EXCEPTIONAL COLLECTIONS, AND THE NÉRON–SEVERI LATTICE FOR SURFACES

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ABSTRACT. We work out properties of smooth projective varieties X over a (not necessarily algebraically closed) field k that admit collections of objects in the bounded derived category of coherent sheaves $D^b(X)$ that are either full exceptional, or numerically exceptional of maximal length. Our main result gives a necessary and sufficient condition on the Néron–Severi lattice for a smooth projective surface S with $\chi(O_S) = 1$ to admit a numerically exceptional collection of maximal length, consisting of line-bundles. As a consequence we determine exactly which complex surfaces with $p_g = q = 0$ admit a numerically exceptional collection of maximal length. Another consequence is that a minimal geometrically rational surface with a numerically exceptional collection of maximal length is rational.

INTRODUCTION

Recently, a substantial amount of work [1, 7, 8, 16] was carried out in order to exhibit exceptional collections of line-bundles of maximal length on complex surfaces of general type with $p_g = q = 0$, motivated by the will to exhibit geometric (quasi)-phantom triangulated categories, *i.e.*, categories with trivial or torsion Grothendieck group K_0; see also [17]. Kuznetsov's recent ICM address [29] sets the notion of exceptional collection into the wider picture of semi-orthogonal decompositions for bounded derived categories of smooth projective varieties.

The purpose of this work is twofold: we give arithmetic, and geometric, constraints for the existence of exceptional collections of maximal length on smooth projective surfaces over a field. For instance, on the geometric side, we show that there are no numerical obstructions to the existence of exceptional collections of maximal length on complex surfaces of general type with $p_g = q = 0$ (in fact we give in Theorem 3.10 a complete classification of complex surfaces with $p_g = q = 0$ that admit a numerically exceptional collection of maximal length). On the arithmetic side, we show that a minimal geometrically rational surface over a perfect field k that admits a numerically exceptional collection of maximal length is rational (Theorem 3.7). We also show that a numerically exceptional collection of maximal length, consisting of line-bundles, on a surface S defined over an arbitrary field k remains of maximal length after any field extension (Theorem 3.3). These results are deduced from a general criterion (the main Theorem 3.1) that gives, for a smooth projective surface S defined over an arbitrary field k, a necessary and sufficient condition on the Néron–Severi lattice of S for S to admit a numerically exceptional collection of maximal length, consisting of line-bundles.

Along the way, we find that if a surface admits a full exceptional collection, then its integral Chow motive is a direct sum of Lefschetz motives (Theorem 2.7). On a slightly unrelated note, we also provide a new characterization of projective space (Theorem 1.2).

Derived category of coherent sheaves and exceptional objects. Let k be a field and let $\mathcal{T}$ be a k-linear triangulated category. The typical example of triangulated category that we have in mind is given by the bounded derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X defined over k. Given a morphism $f : T_1 \to T_2$ between two objects T_1 and

T_2 of $\mathcal{T}$, there is a distinguished triangle

$$T_1 \xrightarrow{f} T_2 \longrightarrow \mathrm{cone}(f) \longrightarrow T_1[1].$$

If $\mathcal{A}$ and $\mathcal{B}$ are two strictly full triangulated subcategories of $\mathcal{T}$ (we mean that $\mathcal{A}$ and $\mathcal{B}$ are closed under shifts and cones), then

$$\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$$

is a *semi-orthogonal decomposition* if

  (i) $\mathrm{Hom}(B, A) = 0$ for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (note that since $\mathcal{A}$ and $\mathcal{B}$ are closed under shifts, we in fact have $\mathrm{Ext}^i(B, A) = 0$ for all integers $i \in \mathbb{Z}$);
 (ii) $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{T}$: for all objects $T \in \mathcal{T}$, T fits into a distinguished triangle $T_\mathcal{B} \to T \to T_\mathcal{A} \to T_\mathcal{B}[1]$ for some objects $T_\mathcal{A}$ of $\mathcal{A}$ and $T_\mathcal{B}$ of $\mathcal{B}$.

More generally,

$$\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$$

is a *semi-orthogonal decomposition* if

  (i) $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for all $i > j$;
 (ii) For all $T \in \mathcal{T}$, there exist $T_i \in \mathcal{T}$ and a sequence $0 = T_n \to T_{n-1} \to \ldots \to T_1 \to T_0 = T$ such that $\mathrm{cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$ for all i.

The simplest triangulated category is probably the bounded derived category $D^b(k-\mathrm{vs})$ of k-vector spaces. Given a triangulated category $\mathcal{T}$, it is natural to be willing to split off copies of $D^b(k-\mathrm{vs})$ inside $\mathcal{T}$, in the sense of semi-orthogonal decompositions. Consider then a functor $D^b(k-\mathrm{vs}) \to \mathcal{T}$. Such a functor is determined by the image $E \in \mathcal{T}$ of the one-dimensional k-vector space placed in degree 0. Let us denote this functor φ_E; then, for any complex $V^\bullet \in D^b(k-\mathrm{vs})$, we have $\varphi_E(V^\bullet) = V^\bullet \otimes E$. A right-adjoint functor is given by $\varphi_E^!(F) = \mathrm{Hom}^\bullet(E, F)$, so that $\varphi_E^! \varphi_E(V^\bullet) = \mathrm{Hom}^\bullet(E, E) \otimes V^\bullet$. Thus φ_E is fully faithful if and only if $\mathrm{Hom}^\bullet(E, E) = k$ placed in degree 0.

Definition 1. *An object $E \in \mathcal{T}$ is* exceptional *if*

$$\mathrm{Hom}(E, E[l]) = \begin{cases} k & \textit{if } l = 0; \\ 0 & \textit{otherwise.} \end{cases}$$

An exceptional collection *is a collection of exceptional objects $E_1, \ldots, E_n \in \mathcal{T}$ such that*

$$\mathrm{Hom}(E_i, E_j[l]) = 0 \quad \textit{for all } i > j \textit{ and all } l \in \mathbb{Z}.$$

An important feature of exceptional objects is that strictly full triangulated subcategories generated by exceptional collections are *admissible*, meaning that the inclusion functor admits a left and a right adjoint. Consequently, given an exceptional collection $(E_1, \ldots, E_n)$ of objects in $\mathcal{T}$, we have a semi-orthogonal decomposition

$$\mathcal{T} = \langle E_1, \ldots, E_n, \mathcal{A} \rangle,$$

where $\mathcal{A} = \{T \in \mathcal{T} : \mathrm{Hom}(T, E_i) = 0 \text{ for all } 1 \le i \le n\}$, and where, by abuse, we have denoted E_i the subcategory generated by E_i.
When $\mathcal{A} = 0$, the semi-orthogonal decomposition

$$\mathcal{T} = \langle E_1, \ldots, E_n \rangle$$

is said to be a *full exceptional collection.*

Let us now give some examples of smooth projective varieties X whose bounded derived category of coherent sheaves admits a full exceptional collection.

  • $X = \mathbb{P}^n$: we have the Beilinson sequence [4] (this is perhaps the most famous example of a full exceptional collection)

$$D^b(\mathbb{P}^n) = \langle O, O(1), \ldots, O(n) \rangle.$$

- $X = \widetilde{\mathbb{P}}^2$ the blow-up of $\mathbb{P}^2$ at a point: if E denotes the exceptional divisor, then

$$D^b(\widetilde{\mathbb{P}}^2) = \langle O_E(1), O, O(1), O(2) \rangle$$

is a full exceptional collection. More generally, there is a blow-up formula due to Orlov [38]. Note that, by *mutating* $O_E(1)$ to the right, one obtains a full exceptional collection consisting of line-bundles:

$$D^b(\widetilde{\mathbb{P}}^2) = \langle O, O(E), O(1), O(2) \rangle.$$

- $X = Q^n \subset \mathbb{P}^{n+1}_{\mathbb{C}}$: a smooth complex quadric. Kapranov [22] showed that

$$D^b(Q^n) = \begin{cases} \langle S, O, O(1), \ldots, O(n-1) \rangle & \text{if } n \text{ is odd} \\ \langle S^-, S^+, O, O(1), \ldots, O(n-1) \rangle & \text{if } n \text{ is even} \end{cases}$$

is a full exceptional collection. Here, S, S^- and S^+ are certain spinor bundles.

Other examples of varieties admitting exceptional collections include complex Grassmannians (Kapranov *loc. cit.*), and several other complex rational homogeneous spaces. In fact, it is expected that if G is a semi-simple algebraic group over an algebraically closed field of characteristic zero and $P \subset G$ is a parabolic subgroup, then there is a full exceptional collection of vector bundles in $D^b(G/P)$; see [30, Conjecture 1.1] and the results in that direction therein. The projective space $\mathbb{P}^n$ admits a full exceptional collection consisting of $n+1$ *line-bundles*. At least in characteristic zero, this property characterizes completely the projective space among n-dimensional smooth projective varieties:

Theorem 1 (Theorem 1.2)**.** *Let X be a smooth projective variety of dimension n over a field k of characteristic zero. Assume that $\langle L_0, \ldots, L_n \rangle$ is a full exceptional collection of $D^b(X)$ for some line-bundles $L_0, \ldots, L_n$. Then X is isomorphic to the projective space $\mathbb{P}^n$.*

As is apparent, the class of varieties admitting full exceptional collections is rather restricted. Perhaps the simplest constraint for a smooth projective variety to admit a full exceptional collection is the following. If we have a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$, then $K_0(\mathcal{T}) = K_0(\mathcal{A}) \oplus K_0(\mathcal{B})$. Since $K_0(D^b(k\text{–vs})) = \mathbb{Z}$, we see that if $D^b(X) = \langle E_1, \ldots, E_r \rangle$ is a full exceptional collection, then $K_0(X) = \mathbb{Z}^r$. As will be explained in the introduction of Section 2, this implies that the Chow motive of X with rational coefficients is a direct sum of Lefschetz motives. Such a constraint was originally obtained *via* the theory of non-commutative motives by Marcolli and Tabuada [34].

Chow motives and Lefschetz motives. Let R be a ring. The category of Chow motives with R-coefficients over k is constructed as follows. First one linearizes the category of smooth projective varieties over k by declaring that $\mathrm{Hom}(X, Y) = \mathrm{CH}^d(X \times Y) \otimes_{\mathbb{Z}} R$, that is, by declaring that the morphism between X and Y are given by correspondences with R-coefficients modulo rational equivalence. Here, X is assumed to be of pure dimension d (otherwise one works component-wise) and the composition law is given by

$$\beta \circ \alpha = (p_{XZ})_*(p_{XY}^* \alpha \cdot p_{YZ}^* \beta) \in \mathrm{CH}^d(X \times Z),$$

for all $\alpha \in \mathrm{CH}^d(X \times Y)$ and all $\beta \in \mathrm{CH}^e(Y \times Z)$, where e is the dimension of Y, and p_{XZ}, p_{XY}, p_{YZ} are the projections from $X \times Y \times Z$ onto $X \times Z, X \times Y, Y \times Z$, respectively. This R-linear category is far from being abelian, so that one formally adds to this R-linear category the images of idempotents. This is called taking the pseudo-abelian, or Karoubi, envelope. This new category is called the category of effective Chow motives, and objects are pairs (X, p), where X is a smooth projective variety of dimension d and $p \in \mathrm{CH}^d(X \times X) \otimes_{\mathbb{Z}} R$ is an idempotent. When p is the class of the diagonal Δ_X in $\mathrm{CH}^d(X \times X)$, we write $\mathfrak{h}(X)$ for (X, Δ_X). In general, the object (X, p) should be thought of as the image of p acting on the motive $\mathfrak{h}(X)$ of X. For example, in the category of effective Chow motives, the motive $\mathfrak{h}(\mathbb{P}^1)$ of the projective line becomes isomorphic to $(\mathbb{P}^1, p) \oplus (\mathbb{P}^1, q)$, where $p := \{0\} \times \mathbb{P}^1$ and $q := \mathbb{P}^1 \times \{0\}$ are idempotents in $\mathrm{Hom}(\mathfrak{h}(\mathbb{P}^1), \mathfrak{h}(\mathbb{P}^1)) := \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$. The object $(\mathbb{P}^1, p)$ is isomorphic to $\mathbb{1} := \mathfrak{h}(\mathrm{Spec}\, k)$, and the motive $(\mathbb{P}^1, p)$ is called the *Lefschetz motive* and is written $\mathbb{1}(-1)$. The fiber product of two smooth projective varieties induces a tensor product in the category of effective Chow motives,

for which $\mathbb{1}$ is a unit. The category of Chow motives with R-coefficients is then obtained by formally inverting the Lefschetz motive.

Concretely, a Chow motive with R-coefficients is a triple (X, p, n) consisting of a smooth projective variety X of pure dimension d over k, of a correspondence $p \in \mathrm{CH}^d(X \times X) \otimes_{\mathbb{Z}} R$ such that $p \circ p = p$, and of an integer n. A morphism $\gamma \in \mathrm{Hom}((X, p, n), (Y, q, m))$ is an element of $q \circ (\mathrm{CH}^{d-n+m}(X \times Y) \otimes_{\mathbb{Z}} R) \circ p$.

The simplest motives are the motive $\mathbb{1} := \mathfrak{h}(\mathrm{Spec}\, k)$ of a point $\mathrm{Spec}\, k$ and its Tate twists, that is, the motives $\mathbb{1}(n) = (\mathrm{Spec}\, k, \mathrm{id}, n)$ for $n \in \mathbb{Z}$. These are called the *Lefschetz motives*. Note that

$$\mathrm{Hom}(\mathbb{1}(-n), \mathfrak{h}(X)) = \mathrm{CH}^n(X) \otimes_{\mathbb{Z}} R.$$

As in the case of triangulated categories, it is natural, given a motive M, to split off copies of Lefschetz motives. Given a morphism $\gamma \in \mathrm{Hom}(\mathbb{1}(-n), \mathfrak{h}(X))$, there is an obvious obstruction to the existence of a splitting to that morphism: if $\gamma \in \mathrm{CH}^n(X)$ is a non-zero numerically trivial cycle, then γ does not admit a left-inverse. Even if γ is not numerically trivial, the existence of a left-inverse is in general a problem of existence of algebraic cycles (in this case, the existence of a cycle $\gamma' \in \mathrm{CH}^{d-n}(X)$ such that $\deg(\gamma \cdot \gamma') = 1$). In Section 2, we prove:

Theorem 2 (Theorem 2.7). *Let S be a smooth projective surface over a field k. Assume that S has a full exceptional collection. Then the integral Chow motive of S is isomorphic to a direct sum of Lefschetz motives.*

One may naturally ask if the converse to Theorem 2.7 holds. In Remark 2.9, we give evidence that the converse should fail to be true.

Numerical constraints. Although the problem of classifying smooth projective complex surfaces that admit a full exceptional collection seems out of reach at present (it is conjectured that only the surfaces that are rational have a full exceptional collection), a fair amount of work [1, 7, 8, 16] has been carried out in order to construct exceptional collections of maximal length on complex surfaces with $p_g = q = 0$. (As usual, for a smooth projective surface S, the geometric genus is $p_g := h^0(\Omega_S^2) = h^2(O_S)$ and the irregularity is $q := h^1(O_S)$.) A first step in constructing exceptional collections consists in constructing numerically exceptional collections.

Recall that, given a k-linear triangulated category $\mathcal{T}$ and two objects E and F in $\mathcal{T}$, the *Euler pairing* χ is the integer

$$\chi(E, F) := \sum_l (-1)^l \dim_k \mathrm{Hom}(E, F[l]).$$

The Euler pairing defines a bilinear pairing on the Grothendieck groups $K_0(\mathcal{T})$ that we still denote χ.

