MOTIVIC HYPERKÄHLER RESOLUTION CONJECTURE : I. GENERALIZED KUMMER VARIETIES

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Abstract. Given a smooth projective variety M endowed with a faithful action of a finite group G, following Jarvis–Kaufmann–Kimura [34] and Fantechi–Göttsche [26], we define the orbifold motive (or Chen–Ruan motive) of the quotient stack $[M/G]$ as an algebra object in the category of Chow motives. Inspired by Ruan [48], one can formulate a motivic version of his Cohomological HyperKähler Resolution Conjecture (CHRC). We prove this motivic version, as well as its K-theoretic analogue conjectured in [34], in two situations related to an abelian surface A and a positive integer n. Case (A) concerns Hilbert schemes of points of A : the Chow motive of $A^{[n]}$ is isomorphic as algebra objects, up to a suitable sign change, to the orbifold motive of the quotient stack $[A^n/\mathfrak{S}_n]$. Case (B) for generalized Kummer varieties : the Chow motive of the generalized Kummer variety $K_n(A)$ is isomorphic as algebra objects, up to a suitable sign change, to the orbifold motive of the quotient stack $[A_0^{n+1}/\mathfrak{S}_{n+1}]$, where A_0^{n+1} is the kernel abelian variety of the summation map $A^{n+1} \to A$. In particular, these results give complete descriptions of the Chow motivic algebras (resp. Chow rings) of $A^{[n]}$ and $K_n(A)$ in terms of the first Chow motive $\mathfrak{h}^1(A)$ of A (resp. $\mathrm{CH}^*(A^k)$ the Chow ring of self-powers of A). As a byproduct, we prove the original Cohomological HyperKähler Resolution Conjecture for generalized Kummer varieties. As an application, we provide multiplicative Chow–Künneth decompositions for Hilbert schemes of abelian surfaces and for generalized Kummer varieties. In particular, we have a multiplicative direct sum decomposition of their Chow rings with rational coefficients, which are expected to be the splitting of the conjectural Bloch–Beilinson–Murre filtration. The existence of such a splitting for holomorphic symplectic varieties is conjectured by Beauville [10]. Finally, as another application analogous to Voisin's result in [56], we prove that over a non-empty Zariski open subset of the base, there exists a decomposition isomorphism $R\pi_*\mathbf{Q} \simeq \oplus R^i\pi_*\mathbf{Q}[-i]$ in $D_c^b(B)$ which is compatible with the cup-products on both sides, where $\pi : \mathcal{K}_n(\mathcal{A}) \to B$ is the relative generalized Kummer variety associated to a (smooth) family of abelian surfaces $\mathcal{A} \to B$.

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1. Introduction

1.1. **Motivation 1 : Ruan's hyperKähler resolution conjectures.** In [17], Chen and Ruan construct the orbifold cohomology ring $H^*_{orb}(\mathcal{X})$ for any complex orbifold $\mathcal{X}$. As a $\mathbf{Q}$-vector space, it is defined to be the cohomology of its inertia variety $H^*(I\mathcal{X})$ (with degrees shifted by some rational numbers called *age*), but is endowed with a highly non-trivial ring structure coming from moduli spaces of curves mapping to X. An algebro-geometric treatment is contained in Abramovich–Graber–Vistoli's work [1], based on the construction of moduli stack of twisted stable maps in [2]. In the global quotient case[1], some equivalent definitions are available : see for example [26], [34], [36] and §2.

Originating from the topological string theory of orbifolds in [23], [24], one observes that the stringy topological invariants of an orbifold, *e.g.* the orbifold Euler number and the orbifold Hodge numbers, should be related to the corresponding invariants of a crepant resolution ([4], [5], [39]). A much deeper relation was brought forward by Ruan, who made, among others, the following Cohomological HyperKähler Resolution Conjecture (CHRC) in [48]. For more general and sophisticated versions of this conjecture, see [49], [15], [18].

Conjecture 1.1 (Ruan's CHRC)**.** *Let $\mathcal{X}$ be a compact complex orbifold with underlying variety X being Gorenstein. If there is a crepant resolution $Y \to X$ with Y being hyperKähler, then we have an isomorphism of graded commutative $\mathbf{C}$-algebras : $H^*(Y, \mathbf{C}) \simeq H^*_{orb}(\mathcal{X}, \mathbf{C})$.*

As the construction of orbifold product can be expressed using algebraic correspondences (*cf.* [1] and §2), one has the analogous definition of the orbifold Chow ring $\mathrm{CH}_{orb}(\mathcal{X})$ (see Definition 2.6 for the global quotient case) of a smooth proper Deligne–Mumford stack $\mathcal{X}$. Motivated by the study of algebraic cycles on hyperKähler varieties, we propose to investigate the Chow-theoretic analogue of Conjecture 1.1. For reasons which will become clear shortly, it is more powerful and fundamental to consider the following *motivic* version of Conjecture 1.1. Let $\mathrm{CHM}_{\mathbf{C}}$ be the category of Chow motives with complex coefficients and $\mathfrak{h}$ be the (contravariant) functor that associates to a smooth projective variety its Chow motive.

Meta-conjecture 1.2 (MHRC)**.** *Let $\mathcal{X}$ be a smooth proper complex Deligne–Mumford stack with underlying coarse moduli space X being a (singular) symplectic variety. If there is a symplectic resolution $Y \to X$, then we have an isomorphism $\mathfrak{h}(Y) \simeq \mathfrak{h}_{orb}(\mathcal{X})$ as commutative algebra objects in* $\mathrm{CHM}_{\mathbf{C}}$*, hence in particular an isomorphism of graded $\mathbf{C}$-algebras :* $\mathrm{CH}^*(Y)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}(\mathcal{X})_{\mathbf{C}}$.

See Definition 3.1 for generalities on symplectic singularities and symplectic resolutions. The reason why it is only a *meta*-conjecture is that the definition of orbifold Chow motive for a smooth proper Deligne–Mumford stack in general is not available in the literature and we will not develop the theory in this generality in this paper (see however Remark 2.9). From now on, let us restrict ourselves to the case where the Deligne–Mumford stack in question is of the form of a global quotient $\mathcal{X} = [M/G]$, where M is a smooth projective variety with a faithful action of a finite group G, in which case we will define the *orbifold Chow motive* $\mathfrak{h}_{orb}(\mathcal{X})$ in a very explicit way in Definition 2.5.

The *Motivic HyperKähler Resolution Conjecture* that we are interested in is the following more precise statement, which would contain all situations considered in this paper and its sequel.

Conjecture 1.3 (MHRC : global quotient case)**.** *Let M be a smooth projective holomorphic symplectic variety equipped with an action of a finite group G by symplectic automorphisms of M. If Y is a symplectic*

[1] In this paper, by 'global quotient', we always mean the quotient of a smooth projective variety by a *finite* group.

resolution of the quotient variety M/G, then we have an isomorphism of (commutative) algebra objects in the category of Chow motives with complex *coefficients :*

$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{orb}([M/G]) \text{ in } \mathrm{CHM}_{\mathbf{C}}.$$

In particular, we have an isomorphism of graded $\mathbf{C}$*-algebras*

$$\mathrm{CH}^*(Y)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([M/G])_{\mathbf{C}}.$$

The definition of the *orbifold motive* of $[M/G]$ as a *(commutative) algebra object* in the category of Chow motives with rational coefficients[2] is particularly down-to-earth ; it is the G-invariant sub-algebra object of some explicit algebra object :

$$\mathfrak{h}_{orb}([M/G]) := \left(\bigoplus_{g\in G} \mathfrak{h}(M^g)(-\operatorname{age}(g)), \star_{orb}\right)^G,$$

where for each $g \in G$, M^g is the fixed subvariety of g and the orbifold product $\star_{orb}$ is defined by using natural inclusions and Chern classes of normal bundles of various fixed loci ; see Definition 2.5 (or (2) below) for the precise formula of $\star_{orb}$ as well as the Tate twists by age (2.3) and the G-action. The orbifold Chow ring[3] is then defined as the following commutative algebra

$$\mathrm{CH}^*_{orb}([M/G]) := \bigoplus_i \mathrm{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), \mathfrak{h}_{orb}([M/G])),$$

or equivalently more explicitly :

$$\mathrm{CH}^*_{orb}([M/G]) := \left(\bigoplus_{g\in G} \mathrm{CH}^{*-\operatorname{age}(g)}(M^g), \star_{orb}\right)^G, \tag{1}$$

where $\star_{orb}$ is as follows : for two elements $g, h \in G$ and $\alpha \in \mathrm{CH}^{i-\operatorname{age}(g)}(M^g)$, $\beta \in \mathrm{CH}^{j-\operatorname{age}(h)}(M^h)$, their orbifold product is the following element in $\mathrm{CH}^{i+j-\operatorname{age}(gh)}(M^{gh})$:

$$\alpha \star_{orb} \beta := \iota_*\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{top}(F_{g,h})\right), \tag{2}$$

where $M^{<g,h>} := M^g \cap M^h$, $\iota : M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion and $F_{g,h}$ is the obstruction bundle. This construction is completely parallel to the construction of orbifold cohomology due to Fantechi–Göttsche [26] which is further simplified in Jarvis–Kaufmann–Kimura [34].

With the orbifold Chow theory briefly reviewed above, we see that in Conjecture 1.3, the fancy side of $[M/G]$ is actually the easier side which can be used to study the motive and cycles of the hyper-Kähler variety Y. Let us turn this idea into the following working principle, which will be illustrated repeatedly in examples in the rest of the introduction.

Slogan : *The cohomology theories[4] of a holomorphic symplectic variety can be understood via the hidden stack structure of its singular symplectic models.*

[2]Strictly speaking, the orbifold Chow motive of $[M/G]$ in general lives in the larger category of Chow motives with *fractional* Tate twists. However, in our cases of interest, namely when there exists a crepant resolution, for the word 'crepant resolution' to make sense we understand that the underlying singular variety M/G is at least Gorenstein, in which case all age shiftings are integers and we stay in the usual category of Chow motives. See Definitions 2.1 and 2.5 for the general notions.

[3]The definition of the orbifold Chow ring has already appeared in Page 211 of Fantechi–Göttsche [26] and proved to be equivalent to Abramovich–Grabber–Vistoli's construction in [1] by Jarvis–Kaufmann–Kimura in [34] .

[4]In the large sense: Weil cohomology, Chow rings, K-theory, motivic cohomology, *etc.* and finally, motives.

Interesting examples of symplectic resolutions appear when considering the Hilbert–Chow morphism of a smooth projective surface. More precisely, in his fundamental paper [7], Beauville provides such examples :

Example 1.
Let S be a complex projective K3 surface or an abelian surface. Its Hilbert scheme of length-n subschemes, denoted by $S^{[n]}$, is a symplectic crepant resolution of the symmetric product $S^{(n)}$ via the Hilbert–Chow morphism. The corresponding Cohomological HyperKähler Resolution Conjecture was proved independently by Fantechi–Göttsche in [26] and Uribe in [51] making use of Lehn–Sorger's work [38] computing the ring structure of $H^*(S^{[n]})$. The Motivic HyperKähler Resolution Conjecture 1.3 in the case of K3 surfaces will be proved in [30] and the case of abelian surfaces is the following theorem.

Theorem 1.4 (MHRC for $A^{[n]}$). *Let A be an abelian surface and $A^{[n]}$ be its Hilbert scheme as before. Then we have an isomorphism of commutative algebra objects in the category* CHM *of Chow motives with rational coefficients :*

$$\mathfrak{h}\left(A^{[n]}\right) \simeq \mathfrak{h}_{orb,dt}([A^n/\mathfrak{S}_n]),$$

where on the left hand side, the product structure is given by the small diagonal of $A^{[n]}\times A^{[n]}\times A^{[n]}$ while on the right hand side, the product structure is given by the orbifold product $\star_{orb}$ with a suitable sign change, called discrete torsion *in 3.5. In particular, we have an isomorphism of commutative graded $\mathbf{Q}$-algebras :*

$$\mathrm{CH}^*\left(A^{[n]}\right)_{\mathbf{Q}} \simeq \mathrm{CH}^*_{orb,dt}([A^n/\mathfrak{S}_n]). \tag{3}$$

Example 2.
Let A be a complex abelian surface. The composition of the Hilbert–Chow morphism followed by the sum map $A^{[n+1]} \to A^{(n+1)} \to A$ is an isotrivial fibration. The *generalized Kummer variety $K_n(A)$* is by definition the fiber of this morphism over the origin of A. It is a hyperKähler resolution of the quotient $A_0^{n+1}/\mathfrak{S}_{n+1}$, where A_0^{n+1} is the kernel abelian variety of the summation map $A^{n+1} \to A$. The main result of the paper is the following theorem confirming the Motivic HyperKähler Resolution Conjecture 1.3 in this situation.

Theorem 1.5 (MHRC for $K_n(A)$). *Let $K_n(A)$ be the $2n$-dimensional generalized Kummer variety associated to an abelian surface A. Let $A_0^{n+1} := \mathrm{Ker}\left(+ : A^{n+1} \to A\right)$ endowed with the natural $\mathfrak{S}_{n+1}$-action. Then we have an isomorphism of commutative algebra objects in the category* CHM *of Chow motives with rational coefficients :*

$$\mathfrak{h}\left(K_n(A)\right) \simeq \mathfrak{h}_{orb,dt}([A_0^{n+1}/\mathfrak{S}_{n+1}]),$$

where on the left hand side, the product structure is given by the small diagonal while on the right hand side, the product structure is given by the orbifold product $\star_{orb}$ with the sign change given by discrete torsion in 3.5. In particular, we have an isomorphism of commutative graded $\mathbf{Q}$-algebras :

$$\mathrm{CH}^*\left(K_n(A)\right)_{\mathbf{Q}} \simeq \mathrm{CH}^*_{orb,dt}([A_0^{n+1}/\mathfrak{S}_{n+1}]). \tag{4}$$

1.2. **Consequences.**

We get some by-products of our main results.

Taking the Betti cohomological realization, we confirm Ruan's original Cohomological HyperKähler Resolution Conjecture 1.1 in this case :

Theorem 1.6 (CHRC for $K_n(A)$)**.** *Let notation be as in Theorem 1.5. We have an isomorphism of graded commutative* $\mathbf{Q}$*-algebras :*

$$H^*\left(K_n(A)\right)_{\mathbf{Q}} \simeq H^*_{orb,dt}([A_0^{n+1}/\,\mathfrak{S}_{n+1}]).$$

The CHRC has never been proved in the case of generalized Kummer varieties in the literature. Related work on the CHRC in this case are Nieper–Wisskirchen's description of the cohomology ring $H^*(K_n(A), \mathbf{C})$ in [44], which plays an important rôle in our proof ; and Britze's thesis [14] comparing $H^*(A \times K_n(A), \mathbf{C})$ and the computation of the orbifold cohomology ring of $[A \times A_0^{n+1}/\,\mathfrak{S}_{n+1}]$ in Fantechi–Göttsche [26]. See however Remark 6.16.

From the K-theoretic point of view, we also have the following closely related conjecture (KHRC) in [34, Conjecture 1.2], where the *orbifold K-theory* is defined in a similar way with top Chern class in (2) replaced by the K-theoretic Euler class ; see Definition 2.7 for details.

Conjecture 1.7 (K-theoretic HyperKähler Resolution Conjecture [34])**.** *In the same situation as in Conjecture 1.2, we have isomorphisms of* $\mathbf{C}$*-algebras :*

$$K(Y)_{\mathbf{C}} \simeq K_{orb}(\mathcal{X})_{\mathbf{C}}\,;$$
$$K^{top}(Y)_{\mathbf{C}} \simeq K^{top}_{orb}(\mathcal{X})_{\mathbf{C}}.$$

Using Theorems 1.4 and 1.5, we can confirm Conjecture 1.7 in the two cases considered here :

Theorem 1.8 (KHRC for $A^{[n]}$ and $K_n(A)$)**.** *Let A be an abelian surface and n a natural number. There are isomorphisms of commutative* $\mathbf{C}$*-algebras :*

$$\begin{aligned}
K\left(A^{[n]}\right)_{\mathbf{C}} &\simeq K_{orb}([A^n/\,\mathfrak{S}_n])_{\mathbf{C}}\,;\\
K^{top}\left(A^{[n]}\right)_{\mathbf{C}} &\simeq K^{top}_{orb}([A^n/\,\mathfrak{S}_n])_{\mathbf{C}}\,;\\
K\left(K_n(A)\right)_{\mathbf{C}} &\simeq K_{orb}([A_0^{n+1}/\,\mathfrak{S}_{n+1}])_{\mathbf{C}}\,;\\
K^{top}\left(K_n(A)\right)_{\mathbf{C}} &\simeq K^{top}_{orb}([A_0^{n+1}/\,\mathfrak{S}_{n+1}])_{\mathbf{C}}.
\end{aligned}$$

1.3. **On explicit descriptions of Chow rings.** Let us make some remarks on the way we understand Theorem 1.4 and Theorem 1.5. For each of them, the seemingly fancy right hand side of (3) and (4) given by orbifold Chow ring is actually very concrete (see (1)) : as groups, since all fixed loci are just various diagonals, they are the Chow groups of products of the abelian surface A, which can be handled by Beauville's decomposition of Chow rings of abelian varieties [8] ; while the ring structures are given by the orbifold product which is extremely simplified in our cases (see (2)) : all obstruction bundles $F_{g,h}$ are trivial and hence the orbifold products are either the intersection product pushed forward by inclusions or simply zero.

In short, given an abelian surface A, Theorem 1.4 and Theorem 1.5 provide an explicit description of the Chow rings of $A^{[n]}$ and of $K_n(A)$ in terms of Chow rings of products of A (together with some combinatoric rules specified by the orbifold product). To illustrate how explicit it is, we work out two simple examples in §3.2 : the Chow ring of the Hilbert square of a K3 surface or an abelian surface and the Chow ring of the Kummer K3 surface associated to an abelian surface.

1.4. **Motivation 2 : Beauville's splitting property.** The original motivation for the authors to study the Motivic HyperKähler Resolution Conjecture 1.2 was to understand the (rational) Chow rings, or more generally the Chow motives, of smooth projective holomorphic symplectic varieties, that is, an even-dimensional projective manifold carrying a holomorphic 2-form which is symplectic (*i.e.* non-degenerate at each point). As an attempt to unify his work on algebraic cycles on abelian varieties [8] and his result with Voisin on Chow rings of K3 surfaces [11], Beauville conjectured in [10], under the name of *the splitting property*, that for a smooth projective holomorphic symplectic variety X, there exists a canonical multiplicative splitting of the conjectural Bloch–Beilinson–Murre filtration of the rational Chow ring (see Conjecture 7.1 for the precise statement). In this paper, we will understand the splitting property as in the following motivic version (see Definition 7.2 and Conjecture 7.4) :

Conjecture 1.9 (Beauville's Splitting Property : motives)**.** *Let X be a smooth projective holomorphic symplectic variety of dimension $2n$. Then we have a canonical multiplicative Chow–Künneth decomposition of $\mathfrak{h}(X)$ of Bloch–Beilinson type, that is, a direct sum decomposition in the category of rational Chow motives :*

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} h^i(X) \tag{5}$$

satisfying the following properties :

(1) *(Chow–Künneth) The cohomology realization of the decomposition gives the Künneth decomposition : for each $0 \le i \le 4n$, $H^*(\mathfrak{h}^i(X)) = H^i(X)$.*
(2) *(Multiplicativity) The product $\mu : \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ given by the small diagonal $\delta_X \subset X \times X \times X$ respects the decomposition : the restriction of μ on the summand $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$ factorizes through $\mathfrak{h}^{i+j}(X)$.*
(3) *(Bloch–Beilinson–Murre) For any $i, j \in \mathbf{N}$,*
   - $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if* $j < i$;
   - $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if* $j > 2i$;
   - *the realization induces an injective map* $\mathrm{Hom}_{\mathrm{CHM}}\left(\mathbb{1}(-i), \mathfrak{h}^{2i}(X)\right) \to \mathrm{Hom}_{\mathbf{Q}-HS}\left(\mathbf{Q}(-i), H^{2i}(X)\right)$.

Such a decomposition naturally induces a (multiplicative) bigrading on the Chow ring $\mathrm{CH}^*(X) = \oplus_{i,s} \mathrm{CH}^i(X)_s$ by setting :

$$\mathrm{CH}^i(X)_s := \mathrm{Hom}_{\mathrm{CHM}}\left(\mathbb{1}(-i), \mathfrak{h}^{2i-s}(X)\right),$$

which is the original splitting that Beauville envisaged.

Our main results Theorem 1.4 and Theorem 1.5 allow us, for X being a Hilbert scheme of an abelian surface or a generalized Kummer variety, to achieve in Theorem 1.10 below partially the goal Conjecture 1.9 : we construct the candidate direct sum decomposition (5) satisfying the first two conditions (1) and (2) in Conjecture 1.9, namely a *self-dual multiplicative Chow–Künneth decomposition* (see Definition 7.2, *cf.* [50]). The remaining Condition (3) on Bloch–Beilinson–Murre properties is very much related to Beauville's *Weak Splitting Property*, which has already been proved in [29] for the case of generalized Kummer varieties ; see [10], [55], [59], [47] for the complete story and more details.

