REMARKS ON MOTIVES OF ABELIAN TYPE

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Abstract. A motive over a field k is of abelian type if it belongs to the thick and rigid subcategory of Chow motives spanned by the motives of abelian varieties over k. This paper contains three sections of independent interest. First, we show that a motive which becomes of abelian type after a base field extension of algebraically closed fields is of abelian type. Given a field extension K/k and a motive M over k, we also show that M is finite-dimensional if and only if M_K is finite-dimensional. As a corollary, we obtain Chow–Künneth decompositions for varieties that become isomorphic to an abelian variety after some field extension. Second, let Ω be a universal domain containing k. We show that Murre's conjectures for motives of abelian type over k reduce to Murre's conjecture (D) for products of curves over Ω. In particular, we show that Murre's conjecture (D) for products of curves over Ω implies Beauville's vanishing conjecture on abelian varieties over k. Finally, we give criteria on Chow groups for a motive to be of abelian type. For instance, we show that M is of abelian type if and only if the total Chow group of algebraically trivial cycles $\mathrm{CH}_*(M_\Omega)_{\mathrm{alg}}$ is spanned, via the action of correspondences, by the Chow groups of products of curves. We also show that a morphism of motives $f : N \to M$, with N finite-dimensional, which induces a surjection $f_* : \mathrm{CH}_*(N_\Omega)_{\mathrm{alg}} \to \mathrm{CH}_*(M_\Omega)_{\mathrm{alg}}$ also induces a surjection $f_* : \mathrm{CH}_*(N_\Omega)_{\mathrm{hom}} \to \mathrm{CH}_*(M_\Omega)_{\mathrm{hom}}$ on homologically trivial cycles.

Introduction

Let k be a field. A *motive over* k is a motive for rational equivalence defined over k with rational coefficients. The motive of a smooth projective variety X over k is denoted by $\mathfrak{h}(X)$. We refer to [18] for definitions and basic properties. Our notations will only differ from loc. cit. by the use of a covariant set-up rather than a contravariant one. For instance, with our conventions, $\mathfrak{h}(\mathbb{P}^1_k) = \mathbb{1} \oplus \mathbb{1}(1)$. A motive M is said to be *effective* if it is isomorphic to the direct summand of the motive of a smooth projective variety, equivalently if it is isomorphic to a motive of the form (X, p, n) for some smooth projective variety X, some idempotent $p \in \mathrm{End}(\mathfrak{h}(X)) := \mathrm{CH}_{\dim X}(X \times X)$ and some integer $n \geq 0$. A motive M is said to be of *abelian type* if it is isomorphic to a motive that belongs to the thick and rigid subcategory of motives over k spanned by the motives of abelian varieties. Equivalently, M is of abelian type if one of its twist $M(n) := M \otimes \mathbb{1}(n)$ is isomorphic to the direct summand of the motive of a product of curves. Finally, we refer to Kimura [15] for the notion of *finite-dimensionality* of motives. Motives of curves are finite-dimensional and finite-dimensionality is stable under tensor product, direct sum and direct summand. As such, motives of abelian type are finite-dimensional. It is conjectured that all motives are finite-dimensional. Given a finite-dimensional motive M, a crucial result of Kimura [15, Proposition 7.5] states that $\mathrm{Ker}\,\big(\mathrm{End}(M) \to \mathrm{End}(\overline{M})\big)$, where $\overline{M}$ denotes the reduction modulo numerical equivalence of M, is a nilpotent ideal of $\mathrm{End}(M)$.

This paper contains three independent sections.

0.1. Let $\overline{k}$ be an algebraic closure of k and let K/k be an extension of k which is algebraically closed. Our first result is the following rigidity property for motives of abelian type.

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Theorem 1. *Let M be a motive over k. Then $M_{\overline{k}}$ is of abelian type if and only if M_K is of abelian type.*

The proof, which is given in Subsection 1.1, proceeds through a standard specialization argument. More interesting is the following descent theorem the proof of which is given in Subsection 1.2.

Theorem 2. *Let M be a motive over k. Then M is finite-dimensional if and only if M_K is finite-dimensional.*

The main point in the proof of Theorem 2 consists, assuming the existence of a decomposition of M_K into a direct sum of an odd-dimensional part and an even-dimensional part, in exhibiting a similar decomposition of M. This is *a priori* not obvious as the size of the Chow groups of a variety might strictly increase after base-change to a field extension. Abelian varieties (and, more generally, motives of abelian type) are known [8] to have a Chow–Künneth decomposition. As a consequence of Theorem 2, we obtain in Corollary 1.9 the existence of a Chow–Künneth decomposition for varieties over k that become isomorphic to an abelian variety after some field extension, thereby generalizing the case of abelian varieties which was taken care of by Deninger–Murre [8].

0.2. Let now Ω be a universal domain, that is, an algebraically closed field of infinite transcendence degree over its prime subfield, containing k. Our second main result is concerned with Murre's conjectures (recalled in Conjecture 2.1) in the case of motives of abelian type. It is known that any abelian variety A over k is endowed with a Chow–Künneth decomposition which diagonalizes the induced action of the multiplication-by-m maps on the Chow groups of A; see Remark 2.10. Proposition 2.7 shows that Murre's conjecture (B) does not depend on the choice of a Chow–Künneth decomposition for A, so that Beauville's vanishing conjecture [4] on abelian varieties is equivalent to Murre's conjecture (B). Proposition 2.8 shows that Murre's conjecture (B) for motives of abelian type implies Murre's conjecture (C) for motives of abelian type. Finally, that Murre's conjecture (D) for motives of abelian type implies Murre's conjecture (B) for motives of abelian type relies on the main theorem of K. Xu and Z. Xu [23] which we reproduce as Theorem 2.6. A combination of Theorems 2.9 and Remark 2.10 is the following.

Theorem 3. *Assume Murre's conjecture (D) for product of curves defined over Ω. Then Murre's conjectures (A), (B), (C) and (D) hold for all motives of abelian type over k. In particular, Beauville's vanishing conjecture* [4] *on abelian varieties over k holds.*

In particular, as is explained in Section 2, Beauville's vanishing conjectures on abelian varieties over k reduce to Beauville's conjecture that the cycle class map $\mathrm{CH}_l(A_\Omega) \to H^{2\dim A-2l}(A_\Omega, \mathbf{Q}_\ell)$ be injective when restricted to $\mathrm{CH}_l^{(2l)}(A_\Omega)$ for all l and for all abelian varieties A over k.

Theorem 3 confirms that conjecture (D) plays a prominent role among Murre's conjectures. For instance, given a smooth projective surface S over k with $H^1(S) = 0$ and $H^2(S)$ supported on a divisor, it is well known that Murre's conjecture (D) for $S \times S$ implies that $\mathrm{CH}_0(S) = \mathbf{Q}$. Here $H^\bullet$ denotes any Weil cohomology theory, e.g. ℓ-adic cohomology $H^\bullet(S_{\overline{k}}, \mathbf{Q}_\ell)$. In particular, Murre's conjecture (D) for fourfolds implies Bloch's conjecture on surfaces.

0.3. Let M be a motive over Ω. Given an adequate equivalence relation "$\sim$" on cycles (e.g. $\sim$ could be trivial, numerical, homological, smash-nilpotent, algebraic, or, when $k = \mathbf{C}$, Abel-Jacobi equivalence), we denote by $\mathrm{CH}_l(M)_\sim$ the sub-group of $\mathrm{CH}_l(M)$ consisting of cycles that are ~ 0. In [21], we proved that if $\mathrm{CH}_*(M)_{\mathrm{alg}}$ is generated by the Chow groups of zero-cycles on a (possibly non-connected) curve, then M splits as a direct sum of direct summands of twisted motives of curves. The following theorem generalizes the main result of loc. cit. and gives a criterion for a motive to be of abelian type.

Theorem 4. *Let $\sim$ be an adequate equivalence relation on cycles which, when restricted to 0-cycles, is coarser than albanese equivalence on 0-cycles. Let M be a motive over Ω. Then M is of abelian type if and only if $\mathrm{CH}_*(M)_\sim$ is generated, via the action of correspondences, by the Chow groups of products of curves.*

Theorem 4 is proved in Subsection 3.4. There we actually state and prove Theorem 3.11 which is the main theorem of Section 3 and from which is derived Theorem 4. Here is an outline of the proof. Let $f : N \to M$ be a morphism of motives. By Jannsen's semi-simplicity theorem [13], the numerical motive $\overline{M}$ splits as $\overline{M}_1 \oplus \overline{M}_2$ where $\overline{M}_1$ is the image of $\overline{f}$. If N is of abelian type, then, by finite-dimensionality of N, this splitting lifts to a splitting $M_1 \oplus M_2$ of M such that M_1 isomorphic to a direct summand of N, and such that the induced map $N \to M_2$ is numerically trivial and such that $\mathrm{CH}_*(N)_\sim \to \mathrm{CH}_*(M_2)_\sim$ is surjective. Writing $M_2 = (X, p, n)$, we then proceed by induction on the dimension of X to show that M_2 is of abelian type. Lemma 3.10 is the key lemma for that matter. Its proof relies on a refined version of a theorem of Bloch and Srinivas [5] which is expounded in Subsection 3.2; see Proposition 3.5. Note that when $\sim$ is the trivial equivalence relation, Theorem 4 can be proved without using the finite-dimensionality of the motives of curves; see Theorem 3.18.

When M is the motive of a smooth projective variety X over Ω, Theorem 4 can be made more precise and one need not consider the Chow groups of X in all degrees. For instance, we have the following two theorems.

Theorem 5. *Let $\sim$ be as in Theorem 4. Let X be a smooth projective variety of dimension $2n$ or $2n+1$ over Ω. Then the motive of X is of abelian type if and only if $\mathrm{CH}_0(X)_\sim, \dots, \mathrm{CH}_{n-1}(X)_\sim$ are generated, via the action of correspondences, by the Chow groups of products of curves.*

Theorem 6. *Let $\sim$ be as in Theorem 4. Let X be a smooth projective variety of dimension $2n-1$ or $2n$ over Ω. Assume that $\mathrm{CH}_0(X)_\sim, \dots, \mathrm{CH}_{n-2}(X)_\sim$ are generated, via the action of correspondences, by the Chow groups of products of curves. Then X has a Chow–Künneth decomposition.*

Some applications of these two theorems are discussed in Subsections 3.5 and 3.6. For instance, we consider X a smooth projective variety rationally dominated by a product of curves and we show that if $\dim X \le 4$, then X has a Chow–Künneth decomposition; and if $\dim X \le 3$, then X is finite-dimensional in the sense of Kimura. We also show, based on a classification result of Demailly–Peternell–Schneider [7], that a complex fourfold with a nef tangent bundle has a Chow–Künneth decomposition.

