{X}^\circ)_{\mathbf{Q}}$ and any $b \in B^\circ$, the restriction of z to the fiber X_b is zero if and only if it is numerically trivial.*

1.2. **Beauville–Donagi family.** For the universal family of Fano varieties of lines of cubic fourfolds, which are hyper-Kähler fourfolds of $K3^{[2]}$-type ([4]), we have the following slightly stronger result than predicted by Conjecture 1.3 :

Theorem 1.5. *Let B° be the parameter space of smooth cubic fourfolds, $\mathcal{X}^\circ \to B^\circ$ be the universal family and $\mathcal{F}^\circ \to B^\circ$ be the universal family of Fano varieties of lines of the fibers of $\mathcal{X}^\circ/B^\circ$. Then for any $i \in \mathbf{N}$, any $z \in \mathrm{CH}^i(\mathcal{F}^\circ)_{\mathbf{Q}}$ and any $b \in B^\circ$, the restriction of z to the fiber F_b is numerically trivial if and only if it is (rationally equivalent to) zero.* [1]

In order to study the next case (Theorem 1.7), we also prove the following analogous result on the 'double' universal family of Fano varieties of lines :

Theorem 1.6. *Notation is as in Theorem 1.5. Then for any $i \neq 5$ or 6, $z \in \mathrm{CH}^i(\mathcal{F}^\circ \times_{B^\circ} \mathcal{F}^\circ)_{\mathbf{Q}}$ and any $b \in B^\circ$, the restriction of z to the fiber $F_b \times F_b$ is numerically trivial if and only if it is (rationally equivalent to) zero.*[2]

[1] In fact, we show that the restriction of $\mathrm{CH}^*(\mathcal{F}^\circ)_{\mathbf{Q}}$ to $\mathrm{CH}^*(F_b)_{\mathbf{Q}}$ is the *tautological* subring, which is the $\mathbf{Q}$-subalgebra generated by the Plüker polarization of F_b and by the Chern classes of F_b, see Remark 3.3.

[2] We actually show that the restriction of $\mathrm{CH}^i(\mathcal{F}^\circ \times_{B^\circ} \mathcal{F}^\circ)_{\mathbf{Q}}$ to $\mathrm{CH}(F_b \times F_b)_{\mathbf{Q}}$ is the *tautological* subring, which is the $\mathbf{Q}$-subalgebra generated by the tautological subrings of two factors together with the classes of the diagonal and the incidence subvariety. See Proposition 5.3.

Moreover, assuming the following conjectured relation (see Conjecture 5.4) in $\mathrm{CH}(F_b \times F_b)$

$$\Delta_*(g) = g_1^2 g_2 I + g_1 g_2^2 I + Q(g_1, g_2, c_1, c_2),$$

the same conclusion holds also for $i = 5$ *and 6.*

1.3. **Lehn–Lehn–Sorger–van Straten family.** Similarly to the Fano varieties of lines of cubic fourfolds, Lehn–Lehn–Sorger–van Straten (LLSvS) consider in [15] the twisted cubic curves on a cubic fourfold not containing a plane and show that the base of the maximal rationally connected quotient of the moduli space of such curves is a hyper-Kähler eightfold. Later Addington and M. Lehn show in [1] that this hyper-Kähler eightfold is of $K3^{[4]}$-deformation type. For the universal family of LLSvS hyper-Kähler eightfolds, we have the following result, which confirms the zero-cycle and codimension 2 cycle cases of the Generalized Franchetta Conjecture 1.3.

Theorem 1.7. *Let* $B^{\circ\circ}$ *be the parameter space of smooth cubic fourfolds not containing a plane and* $\mathcal{Z} \to B^{\circ\circ}$ *be the universal family of LLSvS hyper-Kähler eightfolds (*[15]*). Then*

(i) for any $b \in B^{\circ\circ}$ *and for any* $\gamma \in \mathrm{CH}^8(\mathcal{Z})$ *which is fiber-wise of degree 0, the restriction of* γ *to the fiber* Z_b *is (rationally equivalent to) zero.*
(ii) for any $b \in B^{\circ\circ}$ *and for any* $\gamma \in \mathrm{CH}^2(\mathcal{Z})$*, its restriction to the fiber* Z_b *is zero if and only if its cohomology class vanishes.*

As a consequence, we deduce a part of the Beauville–Voisin conjecture 2.3 as well as the refined Conjecture 2.4 for LLSvS eightfolds :

Corollary 1.8. *Given any smooth cubic fourfold X which does not contain a plane, let Z be the LLSvS hyper-Kähler eightfold associated to X. Denote by h the polarization class. Then the classes*

$$h^8, c_2h^6, c_2^2h^4, c_2^3h^2, c_2^4, c_4h^4, c_2c_4h^2, c_2^2c_4, c_6h^2, c_2c_6, c_4^2, c_8 \in \mathrm{CH}_0(Z)_{\mathbf{Q}}$$

are all proportional, where $c_i := c_i(T_Z)$ *is the i-th (Chow-theoretic) Chern class of the tangent bundle of Z. We call the generator of degree 1 in this one-dimensional subspace the* canonical 0-cycle class *or the* Beauville–Voisin class *of Z, denoted by* $\mathfrak{o}_Z$*.*
More strongly, let $R(Z)$ *be the* $\mathbf{Q}$*-subalgebra generated by the polarization class h, the Chern classes* c_i *together with the following classes of coisotropic subvarieties of Z :*

- *the embedded cubic fourfold* $X \subset Z$ *(*[15]*) ;*
- *the space of twisted cubic contained in a general hyperplane section of X (*[21]*) ;*
- *the fixed locus of the anti-symplectic involution* ι *of Z (*[14]*) ;*
- *the images by* ι *of all the above subvarieties.*

Then $R^8(Z) = \mathbf{Q} \cdot \mathfrak{o}_Z$*.*

Convention : All algebraic varieties are over the field of complex numbers. We work with Chow groups with rational coefficients.