Definition 2. *An object E is said to be* numerically exceptional *if*

$$\chi(E, E) = 1.$$

A collection $(E_1, \ldots, E_r)$ of numerically exceptional objects in $\mathcal{T}$ is called a numerically exceptional collection *if*

$$\chi(E_j, E_i) = 0 \quad \text{for all } j > i.$$

A numerically exceptional collection $(E_1, \ldots, E_r)$ on a smooth projective variety X over k is said to be of maximal length *if $E_1, \ldots, E_r$ span the numerical Grothendieck group, or equivalently if r is equal to the rank of $K_0^{\mathrm{num}}(X)$. (Here, $K_0^{\mathrm{num}}(X) := K_0(X)/(\ker \chi)$; note that the left and right kernels of χ are the same so that the notation $\ker \chi$ is unambiguous.)*

Obviously, an exceptional object is numerically exceptional, an exceptional collection is a numerically exceptional collection, and a full exceptional collection is a numerically exceptional collection of maximal length.

In this work, we give a complete classification of smooth projective complex surfaces, with $p_g = q = 0$, that admit numerically exceptional collections of maximal length:

Theorem 3 (Theorem 3.10). *Let S be a smooth projective complex surface, with $p_g = q = 0$. As usual, $\kappa(S)$ denotes the Kodaira dimension of S.*

- *If S is not minimal, then S has a numerically exceptional collection of maximal length.*

Assume now that S is minimal.

- *If $\kappa(S) = -\infty$, then S has a numerically exceptional collection of maximal length.*
- *If $\kappa(S) = 0$, then S is an Enriques surface and it does not have a numerically exceptional collection of maximal length.*
- *If $\kappa(S) = 1$, then S is a Dolgachev surface $X_9(p_1, \ldots, p_n)$, and S has a numerically exceptional collection of maximal length if and only if S is one of $X_9(2,3)$, $X_9(2,4)$, $X_9(3,3)$, $X_9(2,2,2)$. (We refer to paragraph 3.5 for the notations.)*
- *If $\kappa(S) = 2$, then S has a numerically exceptional collection of maximal length.*

In particular, for surfaces of general type with $p_g = q = 0$, there is no numerical obstruction to the existence of exceptional collections of maximal length. In the case of Enriques surfaces, a general Enriques surface admits ten different elliptic pencils $|2F_1|, \ldots |2F_{10}|$ and the line-bundles $O(F_1), \ldots, O(F_{10})$ provide an exceptional collection of length 10; see Zube [47]. (Any reordering of this length-10 exceptional collection is still an exceptional collection.) Our Theorem 3.10 says in particular that it is not possible to find an exceptional collection consisting of 12 exceptional objects. On the other hand, Theorem 3.10 says that an Enriques surface blown up at a point admits a numerically exceptional collection of maximal length, that is, there is no numerical obstruction to the existence of an exceptional collection of maximal length on an Enriques surface blown up at a point; *cf.* Remark 3.12.

So far, exceptional collections have almost only be considered for varieties defined over algebraically closed fields. Our analysis of numerically exceptional collections of maximal length leads, intuitively, to the conclusion that exceptional collections of maximal length, consisting of rank one objects, do not exist for surfaces that are not "split". First, we have a general result:

Theorem 4 (Theorem 3.3). *Let S be a smooth projective surface over a field k, with $\chi(O_S) = 1$ and with $\mathrm{H}^1_{et}(S_{\bar{k}}, \mathbb{Q}_\ell) = 0$. Here, $\bar{k}$ is a separable closure of k, $S_{\bar{k}} = S \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$, and ℓ is a prime $\neq \mathrm{char}\, k$. Assume that S admits a numerically exceptional collection $(E_0, E_1, \ldots, E_{n+1})$ of maximal length, consisting of rank one objects. Then the cycle class map*

$$\mathrm{CH}^1(S) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{et}(S_{\bar{k}}, \mathbb{Z}_\ell(1))$$

is surjective modulo torsion, that is, it induces a surjective map

$$\mathrm{CH}^1(S) \otimes \mathbb{Z}_\ell \twoheadrightarrow \mathrm{H}^2_{et}(S_{\bar{k}}, \mathbb{Z}_\ell(1))/torsion.$$

In particular, for all field extensions K/k, the collection $((E_0)_K, (E_1)_K, \ldots, (E_{n+1})_K)$ for $S_K = S \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$ is numerically exceptional of maximal length, and the base-change

$$\mathrm{N}^1(S) \xrightarrow{\simeq} \mathrm{N}^1(S_K)$$

is an isometry.

Second, we find arithmetic obstructions for geometrically rational surfaces to admit a numerically exceptional collection of maximal length:

Theorem 5 (Theorem 3.7). *Let S be a minimal smooth projective surface defined over a perfect field k, such that $S_{\bar{k}}$ is rational. If S admits a numerically exceptional collection of maximal length, then S is rational.*

In particular, the folklore conjecture of Orlov stating that a surface over an algebraically closed field admits a full exceptional collection only if it is rational can be extended to surfaces defined over arbitrary fields. Note however that a surface may be rational but not admit a numerically exceptional collection of maximal length; see Remark 3.9.

Surprisingly, Theorems 2, 3, 4 and 5 are obtained essentially by exploiting the linear algebra constraints on the Néron–Severi lattice of S imposed by the existence of a numerically exceptional collection of maximal length. The rich linear algebra stemming from the Riemann–Roch formula

is treated independently in the appendix. In this work, the *Néron–Severi lattice* of a smooth projective surface S over a field k, denoted $\mathrm{N}^1(S)$, refers to the group of codimension-1 cycles of S modulo numerical equivalence. Note that, by definition of numerical equivalence, $\mathrm{N}^1(S)$ is torsion-free.

Our main theorem, from which all theorems stated above arise from, is:

Theorem 6 (Main Theorem 3.1). *Let S be a smooth projective surface over a field k, with $\chi(O_S) = 1$. The following statements are equivalent:*

(i) *S admits a numerically exceptional collection $(L_0, L_1, \ldots, L_{n+1})$ of line-bundles which is of* maximal length, *that is, $n = \mathrm{rk}\,\mathrm{N}^1(S)$.*

(ii) *S admits a numerically exceptional collection $(E_0, E_1, \ldots, E_{n+1})$ of rank one objects in $D^b(S)$ which is of* maximal length, *that is, $n = \mathrm{rk}\,\mathrm{N}^1(S)$.*

(iii) *We have $(K_S)^2 = 10 - \mathrm{rk}\,\mathrm{N}^1(S)$, and the lattice $\mathrm{N}^1(S)$ and the canonical divisor K_S, when seen as an element of $\mathrm{N}^1(S)$, satisfy one of the following properties:*
  - *$\mathrm{N}^1(S)$ is unimodular of rank 1, and $K_S = 3D$ for some primitive divisor D;*
  - *$\mathrm{N}^1(S)$ is the hyperbolic plane, and $K_S = 2D$ for some primitive divisor D;*
  - *$\mathrm{N}^1(S)$ is unimodular and odd of rank > 1, and K_S is a primitive divisor.*

Note that a complex smooth projective surface with $p_g = q = 0$ always satisfies the equation $K_S^2 = 10 - \mathrm{rk}\,\mathrm{N}^1(S)$. It is thus surprising that, over any field, the existence of a numerically exceptional collection of line-bundles of maximal length imposes that equation.

A recent result of Perling [40] (see Theorem 2.2) implies that, for complex surfaces with $p_g = q = 0$, the conditions of Theorem 3.1 are further equivalent to the existence of a numerically exceptional collection of maximal length (without any assumptions on the ranks of the objects of the collection). Theorem 3.1 suggests then that if an exceptional collection of maximal length exists on a surface over an algebraically closed field, then an exceptional collection of maximal length that consists of line-bundles should exist. (As far as I am aware, all examples of complex surfaces that admit an exceptional collections of maximal length admit such a collection consisting of line-bundles.)

Finally, given a smooth projective variety X, note that if X has an exceptional object of non-zero rank or a numerically exceptional line-bundle, then the structure sheaf O_X is numerically exceptional, that is, $\chi(O_X) = 1$. (Note also that, since the classes of rank 0 objects sit in a codimension 1 subspace of $K_0(X)$, if X has an exceptional collection of maximal length, then X has an exceptional object of non-zero rank and hence $\chi(O_X) = 1$.) Therefore, the running assumption that $\chi(O_S) = 1$ is innocuous.

Notations. Given a smooth projective surface S defined over a field k, $\mathrm{N}^1(S)$ denotes the Néron–Severi lattice of S. The following numerical invariants of S will be used:

$\rho = \mathrm{rk}\,\mathrm{N}^1(S)$, the Picard rank of S.

$q = \dim_k H^1(S, O_S)$, the irregularity of S.

$p_g = \dim_k H^2(S, O_S)$, the geometric genus of S.

$b_i = \dim_{\mathbb{Q}_\ell} H^i_{et}(S_{\bar{k}}, \mathbb{Q}_\ell)$, the Betti numbers of S, where ℓ is a prime $\neq \mathrm{char}\,k$.

1. A CHARACTERIZATION OF PROJECTIVE SPACE

In this section, upon which the rest of this paper does not depend, the base field k is assumed to be of characteristic zero. Galkin, Katzarkov, Mellit, and Shinder [15] have recently considered so-called *minifolds*. These are smooth projective varieties of dimension n whose bounded derived category $D^b(X)$ admits a full exceptional collection of objects in $D^b(X)$ of length $n + 1$.

Theorem 1.1. *[Galkin, Katzarkov, Mellit, and Shinder] Assume that the base field k is the field of complex numbers. Then*

(1) *The only two-dimensional minifold is $\mathbb{P}^2$.*
(2) *The minifolds of dimension 3 are: the projective space $\mathbb{P}^3$, the quadric three-fold, the del Pezzo quintic three-fold, and a six-dimensional family of three-folds V_{22}; see [15] for more details.*
(3) *The only four-dimensional Fano minifold is $\mathbb{P}^4$.*

For other examples of minifolds, we refer to [42, 26]. Here, although we allow the base-field k to be non-algebraically closed, we consider a more restrictive class of varieties, namely smooth projective varieties of dimension n whose bounded derived category $D^b(X)$ admits a full exceptional collection of line-bundles of length $n + 1$. We show in Theorem 1.2 below that such a property characterizes completely projective space. As Theorem 1.1 shows, it is important to consider full exceptional collections of line-bundles rather than full exceptional collections consisting of objects in the derived category $D^b(X)$. In fact, it is important to consider full exceptional collections of line-bundles, rather than merely full exceptional collections of pure sheaves or even vector-bundles. For example, Kapranov [22] showed that quadrics of odd dimension, say d, over an algebraically closed field have a full exceptional collection consisting of $d + 1$ vector-bundles. On a slightly different perspective, Bernardara [6] showed that a Severi–Brauer variety X of dimension r has a full semi-orthogonal collection of objects E_i, $0 \leq i \leq r$, such that $\operatorname{Hom}(E_i, E_i[l]) = 0$ for all $l \neq 0$ and $\operatorname{Hom}(E_i, E_i) = A^{\otimes i}$ for the central division algebra A over k that has same class as X in the Brauer group $\operatorname{Br}(k)$.

Theorem 1.2. *Let X be a smooth projective variety of dimension n over a field k of characteristic zero. Assume that $\langle L_0, \ldots, L_n \rangle$ is a full exceptional collection of $D^b(X)$ for some line-bundles $L_0, \ldots, L_n$. Then X is isomorphic to the projective space $\mathbb{P}^n$.*

Proof. First, note that the assumption implies that $\operatorname{Pic} X$ is torsion-free (see Lemma 2.6 below) and of rank 1, so that $\operatorname{Pic} X = \mathbb{Z}H$ for some divisor H. By Bondal–Polishchuk [9, Theorem 3.4], a d-dimensional smooth projective variety with a full exceptional collection consisting of $d + 1$ pure sheaves is necessarily Fano, that is, $-K_X$ is ample. The theorem then follows from Proposition 1.3 below. $\square$

Proposition 1.3. *Let X be a smooth projective variety of dimension n with $\operatorname{Pic} X = \mathbb{Z}H$. Assume either that the dimension n of X is odd, or that X is Fano. If X admits a numerically exceptional collection $(L_0, \ldots, L_n)$ of line-bundles, then $X \cong \mathbb{P}^n$.*

Proof. Since $\operatorname{Pic} X = \mathbb{Z}H$, we may write $L_i = O_X(a_i H)$ for some integers a_i. By Riemann–Roch, $\chi(O_X(aH))$ is a polynomial P with rational coefficients of degree n in the variable a. Since $\chi(O_X) = 1$, this polynomial vanishes at most n times. We know, by semi-orthogonality, that $\chi(L_j, L_i) = \chi(L_i \otimes (L_j)^{-1}) = P(a_i - a_j) = 0$ for all $i < j$. Therefore the set $\{a_i - a_j : 0 \leq i < j \leq n\}$ has order at most n. Moreover, $a_i \neq a_j$ for all $i \neq j$, because otherwise $\chi(O_X(a_i H - a_j H)) = \chi(O_X) = 1$ would not be zero. This easily implies that there exist an integer m and a non-zero integer k such that $a_i = m + ki$ for all $0 \leq i \leq n$. Up to replacing H with $-H$ if necessary, we may assume that k is a positive integer. Now the polynomial P vanishes exactly at $-lk$ for $1 \leq l \leq n$. By Riemann–Roch, we have

$$(1) \qquad P(a) = \chi(O_X(aH)) = \frac{\deg(H^n)}{n!} a^n + \frac{\deg(H^{n-1} \cdot c_1(X))}{2(n-1)!} a^{n-1} + \cdots + \chi(O_X).$$

Therefore, $n! = \deg(H^n) \cdot k \cdot (2k) \cdots (nk) = \deg(H^n)k^n n!$. Having in mind that k is positive, it follows that $k = 1$ and then that $\deg(H^n) = 1$.

We can also compute $\deg(H^{n-1} \cdot c_1(X))$: by looking at the coefficient of a^{n-1} in (1), we find $\sum_{l=1}^{n} l = \frac{n}{2} \deg(H^{n-1} \cdot c_1(X))$, which gives $\deg(H^{n-1} \cdot c_1(X)) = n + 1$. Therefore $c_1(X) = (n + 1)H$. We now distinguish whether X is odd-dimensional, or Fano. In the latter case, by definition, $c_1(X)$ is ample. In the odd-dimensional case, since either H or $-H$ is ample and since $\deg(H^n) = 1$, we find that H is ample. In both cases, $c_1(X)$ is $n + 1$ times an ample divisor. By [23], it follows that the base-change of X to the complex numbers is isomorphic to

$\mathbb{P}^n_{\mathbb{C}}$. Thus X is a Severi–Brauer variety. It has a zero-cycle of degree 1, namely H^n. Therefore it is split, *i.e.*, X is isomorphic to $\mathbb{P}^n$.   □

Remark 1.4. When n is odd, Proposition 1.3 shows that, provided $\mathrm{Pic}(X)$ is torsion-free of rank 1, the assumptions of Theorem 1.2 can be relaxed, essentially by dropping the condition that $\langle L_0, \ldots, L_n \rangle$ is full. This is possibly related to the fact that there are no fake projective spaces of odd dimension; *cf.* [41]. As a corollary to Proposition 1.3, one sees that a quadric hypersurface of odd dimension $n \geq 3$ or a non-split Severi–Brauer variety of odd dimension does not admit a numerically exceptional collection of line-bundles of length $n + 1$.