Theorem 1.10 (=Theorem 7.9 + Proposition 7.13)**.** *Let A be an abelian surface and n be a positive integer. Let X be the corresponding $2n$-dimensional Hilbert scheme $A^{[n]}$ or generalized Kummer variety $K_n(A)$. Then X has a canonical self-dual multiplicative Chow–Künneth decomposition*

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(X),$$

where in the two respective cases we have

$$\begin{aligned}\mathfrak{h}^i(A^{[n]}) &:= \left(\bigoplus_{g\in\mathfrak{S}_n}\mathfrak{h}^{i-2\operatorname{age}(g)}((A^n)^g)(-\operatorname{age}(g))\right)^{\mathfrak{S}_n};\\ \mathfrak{h}^i(K_n(A)) &:= \left(\bigoplus_{g\in\mathfrak{S}_{n+1}}\mathfrak{h}^{i-2\operatorname{age}(g)}((A_0^{n+1})^g)(-\operatorname{age}(g))\right)^{\mathfrak{S}_{n+1}}.\end{aligned}$$

In particular, we have a canonical multiplicative bigrading on the (rational) Chow ring given by

$$\mathrm{CH}^i(X)_s := \mathrm{CH}^i(\mathfrak{h}^{2i-s}(X)).$$

Moreover, the i-th Chern class of X is in $\mathrm{CH}^i(X)_0$ *for any i.*

The associated filtration $F^j\,\mathrm{CH}^i(X) := \oplus_{s\geq j}\mathrm{CH}^i(X)_s$ is supposed to satisfy the Bloch–Beilinson–Murre conjecture (see Conjecture 7.11). We point out in Remark 7.12 that Beauville's Conjecture 7.5 on abelian varieties implies for X in our two cases some Bloch–Beilinson–Murre properties: $\mathrm{CH}^*(X)_s = 0$ for $s < 0$ and the cycle class map restricted to $\mathrm{CH}^*(X)_0$ is injective.

See Remark 7.10 for previous related results.

1.5. **Cup products *vs.* decomposition theorem.** For a smooth projective morphism $\pi : \mathcal{X} \to B$ Deligne shows in [21] that one has an isomorphism

$$R\pi_*\mathbf{Q} \cong \bigoplus_i R^i\pi_*\mathbf{Q}[-i],$$

in the derived category of sheaves of $\mathbf{Q}$−vector spaces on B. Voisin [56] remarks that this isomorphism cannot be made compatible with the product structures on both sides even after shrinking B to a Zariski open subset and shows that it can be made so if π is a smooth family of projective K3 surfaces. Her result is extended in [52] to relative Hilbert schemes of finite lengths of a smooth family of projective K3 surfaces or abelian surfaces. As a by-product of our main result in this paper, we can similarly prove the case of generalized Kummer varieties.

Theorem 1.11 (=Corollary 8.4). *Let* $\mathcal{A} \to B$ *be an abelian surface over B. Consider* $\pi : K_n(\mathcal{A}) \to B$ *the relative generalized Kummer variety. Then there exist a decomposition isomorphism*

$$R\pi_*\mathbf{Q} \cong \bigoplus_i R^i\pi_*\mathbf{Q}[-i], \tag{6}$$

and a nonempty Zariski open subset U of B, such that this decomposition becomes multiplicative for the restricted family over U.

Convention and notation. Throughout the paper, all varieties are defined over the field of complex numbers.

- The notation CH (*resp.* $\mathrm{CH}_{\mathbf{C}}$) means Chow groups with *rational* (*resp. complex*) coefficients. CHM is the category of Chow motives over the complex numbers with *rational* coefficients.
- For a variety X, its small diagonal, always denoted by δ_X, is $\{(x,x,x) \mid x \in X\} \subset X \times X \times X$.
- For a smooth surface X, its Hilbert scheme of length-n subschemes is always denoted by $X^{[n]}$, which is smooth of dimension $2n$ by [27].

- An (even) dimensional smooth projective variety is *holomorphic symplectic* if it has a holomorphic symplectic (*i.e.* non-degenerate at each point) 2-form. When talking about resolutions, we tend to use the word *hyperKähler* as its synonym, which usually (but not in this paper) requires also the 'irreducibility', that is, the simple connectedness of the variety and the uniqueness up to scalars of the holomorphic symplectic 2-form. In particular, punctual Hilbert schemes of abelian surfaces are examples of holomorphic symplectic varieties.
- An abelian variety is always supposed to be connected. Its non-connected generalization causes extra difficulty and is dealt with in §6.2.
- When working with 0-cycles on an abelian variety A, to avoid confusion, for a collection of points $x_1, \dots, x_m \in A$, we will write $[x_1] + \cdots + [x_m]$ for the 0-cycle of degree m (or equivalently, a point in $A^{(m)}$, the m-th symmetric product of A) and $x_1 + \cdots + x_m$ will stand for the usual sum using the group law of A, which is therefore a point in A.

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2. Orbifold motives and orbifold Chow rings

To fix the notation, we start by a brief reminder of the construction of pure motives (*cf.* [3]). In order to work with Tate twists by age functions (2.3), we have to extend slightly the usual notion of pure motives by allowing twists by a rational number.

Definition 2.1 (Chow motives with fractional Tate twists)**.** The category of *Chow motives with fractional Tate twists* with rational coefficients, denoted by $\widetilde{\mathrm{CHM}}$, has as objects finite direct sums of triples of the form (X, p, n) with X a connected smooth projective variety, $p \in \mathrm{CH}^{\dim X}(X \times X)$ a projector and $n \in \mathbf{Q}$ a *rational* number. Given two objects (X, p, n) and (Y, q, m), the morphism space between them consists of correspondences :

$$Hom_{\widetilde{\mathrm{CHM}}}((X, p, n), (Y, q, m)) := q \circ \mathrm{CH}^{\dim X + m - n}(X \times Y) \circ p,$$

where we simply impose that all Chow groups of a variety with non-integer codimension are zero. The composition law of correspondences is the usual one. Identifying $(X, p, n) \oplus (Y, q, n)$ with $(X \coprod Y, p \coprod q, n)$ makes CHM a $\mathbf{Q}$-linear category. Moreover, CHM is a rigid symmetric monoïdal category with unit $\mathbb{1} := (\operatorname{Spec} \mathbf{C}, \operatorname{Spec} \mathbf{C}, 0)$, tensor product defined by $(X, p, n) \otimes (Y, q, m) := (X \times Y, p \times q, n + m)$ and duality given by $(X, p, n)^\vee := \left(X, {}^t p, \dim X - n\right)$. There is a natural contravariant functor $\mathfrak{h} : \mathrm{SmProj}^{op} \to \mathrm{CHM}$ sending a smooth projective variety X to its Chow motive $\mathfrak{h}(X) = (X, \Delta_X, 0)$ and a morphism $f : X \to Y$ to its transposed graph ${}^t\Gamma_f \in \mathrm{CH}^{\dim Y}(Y \times X) = \mathrm{Hom}_{\mathrm{CHM}}(\mathfrak{h}(Y), \mathfrak{h}(X))$.

Remarks 2.2.
1. The category $\widetilde{\mathrm{CHM}}_{\mathbf{C}}$ of *Chow motives with fractional Tate twists* with complex coefficients is defined similarly by replacing all Chow groups with rational coefficients CH by Chow groups with complex coefficients $\mathrm{CH}_{\mathbf{C}}$ in the above definition.
2. The usual category of Chow motives with rational (*resp.* complex) coefficients CHM (*resp.* $\mathrm{CHM}_{\mathbf{C}}$, *cf.* [3]) is identified with the full subcategory of $\widetilde{\mathrm{CHM}}$ (*resp.* $\widetilde{\mathrm{CHM}}_{\mathbf{C}}$) consisting of objects (X, p, n) with $n \in \mathbf{Z}$.
3. The above construction works for any adequate equivalence relation and gives corresponding categories of pure motives (with fractional Tate twists) by replacing CH by the group

of algebraic cycles modulo the chosen adequate equivalence relation (*cf.* [3]). In particular, we can talk about the category of *numerical motives* NumM and *homological motives*[5] HomM with rational or complex coefficients as well as their variants with fractional Tate twists, *etc.*.

(4) Thanks to the extension of the intersection theory (with rational coefficients) of Fulton [31] to the so-called **Q**-*varieties* by Mumford [40], the motive functor $\mathfrak{h}$ defined above can actually be extended to the larger category of finite group quotients of smooth projective varieties, or more generally to **Q**-varieties with global Cohen-Macaulay cover, see for example [20, §§2.2-2.3]. Indeed, for global quotients one defines $\mathfrak{h}(M/G) := (M, \frac{1}{|G|}\sum_{g\in G} {}^t\Gamma_g, 0)$. Note that it is essential to work with rational (or complex) coefficients.

Let M be an m-dimensional smooth projective complex variety equipped with an action of a finite group G. We adapt the constructions in [26] and [34] to define the orbifold motive of the smooth proper Deligne–Mumford stack $[M/G]$. For any $g \in G$, $M^g := \{x \in M \mid gx = x\}$ is the fixed locus of the automorphism g, which is a smooth subvariety of M. The following notion is due to Reid (see [46]).

Definition 2.3 (Age)**.** Given an element $g \in G$, let $r \in \mathbf{N}$ be its order. The *age* of g, denoted by $\operatorname{age}(g)$, is the locally constant $\mathbf{Q}_{\geq 0}$-valued function on M^g defined as follows. Let Z be a connected component of M^g. Choosing any point $x \in Z$, we have the induced automorphism $g_* \in GL(T_xM)$, whose eigenvalues, repeated according to multiplicities, are

$$\left\{e^{2\pi\sqrt{-1}\frac{\alpha_1}{r}}, \cdots, e^{2\pi\sqrt{-1}\frac{\alpha_m}{r}}\right\},$$

with $0 \leq \alpha_i \leq r-1$. One defines

$$\operatorname{age}(g)|_Z := \frac{1}{r}\sum_{i=1}^m \alpha_i.$$

It is obvious that the value of $\operatorname{age}(g)$ on Z is independent of the choice of $x \in Z$ and it takes values in $\mathbf{N}$ if $g_* \in \mathrm{SL}(T_xM)$. Also immediate from the definition, we have $\operatorname{age}(g) + \operatorname{age}(g^{-1}) = \operatorname{codim}(M^g \subset M)$ as locally constant functions. Thanks to the natural isomorphism $h : M^g \to M^{hgh^{-1}}$ sending x to $h.x$, for any $g, h \in G$, the age function is invariant under conjugation.

Example 2.4. Let S be a smooth projective variety of dimension d and n a positive integer. The symmetric group $\mathfrak{S}_n$ acts by permutation on $M = S^n$. For each $g \in \mathfrak{S}_n$, a straightforward computation shows that $\operatorname{age}(g)$ is the constant function $\frac{d}{2}(n - |O(g)|)$, where $O(g)$ is the set of orbits of g as a permutation of $\{1, \ldots, n\}$. For example, when S is a surface (*i.e.* , $d = 2$), the age in this case is always a non-negative integer and we have $\operatorname{age}(\mathrm{id}) = 0$, $\operatorname{age}(12\cdots r) = r-1$, $\operatorname{age}(12)(345) = 3$ *etc.*.

Recall that an *algebra object* in a symmetric monoïdal category $(\mathcal{M}, \otimes, \mathbb{1})$ (for example, CHM, $\widetilde{\mathrm{CHM}}$ *etc.*) is an object $A \in \operatorname{Obj}\mathcal{M}$ together with a morphism $\mu : A \otimes A \to A$ in $\mathcal{M}$, called the *multiplication* or *product structure*, satisfying the associativity axiom $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. An algebra object A in $\mathcal{M}$ is called *commutative* if $\mu \circ \iota = \mu$, where $\iota : A \otimes A \to A \otimes A$ is the structural symmetry isomorphism of $\mathcal{M}$. For each smooth projective variety X, its Chow motive $\mathfrak{h}(X)$ is naturally a commutative algebra object in CHM (hence in $\widetilde{\mathrm{CHM}}$, $\widetilde{\mathrm{CHM}}_{\mathbf{C}}$, *etc.*) whose multiplication is given by the small diagonal $\delta_X \in \mathrm{CH}^{2\dim X}(X \times X \times X) = \operatorname{Hom}_{\mathrm{CHM}}(\mathfrak{h}(X) \otimes \mathfrak{h}(X), \mathfrak{h}(X))$.

[5] One has to choose a Weil cohomology theory when talking about homological motives. In this paper, however, we always use the Betti cohomology and make the choice implicit.

Definition 2.5 (Orbifold Chow motive)**.** We define first of all an auxiliary (in general non-commutative) algebra object $\mathfrak{h}(M,G)$ of $\widetilde{\mathrm{CHM}}$ in several steps :

(1) As a Chow motive with fractional twists, $\mathfrak{h}(M,G)$ is defined to be the direct sum over G, of the motives of fixed loci twisted à la Tate by $-$ age :

$$\mathfrak{h}(M,G) := \bigoplus_{g\in G} \mathfrak{h}(M^g)\,(-\operatorname{age}(g))\,.$$

(2) $\mathfrak{h}(M,G)$ is equipped with a natural G-action : each element $h\in G$ induces for each $g\in G$ an isomorphism $h : M^g \to M^{hgh^{-1}}$ by sending x to $h.x$, hence an isomorphism between the direct factors $\mathfrak{h}(M^g)(-\operatorname{age}(g))$ and $\mathfrak{h}(M^{hgh^{-1}})(-\operatorname{age}(hgh^{-1}))$ by the conjugation invariance of the age function.

(3) For any $g\in G$, let r be its order. We have a natural automorphism g_* of the vector bundle $TM|_{M^g}$. Consider its eigen-subbundle decomposition :

$$TM|_{M^g} = \bigoplus_{j=0}^{r-1} W_{g,j},$$

where $W_{g,j}$ is the subbundle associated to the eigenvalue $e^{2\pi\sqrt{-1}\frac{j}{r}}$. Define

$$S_g := \sum_{j=0}^{r-1} \frac{j}{r}[W_{g,j}] \in K_0(M^g)_{\mathbf{Q}}.$$

Note that the virtual rank of S_g is nothing but $\operatorname{age}(g)$ by Definition 2.3.

(4) For any $g_1, g_2\in G$, let $M^{<g_1,g_2>} = M^{g_1}\cap M^{g_2}$ and $g_3 = g_2^{-1}g_1^{-1}$. Define the following element in $K_0(M^{<g_1,g_2>})_{\mathbf{Q}}$:

$$F_{g_1,g_2} := S_{g_1}\big|_{M^{<g_1,g_2>}} + S_{g_2}\big|_{M^{<g_1,g_2>}} + S_{g_3}\big|_{M^{<g_1,g_2>}} + TM^{<g_1,g_2>} - TM|_{M^{<g_1,g_2>}}\,.$$

Note that its virtual rank is

$$\operatorname{rk} F_{g_1,g_2} = \operatorname{age}(g_1) + \operatorname{age}(g_2) - \operatorname{age}(g_1g_2) - \operatorname{codim}(M^{<g_1,g_2>}\subset M^{g_1g_2}). \tag{7}$$

In fact, this class in the Grothendieck group is represented by a genuine obstruction vector bundle constructed in [26] (*cf.* [34]). In particular, $\operatorname{age}(g_1)+\operatorname{age}(g_2)-\operatorname{age}(g_1g_2)$ is always an integer.

(5) The product structure $\star_{orb}$ on $\mathfrak{h}(M,G)$ is defined to be multiplicative with respect to the G-grading and for each $g_1, g_2\in G$, the *orbifold product*

$$\star_{orb} : \mathfrak{h}(M^{g_1})(-\operatorname{age}(g_1))\otimes \mathfrak{h}(M^{g_2})(-\operatorname{age}(g_2)) \to \mathfrak{h}(M^{g_1g_2})(-\operatorname{age}(g_1g_2))$$

is the correspondence determined by the algebraic cycle

$$\delta_*(c_{top}(F_{g_1,g_2})) \in \mathrm{CH}^{\dim M^{g_1}+\dim M^{g_2}+\operatorname{age}(g_1)+\operatorname{age}(g_2)-\operatorname{age}(g_1g_2)}(M^{g_1}\times M^{g_2}\times M^{g_1g_2}),$$

where $\delta : M^{<g_1,g_2>}\to M^{g_1}\times M^{g_2}\times M^{g_1g_2}$ is the natural morphism sending x to (x,x,x) and c_{top} means the top Chern class of F_{g_1,g_2}. One can check easily that the product structure $\star_{orb}$ is invariant under the action of G.

(6) The associativity of $\star_{orb}$ is non-trivial. The proof in [34, Lemma 5.4] is completely algebraic hence also works in our motivic case.

(7) Finally, the *orbifold Chow motive* of $[M/G]$, denoted by $\mathfrak{h}_{orb}([M/G])$, is the G-invariant subalgebra object[6] of $\mathfrak{h}(M,G)$, which turns out to be a commutative algebra object in $\widetilde{\mathrm{CHM}}$:

$$\mathfrak{h}_{orb}([M/G]) := \mathfrak{h}(M,G)^G = \left(\bigoplus_{g\in G} \mathfrak{h}(M^g)\left(-\operatorname{age}(g)\right), \star_{orb}\right)^G \tag{8}$$

We still use $\star_{orb}$ to denote the orbifold product on this sub-algebra object $\mathfrak{h}_{orb}([M/G])$.

By replacing the rational equivalence relation by another adequate equivalence relation, the same construction gives the *orbifold homological motives*, *orbifold numerical motives*, *etc.* associated to a global quotient smooth proper Deligne–Mumford stack as algebra objects in the corresponding categories of pure motives (with fractional Tate twists).

The definition of the orbifold Chow ring then follows in the standard way and agrees with the one in [26], [34] and [1].

Definition 2.6 (Orbifold Chow ring)**.** The *orbifold Chow ring* of $[M/G]$ is the commutative $\mathbf{Q}_{\geq 0}$-graded $\mathbf{Q}$-algebra $\mathrm{CH}^*_{orb}([M/G]) := \bigoplus_{i\in\mathbf{Q}_{\geq 0}} \mathrm{CH}^i_{orb}([M/G])$ with

$$\mathrm{CH}^i_{orb}([M/G]) := \operatorname{Hom}_{\widetilde{\mathrm{CHM}}}(\mathbb{1}(-i), \mathfrak{h}_{orb}([M/G])) \tag{9}$$

The ring structure on $\mathrm{CH}^*_{orb}([M/G])$, called *orbifold product*, denoted again by $\star_{orb}$, is determined by the product structure $\star_{orb} : \mathfrak{h}_{orb}([M/G]) \otimes \mathfrak{h}_{orb}([M/G]) \to \mathfrak{h}_{orb}([M/G])$ in Definition 2.5.
More concretely, $\mathrm{CH}^*_{orb}([M/G])$ is the G-invariant sub-$\mathbf{Q}$-algebra of an auxiliary (non-commutative) finitely $\mathbf{Q}_{\geq 0}$-graded $\mathbf{Q}$-algebra $\mathrm{CH}^*(M,G)$, which is defined by

$$\mathrm{CH}^*(M,G) := \left(\bigoplus_{g\in G} \mathrm{CH}^{*-\operatorname{age}(g)}(M^g), \star_{orb}\right),$$

where for two elements $g,h\in G$ and $\alpha\in \mathrm{CH}^{i-\operatorname{age}(g)}(M^g)$, $\beta\in\mathrm{CH}^{j-\operatorname{age}(h)}(M^h)$, their orbifold product is the following element in $\mathrm{CH}^{i+j-\operatorname{age}(gh)}(M^{gh})$:

$$\alpha\star_{orb}\beta := \iota_*\left(\alpha|_{M^{<g,h>}}\cdot\beta|_{M^{<g,h>}}\cdot c_{top}(F_{g,h})\right), \tag{10}$$

where $\iota : M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion.

Similarly, we have the orbifold K-theory:

Definition 2.7 (Orbifold K-theory)**.** The *orbifold K-theory* of $[M/G]$, denoted by $K_{orb}([M/G])$, is the G-invariant sub-algebra of the $\mathbf{Q}$-algebra $K(M,G)$, which is defined by

$$K(M,G) := \left(\bigoplus_{g\in G} K(M^g), \star_{orb}\right),$$

where for two elements $g,h\in G$ and $\alpha\in K(M^g)$, $\beta\in K(M^h)$, their orbifold product is the following element in $K(M^{gh})$:

$$\alpha\star_{orb}\beta := \iota_*\left(\alpha|_{M^{<g,h>}}\cdot\beta|_{M^{<g,h>}}\cdot\lambda_{-1}(F^{\vee}_{g,h})\right), \tag{11}$$

where $\iota : M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion.