2. General remarks

2.1. **Generic fiber *vs.* geometric fibers.** There is the following slightly different version of the generalized Franchetta conjecture for hyper-Kähler varieties :

Conjecture 2.1. *Let* $\mathcal{F}$ *be the moduli space of polarized hyper-Kähler varieties and* $\pi : \mathcal{X} \to \mathcal{F}$ *be the universal family. Denote by* $\mathcal{X}_\eta$ *the generic fiber of* π*, where* $\eta = \mathrm{Spec}(\mathbf{C}(\mathcal{F}))$*. Then the group* $\mathrm{CH}^*(\mathcal{X}_\eta)_{\mathrm{hom}}$ *is torsion.*

Here homological equivalence is with respect to some classical Weil cohomology, for instance, étale cohomology or de Rham cohomology.

Lemma 2.2. *Conjecture 1.3 implies Conjecture 2.1.*

Proof. Since $\mathrm{CH}(\mathcal{X}_\eta) = \varinjlim \mathrm{CH}(\mathcal{X}_U)$, where U runs through all non-empty Zariski open subset of the moduli space $\mathcal{F}$, for a given cycle $z_\eta \in \mathrm{CH}(\mathcal{X}_\eta)$, we can assume it comes from the universal family: $z \in \mathrm{CH}(\mathcal{X}^\circ)$. Using [22, Lemma 2.1], the hypothesis that the restriction of z to the geometric generic fiber is homologically trivial implies that the restriction of z to every very general geometric fiber is also trivial. Now the conclusion of Conjecture 1.3 says that the restriction of z to a very general geometric fiber is (rationally equivalent to) zero. By the standard argument of decomposition of the diagonal ([7], [23], [26]), this implies the existence of a Zariski open dense subset $U \subset \mathcal{F}$, such that $z|_{\mathcal{X}_U}$ is zero. In particular, z_η is rationally equivalent to zero. □

By Lemma 2.2, the previous version of the Franchetta conjecture is *a priori* weaker ; we will therefore focus in this paper on Conjecture 1.3.

2.2. **Relation to Beauville–Voisin conjecture.** As is mentioned in the introduction, the Generalized Franchetta Conjecture 1.3 is very much related to the following Beauville–Voisin conjecture :

Conjecture 2.3 (Beauville–Voisin Conjecture, [3], [24])**.** *Let X be a projective hyper-Kähler variety. Let the* Beauville–Voisin subring $\langle c_i(X), Pic(X)\rangle$ *be the* $\mathbf{Q}$*-subalgebra of* $\mathrm{CH}^*(X)$ *generated by line bundles and all (Chow theoretic) Chern classes of T_X. Then the restriction of the cycle class map to the Beauville–Voisin subring is injective. In other words, any polynomial of line bundles and Chern classes of X is homologically equivalent to zero if and only if it is rationally equivalent to zero.*

The original version due to Beauville in [3], under the name of *weak splitting property*, contains only line bundles ; the Chern classes of the tangent bundle are introduced by Voisin in [24]. There are recently some active progress towards this conjecture : see [3], [24], [10], [28], [19], [12] for the known results and more details. More recently, Voisin [27] proposes the following stronger version of Conjecture 2.3 involving the coisotropic subvarieties (in particular lagrangian subvarieties):

Conjecture 2.4 (Voisin's refinement [27])**.** *Let X be a projective hyper-Kähler variety. Then the restriction of the cycle class map to the* $\mathbf{Q}$*-subalgebra of* $\mathrm{CH}(X)_{\mathbf{Q}}$ *generated by line bundles, Chern classes of T_X and coisotropic subvarieties, is injective.*

We would like to point out that the generalized Franchetta conjecture implies the part of the Beauville–Voisin conjecture involving only the Chern classes of the tangent bundle and the polarization class. More generally it actually implies part of the refined Conjecture 2.4 once taking into account coisotropic subvarieties which are defined universally over the moduli space (see Corollary 1.8 for an example) :

Proposition 2.5. *Let $\mathcal{F}$ be a moduli space of polarized hyper-Kähler varieties. If Conjecture 1.3 holds true for the universal family over $\mathcal{F}$, then for any member X of this family, the cycle class map restricted to the* $\mathbf{Q}$*-subalgebra generated by the polarization line bundle and the Chern classes of X, is injective.*
More generally, still assuming Conjecture 1.3, for any member X of this family, the cycle class map restricted to the $\mathbf{Q}$*-subalgebra generated by the algebraic cycles of X that exist universally over the moduli space, is injective.*

Proof. Let $\mathcal{X}^\circ \to \mathcal{F}^\circ$ be the universal family, where $\mathcal{F}^\circ \subset \mathcal{F}$ is an open subset. For any member X and any given polynomial of polarization line bundle and Chern classes of the tangent bundle

$z := P(h, c_i(T_X)) \in \mathrm{CH}(X)_{\mathbf{Q}}$ such that the cohomology class of z vanishes, we want to show that $z = 0$. Consider $\gamma := P(h, c_i(T_{\mathcal{X}^\circ/\mathcal{F}^\circ})) \in \mathrm{CH}(\mathcal{X}^\circ)$. If X belongs to $\mathcal{F}^\circ$, then clearly $\gamma|_X = z$ and hence γ has fiber-wise vanishing cohomology class. Then the generalized Franchetta conjecture 1.3 says exactly that z is rationally equivalent to zero. If X does not belong to $\mathcal{F}^\circ$, the specialization argument for algebraic cycles allows us to conclude. The last assertion is more or less tautological. □

2.3. **Moduli space *vs.* parameter space.**

Remark 2.6. To show the Generalized Franchetta Conjecture 1.3 in some cases, it will be convenient to work over some parameter space which dominates the moduli space, instead of the moduli space itself. More precisely, keep the same notation as in Conjecture 1.3 and let $B \to U$ be a surjective morphism to some Zariski dense open subset U of the moduli space $\mathcal{F}^\circ$. Denote by $\mathcal{Y} \to B$ the pulled-back family of the universal family $\mathcal{X}^\circ \to \mathcal{F}^\circ$. Then the generalized Franchetta conjecture for $\mathcal{Y} \to B$ implies the generalized Franchetta conjecture for $\mathcal{X}^\circ \to \mathcal{F}^\circ$.