2. Full exceptional collections on surfaces and Lefschetz motives

Let X be a smooth projective variety defined over a field k. Assume that X has a full exceptional collection. Then, by flat base-change [27], for a universal domain Ω containing k (this means that Ω is algebraically closed of infinite transcendence degree over its prime subfield), the pull-back X_Ω of X along $\mathrm{Spec}\,\Omega \to \mathrm{Spec}\,k$ also has a full exceptional collection. It follows that $K_0(X_\Omega)$ is of finite rank (see e.g. Lemma 2.6), and hence by applying the Chern character that the Chow ring $\mathrm{CH}^*(X_\Omega) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite-dimensional vector space over $\mathbb{Q}$. By Kimura [25], it follows that the Chow motive with rational coefficients of X_Ω is isomorphic to a direct sum of Lefschetz motives. (In fact, one may choose such an isomorphism to be defined over the base field k, so that the Chow motive with rational coefficients of X is isomorphic to a sum of Lefschetz motives; see [45, Corollary 3.5]). Such a result was also obtained in [34], via the theory of non-commutative motives. The main result of this section is Theorem 2.7: we show that when S is a surface with a full exceptional collection, then the result above can be improved by showing that the *integral* Chow motive of S is isomorphic to a sum of Lefschetz motives. The question of understanding the links between the derived category of coherent sheaves on X and the Chow motive of X is of course not new and we refer to Orlov's [39].

2.1. **The Riemann–Roch formula for surfaces.** In this section, S is a smooth projective surface defined over a field k. Let K_S be the canonical divisor on S and let D be a divisor on S. The Riemann–Roch formula is

$$\chi(O_S(D)) = \frac{1}{2} D \cdot (D - K_S) + \chi(O_S).$$

More generally, there is a Riemann–Roch formula for any object E in $D^b(S)$. The rank of an object E in $D^b(S)$ is defined as follows: if $E^\bullet$ is a complex representing E, then $\mathrm{rk}\,E := \sum_i (-1)^i \,\mathrm{rk}\,E^i$. If E and F are two objects in $D^b(S)$ of respective ranks e and f, then the Riemann–Roch formula is

$$\chi(E, F) = ef\chi(O_S) + \frac{1}{2}\left(fc_1(E)^2 + ec_1(F)^2 - 2c_1(E)c_1(F)\right) - \frac{1}{2}K_S \cdot (ec_1(F) - fc_1(E)) - (fc_2(E) + ec_2(F)).$$

Assume from now on that $\chi(O_S) = 1$. If E is an object in $D^b(S)$ of rank 1 that is numerically exceptional, that is $\chi(E, E) = 1$, then the above Riemann–Roch formula gives $c_2(E) = 0$. This justifies referring to a numerically exceptional object of rank one as a *numerical line-bundle*. Finally, note that if E and F are two numerical line-bundles in $D^b(S)$, then the Riemann–Roch formula takes the simple form:

$$(2) \qquad \chi(E, F) = \frac{1}{2}\left(c_1(F) - c_1(E)\right)^2 - \frac{1}{2}K_S \cdot (c_1(F) - c_1(E)) + 1.$$

2.2. **Numerically exceptional collections of line-bundles and the Riemann–Roch formula.** If $(E_0, E_1, \ldots, E_{r+1})$ is a numerically exceptional collection, then by definition the matrix $\left(\chi(E_i, E_j)\right)_{0 \leq i,j \leq r+1}$ is upper-triangular with 1's as diagonal entries. If moreover one assumes that the objects E_i have rank 1 for all i, then the Riemann–Roch formula relates the upper-triangularity of the matrix $\left(\chi(E_i, E_j)\right)_{0 \leq i,j \leq r+1}$ to the trigonality of the matrix $(D_i \cdot D_j)_{1 \leq i,j \leq r+1}$, where $D_i := c_1(E_i) - c_1(E_{i-1})$:

Proposition 2.1. *Let S be a smooth projective surface with $\chi(O_S) = 1$ and let r be a non-negative integer. The following statements are equivalent.*

(i) *There exists a collection $(E_0, E_1, \ldots, E_{r+1})$ of line-bundles which is a numerically exceptional collection in $D^b(S)$.*

(ii) *There exists a collection $(E_0, E_1, \ldots, E_{r+1})$ of numerical line-bundles in $D^b(S)$ which is a numerically exceptional collection.*

(iii) *There exist divisors $D_1, \ldots, D_{r+1} \in \mathrm{CH}^1(S)$ such that $K_S \cdot D_i = -2 - (D_i)^2$ for all i and such that the intersection matrix $(D_i \cdot D_j)_{1 \leq i,j \leq r+1}$ has the trigonal form*

$$
\left(D_i \cdot D_j\right)_{1 \leq i,j \leq r+1} = \begin{pmatrix} a_1 & 1 & & \\ 1 & a_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & a_{r+1} \end{pmatrix}
$$

where all blank entries consist of zeroes.

Proof. $(i) \Rightarrow (ii)$: This is obvious.

$(ii) \Rightarrow (iii)$: Assume that the collection of numerical line-bundles $(E_0, E_1, \ldots, E_{r+1})$ is numerically exceptional and let us define, for $0 < i \leq r + 1$, $D_i := c_1(E_i) - c_1(E_{i-1}) \in \mathrm{CH}^1(S)$. By orthogonality, we have, for all $0 \leq i < j \leq r + 1$, $\chi(E_j, E_i) = 0$. By Riemann–Roch (2), we find for all $0 \leq i < j \leq r + 1$,

$$
(D_{i+1} + D_{i+2} + \cdots + D_j)^2 + K_S \cdot (D_{i+1} + D_{i+2} + \cdots + D_j) = -2.
$$

Taking $j = i + 1$ yields $K_S \cdot D_i = -2 - (D_i)^2$ for all $1 \leq i \leq r + 1$. Taking then $j = i + 2$ yields $D_i \cdot D_j = 1$ for all $1 \leq i, j \leq r + 1$ such that $|i - j| = 1$. Finally, taking $j = i + 3$, $j = i + 4$, and so on, gives $D_i \cdot D_j = 0$ for all $1 \leq i, j \leq r + 1$ such that $|i - j| > 1$.

$(iii) \Rightarrow (i)$: We define $E_0 := O_S$, and $E_i := O_S(D_1 + \cdots + D_i)$ for all $1 \leq i \leq r + 1$. Since $\chi(O_S) = 1$, we have $\chi(E_i, E_i) = 1$ for all i. On the other hand, by Riemann–Roch (2), one immediately finds that $\chi(E_j, E_i) = 0$ for $0 \leq i < j \leq r + 1$, thus showing that the collection $(E_0, E_1, \ldots, E_{r+1})$ is numerically exceptional. □

2.3. **Numerically exceptional collections of maximal length and the Néron–Severi lattice.** In order to prove Theorem 2.7, we will have to prove that the existence of a (numerically) exceptional collection of maximal length consisting of objects in $D^b(S)$ implies that the Néron–Severi lattice $N^1(S)$ is unimodular. For that matter, M. Perling [40, Corollary 10.9 & Remark 10.12] recently proved:

Theorem 2.2 (Perling [40]). *Let S be a smooth projective surface with $\chi(O_S) = 1$. Then any numerically exceptional collection of maximal length on S can be transformed by mutations into a numerically exceptional collection of maximal length $(Z_0, Z_1, \ldots, Z_t, F_0, \ldots, F_{\rho-t})$, where $\mathrm{rk}\, Z_i = 0$ for $0 \leq i \leq t$ and $\mathrm{rk}\, F_j = 1$ for $0 \leq j \leq \rho - t$, and $\rho - t = 2$ or 3.*

Moreover, if $p_g = q = 0$ and $\rho = b_2$ (or more generally if $\rho = b_2 - 2b_1$), then any numerically exceptional collection of maximal length on S can be transformed by mutations into a numerically exceptional collection of maximal length that consists of rank one objects.

From the arguments developed by Perling, one can extract the following:

Proposition 2.3. *Let S be a smooth projective surface with $\chi(O_S) = 1$. Assume that S has a numerically exceptional collection of maximal length. Then the Néron–Severi lattice $N^1(S)$ is unimodular, and $\rho = b_2 - 2b_1$ modulo 8.*

Proof. By Theorem 2.2, it suffices to prove that if $(Z_0, Z_1, \ldots, Z_t, F_0, \ldots, F_s)$ is a numerically exceptional collection of maximal length, with $\mathrm{rk}\, Z_i = 0$ for $0 \leq i \leq t$ and $\mathrm{rk}\, F_j = 1$ for $0 \leq j \leq s$, then the Néron–Severi lattice is unimodular.

Suppose Z is a numerically exceptional object of rank zero. From the Riemann–Roch formula, one immediately sees that $c_1(Z)^2 = -1$. Suppose now that (Z_1, Z_2) is a numerically exceptional

collection consisting of objects of rank zero. From the Riemann–Roch formula again, one finds that $c_1(Z_1) \cdot c_1(Z_2) = 0$.

Consider now a numerically exceptional collection $(Z_0, Z_1, \ldots, Z_t, F_0, \ldots, F_s)$, where $\operatorname{rk} Z_i = 0$ for $0 \leq i \leq t$. The Riemann–Roch formula gives for all i and all j

$$2c_1(Z_i) \cdot c_1(F_j) = -\operatorname{rk}(F_j)\big(K_X \cdot c_1(Z_i) + 1 + 2c_2(Z_i)\big).$$

Thus we see that

  (i) $c_1(Z_i)^2 = -1$ for all i ;

  (ii) $c_1(Z_i) \cdot c_1(Z_j) = 0$ for all $i \neq j$;

  (iii) $c_1(Z_i) \cdot (c_1(F_j) - c_1(F_k)) = (\operatorname{rk} F_k - \operatorname{rk} F_j)\big(K_X \cdot c_1(Z_i) + 1 + 2c_2(Z_i)\big)$ for all i, j, and k.

Therefore, if $(Z_0, Z_1, \ldots, Z_t, F_0, \ldots, F_s)$ is of maximal length, with $\operatorname{rk} Z_i = 0$ for $0 \leq i \leq t$ and $\operatorname{rk} F_j = 1$ for $0 \leq j \leq s$, then the subspace of $\mathrm{N}^1(S)$ spanned by $c_1(F_0), \ldots, c_1(F_s)$ has rank $s-1$. Thus, denoting $D_j := c_1(F_j) - c_1(F_{j-1})$, we see that the matrix $\big(D_i \cdot D_j\big)_{1 \leq i,j \leq s}$, which is trigonal by the proof of Proposition 2.1, is degenerate. By Proposition A.3, $\big(D_i \cdot D_j\big)_{1 \leq i,j \leq s-1}$ is unimodular. We conclude that the lattice

$$\mathbb{Z}c_1(Z_0) \oplus \cdots \oplus \mathbb{Z}c_1(Z_t) \oplus \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_{s-1}$$

is unimodular, and therefore that $\mathrm{N}^1(S)$ is unimodular.

Finally, we show that $\rho = b_2 - 2b_1$ modulo 8. Noether's formula $K_S^2 = 12\chi(O_S) - c_2(S)$ gives $K_S^2 = 12 - 2b_0 + 2b_1 - b_2 = 10 + 2b_1 - b_2$. On the other hand, since the lattice $\mathrm{N}^1(S)$ is unimodular, van der Blij's lemma (see [20, Lemma II.(5.2)], or Remark A.11), together with the Hodge index theorem, gives $K_S^2 = 2 - \rho$ modulo 8. This concludes the proof. $\qquad\square$

Remark 2.4. At least in the case when the base field k is algebraically closed, Sasha Kuznetsov has mentioned to me the following geometric argument, which is directly inspired from [5, Theorem 4.7], showing that the existence of a numerically exceptional collection of maximal length implies the unimodularity of the Néron–Severi lattice. Choose a basis of $\mathrm{N}^1(S)$ consisting of classes of smooth curves C_i which intersect pairwise transversally. Let F_i be a theta-characteristic on C_i, considered as a torsion sheaf on S supported on C_i. Then it is clear that $\dim \operatorname{Ext}^p(F_i, F_j) = [C_i] \cdot [C_j]$ for $p = 1$ and 0 otherwise, so $\chi(F_i, F_j) = -[C_i] \cdot [C_j]$. Moreover, clearly $\chi(O_S, F_i) = 0$ for all i, and if P is a general point (not lying on any of C_i) then $\chi(O_S, O_P) = \chi(O_P, O_S) = 1$, $\chi(O_P, O_P) = 0$, and $\chi(F_i, O_P) = \chi(O_P, F_i) = 0$. This shows that the bilinear form χ expressed in the basis $(O_S, O_P, F_1, \ldots, F_n)$ is block upper-triangular with the first of the diagonal blocks being a 2-by-2 matrix

$$\begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix}$$

and the second diagonal block being the matrix $(-[C_i] \cdot [C_j])_{1 \leq i,j \leq n}$. Thus its determinant is $(-1)^{n+1} \det([C_i] \cdot [C_j])$. On the other hand, if there is a numerically exceptional collection then the determinant is 1, so one can conclude that the determinant of the intersection form is $(-1)^{n+1}$.

A straightforward corollary to Proposition 2.3 is :

Corollary 2.5. *Let S be a smooth projective surface with $\chi(O_S) = 1$. Assume that the surface S admits a numerically exceptional collection of maximal length. Then S has a zero-cycle of degree 1.* $\qquad\square$

For example, a smooth quadric surface in $\mathbb{P}^3$ with no rational point does not admit a numerically exceptional collection of maximal length, and hence does not admit a full exceptional collection. (See Theorem 3.6 and Remark 3.9 for stronger statements.)

Note that the intersection pairing on $\mathrm{N}^1(S)$ for a smooth projective complex surface S with $p_g = 0$ is always unimodular by Poincaré duality and by the Lefschetz $(1,1)$-theorem. Thus Proposition 2.3 provides an arithmetic obstruction for a surface defined over a non-algebraically closed field to admit a numerically exceptional collection of maximal length.

2.4. Exceptional collections of maximal length and Chow motives.

Lemma 2.6. *Let X be a smooth projective variety. Assume that $K_0(X)$ is torsion-free. Then $\mathrm{CH}^1(X)$ is torsion-free. If additionally X is a smooth projective surface, then the Chow ring $\mathrm{CH}^*(X)$ is torsion-free.*

Proof. First, we note that $\mathrm{CH}^0(X) = \mathbb{Z}[X]$ is torsion-free for all varieties X.

We show that if a smooth variety X has torsion-free $K_0(X)$, then $\mathrm{CH}^1(X)$ is torsion-free. This was already proved in [15, Lemma 2.2], and we reproduce their proof for the sake of completeness. Recall that the first Chern class provides a group isomorphism $\mathrm{Pic}(X) \cong \mathrm{CH}^1(X)$, so that it is equivalent to show that $\mathrm{Pic}(X)$ is torsion-free. Assume that L is a line-bundle with $L^{\otimes r} = 0$ for some integer $r \geq 2$. The element $[L] - 1 \in K_0(X)$ has rank zero and thus belongs to $F^1 K_0(X)$, where $F^\bullet$ denotes the topological filtration on K-groups. By multiplicativity of the topological filtration, we find that $([L] - 1)^{\dim X + 1} = 0 \in K_0(X)$. Let N be the smallest integer such that $([L] - 1)^N = 0$. If $N = 1$, then $[L] = 1$ and we get $L \cong O_X$. If $N \geq 2$, we have

$$1 = [L^{\otimes r}] = [L]^r = (1 + ([L] - 1))^r = 1 + r([L] - 1) + \alpha([L] - 1)^2 \ \in K_0(X),$$

for some $\alpha \in K_0(X)$. Multiplying by $([L] - 1)^{N-2}$ yields

$$r([L] - 1)^{N-1} = 0 \ \in K_0(X),$$

that is, $([L] - 1)^{N-1}$ is a non-zero torsion element in $K_0(X)$.