[6] Here we use the fact that the category $\widetilde{\mathrm{CHM}}$ is $\mathbf{Q}-$linear and pseudo-abelian to define the G-invariant part A^G of a G-object A as the image of the projector $\frac{1}{|G|}\sum_{g\in G} g \in End(A)$.

Remark 2.8. The main interest of the paper lies in the situation when the underlying singular variety of the orbifold has at worst Gorenstein singularities. Recall that an algebraic variety X is *Gorenstein* if it is Cohen–Macaulay and the dualizing sheaf is a line bundle, denoted ω_X. In the case of a global quotient M/G, being Gorenstein is the same thing as the local G-triviality of the canonical bundle ω_M, which is again equivalent to the condition that the stabilizer of each point $x \in M$ is contained in $\mathrm{SL}(T_xM)$. In this case, it is straightforward to check that the Gorenstein condition implies that the age function actually takes values in the integers $\mathbf{Z}$ and therefore that the orbifold motive lies in the usual category of pure motives (without fractional twists) CHM. In particular, the orbifold Chow ring and orbifold cohomology ring are $\mathbf{Z}$-graded. Example 2.4 exhibits a typical situation that we would like to study ; see also Remark 3.2.

Remark 2.9 (Non-global quotients)**.** In the broader setting of smooth proper Deligne–Mumford stacks which are not necessarily finite group global quotients, the orbifold Chow ring is still well-defined in [1] but the down-to-earth construction as above, which is essential for the applications (*cf.* our slogan in §1), is lost (see however the equivariant treatment [25]). Another problem is that the definition of the orbifold Chow motive in this general setting is neither available in the literature nor covered in this paper. We expect that the techniques in [25] would suffice to produce such a theory at least for Deligne–Mumford stacks that are global quotients of smooth projective varieties by linear algebraic groups.

3. Motivic HyperKähler Resolution Conjecture

3.1. **A motivic version of the Cohomological HyperKähler Resolution Conjecture.** In [48], as part of the broader picture of *stringy geometry and topology of orbifolds*, Yongbin Ruan proposed the Cohomological HyperKähler Resolution Conjecture (CHRC) which says that the orbifold cohomology ring of a compact Gorenstein orbifold is isomorphic to the Betti cohomology ring of a hyperKähler crepant resolution of the underlying singular variety if one takes $\mathbf{C}$ as coefficients ; see Conjecture 1.1 in the introduction for the statement. As explained in Ruan [49], the plausibility of CHRC is justified by some considerations from theoretical physics as follows. Topological string theory predicts that the quantum cohomology theory of an orbifold should be equivalent to the quantum cohomology theory of a/any crepant resolution of (possibly some deformation of) the underlying singular variety. On the one hand, the orbifold cohomology ring constructed by Chen–Ruan [17] is the classical part (genus zero with three marked points) of the quantum cohomology ring of the orbifold (see [16]) ; on the other hand, the classical limit of the quantum cohomology of the resolution is the so-called *quantum corrected* cohomology ring ([49]). However if the crepant resolution has a hyperKähler structure, then all its Gromov–Witten invariants as well as the quantum corrections vanish and one expects therefore an equivalence, *i.e.* an isomorphism of $\mathbf{C}$-algebras, between the orbifold cohomology of the orbifold and the usual Betti cohomology of the hyperKähler crepant resolution.

Before moving on to a more algebro-geometric study, we have to recall some standard definitions and facts on (possibly singular) symplectic varieties (*cf.* [9], [43]) :

Definition 3.1.
- A *symplectic form* on a smooth complex algebraic variety is a closed holomorphic 2-form that is non-degenerate at each point. A smooth variety is called *holomorphic symplectic* or just *symplectic* if it admits a symplectic form. Projective examples include deformations of Hilbert schemes of K3 surfaces and abelian surfaces and generalized Kummer varieties *etc.*. A typical non-projective example is provided by the cotangent bundle of a smooth variety.

- A (possibly singular) *symplectic variety* is a normal complex algebraic variety such that its smooth part admits a symplectic form whose pull-back to a/any resolution extends to a holomorphic 2-form. A germ of such a variety is called a *symplectic singularity*. Such singularities are necessarily rational Gorenstein [9] and conversely, by a result of Namikawa [43], a normal variety is symplectic if and only if it has rational Gorenstein singularities and its smooth part admits a symplectic form. The main examples that we are dealing with are of the form of a quotient by a finite group of symplectic automorphisms of a smooth symplectic variety, *e.g.*, the symmetric products $S^{(n)} = S^n/\mathfrak{S}_n$ of smooth algebraic surfaces S with trivial canonical bundle.
- Given a singular symplectic variety X, a *symplectic resolution* or *hyperKähler resolution* is a resolution $f: Y \to X$ such that the pull-back of a symplectic form on the smooth part X_{reg} extends to a symplectic form on Y. Note that a resolution is symplectic if and only if it is *crepant*: $f^*\omega_X = \omega_Y$. The definition is independent of the choice of symplectic form on X_{reg}. A symplectic resolution is always semi-small. The existence of symplectic resolutions and the relations between them form a highly attractive topic in holomorphic symplectic geometry. An interesting situation, which will not be touched upon in this paper however, is the normalization of the closure of a nilpotent orbit in a complex semi-simple Lie algebra, whose symplectic resolutions are extensively studied in the literature (see [28], [13]). For examples relevant to this paper, see Examples 3.4.

Returning to the story of the HyperKähler Resolution Conjecture, in order to study algebraic cycles and motives of holomorphic symplectic varieties, especially with a view towards Beauville's splitting property conjecture [10] (see §7), we would like to propose the motivic version of the CHRC ; see Meta-Conjecture 1.2 in the introduction for the general statement. As we are dealing exclusively with the global quotient case in this paper and its sequel, we will concentrate on this more restricted case and on the more precise formulation Conjecture 1.3 in the introduction.

Remark 3.2 (Integral grading)**.** Notation as in Conjecture 1.3, since G preserves a symplectic form (hence a canonical form) of M, the quotient variety M/G has at worst Gorenstein singularities. As is pointed out in Remark 2.8, this implies that the age functions take values in $\mathbf{Z}$, the orbifold motive $\mathfrak{h}_{orb}([M/G])$ is in CHM, the usual category of (rational) Chow motives and the orbifold Chow ring $\mathrm{CH}^*_{orb}([M/G])$ is integrally graded.

Remark 3.3 (K-theoretic analogue)**.** As is mentioned in the Introduction (Conjecture 1.7), we are also interested in the K-theoretic version of the Hyper-Kähler Resolution Conjecture (KHRC) proposed in [34, Conjecture 1.2]. We want to point out that in Conjecture 1.3 above, the statement for Chow rings is more or less equivalent to KHRC ; however, the full formulation for Chow motivic algebras is, on the other hand, strictly richer. In fact, in all cases that we are able to prove KHRC, in this paper as well as in the upcoming one [30], we have to first solve MHRC on the motive level and deduce KHRC as a consequence. See §4 for the proof of Theorem 1.8.

Examples 3.4. All examples studied in this paper are in the following situation : let M and G be as in Conjecture 1.3 and Y be (the principal component of) the G-Hilbert scheme $G-\mathrm{Hilb}(M)$ of *G-clusters* of M, that is, a 0-dimensional G-invariant subscheme of M whose global functions form the regular G-representation (*cf.* [33], [42]). In some interesting cases, Y gives a symplectic resolution of M/G :

- Let S be a smooth algebraic surface and $G = \mathfrak{S}_n$ act on $M = S^n$ by permutation. By the result of Haiman [32], $Y = \mathfrak{S}_n-\mathrm{Hilb}(S^n)$ is isomorphic to the n-th punctual Hilbert scheme $S^{[n]}$, which is a crepant resolution, hence symplectic resolution if S has trivial canonical bundle, of $M/G = S^{(n)}$, the n-th symmetric product.

- Let A be an abelian surface, M be the kernel of the sum map $s : A^{n+1} \to A$ and $G = \mathfrak{S}_{n+1}$ acts on M by permutations, then $Y = G\text{--Hilb}(M)$ is isomorphic to the generalized Kummer variety $K_n(A)$ and is a symplectic resolution of M/G.

Although both sides of the isomorphism in Conjecture 1.3 are in the category CHM of motives with *rational* coefficients, it is in general necessary to make use of roots of unity to realize such an isomorphism of algebra objects. However, in some situation, it is possible to stay in CHM by making some sign change, which is related to the notion of *discrete torsion* in theoretical physics :

Definition 3.5 (Discrete torsion)**.** For any $g, h \in G$, let

$$\epsilon(g,h) := \frac{1}{2}\left(\operatorname{age}(g) + \operatorname{age}(h) - \operatorname{age}(gh)\right). \tag{12}$$

It is easy to check that

$$\epsilon(g_1, g_2)\epsilon(g_1g_2, g_3) = \epsilon(g_1, g_2g_3)\epsilon(g_2, g_3). \tag{13}$$

In the case when $\epsilon(g,h)$ is an integer for all $g, h \in G$, we can define the *orbifold Chow motive with discrete torsion* of a global quotient stack $[M/G]$, denoted by $\mathfrak{h}_{orb,dt}([M/G])$, by the following simple change of sign in Step (5) of Definition 2.5 : the orbifold product with discrete torsion

$$\star_{orb,dt} : \mathfrak{h}(M^{g_1})(-\operatorname{age}(g_1)) \otimes \mathfrak{h}(M^{g_2})(-\operatorname{age}(g_2)) \to \mathfrak{h}(M^{g_1g_2})(-\operatorname{age}(g_1g_2))$$

is the correspondence determined by the algebraic cycle

$$\epsilon(g_1, g_2) \cdot \delta_*(c_{top}(F_{g_1,g_2})) \in \mathrm{CH}^{\dim M^{g_1} + \dim M^{g_2} + \operatorname{age}(g_1) + \operatorname{age}(g_2) - \operatorname{age}(g_1g_2)}(M^{g_1} \times M^{g_2} \times M^{g_1g_2}).$$

Thanks to (13), $\star_{orb,dt}$ is still associative. Similarly, the *orbifold Chow ring with discrete torsion* of $[M/G]$ is obtained by replacing Equation (11) in Definition 2.6 by

$$\alpha \star_{orb,dt} \beta := \epsilon(g,h) \cdot i_*\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{top}(F_{g,h})\right), \tag{14}$$

which is again associative by (13).

Thanks to the notion of discrete torsions, we can have the following version of Motivic HyperKähler Resolution Conjecture, which takes place in the category of rational Chow motives and involves only rational Chow groups.

Conjecture 3.6 (MHRC : global quotient case with discrete torsion)**.** *In the same situation as Conjecture 1.3, suppose that $\epsilon(g,h)$ of Definition 3.5 is an integer for all $g, h \in G$. Then we have an isomorphism of (commutative) algebra objects in the category of Chow motives with* rational *coefficients :*

$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{orb,dt}([M/G]) \text{ in } \mathrm{CHM}.$$

In particular, we have an isomorphism of graded $\mathbf{Q}$*-algebras*

$$\mathrm{CH}^*(Y) \simeq \mathrm{CH}^*_{orb,dt}([M/G]).$$

It is easy to see that Conjecture 3.6 implies Conjecture 1.3 : to get rid of the discrete torsion sign change $\epsilon(g,h)$, it suffices to multiply the isomorphism to each factor $\mathfrak{h}(M^g)(-\operatorname{age}(g))$, or $\mathrm{CH}(M^g)$ by $\sqrt{-1}^{\operatorname{age}(g)}$, which involves of course the complex numbers (roots of unities at least).

3.2. **Toy examples.** To better illustrate the conjecture as well as the proof in the next section, we present in this subsection the explicit computation for two simplest nontrivial cases of MHRC.

3.2.1. *Hilbert squares of K3 surfaces.* Let S be a K3 surface or an abelian surface. Consider the involution f on $S \times S$ flipping the two factors. The relevant DM stack is $[S^2/f]$; its underlying singular symplectic variety is the second symmetric product $S^{(2)}$, and $S^{[2]}$ is its symplectic resolution. Let $\widetilde{S^2}$ be the blowup of S^2 along its diagonal Δ_S:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \widetilde{S^2} \arrow[d, "\epsilon"] \\ \Delta_S \arrow[r, "i"'] & S \times S \end{tikzcd}$$

Then f lifts to a natural involution on $\widetilde{S^2}$ and the quotient is

$$q : \widetilde{S^2} \twoheadrightarrow S^{[2]}.$$

On the one hand, $\mathrm{CH}^*(S^{[2]})$ is identified, *via* q^*, with the invariant part of $\mathrm{CH}^*(\widetilde{S^2})$; on the other hand, by Definition 2.6, $\mathrm{CH}^*_{orb}([S^2/\,\mathfrak{S}_2]) = \mathrm{CH}^*(S^2, \mathfrak{S}_2)^{inv}$. Therefore to check the MHRC 1.3 or 3.6 in this case, we only have to show the following

Proposition 3.7. *We have an isomorphism of* $\mathbf{C}$*-algebras:* $\mathrm{CH}^*(\widetilde{S^2})_{\mathbf{C}} \simeq \mathrm{CH}^*(S^2, \mathfrak{S}_2)_{\mathbf{C}}$. *If one makes a sign change in the orbifold product on the right hand side, there is an isomorphism of* $\mathbf{Q}$*-algebras of these two Chow rings with rational coefficients.*

Proof. A straightforward computation using (3) and (4) of Definition 2.5 shows that all obstruction bundles are trivial (at least in the Grothendieck group). Hence by Definition 2.6,

$$\mathrm{CH}^*(S^2, \mathfrak{S}_2) = \mathrm{CH}^*(S^2) \oplus \mathrm{CH}^{*-1}(\Delta_S)$$

whose ring structure is explicitly given by

- For any $\alpha \in \mathrm{CH}^i(S^2), \beta \in \mathrm{CH}^j(S^2)$, $\alpha \star_{orb} \beta = \alpha \bullet \beta \in \mathrm{CH}^{i+j}(S^2)$;
- For any $\alpha \in \mathrm{CH}^i(S^2), \beta \in \mathrm{CH}^j(\Delta_S)$, $\alpha \star_{orb} \beta = \alpha|_\Delta \bullet \beta \in \mathrm{CH}^{i+j}(\Delta_S)$;
- For any $\alpha \in \mathrm{CH}^i(\Delta_S), \beta \in \mathrm{CH}^j(\Delta_S)$, $\alpha \star_{orb} \beta = \Delta_*(\alpha \bullet \beta) \in \mathrm{CH}^{i+j+2}(S^2)$.

The blow-up formula (*cf.* for example, [54, Theorem 9.27]) provides an *a priori* only additive isomorphism

$$(\epsilon^*, j_*\pi^*) : \mathrm{CH}^*(S^2) \oplus \mathrm{CH}^{*-1}(\Delta_S) \xrightarrow{\simeq} \mathrm{CH}^*(\widetilde{S^2}),$$

whose inverse is given by $(\epsilon_*, -\pi_* j^*)$.
With everything given explicitly as above, it is straightforward to check that this isomorphism respects also the multiplication up to a sign change:

- For any $\alpha \in \mathrm{CH}^i(S^2), \beta \in \mathrm{CH}^j(S^2)$, one has $\epsilon^*(\alpha \star_{orb} \beta) = \epsilon^*(\alpha \bullet \beta) = \epsilon^*(\alpha) \bullet \epsilon^*(\beta)$;
- For any $\alpha \in \mathrm{CH}^i(S^2), \beta \in \mathrm{CH}^j(\Delta_S)$, the projection formula yields

$$j_*\pi^*(\alpha \star_{orb} \beta) = j_*\pi^*(\alpha|_\Delta \bullet \beta) = j_*\left(j^*\epsilon^*(\alpha) \bullet \pi^*\beta\right) = \epsilon^*(\alpha) \bullet j_*\pi^*(\beta);$$

- For any $\alpha \in \mathrm{CH}^i(\Delta_S), \beta \in \mathrm{CH}^j(\Delta_S)$, we make a sign change: $\alpha \star_{orb,dt} \beta = -\Delta_*(\alpha \bullet \beta)$ and we get

$$j_*\pi^*(\alpha) \bullet j_*\pi^*(\beta) = j_*\left(j^* j_*\pi^*\alpha \bullet \pi^*\beta\right) = j_*\left(c_1(N_{E/\widetilde{S^2}}) \bullet \pi^*\alpha \bullet \pi^*\beta\right) = -\epsilon^*\Delta_*(\alpha \bullet \beta) = \epsilon^*(\alpha \star_{orb,dt} \beta),$$

where in the last but one equality one uses the excess intersection formula for the blowup diagram together with the fact that $N_{E/\widetilde{S^2}} = \mathscr{O}_\pi(-1)$ while the excess normal bundle is

$$\pi^* T_S/\mathscr{O}_\pi(-1) \simeq T_\pi \otimes \mathscr{O}_\pi(-1) \simeq \mathscr{O}_\pi(1),$$

where one uses the assumption that $K_S = 0$ to deduce that $T_\pi \simeq \mathscr{O}_\pi(2)$.

As the sign change is exactly the one given by discrete torsion in Definition 3.5, we have an isomorphism of $\mathbf{Q}$-algebras

$$\mathrm{CH}^*(S^{[2]}) \simeq \mathrm{CH}^*_{orb,dt}([S^2/\mathfrak{S}_2]).$$

Without making any sign change, the above computation shows that

$$(\epsilon^*, \sqrt{-1}\cdot j_*\pi^*) : \mathrm{CH}^*(S^2)_{\mathbf{C}} \oplus \mathrm{CH}^{*-1}(\Delta_S)_{\mathbf{C}} \xrightarrow{\simeq} \mathrm{CH}^*(\widetilde{S^2})_{\mathbf{C}}$$

is an isomorphism of $\mathbf{C}$-algebras, whose inverse is given by $(\epsilon_*, \sqrt{-1}\cdot\pi_* j^*)$. Hence the isomorphism of $\mathbf{C}$-algebras:

$$\mathrm{CH}^*(S^{[2]})_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([S^2/\mathfrak{S}_2])_{\mathbf{C}}.$$

□

3.2.2. *Kummer K3 surfaces.* Let A be an abelian surface. We always identify $A^2_0 := \mathrm{Ker}\left(A\times A \xrightarrow{+} A\right)$ with A by $(x,-x)\mapsto x$, under which the associated Kummer K3 surface $S := K_1(A)$ is a hyperKähler crepant resolution of the symplectic quotient A/f, where f is the involution of multiplication by -1 on A. Consider the blow-up of A along the fixed locus F which is the set of 2-torsion points of A:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \widetilde{A} \arrow[d, "\epsilon"] \\ F \arrow[r, "i"'] & A. \end{tikzcd}$$

Then S is the quotient of $\widetilde{A}$ by $\widetilde{f}$, the lifting of the involution f. As in the previous toy example, the MHRC in the present situation is reduced to the following

Proposition 3.8. *We have an isomorphism of* $\mathbf{C}$*-algebras:* $\mathrm{CH}^*(\widetilde{A})_{\mathbf{C}} \simeq \mathrm{CH}^*(A,\mathfrak{S}_2)_{\mathbf{C}}$. *If one makes a sign change in the orbifold product on the right hand side, there is an isomorphism of* $\mathbf{Q}$*-algebras of these two Chow rings with rational coefficients.*

Proof. As the computation is quite similar to that of Proposition 3.7, we only give a sketch. By Definition 2.6, $\mathrm{age}(\mathrm{id}) = 0$, $\mathrm{age}(\widetilde{f}) = 1$ and $\mathrm{CH}^*(A,\mathfrak{S}_2) = \mathrm{CH}^*(A)\oplus\mathrm{CH}^{*-1}(F)$ whose ring structure is given by

- For any $\alpha\in\mathrm{CH}^i(A), \beta\in\mathrm{CH}^j(A)$, $\alpha \star_{orb}\beta = \alpha\bullet\beta\in\mathrm{CH}^{i+j}(A)$;
- For any $\alpha\in\mathrm{CH}^i(A), \beta\in\mathrm{CH}^0(F)$, $\alpha \star_{orb}\beta = \alpha|_F\bullet\beta\in\mathrm{CH}^{i}(F)$;
- For any $\alpha\in\mathrm{CH}^0(F), \beta\in\mathrm{CH}^0(F)$, $\alpha \star_{orb}\beta = i_*(\alpha\bullet\beta)\in\mathrm{CH}^{2}(A)$.

Again by the blow-up formula, we have an isomorphism

$$(\epsilon^*, j_*\pi^*) : \mathrm{CH}^*(A)\oplus\mathrm{CH}^{*-1}(F) \xrightarrow{\simeq} \mathrm{CH}^*(\widetilde{A}),$$

whose inverse is given by $(\epsilon_*, -\pi_* j^*)$. It is now straightforward to check that they are moreover ring isomorphisms with the left-hand side equipped with the orbifold product. The sign change comes from the negativity of the self-intersection of (the components of) the exceptional divisor. □

4. Main results and steps of the proofs

The main results of the paper are the verification of Conjecture 3.6, hence Conjecture 1.3 in the following two cases (A) and (B). See Theorem 1.4 and Theorem 1.5 in the introduction for the precise statements. These two theorems are proved in §5 and §6 respectively.