An interesting consequence of Theorem 3.11 is that, among finite-dimensional motives, a motive M is entirely determined, up to direct factors isomorphic to Lefschetz motives, by its Chow groups of algebraically trivial cycles. Another consequence of Theorem 3.11 and its proof is the following theorem which is concerned with Griffiths groups. We write $\mathrm{Griff}_i(X)$ for $\mathrm{CH}_i(X)_{\mathrm{hom}} / \mathrm{CH}_i(X)_{\mathrm{alg}}$.

Theorem 7. *Let $\sim$ be as in Theorem 4. Let $f : N \to M$ be a morphism of motives. Assume that N is finite-dimensional and that there is an integer l such that $f_* : \mathrm{CH}_i(N_\Omega)_\sim \to \mathrm{CH}_i(M_\Omega)_\sim$ is surjective for all $i < l$. Then $f_* : \mathrm{Griff}_i(N) \to \mathrm{Griff}_i(M)$ is surjective for all $i \le l$. If moreover $\sim$ is algebraic equivalence, then $f_* : \mathrm{CH}_i(N_\Omega)_{\mathrm{hom}} \to \mathrm{CH}_i(M_\Omega)_{\mathrm{hom}}$ is surjective for all $i < l$.* $\square$

In the spirit of Theorem 7, we are able to extend a result of R. Sebastian [19]. We show in Theorem 3.17 that if M is an effective motive over k such that $\mathrm{CH}_0(M_\Omega)$ is spanned via the action of correspondences by 0-cycles on products of curves, then numerical equivalence agrees with smash-nilpotence equivalence on 1-cycles on M.

Finally, Theorem 4 can be viewed as an analogue modulo rational equivalence of the following result which gives yet another characterization of motives of abelian type.

Theorem 8. *Let X be a smooth projective complex variety, the motive of which is finite-dimensional and the cohomology of which is spanned, via the action of correspondences, by the cohomology of products of curves. Then X is of abelian type.*

This result is essentially due to Arapura [3]: a homological motive whose cohomology is spanned by the cohomology of curves is "motivated" by the homological motives of curves. A standard lifting argument for finite-dimensional motives then proves Theorem 8. We however include a paragraph to prove this result as we slightly improve on Arapura's result; see §3.8.

1. Descent and motives of abelian type
Proof of Theorems 1 and 2

Let's first consider the following situation. Consider a scheme X over an algebraically closed field k such that X_K is K-isomorphic to an abelian variety A over K for some field extension K/k. The K-isomorphism $f : X_K \to A$ is defined over a subfield of K which is finitely generated over k. Therefore, we may assume that K is finitely generated over k and that there is a smooth irreducible variety U over k with function field K such that A spreads to an abelian scheme $\mathscr{A}$ over U and such that f spreads to a U-isomorphism $f_U : X \times_k U \to \mathscr{A}$. Specializing at a closed point u of U, i.e., pulling back along the closed immersion $u \to U$, the U-isomorphism f_U gives a k-isomorphism $f_u : X = X_u \to \mathscr{A}_u$. Thus, X is isomorphic to an abelian variety.

1.1. **Proof of Theorem 1.** Let k be an algebraically closed field and let M be a motive over k, say $M = (X, p, n)$. We assume that there is a field K/k such that M_K is isomorphic over K to a motive of abelian type. We want to show that M is of abelian type. Since M becomes of abelian type over K, it actually becomes of abelian type over a subfield of K which is finitely generated over k. We can thus assume that K is finitely generated over k. Let then A be an abelian variety over K such that we have a K-isomorphism $M_K = (X_K, p_K, n) \cong (A, q, m)$. Up to tensoring with the Lefschetz motive, which can be thought of as a direct summand of the motive of an elliptic curve defined over k, we may assume that $n = m = 0$. Let Y be a smooth quasi-projective variety defined over k with function field K. Let then U be a Zariski-open subset of Y such that A spreads to an abelian scheme $\mathscr{A} \to U$, $q \in \mathrm{CH}^{\dim A}(A \times_K A)$ spreads to a relative idempotent $\kappa \in \mathrm{CH}^{\dim A}(\mathscr{A} \times_U \mathscr{A})$, and such that the K-isomorphism $(X_K, p_K) \cong (A, q)$ spreads to an isomorphism $(X_U, p_U) \cong (\mathscr{A}, \kappa)$ of relative motives over U. Here, (X_U, p_U) denotes the constant motive over U whose closed fibers are (X, p). A Zariski-open subset of Y that satisfies the last two properties exists by the localization exact sequence for Chow groups.

Let t be a closed point of U. By assumption, t is a smooth point and the inclusion $j_t : t \hookrightarrow U$ is thus a regular embedding. Therefore, by [9, §6], there is a Gysin morphism j_t^* defined on Chow groups which commutes with flat pull-backs, proper push-forwards and intersection products. It follows that relative idempotents over U specialize to idempotents and that the relative isomorphism $(X_U, p_U) \cong (\mathscr{A}, \kappa)$ specializes to an isomorphism $(X, p) = (X_t, (j_t, j_t)^* p_U) \cong (\mathscr{A}_t, (j_t, j_t)^* \kappa)$ defined over k. $\square$

1.2. **Finite-dimensionality of motives is stable under descent.** Let k be a field and let K/k be a field extension. In this paragraph, we wish to study the stability under descent of two notions attached to motives: finite-dimensionality and Chow–Künneth decompositions.

Definition 1.1. A motive M over k is said to be *finite-dimensional* if there exists a splitting $M = M^+ \oplus M^-$ such that $S^n M^- = \Lambda^n M^+ = 0$ for $n >> 0$. A motive M^- whose symmetric powers $S^n M^-$ vanish for $n >> 0$ is said to be oddly finite-dimensional and a motive M^+ whose exterior powers $\Lambda^n M^+$ vanish for $n >> 0$ is said to be evenly finite-dimensional. The notion of finite-dimensionality is due independently to Kimura [15] and O'Sullivan.

The motive M is said to have a *Künneth decomposition* if there is a finite-sum decomposition of the homological motive $M^{\mathrm{hom}} = \bigoplus_{i \in \mathbf{Z}} (M^{\mathrm{hom}})_i$ such that $H_*((M^{\mathrm{hom}})_i) = H_i(M^{\mathrm{hom}})$.

The motive M is said to have a *Chow–Künneth decomposition* if there is a finite-sum decomposition $M = \bigoplus_{i \in \mathbf{Z}} M_i$ such that this decomposition defines modulo homological equivalence a Künneth decomposition of M.

First recall the following lemma.

Lemma 1.2. *Let X be a scheme over k. Then the map $\mathrm{CH}_*(X) \to \mathrm{CH}_*(X_K)$ induced by base-change is injective. If K/k is finite, then the composite with proper push-forward $\mathrm{CH}_*(X) \to \mathrm{CH}_*(X_K) \to \mathrm{CH}_*(X)$ is multiplication by $[K : k]$. Moreover, if K/k is a purely inseparable extension, then $\mathrm{CH}_*(X) \to \mathrm{CH}_*(X_K)$ and $\mathrm{CH}_*(X_K) \to \mathrm{CH}_*(X)$ are both isomorphisms.* □

Proof. This is classical; see for instance [6, Lemma 1.A.3]. When K/k is finite, the proof is immediate by definition of flat pull-back and proper push-forward of cycles; see [9, Example 1.7.4]. As for the purely inseparable case, consider a cycle $\gamma \in \mathrm{CH}_*(X_K)$ defined over a finite purely inseparable extension K of k of degree p^r, say. The cycle $\frac{1}{p^r}\gamma$ is then defined over k. Thus, γ is the image of the cycle $p^r \cdot (\frac{1}{p^r}\gamma)$ under the map $\mathrm{CH}_*(X) \to \mathrm{CH}_*(X_K)$. □

Let us mention the basic fact that a motive that becomes zero after base-change is zero.

Proposition 1.3. *Let M be a motive over k. Then $M = 0$ if and only if $M_K = 0$.*

Proof. By definition, a motive (X, p, n) is zero if and only if $p = 0 \in \mathrm{CH}_*(X \times X)$. The proposition then follows from Lemma 1.2. □

Theorem 1.4. *Let M be a motive over k. Then M is finite-dimensional if and only if M_K is finite-dimensional.*

Proof. If M is finite-dimensional then it is clear that M_K is finite-dimensional. Indeed, if M splits as $M^+ \oplus M^-$ with $S^n M^- = \Lambda^n M^+ = 0$ for some n, then we have that M_K splits as $(M^+)_K \oplus (M^-)_K$ with $S^n(M^-)_K = \Lambda^n(M^+)_K = 0$ in view of Proposition 1.3 and the fact that symmetric powers and exterior powers of motives commute with base-change,.