It remains to see that if S is a smooth projective surface such that $K_0(S)$ is torsion-free, then $\mathrm{CH}^2(S)$ is torsion-free. This follows immediately from the fact [14, Ex. 15.3.6] that the second Chern class induces an isomorphism $c_2 : F^2 K_0(S) \xrightarrow{\simeq} \mathrm{CH}^2(S)$ (here, $F^2 K_0(S)$ is the subgroup of $K_0(S)$ spanned by coherent sheaves supported in codimension 2). Since $K_0(S)$ is assumed to be torsion-free, we obtain that $\mathrm{CH}^2(S)$ is torsion-free. $\square$

Theorem 2.7. *Let S be a smooth projective surface over a field k, with $\chi(O_S) = 1$. Assume that S has a numerically exceptional collection of maximal length. Then $\mathbb{1} \oplus \mathbb{1}(-1)^{\oplus \rho} \oplus \mathbb{1}(-2)$ is a direct summand of the integral Chow motive of S. Moreover, if S has a full exceptional collection, then the integral Chow motive of S is isomorphic to a sum of Lefschetz motives.*

Proof. First assume that S has a numerically exceptional collection $(E_0, \ldots, E_{n+1})$ of maximal length. By Proposition 2.3, there exists a n-tuple $(D_1, \ldots, D_n)$ of elements of $\mathrm{CH}^1(S)$, such that the matrix $M = (D_i \cdot D_j)_{1 \leq i,j \leq n}$ is unimodular. (In particular, $\mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_n$ is a sub-group of $\mathrm{CH}^1(S)$.) Let then $(D_1^\vee, \ldots, D_n^\vee)$ be the basis of $\mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_n$ that is dual to the basis $(D_1, \ldots, D_n)$ with respect to the intersection pairing and let, by Corollary 2.5, $a \in \mathrm{CH}_0(S)$ be a zero-cycle of degree 1 on S. We define the correspondences $\pi^0 := a \times S$, $\pi^4 := S \times a$, and

$$p_i := D_i^\vee \times D_i, \quad \text{for all } 1 \leq i \leq n.$$

These define mutually orthogonal idempotents in the correspondence ring $\mathrm{CH}^2(S \times S)$. For instance $p_i \circ p_j = (D_j \cdot D_i^\vee) D_j^\vee \times D_i = \delta_{i,j} D_j^\vee \times D_i$, where $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$. It is also clear that $\pi^0 \circ \pi^0 = \pi^0$, $\pi^4 \circ \pi^4 = \pi^4$, and that $\pi^0 \circ \pi^4 = \pi^4 \circ \pi^0 = \pi^4 \circ p_i = p_i \circ \pi^4 = \pi^0 \circ p_i = p_i \circ \pi^0 = 0$. Moreover, we have $(S, \pi^0) \cong \mathbb{1}$, $(S, \pi^4) \cong \mathbb{1}(-2)$, and $(S, p_i) \cong \mathbb{1}(-1)$ for all i, and this proves that $\mathbb{1} \oplus \mathbb{1}(-1)^{\oplus \rho} \oplus \mathbb{1}(-2)$ is a direct summand of the integral Chow motive of S.

Assume now that the collection $(E_0, \ldots, E_{n+1})$ is full exceptional. It is enough to show that the idempotent correspondence

$$\Gamma := \Delta_S - \pi^0 - \pi^4 - \sum_i p_i \in \mathrm{CH}^2(S \times S)$$

is equal to 0. The key is to show that Γ acts as zero on the integral Chow groups $\mathrm{CH}^i(S)$ for $i = 0, 1$ and 2. For this, it is enough to show that $\mathrm{CH}^1(S) = \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_n$ and that $\mathrm{CH}^2(S) = \mathbb{Z}a$. First recall that, in general, if X is a smooth projective variety with a full exceptional collection that consists of N exceptional objects, then $K_0(X)$ is free of rank N. The Chern character induces an isomorphism $K_0(S) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \mathrm{CH}^*(S) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\mathrm{rk}\,\mathrm{CH}^0(S) = 1$

and since $\operatorname{rk} \operatorname{CH}^2(S) \geq 1$, we get that $\operatorname{rk} \operatorname{CH}^1(S) \leq n$. Since the intersection pairing restricted to the subspace of $\operatorname{CH}^1(S)$ spanned by $D_1, \dots, D_n$ is unimodular, and since $\operatorname{CH}^1(S)$ is torsion-free by Lemma 2.6, we find that $(D_1, \dots, D_n)$ forms a $\mathbb{Z}$-basis of $\operatorname{CH}^1(S)$. It follows that $\operatorname{CH}^2(S)$ has rank 1. Since $\operatorname{CH}^2(S)$ is also torsion-free by Lemma 2.6, we find that $\operatorname{CH}^2(S)$ is spanned by any zero-cycle of minimal positive degree and thus that $\operatorname{CH}^2(S) = \mathbb{Z}a$.

Finally, by flat base-change [27], the sequence $\langle (E_0)_K, (E_1)_K, \dots, (E_{n+1})_K \rangle$ is a full exceptional collection for $S_K = S \times_{\operatorname{Spec} k} \operatorname{Spec} K$ for all field extensions K/k. Here $(E_i)_K$ is the pull-back of the object E_i along the projection $S_K \to S$. Hence, the arguments above show that $(\Gamma_K)_* \operatorname{CH}^*(S_K) = 0$ for all field extensions K/k. Therefore the correspondence Γ is nilpotent; see e.g. [43, Proposition 3.2]. Because Γ is an idempotent, we conclude that $\Gamma = 0$. $\qquad\square$

Remark 2.8. One may in fact prove the following generalization of Theorem 2.7: If S is a surface such that the base-change $K_0(S) \to K_0(S_K)$ is surjective for all field extensions K/k, then the integral Chow motive of S is isomorphic to a direct sum of Lefschetz motives. (This is indeed a generalization of Theorem 2.7 because of the base-change theorem [27] for full exceptional collections.) For that matter, one uses a recent result of Totaro [44, Theorem 4.1], combined with the integral version of the Riemann–Roch formula as used in the proof of Lemma 2.6.

Remark 2.9. It seems that the converse to Theorem 2.7 is not true, that is, for a surface to have a full exceptional collection seems more restrictive than its integral Chow motive being isomorphic to a direct sum of Lefschetz motive. Consider for instance a complex Barlow surface S. On the one hand, it is proved in [2, Proposition 1.9] and [46, Corollary 2.2] that the Chow group of zero-cycles of S is universally trivial, in the sense that $\operatorname{CH}^2(S_K) = \mathbb{Z}$ for all field extensions $K/\mathbb{C}$. By [44, Theorem 4.1], it follows that the integral Chow motive of S is a direct sum of Lefschetz motives. On the other hand, Böhning, Graf von Bothmer, Katzarkov, and Sosna [8] have exhibited a complex Barlow surface S (a determinantal Barlow surface) with an exceptional collection whose orthogonal complement is a *phantom* category, that is, a non-trivial strictly full triangulated category with vanishing K_0. Of course this does not say that the Barlow surface S does not admit a full exceptional collection, but it looks like a possibility that it won't. Finally, as yet another reason why having a full exceptional collection is stronger than having an integral motive isomorphic to a direct sum of Lefschetz motives, it is believed and conjectured that a surface that admits a full exceptional collection must be rational.

3. Numerically exceptional collections of maximal length on surfaces

In the previous section, we saw that the existence of a full exceptional collection for a surface S gives serious constraints on the integral motive of S. In this section, we show that, for a surface S, the weaker condition of having a numerically exceptional collection of maximal length, consisting of rank one objects, is still very restrictive. The main result, which builds up on Proposition 3.2, is Theorem 3.1: we give a necessary and sufficient condition for a smooth projective surface S defined over a field k, with $\chi(O_S) = 1$, to admit a numerically exceptional collection of maximal length, consisting of line-bundles. Although its proof consists mostly of elementary linear algebra and lattice theory, Theorem 3.1 has surprising consequences. On an arithmetic perspective, Theorem 3.3 roughly says that a "non-split" surface over a field k (e.g. a surface that is not rational over k but that becomes rational after some field extension; see Theorem 3.7) does not admit a numerically exceptional collection of maximal length, consisting of line-bundles. On a geometric perspective, Theorem 3.10 determines exactly which complex surfaces with $p_g = q = 0$ admit a numerically exceptional collection of maximal length.

3.1. **Main theorem.** Before we proceed to the statement of Theorem 3.1, let us recall some facts about lattices for which we refer to [20]. A lattice Λ is a free $\mathbb{Z}$-module of finite rank equipped with a symmetric bilinear form $b : \Lambda \times \Lambda \to \mathbb{Z}$. A lattice is said to be *even* if the norm of every vector is even; it is said to be *odd* otherwise. A lattice is said to be *unimodular* if the determinant of its bilinear form (expressed in any $\mathbb{Z}$-basis) is equal to ± 1. An odd unimodular lattice of signature $(1, N)$ is always isomorphic to the lattice $\langle -1 \rangle^{\oplus N} \oplus \langle 1 \rangle$. Here, $\langle \pm 1 \rangle$ denotes

the lattice of rank 1 with generator of norm equal to ± 1, and the direct sum is understood as being orthogonal. An even unimodular lattice of signature $(1, N)$ exists only when $N - 1$ is divisible by 8, in which case it is isomorphic to $U \oplus E_8(-1)^{\oplus \frac{N-1}{8}}$. Here, U is the hyperbolic plane and $E_8(-1)$ is the opposite of the E_8-lattice.

Theorem 3.1. *Let S be a smooth projective surface over a field k, with $\chi(O_S) = 1$. The following statements are equivalent:*

 (i) *S admits a numerically exceptional collection $(L_0, L_1, \ldots, L_{n+1})$ of line-bundles which is of maximal length, that is, $n = \mathrm{rk}\, \mathrm{N}^1(S)$.*
 (ii) *S admits a numerically exceptional collection $(E_0, E_1, \ldots, E_{n+1})$ of numerical line-bundles in $D^b(S)$ which is of maximal length, that is, $n = \mathrm{rk}\, \mathrm{N}^1(S)$.*
 (iii) *We have $(K_S)^2 = 10 - \mathrm{rk}\, \mathrm{N}^1(S)$, and the lattice $\mathrm{N}^1(S)$ and the canonical divisor K_S, when seen as an element of $\mathrm{N}^1(S)$, satisfy one of the following properties:*
   - *$\mathrm{N}^1(S) \cong \langle 1 \rangle$ and $K_S = 3D$ for some primitive divisor D;*
   - *$\mathrm{N}^1(S) \cong U$ and $K_S = 2D$ for some primitive divisor D;*
   - *$\mathrm{N}^1(S) \cong \langle 1 \rangle \oplus \langle -1 \rangle^{\oplus n}$ with $n > 0$ and K_S is a primitive divisor.*

A first step towards proving the theorem consists in characterizing the Néron–Severi lattice, together with the way the canonical divisor sits in it, of surfaces that admit a numerically exceptional collection of maximal length:

Proposition 3.2. *Let S be a smooth projective surface with $\chi(O_S) = 1$. The following statements are equivalent.*

 (i) *S admits a numerically exceptional collection $(E_0, E_1, \ldots, E_{n+1})$ of numerical line-bundles, which is of maximal length, that is, $n = \mathrm{rk}\, \mathrm{N}^1(S)$.*
 (ii) *The Néron–Severi lattice $\mathrm{N}^1(S)$ is trigonal and unimodular, and K_S is a special characteristic element in the sense of Definition A.2.*

Proof. By Proposition A.3, (ii) is equivalent to the existence of $n + 1$ divisors $D_1, \ldots, D_{n+1}$ in $\mathrm{N}^1(S)$ such that the matrix $(D_i \cdot D_j)_{1 \le i, j \le n+1}$ is trigonal and such that $K_S \cdot D_i = -2 - (D_i)^2$ for all i. This in turn is equivalent, by Proposition 2.1, to the existence of a numerically exceptional collection of maximal length consisting of numerical line-bundles. $\square$

Proof of Theorem 3.1. $(i) \Rightarrow (ii)$: This is obvious.

$(ii) \Rightarrow (iii)$: By Proposition 3.2, the Néron–Severi lattice $\mathrm{N}^1(S)$ is trigonal and unimodular, and K_S is a *special* characteristic element in the sense of Definition A.2, *i.e.*, $K_S \cdot c_1(E_i) = -c_1(E_i)^2 - 2$ for all $0 \le i \le n+1$. We can conclude by invoking Theorem A.4 (and specifically the implication $(i) \Rightarrow (ii)$ therein).

$(iii) \Rightarrow (i)$: This is the conjunction of Proposition 3.2, and of the implication $(ii) \Rightarrow (i)$ of Theorem A.4. Actually, thanks to item (iii) of Theorem A.4, we can be more precise: we can find an orthogonal basis of $\mathrm{N}^1(S)$ in which K_S has a nice expression, and we can express explicitly a numerically exceptional collection of maximal length in terms of that orthogonal basis.

Assume that $\mathrm{N}^1(S)$ is an odd unimodular lattice of rank n. Since it has signature $(1, n-1)$ and since $(K_S)^2 = 10 - n$ by assumption, Theorem A.4 implies that there exists a $\mathbb{Z}$-basis $(D_1, \ldots, D_n)$ such that $K_S = D_1 + D_2 + \cdots + D_{n-1} - 3D_n$, $D_i \cdot D_j = 0$ for $i \ne j$, $(D_i)^2 = -1$ for $i \le n - 1$ and $(D_n)^2 = 1$. Let us then define $D_{n+1} = 2D_n$. Then we easily check that the collection $(O_S, O_S(D_1), \ldots, O_S(D_{n+1}))$ is numerically exceptional. (Note the analogy: if S is $\mathbb{P}^2$ blown up at $n - 1$ points, then the exceptional divisors $E_1, \ldots, E_{n-1}$ together with $H := c_1(O(1))$ provide an orthogonal basis of $\mathrm{N}^1(S)$ such that $(E_i)^2 = -1$ for all i, $H^2 = 1$, and $K_S = E_1 + \cdots + E_{n-1} - 3H$; moreover the collection $(O_S, O_S(E_1), \ldots, O_S(E_{n-1}), O_S(1), O_S(2))$ is numerically exceptional – in fact, full exceptional.)

Likewise, if $\mathrm{N}^1(S)$ is isomorphic to the hyperbolic plane and if K_S is twice a primitive divisor, then, because $(K_S)^2 = 8$ by assumption, Theorem A.4 gives a $\mathbb{Z}$-basis (D_1, D_2) of $\mathrm{N}^1(S)$ such that $(D_1)^2 = (D_2)^2 = 0$, $D_1 \cdot D_2 = 1$, and $K_S = -2D_1 - 2D_2$. We then define $D_3 := D_1 + D_2$.

Again it is straightforward to check that the collection $(O_S, O_S(D_1), O_S(D_2), O_S(D_1 + D_2))$ is numerically exceptional. (Note the analogy: suppose S is the Hirzebruch surface Σ_n, $n \geq 0$, that is the projective bundle $\mathbb{P}(O \oplus O(-n))$ over $\mathbb{P}^1$, and let F be a fiber and C the zero section. In particular we have $F^2 = 0, F \cdot C = 0, C^2 = -n$ and $K_S = -(2 + n)F - 2C$. If n is even, say $n = 2m$, then $D_1 := F$ and $D_2 := C + mF$ is a hyperbolic basis of $\mathrm{N}^1(\Sigma_{2m})$ such that $K_S = -2D_1 - 2D_2$; moreover the collection $(O_S, O_S(F), O_S(C + mF), O_S(C + (m + 1)F))$ is numerically exceptional – in fact, full exceptional. If n is odd, say $n = 2m - 1$, then the Néron–Severi lattice is odd and we are in the previous situation. The divisors $D_1 := C + (m - 1)F$ and $D_2 := C + mF$ provide an orthogonal basis such that $(D_1)^2 = -1$ and $(D_2)^2 = 1$ and such that $K_S = D_1 - 3D_2$; moreover the collection $(O_S, O_S(C + (m - 1)F), O_S(C + mF), O_S(2C + 2mF))$ is numerically exceptional – in fact, full exceptional.) $\qquad\square$

3.2. **Consequence for the cycle class map.** Let k be a field and denote $\bar{k}$ a separable closure. Given a field extension K/k and a scheme X over k, we write $X_K := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$.