Let A be an abelian surface and n be a positive integer.

Case (A) (Hilbert schemes of abelian surfaces)
$M = A^n$ endowed with the natural action of $G = \mathfrak{S}_n$. The symmetric product $A^{(n)} = M/G$ is a singular symplectic variety and the Hilbert–Chow morphism

$$\rho : Y = A^{[n]} \to A^{(n)}$$

gives a symplectic resolution.

Case (B) (Generalized Kummer varieties)
$M = A_0^{n+1} := \operatorname{Ker}\left(A^{n+1} \xrightarrow{s} A\right)$ endowed with the natural action of $G = \mathfrak{S}_{n+1}$. The quotient $A_0^{n+1}/\,\mathfrak{S}_{n+1} = M/G$ is a singular symplectic variety. Recall that the *generalized Kummer variety* $K_n(A)$ is the fiber over O_A of the isotrivial fibration $A^{[n+1]} \to A^{(n+1)} \xrightarrow{s} A$. The restriction of the Hilbert–Chow morphism

$$Y = K_n(A) \to A_0^{n+1}/\,\mathfrak{S}_{n+1}$$

gives a symplectic resolution.

Let us deduce the KHRC 1.7 in these two cases from our main results[7] :

Proof of Theorem 1.8. Let M and G be either as in Case (A) or Case (B) above. Without using discrete torsions, we have an isomorphism of $\mathbf{C}$-algebras $\mathrm{CH}^*(M)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([M/G])$. An *orbifold Chern character* is constructed in [34], which by [34, Main result 3] provides an isomorphism of $\mathbf{Q}$-algebras :

$$\mathrm{ch}_{orb} : K_{orb}\left([M/G]\right)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{CH}^*_{orb}\left([M/G]\right)_{\mathbf{Q}}.$$

The desired isomorphism of algebras is then obtained by the composition of ch_{orb} (tensored with $\mathbf{C}$), the Chern character isomorphism $\mathrm{ch} : K(Y)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{CH}^*(Y)_{\mathbf{Q}}$ tensored with $\mathbf{C}$, and the isomorphism $\mathrm{CH}^*(M)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([M/G])_{\mathbf{C}}$ from our main results.
Similarly, for topological K-theory one uses the orbifold topological Chern character, which is also constructed in [34],

$$\mathrm{ch}_{orb} : K^{top}_{orb}\left([M/G]\right)_{\mathbf{Q}} \xrightarrow{\simeq} H^*_{orb}\left([M/G], \mathbf{C}\right),$$

together with $\mathrm{ch} : K^{top}(Y)_{\mathbf{Q}} \xrightarrow{\simeq} H^*(Y, \mathbf{C})$ and the Cohomological Hyper-Kähler Resolution Conjecture :

$$H^*_{orb}\left([M/G], \mathbf{C}\right) \simeq H^*(Y, \mathbf{C}),$$

which is proved in Case (A) in [26] and [51] based on [38] and in Case (B) in Theorem 1.6. □

In the rest of this section, we explain the main steps of the proofs of Theorem 1.4 and Theorem 1.5. For both cases, the proof proceeds in three steps. For each step, Case (A) is quite straightforward and Case (B) requires more subtle and technical arguments.

Step (i).

[7]Note that our proof for KHRC passes through Chow rings, thus a direct geometric (sheaf-theoretic) description of the isomorphism between $K(Y)_{\mathbf{C}}$ and $K_{orb}([M/G])_{\mathbf{C}}$ is still missing.

Recall the notation $\mathfrak{h}(M,G) := \oplus_{g\in G}\,\mathfrak{h}(M^g)(-\operatorname{age}(g))$. Denote by

$$\iota : \mathfrak{h}\,(M,G)^G \hookrightarrow \mathfrak{h}\,(M,G) \text{ and } p : \mathfrak{h}\,(M,G) \twoheadrightarrow \mathfrak{h}\,(M,G)^G$$

the inclusion of and the projection onto the G-invariant part $\mathfrak{h}\,(M,G)^G$, which is a direct factor of $\mathfrak{h}\,(M,G)$ inside CHM. We will first construct a (*a priori* just additive) G-equivariant morphism of Chow motives $\mathfrak{h}(Y) \to \mathfrak{h}(M,G)$, given by some correspondences $\left\{(-1)^{\operatorname{age}(g)}U^g \in \mathrm{CH}(Y\times M^g)\right\}_{g\in G}$ inducing an (additive) isomorphism

$$\phi = p\circ \sum_g (-1)^{\operatorname{age}(g)}U^g : \mathfrak{h}(Y) \xrightarrow{\simeq} \mathfrak{h}_{orb}([M/G]) = \mathfrak{h}(M,G)^G.$$

The isomorphism ϕ will have the property that its inverse is $\psi := (\frac{1}{|G|}\sum_g {}^tU^g)\circ\iota$ (see Proposition 5.2 and Proposition 6.4 for Case (A) and (B) respectively). Our goal is then to prove that these morphisms are moreover multiplicative (after the sign change by discrete torsion), *i.e.* the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{h}(Y)^{\otimes 2} \arrow[r, "\delta_Y"] \arrow[d, "\phi^{\otimes 2}"'] & \mathfrak{h}(Y) \arrow[d, "\phi"] \\ \mathfrak{h}_{orb}([M/G])^{\otimes 2} \arrow[r, "\star_{orb,dt}"'] & \mathfrak{h}_{orb}([M/G]) \end{tikzcd} \tag{15}$$

The main theorem will then be deduced from the following

Proposition 4.1. *Notation being as before, the following two algebraic cycles have the same symmetrization in* $\mathrm{CH}\Big(\big(\coprod_{g\in G} M^g\big)^3\Big)$*:*

- $W := \left(\frac{1}{|G|}\sum_g U^g \times \frac{1}{|G|}\sum_g U^g \times \sum_g (-1)^{\operatorname{age}(g)}U^g\right)_*(\delta_Y)$;
- *The algebraic cycle Z determining the orbifold product (Definition 2.5(5)) with the sign change by discrete torsion (Definition 3.5):*

$$Z|_{M^{g_1}\times M^{g_2}\times M^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1g_2 \\ \epsilon_{g_1,g_2}\cdot \delta_* c_{top}(F_{g_1,g_2}) & \text{if } g_3 = g_1g_2. \end{cases}$$

Here the symmetrization *of a cycle in* $\big(\coprod_{g\in G} M^g\big)^3$ *is the operation*

$$\gamma \mapsto \frac{1}{|G|^3}\sum_{g_1,g_2,g_3\in G} (g_1,g_2,g_3).\gamma.$$

Proposition 4.1 implies Theorem 1.4 and 1.5. The only thing to show is the commutativity of (15), which is of course equivalent to the commutativity of the diagram

$$\begin{tikzcd} \mathfrak{h}(Y)^{\otimes 2} \arrow[r, "\delta_Y"] & \mathfrak{h}(Y) \arrow[d, "\phi"] \\ \mathfrak{h}_{orb}([M/G])^{\otimes 2} \arrow[r, "\star_{orb,dt}"'] \arrow[u, "\psi^{\otimes 2}"] & \mathfrak{h}_{orb}([M/G]) \end{tikzcd}$$

By the definition of ϕ and ψ, we need to show that the following diagram is commutative :

$$\begin{tikzcd} \mathfrak{h}(Y)^{\otimes 2} \arrow[r, "\delta_Y"] & \mathfrak{h}(Y) \arrow[d, "\sum_g (-1)^{\mathrm{age}(g)} U^g"] \\ \mathfrak{h}(M,G)^{\otimes 2} \arrow[u, "(\frac{1}{|G|}\sum_g {}^tU^g)^{\otimes 2}"] & \mathfrak{h}(M,G) \arrow[d, "p"] \\ \mathfrak{h}_{orb}([M/G])^{\otimes 2} \arrow[u, "\iota^{\otimes 2}"] \arrow[r, "\star_{orb,dt}"'] & \mathfrak{h}_{orb}([M/G]) \end{tikzcd} \tag{16}$$

It is elementary to see that the composition $\sum_g (-1)^{\mathrm{age}(g)} U^g \circ \delta_Y \circ (\frac{1}{|G|}\sum_g {}^tU^g)^{\otimes 2}$ is the morphism (or correspondence) induced by the cycle W in Proposition 4.1. On the other hand, $\star_{orb,dt}$ for $\mathfrak{h}_{orb}([M/G])$ is by definition $p \circ Z \circ \iota^{\otimes 2}$. Therefore, the desired commutativity, hence also the main results, amounts to the equality $p \circ W \circ \iota^{\otimes 2} = p \circ Z \circ \iota^{\otimes 2}$, which says exactly that the symmetrizations of W and of Z are equal in $\mathrm{CH}\left(\left(\coprod_{g\in G} M^g\right)^3\right)$. □

One is therefore reduced to show Proposition 4.1 in both cases (A) and (B).

Step (ii).

We prove that W on the one hand and Z on the other hand, as well as their symmetrizations, are both symmetrically distinguished in the sense of O'Sullivan [45] (see Definition 5.4). In Case (B) concerning the generalized Kummer varieties, we have to generalize a little bit the category of abelian varieties and the corresponding notion of symmetrically distinguished cycles, in order to deal with algebraic cycles on 'non-connected abelian varieties' in a canonical way. By the result of O'Sullivan [45] (see Theorem 5.5 and Theorem 5.6), it suffices for us to check that the symmetrizations of W and Z are numerically equivalent.

Step (iii).

Finally, in Case (A), explicit computations of the cohomological realization of ϕ show that the induced (iso-)morphism $\phi : H^*(Y) \to H^*_{orb}([M/G])$ is the same as the one constructed in [38]. While in Case (B), based on the result of [44], one can prove that the cohomological realization of ϕ satisfies Ruan's original Cohomological HyperKähler Resolution Conjecture. Therefore the symmetrizations of W and Z are homologically equivalent, which finishes the proof by Step (ii).

5. Case (A) : Hilbert schemes of abelian surfaces

We prove Theorem 1.4 in this section. Notations are as before : $M := A^n$ with the action of $G := \mathfrak{S}_n$ and the quotient $X := A^{(n)} := M/G$. Then the Hilbert–Chow morphism

$$\rho : A^{[n]} =: Y \to A^{(n)}$$

gives a symplectic resolution.

5.1. **Step (i) – Additive isomorphisms.** In this subsection, we establish an isomorphism between $\mathfrak{h}(Y)$ and $\mathfrak{h}_{orb}([M/G])$ by using results in [19]. First we construct some correspondences similar to the ones used in *loc.cit.* . For each $g \in G = \mathfrak{S}_n$, let $O(g)$ be the set of orbits of g as a permutation of $\{1, 2, \cdots n\}$. It is computed in Example 2.4 that

$$\mathrm{age}(g) = n - |O(g)|.$$

As in [26], we make the natural identification
$$(A^n)^g \simeq A^{O(g)}.$$
Let
$$U^g := (A^{[n]} \times_{A^{(n)}} (A^n)^g)_{red} = \left\{(z, x_1, \cdots, x_n) \in A^{[n]} \times (A^n)^g \,\middle|\, \rho(z) = [x_1] + \cdots + [x_n]\right\} \tag{17}$$
be the incidence variety, where $\rho : A^{[n]} \to A^{(n)}$ is the Hilbert–Chow morphism. As the notation suggests, U^g is the fixed locus of the induced automorphism g on the *isospectral Hilbert scheme*
$$U := U^{\mathrm{id}} = A^{[n]} \times_{A^{(n)}} A^n = \left\{(z, x_1, \cdots, x_n) \in A^{[n]} \times A^n \,\middle|\, \rho(z) = [x_1] + \cdots + [x_n]\right\}.$$
Note that $\dim U^g = n + |O(g)| = 2n - \mathrm{age}(g)$ and $\dim\left(A^{[n]} \times (A^n)^g\right) = 2\dim U^g$. We consider the following correspondence for each $g \in G$,
$$\Gamma_g := (-1)^{\mathrm{age}(g)} U^g \in \mathrm{CH}^{2n-\mathrm{age}(g)}\left(A^{[n]} \times (A^n)^g\right), \tag{18}$$
which defines a morphism of Chow motives :
$$\Gamma := \sum_{g\in G} \Gamma_g : \mathfrak{h}(A^{[n]}) \to \bigoplus_{g\in G} \mathfrak{h}\left((A^n)^g\right)(-\mathrm{age}(g)) =: \mathfrak{h}(A^n, \mathfrak{S}_n), \tag{19}$$
here we used the notation from Definition 2.5.

Lemma 5.1. *The algebraic cycle Γ in (19) defines an $\mathfrak{S}_n$-equivariant morphism with respect to the trivial action on $A^{[n]}$ and the action on $\mathfrak{h}(A^n, \mathfrak{S}_n)$ of Definition 2.5.*

Proof. For each $g, h \in G$, as the age function is invariant under conjugation, it suffices to show that the following composition is equal to $\Gamma_{hgh^{-1}}$:
$$\mathfrak{h}(A^{[n]}) \xrightarrow{\Gamma_g} \mathfrak{h}\left((A^n)^g\right)(-\mathrm{age}(g)) \xrightarrow{h} \mathfrak{h}\left((A^n)^{hgh^{-1}}\right)(-\mathrm{age}(g)),$$
which follows from the commutative diagram :
$$\begin{tikzcd} A^{[n]} & U^g \arrow[l] \arrow[r, "h", "\simeq"'] \arrow[d] & U^{hgh^{-1}} \arrow[d] \\ & (A^n)^g \arrow[r, "h", "\simeq"'] & (A^n)^{hgh^{-1}}. \end{tikzcd}$$
□

As before, $\iota : \mathfrak{h}(A^n, G)^G \hookrightarrow \mathfrak{h}(A^n, G)$ and $p : \mathfrak{h}(A^n, G) \twoheadrightarrow \mathfrak{h}(A^n, G)^G$ are the inclusion of and the projection onto the G-invariant part. Thanks to Lemma 5.1, we obtain the desired morphism
$$\phi := p \circ \Gamma : \mathfrak{h}(A^{[n]}) \to \mathfrak{h}_{orb}([A^n/G]) = \mathfrak{h}(A^n, G)^G, \tag{20}$$
which satisfies $\Gamma = \iota \circ \phi$.

Now one can reformulate the result of de Cataldo–Migliorini [19], which actually works for all surfaces, as follows :

Proposition 5.2. *The morphism ϕ is an isomorphism, whose inverse is given by $\psi := \frac{1}{n!}\left(\sum_{g\in G} {}^tU^g\right) \circ \iota$, where*
$${}^tU^g : \mathfrak{h}\left((A^n)^g\right)(-\mathrm{age}(g)) \to \mathfrak{h}(A^{[n]})$$
is the transposed correspondence of U^g.

Proof. We start by a recollection of some notation from [19]. Let $\mathscr{P}(n)$ be the set of partitions of n. Given such a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l) = (1^{a_1} \cdots n^{a_n})$, where $l := |\lambda|$ is the *length* of λ and $a_i = |\{j \mid 1 \leq j \leq n\,; \lambda_j = i\}|$, we define $\mathfrak{S}_\lambda := \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_n}$. Let A^λ be A^l, equipped with the natural action of $\mathfrak{S}_\lambda$ and with the natural morphism to $A^{(n)}$ by sending $(x_1, \cdots, x_l)$ to $\sum_{j=1}^{l} \lambda_j [x_j]$. Define the incidence subvariety $U^\lambda := (A^{[n]} \times_{A^{(n)}} A^\lambda)_{red}$. Denote the quotient $A^{(\lambda)} := A^\lambda / \mathfrak{S}_\lambda$ and $U^{(\lambda)} := U^\lambda / \mathfrak{S}_\lambda$, where the latter is also regarded as a correspondence between $A^{[n]}$ and $A^{(\lambda)}$.
The main theorem in [19] asserts that the following correspondence is an isomorphism. See Remark 2.2 (4) for the use of Chow motives of global quotients (like $A^{(\lambda)}$).

$$\phi' := \sum_{\lambda \in \mathscr{P}(n)} U^{(\lambda)} : \mathfrak{h}(A^{[n]}) \xrightarrow{\simeq} \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}(A^{(\lambda)})(|\lambda| - n)\,;$$

moreover, the inverse of ϕ' is given by

$$\psi' := \sum_{\lambda \in \mathscr{P}(n)} \frac{1}{m_\lambda} \cdot {}^t U^{(\lambda)} : \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}(A^{(\lambda)})(|\lambda| - n) \xrightarrow{\simeq} \mathfrak{h}(A^{[n]}),$$

where $m_\lambda = (-1)^{n-|\lambda|} \prod_{j=i}^{|\lambda|} \lambda_j$ is a non-zero constant. To relate our morphism ϕ to the above isomorphism ϕ' as well as their inverses, one uses the following elementary

Lemma 5.3. *One has a canonical isomorphism :*

$$\left(\bigoplus_{g \in \mathfrak{S}_n} \mathfrak{h}\left((A^n)^g\right)(-\operatorname{age}(g)) \right)^{\mathfrak{S}_n} \xrightarrow{\simeq} \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}\left(A^{(\lambda)}\right)(|\lambda| - n).$$

Proof. By $g \in \lambda$, we mean that the partition determined by the permutation $g \in \mathfrak{S}_n$ is $\lambda \in \mathscr{P}(n)$. For each partition λ of length l, since for any $g \in \lambda$, the stabilizer of g for the action of $\mathfrak{S}_n$ is isomorphic to the semi-direct product $(\mathbf{Z}/\lambda_1 \times \cdots \times \mathbf{Z}/\lambda_l) \rtimes \mathfrak{S}_\lambda$. Thus

$$\left(\bigoplus_{g \in \lambda} \mathfrak{h}\left((A^n)^g\right) \right)^{\mathfrak{S}_n} \simeq \mathfrak{h}\left((A^n)^g\right)^{(\mathbf{Z}/\lambda_1 \times \cdots \times \mathbf{Z}/\lambda_l) \rtimes \mathfrak{S}_\lambda} \simeq \mathfrak{h}\left((A^n)^g\right)^{\mathfrak{S}_\lambda} \simeq \mathfrak{h}\left(A^{(\lambda)}\right).$$

One concludes by taking the direct sum of this isomorphism over all $\lambda \in \mathscr{P}(n)$. ◻

Now it is easy to conclude the proof of Proposition 5.2. For any $g \in \lambda$, the isomorphism between $(A^n)^g$ and A^λ will identify U^g to U^λ. Hence the composition of ϕ with the isomorphism in Lemma 5.3 is equal to $\sum_{\lambda \in \mathscr{P}(n)} \frac{n!}{m_\lambda} U^{(\lambda)}$, which is an isomorphism since $\phi' = \sum_{\lambda \in \mathscr{P}(n)} U^{(\lambda)}$ is. As a consequence, ϕ itself is an isomorphism and its inverse is the composition of the isomorphism in Lemma 5.3 followed by $\psi' = \sum_{\lambda \in \mathscr{P}(n)} \frac{1}{n!} \cdot {}^t U^{(\lambda)}$, which is $\frac{1}{n!} \left(\sum_{g \in G} {}^t U^g \right) \circ \iota =: \psi$. ◻

Then to show Theorem 1.4, it suffices to prove Proposition 4.1 in this situation, which will be done in the next two steps.

5.2. **Step (ii) – Symmetrically distinguished cycles on abelian varieties.** The following definition is due to O'Sullivan [45]. Recall that all Chow groups are with rational coefficients. As in *loc.cit.* we denote in this section by $\overline{\mathrm{CH}}$ the $\mathbf{Q}$-vector space of algebraic cycles modulo the numerical equivalence relation.

Definition 5.4 (Symmetrically distinguished cycles [45])**.** Let A be an abelian variety and $\alpha \in \mathrm{CH}^i(A)$. For each integer $m \geq 0$, denote by $V_m(\alpha)$ the $\mathbf{Q}$-vector subspace of $\mathrm{CH}(A^m)$ generated by elements of the form

$$p_*(\alpha^{r_1} \times \alpha^{r_2} \cdots \times \alpha^{r_n}),$$

where $n \leq m$, $r_j \geq 0$ are integers, and $p : A^n \to A^m$ is a closed immersion with each component $A^n \to A$ being either a projection or the composite of a projection with $[-1] : A \to A$. Then α is called *symmetrically distinguished* if for every m the restriction of the projection $\mathrm{CH}(A^m) \to \overline{\mathrm{CH}}(A^m)$ to $V_m(\alpha)$ is injective.