Assume now that M_K is finite-dimensional. This means that M_K has a splitting $(M_K)^+ \oplus (M_K)^-$ with $S^n(M_K)^- = \Lambda^n(M_K)^+ = 0$ for some $n >> 0$. Such a splitting is defined over a finitely generated field over k and we may assume that K is finitely generated over k. By a specialization argument as in the proof of Theorem 1, we may even assume that K is a finite extension of k. By Lemma 1.2, we may further assume that K is a finite Galois extension of k with Galois group G, say. Let then $p_K := \mathrm{id}_{M_K} = p^+ + p^- \in \mathrm{End}(M_K)$ be the decomposition corresponding to the decomposition $M_K = (M_K)^+ \oplus (M_K)^-$. The group G acts on $\mathrm{End}(M_K)$ as follows : for all $g \in G$ and all $f \in \mathrm{End}(M_K)$ we have $g \cdot f := g \circ f \circ g^{-1}$. This is well-defined since p_K is defined over k (so that $p_K \circ g = g \circ p_K$). Consider the G-invariant correspondence

$$\tilde{p}^+ := \frac{1}{|G|} \sum_{g \in G} g \cdot p^+ \in \mathrm{End}(M_K).$$

The correspondence p^+ defines in $\mathrm{End}(M_K^{\mathrm{hom}})$ the projector on the even-degree homology of M_K and since M_K is defined over k, the ℓ-adic cohomology class of p^+ is clearly invariant under the Galois group of k. This yields that $\tilde{p}^+$ and p^+ are homologically equivalent. We can thus write

$$\tilde{p}^+ \circ \tilde{p}^+ = \tilde{p}^+ + n$$

for some correspondence $n \in \mathrm{End}(M_K)$ that is homologically trivial, and hence nilpotent by finite-dimensionality of M_K [15, Prop. 7.5]. Since $\tilde{p}^+$ is G-invariant, it follows that $\tilde{p}^+ \circ \tilde{p}^+$ is G-invariant and hence that n is G-invariant. Looking at $\tilde{p}^+ \circ \tilde{p}^+ \circ \tilde{p}^+$, we see that $n \circ \tilde{p}^+ = \tilde{p}^+ \circ n$. Following Beilinson, we compute

$$\left(\tilde{p}^+ + (1 - 2\tilde{p}^+) \circ n\right)^{\circ 2} = \tilde{p}^+ + (1 - 2\tilde{p}^+) \circ n + n^{\circ 2} \circ (4n - 3).$$

A straightforward descending induction on the nilpotence index of n shows that there is a homologically trivial correspondence m such that $q^+ := \tilde{p}^+ + m$ is a G-invariant idempotent in

$\mathrm{End}(M_K)$. Thus p^+ is homologically equivalent to the G-invariant idempotent q^+. Likewise, p^- is homologically equivalent to the G-invariant idempotent $q^- := \mathrm{id}_{M_K} - q^+$. By finite-dimensionality of M_K, it follows that $\mathrm{Im}\, q^+$ and $\mathrm{Im}\, q^-$, which are motives defined over k, are respectively isomorphic over K to $(M_K)^+$ and $(M_K)^-$. In particular, we obtain that $S^n(\mathrm{Im}\, q^-) = \Lambda^n(\mathrm{Im}\, q^+) = 0$ over K for $n >> 0$. By Proposition 1.3, we conclude that $\mathrm{Im}\, q^+ \oplus \mathrm{Im}\, q^-$ is a decomposition of M into an evenly finite-dimensional motive and an oddly finite-dimensional motive. □

Proposition 1.5. *Let M be a motive over k. Then $M_{\overline{k}}$ has a Chow–Künneth decomposition if and only if there exists a field extension K/k such that M_K has a Chow–Künneth decomposition.*

Proof. The "only if" part of the proposition is obvious and the "if" part follows from a specialization argument as in the proof of Theorem 1 together with the compatibility of specialization with the cycle class map [9, §20.3]. □

Question 1.6. *Let M be a motive over k. Assume that M_K has a Chow–Künneth decomposition. Then does M have a Chow–Künneth decomposition?*

This question has a positive answer modulo homological equivalence, as is observed in the following proposition.

Proposition 1.7. *Let M be a motive over k. Then M^{hom} has a Künneth decomposition if and only if M_K has a Künneth decomposition.*

Proof. The "only if" part of the proposition is obvious. Recall that, for any extension F'/F of algebraically closed fields and for any smooth projective variety X over F, the base-change map $\mathrm{CH}_*(X) \to \mathrm{CH}_*(X_{F'})$ is an isomorphism modulo homological equivalence. If M_K has a Künneth decomposition, then it follows by specialization that, for some finite extension l/k, M_l has a Künneth decomposition. By Lemma 1.2, we may assume that l/k is Galois. Let us write $M = (X, p, n)$ with $d = \dim X$. Since M_l is defined over k, the Künneth projectors are invariant in $\mathrm{End}(M_l^{\mathrm{hom}}) \subseteq H^{2d}_{et}(X_{\bar{k}} \times_{\bar{k}} X_{\bar{k}}, \mathbf{Q}_\ell(d))$ under the action of the Galois group of k. It follows that $\mathrm{Gal}(l/k)$ acts trivially on those. Thus, the Künneth decomposition of M_l is defined over k and hence defines a Künneth decomposition of M. □

The following theorem shows that, assuming finite-dimensionality for M, Question 1.6 has a positive answer.

Theorem 1.8. *Let M be a motive over k. If M_K is finite-dimensional and has a Künneth decomposition, then M is finite-dimensional and has a Chow–Künneth decomposition.*

Proof. By Theorem 1.4, M is finite-dimensional; and by [15, Prop. 7.5], it follows that the kernel of $\mathrm{End}(M) \to \mathrm{End}(M^{\mathrm{hom}})$ is nilpotent. Therefore, by [12, Lemma 5.4], a sum of idempotents in $\mathrm{End}(M^{\mathrm{hom}})$ that adds to the identity lifts to a sum of idempotents in $\mathrm{End}(M)$ that adds to the identity in $\mathrm{End}(M)$. Now M has a Künneth decomposition by Proposition 1.7. We thus find that this Künneth decomposition lifts to a Chow–Künneth decomposition for M. □

For instance, if M is a motive such that $M_{\overline{k}}$ is of abelian type, then M has a Chow–Künneth decomposition. In particular, we obtain the following result that extends the classical result of Deninger–Murre [8] according to which every abelian variety has a Chow–Künneth decomposition.

Corollary 1.9. *Let X be a variety over k such that $X_{\overline{k}}$ has the structure of an abelian variety. Then X has a Chow–Künneth decomposition.* □

2. Murre's conjectures and motives of abelian type Proof of Theorem 3

Murre's conjectures [17] were originally stated for smooth projective varieties. Here, we give a statement for motives which contains the original statement of Murre for smooth projective varieties.

Conjecture 2.1 (Murre [17]). *Let M be a motive defined over a field k.*

(A) M has a Chow–Künneth decomposition : M splits as a finite direct sum $\bigoplus_{i\in\mathbf{Z}} M_i$, where $H_*(M_i) = H_i(M)$ for all i.

(B) $\mathrm{CH}_l(M_i) = 0$ for $i < 2l$.

(C) $F^\nu \mathrm{CH}_l(M) := \bigoplus_{i\geq 2l+\nu} \mathrm{CH}_l(M_i)$ does not depend on the choice of a Chow–Künneth decomposition.

(D) $\mathrm{CH}_l(M_{2l})_{\mathrm{hom}} = 0$ for all l.

There are several remarks to be made about the formulation of Murre's conjectures given above; see Remarks 2.2, 2.3 and 2.4. On the behavior of those conjectures with respect to field extensions, we refer to Propositions 2.5 and 2.5. On the independence of conjectures (B) and (D) with respect to the choice of a Chow–Künneth decomposition as in (A), we refer to Proposition 2.7 for a partial answer. Finally, on links between conjectures (B) and (C), we refer to Proposition 2.8.

Remark 2.2 (On conjecture (B)). Usually, conjecture (B) for varieties is stated in a stronger form which takes the following form for motives: if $M = (X, p, n)$ has a Chow–Künneth decomposition $M = \bigoplus_{i\in\mathbf{Z}} M_i$, then $\mathrm{CH}_l(M_i) = 0$ for $i < 2l$ and for $i > l - n + \dim X$. However, a combination of Murre's conjectures with the Lefschetz standard conjecture for M implies the strong form of conjecture (B). In particular, if $M = (X, p, n)$ is of abelian type, then it is known that $\mathrm{CH}_l(M_i) = 0$ for $i > l - n + \dim X$. Also, the formulation given in Conjecture 2.1 has the advantage of not involving a variety X and an integer n such that $M = (X, p, n)$.

Remark 2.3 (On conjecture (C)). Let G be the filtration on $\mathrm{CH}_l(M)$ induced by a Chow–Künneth decomposition $M = \bigoplus M'_i$. In (C), $F^\nu \mathrm{CH}_l(M)$ and $G^\nu \mathrm{CH}_l(M)$ are meant to coincide as sub-vector spaces of $\mathrm{CH}_l(M)$ (not to be merely isomorphic, as would be the case, for instance, were M finite-dimensional).

Remark 2.4 (On conjecture (D)). First, for a motive N, the notation $\mathrm{CH}_l(N)_{\mathrm{hom}}$ is unambiguous: if $N = (Y, q, n)$, then $q_*\big(\mathrm{CH}_{l-n}(Y)_{\mathrm{hom}}\big) = \big(q_* \mathrm{CH}_{l-n}(Y)\big)_{\mathrm{hom}}$. Indeed, the inclusion $\subseteq$ is obvious because the action of correspondences preserves homological equivalence of cycles. The inclusion $\supseteq$ follows from the fact that q is an idempotent.

Secondly, given an integer l, $\mathrm{CH}_l(M_{2l})_{\mathrm{hom}}$ vanishes if and only if $\mathrm{Ker}\,\big((p_{2l})_* : \mathrm{CH}_l(M) \to \mathrm{CH}_l(M)\big) = \mathrm{CH}_l(M)_{\mathrm{hom}}$. As such, our formulation really is equivalent to Murre's original formulation [17] of conjecture (D) for smooth projective varieties. To see this, recall that the idempotent p_{2l} has homology class the central projection on $H_{2l}(M)$ and, as such, acts as the identity on $H_{2l}(M)$. Then note that, by functoriality of the cycle class map with respect to the action of correspondences, we always have $\mathrm{Ker}\,\big((p_{2l})_* : \mathrm{CH}_l(M) \to \mathrm{CH}_l(M)\big) \subseteq \mathrm{CH}_l(M)_{\mathrm{hom}}$. It is obvious that $\mathrm{Ker}\,\big((p_{2l})_* : \mathrm{CH}_l(M) \to \mathrm{CH}_l(M)\big) = \mathrm{CH}_l(M)_{\mathrm{hom}}$ implies that $\mathrm{CH}_l(M_{2l})_{\mathrm{hom}}$ vanishes. Conversely, $\mathrm{CH}_l(M_{2l})_{\mathrm{hom}} = 0$ clearly implies that $\mathrm{Ker}\,\big((p_{2l})_* : \mathrm{CH}_l(M) \to \mathrm{CH}_l(M)\big) \supseteq \mathrm{CH}_l(M)_{\mathrm{hom}}$.