Theorem 3.1 gives constraints of arithmetic nature for the existence of numerically exceptional collections, consisting of numerical line-bundles, of maximal length:

Theorem 3.3. *Let S be a smooth projective surface over a field k, with $\chi(O_S) = 1$ and with first Betti number $b_1 = 0$. Assume that S admits a numerically exceptional collection $(E_0, E_1, \ldots, E_{n+1})$ of maximal length, consisting of numerical line-bundles. Then, for all primes ℓ not dividing $\mathrm{char}\, k$, the cycle class map*

$$\mathrm{CH}^1(S) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{et}(S_{\bar{k}}, \mathbb{Z}_\ell(1))$$

is surjective modulo torsion, that is, it induces a surjective map

$$\mathrm{CH}^1(S) \otimes \mathbb{Z}_\ell \twoheadrightarrow \mathrm{H}^2_{et}(S_{\bar{k}}, \mathbb{Z}_\ell(1))/torsion.$$

In particular, the collection $((E_0)_K, (E_1)_K, \ldots, (E_{n+1})_K)$ for S_K is numerically exceptional of maximal length, and the base-change

$$\mathrm{N}^1(S) \xrightarrow{\simeq} \mathrm{N}^1(S_K)$$

is an isometry for all field extensions K/k.

In other words, in the same way a full exceptional collection remains full exceptional after extension of the base field [27], a numerically exceptional collection of numerical line-bundles of maximal length on a surface S with $p_g = q = 0$ remains of maximal length after any field extension. Note that, by Theorem 2.7, if $\langle E_0, E_1, \ldots, E_{n+1} \rangle$ is full exceptional, then the cycle class map $\mathrm{CH}^1(S) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{et}(S_{\bar{k}}, \mathbb{Z}_\ell(1))$ is surjective.

Note that surfaces S with $\chi(O_S) = 1$ and $b_1 = 0$ include surfaces with $p_g = q = 0$; see *e.g.* [32, §3.4]. These conditions are equivalent in characteristic zero, or if the surface lifts to characteristic zero. However, in positive characteristic, there are examples of surfaces with $b_1 = 0$ and $p_g = q > 0$, *e.g.*, non-classical Godeaux surfaces [31].

Proof of Theorem 3.3. By Noether's formula, $(K_S)^2 = 10 - b_2$. By Theorem 3.1, we also have $(K_S)^2 = 10 - \rho$. This implies that $\rho = b_2$. On the other hand, Theorem 3.1 also says that the intersection pairing on $\mathrm{N}^1(S)$ is unimodular. This finishes the proof of the theorem. $\qquad\square$

3.3. **On a result of Hille and Perling** [19]. The aim of this paragraph is to extend the main result of Hille–Perling [19] to surfaces that are defined over non-algebraically closed fields and that admit numerically exceptional collections of maximal length (rather than full exceptional) consisting of line-bundles. Hille and Perling proved the following (we refer to [40, Theorem 11.3] for a precise statement):

Theorem 3.4 (Hille–Perling [19]). *Let S be a smooth projective surface defined over an algebraically closed field k. Assume that S admits a full exceptional collection $(E_0, \ldots, E_{n+1})$ consisting of line-bundles. Set $E_{n+2} := E_0(-K_S)$. Then to this sequence there is associated in a canonical way a smooth complete toric surface with torus invariant prime divisors $\Delta_0, \ldots, \Delta_{n+1}$ such that $\Delta_i^2 + 2 = \chi(E_{i+1} \otimes E_i^{-1})$ for all $0 \leq i \leq n + 1$.*

In light of Theorem 3.1, the assumption that the base field k is algebraically closed in Theorem 3.4 can be lifted:

Theorem 3.5. *Let S be a smooth projective surface defined over a field k. Assume that $\chi(O_S) = 1$ and that S admits a numerically exceptional collection $(E_0, \ldots, E_{n+1})$ of maximal length, consisting of line-bundles. Then the conclusion of Theorem 3.4 holds.*

Proof. Let us define as before, for $1 \le i \le n+1$, $D_i := c_1(E_i) - c_1(E_{i-1})$. Let us also define, following Hille and Perling [19, p. 1242], $D_0 := -K_S - \sum_{i=1}^{n+1} D_i$ (compare with Proposition A.3). By convention, we set $D_{i+n+2} := D_i$. Then, by Proposition 2.1, we get

(i) $D_i \cdot D_{i+1} = 1$ for all i;

(ii) $D_i \cdot D_j = 0$ for $i \ne j$ and $\{i, j\} \ne \{l, l+1\}$ for all $0 \le l \le n+1$;

(iii) $\sum_{i=1}^{n+2} D_i = -K_S$.

The data consisting of $\{D_i, 1 \le i \le n+2\}$ will define an *abstract toric system* in the sense of [19, Definition 2.6] if the extra condition

(iv) $\sum_{i=1}^{n+2} (D_i)^2 = 12 - 3(n+2)$

holds. Conditions (i), (ii) and (iii) yield $(K_S)^2 = 2(n+2) + \sum_{i=1}^{n+2} (D_i)^2$. But then, by our main Theorem 3.1, we have $(K_S)^2 = 10 - n$. Therefore (iv) does indeed hold, so that $\{D_i, 1 \le i \le n\}$ does define an abstract toric system. The theorem then follows because the proof of [19, Theorem 3.5] depends only on the combinatorial data of an abstract toric system. $\square$

3.4. **Exceptional collections and rational surfaces.** The following theorem is due to Manin [33] and Iskovskikh [21]; see also [18, Theorem 3.9].

Theorem 3.6. *Let S be a smooth projective minimal surface defined over a perfect field k. Assume that $S_{\bar{k}}$ is rational. Then S is one of the following:*

- $\mathbb{P}^2$;
- $S \subset \mathbb{P}^3$ *a smooth quadric with* $\mathrm{Pic}(S) = \mathbb{Z}$;
- *a del Pezzo surface with* $\mathrm{Pic}(S) = \mathbb{Z}K_S$;
- *a conic bundle* $f : S \to C$ *over a conic, with* $\mathrm{Pic}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$.

The following theorem is a consequence of Theorem 3.1; it shows that geometrically rational, but non-rational, minimal surfaces defined over a perfect field do not admit an exceptional collection of maximal length.

Theorem 3.7. *Let S be a geometrically rational, smooth projective surface defined over a perfect field k that admits a numerically exceptional collection $(E_0, \ldots, E_{n+1})$ of maximal length. Assume either that S is minimal, or that the objects E_i are numerical line-bundles. Then S is rational.*

Lemma 3.8. *Let S be a smooth projective surface over a perfect field k, and denote Σ a minimal model of S. Assume that the Néron–Severi lattice $\mathrm{N}^1(S)$ is unimodular. Then S is obtained from Σ by successively blowing up rational k-points on Σ. Moreover, $\mathrm{N}^1(\Sigma)$ is unimodular.*

Proof. First note that S is obtained from Σ by successively blowing up Galois-invariant closed points; see for instance [18]. Let $\tilde{T}$ be the blow-up of a smooth projective surface T over k along a Galois-invariant closed point of degree $d \ge 1$. Such a blow-up produces a Galois-invariant collection of pairwise disjoint (-1)-curves, say $E_1, \ldots, E_d$, and the Néron–Severi lattice of $\tilde{T}$ splits orthogonally as $\langle E \rangle \oplus N'$, where $E = E_1 + \cdots + E_d$ and $E^2 = -d$, for some lattice N'. This establishes the lemma. $\square$

Proof of Theorem 3.7. Let S be a geometrically rational, smooth projective surface defined over a perfect field k that admits a numerically exceptional collection of maximal length. By Proposition 2.3, its Néron–Severi lattice is unimodular. It follows from Lemma 3.8 that S is obtained from one of the minimal surfaces listed in Theorem 3.6 by successively blowing up rational points. Denote Σ a minimal model for S; $\mathrm{N}^1(\Sigma)$ is unimodular.

If $\Sigma = \mathbb{P}^2$, then S is obviously rational.

If Σ is a smooth quadric in $\mathbb{P}^3$ with $\mathrm{Pic}(\Sigma) = \mathbb{Z}$, then $\rho(\Sigma) \neq b_2(\Sigma) = 2$ modulo 8. It follows that $\rho(S) \neq b_2(S)$ modulo 8. Therefore, by Proposition 2.3, S does not admit a numerically exceptional collection of maximal length and we get a contradiction.

If Σ is a conic bundle $f : \Sigma \to C$ over a conic, with $\mathrm{Pic}(\Sigma) \cong \mathbb{Z} \oplus \mathbb{Z}$, then by the unimodularity Σ has a degree-one zero-cycle and hence $C = \mathbb{P}^1$. Note that $\mathrm{N}^1(\Sigma)$ is spanned by a fiber F and by a multi-section D. Indeed $\mathrm{N}^1(\Sigma)$ cannot be spanned by two vertical components, *i.e.*, by irreducible components of some fibers, since otherwise the intersection pairing on $\mathrm{N}^1(\Sigma)$ would be negative, contradicting the Hodge index theorem. Suppose now that $\mathrm{N}^1(\Sigma)$ is spanned by two multi-sections D and D'. Then there exist co-prime integers u and v such that $uD + vD'$ is torsion in $\mathrm{CH}^1(\Sigma_\eta)$, where Σ_η is the conic that is the generic fiber of f. Thus, by localization for Chow groups, we see that $\mathrm{N}^1(\Sigma)$ is spanned by D and by a vertical component. But since $\mathrm{N}^1(\Sigma)$ has rank two we see [18, §3.2] that in fact $\mathrm{N}^1(\Sigma)$ is spanned by D and a fiber F, as claimed. Since $F^2 = 0$, the unimodularity yields $D \cdot F = 1$, and hence that D is in fact a section. We conclude by showing that this implies that f is a smooth $\mathbb{P}^1$-bundle : let F' be a fiber of f and denote k_1/k the field of definition of F'. Since $D \cdot F' = 1$, we see that F' has a k_1-rational point and that F' is smooth (since otherwise the two geometric components of F' would be defined over k_1 and thus would not be in the same Galois orbit and hence $\mathrm{N}^1(\Sigma)$ would have rank ≥ 3). This implies that $F' = \mathbb{P}^1_{k_1}$. This proves that Σ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, and hence that S is rational.

If Σ is a del Pezzo surface with $\mathrm{Pic}(\Sigma) = \mathbb{Z}K_\Sigma$, then we distinguish between two cases. Assume S admits a numerically exceptional collection of maximal length, consisting of numerical line-bundles. Note that if a surface S is obtained from a surface Σ by successively blowing up k-rational points, then, since each such blowup increases $\mathrm{rk}(\mathrm{N}^1)$ by 1 and decreases K^2 by 1, $(K_S)^2 = 10 - \mathrm{rk}\,\mathrm{N}^1(S)$ if and only if $(K_\Sigma)^2 = 10 - \mathrm{rk}\,\mathrm{N}^1(\Sigma)$. Therefore, by Theorem 3.1, we have on the one hand that $\mathrm{N}^1(\Sigma)$ is unimodular and hence $K_\Sigma^2 = 1$, and on the other hand that $K_\Sigma^2 = 10 - \mathrm{rk}\,\mathrm{N}^1(\Sigma) = 9$, thus yielding a contradiction. Assume now that $S = \Sigma$ (*i.e.*, S is minimal). Then Theorem 2.2 gives a numerically exceptional collection of maximal length, consisting of 3 numerical line-bundles, for S. We conclude to a contradiction as before.   □

Remark 3.9. It should be noted that the converse to Theorem 3.7 does not hold : a rational surface over a field k need not admit a numerically exceptional collection of maximal length. Consider for example the rational surface X defined over a non-algebraically closed field obtained as the blow-up of the projective plane $\mathbb{P}^2$ along a non-rational Galois-invariant closed point. Then the Néron–Severi lattice of X is not unimodular and, by virtue of Theorem 3.1, X does not admit a numerically exceptional collection of maximal length.

In fact, a *minimal* rational surface over a field k need not admit a numerically exceptional collection of maximal length. Indeed, a smooth quadric Q with a rational point is rational, but, if $\mathrm{Pic}(Q) = \mathbb{Z}$, then, by Proposition 2.3, Q does not admit a numerically exceptional collection of maximal length since $\rho = b_2 - 1$.

3.5. Numerically exceptional collections of maximal length for complex surfaces.
Let us now turn to geometric consequences. Until the end of this paragraph, the base field is the field of complex numbers. In Theorem 3.10, we determine exactly which smooth projective complex surfaces with $p_g = q = 0$ admit a numerically exceptional collection of maximal length.

Let S be a smooth minimal projective complex surface with $q = p_g = 0$. Then, by the Enriques–Kodaira classification of compact complex surfaces according to their Kodaira dimension κ, S is one of the following (see e.g. [24]) :

• $\kappa = -\infty$, a minimal rational surface. The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces Σ_n, $n = 0, 2, 3, 4, \ldots$, where Σ_n is the $\mathbb{P}^1$-bundle $\mathbb{P}(O \oplus O(-n))$ over $\mathbb{P}^1$. For instance, $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Note that Σ_1 is not minimal, it is $\mathbb{P}^2$ blown up once. Note also that $K^2 = 9$ for $\mathbb{P}^2$ and $K^2 = 8$ for Σ_n.

• $\kappa = 0$ or 1, a minimal properly elliptic surface. Let X_9 be the rational elliptic surface obtained from $\mathbb{P}^2$ by blowing up the nine base points of a generic cubic pencil. Then Dolgachev

[13] proved that the minimal complex surfaces with $p_g = q = 0$ of Kodaira dimension 0 or 1 are obtained from X_9 by performing logarithmic transformations on at least two different smooth fibers. We denote the surfaces obtained in this way by $X_9(p_1, ..., p_n)$ $(p_1 \leq p_2 \leq \ldots \leq p_n)$, where the p_i are the multiplicities of the logarithmic transformations, and call them Dolgachev surfaces. (Some authors reserve this name for the case when there are only two multiple fibers and their multiplicities are relatively prime.) Now $X_9(2, 2)$ is the Enriques surface; it is the only Dolgachev surface with Kodaira dimension 0. Note that $K_S^2 = 0$ for all of these surfaces.

• $\kappa = 2$, a minimal surface of general type. For a minimal surface of general type, we have $K_S^2 > 0$, as well as Castelnuovo's inequality $c_2 > 0$. If in addition $q = p_g = 0$, then $K_S^2 + c_2 = 12$ by Noether's formula, and $K_S^2 \leq 9$.

In fact, unless S has Kodaira dimension $= 0$ or 1, there are no obstructions to the existence of a numerically exceptional collection of maximal length:

Theorem 3.10. *Let S be a smooth projective complex surface with $p_g = q = 0$.*

• *If S is not minimal, then S has a numerically exceptional collection of maximal length.*

Assume now that S is minimal.

• *If $\kappa(S) = -\infty$, then S has a numerically exceptional collection of maximal length.*
• *If $\kappa(S) = 0$, then S is an Enriques surface and it does not have a numerically exceptional collection of maximal length.*
• *If $\kappa(S) = 1$, then S is a Dolgachev surface $X_9(p_1, \ldots, p_n)$, and S has a numerically exceptional collection of maximal length if and only if S is one of $X_9(2, 3)$, $X_9(2, 4)$, $X_9(3, 3)$, $X_9(2, 2, 2)$.*
• *If $\kappa(S) = 2$, then S has a numerically exceptional collection of maximal length.*

Proof. First note that under the condition $p_g = q = 0$, we have $\chi(O_S) = 1$, $b_2 = \rho$, and the intersection pairing on $N^1(S)$ is unimodular. Indeed, the first Chern class induces an isomorphism $\text{Pic}(S) \cong H^2(S, \mathbb{Z})$, so that $N^1(S) \cong H^2(S, \mathbb{Z})/torsion$; furthermore, by Poincaré duality, it follows that the intersection pairing on $N^1(S)$ is unimodular.