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}^\circ \arrow[d] \\ B \arrow[r] & U \arrow[r, hook] & \mathcal{F}^\circ \end{tikzcd}$$

Indeed, for any $z \in \mathrm{CH}(\mathcal{X}^\circ)$, denote by $z' \in \mathrm{CH}(\mathcal{Y})$ its pulled-back. Obviously, the hypothesis that the restriction of z to a very general fiber of $\mathcal{X}^\circ/\mathcal{F}^\circ$ is homologically trivial implies the same thing for the restriction of z' to the fibers of $\mathcal{Y}/B$. The generalized Franchetta conjecture for $\mathcal{Y}/B$ then implies that z' restricts to zero on each fiber of $\mathcal{Y}/B$. Hence so is z for each fiber of $\mathcal{X}_U \to U$. A specialization argument shows that the same thing holds for each fiber of $\mathcal{X}^\circ \to \mathcal{F}^\circ$.

3. Fano varieties of lines of cubic fourfolds

In this section, we prove Theorem 1.5, which by Remark 2.6 confirms the Generalized Franchetta Conjecture 1.3 for the 20-dimensional complete family of hyper-Kähler fourfolds constructed by Beauville–Donagi in [4]. The key idea of the proof is as in [25] and [18] : the universal family has very simple Chow groups.

We start by setting up some notations. Let V be a 6-dimensional vector space and $\mathbf{P}^5 = \mathbf{P}(V)$ be its projectivization. The parameter space of possibly singular cubic fourfolds is given by the following projective space :

$$B := \mathbf{P}\left(H^0(\mathbf{P}^5, \mathcal{O}(3))\right) = \mathbf{P}(\mathrm{Sym}^3 V^\vee) \simeq \mathbf{P}^{55}.$$

Let $B^\circ \subset B$ be the open subset parameterizing smooth cubic fourfolds. We thus have the universal family $\mathcal{X} \to B$ as well as the smooth family $\mathcal{X}^\circ \to B^\circ$ by base-change.

Let $G := \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5) \simeq \mathrm{Gr}(2, 6)$ be the Grassmannian variety parameterizing all projective lines in $\mathbf{P}^5$. Denote by S (*resp.* Q) the tautological rank 2 subbundle (*resp.* rank 4 quotient bundle), fitting into the following short exact sequences of vector bundles over G :

$$0 \to S \to \mathcal{O}_G \otimes V \to Q \to 0.$$

Note that for any equation $f \in \mathrm{Sym}^3 V^\vee$, the above short exact sequence gives a section s_f of the vector bundle $\mathrm{Sym}^3 S^\vee$, whose zero locus $(s_f = 0)$ is exactly the Fano variety of lines of the cubic fourfold defined by f.

Consider the incidence subvariety $\mathcal{F}$ in $B \times G$ defined by

$$\mathcal{F} := \{([f], l) \in B \times G \mid f|_l = 0\},$$

together with the two natural projections:

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ B & & G \end{tikzcd}$$

It is easy to see that $\pi : \mathcal{F} \to B$ is the universal Fano varieties of lines of fibers of $\mathcal{X}/B$ and that $p : \mathcal{F} \to G$ is a projective bundle whose fiber over a line $l \in G$ parametrizes all (possibly singular) cubic fourfolds passing through l.

As in [18, Lemma 2.1], we have the following:

Lemma 3.1. *For any $b \in B$, the following two images of restriction maps are the same:*

$$\mathrm{Im}\,(\mathrm{CH}(\mathcal{F}) \to \mathrm{CH}(F_b)) = \mathrm{Im}\,(\mathrm{CH}(G) \to \mathrm{CH}(F_b))\,.$$

Proof. The inclusion "$\supset$" is trivial (we have the factorization $F_b \hookrightarrow \mathcal{F} \to G$).
Let us show the inverse inclusion. Given any cycle $z \in \mathrm{CH}(\mathcal{F})$, by the projective bundle formula,

$$z = \sum_{k \geq 0} p^*(z_k) \cdot \xi^k,$$

where $z_k \in \mathrm{CH}(G)$ and $\xi = c_1\left(\mathcal{O}_p(1)\right)$. As in [18, Lemma 2.1], we easily check that ξ is a linear combination of cycles pulled back from B by π and cycles pulled back from G by p. Hence z is a polynomial of cycles of the form $p^*(\alpha)$ and $\pi^*(\beta)$. The latter type being zero when restricted to any fiber F_b, the restriction of z to F_b is therefore the restriction of some cycle of G. □

Lemma 3.2. *For any $b \in B^\circ$,*

$$\mathrm{Im}\,(\mathrm{CH}(G) \to \mathrm{CH}(F_b)) \subset \langle c_i(F_b), \mathrm{Pic}(F_b)\rangle,$$

where the right hand side is the Beauville–Voisin ring of $\mathrm{CH}(F_b)$ *generated (as a* $\mathbf{Q}$*-algebra) by line bundles and Chern classes of the tangent bundle.*

Proof. Since $\mathrm{CH}(G)$ is generated (as a $\mathbf{Q}$-algebra) by $c_1(S^\vee)$ and $c_2(S^\vee)$, it suffices to show that both of their restrictions to F_b lies in the Beauville–Voisin ring. The first one being a line bundle, it remains to show $c_2(S^\vee|_{F_b}) \in \langle c_i(F_b), \mathrm{Pic}(F_b)\rangle$. However, using the short exact sequence

$$0 \to T_{F_b} \to T_G|_{F_b} \to \mathrm{Sym}^3 S^\vee|_{F_b} \to 0$$

together with $T_G \simeq S^\vee \otimes Q$, one finds that

$$ch(T_{F_b}) = ch(S^\vee|_{F_b})\left(6 - ch(S|_{F_b})\right) - ch\left(\mathrm{Sym}^3 S^\vee|_{F_b}\right),$$

and hence $c_2(T_{F_b}) = -ch_2(T_{F_b}) = 5c_1(S^\vee|_{F_b})^2 - 8c_2(S^\vee|_{F_b})$. Therefore, $c_2(S^\vee|_{F_b})$ also belongs to the Beauville–Voisin ring.[3] □

We can now easily conclude:

[3]The class $c_2(S^\vee|_{F_b})$ is the class that Claire Voisin calls c in [24].