Proposition 2.5. *Let K/k be a field extension and let N be a motive over k. Assume that N has a Chow–Künneth decomposition $\bigoplus_{i\in\mathbf{Z}} N_i$ and that N_K is endowed with the induced Chow–Künneth decomposition $\bigoplus_{i\in\mathbf{Z}} (N_i)_K$. Consider the following statements :*

(1) N satisfies Murre's conjecture (B) ;

(2) N_K satisfies Murre's conjecture (B) ;

(3) N satisfies Murre's conjecture (D) ;

(4) N_K satisfies Murre's conjecture (D).

Then (2) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (3).
Moreover, if k is a universal domain, then (2) $\Leftrightarrow$ (1) *and* (4) $\Leftrightarrow$ (3).

Proof. That (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) follows immediately from the fact that the base-change map $\mathrm{CH}_l(N) \to \mathrm{CH}_l(N_K)$ is injective for all l; see Lemma 1.2.

Assume now that k is a universal domain. Consider $F \subset k$ a field of definition of N and of its Chow–Künneth decomposition $\bigoplus_{i\in \mathbf{Z}} N_i$ which is finitely generated. Let $\overline{K}/K$ be an algebraic closure of K. By the above, it is enough to show that if N satisfies Murre's conjecture (B) or (D), then $N_{\overline{K}}$ endowed with the induced Chow–Künneth decomposition $\bigoplus_{i\in\mathbf{Z}}(N_i)_{\overline{K}}$ satisfies Murre's conjecture (B) or (D), respectively. Fix a field isomorphism $\overline{K} \cong k$ which restricts to the identity on F. Pulling back along that isomorphism, we get isomorphisms $\mathrm{CH}_l(N_i) \cong \mathrm{CH}_l((N_i)_{\overline{K}})$ and $\mathrm{CH}_l(N_i)_{\mathrm{hom}} \cong \mathrm{CH}_l((N_i)_{\overline{K}})_{\mathrm{hom}}$ for all l and all i. This finishes the proof of the proposition. □

Theorem 2.6 (K. Xu & Z. Xu [23])**.** *Assume that k is algebraically closed. Let X be a smooth projective variety over k and let C be a smooth projective curve over k with function field $K = k(C)$. Assume that X has a Chow–Künneth decomposition and that X_K endowed with the induced Chow–Künneth decomposition satisfies Murre's conjectures (B) and (D). Then $X \times C$ has a Chow–Künneth decomposition that satisfies (B).*

Proof. Let's consider a Chow–Künneth component $M = (X, p)$ of $\mathfrak{h}(X)$ of weight j such that M_K satisfies Murre's conjectures (B) and (D). Consider a closed point c of C. The idempotents $[C\times\{c\}]$ and $[\{c\}\times C]$ in $\mathrm{End}(\mathfrak{h}(C))$ induce a Chow–Künneth decomposition $\mathfrak{h}(C) = \mathbb{1} \oplus \mathfrak{h}_1(C) \oplus \mathbb{1}(1)$, where $\mathfrak{h}_1(C) = (C, \pi_1)$ and $\pi_1 := \Delta_C - [C \times \{c\}] - [\{c\} \times C]$. It is clear that $M \otimes \mathbb{1}$ and $M \otimes \mathbb{1}(1)$ satisfy (B). The motive $M \otimes \mathfrak{h}_1(C) = (X \times C, p \times \pi_1)$ has weight $j+1$ and, therefore, we only have to prove that $\mathrm{CH}_l(M \otimes \mathfrak{h}_1(C)) = 0$ for $j+1 < 2l$ (and for $j+1 > d+l+1$, if one cares about the stronger form of (B); see Remark 2.2). In other words, we have to show that $p \times \pi_1$ acts trivially on $\mathrm{CH}_l(X \times C)$ for $j+1 < 2l$ (and for $j+1 > d+l+1$).

Let $\gamma \in \mathrm{CH}_l(X \times C)$. Since π_1 acts trivially on $[C] \in \mathrm{CH}_1(C)$, we see that $p \times \pi_1$ acts trivially on cycles of the form $\alpha \times [C]$ where $\alpha \in \mathrm{CH}_{l-1}(X)$. Because

$$\gamma = (\gamma - \gamma|_{X\times c} \times [C]) + \gamma|_{X\times c} \times [C],$$

we may assume that $\gamma|_{X\times c} = 0$. It is then enough to show that $p \times \Delta_C$ acts trivially on cycles $\gamma \in \mathrm{CH}_l(X \times C)$ such that $\gamma|_{X\times c} = 0$ for $j+1 < 2l$ (and for $j+1 > d+l+1$).

Let η be the generic point of C. The cycle $\gamma|_{X\times\eta} \in \mathrm{CH}_{l-1}(X_K)$ is then algebraically equivalent to the cycle $\gamma|_{X\times c_K} \in \mathrm{CH}_{l-1}(X_K)$. The latter cycle is obtained as the image of $\gamma|_{X\times c}$ by the base-change map $\mathrm{CH}_l(X) \to \mathrm{CH}_l(X_K)$ and is thus zero by assumption. Therefore, $\gamma|_{X\times\eta} \in \mathrm{CH}_{l-1}(X_K)$ is algebraicallly trivial and hence homologically trivial. Since (X_K, p_K) is assumed to satisfy Murre's conjecture (D), it follows that $(p_K)_*(\gamma|_{X\times\eta}) = 0$ if $j = 2l-2$; and since (X_K, p_K) is assumed to satisfy Murre's conjecture (B), it follows that $(p_K)_*(\gamma|_{X\times\eta}) = 0$ if $j < 2l-2$ (and if $j > d+l-1$).

Now, we have the following key formula; see the proof of Lemma 3.4 or [23, Lemma 3.2(ii)].

$$\big((p \times \Delta_C)_*\gamma\big)|_{X\times\eta} = (p_K)_*(\gamma|_{X\times\eta}).$$

We deduce, from the localization exact sequence ($C^{(1)}$ denotes the set of closed points of C)

$$\bigoplus_{d\in C^{(1)}} \mathrm{CH}_l(X \times d) \longrightarrow \mathrm{CH}_l(X \times C) \longrightarrow \mathrm{CH}_{l-1}(X \times \eta) \longrightarrow 0,$$

that, for $j < 2l-1$ (and for $j > d+l-1$),

$$(p \times \Delta_C)_*\gamma = \sum_i \gamma_i \times [d_i]$$

for finitely many cycles $\gamma_i \in \mathrm{CH}_l(X)$ and for finitely many closed points d_i in C. The correspondence $p \times \Delta_C$ is an idempotent and this yields that

$$(p \times \Delta_C)_*\gamma = \sum_i (p_*\gamma_i) \times [d_i].$$

But then, because (X, p) is pure of weight j and because (X_K, p_K) satisfies Murre's conjecture (B), it follows from Proposition 2.5 that (X, p) satisfies Murre's conjecture (B), i.e., that $p_* \mathrm{CH}_l(X) = 0$ for $j < 2l$ (and for $j > d + l$). Thus, $(p \times \Delta_C)_*\gamma = 0$ for $j < 2l - 1$ (and for $j > d + l$). □

Proposition 2.7. *Let M be a motive over k which is finite-dimensional. Assume that M has a Chow–Künneth decomposition that satisfies Murre's conjecture (B) or (D). Then any other Chow–Künneth decomposition of M satisfies Murre's conjecture (B) or (D), respectively. Moreover, if N is a direct summand of M, then N satisfies Murre's conjecture (B) or (D), respectively.*

Proof. Let $\bigoplus_{i\in\mathbf{Z}} M_i$ and $\bigoplus_{i\in\mathbf{Z}} M'_i$ be two Chow–Künneth decompositions for M. By definition of a Chow–Künneth decomposition, M_i and M'_i are isomorphic modulo homological equivalence for all i. Therefore, by finite-dimensionality, M_i is isomorphic to M'_i for all i. It follows that $\mathrm{CH}_l(M_i) = 0$ if and only if $\mathrm{CH}_l(M'_i) = 0$, and that $\mathrm{CH}_l(M_{2l})_{\mathrm{hom}} = 0$ if and only if $\mathrm{CH}_l(M'_{2l})_{\mathrm{hom}} = 0$.

The Chow–Künneth decomposition $\bigoplus_{i\in\mathbf{Z}} M_i$ of M defines a Künneth decomposition of M modulo homological equivalence. The Künneth projectors are central in $\mathrm{End}(M^{\mathrm{hom}})$. Therefore, as a direct summand of M, the motive N has a Künneth decomposition. Thus, by finite-dimensionality, N has a Chow–Künneth decomposition $\bigoplus_{i\in\mathbf{Z}} N_i$, where each N_i is isomorphic to a direct summand of M_i. This yields the proposition. □

The following proposition and its proof are very similar to [22, Proposition 3.1].

Proposition 2.8. *Let M be a motive over k which has a Chow–Künneth decomposition. Assume that Murre's conjecture (B) holds for M and for $M \otimes M^\vee$ with respect to any choice of Chow–Künneth decomposition. Then M satisfies Murre's conjecture (C).*

Proof. Let $\bigoplus_{i\in\mathbf{Z}} M_i$ and $\bigoplus_{i\in\mathbf{Z}} M'_i$ be two Chow–Künneth decompositions for M, and let F and F' be the induced filtrations on $\mathrm{CH}_l(M)$. We first show that $\mathrm{Hom}(M_i, M'_j) = 0$ for all $i > j$. Indeed, $(M \otimes M^\vee)_k := \bigoplus_{k=j-i} M'_j \otimes M_i^\vee$ defines a Chow–Künneth decomposition for $M \otimes M^\vee$. By assumption, $\mathrm{CH}_0\big((M \otimes M^\vee)_k\big) = 0$ for all $k < 0$. Therefore,

$$\mathrm{Hom}(M_i, M'_j) := \mathrm{CH}_0(M'_j \otimes M_i^\vee) = 0 \text{ for all } i > j.$$

This implies that F is finer than F', i.e., that $F \subseteq F'$. By symmetry, we conclude that $F = F'$. □

We are now in a position to prove the main result of this section.