If S is not minimal, that is if S is the blow-up of a smooth projective surface, then $N^1(S)$ is odd of rank ≥ 2 and K_S is clearly primitive. Hence, by Theorem 3.1, if S is not minimal, then S admits a numerically exceptional collection of line-bundles of maximal length.

From now on, we assume that S is a minimal surface. By Theorem 2.2 and Theorem 3.1, note that, since $b_2 = \rho$, S has a numerically exceptional collection of maximal length if and only if it has one consisting of line-bundles.

• If $\kappa = -\infty$, then in fact S has a full exceptional collection of line-bundles: if $S = \mathbb{P}^2$, then the Beilinson collection $\langle O_S, O_S(1), O_S(2) \rangle$ is full exceptional; if $S = \Sigma_n$, $n = 0, 2, 3, 4, \ldots$, is a Hirzebruch surface, then denoting respectively F and C a fiber and a section of the corresponding $\mathbb{P}^1$-bundle (so that $F^2 = 0, F \cdot C = 1$ and $C^2 = -n$), we have that $\langle O_S, O_S(F), O_S(C + sF), O_S(C + (s + 1)F) \rangle$ is a full exceptional collection for all integers s (this is essentially contained in [38]).

• If $\kappa = 2$, then we know that $K_S^2 \in \{1, 2, \ldots, 9\}$. Thus if $K_S = rD$ for some positive integer r and some primitive divisor D, then $r \in \{1, 2, 3\}$. Also, by Noether's formula, the Néron–Severi lattice $N^1(S)$ has rank $\in \{1, 2, \ldots, 9\}$, and, by the classification of unimodular lattices of signature $(1, n - 1)$, we see that $N^1(S)$ is even if and only if it is isomorphic to the hyperbolic plane U. Now, we have

Lemma 3.11. *Let S be a complex surface with $p_g = q = 0$. Then, the Néron–Severi lattice $N^1(S)$ is even if and only if $K_S = 2D$ for some divisor $D \in N^1(S)$.*

Proof of the lemma. Recall that K_S is a characteristic element in $N^1(S)$, that is, $E \cdot (E - K_S)$ is even for all $E \in N^1(S)$ (this goes by the name of Wu's formula; it follows from the Riemann–Roch formula (2) whereby $E \cdot (E - K_S) = 2(\chi(O_S(E)) - \chi(O_S)))$. Therefore E^2 is even for all $E \in N^1(S)$ if and only if $K_S \cdot E$ is even for all $E \in N^1(S)$. Thus, if $K_S = 2D$, then $N^1(S)$ is even. Conversely, the pairing on $N^1(S)$ is unimodular by Poincaré duality, and hence induces

an isomorphism from $\mathrm{N}^1(S)$ to its dual. Since K_S is characteristic and $\mathrm{N}^1(S)$ is assumed to be even, the element $K_S \in \mathrm{N}^1(S)$ is mapped to $2w$, for some $w \in \mathrm{N}^1(S)^\vee$, under this isomorphism. It is then apparent that $K_S = 2D$ for some divisor $D \in \mathrm{N}^1(S)$. $\square$

By Lemma 3.11, if the intersection pairing on $\mathrm{N}^1(S)$ is odd, then K_S is either primitive or equal to $3D$ for some primitive divisor D. In order to conclude, we need to show that $K_S = 3D$ if and only if $n = 1$, which is further equivalent by Noether's formula to $K_S^2 = 9$. Clearly if $K_S = 3D$, then $K_S^2 = 9D^2$ so that K_S^2 must be equal to 9. If now $n = 1$, then $\mathrm{N}^1(S) = \mathbb{Z}H$ for some divisor H, which by Poincaré duality satisfies $H^2 = 1$. The canonical divisor K_S is then equal to aH for some integer a. Since $K_S^2 = 9$, we find that $a = \pm 3$ and we are done. By Theorem 3.1, we deduce that every minimal smooth projective complex surface of general type with $p_g = q = 0$ admits a numerically exceptional collection of maximal length.

• If $\kappa = 0$ or 1, then S is a Dolgachev surface. The Néron–Severi lattice of a Dolgachev surface S has rank 10, so that by Theorem 3.1 (combined with Lemma 3.11) S admits a numerically exceptional collection of maximal length if and only if K_S is primitive.

Let us first consider the case $\kappa = 0$, that is, the case where S is a classical Enriques surface. It is known that the canonical sheaf ω_S is 2-torsion, and hence that $K_S = 0$ in $\mathrm{N}^1(S)$. Therefore, S does not admit a numerically exceptional collection of maximal length. (Recall also that the Néron–Severi lattice of an Enriques surface is $U \oplus E_8(-1)$; it is even of rank 10.)

Consider now a Dolgachev surface $S = X_9(p_1, \ldots, p_n)$ of Kodaira dimension 1 (*i.e.*, which is not $X_9(2,2)$). Its canonical divisor K_S is not torsion and is given by [13, p. 129]

$$K_S = (n-1)F - \sum_{i=1}^n F_i \ \in \mathrm{Pic}(S),$$

where F is the class of a general fiber and F_i is the class of the multiple fiber corresponding to p_i (in particular, $p_i F_i = F \in \mathrm{Pic}(S)$). The canonical divisor K_S may or may not be primitive. First we show by elementary arithmetic that if S is not one of $X_9(2,3)$, $X_9(2,4)$, $X_9(3,3)$, $X_9(2,2,2)$, then K_S is not primitive. Let us assume that S is $X_9(p_1, \ldots, p_n)$ with $2 \le p_1 \le p_2 \le \ldots \le p_n$ and with distinct multiple fibers F_i such that $p_i F_i = F$. Let us write $c := \gcd(p_1, p_2)$, $d := \mathrm{lcm}(p_1, p_2)$, and let u and v be integers such that $up_1 + vp_2 = c$. Consider $G := vF_1 + uF_2 \in \mathrm{Pic}(S)$; then note that

$$dG = \frac{vp_1p_2}{c}F_1 + \frac{up_1p_2}{c}F_2 = \frac{vp_2}{c}F + \frac{up_1}{c}F = F.$$

Note also that, in the Néron–Severi lattice $\mathrm{N}^1(S)$, the fibers F, F_1, F_2 are integral multiples of G (namely, $F = dG$, $F_1 = q_2 G$ and $F_2 = q_1 G$, where $p_1 = cq_1$ and $p_2 = cq_2$) and the fibers $F_3, \ldots, F_n$ are all rational multiples of G.

If $n = 2$, we claim that K_S is primitive only if (p_1, p_2) is one of $(2,3)$, $(2,4)$ or $(3,3)$. In that case, we have

$$(3) \qquad\qquad K_S = F - F_1 - F_2 = (d - q_2 - q_1)G = (cq_1q_2 - q_1 - q_2)G.$$

Note that $cq_1q_2 - q_1 - q_2 = (c-1)q_1q_2 + (q_1 - 1)(q_2 - 1) - 1$. Assume this is equal to 1. If $q_1 = 1$ then $(c-1)q_2 = 2$, hence $c = 2, q_2 = 2$; or $c = 3, q_2 = 1$. The first gives $(2,4)$, the second gives $(3,3)$. If $q_1, q_2 \ge 2$, then the second summand is ≥ 1, hence the first is ≤ 1, hence $c = 1$, hence $(q_1 - 1)(q_2 - 1) = 2$. This gives $(2,3)$.

If $n > 2$, then K_S is primitive only if $n = 3$ and $(p_1, p_2, p_3) = (2,2,2)$. Indeed, we have

$$K_S = (2F - F_1 - F_2) + (F - F_3) + \cdots (F - F_{n-1}) - F_n$$
$$= (2d - q_1 - q_2)G + (F - F_3) + \cdots (F - F_{n-1}) - F_n.$$

On the one hand, we have (note that $q_1 \le q_2$)

$$2d - q_1 - q_2 = 2p_1q_2 - q_1 - q_2 = (2p_1 - 1)q_2 - q_1 \ge 3q_2 - q_1 \ge 2q_2$$

with equality if and only if $p_1 = p_2 = 2$. On the other hand, each divisor $F - F_i$ is a positive rational multiple of G. Together with the inequality $(2d - q_1 - q_2) \geq 2q_2 \geq 2$, it follows that

$$K_S \geq 2G - F_n,$$

and equality holds only if $n = 3$ and $p_1 = p_2 = 2$. If we can write $K_S = \lambda G$ in $\mathrm{N}^1(S)$ for some rational number $\lambda > 1$, then K_S cannot be primitive. We deduce that K_S can only be primitive when $n = 3$, $p_1 = p_2 = 2$ and $F_3 = G$, that is, when $S = X_9(2, 2, 2)$.

Finally we check that K_S is primitive for the Dolgachev surfaces $X_9(2, 3)$, $X_9(2, 4)$, $X_9(3, 3)$ and $X_9(2, 2, 2)$. In the first three cases, by (3), we have $K_S = G$. More generally, when $n = 2$, we claim that the divisor G is primitive. We proceed as in [3, p. 384], by contradiction. If, for some rational number $0 < \lambda < 1$, the class λG is represented by a divisor, then by the Riemann-Roch formula, either λG or $K_S - \lambda G$ is effective. Note then that an effective divisor D such that $\deg(D \cdot F) = 0$ is linearly equivalent to $a_1 F_1 + a_2 F_2$ for some non-negative integers a_1 and a_2. Since clearly λG is not effective, $K_S - \lambda G$ must be effective, that is, we can write $K_S - \lambda G = a_1 F_1 + a_2 F_2$ for some non-negative integers a_1 and a_2. But then we obtain

$$(d - q_1 - q_2 - \lambda)G = a_1 F_1 + a_2 F_2 = (a_1 q_2 + a_2 q_1)G$$

and hence we find that λ is an integer, which gives a contradiction.

In the last case $(S = X_9(2, 2, 2))$, we have $K_S = 2F - F_1 - F_2 - F_3 = F_1 + F_2 - F_3$. Again, if, for some rational number $0 < \lambda < 1$, the class λK_S is represented by a divisor, then by the Riemann–Roch formula, either λK_S or $(1 - \lambda)K_S$ is effective. Assume that μK_S is effective for some $0 < \mu < 1$. Then there exist non-negative integers a_1, a_2, a_3 such that $\mu K_S = a_1 F_1 + a_2 F_2 + a_3 F_3$. Since F_1, F_2 and F_3 are numerically equivalent, we find that $\mu = a_1 + a_2 + a_3$, in particular we find that μ is an integer. Theorem 3.10 is now proved. □

Remark 3.12. Although an Enriques surface does not admit an exceptional collection of maximal length, it would be very interesting to decide whether or not an Enriques surface blown up at a point admits an exceptional collection of maximal length. This would give an example of a triangulated category with an exceptional collection of maximal length that admits an exceptional object whose orthogonal complement does not admit an exceptional collection of maximal length. Indeed, denoting $p : \tilde{S} \to S$ the blow-up of S along a point P and denoting E the exceptional divisor, we have by Orlov's blow-up formula a semi-orthogonal decomposition $D^b(\tilde{S}) \cong \langle O_E(-1), p^* D^b(S) \rangle$. Then the right-orthogonal complement of the exceptional object $O_E(-1)$ in $D^b(\tilde{S})$ does not admit an exceptional collection of maximal length by Theorem 3.1. This is related to the failure of the Jordan–Hölder property for semi-orthogonal decompositions; *cf.* [28].

It would also be interesting to exhibit exceptional collections of maximal length for the Dolgachev surfaces $X_9(2, 3)$, $X_9(2, 4)$, $X_9(3, 3)$ and $X_9(2, 2, 2)$. The orthogonal of such collections would yield new examples of quasi-phantom categories (triangulated categories with torsion K_0) in the case of the Dolgachev surfaces $X_9(2, 4)$, $X_9(3, 3)$ and $X_9(2, 2, 2)$. In the case of $X_9(2, 3)$, it would yield (if one believes in Orlov's conjecture) a new example of phantom category (a non-zero triangulated category with vanishing K_0).

N.B. Cho and Lee [10] have recently constructed exceptional collections on some Dolgachev surfaces of type $X_9(2, 3)$ of maximal length whose orthogonal complements provide examples of phantom categories.

Appendix A. On trigonal unimodular lattices

The main result is Theorem A.4. The equivalence $(i) \Leftrightarrow (ii)$ therein reduces the equivalence $(i) \Leftrightarrow (iii)$ of Theorem 3.1 to a purely linear algebraic statement.

We refer to [20] for the basics of lattice theory. A lattice (Λ, b) is a free $\mathbb{Z}$-module Λ of finite rank equipped with a symmetric bilinear form $b : \Lambda \times \Lambda \to \mathbb{Z}$. The *norm* of a vector $x \in \Lambda$ is $b(x, x) \in \mathbb{Z}$. We denote $\langle a \rangle$ the rank-one lattice $\Lambda = \mathbb{Z}\lambda$ such that $b(\lambda, \lambda) = a$. The *signature* of a lattice (Λ, b) is (n^+, n^-, n^0) if $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ splits as the orthogonal sum $\langle 1 \rangle^{\oplus n^+} \oplus \langle -1 \rangle^{\oplus n^-} \oplus \langle 0 \rangle^{\oplus n^0}$. If $n^0 = 0$, that is, if Λ is non-degenerate, we will omit the term n^0 from the signature of Λ.

A lattice is said to be *even* if the norm of every vector is even; it is said to be *odd* otherwise. A lattice is said to be *unimodular* if the determinant of its bilinear form (expressed in any $\mathbb{Z}$-basis) is equal to ± 1. Let us denote U the hyperbolic lattice, that is the lattice $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $b(e_1, e_2) = 1$ and $b(e_i, e_i) = 0$ for $i = 1, 2$; it is up to isomorphism the only even unimodular lattice of rank 2.

Definition A.1 (Trigonal lattices). We will say that a lattice (Λ, b) of rank n is *trigonal* if there exists a $\mathbb{Z}$-basis $(e_1, \ldots, e_n)$ of Λ such that the matrix of b expressed in that basis has the trigonal form

$$(4) \qquad M := \mathrm{Mat}_{(e_1, \ldots, e_n)}(b) = \begin{pmatrix} a_1 & 1 & & & & \\ 1 & a_2 & 1 & & & \\ & 1 & a_3 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & a_n \end{pmatrix}$$

where the entries outside the 3 diagonals consist solely of zeroes. For simplicity, we will write

$$M = [a_1, a_2, \ldots, a_n]$$

and sometimes $M = \mathrm{trig}(a_1, a_2, \ldots, a_n)$, for clarity. A $\mathbb{Z}$-basis in which b takes a trigonal form will be called a *trigonal basis* for b. Such a trigonal reduction is fairly special for unimodular matrices; see Remark A.11, but also [35] where it is shown that for any unimodular bilinear form over $\mathbb{Z}$ there is a basis in which its matrix takes the form (4) with the $(n-1, n)$ and $(n, n-1)$ entries replaced by some positive integer d.

Definition A.2 (Special characteristic elements). Recall that an element ω in a lattice (Λ, b) is said to be *characteristic* if $b(\omega, \lambda) = b(\lambda, \lambda) \pmod 2$ for all $\lambda \in \Lambda$. We will say that a characteristic element ω in a trigonal lattice (Λ, b) is *special* if there exists a trigonal basis $(e_1, \ldots, e_n)$ of (Λ, b) such that

$$b(\omega, e_i) = -b(e_i, e_i) - 2, \quad \text{for all } 1 \le i \le n.$$

(Note that such a special characteristic element always exists if the trigonal lattice Λ is unimodular.) The motivation for introducing special characteristic elements comes from Proposition 2.1.