Despite their seemingly complicated definition, symmetrically distinguished cycles behave very well. More precisely, we have

Theorem 5.5 (O'Sullivan [45])**.** *Let A be an abelian variety.*

(1) *The symmetric distinguished cycles in $\mathrm{CH}^i(A)$ form a sub-$\mathbf{Q}$-vector space.*
(2) *The fundamental class of A is symmetrically distinguished and the intersection product of two symmetrically distinguished cycles is symmetrically distinguished. They form therefore a graded sub-$\mathbf{Q}$-algebra of $\mathrm{CH}^*(A)$.*
(3) *Let $f : A \to B$ be a morphism of abelian varieties, then $f_* : \mathrm{CH}(A) \to \mathrm{CH}(B)$ and $f^* : \mathrm{CH}(B) \to \mathrm{CH}(A)$ preserve symmetrically distinguished cycles.*

The reason why this notion is very useful in practice is that it allows us to conclude an equality of algebraic cycles modulo rational equivalence from an equality modulo numerical equivalence (or, *a fortiori*, modulo homological equivalence) :

Theorem 5.6 (O'Sullivan [45])**.** *The composition $\mathrm{CH}(A)_{sd} \hookrightarrow \mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$ is an isomorphism of $\mathbf{Q}$-algebras, where $\mathrm{CH}(A)_{sd}$ is the sub-algebra of symmetrically distinguished cycles. In other words, in each numerical class of algebraic cycle on A, there exists a unique symmetrically distinguished algebraic cycle modulo rational equivalence. In particular, a (polynomial of) symmetrically distinguished cycles is trivial in $\mathrm{CH}(A)$ if and only if it is numerically trivial.*

Returning to the proof of Theorem 1.4, it remains to prove Proposition 4.1. Keep the same notation as in Step (i), we first prove that in our situation the two cycles in Proposition 4.1 are symmetrically distinguished.

Proposition 5.7. *The following two algebraic cycles, as well as their symmetrizations,*

- $W := \left(\frac{1}{|G|}\sum_g U^g \times \frac{1}{|G|}\sum_g U^g \times \sum_g (-1)^{\mathrm{age}(g)} U^g\right)_* (\delta_{A^{[n]}})$;
- *The algebraic cycle Z determining the orbifold product (Definition 2.5(5)) with the sign change by discrete torsion (Definition 3.5) :*

$$Z|_{M^{g_1}\times M^{g_2}\times M^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1g_2 \\ \epsilon_{g_1,g_2} \cdot \delta_* c_{top}(F_{g_1,g_2}) & \text{if } g_3 = g_1g_2. \end{cases}$$

are symmetrically distinguished in $\mathrm{CH}\left(\left(\coprod_{g\in G}(A^n)^g\right)^3\right)$.

Proof. For W, it amounts to show that for any $g_1, g_2, g_3 \in G$, we have that $(U^{g_1} \times U^{g_2} \times U^{g_3})_* (\delta_{A^{[n]}})$ are symmetrically distinguished in $\mathrm{CH}\left((A^n)^{g_1} \times (A^n)^{g_2} \times (A^n)^{g_3}\right)$. Indeed, by [57, Proposition 5.6], $(U^{g_1} \times U^{g_2} \times U^{g_3})_* (\delta_{A^{[n]}})$ is a polynomial of big diagonals of $(A^n)^{g_1} \times (A^n)^{g_2} \times (A^n)^{g_3} =: A^N$. However, all big diagonals of A^N are clearly symmetrically distinguished since $\Delta_A \in \mathrm{CH}(A\times A)$ is. By Theorem

5.5, W is symmetrically distinguished.
As for Z, for any fixed $g_1, g_2 \in G$, F_{g_1,g_2} is easily seen to always be a trivial vector bundle, at least virtually, hence its top Chern class is either 0 or 1 (the fundamental class), which is of course symmetrically distinguished. Also recall that (Definition 2.5)

$$\delta : (A^n)^{<g_1,g_2>} \hookrightarrow (A^n)^{g_1} \times (A^n)^{g_2} \times (A^n)^{g_1g_2},$$

which is a (partial) diagonal inclusion, in particular a morphism of abelian varieties. Therefore $\delta_*(c_{top}(F_{g_1,g_2}))$ is symmetrically distinguished by Theorem 5.5, hence so is Z.
Finally, since any automorphism in $G \times G \times G$ preserves symmetrically distinguished cycles, symmetrizations of Z and W remain symmetrically distinguished. □

By Theorem 5.6, in order to show Proposition 4.1, it suffices to show on the one hand that the symmetrizations of Z and W are both symmetrically distinguished, and on the other hand that they are numerically equivalent. The first part is exactly the previous Proposition 5.7 and we now turn to an *a priori* stronger version of the second part in the following final step.

5.3. **Step (iii) – Cohomological realizations.** We will show in this subsection that the symmetrizations of the algebraic cycles W and Z have the same (rational) cohomology class. To this end, it is enough to show the following

Proposition 5.8. *The cohomology realization of the (additive) isomorphism*

$$\phi : \mathfrak{h}(A^{[n]}) \xrightarrow{\simeq} \left(\oplus_{g\in G}\, \mathfrak{h}((A^n)^g)(-\operatorname{age}(g))\right)^{\mathfrak{S}_n}$$

is an isomorphism of $\mathbf{Q}$*-algebras*

$$\overline{\phi} : H^*(A^{[n]}) \xrightarrow{\simeq} H^*_{orb,dt}([A^n/\,\mathfrak{S}_n]) = \left(\oplus_{g\in G} H^{*-2\operatorname{age}(g)}((A^n)^g), \star_{orb,dt}\right)^{\mathfrak{S}_n}.$$

In other words, $\operatorname{Sym}(W)$ *and* $\operatorname{Sym}(Z)$ *are homologically equivalent.*

Proof. The existence of isomorphism of $\mathbf{Q}$-algebras between $H^*(A^{[n]})$ and $H^*_{orb,dt}([A^n/\,\mathfrak{S}_n])$ is established by Fantechi and Göttsche [26, Theorem 3.10] based on the work of Lehn and Sorger [38]. Therefore by the definition of ϕ in Step (i), it suffices to show that the cohomological correspondence

$$\Gamma_* := \sum_{g\in\mathfrak{S}_n} (-1)^{\operatorname{age}(g)} U^g_* : H^*(A^{[n]}) \to \bigoplus_{g\in\mathfrak{S}_n} H^{*-2\operatorname{age}(g)}\left((A^n)^g\right)$$

coincides with the following inverse of the isomorphism Ψ used in Fantechi–Göttsche [26, Theorem 3.10]

$$\begin{array}{rcl} \Phi : H^*(A^{[n]}) & \to & \bigoplus_{g\in\mathfrak{S}_n} H^{*-2\operatorname{age}(g)}\left((A^n)^g\right) \\ \mathfrak{p}_{\lambda_1}(\alpha_1)\cdots\mathfrak{p}_{\lambda_l}(\alpha_l)\,\mathbb{1} & \mapsto & n!\cdot \operatorname{Sym}(\alpha_1\times\cdots\times\alpha_l), \end{array}$$

Let us explain the notations from [26] in the above formula: $\alpha_1, \dots, \alpha_l \in H^*(A)$, $\times$ stands for the exterior product $\prod \operatorname{pr}_i^*(-)$, $\mathfrak{p}$ is the Nakajima operator (*cf.* [41]), $\mathbb{1} \in H^0(A^{[0]}) \simeq \mathbf{Q}$ is the fundamental class of the point, $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition of n, $g \in \mathfrak{S}_n$ is a permutation of type λ with a numbering of orbits of g (as a permutation) chosen: $\{1, \dots, l\} \xrightarrow{\sim} O(g)$, such that λ_j is the length of the j-th orbit, then the class $\alpha_1 \times \cdots \times \alpha_l$ is placed in the direct summand indexed by g and Sym means the symmetrization operation $\frac{1}{n!}\sum_{h\in\mathfrak{S}_n} h$. Note that $\operatorname{Sym}(\alpha_1\times\cdots\times\alpha_l)$ is independent of the choice of g, numbering *etc.*

Recall that the Nakajima operator $\mathfrak{p}_k(\alpha) : H^*(A^{[l]}) \to H^*(A^{[l+k]})$ for any l is by definition $\beta \mapsto {I_{l,l+k}}^*(\beta \times \alpha) = q_*p^*(\beta \times \alpha)$ using the following correspondence :

$$\begin{tikzcd}[column sep=small] & I_{l,l+k} := \left\{(\xi', x, \xi) \in A^{[l+k]} \times A \times A^{[l]} \;\middle|\; \xi \subset \xi' \;;\; \rho(\xi') = \rho(\xi) + k[x]\right\} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A^{[l+k]} & & A^{[l]} \times A \end{tikzcd}$$

Here and in the sequel, ρ is always the Hilbert–Chow morphism.
By a repeated (but straightforward) use of the projection formula, one has

$$\mathfrak{p}_{\lambda_1}(\alpha_1) \cdots \mathfrak{p}_{\lambda_l}(\alpha_l)\, \mathbb{1} = {I_\lambda}^*(\alpha_1 \times \cdots \times \alpha_l) = q_*p^*(\alpha_1 \times \cdots \times \alpha_l)$$

using the following correspondence :

$$\begin{tikzcd}[column sep=small] & I_\lambda := \left\{(x_1, \ldots, x_l, \xi_1, \ldots, \xi_l) \;\middle|\; \substack{x_i \in A \;;\; \xi_1 \subset \cdots \subset \xi_l \;;\\ \rho(\xi_i) = \rho(\xi_{i-1}) + \lambda_i[x_i]}\right\} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A^{[n]} & & A^\lambda := A^l \end{tikzcd}$$

where $(x_1, \ldots, x_l, \xi_1, \ldots, \xi_l)$ is sent to $(x_1, \ldots, x_l)$ by p and to $\xi_l \in A^{[n]}$ by q. It is easy to see that the natural morphism $I_\lambda \to U^\lambda = \left\{(\xi, x_1, \ldots, x_l) \in A^{[n]} \times A^l \mid \rho(\xi) = \sum_{i=1}^{l} \lambda_i[x_i]\right\}$ forgetting the subschemes $\xi_1, \ldots, \xi_{l-1}$, is a birational morphism. Therefore

$$\mathfrak{p}_{\lambda_1}(\alpha_1) \cdots \mathfrak{p}_{\lambda_l}(\alpha_l)\, \mathbb{1} = {U^\lambda}^*(\alpha_1 \times \cdots \times \alpha_l)$$

and one only has to show that

$$\sum_{g \in \mathfrak{S}_n} (-1)^{\mathrm{age}(g)} {U^g}_* {U^\lambda}^*(\alpha_1 \times \cdots \times \alpha_l) = n! \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l). \tag{21}$$

Indeed, for a given $g \in G$, if g in not of type λ, then by [19, Proposition 5.1.3], we know that ${U^g}_* \circ {U^\lambda}^* = 0$. For any $g \in G$ of type λ, fix a numbering $\varphi : \{1, \ldots, l\} \xrightarrow{\sim} O(g)$ such that $|\varphi(j)| = \lambda_j$ and let $\widetilde{\varphi} : A^\lambda = A^l \to A^{O(g)}$ be the induced isomorphism. Then denoting by $q : A^\lambda \twoheadrightarrow A^{(\lambda)}$ the quotient map by $\mathfrak{S}_\lambda$, the computation [19, Proposition 5.1.4] implies that for such $g \in \lambda$,

$$\begin{aligned} & {U^g}_* \circ {U^\lambda}^*(\alpha_1 \times \cdots \times \alpha_l) \\ = \; & \widetilde{\varphi}_* \circ U^\lambda_* \circ {U^\lambda}^*(\alpha_1 \times \cdots \times \alpha_l) \\ = \; & (-1)^{n-|\lambda|} \prod_{i=1}^{|\lambda|} \lambda_i \cdot \widetilde{\varphi}_* \circ q^* \circ q_*(\alpha_1 \times \cdots \times \alpha_l) \\ = \; & (-1)^{n-|\lambda|} \prod_{i=1}^{|\lambda|} \lambda_i \cdot \deg(q) \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l) \\ = \; & (-1)^{n-|\lambda|} \prod_{i=1}^{|\lambda|} \lambda_i \cdot |\, \mathfrak{S}_\lambda \,| \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l) \end{aligned}$$

Putting those together, we have

$$\begin{aligned}
&\sum_{g\in\mathfrak{S}_n}(-1)^{\mathrm{age}(g)}U^g{}_*U^{\lambda^*}(\alpha_1\times\cdots\times\alpha_l)\\
=\ &\sum_{g\in\lambda}(-1)^{n-|\lambda|}U^g{}_*U^{\lambda^*}(\alpha_1\times\cdots\times\alpha_l)\\
=\ &\sum_{g\in\lambda}\left(\prod_{i=1}^{|\lambda|}\lambda_i\right)\cdot|\mathfrak{S}_\lambda|\cdot\mathrm{Sym}(\alpha_1\times\cdots\times\alpha_l)\\
=\ &n!\cdot\mathrm{Sym}(\alpha_1\times\cdots\times\alpha_l),
\end{aligned}$$

where the last equality uses the simple counting of conjugacy class : the number of permutations of type λ is $\frac{n!}{\left(\prod_{i=1}^{|\lambda|}\lambda_i\right)\cdot|\mathfrak{S}_\lambda|}$. The desired equality (21), hence also the Proposition, is proved. □

As explained in §4, the proof of Theorem 1.4 is now complete : Proposition 5.7 and Proposition 5.8 together imply that $\mathrm{Sym}(W)$ and $\mathrm{Sym}(Z)$ are rationally equivalent using Theorem 5.6. Therefore Proposition 4.1 holds in our situation Case (A), which means exactly that the isomorphism ϕ in Proposition 5.2 (defined in (20)) is also multiplicative with respect to the product structure on $\mathfrak{h}(A^{[n]})$ given by the small diagonal and the orbifold product with sign change by discrete torsion on $\mathfrak{h}(A^n,\mathfrak{S}_n)^{\mathfrak{S}_n}$.

6. Case (B) : Generalized Kummer varieties

We prove Theorem 1.5 in this section. Notation is as in the beginning of §4 :

$$M=A_0^{n+1}:=\mathrm{Ker}\left(A^{n+1}\xrightarrow{+}A\right)$$

which is non-canonically isomorphic to A^n, with the action of $G=\mathfrak{S}_{n+1}$ and the quotient $X:=A_0^{(n+1)}:=M/G$. Then the restriction of the Hilbert–Chow morphism to the *generalized Kummer variety*

$$K_n(A)=:Y\xrightarrow{f}A_0^{(n+1)}$$

is a symplectic resolution.

6.1. **Step (i) – Additive isomorphisms.** We use the result in [20] to establish an additive isomorphism $\mathfrak{h}(Y)\xrightarrow{\simeq}\mathfrak{h}_{orb}([M/G])$.

Recall that a morphism $f:Y\to X$ is called *semi-small* if for all integer $k\geq 0$, the codimension of the locus $\left\{x\in X\mid \dim f^{-1}(x)\geq k\right\}$ is at least $2k$. In particular, f is generically finite. Consider a (finite) Whitney stratification $X=\coprod_a X_a$ by connected strata, such that for any a, the restriction $f|_{f^{-1}(X_a)}:f^{-1}(X_a)\to X_a$ is a topological fiber bundle of fiber dimension d_a. Then the semismallness condition says that $\operatorname{codim}X_a\geq 2d_a$ for any a. In that case, a stratum X_a is said to be *relevant* if the equality holds : $\operatorname{codim}X_a=2d_a$. Here is the key result we are using :

Theorem 6.1 (de Cataldo – Migliorini [20])**.** *Let $f:Y\to X$ be a semi-small morphism of complex projective varieties with Y being smooth. For each connected relevant stratum X_a of codimension $2d_a$ (and fiber dimension d_a), let $Z_a\to X_a$ be the (not necessarily connected) étale cover corresponding to the $\pi_1(X_a)$-set of maximal (=d_a) dimensional irreducible components of fibers. Assume $\overline{Z}_a\to\overline{X}_a$ is a finite projective compactification with $\overline{Z}_a$ admitting a stratification with strata being finite group quotients of* *smooth varieties. Then (the closure of) the incidence subvarieties between $\overline{Z}_a$ and Y induce an isomorphism of Chow motives :*

$$\bigoplus_a \mathfrak{h}(\overline{Z}_a)(-d_a) \simeq \mathfrak{h}(Y).$$

Moreover, the inverse isomorphism is again given by the incidence subvarieties but with different non-zero coefficients.

Remark 6.2.
- The compactifications $\overline{Z}_a$ are singular, but they are $\mathbf{Q}$-varieties, for which the usual intersection theory works with rational coefficients (see Remark 2.2).
- The statement about the correspondence inducing isomorphisms as well as the (non-zero) coefficients of the inverse correspondence is contained in [20, §2.5].
- Since any symplectic resolution of a (singular) symplectic variety is semi-small, the previous theorem applies to the situation of Conjecture 1.3 and 3.6.
- Note that the correspondence in [19] which is used in §5 for Case (A) is a special case of Theorem 6.1.

Let us start by making precise a Whitney stratification for the (semi-small) symplectic resolution $Y = K_n(A) \to X = A_0^{(n+1)}$ as follows. The notation is as in the proof of Proposition 5.2. Let $\mathscr{P}(n+1)$ be the set of partitions of $n+1$, then

$$X = \coprod_{\lambda \in \mathscr{P}(n+1)} X_\lambda,$$

where the locally closed strata are defined by

$$X_\lambda := \left\{ \sum_{i=1}^{|\lambda|} \lambda_i [x_i] \in A^{(n+1)} \;\middle|\; \substack{\sum_{i=1}^{|\lambda|} \lambda_i x_i = 0 \\ x_i \text{ distinct}} \right\},$$

with normalization of closure being

$$\overline{Z}_\lambda = \overline{X}_\lambda^{\,norm} = A_0^{(\lambda)} := A_0^\lambda / \mathfrak{S}_\lambda,$$

where

$$A_0^\lambda = \left\{ (x_1, \ldots, x_{|\lambda|}) \in A^\lambda \;\middle|\; \sum_{i=1}^{|\lambda|} \lambda_i x_i = 0 \right\}. \tag{22}$$

It is easy to see that $\dim X_\lambda = \dim A_0^\lambda = 2(|\lambda| - 1)$ while the fibers over X_λ are isomorphic to a product of Briançon varieties ([12]) $\prod_{i=1}^{|\lambda|} \mathfrak{B}_{\lambda_i}$, which is irreducible of dimension $\sum_{i=1}^{|\lambda|}(\lambda_i - 1) = n + 1 - |\lambda| = \frac{1}{2} \operatorname{codim} X_\lambda$.

In conclusion, $f : K_n(A) \to A_0^{(n+1)}$ is a semi-small morphism with all strata being relevant and all fibers over strata being irreducible. One can therefore apply Theorem 6.1 to get the following

Corollary 6.3. *For each $\lambda \in \mathscr{P}(n+1)$, let*

$$V^\lambda := \left\{ (\xi, x_1, \ldots, x_{|\lambda|}) \;\middle|\; \substack{\rho(\xi) = \sum_{i=1}^{|\lambda|} \lambda_i [x_i]\,; \\ \sum_{i=1}^{|\lambda|} \lambda_i x_i = 0} \right\} \subset K_n(A) \times A_0^\lambda$$

be the incidence subvariety, whose dimension is $n-1+|\lambda|$. Then the quotients $V^{(\lambda)} := V^\lambda / \mathfrak{S}_\lambda \subset K_n(A) \times A_0^{(\lambda)}$ induce an isomorphism of rational Chow motives :

$$\phi' : \mathfrak{h}(K_n(A)) \xrightarrow{\simeq} \bigoplus_{\lambda \in \mathscr{P}(n+1)} \mathfrak{h}\left(A_0^{(\lambda)}\right)(|\lambda| - n - 1).$$

Moreover, the inverse $\psi' := \phi'^{-1}$ *is induced by* $\sum_{\lambda\in\mathscr{P}(n+1)}\frac{1}{m_\lambda}V^{(\lambda)}$*, where* $m_\lambda = (-1)^{n+1-|\lambda|}\prod_{i=1}^{|\lambda|}\lambda_i$ *is a non-zero constant.*

Similar to Proposition 5.2 for Case (A) , the previous Corollary 6.3 allows us to establish an additive isomorphism between $\mathfrak{h}(K_n(A))$ and $\mathfrak{h}_{orb}([A_0^{n+1}/\mathfrak{S}_{n+1}])$:

Proposition 6.4. *Let* $M = A_0^{n+1}$ *with the action of* $G = \mathfrak{S}_{n+1}$*. Let* p *and* ι *denote the projection onto and the inclusion of the G-invariant part of* $\mathfrak{h}(M, G)$*. For each* $g \in G$*, let*

$$V^g := (K_n(A) \times_{A_0^{(n+1)}} M^g)_{red} \subset K_n(A) \times M^g \tag{23}$$

be the incidence subvariety. Then they induce an isomorphism of rational Chow motives :

$$\phi := p \circ \sum_{g\in G}(-1)^{\mathrm{age}(g)}V^g : \mathfrak{h}\left(K_n(A)\right) \xrightarrow{\simeq} \left(\bigoplus_{g\in G}\mathfrak{h}\left(M^g\right)\left(-\,\mathrm{age}(g)\right)\right)^G .$$

Moreover, its inverse ψ *is given by* $\frac{1}{(n+1)!}\cdot\sum_{\lambda\in\mathscr{P}(n+1)}{}^tV^g\circ\iota$*.*

Proof. The proof goes exactly as for Proposition 5.2, with Lemma 5.3 replaced by the following canonical isomorphism :

$$\left(\bigoplus_{g\in\mathfrak{S}_{n+1}}\mathfrak{h}\left((A_0^{n+1})^g\right)\left(-\,\mathrm{age}(g)\right)\right)^{\mathfrak{S}_{n+1}} \xrightarrow{\simeq} \bigoplus_{\lambda\in\mathscr{P}(n+1)}\mathfrak{h}\left(A_0^{(\lambda)}\right)(|\lambda|-n-1).$$

Indeed, let λ be the partition determined by g, then it is easy to compute $\mathrm{age}(g) = n+1-|O(g)| = n+1-|\lambda|$ and moreover the quotient of $(A_0^{n+1})^g$ by the centralizer of g, which is $\prod_{i=1}^{|\lambda|}\mathbf{Z}/\lambda_i\mathbf{Z}\rtimes\mathfrak{S}_\lambda$, is exactly $A_0^{(\lambda)}$. ◻

To show Theorem 1.5, it remains to show Proposition 4.1 in this situation (where all cycles U are actually V of Proposition 6.4).