Theorem 2.9. *Let Ω be a universal domain that contains k. Murre's conjecture (D) for products of curves over Ω implies Murre's conjectures (A), (B), (C) and (D) for motives over k which are of abelian type.*

Proof. Let M be a motive over k which is of abelian type. Then M_Ω is isomorphic to a direct summand of the motive of a product of smooth projective curves over Ω. We claim that, under the assumption that products of curves over Ω satisfy Murre's conjecture (D), products of curves over Ω satisfy Murre's conjectures (A)–(D). Indeed, a product of smooth projective curves over Ω is finite-dimensional [15, Corollaries 4.4 & 5.11] and has a Chow–Künneth decomposition. By (3) ⇒ (4) in Proposition 2.5, products of curves defined over the function field of a curve over Ω satisfy Murre's conjecture (D). It follows from Theorem 2.6, by a straightforward induction on the number of curves involved in the product, that Murre's conjecture (B) holds for products of curves over Ω.

Now, by Proposition 2.7, we find that M_Ω satisfies Murre's conjectures (A), (B) and (D). A combination of Propositions 2.5 and 2.7 shows that M satisfies Murre's conjectures (A), (B) and (D). Finally, the motive $M \otimes M^\vee$ is also of abelian type over k. Therefore Murre's conjecture (B) also holds for $M \otimes M^\vee$. Thus, thanks to Proposition 2.8, M also satisfies Murre's conjecture (C). □

Remark 2.10. Let A be an abelian variety of dimension d over k. Let m be an integer and let $[m] : A \to A$ denote the multiplication-by-m endomorphism of A. Then there exists a Chow–Künneth decomposition $\{\Pi_i\}$ for A such that $\Pi_i \circ \Gamma_{[m]} = \Gamma_{[m]} \circ \Pi_i = m^i \cdot \Pi_i \in \mathrm{CH}_d(A \times A)$; see [8]. Moreover, there is a decomposition

$$\mathrm{CH}_l(A) = \textstyle\bigoplus_l^{l+d} \mathrm{CH}_l^{(i)}(A), \text{ where } \mathrm{CH}_l^{(i)}(A) := \{\alpha \in \mathrm{CH}_l(A) \,|\, [m]_*\alpha = m^i\alpha, \forall m \in \mathbf{Z}\}.$$

Beauville [4] conjectured that $\mathrm{CH}_l^{(i)}(A) = 0$ for $i < 2l$, and Murre checked [17, Lemma 2.5.1] that

$$(\Pi_i)_* \mathrm{CH}_l(A) = \mathrm{CH}_l^{(i)}(A).$$

We thus see, thanks to Proposition 2.7, that Beauville's conjecture for A equipped with the Chow–Künneth decomposition $\{\Pi_i\}$ above is equivalent to Murre's conjecture (B) for A equipped with any Chow–Künneth decomposition.

Beauville [4] also conjectured that the cycle class map $\mathrm{CH}_l(A) \to H^{2\dim A - 2l}(A_{\overline{k}}, \mathbf{Q}_\ell)$ to ℓ-adic cohomology is injective when restricted to $\mathrm{CH}_l^{(2l)}(A)$. The identity $(\Pi_i)_* \mathrm{CH}_l(A) = \mathrm{CH}_l^{(i)}(A)$ shows that this conjecture is actually equivalent to Murre's conjecture (D) for A. Since motives of abelian type are spanned either by motives of curves or by motives of abelian varieties, Proposition 2.7 implies that Murre's conjecture (D) for products of curves is equivalent to Beauville's conjecture that $\mathrm{CH}_l^{(2l)}(A) \to H^{2\dim A - 2l}(A_{\overline{k}}, \mathbf{Q}_\ell)$ is injective for all l and all abelian varieties.

We have thus showed that if $\mathrm{CH}_l^{(2l)}(A_\Omega) \to H^{2\dim A - 2l}(A_\Omega, \mathbf{Q}_\ell)$ is injective for all l and all abelian varieties A over k, then Beauville's vanishing conjecture holds, i.e., $\mathrm{CH}_l^{(i)}(A) = 0$ for all abelian varieties A over k and all $i < 2l$.

3. Chow groups and motives of abelian type
Proof of Theorem 4

3.1. **Chow groups and field extensions.** The following lemma is certainly well known.

Lemma 3.1. *Let $f : M \to N$ be a morphism of motives defined over k. Assume that, for some field extension K/k, $(f_K)_* : \mathrm{CH}_0(M_K) \to \mathrm{CH}_0(N_K)$ is surjective or injective. Then $f_* : \mathrm{CH}_0(M) \to \mathrm{CH}_0(N)$ is surjective or injective, respectively.*

Proof. Given a smooth projective variety X over k, by Lemma 1.2, the base-change map $\mathrm{CH}_l(X) \to \mathrm{CH}_l(X_K)$ is injective for all l. Besides, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{CH}_0(M) \arrow[r] \arrow[d, "f_*"] & \mathrm{CH}_0(M_K) \arrow[d, "(f_K)_*"] \\ \mathrm{CH}_0(N) \arrow[r] & \mathrm{CH}_0(N_K). \end{tikzcd}$$

It is then straightforward to see that if $(f_K)_*$ is injective, then f_* is injective.

Let's now assume that there is a field K/k such that $(f_K)_* : \mathrm{CH}_0(M_K) \to \mathrm{CH}_0(N_K)$ is surjective. Let's pick a cycle $\gamma \in \mathrm{CH}_0(N)$ and let's denote by γ_K its image in $\mathrm{CH}_0(N_K)$. There is a cycle $\beta \in \mathrm{CH}_0(M_K)$ such that $(f_K)_*\beta = \gamma_K$. The cycle β is defined over a finitely generated extension of k. We may therefore assume that K is finitely generated over k and that it is the function field of a smooth quasi-projective variety Y over k. Let y be a closed point in Y and let $k(y)/k$ be its residue field. Such a point defines a regular embedding $y \hookrightarrow Y$, so that, for any smooth projective variety X over k and for any integer l, there is a specialization map $\sigma : \mathrm{CH}_l(X_K) \to \mathrm{CH}_l(X_{k(y)})$ which commutes with flat pull-backs, proper push-forwards and intersection product; see [9, §§6 & 20.3]. Moreover, for a cycle $\alpha \in \mathrm{CH}_l(X)$, we have $\sigma(\alpha_K) = \alpha_{k(y)}$ because $\sigma(\alpha_K)$ is obtained as the intersection of $\alpha \times Y$ with $X \times k(y)$. It immediately follows, after specialization, that $(f_{k(y)})_*\sigma(\beta) = \alpha_{k(y)}$. Now, for any smooth projective variety X over k, the composite map $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_{k(y)}) \to \mathrm{CH}_0(X)$ is multiplication by $[k(y) : k]$; see Lemma 1.2. These maps commute with the action of correspondences and this yields that α lies in the image of $f_* : \mathrm{CH}_0(M) \to \mathrm{CH}_0(N)$. □

The converse to Lemma 3.1 is not true in general. Consider for instance a smooth projective curve C over a finite field F with positive genus, a closed point c on C, and the correspondence $f := [C \times c] \in \mathrm{CH}_1(C \times C) = \mathrm{End}(\mathfrak{h}(C))$. Then $f_* : \mathrm{CH}_0(C) \to \mathrm{CH}_0(C)$ is an isomorphism (both Chow groups are spanned by $[c]$). However, $(f_K)_* : \mathrm{CH}_0(C_K)_{\mathrm{hom}} \to \mathrm{CH}_0(C_K)_{\mathrm{hom}}$ is zero for all field extensions K/F, and $\mathrm{CH}_0(C_K)_{\mathrm{hom}} \neq 0$ for some extension K/F (for instance, a finite extension of $F(t)$ over which $\mathrm{Pic}^0(C_K)$ acquires a non-torsion rational point). Thus, for such a choice of field K, $(f_K)_* : \mathrm{CH}_0(C_K) \to \mathrm{CH}_0(C_K)$ is neither injective, nor surjective.

Anyhow, when the base-field k is a universal domain, the converse does hold:

Lemma 3.2. *Let $f : M \to N$ be a morphism of motives defined over Ω. Then, for all fields K over Ω, $f_* : \mathrm{CH}_0(M) \to \mathrm{CH}_0(N)$ is surjective or injective if and only if $(f_K)_* : \mathrm{CH}_0(M_K) \to \mathrm{CH}_0(N_K)$ is surjective or injective, respectively.*

Proof. Let us first prove the lemma for fields K that have same cardinality as Ω. Let $k \subset \Omega$ be a field of definition of $f : M \to N$ which is finitely generated. Let $\overline{K}$ be an algebraic closure of K and fix a field isomorphism $\sigma : \overline{K} \xrightarrow{\simeq} \Omega$ which restricts to the identity on k. We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}_0(M) & \longrightarrow & \mathrm{CH}_0(M_K) & \longrightarrow & \mathrm{CH}_0(M_{\overline{K}}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (f_K)_*} & & \downarrow{\scriptstyle (f_{\overline{K}})_*} \\ \mathrm{CH}_0(N) & \longrightarrow & \mathrm{CH}_0(N_K) & \longrightarrow & \mathrm{CH}_0(N_{\overline{K}}) \end{array}$$

where the horizontal arrows are induced by base-change and are therefore injective by Lemma 1.2. We note, by pulling back along σ or σ^{-1}, that f_* is surjective or injective if and only if $(f_{\overline{K}})_*$ is surjective or injective, respectively. The lemma then follows from Lemma 3.1.

Now, assume that K/Ω is any field extension and that f_* is surjective or injective, respectively. Let α be a cycle in $\mathrm{CH}_*(N_K)$ and let β be a cycle in $\mathrm{Ker}\,(f_K)_*$. These cycles are defined over a subfield L of K which is finitely generated over Ω. By the above, $(f_L)_*$ is surjective or injective, respectively. It is then straightforward to see that $\alpha \in \mathrm{Im}\,(f_K)_*$ and that $\beta = 0$. Thus, $(f_K)_*$ is surjective or injective, respectively. □

3.2. **A refinement of a theorem of Bloch and Srinivas.** In order to prove the key Lemma 3.10, we need a slight refinement of the decomposition of the diagonal argument of Bloch and Srinivas which appears in [5]. This is embodied in Proposition 3.5.