Proof of Theorem 1.5. For any $z \in \mathrm{CH}(\mathcal{F}^\circ)$, by surjectivity of $\mathrm{CH}(\mathcal{F}) \to \mathrm{CH}(\mathcal{F}^\circ)$, we can actually assume $z \in \mathrm{CH}(\mathcal{F})$. For any $b \in B^\circ$, thanks to Lemma 3.1, $z|_{F_b}$ is the restriction of some cycle from G, which must lie in the Beauville–Voisin ring $\langle c_i(F_b), \mathrm{Pic}(F_b)\rangle$ by Lemma 3.2. Now the equivalence between homological triviality and rational triviality of $z|_{F_b}$ is a consequence of Voisin's result [24, Theorem 1.4(ii)] saying that the cycle class map restricted to the Beauville–Voisin ring is injective. Finally, numerical equivalence and homological equivalence coincide for Fano varieties of lines of cubic fourfolds by [8]. □

Remark 3.3. In fact, the above proof shows that the restriction of a cycle $z \in \mathrm{CH}(\mathcal{F}^\circ)$ to a fiber Fano variety of lines F is in the so-called *tautological* ring $R^*(F)$, which is the $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(F)$, in general smaller than the Beauville–Voisin ring, generated by the Plücker polarization class g and the Chern classes of F. In particular,

- $R^1(F) = \mathbf{Q} \cdot g$;
- $R^2(F) = \mathbf{Q} \cdot g^2 \oplus \mathbf{Q} \cdot c_2$;
- $R^3(F) = \mathbf{Q} \cdot g^3$ (by [24, Lemma 3.5] gc_2 and g^3 are proportional) ;
- $R^4(F) = \mathbf{Q} \cdot \mathfrak{o}_F$, where $\mathfrak{o}_F$ is the canonical 0-cycle class and c_2^2, c_4, g^4, g^2c_2 are all proportional to it by [24, Lemma 3.2].

4. Hilbert squares of complete intersection K3 surfaces

In this section, we prove Theorem 1.4. There are three families of complete intersection K3 surfaces, namely, quartic surfaces in $\mathbf{P}^3$, complete intersections of quadrics and cubics in $\mathbf{P}^4$ and complete intersections of three quadric hypersurfaces in $\mathbf{P}^5$.

Let us fix some notations : in each of the three cases

- $\mathbf{P} := \mathbf{P}^3, \mathbf{P}^4$ *resp.* $\mathbf{P}^5$ is the ambient projective space ;
- $E := \mathcal{O}_{\mathbf{P}}(4), \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(3)$, *resp.* $\mathcal{O}_{\mathbf{P}}(2)^{\oplus 3}$ is the relevant vector bundle ;
- $B := \mathbf{P}H^0(\mathbf{P}, E)$ is the parameter (projective) space and B° is the open subset parametrizing smooth K3 surfaces.
- $\mathcal{S} := \{(x, [s]) \in \mathbf{P} \times B \mid s(x) = 0\}$ be the universal family.

We have therefore the natural projections, where p is clearly a projective bundle :

$$\begin{tikzcd} \mathcal{S} \arrow[r, "p"] \arrow[d, "\pi"'] & \mathbf{P} \\ B & \end{tikzcd} \tag{1}$$

Similarly, the relative square and the open complement of the relative diagonal in it fit into the following diagram

$$\begin{tikzcd} \mathcal{S}\times_{\mathcal{B}}\mathcal{S}\backslash\Delta_{\mathcal{S}/B} \arrow[r, "q'"] \arrow[d, hook, "j"'] & \mathbf{P}\times\mathbf{P}\backslash\Delta_{\mathbf{P}} \arrow[d, hook] \\ \mathcal{S}\times_{\mathcal{B}}\mathcal{S} \arrow[r, "q:=(p,p)"] \arrow[d, "\pi_2:=(\pi,\pi)"'] & \mathbf{P}\times\mathbf{P} \\ B & \end{tikzcd} \tag{2}$$

Note that although q itself is not a projective bundle, its restriction q' is. Let ξ be the first Chern class of $\mathcal{O}_{q'}(1)$. The relative diagonal $\Delta_{\mathcal{S}/B}$ being of codimension 2, ξ extends uniquely to the whole $\mathcal{S}\times_B\mathcal{S}$, which we still denote by ξ by abuse of notation.

We can show the analogue of Lemma 3.1 in our case :

Proposition 4.1. *For any $b\in B$, we have:*

$$\mathrm{Im}\,(\mathrm{CH}(\mathcal{S}\times_B\mathcal{S})\to\mathrm{CH}(S_b\times S_b))=\mathrm{Im}\,(\mathrm{CH}(\mathbf{P}\times\mathbf{P})\to\mathrm{CH}(S_b\times S_b))+\Delta_*\,\mathrm{Im}\,(\mathrm{CH}(\mathbf{P})\to\mathrm{CH}(S_b))\,,$$

where $\Delta: S_b\hookrightarrow S_b\times S_b$ is the diagonal embedding.