First we characterize trigonal unimodular lattices (endowed with a special characteristic element):

Proposition A.3. *Let (Λ, b) be a lattice of rank n and signature (n^+, n^-, n^0). The following statements are equivalent.*

(i) *(Λ, b) is trigonal and unimodular.*
(ii) *There exist $\lambda_1, \ldots, \lambda_{n+1}$ in Λ such that $\big(b(\lambda_i, \lambda_j)\big)_{1 \le i,j \le n+1}$ is a trigonal matrix.*
(iii) *There exist $\lambda_0, \lambda_1, \ldots, \lambda_{n+1}$ in Λ such that*

$$\big(b(\lambda_i, \lambda_j)\big)_{0 \le i,j \le n+1} = \begin{pmatrix} a_0 & 1 & & & (-1)^{n^+ - 1} \\ 1 & a_1 & 1 & & \\ & 1 & a_2 & \ddots & \\ & & \ddots & \ddots & 1 \\ (-1)^{n^+ - 1} & & & 1 & a_{n+1} \end{pmatrix}.$$

Moreover, assuming (Λ, b) is trigonal and unimodular, a characteristic element $\omega \in \Lambda$ is special if and only if there exist $\lambda_0, \lambda_1, \ldots, \lambda_{n+1}$ in Λ as in (iii) with the additional property that $\omega = -\sum_{i=0}^{n+1} \lambda_i$.

Proof. $(i) \Rightarrow (iii)$: Let $(e_1, \ldots, e_n)$ be a trigonal basis for (Λ, b). Since (Λ, b) is assumed to be unimodular, b identifies naturally Λ with its dual. We define e_0 (resp. e_{n+1}) to be the dual of e_1 (resp. e_n), that is, e_0 (resp. e_{n+1}) is the element of Λ such that $b(e_0, e_i) = 1$ if $i = 1$ and 0 otherwise (resp. such that $b(e_0, e_i) = 1$ if $i = n$ and 0 otherwise). We claim that the matrix $\big(b(e_i, e_j)\big)_{0 \leq i,j \leq n+1}$ is as in (iii). It is enough to check that $b(e_0, e_{n+1}) = (-1)^{n^+ - 1}$. We have $b(e_0, e_{n+1}) = b^{-1}(e_1, e_n)$, where b^{-1} is the symmetric bilinear form on Λ whose matrix expressed in the basis $(e_1, \ldots, e_n)$ is the inverse of $M := \big(b(e_i, e_j)\big)_{1 \leq i,j \leq n}$. Thus, denoting $m_{1,n}$ the $(1, n)^{\text{th}}$ minor of M (that is, the determinant of the submatrix formed by deleting the 1st row and n^{th} column of M), we have

$$b(e_0, e_{n+1}) = (-1)^{n+1}(\det M)^{-1} m_{1,n} = (-1)^{n+1}(-1)^{n^-} = (-1)^{n^+ - 1},$$

where in the second equality we have used that $\det M = (-1)^{n^-}$ and $m_{1,n} = 1$.

$(iii) \Rightarrow (ii)$: This is obvious.

$(ii) \Rightarrow (i)$: We are going to show that the determinant of $\big(b(\lambda_i, \lambda_j)\big)_{1 \leq i,j \leq n} = [a_1, \ldots, a_n]$ is equal to $(-1)^{n^-}$. Since Λ has rank n, this will prove that (Λ, b) is unimodular and that $(\lambda_1, \ldots, \lambda_n)$ provides a trigonal basis of Λ. Let us consider, for $m \leq n+1$, the $(m \times m)$-trigonal matrix $[a_1, \ldots, a_m]$, and let us denote

$$d_m := \det\big(\text{trig}(a_1, \ldots, a_m)\big).$$

It is easy to see that d_m satisfies the Fibonacci type recurrence relation

$$d_m = a_m d_{m-1} - d_{m-2}, \quad \text{for all } m > 1,$$

with $d_0 = 1$. Note that since $\text{rk}\,\Lambda = n$, we have $d_{n+1} = 0$. From this formula, we derive two things: first that $d_n \neq 0$ (otherwise d_m would vanish for all $m \leq n+1$, but $d_0 = 1$); second that $\gcd(d_m, d_{m-1}) = \gcd(d_{m-1}, d_{m-2})$ for all $2 \leq m \leq n+1$. Because $d_0 = 1$ and $d_{n+1} = 0$, we see that $d_n = \pm 1$ (and in fact, since the signature is then $(n^+, n^-, 0)$, $d_n = (-1)^{n^-}$).

Let us now assume that there are $\lambda_0, \lambda_1, \ldots, \lambda_{n+1}$ as in (iii), with $\omega = -\sum_{i=0}^{n+1} \lambda_i$. The proof of $(iii) \Rightarrow (i)$ shows that in fact $(\lambda_1, \ldots, \lambda_n)$ is a trigonal basis of Λ. Furthermore, $b(\omega, \lambda_i) = -b(\lambda_i, \lambda_i) - 2$ for all $1 \leq i \leq n$. Therefore ω is a special characteristic element in Λ.

Conversely, pick a trigonal basis $(e_1, \ldots, e_n)$ of the unimodular lattice (Λ, b) such that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for all $1 \leq i \leq n$. We note that $\omega + e_1 + \cdots + e_n$ identifies with the dual of $-(e_1 + e_n)$ with respect to the unimodular symmetric bilinear form b. Therefore, for e_0 and e_n as defined in the proof of $(i) \Rightarrow (iii)$, $\omega = -(e_0 + \cdots + e_n)$. This concludes the proof of the proposition. $\square$

We now state our main result; it gives a characterization of pairs consisting of a trigonal unimodular lattice of signature $(1, n-1)$ endowed with a special characteristic element.

Theorem A.4. *Let (Λ, b) be a lattice of signature $(1, n-1)$ and let ω be a vector in Λ. The following statements are equivalent:*

(i) (Λ, b) is unimodular and trigonal, and ω is a special characteristic element;

(ii) The vector ω is characteristic of norm $b(\omega, \omega) = 10 - n$ and the pair (Λ, ω) satisfies one of the following properties:
  - *$\Lambda \cong \langle 1 \rangle$ and $\omega = 3\lambda$ for some primitive vector λ;*
  - *$\Lambda \cong U$ and $\omega = 2\lambda$ for some primitive vector λ;*
  - *$\Lambda \cong \langle 1 \rangle \oplus \langle -1 \rangle^{\oplus n}$ with $n > 0$ and ω is primitive.*

(iii) There exists a $\mathbb{Z}$-basis $(e_1, \ldots, e_n)$ of Λ such that:
  - *$\text{Mat}_{(e_i)}(b) = \text{diag}(1, -1, \ldots, -1)$ if b is odd;*
  - *$\text{Mat}_{(e_i)}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if b is even;*
  and such that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for all $1 \leq i \leq n$.

Before we prove the theorem, we state and prove a few useful lemmas that give constraints on the coefficients of trigonal unimodular matrices.

Lemma A.5. *Let $n \geq 2$ and let $a_1, a_2, \ldots, a_n$ be real numbers. Consider a trigonal matrix $M = [a_1, a_2, \ldots, a_n]$ as in (4). Assume that $\det M = \pm 1$. Then there exists an index i such that $|a_i| < 2$.*

Proof. Assume for contradiction that for all i we have $|a_i| \geq 2$. Let V be an n-dimensional $\mathbb{R}$-vector space with basis $(e_1, \ldots, e_n)$; we view M as the matrix of a symmetric bilinear form b on V expressed in the basis $(e_1, \ldots, e_n)$. Since $|a_i| \geq 2$ for all i, it is possible to define inductively $b_1 := a_1$ and $b_i := a_i - \frac{1}{b_{i-1}}$ for $i \geq 2$; in fact one sees by induction that $|b_i| > 1$ for all i. Let us consider the following volume-preserving change of basis for V: $e'_1 := e_1$, and $e'_i := e_i - \frac{1}{b_{i-1}} e'_{i-1}$. Expressed in the basis $(e'_1, \ldots, e'_n)$, the matrix M' of the bilinear form b has determinant $\det M' = \det M = \pm 1$. Moreover, M' is diagonal with diagonal terms given by the real numbers b_i. Thus $|\det M'| = \prod_i |b_i| > 1$, which is a contradiction. $\qquad \square$

Observations A.6. Let (Λ, b) be a trigonal unimodular lattice of rank n equipped with a special characteristic element ω. Consider a basis $(e_1, \ldots, e_n)$ of Λ such that $\mathrm{Mat}_{(e_i)}(b) = [a_1, \ldots, a_n]$ for some integers a_i, and such that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for all $1 \leq i \leq n$. We make the following observations:

(i) Suppose there exists $j < n$ such that $a_j = 0$ and, given an integer x, consider the new basis
$$(e'_i)_{1 \leq i \leq n} := (e_1, \ldots, e_j, e_{j+1} + x e_j, e_{j+2}, \ldots, e_n).$$
Then that new basis is trigonal for b; in fact the matrix of b in that basis is
$$\mathrm{Mat}_{(e'_i)}(b) = [a_1, \ldots, a_{j-1}, 0, a_{j+1} + 2x, a_{j+2}, \ldots, a_n].$$
Moreover, one readily checks that ω satisfies $b(\omega, e'_i) = -b(e'_i, e'_i) - 2$ for all $1 \leq i \leq n$.

(ii) Suppose there exists $j < n$ such that $a_j = -1$ and consider the new basis
$$(e'_i)_{1 \leq i \leq n} := (e_1, \ldots, e_j, e_j + e_{j+1}, e_{j+2}, \ldots, e_n).$$
Then in that new basis b splits as the direct orthogonal sum of two trigonal lattices; precisely the matrix of b in that basis is
$$\mathrm{Mat}_{(e'_i)}(b) = [a_1, \ldots, a_{j-1}, -1] \oplus [a_{j+1} + 1, a_{j+2}, \ldots, a_n].$$
Moreover, one readily checks that ω satisfies $b(\omega, e'_i) = -b(e'_i, e'_i) - 2$ for all $1 \leq i \leq n$.

Lemma A.7. *Suppose (Λ, b) is a trigonal unimodular lattice and let $(e_i)_{1 \leq i \leq n}$ be a trigonal basis for b.*

(i) *If b has signature $(1, n-1)$, then there exists an index i such that $a_i := b(e_i, e_i) = -1$ or 0;*
(ii) *If b is positive definite, then there exists an index i such that $a_i := b(e_i, e_i) = 1$;*
(iii) *If b is negative definite, then there exists an index i such that $a_i := b(e_i, e_i) = -1$.*

Proof. By Lemma A.5, there exists an index i such that $a_i = -1, 0$ or 1. The assertions *(ii)* *(iii)* are then clear. Suppose now that b has signature $(1, n-1)$. Assume for contradiction that there is no index i for which $a_i = -1$ or 0. Then by Lemma A.5 there is an index i for which $a_i = 1$. The $(n-1)$-tuple $(e_1, \ldots, e_{i-2}, e_{i-1} - e_i, e_i - e_{i+1}, -e_{i+2}, \ldots, -e_n)$ gives a $\mathbb{Z}$-basis of the orthogonal complement $\langle e_i \rangle^\perp$ in Λ of the sub-lattice spanned by e_i. The matrix of the bilinear form $b|_{\langle e_i \rangle^\perp}$ expressed in that basis is
$$[a_1, \ldots, a_{i-2}, a_{i-1} - 1, a_{i+1} - 1, a_{i+2}, \ldots, a_n].$$
Moreover, $b|_{\langle e_i \rangle^\perp}$ is unimodular and negative definite. Therefore, by *(iii)*, we have $-1 \in \{a_1, \ldots, a_{i-2}, a_{i-1} - 1, a_{i+1} - 1, a_{i+2}, \ldots, a_n\}$, and this shows that there is an index j such that $a_j = -1$ or 0, which contradicts our assumption. $\qquad \square$

Lemma A.8. *Let (Λ, b) be an even trigonal unimodular lattice of rank n. Then n is even and $\Lambda \cong U^{\oplus m}$, where $2m = n$. Moreover, there exists a trigonal $\mathbb{Z}$-basis $(e_i)_{1 \leq i \leq 2m}$ such that $\mathrm{Mat}_{(e_i)}(b) = [0, 0, \ldots, 0]$.*

Proof. Let $(e_i)_{1 \le i \le n}$ be a trigonal basis for b and denote $a_i := b(e_i, e_i)$. By Lemma A.5, for the bilinear form b to be unimodular, one of the a_j has to be equal to $-1, 0$ or 1. The pairing is assumed to be even, so that one of the a_j is equal to 0. The integer a_{j+1} is even. Consider, as in Observation A.6(i), the change of $\mathbb{Z}$-basis $e_i \mapsto e_i$ for $i \ne j+1$ and $e_{j+1} \mapsto e_{j+1} - \frac{a_{j+1}}{2} e_j$. In that new basis, the matrix of the form b is $[a_1, \dots, a_{j-1}, 0, 0, a_{j+2}, \dots, a_{2n}]$. Performing several similar changes of bases shows that there is a basis $(e_1', \dots, e_{2n}')$ of Λ such that the matrix of the form b expressed in that basis is $[0, 0, \dots, 0]$. A straightforward calculation shows that $\det b = 0$ if n is odd, and that $\det b = (-1)^m$ if $n = 2m$ for some integer m. Thus n is even. Consider then the basis $(e_1', e_2', e_3' - e_1', e_4', e_5' - e_3', e_6', \dots, e_{2m-1}' - e_{2m-3}', e_{2m}')$ of Λ. It becomes apparent that $\Lambda \cong U^{\oplus m}$. $\qquad\square$

The following proposition and its corollary prove $(i) \Rightarrow (ii)$ of Theorem A.4, and is the heart of the proof of $(ii) \Rightarrow (iii)$ of our Main Theorem 3.1 (and hence of the arithmetic application thereof given by Theorem 3.6).

Proposition A.9. *Let (Λ, b) be a trigonal unimodular lattice of signature $(1, n-1)$ and let ω be a special characteristic element in Λ. Then there exists a $\mathbb{Z}$-basis $(e_1, \dots, e_n)$ of Λ such that*

$$b(\omega, e_i) = -b(e_i, e_i) - 2 \text{ for all } 1 \le i \le n \quad \text{and} \quad \Lambda = \begin{cases} \text{trig}(0,0) & \text{if } \Lambda \text{ is even;} \\ \langle 1 \rangle \oplus \langle -1 \rangle^{\oplus n-1} & \text{if } \Lambda \text{ is odd.} \end{cases}$$

Proof. The case $n = 1$ is trivial. Assume that $n = 2$; in that case any trigonal basis (e_1, e_2) for b is such that $\text{Mat}_{(e_i)}(b) = [a_1, a_2]$ with $a_1 a_2 = 0$ (since $\det b = -1$). Thus, by Observation A.6(i), there is a basis (e_1, e_2) for b is such that $\text{Mat}_{(e_i)}(b) = [0, a]$ with $a = 0$ or -1, and such that ω is special with respect to that basis. The former case is the case where Λ is even, while in the latter case Observation A.6(ii) gives us a new basis, namely $(e_1' + e_2', e_2')$, in which the matrix of b is $\text{diag}(1, -1)$ and with respect to which ω is special.

For the sake of the induction argument to come, let us consider a negative definite trigonal unimodular lattice (Λ, b) of rank 2. Let (e_1, e_2) be a trigonal basis; the unimodularity of b shows that up to reordering e_1 and e_2, we have $\text{Mat}_{(e_i)}(b) = [-2, -1]$. Assume that ω is such that $b(\omega, e_i) = -b(e_i, e_i) - 2$. Then Observation A.6(ii) says that in the basis $(e_1', e_2') := (e_1 + e_2, e_2)$ we have $\text{Mat}_{(e_i')}(b) = \text{diag}(-1, -1)$ and $b(w, e_i') = -b(e_i, e_i) - 2 = -1$.