First we prove a lemma which seems to be known as Lieberman's lemma and which is quoted in [2, 3.1.4] and [18, 1.10] without proof. Let X, X', Y and Y' be smooth projective varieties over a field k. Let $a \in \mathrm{CH}_p(X' \times X)$, $b \in \mathrm{CH}_q(Y \times Y')$ and $\gamma \in \mathrm{CH}_r(X \times Y)$ be correspondences. Let's write $({}^t a, b) := \tau_*({}^t a \times b)$, where $\tau : X \times X' \times Y \times Y' \to X \times Y \times X' \times Y'$ is the map permuting the two middle factors.

Lemma 3.3. *We have the formula*

$$({}^t a, b)_* \gamma = b \circ \gamma \circ a \ \in \mathrm{CH}_{p+q+r-\dim X - \dim Y}(X' \times Y').$$

Proof. The lemma is proved in [9, 16.1.1] in the case when a and b are either graphs of morphisms or the transpose thereof. We reduce to this case by showing that every correspondence is the sum of the composite of graphs of morphisms and their transpose.

If X_1, X_2, X_3 are varieties, let's write, for $i \in \{1,2\}$, $p_{X_i}^{X_1X_2}$ for the projection $X_1 \times X_2 \to X_i$; let's also write, for $i \neq j \in \{1,2,3\}$, $p_{X_iX_j}^{X_1X_2X_3}$ for the projection $X_1 \times X_2 \times X_3 \to X_i \times X_j$.

Let's consider the case $X = X'$ and $b = [W]$, where W is an irreducible subvariety of $Y \times Y'$ of dimension q. We are going to prove that b is the composite of the graph of a morphism with the transpose of the graph of another morphism. For this purpose, let's consider an alteration $\sigma : \widetilde{W} \to W$ and let's define the composite morphism $h : \widetilde{W} \to W \hookrightarrow Y \times Y'$. Then, by proper pushforward, we have $\deg(\sigma) \cdot b = h_*[\widetilde{W}]$. Now,

$$\begin{aligned}
\deg(\sigma) \cdot b \circ \gamma &= (p_{XY'})_* \big((p_{YY'}^* h_* [\widetilde{W}]) \cap p_{XY}^* \gamma \big) \\
&= (p_{XY'})_* \big((\mathrm{id}_X \times h)_* (p_{Y'}^{XY'})^* [\widetilde{W}] \cap p_{XY}^* \gamma \big) \\
&= (p_{XY'})_* \big((\mathrm{id}_X \times h)_* (\mathrm{id}_X \times h)^* p_{XY}^* \gamma \big) \\
&= (\mathrm{id}_X \times (p_{Y'}^{YY'} \circ h))_* (\mathrm{id}_X \times (p_Y^{YY'} \circ h))^* \gamma \\
&= (\Delta_X, \Gamma_{p_{Y'}^{YY'} \circ h})_* (\Delta_X, {}^t\Gamma_{p_Y^{YY'} \circ h})_* \gamma \\
&= \Gamma_{p_{Y'}^{YY'} \circ h} \circ {}^t\Gamma_{p_Y^{YY'} \circ h} \circ \gamma.
\end{aligned}$$

Here, we have omitted the superscript "XYY'". The first equality is by definition of the composition law for correspondences. The second equality follows from the fibre square

$$\begin{tikzcd}
X \times W \arrow[r, "\mathrm{id}_X \times h"] \arrow[d, "p_W^{XW}"'] & X \times Y \times Y' \arrow[d, "p_{YY'}"'] \\
W \arrow[r, "h"] & Y \times Y'.
\end{tikzcd}$$

The third equality follows from the projection formula. The fourth equality follows from the equalities $p_{XY} \circ (\mathrm{id}_X \times h) = \mathrm{id}_X \times (p_Y^{YY'} \circ h)$ and $p_{XY'} \circ (\mathrm{id}_X \times h) = \mathrm{id}_X \times (p_{Y'}^{YY'} \circ h)$. The fifth equality is [9, 16.1.2.(c)]. Finally, the last equality follows from [9, 16.1.1].

This last equality holds for all smooth projective varieties X, all integers r and all correspondences $\gamma \in \mathrm{CH}_r(X \times Y)$. By Manin's identity principle [2, 4.3.1], we get

$$\deg(\sigma) \cdot b = \Gamma_{p_{Y'}^{YY'} \circ h} \circ {}^t\Gamma_{p_Y^{YY'} \circ h}.$$

Thus, as claimed, every correspondence is the sum of the composite of graphs of morphisms and their transpose. □

Lemma 3.4. *Let X and Y be smooth projective varieties over a field k. Let $\Gamma \in \mathrm{CH}_n(X \times Y)$ be a correspondence. Let η_X be the generic point of X and let $[\eta_X] \in \mathrm{CH}_0(X_{k(X)})$ be the class of η_X viewed as a rational point of $X_{k(X)}$. Then, under the natural map $\mathrm{CH}_n(X \times Y) \to \mathrm{CH}_{n-\dim X}(k(X) \times Y)$, Γ is mapped to $(\Gamma_{k(X)})_*[\eta_X]$.*

Proof. Let $d := \dim X$. Since the map $\mathrm{CH}_d(X \times X) \to \mathrm{CH}_0(k(X) \times X)$ is obtained as the direct limit, ranging over the open subsets U of X, of the flat pullback maps $\mathrm{CH}_d(X \times X) \to \mathrm{CH}_0(U \times X)$, we see that $[\Delta_X] \in \mathrm{CH}_d(X \times X)$ is mapped to $[\eta_X] \in \mathrm{CH}_0(k(X) \times X)$. Besides, by Lemma 3.3, $(\Delta_X, \Gamma)_* \Delta_X = \Gamma \circ \Delta_X = \Gamma$. The lemma then follows by commutativity, for all integer r, of the diagram

$$\begin{tikzcd}
\mathrm{CH}_r(X \times X) \arrow[r] \arrow[d, "(\Delta_X,\Gamma)_*"] & \mathrm{CH}_{r-d}(k(X) \times X) \arrow[r] \arrow[d, "(\Gamma_{k(X)})_*"] & 0 \\
\mathrm{CH}_{n+r-d}(X \times Y) \arrow[r] & \mathrm{CH}_{n+r-2d}(k(X) \times Y) \arrow[r] & 0.
\end{tikzcd}$$

Let us prove commutativity of the diagram. It is obtained as the direct limit over the open inclusions $j_U : U \hookrightarrow X$ of the diagrams

$$\begin{tikzcd}
\mathrm{CH}_r(X \times X) \arrow[r, "({}^t\Gamma_{j_U},\Delta_X)_*"] \arrow[d, "(\Delta_X,\Gamma)_*"] & \mathrm{CH}_r(U \times X) \arrow[d, "(\Delta_U,\Gamma)_*"] \\
\mathrm{CH}_{n+r-d}(X \times Y) \arrow[r, "({}^t\Gamma_{j_U},\Delta_Y)_*"] & \mathrm{CH}_{n+r-d}(U \times Y).
\end{tikzcd}$$

The action of (Δ_U, Γ) on $\mathrm{CH}_r(U \times X)$ is well-defined because (Δ_U, Γ) has a representative whose support is proper over $U \times Y$, cf. [9, Remark 16.1]. These diagrams commute for all U for the following reason. Let U, V and W be nonsingular open varieties and let $\alpha \in \mathrm{CH}_i(U \times V)$ (resp. $\beta \in \mathrm{CH}_j(V \times W)$) be a correspondence which has a representative which is proper over U and V (resp. V and W). Then, by loc. cit. , it is possible to define the composite $\beta \circ \alpha$ and to show as in [9, Proposition 16.1.2(a)] that $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ on cycles. We may now conclude that the diagram is commutative by checking that $(\Delta_U, \Gamma) \circ ({}^t\Gamma_{j_U}, \Delta_X) = ({}^t\Gamma_{j_U}, \Delta_Y) \circ (\Delta_X, \Gamma) = ({}^t\Gamma_{j_U}, \Gamma)$. □

Proposition 3.5. *Let $M = (X, p)$ and $N = (Y, q, n)$ be two motives over a field k and let $\varphi : N \to M$ be a morphism. Suppose that $(\varphi_{k(X)})_* : \mathrm{CH}_0(N_{k(X)}) \to \mathrm{CH}_0(M_{k(X)})$ is surjective. Then there exist a morphism $\Gamma_1 : M \to N$, a smooth projective variety Z of dimension $\dim X - 1$ over k and a morphism $\Gamma_2 : M \to M$ that factors through the motive $(Z, \Delta_Z, 1)$ such that*

$$p = \varphi \circ \Gamma_1 + \Gamma_2.$$

Proof. As in the proof of Lemma 3.4, we have the following commutative diagram whose rows are exact by localization.

$$\begin{tikzcd}
\mathrm{CH}_{d-n}(X \times Y) \arrow[r] \arrow[d, "(\Delta_X,\varphi)_*"] & \mathrm{CH}_{-n}(k(X) \times Y) \arrow[r] \arrow[d, "(\varphi_{k(X)})_*"] & 0 \\
\mathrm{CH}_d(X \times X) \arrow[r] & \mathrm{CH}_0(k(X) \times X) \arrow[r] & 0.
\end{tikzcd}$$

Let η_X be the generic point of X and view it as a rational point of $k(X) \times X$ over $k(X)$. Then $p \in \mathrm{CH}_d(X \times X)$ maps to $(p_{k(X)})_*[\eta_X] \in \mathrm{CH}_0(k(X) \times X)$ by Lemma 3.4. Because $(\varphi_{k(X)})_* : \mathrm{CH}_0(N_{k(X)}) \to \mathrm{CH}_0(M_{k(X)})$ is surjective, there exists $y \in \mathrm{CH}_{-n}(k(X) \times Y)$ such that $(\varphi_{k(X)})_* y = (p_{k(X)})_*[\eta_X]$. Let α be a lift of y in $\mathrm{CH}_{d-n}(X \times Y)$. By commutativity of the diagram, we have that $\Gamma_2 := p - (\Delta_X, \varphi)_*\alpha$ maps to zero in $\mathrm{CH}_0(k(X) \times X)$. Therefore, by the localization exact sequence for Chow groups, Γ_2 is supported on $D \times X$ for some codimension-one closed subscheme D of X, i.e., there is a $\beta \in \mathrm{CH}_d(D \times X)$ that maps to Γ_2. Let's write $\iota : D \hookrightarrow X$ for the inclusion map. Consider then $\sigma : Z \to D$ an alteration of D, that is, σ is a generically finite morphism with Z smooth. Such a morphism exists for any variety D over k by de Jong's alteration theorem. Using the alteration $\sigma : Z \to D$, we see that there is a cycle $\gamma \in \mathrm{CH}_d(Z \times X)$ that maps to Γ_2 under the natural map $\mathrm{CH}_d(Z \times X) \to \mathrm{CH}_d(X \times X)$. By Lemma 3.3, we then have $\Gamma_2 = ((\iota \circ \sigma) \times \mathrm{id}_X)_*\gamma = \Gamma_{\iota\circ\sigma} \circ \gamma$, so that Γ_2 factors through $(Z, \Delta_Z, 1)$. By Lemma 3.3 again, we have $(\Delta_X, \varphi)_*\alpha = \varphi \circ \alpha$, so that if we set $\Gamma_1 := q \circ \alpha \circ p$, then we have $p = \varphi \circ \Gamma_1 + \Gamma_2$. □

3.3. **Three lemmas.** The following three lemmas are the building blocks to the proof of Theorem 3.11 which is the main theorem of Section 3.