6.2. **Step (ii) – Symmetrically distinguished cycles on abelian torsors with torsion structures.** Observe that we have the extra technical difficulty that $(A_0^{n+1})^g$ is in general an extension of a finite abelian group by an abelian variety, thus non-connected. To deal with algebraic cycles on not necessarily connected 'abelian varieties' in a canonical way as well as the property of being symmetrically distinguished, we would like to introduce the following category. Roughly speaking, this is the category of *abelian varieties with origin fixed only up to torsion*. It is between the notion of abelian varieties (with origin fixed) and the notion of abelian torsors (*i.e.* a variety isomorphic to an abelian variety thus without a chosen origin).

Definition 6.5 (Abelian torsors with torsion structure)**.** One defines the following category $\mathscr{A}$.
An object of $\mathscr{A}$, called an *abelian torsor with torsion structure*, or an *a.t.t.s.* , is a pair (X, Q_X) where X is a connected smooth projective variety and Q_X is a subset of X such that there exists an isomorphism, as complex algebraic varieties, $f : X \to A$ from X to an abelian variety A which induces a bijection between Q_X and $\mathrm{Tor}(A)$, the set of all torsion points of A. The point here is that the isomorphism f, called a *marking*, usually being non-canonical in practice, is not part of the data.
A morphism between two objects (X, Q_X) and (Y, Q_Y) is a morphism of complex algebraic varieties $\phi : X \to Y$ such that $\phi(Q_X) \subset Q_Y$. Compositions of morphisms are defined in the natural way. Note that by choosing markings, a morphism between two objects in $\mathscr{A}$ is essentially the composition

of a morphism between two abelian varieties followed by a torsion translation.
Denote by $\mathscr{AV}$ the category of abelian varieties. Then there is a natural functor $\mathscr{AV} \to \mathscr{A}$ sending an abelian variety A to $(A, \mathrm{Tor}(A))$.

The following elementary lemma provides the kind of examples that we will be considering :

Lemma 6.6 (Constructing *a.t.t.s.* and compatibility)**.** *Let A be an abelian variety. Let $f : \Lambda \to \Lambda'$ be a morphism of lattices*[8] *and $f_A : A \otimes_{\mathbf{Z}} \Lambda \to A \otimes_{\mathbf{Z}} \Lambda'$ be the induced morphism of abelian varieties.*

(1) *Then* $\mathrm{Ker}(f_A)$ *is canonically a disjoint union of a.t.t.s. such that* $Q_{\mathrm{Ker}(f_A)} = \mathrm{Ker}(f_A) \cap \mathrm{Tor}(A \otimes_{\mathbf{Z}} \Lambda)$.
(2) *If one has another morphism of lattices $g : \Lambda' \to \Lambda''$ inducing morphism of abelian varieties $g_A : A \otimes_{\mathbf{Z}} \Lambda' \to A \otimes_{\mathbf{Z}} \Lambda''$. Then the natural inclusion* $\mathrm{Ker}(f_A) \hookrightarrow \mathrm{Ker}(g_A \circ f_A)$ *is a morphism of a.t.t.s. (on each component).*

Proof. For (1), we have the following two short exact sequences of abelian groups :

$$0 \to \mathrm{Ker}(f) \to \Lambda \xrightarrow{\pi} \mathrm{Im}(f) \to 0\,;$$
$$0 \to \mathrm{Im}(f) \to \Lambda' \to \mathrm{Coker}(f) \to 0,$$

with $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ being lattices. Tensoring them with A, one has exact sequences

$$0 \to A \otimes_{\mathbf{Z}} \mathrm{Ker}(f) \to A \otimes_{\mathbf{Z}} \Lambda \xrightarrow{\pi_A} A \otimes_{\mathbf{Z}} \mathrm{Im}(f) \to 0\,;$$
$$0 \to \mathrm{Tor}^{\mathbf{Z}}(A, \mathrm{Coker}(f)) =: T \to A \otimes_{\mathbf{Z}} \mathrm{Im}(f) \to A \otimes_{\mathbf{Z}} \Lambda',$$

where $T = \mathrm{Tor}^{\mathbf{Z}}(A, \mathrm{Coker}(f))$ is a finite abelian group consisting of some torsion points of $A \otimes_{\mathbf{Z}} \mathrm{Im}(f)$. Then

$$\mathrm{Ker}(f_A) = \pi_A^{-1}(T)$$

is an extension of the finite abelian group T by the abelian variety $A \otimes_{\mathbf{Z}} \mathrm{Ker}(f)$. Choosing a section of π makes $A \otimes_{\mathbf{Z}} \Lambda$ the product of $A \otimes_{\mathbf{Z}} \mathrm{Ker}(f)$ and $A \otimes_{\mathbf{Z}} \mathrm{Im}(f)$, inside of which $\mathrm{Ker}(f_A)$ is the product of $A \otimes_{\mathbf{Z}} \mathrm{Ker}(f)$ and the finite subgroup T of $A \otimes_{\mathbf{Z}} \mathrm{Im}(f)$. This shows that $Q_{\mathrm{Ker}(f_A)} := \mathrm{Ker}(f_A) \cap \mathrm{Tor}(A \otimes_{\mathbf{Z}} \Lambda)$, which is independent of the choice of the section, makes $\mathrm{Ker}(f_A)$ an *a.t.t.s.*
With (1) being proved, (2) is trivial : the torsion structures on $\mathrm{Ker}(f_A)$ and on $\mathrm{Ker}(g_A \circ f_A)$ are both defined by claiming that a point is torsion if it is a torsion point in $A \otimes_{\mathbf{Z}} \Lambda$. □

Before generalizing the notion of symmetrically distinguished cycles to the new category $\mathscr{A}$, we have to first prove the following well-known fact.

Lemma 6.7. *Let A be an abelian variety, $x \in Tor(A)$ be a torsion point. Then the corresponding torsion translation*

$$\begin{array}{rcl} t_x : A & \to & A \\ y & \mapsto & x + y \end{array}$$

acts trivially on $\mathrm{CH}(A)$.

Proof. We follow the proof in [35, Lemma 2.1]. Let m be the order of x. Let Γ_{t_x} be the graph of t_x, then one has $m^*(\Gamma_{t_x}) = m^*(\Delta_A)$ in $\mathrm{CH}(A \times A)$, where m is the multiplication by m map of $A \times A$. However, m^* is an isomorphism of $\mathrm{CH}(A \times A)$ by Beauville's decomposition [8]. We conclude that $\Gamma_{t_x} = \Delta_A$, hence the induced correspondences are the same, which are t_x^* and the identity respectively. □

[8] A *lattice* is a free abelian group of finite rank.

Definition 6.8 (Symmetrically distinguished cycles in $\mathscr{A}$)**.** Given an *a.t.t.s.* $(X, Q_X) \in \mathscr{A}$ (see Definition 6.5), an algebraic cycle $\gamma \in \mathrm{CH}(X)$ is called *symmetrically distinguished*, if for a marking $f : X \to A$, the cycle $f_*(\gamma) \in CH(A)$ is symmetrically distinguished in the sense of O'Sullivan (Definition 5.4). By Lemma 6.7, this definition is independent of the choice of marking. An algebraic cycle on a disjoint union of *a.t.t.s.* is symmetrically distinguished if it is so on each component. We denote $\mathrm{CH}(X)_{sd}$ the subspace consisting of symmetrically distinguished cycles.

The following proposition is clear from Theorem 5.5 and Theorem 5.6.

Proposition 6.9. *Let $(X, Q_X) \in \mathrm{Obj}(\mathscr{A})$ be an a.t.t.s. .*

(1) *The space of symmetric distinguished cycles $\mathrm{CH}^*(X)_{sd}$ is a graded sub-$\mathbf{Q}$-algebra of $\mathrm{CH}^*(X)$.*
(2) *Let $f : (X, Q_X) \to (Y, Q_Y)$ be a morphism in $\mathscr{A}$, then $f_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ and $f^* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$ preserve symmetrically distinguished cycles.*
(3) *The composition $\mathrm{CH}(X)_{sd} \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ is an isomorphism. In particular, a (polynomial of) symmetrically distinguished cycles is trivial in $\mathrm{CH}(X)$ if and only if it is numerically trivial.*

We will need the following easy fact to prove that some cycles on an *a.t.t.s.* are symmetrically distinguished by checking it in an ambient abelian variety.

Lemma 6.10. *Let $i : B \hookrightarrow A$ be a morphism of a.t.t.s. which is a closed immersion. Let $\gamma \in \mathrm{CH}(B)$ be an algebraic cycle. Then γ is symmetrically distinguished in B if and only if $i_*(\gamma)$ is so in A.*

Proof. One implication is clear from Proposition 6.9 (2). For the other one, assuming $i_*(\gamma)$ is symmetrically distinguished in A. By choosing markings, one can suppose that A is an abelian variety and B is a torsion translation by $\tau \in Tor(A)$ of a sub-abelian variety of A. Thanks to Lemma 6.7, changing the origin of A to τ does not change the cycle class $i_*(\gamma) \in \mathrm{CH}(A)$, hence one can further assume that B is a sub-abelian variety of A. By Poincaré reducibility, there is a sub-abelian variety $C \subset A$, such that the natural morphism $\pi : B \times C \to A$ is an isogeny. We have the following diagram :

$$\begin{tikzcd}
& B \times C \arrow[dl, bend right, "\mathrm{pr}_1"'] \arrow[d, "\pi"] \\
B \arrow[ur, bend right, "j"'] \arrow[r, "i"'] & A
\end{tikzcd}$$

As $\pi^* : \mathrm{CH}(A) \to \mathrm{CH}(B \times C)$ is an isomorphism with inverse $\frac{1}{\deg(\pi)}\pi_*$, we have

$$\gamma = \mathrm{pr}_{1*} \circ j_*(\gamma) = \mathrm{pr}_{1*} \circ \pi^* \circ \frac{1}{\deg(\pi)}\pi_* \circ j_*(\gamma) = \frac{1}{\deg(\pi)}\mathrm{pr}_{1*} \circ \pi^* \circ i_*(\gamma).$$

Since π and pr_1 are morphisms of abelian varieties, the hypothesis that $i_*(\gamma)$ is symmetrically distinguished implies that γ is also symmetrically distinguished by Proposition 6.9 (3). □

We now turn to the proof of Proposition 4.1 in Case (B), which takes the following form. As is explained in §4, with Step (i) being done (Proposition 6.4), this would finish the proof of Theorem 1.5.

Proposition 6.11 (=Proposition 4.1 in Case (B))**.** *In $\mathrm{CH}\left(\left(\coprod_{g\in G} M^g\right)^3\right)$, the symmetrizations of the following two algebraic cycles are rationally equivalent :*

- $W := \left(\frac{1}{|G|}\sum_g V^g \times \frac{1}{|G|}\sum_g V^g \times \sum_g (-1)^{\mathrm{age}(g)} V^g\right)_* \left(\delta_{K_n(A)}\right)$;

- *Z is the cycle determining the orbifold product (Definition 2.5(5)) with the sign change by discrete torsion (Definition 3.5):*

$$Z|_{M^{g_1}\times M^{g_2}\times M^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1g_2 \\ \epsilon_{g_1,g_2}\cdot \delta_* c_{top}(F_{g_1,g_2}) & \text{if } g_3 = g_1g_2. \end{cases}$$

To this end, we apply Proposition 6.9(3) by proving in this subsection that they are both symmetrically distinguished (Proposition 6.12) and then verifying in the next one §6.3 that they are homologically equivalent (Proposition 6.13).

Let M be the abelian variety $A_0^{n+1} = \left\{(x_1,\cdots,x_{n+1}) \in A^{n+1} \;\middle|\; \sum_i x_i = 0\right\}$ as before. For any $g\in G$, the fixed locus

$$M^g = \left\{(x_1,\cdots,x_{n+1})\in A^{n+1} \;\middle|\; \sum_i x_i = 0\,;\; x_i = x_{g.i}\ \forall i\right\}$$

has the following decomposition into connected components:

$$M^g = \coprod_{\tau\in A[d]} M^g_\tau, \tag{24}$$

where $d := \gcd(g)$ is the greatest common divisor of the lengths of orbits of the permutation g, $A[d]$ is the set of d-torsion points and the connected component M^g_τ is described as follows.
Let $\lambda\in\mathscr{P}(n+1)$ be the partition determined by g and $l := |\lambda|$ be its length. Choose a numbering $\varphi : \{1,\cdots,l\} \xrightarrow{\simeq} O(g)$ of orbits such that $|\varphi(i)| = \lambda_i$. Then $d = \gcd(\lambda_1,\cdots,\lambda_l)$ and φ induces an isomorphism

$$\widetilde{\varphi} : A_0^\lambda \xrightarrow{\simeq} M^g, \tag{25}$$

sending $(x_1,\cdots,x_l)$ to $(y_1,\cdots,y_{n+1})$ with $y_j = x_i$ if $j\in\varphi(i)$. Here A_0^λ is defined in (22), which has obviously the following decomposition into connected components:

$$A_0^\lambda = \coprod_{\tau\in A[d]} A_\tau^{\lambda/d}, \tag{26}$$

where

$$A_\tau^{\lambda/d} = \left\{(x_1,\cdots,x_l)\in A^\lambda \;\middle|\; \sum_{i=1}^l \frac{\lambda_i}{d}x_i = \tau\right\}$$

is connected (non-canonically isomorphic to A^{l-1} as varieties) and is equipped with a canonical *a.t.t.s.* (Definition 6.5) structure, namely, a point of $A_\tau^{\lambda/d}$ is defined to be of torsion (*i.e.* in $Q_{A_\tau^{\lambda/d}}$) if and only if it is a torsion point (in the usual sense) in the abelian variety A^λ. The decomposition (24) of M^g is the transportation of the decomposition (26) of A_0^λ via the isomorphism (25): $A_\tau^{\lambda/d} \underset{\simeq}{\xrightarrow{\widetilde{\varphi}}} M^g_\tau$. The component M^g_τ hence acquires a canonical structure of *a.t.t.s.* It is clear that the decomposition (24) and the *a.t.t.s.* structure on components are both independent of the choice of φ. One can also define the *a.t.t.s.* structure on M^g by using Lemma 6.6.

Similar to Proposition 5.7, here is the main result of this subsection:

Proposition 6.12. *Notation is as in Proposition 6.11. W and Z, as well as their symmetrizations, are symmetrically distinguished in* $\mathrm{CH}\left(\left(\coprod_{g\in G} M^g\right)^3\right)$*, where* M^g *is viewed as a disjoint union of a.t.t.s. as in (24) and symmetrical distinguishedness is in the sense of Definition 6.8.*

Proof. For W, it is enough to show that for any $g_1, g_2, g_3 \in G$, $q_* \circ p^* \circ \delta_*(\mathbb{1}_{K_n(A)})$ is symmetrically distinguished, where the notation is explained in the following commutative diagram, whose squares are all cartesian and without excess intersections.

$$
\begin{tikzcd}
 & (A^{[n+1]})^3 \arrow[dr, phantom, "\square"] & U^{g_1} \times U^{g_2} \times U^{g_3} \arrow[l, "p''"'] \arrow[r, "q''"] \arrow[dr, phantom, "\square"] & (A^{n+1})^{g_1} \times (A^{n+1})^{g_2} \times (A^{n+1})^{g_3} \\
A^{[n+1]} \arrow[ur, hook, "\delta''"] \arrow[r, hook, "\delta'"] \arrow[dr, phantom, "\square"] & (A^{[n+1]})^{3/A} \arrow[u, hook] \arrow[dr, phantom, "\square"] & U^{g_1} \times_A U^{g_2} \times_A U^{g_3} \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[u, hook] \arrow[dr, phantom, "\square"] & (A^{n+1})^{g_1} \times_A (A^{n+1})^{g_2} \times_A (A^{n+1})^{g_3} \arrow[u, hook, "j"'] \\
K_n(A) \arrow[u, hook] \arrow[r, hook, "\delta"] & K_n(A)^3 \arrow[u, hook] & V^{g_1} \times V^{g_2} \times V^{g_3} \arrow[l, "p"'] \arrow[r, "q"] \arrow[u, hook] & M^{g_1} \times M^{g_2} \times M^{g_3} \arrow[u, hook, "i"']
\end{tikzcd}
\tag{27}
$$

where the incidence subvarieties U^g's are defined in §5.1 (17) (with n replaced by $n+1$) ; all fiber products in the second row are over A ; the second row is the base change by the inclusion of small diagonal $A \hookrightarrow A^3$ of the first row ; the third row is the base change by $O_A \hookrightarrow A$ of the second the row ; finally, $\delta, \delta', \delta''$ are various (absolute or relative) small diagonals.

Observe that the two inclusions i and j are in the situation of Lemma 6.6 : let

$$\Lambda := \mathbf{Z}^{O(g_1)} \oplus \mathbf{Z}^{O(g_2)} \oplus \mathbf{Z}^{O(g_3)},$$

which admits a natural morphism u to $\Lambda' := \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ by weighted sum on each factor (with weights being the lengths of orbits). Let $v : \Lambda' \to \Lambda'' := \mathbf{Z} \oplus \mathbf{Z}$ be $(m_1, m_2, m_3) \mapsto (m_1 - m_2, m_1 - m_3)$. Then it is clear that i and j are identified with the following inclusions

$$\mathrm{Ker}(u_A) \overset{i}{\hookrightarrow} \mathrm{Ker}(v_A \circ u_A) \overset{j}{\hookrightarrow} A \otimes_{\mathbf{Z}} \Lambda.$$

By Lemma 6.6, $(A^{n+1})^{g_1} \times_A (A^{n+1})^{g_2} \times_A (A^{n+1})^{g_3}$ and $M^{g_1} \times M^{g_2} \times M^{g_3}$ are naturally disjoint unions of *a.t.t.s.* and the inclusions i and j are morphisms of *a.t.t.s.* on each component.

Now by functorialities and the base change formula (*cf.* [31, Theorem 6.2]), we have

$$j_* \circ q'_* \circ p'^* \circ \delta'_*(\mathbb{1}_{A^{[n+1]}}) = q''_* \circ p''^* \circ \delta''_*(\mathbb{1}_{A^{[n+1]}}),$$

which is a polynomial of big diagonals of $A^{|O(g_1)|+|O(g_2)|+|O(g_3)|}$ by Voisin's result [57, Proposition 5.6], thus symmetrically distinguished in particular. By Lemma 6.10, $q'_* \circ p'^* \circ \delta'_*(\mathbb{1}_{A^{[n+1]}})$ is symmetrically distinguished on each component of $(A^{n+1})^{g_1} \times_A (A^{n+1})^{g_2} \times_A (A^{n+1})^{g_3}$.
Again by functorialities and the base change formula, we have

$$q_* \circ p^* \circ \delta_*(\mathbb{1}_{K_n(A)}) = i^* \circ q'_* \circ p'^* \circ \delta'_*(\mathbb{1}_{A^{[n+1]}}).$$

Since i is a morphism of *a.t.t.s.* on each component (Lemma 6.6), one concludes that $q_* \circ p^* \circ \delta_*(\mathbb{1}_{K_n(A)})$ is symmetrically distinguished on each component. Hence W, being a linear combination of such cycles, is also symmetrically distinguished.
For Z, as in the Case (A), it is easy to see that all the obstruction bundles F_{g_1,g_2} are (at least virtually) trivial vector bundles because according to Definition 2.5, there are only tangent/normal bundles of/between abelian varieties involved. Therefore the only non-zero case is the push-forward of the fundamental class of $M^{<g_1,g_2>}$ by the inclusion into $M^{g_1} \times M^{g_2} \times M^{g_1 g_2}$, which is obviously symmetrically distinguished. □

6.3. **Step (iii) – Cohomological realizations.** We keep the notation as before. To finish the proof of Proposition 6.11, hence Theorem 1.5, it remains to show that the cohomology classes of the symmetrizations of W and Z are the same. In other words, they have the same realization for Betti cohomology.