Assume now that $n \ge 3$; we are going to proceed by induction. We suppose that for all $m < n$, if (Λ, b) is a trigonal unimodular lattice of signature $(1, m-1)$ or $(0, m)$ endowed with a special characteristic element ω, then there exists a basis $(e_1, \dots, e_m)$ of Λ such that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for all $1 \le i \le m$, and such that $\text{Mat}_{(e_i)}(b)$ is either equal to $[0, 0]$ or to $\text{diag}(\pm 1, -1, \dots, -1)$. We now fix a trigonal unimodular lattice (Λ, b) of signature $(1, n-1)$ or $(0, n)$ endowed with a special characteristic element ω. By Lemma A.8, the trigonal lattice (Λ, b), which has signature $(1, n-1)$ or $(0, n)$, has to be odd. Let $(e_1, \dots, e_n)$ be a trigonal basis of Λ with respect to which ω is special. By Lemma A.7(i), there exists $1 \le j \le n$, such that $b(e_j, e_j) = -1$ or 0. Suppose that $b(e_j, e_j) = 0$. Then, by repeated use of Observation A.6(i), we find a trigonal basis $(e_1', \dots, e_n')$ of Λ with respect to which ω is special, and such that $b(e_k', e_k') = -1$ for some k. Therefore, we may assume that $b(e_j, e_j) = -1$ in the first place. By Observation A.6(ii), we find a basis $(f_1, \dots, f_n)$ of Λ such that

$$b(\omega, f_i) = -b(f_i, f_i) - 2 \quad \text{and}$$
$$\Lambda = [b(f_1, f_1), \dots, b(f_{j-1}, f_{j-1})] \oplus \langle -1 \rangle \oplus [b(f_{j+1}, f_{j+1}), \dots, b(f_n, f_n)].$$

By the induction hypothesis, we obtain a basis $(f_1', \dots, f_n')$ for which $b(\omega, f_i') = -b(f_i', f_i') - 2$ for all $1 \le i \le m$, and for which Λ is either $\langle -1 \rangle^{\oplus n}$, $\langle 1 \rangle \oplus \langle -1 \rangle^{\oplus n-1}$ or $\langle -1 \rangle^{\oplus n-2} \oplus U$. In order to finish off the induction, we note that if there is a basis (e_1, e_2, e_3) of a rank-3 unimodular lattice Λ such that $\text{Mat}_{(e_i)}(b) = \langle -1 \rangle \oplus [0, 0]$ with $b(\omega, e_i) = -b(e_i, e_i) - 2$ for $1 \le i \le 3$, then in the basis

$$(e_i')_{1 \le i \le 3} := (e_2 + e_3 - e_1, e_2 - e_1, e_3 - e_1)$$

we have $\text{Mat}_{(e_i')}(b) = \langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle$ with $b(\omega, e_i') = -b(e_i', e_i') - 2$ for $1 \le i \le 3$. $\qquad\square$

Corollary A.10. *Let* (Λ, b) *be a trigonal unimodular lattice of signature* $(1, n-1)$. *If* ω *is a special characteristic element in* Λ, *then*

$$(5) \qquad\qquad\qquad b(\omega, \omega) = 10 - n.$$

Proof. This follows immediately from Proposition A.9:

- If $\Lambda = [0, 0]$, then $\omega = -2e_1 - 2e_2$ and hence $b(\omega, \omega) = 8$.
- If $\Lambda = \langle 1 \rangle \oplus \langle -1 \rangle^{\oplus n-1}$, then $\omega = -3e_1 - e_2 - \cdots - e_n$ and hence $b(\omega, \omega) = 10 - n$.

$\square$

Remark A.11. In fact, it is possible to generalize Proposition A.9 to unimodular lattices of any signature: it can be shown that if Λ is unimodular and trigonal, then there exists a basis $(e_1, \ldots, e_n)$ of Λ such that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for all $1 \le i \le n$, and such that with respect to that basis Λ splits as a direct orthogonal sum of lattices isomorphic to $[0, 0]$, $[1], [-1]$, $[1, 2]$ and $[1, 3, 1]$. (Note that in particular a unimodular lattice is trigonal if and only if it is isomorphic to a direct sum of copies of U, $\langle 1 \rangle$, and $\langle -1 \rangle$.) As a consequence, if (Λ, b) is a trigonal unimodular lattice of signature (n^+, n^-) and if ω is a special characteristic element in Λ, then

$$(6) \qquad\qquad\qquad b(\omega, \omega) = 8 \left\lfloor \frac{n^+ + 1}{2} \right\rfloor + n^+ - n^-.$$

The formula (6) should be compared to van der Blij's lemma [20, Lemma II.(5.2)]. Let (Λ, b) be a unimodular lattice; it is easy to see that a characteristic element ω always exists since the function $\Lambda \to \mathbb{Z}/2\mathbb{Z}, \lambda \mapsto b(\lambda, \lambda)[\mathrm{mod}\ 2]$ is $\mathbb{Z}/2\mathbb{Z}$-linear. It is also easy to check that the integer $b(\omega, \omega)$ is an invariant modulo 8. Van der Blij's lemma states that in fact $b(\omega, \omega) = n^+ - n^-$ [mod 8]. Thus, for a trigonal unimodular lattice, a special characteristic element ω (that is, the element $\omega \in \Lambda$ such that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for a basis $(e_1, \ldots, e_n)$ of Λ in which the matrix of b is trigonal) can be thought of as an integral characteristic element in the lattice Λ, and (6) gives an integral version of van der Blij's lemma for trigonal unimodular lattices.

The following Witt-type proposition proves $(ii) \Rightarrow (iii)$ of Theorem A.4 for odd unimodular lattices, and is the heart of the proof of $(iii) \Rightarrow (i)$ of our Main Theorem 3.1 (and hence of Theorem 3.10).

Proposition A.12. *Let* (Λ, b) *be an odd unimodular lattice of signature* $(1, n-1)$ *and let* ω *be a characteristic element in* Λ *such that* $b(\omega, \omega) = 10 - n$. *Assume further that* ω *is primitive if* $n \ge 10$. *Then there exists a basis* $(e_1, \ldots, e_n)$ *of* Λ *such that*

$$\mathrm{Mat}_{(e_i)}(b) = \mathrm{diag}(1, -1, \ldots, -1) \quad and \quad \omega = 3e_1 + e_2 + \ldots + e_n.$$

Proof. In the case $n > 10$, that is, in the case $b(\omega, \omega) < 0$, the proposition was already proved in greater generality by Nikulin [36]. Indeed, assume $n > 10$ and let ω be a primitive characteristic element of norm $b(\omega, \omega) = 10 - n$ in the lattice Λ of signature $(1, n-1)$. In particular, $b(\omega, \omega) < 0$ and the restriction of b to the orthogonal complement $\omega^\perp$ of ω is even and indefinite. Therefore, we may invoke [36, Prop. 3.5.1] (which applies since our lattice has rank ≥ 4), which says that there is only one orbit under $O(q)$ of primitive characteristic elements of given negative norm.

Consider now an odd unimodular lattice (Λ, b) of signature $(1, n-1)$. Pick a basis $(e_1, \ldots, e_n)$ of Λ such that $\mathrm{Mat}_{(e_i)}(b) = \mathrm{diag}(1, -1, \ldots, -1)$.

Claim 1. Let ω be an element of Λ such that $b(\omega, \omega) \ge 0$. Then there is an automorphism φ of Λ preserving q (*i.e.*, $\varphi \in O(q)$) such that $\varphi(\omega) = x_1 e_1 + \cdots + x_n e_n$ with

$$(7) \qquad\qquad 0 \le x_n \le x_{n-1} \le \cdots \le x_1 \quad and \quad x_4 + x_3 + x_2 \le x_1.$$

(When $n = 1$ or 2, the latter inequality should be ignored; and when $n = 3$, it should be understood to read $x_3 + x_2 \le x_1$).

In other words, the orbit of any element of non-negative norm under the action of $O(q)$ contains an element whose coordinates satisfy (7).

Proof of Claim 1. Given a vector $v \in \Lambda$ of norm $b(v, v)$ that divides 2, we define the reflection $R_v \in O(q)$ across the hyperplane orthogonal to v by the formula

$$R_v(\lambda) := \lambda - 2\frac{b(\lambda, v)}{b(v, v)}v.$$

Let $\xi := x_1 e_1 + \cdots + x_n e_n$ be an element of Λ. Up to applying the reflections R_{e_i}, we see that all vectors $\pm x_1 e_1 \pm \cdots \pm x_n e_n$ belong to the orbit of ξ under the action of $O(q)$. Consider the action of the symmetric groups $\mathfrak{S}_{n-1}$ on the set $\{2, \ldots, n\}$; this induces an action on the lattice Λ, given by $f_\sigma(\xi) := x_1 e_1 + x_{\sigma^{-1}(2)} e_2 + \cdots + x_{\sigma^{-1}(n)} e_n$ for all $\sigma \in \mathfrak{S}_{n-1}$. Clearly $\sigma \mapsto f_\sigma$ defines a homomorphism $\mathfrak{S}_{n-1} \to O(q)$. Therefore, $f_\sigma(\xi)$ belongs to the orbit of ξ for all $\sigma \in \mathfrak{S}_{n-1}$.

Consider now a non-zero element ξ of non-negative norm. By the above, the vector ξ has in its orbit a vector $x_1 e_1 + \cdots + x_n e_n$ with $0 \le x_1$ and $0 \le x_n \le x_{n-1} \le \cdots \le x_2$. Since $b(\xi, \xi) \ge 0$, we actually have $0 \le x_n \le \cdots \le x_1$. Choose now such a vector in the orbit of ξ with minimal non-negative x_1. Assume that $n \ge 4$ (we indicate how to treat the case $n \le 3$ at the end of the proof). We claim that $x_4 + x_3 + x_2 \le x_1$. If that is not case, consider the reflection R_v with $v := e_1 + e_2 + e_3 + e_4$. Then we have

$$R_v(\xi) = (2x_1 - x_2 - x_3 - x_4)e_1 + (x_1 - x_3 - x_4)e_2 + (x_1 - x_2 - x_4)e_3 + (x_1 - x_2 - x_3)e_4$$
$$+ e_5 + \cdots + e_n.$$

Given that $0 \le x_4 \le x_3 \le x_2 \le x_1$ and $x_4 + x_3 + x_2 > x_1$ (note also that $x_4 < x_1$ because $b(\xi, \xi) \ge 0$), we have

$$-x_1 < 2x_1 - x_2 - x_3 - x_4 < x_1.$$

Therefore, making all the coordinates of $R_v(\xi)$ non-negative and reordering them in decreasing order, we obtain a vector $\xi' := x_1' e_1 + \cdots + x_n' e_n$ in the orbit of ξ with $0 \le x_n' \le \cdots \le x_1' < x_1$, thus yielding a contradiction.
(The cases $n = 1$ and $n = 2$ are obvious, while in the case $n = 3$ one proceeds similarly by considering the reflection R_v with $v := e_1 + e_2 + e_3$.)   □

Claim 2. Assume $n < 10$. Then there is only one orbit of characteristic elements of Λ of norm $10 - n$; it is the orbit of the element $3e_1 + e_2 + \cdots + e_n$.

Proof of Claim 2. For this purpose, given Claim 1, we show that a characteristic vector $\xi := x_1 e_1 + \cdots + x_n e_n$ of norm $b(\xi, \xi) = 10 - n$ and whose coordinates satisfy (7) is necessarily the vector $3e_1 + e_2 + \cdots + e_n$. Squaring the inequality $x_4 + x_3 + x_2 \le x_1$ yields

$$(8) \qquad 2(x_2 x_3 + x_2 x_4 + x_3 x_4) \le x_1^2 - x_2^2 - x_3^2 - x_4^2 = 10 - n + x_5^2 + \cdots + x_n^2.$$

Using the comparison of x_i with x_4, we find $6x_4^2 \le 10 - n + (n - 4)x_4^2$. The assumption $n < 10$ immediately gives $x_4 \le 1$. Since we are assuming that ξ is characteristic, this forces $x_n = \cdots = x_4 = 1$. Therefore we obtain $x_1^2 - x_2^2 - x_3^2 = 7$. Squaring the inequality $x_2 + x_3 < x_1$ then gives $2x_2 x_3 < 7$. Thus (x_2, x_3) is either $(1, 1)$ or $(3, 1)$ (recall that ξ is characteristic so that its coordinates are odd integers). But the latter is not possible since otherwise 17 would be a square. Hence $(x_1, x_2, x_3) = (3, 1, 1)$.   □

Claim 3. Assume $n = 10$. Then there is only one orbit of isotropic primitive characteristic elements of Λ; it is the orbit of the element $3e_1 + e_2 + \cdots + e_n$.

Proof of Claim 3. As in the proof of Claim 2, we obtain the inequality (8). Singling out the term x_{10}^2, we obtain

$$6x_4^2 \le 5x_4^2 + x_{10}^2$$

and hence $x_4^2 \le x_{10}^2$. This proves that $x_4 = x_5 = \cdots = x_{10}$. We are thus reduced to solving the Diophantine equation

$$(9) \qquad x_1^2 = x_2^2 + x_3^2 + 7x_4^2$$

with the constraint that $x_1, \ldots, x_4$ are odd integers, with no common prime factors, satisfying $0 < x_4 \leq x_3 \leq x_2 \leq x_1$ and $x_4 + x_3 + x_2 \leq x_1$. On the one hand, squaring the latter inequality gives

$$2(x_2x_3 + x_2x_4 + x_3x_4) \leq x_1^2 - x_2^2 - x_3^2 - x_4^2 = 6x_4^2.$$

On the other hand, the former inequality gives

$$6x_4^2 \leq 2(x_2x_3 + x_2x_4 + x_3x_4)$$

with equality if and only if $x_2 = x_3 = x_4$. Therefore, we immediately get $x_2 = x_3 = x_4$ and then that $x_1 = 3x_2$. The only primitive solution to (9) with the additional constraint that $0 < x_4 \leq x_3 \leq x_2 \leq x_1$ and $x_4 + x_3 + x_2 \leq x_1$ is then $(3, 1, 1, 1)$.   □

The proof of Proposition A.12 is now complete.   □

Finally we provide a proof of Theorem A.4.

Proof of Theorem A.4. $(i) \Rightarrow (ii)$: Given Proposition A.9 and Corollary A.10, it only remains to see that $\omega = -3e_1$ if $\Lambda = \langle 1 \rangle$, $\omega = -2e_1 - 2e_2$ if $\Lambda = [0, 0]$, and that $\omega = -3e_1 - e_2 - \cdots - e_n$ if $\Lambda = \langle 1 \rangle \oplus \langle -1 \rangle^{\oplus n-1}$.

$(ii) \Rightarrow (iii)$: Given Proposition A.12, it only remains to treat the case where Λ is isomorphic to the hyperbolic plane and ω is twice a primitive vector. Let (e_1, e_2) be a basis of Λ such that the matrix of b is $[0, 0]$. Let us then write $\omega = ae_1 + be_2$. By assumption we have $b(\omega, \omega) = 2ab = 8$. Since ω is twice a primitive vector, we find up to considering the new basis $(\pm e_1, \pm e_2)$ that $\omega = -2e_1 - 2e_2$. It is then apparent that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for $1 \leq i \leq 2$.

$(iii) \Rightarrow (i)$: The even case is obvious, and so is the case where $\Lambda = \langle 1 \rangle$. Let us thus consider an odd unimodular lattice (Λ, b) of signature $(1, n-1)$ with $n > 1$, and let $(e_1, \ldots, e_n)$ be a basis of Λ such that $\mathrm{Mat}_{(e_i)}(b) = \mathrm{diag}(1, -1, \ldots, -1)$. Let ω be the element in Λ such that $b(\omega, e_i) = -b(e_i, e_i) - 2$ for all $1 \leq i \leq n$. Then one readily checks that the vectors

$$\begin{cases} e_i' & = & e_1 - e_{i+1}, & 1 \leq i < n \\ e_n' & = & e_n \end{cases}$$

provide a basis $(e_1', \ldots, e_n')$ of Λ such that $\mathrm{Mat}_{(e_i')}(b) = [-1, 0, \ldots, 0]$ and such that $b(\omega, e_i') = -b(e_i', e_i') - 2$ for all $1 \leq i \leq n$.   □

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