Given a motive M, we denote $\overline{M}$ its reduction modulo numerical equivalence. Recall the following. Let $f : N \to M$ be a morphism of motives and let $\overline{f} : \overline{N} \to \overline{M}$ be its reduction modulo numerical equivalence. If N is finite-dimensional and if $\overline{f}$ has a left-inverse, then f has a left-inverse. Indeed, consider a morphism $h : M \to N$ such that $\overline{h} \circ \overline{f} = \mathrm{id}_{\overline{N}}$. By [15, Proposition 7.5], $h \circ f - \mathrm{id}_N \in \mathrm{End}(N)$ is nilpotent. It is then clear that f has a left-inverse. A similar statement holds if M is finite-dimensional and if "left-inverse" is replaced with "right-inverse".

Lemma 3.6. *Let $f : N \to M$ be a morphism of motives defined over a field k. Assume that N is finite-dimensional in the sense of Definition 1.1. Then M splits as $M_1 \oplus M_2$, where the induced morphism $N \to M_1$ has a right-inverse and where the induced morphism $N \to M_2$ is numerically trivial.*

Proof. The morphism $f : N \to M$ reduces modulo numerical equivalence to a morphism $\overline{f} : \overline{N} \to \overline{M}$. By Jannsen's semi-simplicity Theorem [13], $\overline{N}$ admits a splitting $\overline{N}_1 \oplus \overline{N}_2$ and $\overline{M}$ admits a splitting $\mathrm{Im}\,\overline{f} \oplus \overline{M}'$ such that $\overline{f}$ induces an isomorphism $\overline{N}_1 \to \mathrm{Im}\,\overline{f}$ and such that $\overline{N}_2 \to \overline{M}$ and $\overline{N} \to \overline{M}'$ are zero. By finite-dimensionality of N, $\overline{N}_1$ lifts to a direct summand N_1 of N. Let $s : N_1 \to M$ be the restriction of f to N_1 and let $\overline{r} : \overline{M} \to \overline{N}_1$ be a left-inverse to $\overline{s} : \overline{N}_1 \to \overline{M}$. By finite-dimensionality of N_1, we find a morphism $r : M \to N_1$ which is a left-inverse to $s : N_1 \to M$. The idempotent $s \circ r : M \to M$ thus defines a direct summand M_1 of M which is isomorphic to N_1 and it is clear that the induced morphism $N \to M_1$ has a right-inverse. We can therefore write $M = M_1 \oplus M_2$, where M_2 is defined by the idempotent $\mathrm{id}_M - s \circ r$. That the induced morphism $N \to M_2$ is numerically trivial follows from the facts that $\overline{f}$ is zero on $\overline{N}_1$ and that $\overline{r} \circ \overline{s} = \mathrm{id}_{\overline{N}_1}$. □

For convenience, for a motive M, when we say that there exist a smooth projective variety Z of dimension at most a negative integer over k and an idempotent $q \in \mathrm{End}(\mathfrak{h}(Z))$ such that $M \cong (Z, q, l)$, we mean that $M = 0$.

Lemma 3.7. *Let $M = (X, p, n)$ be a motive over k and let l be an integer greater than n such that $\mathrm{CH}_i(M_\Omega) = 0$ for all $i < l$. Then there exist a smooth projective variety Z of dimension at most $\dim X - l + n$ over k and an idempotent $q \in \mathrm{End}(\mathfrak{h}(Z))$ such that $M \cong (Z, q, l)$.*

Proof. This is due to Kahn–Sujatha [14]. Let's however give a proof. Clearly, we may assume $n = 0$. Let's proceed by induction on l. If $l \leq 0$, there is nothing to prove. Assume that $\mathrm{CH}_0(M_\Omega) = 0$ and hence $\mathrm{CH}_0(M_{k(X)}) = 0$. Then, by Proposition 3.5, there is a smooth projective variety W of dimension $\dim X - 1$ and an idempotent r such that M is isomorphic to $(Z, r, 1)$. Assume now that $\mathrm{CH}_i(M_\Omega) = 0$ for all $i < l$ (and hence that $\mathrm{CH}_i(M_{k(X)}) = 0$ for all $i < l$) and that there is a smooth projective variety Y of dimension $\dim X - l + 1$ and an idempotent q_Y such that M is isomorphic to $(Y, q_Y, l-1)$. The motive $(Y, q_Y, l-1)$ is such that $\mathrm{CH}_0((Y, q_Y)_\Omega) = 0$ and hence $\mathrm{CH}_0((Y, q_Y)_{k(Y)}) = 0$. The case $l = 1$ has been settled above, and we may thus conclude to the existence of a smooth projective variety Z of dimension at most $\dim X - l$ and of an idempotent q such that $M \cong (Z, q, l)$. □

Remark 3.8. Lemma 3.7 gives a necessary and sufficient condition for a motive M over k to be effective, namely that $\mathrm{CH}_l(M_\Omega) = 0$ for all $l < 0$. Indeed, after choosing an embedding $k(X) \hookrightarrow \Omega$, we see that $\mathrm{CH}_l(M_\Omega) = 0$ for all $l < 0$ implies that $\mathrm{CH}_l(M_{k(X)}) = 0$ for all $l < 0$. It then follows from Lemma 3.7 that M is effective. Conversely, if M is effective, then M_Ω is also effective. It is then clear that $\mathrm{CH}_l(M_\Omega) = 0$ for all $l < 0$.

Remark 3.9. Further to Remark 3.8, Lemma 3.7 gives a necessary and sufficient condition for a motive M over k to be isomorphic to a direct summand of a twisted motive of a variety of dimension at most d. Given a motive M over k, there exist a smooth projective variety X of dimension at most d and an integer l such that M is isomorphic to a direct summand of

$\mathfrak{h}(X)(l)$ if and only if there exist integers l and l' with $-l-l'=d$ such that $\mathrm{CH}_i(M_\Omega)=0$ for all $i<l$ and $\mathrm{CH}_j(M_\Omega^\vee)=0$ for all $j<l'$. The condition is indeed clearly sufficient and it is necessary by the following. That $\mathrm{CH}_i(M_\Omega)=0$ for all $i<l$ implies that M is isomorphic to a motive of the form (Y,q,l) for some smooth projective variety Y over k. That $\mathrm{CH}_j(M_\Omega^\vee)=\mathrm{CH}_{j+l+\dim Y}(Y_\Omega,q_\Omega)=0$ for all $j<l'$ implies that $M^\vee$ is isomorphic to a motive of the form $(X,p,l+l'+\dim Y)$ for some smooth projective variety X of dimension at most $-l-l'$ over k and some idempotent $p\in\mathrm{End}(\mathfrak{h}(X))$. Dualizing this isomorphism gives $M\cong(X,{}^tp,l)$.

Lemma 3.10. *Let $f:N\to M$ be a morphism of motives defined over a field k with $M=(X,p)$. Assume that f is numerically trivial and that N is finite-dimensional in the sense of Definition 1.1. If $(f_{k(X)})_*:\mathrm{CH}_0(N_{k(X)})\to\mathrm{CH}_0(M_{k(X)})$ is surjective, then $\mathrm{CH}_0(M_K)=0$ for all field extensions K/k.*

Proof. By Proposition 3.5, we get the existence of $\Gamma_1\in\mathrm{Hom}(M,N)$, and of $\Gamma_2\in\mathrm{End}(\mathfrak{h}(X))$ which factors through $(Z,\Delta_Z,1)$ for some variety Z, such that $p=\gamma\circ\Gamma_1+\Gamma_2$. The arguments below work equally well after base change to K and, without loss of generality, we therefore assume $K=k$. The action of Γ_2 on $\mathrm{CH}_0(X)$ factors through $\mathrm{CH}_0(Z,\Delta_Z,1)=\mathrm{CH}_{-1}(Z)=0$, so that Γ_2 acts trivially on $\mathrm{CH}_0(X)$. Therefore, $p_*x=(\gamma\circ\Gamma_1)_*x$ for all $x\in\mathrm{CH}_0(X)$. Since p is an idempotent, we also get $p_*x=(\gamma\circ\Gamma_1)_*^{\circ n}x$ for all positive integers n and all $x\in\mathrm{CH}_0(X)$. By assumption, γ is numerically trivial. Hence, $\Gamma_1\circ\gamma\in\mathrm{End}(N)$ is numerically trivial. Now, N is finite-dimensional and [15, Proposition 7.5] yields that $\Gamma_1\circ\gamma$ is nilpotent. We have thus proved that $p_*x=0$ for all $x\in\mathrm{CH}_0(X)$, i.e., that $\mathrm{CH}_0(M)=0$. □

3.4. **Proof of the main theorem.** Let X be a smooth projective variety over k. Two 0-cycles α and β on X are said to be *albanese equivalent* if $\alpha-\beta$ has degree zero and lies in the kernel of the albanese map $\mathrm{CH}_0(X)\to\mathrm{Alb}(X)(k)\otimes\mathbf{Q}$. From now on, $\sim$ denotes an adequate equivalence relation on algebraic cycles which is, when restricted to 0-cycles, coarser than albanese equivalence on 0-cycles. For instance, $\sim$ could be homological equivalence, algebraic equivalence, smash-nilpotent equivalence, Abel-Jacobi equivalence (when $k=\mathbf{C}$) or any intersection thereof. By Remark 2.4 applied to $\sim$ instead of homological equivalence, given a motive M, we may consider cycles that are ~ 0 on M and the notation $\mathrm{CH}_i(M)_\sim$ is unambiguous.