Proposition 6.13. *The cohomology realization of the (a priori additive) isomorphism in Proposition 6.4*

$$\phi : \mathfrak{h}(K_n(A)) \xrightarrow{\simeq} \left(\oplus_{g\in G}\, \mathfrak{h}((A_0^{n+1})^g)(-\operatorname{age}(g))\right)^{\mathfrak{S}_{n+1}}$$

is an isomorphism of $\mathbf{Q}$*-algebras*

$$\overline{\phi} : H^*(K_n(A)) \xrightarrow{\simeq} H^*_{orb,dt}([A_0^{n+1}/\,\mathfrak{S}_{n+1}]) = \left(\bigoplus_{g\in\mathfrak{S}_{n+1}} H^{*-2\operatorname{age}(g)}((A_0^{n+1})^g), \star_{orb,dt}\right)^{\mathfrak{S}_n}.$$

In other words, $\operatorname{Sym}(W)$ *and* $\operatorname{Sym}(Z)$ *are homologically equivalent.*

Proof. We use Nieper–Wisskirchen's following description [44] of the cohomology ring $H^*(K_n(A), \mathbf{C})$. Let $s : A^{[n+1]} \to A$ be the composition of the Hilbert–Chow morphism followed by the summation map. Recall that s is an isotrivial fibration. In the sequel, if not specified, all cohomology groups are with complex coefficients. We have a commutative diagram :

$$\begin{tikzcd} H^*(A) \arrow[r, "s^*"] \arrow[d, "\epsilon"'] & H^*(A^{[n]}) \arrow[d, "restr."] \\ \mathbf{C} \arrow[r] & H^*(K_n(A)) \end{tikzcd}$$

where the upper arrow s^* is the pull-back by s, the lower arrow is the unit map sending 1 to the fundamental class $\mathbb{1}_{K_n(A)}$, ϵ is the quotient by the ideal consisting of elements of strictly positive degree and the right arrow is the restriction map. The commutativity comes from the fact that $K_n(A) = s^{-1}(O_A)$ is a fiber. Thus one has a ring homomorphism

$$R : H^*(A^{[n]}) \otimes_{H^*(A)} \mathbf{C} \to H^*(K_n(A)).$$

Then [44, Theorem 1.7] asserts that this is an isomorphism of $\mathbf{C}$-algebras.
Now consider the following diagram :

$$\begin{tikzcd} H^*(A^{[n+1]}) \otimes_{H^*(A)} \mathbf{C} \arrow[r, "R", "\simeq"'] \arrow[d, "\Phi"', "\simeq"] & H^*(K_n(A)) \arrow[d, "\overline{\phi}", "\simeq"'] \\ \left(\oplus_{g\in\mathfrak{S}_{n+1}} H^{*-2\operatorname{age}(g)}((A^{n+1})^g)\right)^{\mathfrak{S}_{n+1}} \otimes_{H^*(A)} \mathbf{C} \arrow[r, "r"'] & \left(\oplus_{g\in\mathfrak{S}_{n+1}} H^{*-2\operatorname{age}(g)}((A_0^{n+1})^g)\right)^{\mathfrak{S}_{n+1}} \end{tikzcd} \tag{28}$$

- As just stated, the upper arrow is an isomorphism of $\mathbf{C}$-algebras, by Nieper–Wisskirchen [44, Theorem 1.7].
- The left arrow Φ comes from the ring isomorphism (which is exactly CHRC 1.1 for Case (A), see §5.3) :
$$H^*(A^{[n+1]}) \xrightarrow{\simeq} \left(\oplus_{g\in\mathfrak{S}_{n+1}} H^{*-2\operatorname{age}(g)}((A^{n+1})^g)\right)^{\mathfrak{S}_{n+1}},$$
established in [26] based on [38]. By (the proof of) Proposition 5.8, this isomorphism is actually induced by $\sum_g (-1)^{\operatorname{age}(g)} \cdot U^g_* : H(A^{[n+1]}) \to \oplus_g H((A^{n+1})^g)$ with U^g the incidence subvariety defined in (17). Note that on the lower-left term of the diagram, the ring homomorphism $H^*(A) \to \left(\oplus_{g\in\mathfrak{S}_{n+1}} H^{*-2\operatorname{age}(g)}((A^{n+1})^g)\right)^{\mathfrak{S}_{n+1}}$ lands in the summand indexed by $g = \mathrm{id}$, and the map $H^*(A) \to H^*(A^{n+1})^{\mathfrak{S}_{n+1}}$ is simply the pull-back by the summation map $A^{(n+1)} \to A$.
- The right arrow is the morphism $\overline{\phi}$ in question. It is already shown in Step (i) Proposition 6.4 to be an isomorphism of vector spaces. The goal is to show that it is also multiplicative.

- The lower arrow r is defined as follows. On the one hand, let the image of the unit $1 \in \mathbf{C}$ be the fundamental class of $A_0^{(n+1)}$ in the summand indexed by $g = \mathrm{id}$. On the other hand, for any $g \in \mathfrak{S}_{n+1}$, we have a natural restriction map $H^{*-2\,\mathrm{age}(g)}((A^{n+1})^g) \to H^{*-2\,\mathrm{age}(g)}((A_0^{n+1})^g)$. They will induce a ring homomorphism $H^*(A^{n+1}, \mathfrak{S}_{n+1})_{\mathbf{C}} \to H^*(A_0^{n+1}, \mathfrak{S}_{n+1})_{\mathbf{C}}$ by Lemma 6.14 below, which is easily seen to be compatible with the $\mathfrak{S}_{n+1}$-action and the ring homomorphisms from $H^*(A)$, hence r is a well-defined homomorphism of $\mathbf{C}$-algebras.
- To show the commutativity of the diagram (28), the case for the unit $1 \in \mathbf{C}$ is easy to check. For the case of $H^*(A^{[n+1]})$, it suffices to remark that for any g the following diagram is commutative

$$\begin{tikzcd} H^*(A^{[n+1]}) \arrow[r, "restr."] \arrow[d, "U^g_*"'] & H^*(K_n(A)) \arrow[d, "V^g_*"] \\ H((A^{n+1})^g) \arrow[r, "restr"'] & H((A_0^{n+1})^g) \end{tikzcd}$$

where V^g is the incidence subvariety defined in (23).

In conclusion, since in the commutative diagram (28), Φ, R are isomorphisms of $\mathbf{C}$-algebras, r is a homomorphism of $\mathbf{C}$-algebra and $\overline{\phi}$ is an isomorphism of vector spaces, we know that they are all isomorphisms of algebras. Thus Proposition 6.13 is proved assuming the following :

Lemma 6.14. *The natural restriction maps $H^{*-2\,\mathrm{age}(g)}((A^{n+1})^g) \to H^{*-2\,\mathrm{age}(g)}((A_0^{n+1})^g)$ for all $g \in \mathfrak{S}_{n+1}$ induce a ring homomorphism $H^*(A^{n+1}, \mathfrak{S}_{n+1}) \to H^*(A_0^{n+1}, \mathfrak{S}_{n+1})$, where their product structures are given by the orbifold product (see Definition 2.5 or 2.6).*

Proof. This is straightforward by definition. Indeed, for any $g_1, g_2 \in \mathfrak{S}_{n+1}$ together with $\alpha \in H((A^{n+1})^{g_1})$ and $\beta \in H((A^{n+1})^{g_2})$, since the obstruction bundle F_{g_1,g_2} is a trivial vector bundle, we have

$$\alpha \star_{orb} \beta = \begin{cases} i_*\left(\alpha|_{(A^{n+1})^{<g_1,g_2>}} \cup \beta|_{(A^{n+1})^{<g_1,g_2>}}\right) & \text{if } \mathrm{rk}\, F_{g_1,g_2} = 0 \\ 0 & \text{if } \mathrm{rk}\, F_{g_1,g_2} \neq 0 \end{cases}$$

where $i : (A^{n+1})^{<g_1,g_2>} \hookrightarrow (A^{n+1})^{g_1g_2}$ is the natural inclusion. Therefore by the base change for the cartesian diagram without excess intersection :

$$\begin{tikzcd} (A_0^{n+1})^{<g_1,g_2>} \arrow[r, hook, "i_0"] \arrow[d, hook] & (A_0^{n+1})^{g_1g_2} \arrow[d, hook] \\ (A^{n+1})^{<g_1,g_2>} \arrow[r, hook, "i"'] & (A^{n+1})^{g_1g_2} \end{tikzcd}$$

we have :

$$\begin{aligned} & \alpha \star_{orb} \beta|_{(A_0^{n+1})^{g_1g_2}} \\ = & \begin{cases} i_{0*}\left(\left(\alpha|_{(A^{n+1})^{<g_1,g_2>}} \cup \beta|_{(A^{n+1})^{<g_1,g_2>}}\right)\Big|_{(A_0^{n+1})^{<g_1,g_2>}}\right) = i_{0*}\left(\alpha|_{(A_0^{n+1})^{<g_1,g_2>}} \cup \beta|_{(A_0^{n+1})^{<g_1,g_2>}}\right) & \text{if } \mathrm{rk}\, F_{g_1,g_2} = 0 \\ 0 & \text{if } \mathrm{rk}\, F_{g_1,g_2} \neq 0 \end{cases} \\ = & \alpha|_{(A_0^{n+1})^{g_1}} \star_{orb} \beta|_{(A_0^{n+1})^{g_2}} \end{aligned}$$

which means that the restriction map is a ring homomorphism. □

The proof of Proposition 6.13 is finished. □

Now the proof of Theorem 1.5 is complete : by Proposition 6.12 and Proposition 6.13, we know that, thanks to Proposition 6.9 (3), the symmetrizations of Z and W in Proposition 6.11 are rationally equivalent, which proves Proposition 4.1 in Case (B). Hence the isomorphism ϕ in Proposition 6.4 is an isomorphism of algebra objects between the motive of the generalized Kummer variety $\mathfrak{h}(K_n(A))$ and the orbifold Chow motive $\mathfrak{h}_{orb}([A_0^{n+1}/\mathfrak{S}_{n+1}])$. □

We would like to note the following corollary obtained by applying the cohomological realization functor to Theorem 1.5.

Corollary 6.15 (CHRC : Kummer case)**.** *The Cohomological HyperKähler Resolution Conjecture is true for Case (B), namely, one has an isomorphism of* **Q***-algebras :*

$$H^*(K_n(A), \mathbf{Q}) \simeq H^*_{orb,dt}\left([A_0^{n+1}/\mathfrak{S}_{n+1}]\right).$$

Remark 6.16. For some reason unclear to the authors, this result has never appeared before in the literature. It is presumably not hard to check CHRC in the case of generalized Kummer varieties directly based on the cohomology result of Nieper–Wisskirchen [44], which is of course one of the key ingredients used in our proof. It is also generally believed that the main result of Britze's Ph.D. thesis [14] should also imply this result. However, the proof of its main result [14, Theorem 40] seems to be flawed : the linear map Θ constructed in the last line of Page 60, which is claimed to be the desired ring isomorphism, is actually the zero map. Nevertheless, the authors believe that it is feasible to check CHRC in this case with the very explicit description of the ring structure of $H^*(K_n(A) \times A)$ obtained in [14].

7. Application 1 : Towards Beauville's splitting property

In this section, a *holomorphic symplectic* variety is always assumed to be smooth projective unless stated otherwise and we require neither the simple connectedness nor the uniqueness up to scalar of the holomorphic symplectic 2-form. Hence examples of holomorphic symplectic varieties include projective deformations of Hilbert schemes of K3 or abelian surfaces, generalized Kummer varieties *etc.*.

7.1. **Beauville's Splitting Property.** Based on [8] and [11], Beauville envisages in [10] the following *Splitting Property* for all holomorphic symplectic varieties.

Conjecture 7.1 (Splitting Property : Chow rings)**.** *Let X be a holomorphic symplectic variety of dimension $2n$. Then one has a canonical bigrading of the rational Chow ring* $\mathrm{CH}^*(X)$*, called* multiplicative splitting of $\mathrm{CH}^*(X)$ of Bloch–Beilinson type *: for any* $0 \le i \le 4n$*,*

$$\mathrm{CH}^i(X) = \bigoplus_{s=0}^{i} \mathrm{CH}^i(X)_s, \tag{29}$$

which satisfies :

- *(Multiplicativity)* $\mathrm{CH}^i(X)_s \bullet \mathrm{CH}^{i'}(X)_{s'} \subset \mathrm{CH}^{i+i'}(X)_{s+s'}$ *;*
- *(Bloch–Beilinson) The associated filtration* $F^j\,\mathrm{CH}^i(X) := \bigoplus_{s\ge j} \mathrm{CH}^i(X)_s$ *satisfies the Bloch–Beilinson conjecture (cf.* [54, Conjecture 11.21] *for example). In particular :*
  - *($F^1 = \mathrm{CH}_{hom}$) The restriction of the cycle class map* $\mathrm{cl} : \bigoplus_{s>0} \mathrm{CH}^i(X)_s \to H^{2i}(X, \mathbf{Q})$ *is zero ;*
  - *(Injectivity) The restriction of the cycle class map* $\mathrm{cl} : \mathrm{CH}^i(X)_0 \to H^{2i}(X, \mathbf{Q})$ *is injective.*

We would like to reformulate (and slightly strengthen) Conjecture 7.1 by using the language of Chow motives as follows, which is, we believe, more fundamental. Let us first of all introduce the following (weaker) notion studied in detail in [50].

Definition 7.2 (Multiplicative Chow–Künneth decomposition)**.** Given a smooth projective variety X of dimension n, a *self-dual multiplicative Chow–Künneth decomposition* is a direct sum decomposition in the category CHM of Chow motives with rational coefficients :

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{2n} \mathfrak{h}^i(X) \tag{30}$$

satisfying the following two properties :

- (Chow–Künneth) The cohomology realization of the decomposition gives the Künneth decomposition : for each $0 \le i \le 2n$, $H^*(\mathfrak{h}^i(X)) = H^i(X)$.
- (Self-duality) The dual motive $\mathfrak{h}^i(X)^\vee$ identifies with $\mathfrak{h}^{2n-i}(X)(n)$.
- (Multiplicativity) The product $\mu : \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ given by the small diagonal $\delta_X \subset X \times X \times X$ respects the decomposition : the restriction of μ on the summand $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$ factorizes through $\mathfrak{h}^{i+j}(X)$.

Such a decomposition induces a (multiplicative) bigrading of the rational Chow ring $\mathrm{CH}^*(X) = \oplus_{i,s} \mathrm{CH}^i(X)_s$ by setting :

$$\mathrm{CH}^i(X)_s := \mathrm{CH}^i(\mathfrak{h}^{2i-s}(X)) := \mathrm{Hom}_{\mathrm{CHM}}\left(\mathbb{1}(-i), \mathfrak{h}^{2i-s}(X)\right). \tag{31}$$

By the definition of motives (*cf.* 2.1), a multiplicative Chow–Künneth decomposition is equivalent to a collection of auto-correspondences $\left\{\pi^0, \cdots, \pi^{2\dim X}\right\}$, where $\pi^i \in \mathrm{CH}^{\dim X}(X \times X)$, satisfying

- $\pi^i \circ \pi^i = \pi^i, \forall i$;
- $\pi^i \circ \pi^j = 0, \forall i \neq j$;
- $\pi^0 + \cdots + \pi^{2\dim X} = \Delta_X$;
- $\mathrm{Im}(\pi^i_* : H^*(X) \to H^*(X)) = H^i(X)$;
- $\pi^k \circ \delta_X \circ (\pi^i \otimes \pi^j) = 0, \forall k \neq i + j$.

The induced multiplicative bigrading on the rational Chow ring $\mathrm{CH}^*(X)$ is given by

$$\mathrm{CH}^i(X)_s := \mathrm{Im}\left(\pi^{2i-s}_* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X)\right).$$

The above Chow–Künneth decomposition is self-dual if the transpose of π^i is equal to $\pi^{2\dim X - i}$.

For later use, we need to generalize the previous notion for Chow motive algebras :

Definition 7.3. Let $\mathfrak{h}$ be an (associative but not-necessarily commutative) algebra object in the category CHM of rational Chow motives. Denote by $\mu : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ its multiplication structure. A *multiplicative Chow–Künneth decomposition* of $\mathfrak{h}$ is a direct sum decomposition

$$\mathfrak{h} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{h}^i,$$

such that

- (Chow–Künneth) the cohomology realization gives the Künneth decomposition : $H^i(\mathfrak{h}) = H^*(\mathfrak{h}^i)$ for all $i \in \mathbf{Z}$;
- (Multiplicativity) the restriction of μ to $\mathfrak{h}^i \otimes \mathfrak{h}^j$ factorizes through $\mathfrak{h}^{i+j}$.

Now one can enhance Conjecture 7.1 to the following :

Conjecture 7.4 (Motivic Splitting Property = Conjecture 1.9)**.** *Let X be a holomorphic symplectic variety of dimension $2n$. Then we have a canonical (Self-dual) multiplicative Chow–Künneth decomposition of $\mathfrak{h}(X)$:*

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} h^i(X)$$

which is moreover of Bloch–Beilinson–Murre type, *that is, for any $i, j \in \mathbf{N}$,*

(1) $\mathrm{CH}^i(h^j(X)) = 0$ *if* $j < i$;
(2) $\mathrm{CH}^i(h^j(X)) = 0$ *if* $j > 2i$;
(3) *the realization induces an injective map* $\mathrm{Hom}_{\mathrm{CHM}}\left(\mathbb{1}(-i), \mathfrak{h}^{2i}(X)\right) \to \mathrm{Hom}_{\mathbf{Q}-HS}\left(\mathbf{Q}(-i), H^{2i}(X)\right)$.

One can deduce Conjecture 7.1 from Conjecture 7.4 via (31). Note that the range of s in (29) follows from the first two Bloch–Beilinson–Murre conditions in Conjecture 7.4.

7.2. **Splitting Property for abelian varieties.** Recall that for an abelian variety B of dimension g, using Fourier transform [6], Beauville [8] constructs a multiplicative bigrading on $\mathrm{CH}^*(B)$:

$$\mathrm{CH}^i(B) = \bigoplus_{s=i-g}^{i} \mathrm{CH}^i(B)_s, \text{ for any } 0 \le i \le g \tag{32}$$

where

$$\mathrm{CH}^i(B)_s := \left\{\alpha \in \mathrm{CH}^i(B) \,\middle|\, \mathbf{m}^*\alpha = m^{2i-s}\alpha\,;\ \forall m \in \mathbf{Z}\right\}, \tag{33}$$

is the simultaneous eigenspace for all $\mathbf{m} : B \to B$, the multiplication by $m \in \mathbf{Z}$ map.

Using similar idea as in *loc.cit.* , Deninger and Murre [22] constructed a multiplicative Chow–Künneth decomposition (Definition 7.2)

$$\mathfrak{h}(B) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(B), \tag{34}$$

with (by [37])

$$\mathfrak{h}^i(B) \simeq \mathrm{Sym}^i(\mathfrak{h}^1(B)). \tag{35}$$

Moreover, one may choose such a multiplicative Chow–Künneth decomposition to be symmetrically distinguished ; see [50, Ch. 7]. This Chow–Künneth decomposition is the candidate decomposition for the analogous Conjecture 7.4 in the case of abelian varieties and induces, via (31), Beauville's bigrading (33) ; the remaining Bloch–Beilinson condition becomes the following conjecture of Beauville [6] on $\mathrm{CH}^*(B)$, which is still largely open.

Conjecture 7.5 (Beauville's conjecture on abelian varieties)**.** *Notation is as above. Then*

- $\mathrm{CH}^i(B)_s = 0$ *for* $s < 0$;
- *The restriction of the cycle class map* $\mathrm{cl} : \mathrm{CH}^i(B)_0 \to H^{2i}(B, \mathbf{Q})$ *is injective.*

Remark 7.6. As torsion translations act trivially on the Chow rings of abelian varieties (Lemma 6.7), the Beauville–Deninger–Murre decompositions (33) and (34) naturally extend to the slightly broader context of *abelian torsors with torsion structure* (see Definition 6.5).

We collect some facts about the Beauville–Deninger–Murre decomposition (34) for the proof of Theorem 7.9 in the next subsection. By choosing markings for *a.t.t.s.* 's, thanks to Lemma 6.7, we see that *a.t.t.s.* 's can be endowed with multiplicative Chow–Künneth decompositions consisting of Chow–Künneth projectors that are symmetrically distinguished, and enjoying the properties embodied in the two following lemmas. Their proofs are reduced immediately to the case of abelian varieties, which are certainly well-known.