Proof. Notation is as in Diagrams (1) and (2). By base-change, it is easy to see that the right-hand side is contained in the left-hand side. Concerning the inverse inclusion, the projective bundle formula gives, for any $z\in\mathrm{CH}(\mathcal{S}\times_B\mathcal{S})_{\mathbf{Q}}$,

$$j^*(z)=\sum_{k\geq 0}q'^*(z_k)\cdot\xi^k,$$

for some cycles $z_k\in\mathrm{CH}(\mathbf{P}\times\mathbf{P}\setminus\Delta_{\mathbf{P}})$. As in Lemma 3.1, it is easy to see that $\xi=j^*\pi_2^*(h)+q'^*(\alpha)$, where $h=c_1(\mathcal{O}_B(1))$ and $\alpha\in\mathrm{CH}(\mathbf{P}\times\mathbf{P}\setminus\Delta_{\mathbf{P}})$. For each k, we denote still by $z_k\in\mathrm{CH}(\mathbf{P}\times\mathbf{P})$ its closure and similarly for α. Therefore, we have

$$z-\sum_k q^*(z_k)\cdot\left(\pi_2^*(h)-q^*(\alpha)\right)^k\in\mathrm{Ker}(j^*).$$

By the localization sequence, there exists $\gamma\in\mathrm{CH}(\mathcal{S})$, such that

$$z-\sum_k q^*(z_k)\cdot\left(\pi_2^*(h)-q^*(\alpha)\right)^k=\Delta_*(\gamma),\tag{3}$$

where $\Delta:\mathcal{S}\hookrightarrow\mathcal{S}\times_B\mathcal{S}$ is the diagonal embedding.

Since $p:\mathcal{S}\to\mathbf{P}$ is also a projective bundle with $c_1(\mathcal{O}_p(1))=\pi^*(h)$, we have

$$\gamma=\sum_l p^*(\gamma_l)\cdot\pi^*(h)^l,$$

for some $\gamma_l\in\mathrm{CH}(\mathbf{P})$. Substituting this into (3), we get

$$z=\sum_k q^*(z_k)\cdot\left(\pi_2^*(h)-q^*(\alpha)\right)^k+\sum_l\Delta_*(p^*(\gamma_l)\cdot\pi^*(h)^l),\tag{4}$$

Now for any $b\in B$, the restriction $z|_{S_b\times S_b}$ is of the desired form simply because the restrictions of $\pi_2^*(h)$ and $p^*(h)$ to the fibers vanish. □

We can now prove Theorem 1.4.

Proof of Theorem 1.4 for relative squares. Keep the same notations as before. Thanks to Proposition 4.1, we only need to show that for any smooth complete intersection K3 surface $S\subset\mathbf{P}$, the cycle class map restricted to $\mathrm{Im}\,(\mathrm{CH}(\mathbf{P}\times\mathbf{P})\to\mathrm{CH}(S\times S))+\Delta_*\,\mathrm{Im}\,(\mathrm{CH}(\mathbf{P})\to\mathrm{CH}(S))$ is injective. Denote $H:=c_1(\mathcal{O}_{\mathbf{P}}(1))$ and $h:=H|_S$. Since $\mathrm{CH}(\mathbf{P}\times\mathbf{P})$ is generated by $\mathrm{pr}_1^*(H)$ and $\mathrm{pr}_2^*(H)$, and $\Delta_*(h)=h\times\mathfrak{o}_S+\mathfrak{o}_S\times h$ (see [5]), it is enough to show that the cycle class map of $S\times S$ restricted to the subalgebra generated by $\mathrm{pr}_1^*(h),\mathrm{pr}_2^*(h)$ and Δ is injective. It is the easiest case of Voisin's [24, Proposition 2.2]. □

Proof of Theorem 1.4 for relative Hilbert squares. Consider the blow-up of $\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ$ along the relative diagonal, the natural involution switching two factors lifts to the blow-up. It is well-known that the Hilbert square is the quotient of this lifted involution and

$$\mathrm{CH}^*(\mathrm{Hilb}^2_{B^\circ}(\mathcal{S}^\circ)) \simeq \mathrm{CH}^*(\mathrm{Bl}_\Delta(\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ)) \simeq \mathrm{CH}^*(\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ)^{inv} \oplus \mathrm{CH}^{*-1}(\mathcal{S}^\circ),$$

where all isomorphisms are compatible with the restriction to the fibers. Therefore for any $b \in B^\circ$ and any $z \in \mathrm{CH}^*(\mathrm{Hilb}^2_{B^\circ}(\mathcal{S}^\circ))$, its restriction to the fiber $z|_{\mathrm{Hilb}^2 S_b}$, viewed as an element in $\mathrm{CH}^*(S_b \times S_b)^{inv} \oplus \mathrm{CH}^{*-1}(S_b)$, lives in $\mathrm{Im}(\mathrm{CH}(\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ)^{inv} \to \mathrm{CH}^*(S_b \times S_b)^{inv}) \oplus \mathrm{Im}(\mathrm{CH}^{*-1}(\mathcal{S}^\circ) \to \mathrm{CH}^{*-1}(S_b))$. We can thus conclude thanks to the established cases of the generalized Franchetta conjecture for the relative squares $\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ$ and for $\mathcal{S}$. □

5. Lehn-Lehn-Sorger-van Straten hyper-Kähler eightfold

In this section we first show Theorem 1.6 and then deduce from it Theorem 1.7.

Keep the same notation as in §3. We still have a correspondence :

$$\begin{tikzcd} & \mathcal{F} \times_B \mathcal{F} \arrow[dl, "\pi_2:=(\pi,\pi)"'] \arrow[dr, "q:=(p,p)"] & \\ B & & G \times G \end{tikzcd}$$

However the problem is that q is no-longer a projective bundle : the fiber of q over a pair of lines (l, l') is the subspace of cubic fourfolds passing through both l and l', whose dimension depends therefore on the relative position of (l, l'). To adapt the same strategy to this case, we use similar techniques as in [25], [11] by studying the various strata of the morphism q. There are three possible relative positions between two projective lines in $\mathbf{P}^5$: identical, intersect but not identical, not intersect.

On the one hand, for a (general) cubic fourfold X with Fano variety of lines F. Let

$$I := \{(l, l') \in F \times F \mid l \cap l' \neq \varnothing\}$$

be the 6-dimensional incidence subvariety of $F \times F$. The incidence subvariety I has two natural projections to F with fiber over $l \in F$ the surface S_l parameterizing lines inside X meeting l. Similarly, we consider the family version of this incidence subvariety inside $\mathcal{F} \times_B \mathcal{F}$:

$$\mathcal{I} := \{(b, l, l') \in F \times_B F \mid l \cap l' \neq \varnothing\} = \{(b, l, l') \in B \times G \times G \mid l, l' \subset X_b\, ; l \cap l' \neq \varnothing\}.$$

On the other hand, we define $J := \{(l, l') \in G \times G \mid l \cap l' \neq \varnothing\}$ to be the incidence subvariety of $G \times G$.