Lemma 7.7 (Künneth)**.** *Let B and B' be two abelian varieties (or more generally a.t.t.s. 's), then the natural isomorphism $\mathfrak{h}(B)\otimes\mathfrak{h}(B') \simeq \mathfrak{h}(B\times B')$ identifies the summand $\mathfrak{h}^i(B)\otimes\mathfrak{h}^j(B)$ as a direct summand of $\mathfrak{h}^{i+j}(B\times B')$ for any $i, j \in \mathbf{N}$.* □

Lemma 7.8. *Let $f : B \to B'$ be a morphism of abelian varieties (or more generally a.t.t.s. 's) of dimension g, g' respectively.*

- *The pull back $f^* := {}^t\Gamma_f : \mathfrak{h}(B') \to \mathfrak{h}(B)$ sends $\mathfrak{h}^i(B')$ to $\mathfrak{h}^i(B)$;*
- *The push forward $f_* := \Gamma_f : \mathfrak{h}(B) \to \mathfrak{h}(B')$ sends $\mathfrak{h}^i(B)$ to $\mathfrak{h}^{i+2g'-2g}(B')$.* □

7.3. **Candidate decompositions in Case (A) and (B).** In the sequel, let A be an abelian surface. We want to do the similar thing as Beauville and Deninger–Murre did for abelian varieties (§7.2) : for the holomorphic symplectic variety X being $A^{[n]}$ or $K_n(A)$, we construct the candidate multiplicative Chow–Künneth decomposition for Conjecture 7.4 thus the candidate bigrading on $\mathrm{CH}^*(X)$ for Conjecture 7.1, then formulate the remaining Bloch–Beilinson condition into a conjecture on the motive of A, which will be investigated upon, especially its relation with Beauville's Conjecture 7.5 on abelian varieties.

Let us start by the existence of multiplicative Chow–Künneth decomposition :

Theorem 7.9. *Given an abelian surface A, let X be*
Case (A) : the $2n$-dimensional Hilbert scheme $A^{[n]}$; or Case (B) : the n-th generalized Kummer variety $K_n(A)$. Then X has a canonical self-dual multiplicative Chow–Künneth decomposition

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(X),$$

where in Case (A) and (B) respectively

$$\mathfrak{h}^i(A^{[n]}) := \left(\bigoplus_{g\in\mathfrak{S}_n} \mathfrak{h}^{i-2\,\mathrm{age}(g)}((A^n)^g)(-\,\mathrm{age}(g))\right)^{\mathfrak{S}_n}; \tag{36}$$

$$\mathfrak{h}^i(K_n(A)) := \left(\bigoplus_{g\in\mathfrak{S}_{n+1}} \mathfrak{h}^{i-2\,\mathrm{age}(g)}((A_0^{n+1})^g)(-\,\mathrm{age}(g))\right)^{\mathfrak{S}_{n+1}}. \tag{37}$$

In particular, a canonical multiplicative bigrading on the (rational) Chow ring given by

$$\mathrm{CH}^i(X)_s := \mathrm{CH}^i(\mathfrak{h}^{2i-s}(X)).$$

Remark 7.10. The existence of a self-dual multiplicative Chow–Künneth decomposition of $A^{[n]}$ is not new : it was previously obtained by Vial in [52]. As for the generalized Kummer varieties, if one ignores the multiplicativity of the Chow–Künneth decomposition, which is of course the key point here, then it follows rather directly from De Cataldo and Migliorini's result [20] as explained in §6.1 (see Corollary 6.3) and is explicitly written down by Z. Xu [58].

Proof. The following proof works for both cases. Let $M := A^n$, $G := \mathfrak{S}_n$, $X := A^{[n]}$ in Case (A) and $M := A_0^{n+1}$, $G := \mathfrak{S}_{n+1}$, $X := K_n(A)$ in Case (B). Thanks to Theorem 1.4 and Theorem 1.5, we have an isomorphism of motive algebras :

$$\mathfrak{h}(X) \xrightarrow{\simeq} \left(\bigoplus_{g \in G} \mathfrak{h}\left(M^g\right)(-\operatorname{age}(g)), \star_{orb,dt} \right)^G$$

whose inverse on each direct factor $\mathfrak{h}\left(M^g\right)(-\operatorname{age}(g))$ is given by a rational multiple of the transpose of the induced morphism $\mathfrak{h}(X) \to \mathfrak{h}\left(M^g\right)(-\operatorname{age}(g))$. It thus suffices to prove that each direct factor has a self-dual Chow–Künneth decomposition in the sense of Definition 7.2, and that the induced Chow–Künneth decomposition on the motive algebra

$$\mathfrak{h} := \bigoplus_{g \in G} \mathfrak{h}\left(M^g\right)(-\operatorname{age}(g)), \text{with } \star_{orb,dt} \text{ as the product,}$$

is multiplicative in the sense of Definition 7.3. To this end, for each $g \in G$, an application of Deninger–Murre's decomposition (34) to M^g, which is an abelian variety in Case (A) and a disjoint union of *a.t.t.s.* in Case (B), gives us a self-dual multiplicative Chow–Künneth decomposition

$$\mathfrak{h}(M^g) = \bigoplus_{i=0}^{2\dim M^g} \mathfrak{h}^i(M^g).$$

Now we define for each $i \in \mathbf{N}$,

$$\mathfrak{h}^i := \bigoplus_{g \in G} \mathfrak{h}^{i-2\operatorname{age}(g)}(M^g)(-\operatorname{age}(g)). \tag{38}$$

Here by convention, $\mathfrak{h}^j(M^g) = 0$ for $j < 0$, hence in (38), $\mathfrak{h}^i = 0$ if $i - 2\operatorname{age}(g) > 2\dim(M^g)$ for any $g \in G$, that is, when $i > \max_{g \in G}\{4n - 2\operatorname{age}(g)\} = 4n$.

Then obviously, as a direct sum of Chow–Künneth decompositions,

$$\mathfrak{h} = \bigoplus_{i=0}^{4n} \mathfrak{h}^i$$

is a Chow–Künneth decomposition. It is self-dual because each M^g has dimension $2n - 2\operatorname{age}(g)$. It remains to show the multiplicativity condition that $\mu : \mathfrak{h}^i \otimes \mathfrak{h}^j \to \mathfrak{h}$ factorizes through $\mathfrak{h}^{i+j}$, which is equivalent to say that for any $i, j \in \mathbf{N}$ and $g, h \in G$, the orbifold product $\star_{orb}$ (discrete torsion only changes a sign thus irrelevant here) restricted to the summand $\mathfrak{h}^{i-2\operatorname{age}(g)}(M^g)(-\operatorname{age}(g)) \otimes \mathfrak{h}^{j-2\operatorname{age}(h)}(M^h)(-\operatorname{age}(h))$ factorizes through $h^{i+j-2\operatorname{age}(gh)}(M^{gh})(-\operatorname{age}(gh))$. Thanks to the fact that the obstruction bundle $F_{g,h}$ is always a trivial vector bundle in both of our cases, we know that (see Definition 2.5) $\star_{orb}$ is either zero when $\operatorname{rk}(F_{g,h}) \neq 0$; or when $\operatorname{rk}(F_{g,h}) = 0$, is defined as the correspondence from $M^g \times M^h$ to M^{gh} given by the following composition

$$\mathfrak{h}(M^g) \otimes \mathfrak{h}(M^h) \xrightarrow{\simeq} \mathfrak{h}(M^g \times M^h) \xrightarrow{\iota_1^*} \mathfrak{h}(M^{<g,h>}) \xrightarrow{\iota_{2*}} \mathfrak{h}(M^{gh})(\operatorname{codim}(\iota_2)), \tag{39}$$

where

$$M^{gh} \xleftarrow{\iota_2} M^{<g,h>} \xrightarrow{\iota_1} M^g \times M^h$$

are morphisms of abelian varieties in Case (A) and morphisms of *a.t.t.s.* 's in Case (B). Therefore, one can suppose further that $\operatorname{rk}(F_{g,h}) = 0$, which implies by using (7) that the Tate twists match :

$$\operatorname{codim}(\iota_2) - \operatorname{age}(g) - \operatorname{age}(h) = -\operatorname{age}(gh).$$

Now Lemma 7.7 applied to the first isomorphism in (39) and Lemma 7.8 applied to the last two morphisms in (39) show that, omitting the Tate twists, the summand $\mathfrak{h}^{i-2\operatorname{age}(g)}(M^g)\otimes\mathfrak{h}^{j-2\operatorname{age}(h)}(M^h)$ is sent by μ inside the summand $h^k(M^{gh})$, with the index

$$k = i+j-2\operatorname{age}(g)-2\operatorname{age}(h)+2\dim(M^{gh})-2\dim(M^{<g,h>}) = i+j-2\operatorname{age}(gh),$$

where the last equality is by equation (7) together with the assumption $\operatorname{rk}(F_{g,h})=0$.
In conclusion, we get a multiplicative Chow–Künneth decomposition $\mathfrak{h}=\bigoplus_{i=0}^{4n}\mathfrak{h}^i$ with $\mathfrak{h}^i$ given in (38) ; hence a multiplicative Chow–Künneth decomposition for its G-invariant part of the sub-motive algebra $\mathfrak{h}(X)$. ☐

The decomposition in Theorem 7.9 is supposed to be Beauville's splitting of the Bloch–Beilinson–Murre filtration on the rational Chow ring of X. In particular,

Conjecture 7.11. *(Bloch–Beilinson for X) Notation is as in Theorem 7.9, then for all $i\in\mathbf{N}$,*

- $\mathrm{CH}^i(X)_s = 0$ *for $s<0$;*
- *The restriction of the cycle class map* $\mathrm{cl}:\mathrm{CH}^i(X)_0\to H^{2i}(X,\mathbf{Q})$ *is injective.*

As a first step towards this conjecture, let us make the following

Remark 7.12. *Beauville's conjecture 7.5 on abelian varieties implies Conjecture 7.11.* Indeed, keep the same notation as before. From (36) (37), we obtain

$$\mathrm{CH}^i(A^{[n]})_s = \mathrm{CH}^i(\mathfrak{h}^{2i-s}(A^{[n]})) = \left(\bigoplus_{g\in\mathfrak{S}_n}\mathrm{CH}^{i-\operatorname{age}(g)}(\mathfrak{h}^{2i-s-2\operatorname{age}(g)}(A^{O(g)}))\right)^{\mathfrak{S}_n} = \bigoplus_{\lambda\in\mathscr{P}(n)}\mathrm{CH}^{i+|\lambda|-n}(A^{\lambda})_s^{\mathfrak{S}_\lambda};$$

$$\mathrm{CH}^i(K_nA)_s = \mathrm{CH}^i(\mathfrak{h}^{2i-s}(K_nA)) = \left(\bigoplus_{g\in\mathfrak{S}_{n+1}}\mathrm{CH}^{i-\operatorname{age}(g)}(\mathfrak{h}^{2i-s-2\operatorname{age}(g)}(A_0^{O(g)}))\right)^{\mathfrak{S}_{n+1}} = \bigoplus_{\lambda\in\mathscr{P}(n+1)}\mathrm{CH}^{i+|\lambda|-n-1}(A_0^{\lambda})_s^{\mathfrak{S}_\lambda},$$

in two cases respectively, whose vanishing ($s<0$) and injectivity into cohomology by cycle class map ($s=0$) follow directly from those of A^λ or A_0^λ.
In fact, [53, Theorem 3] proves more generally that the second point of Conjecture 7.5 (the injectivity of the cycle class map $\mathrm{cl}:\mathrm{CH}^i(B)_0\to H^{2i}(B,\mathbf{Q})$ for all complex abelian varieties) implies Conjecture 7.11 for all smooth projective complex varieties X whose Chow motive is of abelian type, which is the case for a generalized Kummer variety by Proposition 6.4. Of course, one has to check that our definition of $\mathrm{CH}^i(X)_0$ here coincides with the one in [53], which is quite straightforward.

The Chern classes of a (smooth) holomorphic symplectic variety X are also supposed to be in $\mathrm{CH}^i(X)_0$ with respect to Beauville's conjectural splitting. We can indeed check this in both cases considered here :

Proposition 7.13. *Set-up as in Theorem 7.9. The Chern class $c_i(X)$ belongs to $\mathrm{CH}^i(X)_0$ for all i.*

Proof. In Case (A), that is, in the case where X is the Hilbert scheme $A^{[n]}$, this is proved in [52]. Let us now focus on Case (B), that is, on the case where X is the generalized Kummer variety $K_n(A)$. Let $\{\pi^i : 0\le i\le 2n\}$ be the Chow–Künneth decomposition of $K_n(A)$ given by (37). We have to show that $c_i(K_n(A)) = (\pi^{2i})_*c_i(K_n(A))$, or equivalently that $(\pi^j)_*c_i(K_n(A))=0$ as soon as $(\pi^j)_*c_i(K_n(A))$ is homologically trivial. By Proposition 6.4, it suffices to show that for any $g\in G$ $(\pi^j_{M^g})_*((V^g)_*c_i(K_n(A)))=0$ as soon as $(\pi^j_{M^g})_*((V^g)_*c_i(K_n(A)))$ is homologically trivial. Here, recall that

(24) makes M^g a disjoint union of *a.t.t.s.* and that $\pi^j_{M^g}$ is a Chow–Künneth projector on M^g which is symmetrically distinguished on each component of M^g. By Proposition 6.9, it is enough to show that $(V^g)_*(c_i(K_n(A))$ is symmetrically distinguished on each component of M^g. As in the proof of Proposition 6.12, we have for any $g \in G$ the following commutative diagram, whose squares are cartesian and without excess intersections :

$$\begin{tikzcd} A^{[n+1]} & U^g \arrow[l, "p'"'] \arrow[r, "q'"] & (A^{n+1})^g \\ K_n(A) \arrow[u, hook] \arrow[ur, phantom, "\square"] & V^g \arrow[l, "p"'] \arrow[r, "q"] \arrow[u, hook] \arrow[ur, phantom, "\square"] & M^g \arrow[u, hook, "i"'] \end{tikzcd} \tag{40}$$

where the incidence subvariety U^g is defined in §5.1 (17) (with n replaced by $n+1$) and the bottom row is the base change by $O_A \hookrightarrow A$ of the top row. Note that $c_i(K_n(A)) = c_i(A^{[n+1]})|_{K_n(A)}$, since the tangent bundle of A is trivial. Therefore, by functorialities and the base change formula (*cf.* [31, Theorem 6.2]), we have

$$(V^g)_*(c_i(K_n(A)) := q_* \circ p^*(c_i(K_n(A)) = i^* \circ q'_* \circ p'^*(c_i(A^{[n+1]})).$$

By Voisin's result [57, Theorem 5.12], $q'_*\circ p'^*(c_i(A^{[n+1]}))$ is a polynomial of big diagonals of $A^{|O(g)|}$, thus symmetrically distinguished in particular. It follows from Proposition 6.9 that $(V^g)_*(c_i(K_n(A))$ is symmetrically distinguished on each component of M^g. This concludes the proof of the proposition. □

8. Application 2 : Multiplicative decomposition theorem of rational cohomology

Deligne's decomposition theorem states the following :

Theorem 8.1 (Deligne [21])**.** *Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. In the derived category of sheaves of $\mathbf{Q}$-vector spaces on B, there is a decomposition (which is non-canonical in general)*

$$R\pi_*\mathbf{Q} \cong \bigoplus_i R^i\pi_*\mathbf{Q}[-i]. \tag{41}$$

Both sides of (41) carry a cup-product : on the right-hand side the cup-product is the direct sum of the usual cup-products $R^i\pi_*\mathbf{Q} \otimes R^j\pi_*\mathbf{Q} \to R^{i+j}\pi_*\mathbf{Q}$ defined on local systems, while on the left-hand side the derived cup-product $R\pi_*\mathbf{Q} \otimes R\pi_*\mathbf{Q} \to R\pi_*\mathbf{Q}$ is induced by the (derived) action of the relative small diagonal $\delta \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ seen as a relative correspondence from $\mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X}$. As explained in [56], the isomorphism (41) does not respect the cup-product in general. Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, Voisin [56, Question 0.2] asked if there exists a decomposition as in (41) which is multiplicative, *i.e.*, which is compatible with cup-product, maybe over a nonempty Zariski open subset of B. By Deninger–Murre [22], there does exist such a decomposition for an abelian scheme $\pi : \mathcal{A} \to B$. The main result of [56] is :

Theorem 8.2 (Voisin [56])**.** *For any smooth projective family $\pi : \mathcal{X} \to B$ of K3 surfaces, there exist a decomposition isomorphism as in* (41) *and a nonempty Zariski open subset U of B, such that this decomposition becomes multiplicative for the restricted family $\pi|_U : \mathcal{X}|_U \to U$.*

As implicitly noted in [52, Section 4], Voisin's Theorem 8.2 holds more generally for any smooth projective family $\pi : \mathcal{X} \to B$ whose generic fiber admits a multiplicative Chow–Künneth decomposition (K3 surfaces do have a multiplicative Chow–Künneth decomposition ; this follows by suitably reinterpreting, as in [50, Proposition 8.14], the vanishing of the modified diagonal cycle of Beauville–Voisin [11] as the multiplicativity of the Beauville–Voisin Chow–Künneth decomposition.) :

Theorem 8.3. *Let $\pi : \mathcal{X} \to B$ be a smooth projective family, and assume that the generic fiber X of π admits a multiplicative Chow–Künneth decomposition. Then there exist a decomposition isomorphism as in* (41) *and a nonempty Zariski open subset U of B, such that this decomposition becomes multiplicative for the restricted family $\pi|_U : \mathcal{X}|_U \to U$.*

Proof. By spreading out a multiplicative Chow–Künneth decomposition of X, there exist a sufficiently small but nonempty Zariski open subset U of B and relative correspondences $\Pi^i \in \mathrm{CH}^{\dim_B \mathcal{X}}(\mathcal{X}|_U \times_U \mathcal{X}|_U)$, $0 \leq i \leq 2\dim_B \mathcal{X}$, forming a relative Chow–Künneth decomposition, meaning that $\Delta_{\mathcal{X}|_U/U} = \sum_i \Pi^i$, $\Pi^i \circ \Pi^i = \Pi^i$, $\Pi^i \circ \Pi^j = 0$ for $i \neq j$, and Π^i acts as the identity on $R^i(\pi^{[n]}|_U)_*\mathbf{Q}$ and as zero on $R^j(\pi^{[n]}|_U)_*\mathbf{Q}$ for $j \neq i$. By [56, Lemma 2.1], the relative idempotents Π^i induce a decomposition in the derived category

$$R(\pi|_U)_*\mathbf{Q} \cong \bigoplus_{i=0}^{4n} H^i(R(\pi|_U)_*\mathbf{Q})[-i] = \bigoplus_{i=0}^{4n} R^i(\pi|_U)_*\mathbf{Q}[-i]$$

with the property that Π^i acts as the identity on the summand $H^i(R(\pi|_U)_*\mathbf{Q})[-i]$ and acts as zero on the summands $H^j(R(\pi|_U)_*\mathbf{Q})[-j]$ for $j \neq i$. In order to establish the existence of a decomposition as in (41) that is multiplicative and hence to conclude the proof of the theorem, we thus have to show that $\Pi^k \circ \delta \circ (\Pi^i \times \Pi^j)$ acts as zero on $R(\pi|_U)_*\mathbf{Q} \otimes R(\pi|_U)_*\mathbf{Q}$, after possibly further shrinking U, whenever $k \neq i+j$. But more is true : being generically multiplicative, the relative Chow–Künneth decomposition $\{\Pi^i\}$ is multiplicative, that is, $\Pi^k \circ \delta \circ (\Pi^i \times \Pi^j) = 0$ whenever $k \neq i+j$, after further shrinking U if necessary. The theorem is now proved. □

As a corollary, we can extend Theorem 8.2 to families of generalized Kummer varieties :

Corollary 8.4. *Let $\pi : \mathcal{A} \to B$ be an abelian surface over B. Consider Case (A) : $\mathcal{A}^{[n]} \to B$ the relative Hilbert scheme of length-n subschemes on $\mathcal{A} \to B$; or Case (B) : $K_n(\mathcal{A}) \to B$ the relative generalized Kummer variety. Then, in both cases, there exist a decomposition isomorphism as in* (41) *and a nonempty Zariski open subset U of B, such that this decomposition becomes multiplicative for the restricted family over U.*

Proof. The generic fiber of $\mathcal{A}^{[n]} \to B$ (*resp.* $K_n(\mathcal{A}) \to B$) is the $2n$-dimensional Hilbert scheme (*resp.* generalized Kummer variety) attached to the abelian surface that is the generic fiber of π. By Theorem 7.9, it admits a multiplicative Chow–Künneth decomposition. (Strictly speaking, we only established Theorem 7.9 for Hilbert schemes of abelian surfaces and generalized Kummer varieties over the complex number ; however, the proof carries over over any field of characteristic zero.) We conclude by invoking Theorem 8.3. □

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