These incidence subvarieties, together with the diagonals, give the stratification for q :

$$\begin{tikzcd} \Delta_{\mathcal{F}} \arrow[r, hook] \arrow[d, "p"] & \mathcal{I} \arrow[r, hook] \arrow[d, "q|_{\mathcal{I}}"] & \mathcal{F} \times_B \mathcal{F} \arrow[r, "\pi_2"] \arrow[d, "q"] & B \\ \Delta_G \arrow[r, hook] & J \arrow[r, hook] & G \times G & \end{tikzcd}$$

where q is a projective bundle outside of $\mathcal{I}$ and $q|_{\mathcal{I}}$ is also a projective bundle outside of $\Delta_{\mathcal{F}}$.

We have the analogue of Lemma 3.1 and Proposition 4.1 in our case :

Proposition 5.1. *For any $b \in B$, we have*

$$\begin{aligned} & \mathrm{Im}\,(\mathrm{CH}(\mathcal{F} \times_B \mathcal{F}) \to \mathrm{CH}(F_b \times F_b)) \\ = \; & \mathrm{Im}\,(\mathrm{CH}(G \times G) \to \mathrm{CH}(F_b \times F_b)) + i_* \mathrm{Im}\,(\mathrm{CH}(J) \to \mathrm{CH}(I_b)) + \Delta_* \mathrm{Im}\,(\mathrm{CH}(G) \to \mathrm{CH}(F_b)), \end{aligned}$$

where $i : I_b \hookrightarrow F_b \times F_b$ *and* $\Delta : F_b \hookrightarrow F_b \times F_b$ *are the inclusions.*

Proof. The proof is similar to that of Proposition 4.1. $\square$

As the incidence subvariety J is singular along the smaller stratum Δ_G, it is more convenience to work with a natural resolution of singularities. To this end, we define

$$\begin{aligned}
\widetilde{\mathcal{I}} &:= \left\{(b,x,l,l') \in B \times \mathbf{P}^5 \times G \times G \mid l,l' \subset X_b \; ; \; x \in l \cap l'\right\} ; \\
\widetilde{J} &:= \left\{(x,l,l') \in \mathbf{P}^5 \times G \times G \mid x \in l \cap l'\right\} ; \\
\mathcal{P} &:= \left\{(b,x,l) \in B \times \mathbf{P}^5 \times G \mid l \subset X_b \; ; \; x \in l\right\} ; \\
Q &:= \left\{(x,l) \in \mathbf{P}^5 \times G \mid x \in l\right\},
\end{aligned}$$

where $\widetilde{\mathcal{I}}$ (*resp.* $\widetilde{J}$) admits natural birational morphism to $\mathcal{I}$ (*resp.* J), which contracts $\mathcal{P}$ (*resp.* Q) to $\mathcal{F}$ (*resp.* G). We summarize the situation in the following diagram whose squares are all cartesian :

$$\begin{tikzcd}
\mathcal{F} \arrow[d, two heads, "p"] \arrow[dr, phantom, "\square"] & \mathcal{P} \arrow[l, two heads] \arrow[r, hook] \arrow[d, two heads, "q'|_{\mathcal{P}}"] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{I}} \arrow[r] \arrow[d, two heads, "q'"] \arrow[dr, phantom, "\square"] & \mathcal{I} \arrow[r, hook] \arrow[d, two heads, "q|_{\mathcal{I}}"] \arrow[dr, phantom, "\square"] & \mathcal{F} \times_B \mathcal{F} \arrow[d, two heads, "q"] \\
G & Q \arrow[l, two heads] \arrow[r, hook] & \widetilde{J} \arrow[r] & J \arrow[r, hook] & G \times G
\end{tikzcd}$$

Recall that $G = \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$, S is the tautological rank 2 sub-bundle, $g := c_1(S^\vee|_F) \in \mathrm{CH}^1(F)$ is the Plücker polarization class, and $c := c_2(S^\vee|_F) \in \mathrm{CH}^2(F)$. We computed in Lemma 3.2 that $c_2(F) = 5g^2 - 8c$. In $\mathrm{CH}(F \times F)$, $g_i := \mathrm{pr}_i^*(g)$ and $c_i := \mathrm{pr}_i^*(c)$ for $i = 1, 2$.

Definition 5.2 (Tautological ring of $F \times F$)**.** Let X be a smooth cubic fourfold and F be its Fano variety of lines. We define the *tautological ring* of $F \times F$, denoted by $R(F \times F)$, to be the $\mathbf{Q}$-subalgebra of $\mathrm{CH}(F \times F)$ generated by the classes $c_1, c_2, g_1, g_2, \Delta, I$, where Δ and I are the class in $\mathrm{CH}(F \times F)$ of the diagonal Δ_F and the incidence subvariety I respectively.

Proposition 5.3. *For any point* $b \in B^\circ$*, we have*

$$\mathrm{Im}\,(\mathrm{CH}(\mathcal{F} \times_B \mathcal{F}) \to \mathrm{CH}(F_b \times F_b)) \subset R(F_b).$$

Proof. To simplify the notation, let us leave out the subscript b. Thanks to Proposition 5.1, we only need to deal with the following three cases :

- For $\mathrm{Im}\,(\mathrm{CH}(G \times G) \to \mathrm{CH}(F \times F))$, it is enough to observe that $\mathrm{CH}(G \times G)$ satisfies the Künneth formula (since the cycle class map $\mathrm{CH}(G \times G) \to H(G \times G, \mathbf{Q})$ is an isomorphism).
- For $i_* \mathrm{Im}\,(\mathrm{CH}(J) \to \mathrm{CH}(I))$, consider

$$\widetilde{I} := \{(x,l,l') \in X \times G \times G \mid x \in l \cap l'\} \quad \text{and}$$

$$\widetilde{J} := \left\{(x,l,l') \in \mathbf{P}^5 \times G \times G \mid x \in l \cap l'\right\}$$

fitting into the diagram

$$\begin{tikzcd}
\widetilde{I} \arrow[r, "\tau'"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & I \arrow[r, hook, "i"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & F \times F \arrow[d, hook] \\
\widetilde{J} \arrow[r, "\tau"] \arrow[d, "\pi"] & J \arrow[r, hook, "j"] & G \times G \\
\mathbf{P}^5 & &
\end{tikzcd}$$

Denote by $\widetilde{i} = \tau' \circ i$ and $\widetilde{j} = \tau \circ j$. Then any cycle in J can be written as $\tau_*(\alpha)$ for some $\alpha \in \mathrm{CH}(\widetilde{J})$. Observe that $\widetilde{J}$ is a $\mathbf{P}^4 \times \mathbf{P}^4$-bundle over $\mathbf{P}^5$ such that the two relative $\mathcal{O}(1)$ on the fibers are given by $\widetilde{j}^*(g_1)$ and $\widetilde{j}^*(g_2)$, respectively. Therefore α is a linear combination of cycles of the form $\pi^*(h^k)\widetilde{j}^*(c_1^l c_2^m)$ where $k, l, m \in \mathbf{N}$ and $h = \mathcal{O}_{\mathbf{P}^5}(1)$.

$$\begin{aligned}
& i_*(\tau_*(\pi^*(h^k)\widetilde{j}^*(c_1^l c_2^m))|_I) \\
= \; & i_* \circ \tau'_* \left(\pi^*(h^k)\widetilde{j}^*(c_1^l c_2^m)|_{\widetilde{I}}\right) \\
= \; & \widetilde{i}_* \left(\pi^*(h^k)|_{\widetilde{I}} \cdot \widetilde{i}^*(c_1^l c_2^m)\right) \\
= \; & c_1^l c_2^m \cdot i_*(\tau_* \pi^*(h^k)|_I) \\
= \; & c_1^l c_2^m \cdot \Gamma_{h^k}
\end{aligned}$$

where Γ_{h^k}, defined in [20, Appendix A] is the cycle of $F \times F$ represented by the subvariety $\{(l, l') \in F \times F \mid \exists x \in H_1 \cap \cdots \cap H_k \text{ such that } x \in l \cap l'\}$, where $H_1, \cdots, H_k$ are k general hyperplanes in $\mathbf{P}^5$. It is proved in [20, Appendix A] that when $k \geq 1$, Γ_{h^k} is actually a polynomial of c_1, c_2, g_1, g_2, while $\Gamma_{h^0} = I$.

- For $\Delta_* \operatorname{Im}(\mathrm{CH}(G) \to \mathrm{CH}(F))$, let us remark that for any $\alpha \in \mathrm{CH}(F)$, we have $\Delta_*(\alpha) = \Delta \cdot \mathrm{pr}_1^*(\alpha)$. Thus it suffices to see that $\operatorname{Im}(\mathrm{CH}(G) \to \mathrm{CH}(F))$ is generated by g and c.

$\square$

Consequently, in order to prove Theorem 1.6, we need to study the injectivity of the cycle class map restricted to the tautological ring $R(F \times F)$. To this end, we predict the following (new) relation in codimension 5 :

Conjecture 5.4 (A tautological relation)**.** *Notation as before, the following equality holds in* $\mathrm{CH}^5(F \times F)$ *:*

$$\Delta \cdot g_1 = g_1^2 g_2 I + g_1 g_2^2 I + Q(g_1, g_2, c_1, c_2), \tag{5}$$

where Q is a polynomial of weighted degree 5.

Proposition 5.5. *Let X be a smooth cubic fourfold and F be its Fano variety of lines. Then the cycle class map restricted to the tautological ring $R^*(F \times F)$ is injective for $* \neq 5$ or 6.*
If moreover the conjectured relation (5) is true, then the restriction to the whole ring $R^*(F \times F)$ of the cycle class map is injective.

Proof. It suffices to show the proposition for general cubic fourfolds, in which case

$$\mathrm{cl} : R(F \times F) \to Hdg^{2*}(F \times F)_{\mathbf{Q}}$$

is surjective. Let us show it is injective.
Firstly, it is not hard to count the dimensions of the spaces of Hodge classes :

i	0	1	2	3	4	5	6	7	8
$\dim Hdg^{2i}$	1	2	6	8	12	8	6	2	1

It is enough to show that the $R^i(F \times F)$ have the same dimensions.

The following relations in $R^*(F \times F)$ are at our disposal.

(i) $g_1 \cdot \Delta = g_2 \cdot \Delta\,;\, c_1 \cdot \Delta = c_2 \cdot \Delta$.
(ii) For $i = 1, 2$, we have $12 g_i c_i = 5 g_i^3\,; 4\, c_i^2 = g_i^4$.
(iii) Voisin's relation [24][4]:

$$I^2 = 2\Delta + I \cdot (g_1^2 + g_1 g_2 + g_2^2) + \Gamma_2(g_1, g_2, c_1, c_2),$$

where Γ_2 is a polynomial of weighted degree 4.
(iv) In [20, Proposition 17.5],

$$\Delta \cdot I = 6 c_1 \Delta - 3 g_1^2 \Delta.$$

(v) In [20, Lemma 17.6], there is a polynomial P of weighted degree 4 such that

$$c_1 \cdot I = P(g_1, g_2, c_1, c_2)\,;$$

$$c_2 \cdot I = P(g_2, g_1, c_2, c_1).$$

Using these relations, we get easily for each degree a list of generators (as vector-spaces):

- $R^0 = \langle \mathbb{1} \rangle\,;$
- $R^1 = \langle g_1, g_2 \rangle\,;$
- $R^2 = \langle g_1^2, g_1 g_2, g_2^2, c_1, c_2, I \rangle\,;$
- $R^3 = \langle g_1^3, g_1^2 g_2, g_1 g_2^2, g_2^3, g_1 c_2, g_2 c_1, g_1 I, g_2 I \rangle\,;$
- $R^4 = \langle g_1^4, g_1^3 g_2, g_1^2 g_2^2, g_1 g_2^3, g_2^4, g_1^2 c_2, g_2^2 c_1, c_1 c_2, g_1^2 I, g_2^2 I, g_1 g_2 I, \Delta \rangle\,;$
- $R^5 = \langle g_1^4 g_2, g_1^3 g_2^2, g_1^2 g_2^3, g_1 g_2^4, g_1^3 c_2, g_2^3 c_1, g_1^2 g_2 I, g_1 g_2^2 I, g_1 \Delta \rangle\,;$
- $R^6 = \langle g_1^4 g_2^2, g_1^3 g_2^3, g_1^2 g_2^4, g_1^4 c_2, g_2^4 c_1, g_1^2 g_2^2 I, g_1^2 \Delta \rangle\,;$
- $R^7 = \langle g_1^4 g_2^3, g_1^3 g_2^4 \rangle\,;$
- $R^8 = \langle g_1^4 g_2^4 \rangle\,;$

Observe that we have the same number of generators as the dimension of Hdg^{2i} for $i \neq 5$ or 6. Therefore the cycle class map $R^i(F \times F) \to H^{2i}(F \times F, \mathbf{Q})$ is injective for $i = 0, 1, 2, 3, 4, 7, 8$.
As for $i = 5$ (*resp.* $i = 6$) if we assume the conjectured relation (5), the generator $g_1 \Delta$ (*resp.* $g_1^2 \Delta$) is redundant, hence $R^i(F \times F) \to H^{2i}(F \times F)$ is also injective in these two degrees. □

We can now easily conclude Theorem 1.6:

Proof of Theorem 1.6. As the standard conjecture is proved for F_b in [8], thus for $F_b \times F_b$, numerical equivalence coincides with homological equivalence. Since any cycle of $\mathcal{F}^\circ \times_{B^\circ} \mathcal{F}^\circ$ is the restriction of a cycle of $\mathcal{F} \times_B \mathcal{F}$, it is enough to show that for any $b \in B^\circ$, the restriction of a cycle $\gamma \in \mathrm{CH}(\mathcal{F} \times_B \mathcal{F})$ to $F_b \times F_b$ is zero if and only if it is homologically trivial, which is proved by combining Proposition 5.3 and Proposition 5.5. □

[4] The coefficients are made precise by [20, Proposition 17.4].

With Theorem 1.6 being proved, we proceed to study the zero-cycles and codimension 2 cycles of the LLSvS hyper-Kähler eightfolds. The key input is Voisin's degree 6 dominant rational map [27, Proposition 4.8]

$$F \times F \dashrightarrow Z.$$

Consider the family version of Voisin's construction (over $B^{\circ\circ}$) : $\psi : \mathcal{F}^{\circ\circ} \times_{B^{\circ\circ}} \mathcal{F}^{\circ\circ} \dashrightarrow \mathcal{Z}$.

Proof of Theorem 1.7. Take a resolution of indeterminacies[5]:

$$\begin{tikzcd} \mathcal{F}^{\circ\circ} \widetilde{\times_{B^{\circ\circ}}} \mathcal{F}^{\circ\circ} \arrow[d, "\tau"'] \arrow[dr, "f"] & \\ \mathcal{F}^{\circ\circ} \times_{B^{\circ\circ}} \mathcal{F}^{\circ\circ} \arrow[r, dashed, "\psi"'] & \mathcal{Z} \end{tikzcd}$$

For (i), let $\gamma \in \mathrm{CH}^8(\mathcal{Z})_{\mathbf{Q}}$ be a relative zero-cycle whose degree on fibers is zero. Then, for any $b \in B^{\circ\circ}$,

$$(\tau_* f^*(\gamma))\,|_{F_b \times F_b} = \tau_{b*}\left(f^*(\gamma)|_{\widetilde{F_b \times F_b}}\right) = \tau_{b*} f_b^*\left(\gamma|_{Z_b}\right).$$

Thus $\tau_* f^*(\gamma)$ is a relative zero-cycle of fiber degree zero on $\mathcal{F}^{\circ\circ} \times_{B^{\circ\circ}} \mathcal{F}^{\circ\circ}$ and by Theorem 1.6, we know that

$$\tau_{b*} f_b^*\left(\gamma|_{Z_b}\right) = 0 \text{ in } \mathrm{CH}^8(F_b \times F_b).$$

For $b \in B^{\circ\circ}$ general, τ_b is birational hence induces an isomorphism on CH_0, hence $f_b^*\left(\gamma|_{Z_b}\right) = 0$. Moreover, since f_b is generically finite of degree 6 (still under the assumption that b is general), we have

$$\gamma|_{Z_b} = \frac{1}{6} f_{b*} f_b^*(\gamma|_{Z_b}) = 0.$$

A specialization argument shows that $\gamma|_{Z_b} = 0$ for all $b \in B^{\circ\circ}$.
For (ii) codimension 2 cycles : since $H^3(Z_b, \mathbf{Q}) = H^3(F_b \times F_b) = 0$, any cycle in $\mathrm{CH}^2(Z_b)$ or $\mathrm{CH}^2(F_b \times F_b)$ is homologically trivial if and only if its Abel-Jacobi invariant vanishes. Now the same proof as in (i) works because the subspace of Abel-Jacobi kernel for codimension 2 cycles CH^2_{AJ}, just as CH_0, is a birational invariant (for smooth projective varieties), hence $\tau_{b*} : \mathrm{CH}^2(\widetilde{F_b \times F_b})_{hom} \to \mathrm{CH}^2(F_b \times F_b)_{hom}$ is an isomorphism. ◻

Proof of Corollary 1.8. In view of Theorem 1.7, this is just a special case of Proposition 2.5. ◻

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[5]Warning: there is a slight potential conflict of notation, the $\mathcal{F}^{\circ\circ} \widetilde{\times_{B^{\circ\circ}}} \mathcal{F}^{\circ\circ}$ is not necessarily the restriction of the $\widetilde{\mathcal{F} \times_B \mathcal{F}}$ in the proof of Proposition 4.1

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