Theorem 3.11. *Let $f:N\to M=(X,p,n)$ be a morphism of motives over k. Assume that*

- *there exists an integer l such that the induced maps $(f_\Omega)_*:\mathrm{CH}_i(N_\Omega)_\sim\to\mathrm{CH}_i(M_\Omega)_\sim$ are surjective for all $i<l$;*
- *N is finite-dimensional.*

Then M splits as $Q\oplus R(l)$, where

- *Q is isomorphic to a direct summand of $N\oplus\bigoplus_{i=n}^{l-1}\mathfrak{h}(C_i)(i)$ for some curves $C_n,\dots,C_{l-1}$;*
- *R is isomorphic to a direct summand of $\mathfrak{h}(Z)$ for some smooth projective variety Z of dimension at most $\dim X-l+n$ over k.*

If, in addition $M\cong M^\vee(d)$ for some d (e.g. $M=\mathfrak{h}(X)$ with $d=\dim X$), then M splits as $Q'\oplus R'(l)$, where Q' is isomorphic to a direct summand of $N\oplus N^\vee(d)\oplus\bigoplus_i\mathfrak{h}(C_i)(i)$ for some curves C_i, and where R' is isomorphic to a direct summand of $\mathfrak{h}(Z')$ for some smooth projective variety Z' of dimension at most $d-2l$ over k.

Proof. Up to working with each irreducible component of X separately, we may assume that X is irreducible. Up to replacing k with a field of definition of $f:N\to M$ which is finitely generated, we may assume that k is finitely generated and that Ω is not only a universal domain but also a universal domain over k. Let then $k(X)$ be the function field of X and pick an embedding $k(X)\subset\Omega$ which extends that of k. Let us write $M=M_1\oplus M_2$ as in Lemma 3.6, with respect to the morphism $f:N\to M$, so that M_1 is isomorphic to a direct summand of N and the composite morphism $N\to M\to M_2$ is numerically trivial. This latter morphism induces, after base-change to Ω, surjective maps $\mathrm{CH}_i(N_\Omega)_\sim\to\mathrm{CH}_i((M_2)_\Omega)_\sim$ for all

$i < l$. Thus, up to replacing M with M_2, we need only show, provided that f is numerically trivial, that M splits as $Q \oplus R(l)$ as in the theorem with Q isomorphic to a direct summand of $\bigoplus_{i=n}^{l-1} \mathfrak{h}(C_i)(i)$ for some curves $C_n, \ldots, C_{l-1}$.

We proceed by induction on $\dim X$. If $l \leq n$, there is nothing to prove. Up to twisting, we can assume that $n = 0$, and that $l > 0$. We thus have a surjection $\mathrm{CH}_0(N_\Omega)_\sim \to \mathrm{CH}_0(M_\Omega)_\sim = \mathrm{CH}_0(X_\Omega, p_\Omega)_\sim$. By Bertini, let $\iota : C \to X$ be a smooth linear section of dimension 1 of X. By the Lefschetz hyperplane theorem, the induced map $\mathrm{Alb}_C \to \mathrm{Alb}_X$ is surjective. By functoriality of the albanese map, we find that $h := f \oplus g : N \oplus \mathfrak{h}(C) \to M$ induces a surjective map $(f_\Omega \oplus g_\Omega)_* : \mathrm{CH}_0(N_\Omega \oplus \mathfrak{h}(C_\Omega)) \to \mathrm{CH}_0(M_\Omega)$, where $g := p \circ \Gamma_\iota : \mathfrak{h}(C) \to M$. Lemma 3.1 then implies that $(h_{k(X)})_* : \mathrm{CH}_0(N_{k(X)} \oplus \mathfrak{h}(C_{k(X)})) \to \mathrm{CH}_0(M_{k(X)})$ is surjective.

Let us now write $M = M' \oplus M''$ as in Lemma 3.6, with respect to the morphism $g : \mathfrak{h}(C) \to M$, so that M' is isomorphic to a direct summand of $\mathfrak{h}(C)$ and the composite morphism $f \oplus g : N \oplus \mathfrak{h}(C) \to M \to M''$ is numerically trivial. This latter morphism induces, after base-change to $k(X)$, a surjective map $\mathrm{CH}_0(N_{k(X)} \oplus \mathfrak{h}(C_{k(X)})) \to \mathrm{CH}_0(M''_{k(X)})$. Lemma 3.10 implies that $\mathrm{CH}_0(M''_{k(X)}) = 0$. We then deduce, thanks to Lemma 3.7, that there exist a smooth projective variety Z of dimension at most $\dim X - 1$ over k and an idempotent $q \in \mathrm{End}(\mathfrak{h}(Z))$ such that $M'' \cong (Z, q, 1)$. Now the motive (Z, q) is such that there is a numerically trivial morphism $N(-1) \to (Z, q)$ inducing surjective maps $(f_\Omega)_* : \mathrm{CH}_i(N(-1)_\Omega)_\sim \to \mathrm{CH}_i((Z, q)_\Omega)_\sim$ for all $i < l - 1$. We can thus conclude by the induction hypothesis that $M'' = (Z, q, 1)$ splits as $P \oplus R(l)$, where P is isomorphic to a direct summand of $\bigoplus_{i=1}^{l-1} \mathfrak{h}(C_i)(i)$ for some curves $C_1, \ldots, C_{l-1}$ and where R is isomorphic to a direct summand of the motive of a smooth projective variety of dimension at most $\dim X - l$.

Assume now that there is an isomorphism $M \cong M^\vee(d)$. Let $M = Q \oplus R(l)$ be a decomposition as in the first part of the theorem with $R = (Z, r)$. The isomorphism $M \cong M^\vee(d)$ gives a morphism $N \to M \cong M^\vee(d) \to R^\vee(d) \cong (Z, {}^t r, d - l - \dim Z)$ which satisfies the assumptions of the first part of the theorem. Thus, there exist a direct summand S of $N \oplus \bigoplus_j \mathfrak{h}(D_j)(j)$ for some curves D_j, a smooth projective variety Z' of dimension at most $\dim Z - l + (d - l - \dim Z) = d - 2l$ over k and an idempotent $q \in \mathrm{End}(\mathfrak{h}(Z'))$ such that the motive $R^\vee(d)$ splits as $S \oplus (Z', q, l)$. Let us then define $Q' := Q \oplus S^\vee(d)$. This is a direct summand of $N \oplus N^\vee(d) \oplus \mathfrak{h}(C_i)(i)$ for some curves C_i and M is isomorphic to $Q' \oplus (Z', {}^t q, l)$ with $\dim Z' \leq d - 2l$. □

Proof of Theorems 4 and 5. If M is of abelian type over Ω, then a twist of M is isomorphic to a direct summand of the motive of a product of curves $\prod C_i$. Therefore, clearly, $\mathrm{CH}_*(M)_\sim$ is spanned by $\mathrm{CH}_*\left(\prod C_i\right)_\sim$ for any adequate equivalence relation. Conversely, assume that there is a finite-dimensional motive N over Ω (e.g. the motive of a product of curves C_i over Ω) and a correspondence $f : N \to M$ such that $f_* : \mathrm{CH}_*(N)_\sim \to \mathrm{CH}_*(M)_\sim$ is surjective, then Theorem 3.11 shows that M is isomorphic to a direct summand of $N \oplus \bigoplus_j \mathfrak{h}(D_j)(j)$ for some curves D_j, so that if N is the motive of a product of curves then M is of abelian type. As for the proof of Theorem 5, let X be a smooth projective variety of dimension $d = 2n$ or $2n+1$ over Ω and let $f : N \to \mathfrak{h}(X)$ be a morphism inducing surjections $(f_\Omega)_* : \mathrm{CH}_i(N_\Omega)_\sim \to \mathrm{CH}_i(X_\Omega)_\sim$ for all $i \leq n - 1$. Then, since $\mathfrak{h}(X) \cong \mathfrak{h}(X)^\vee(d)$, by Theorem 3.11 $\mathfrak{h}(X)$ splits as $Q \oplus R(n-1)$, where Q is a direct summand of $N \oplus N^\vee(d)$ and where R is a direct summand of the motive of a variety of dimension at most one. In particular, if N is of abelian type, then $\mathfrak{h}(X)$ is of abelian type. □

Proof of Theorem 6. We actually prove the following more general statement. Let X be a smooth projective variety of dimension $2n - 1$ or $2n$ over k and let $f : N \to \mathfrak{h}(X)$ be a morphism inducing surjective maps $(f_\Omega)_* : \mathrm{CH}_i(N_\Omega)_\sim \to \mathrm{CH}_i(X_\Omega)_\sim$ for all $i \leq n - 2$. Assume that N has a Künneth decomposition and is finite-dimensional. Then X has a Chow–Künneth decomposition. Indeed, by Theorem 3.11, $\mathfrak{h}(X)$ splits as a direct sum $Q \oplus R(n-2)$, where Q is the direct summand of $N \oplus N^\vee(d)$, a motive that is finite-dimensional and satisfies the Künneth standard conjecture, and where M_2 is the direct summand of the motive of a curve or surface depending on the parity of $\dim X$. The motive Q has a Chow–Künneth decomposition because it is finite-dimensional and has a Künneth decomposition (see the proof of Theorem 1.8). The motive $R(n-2)$ has a Chow–Künneth decomposition by [20, Theorem 3.5]. □

3.5. **Applications to Chow–Künneth decompositions.** As a straightforward consequence of Theorem 3.11, we deduce finite-dimensionality of the motive of certain 3-folds and the existence of Chow–Künneth decomposition of certain 3- and 4-folds. Combined with results of [22], we also check the validity of some of Murre's conjectures in those cases. The most general statement is the following.

Theorem 3.12. *Let X be a smooth projective variety over a field k. Assume that there exist a smooth projective variety Y whose motive is of abelian type, as well as a correspondence $\Gamma \in \mathrm{CH}_{\dim Y}(Y \times X)$ such that the induced map $(\Gamma_\Omega)_* : \mathrm{CH}_0(Y_\Omega) \to \mathrm{CH}_0(X_\Omega)$ is surjective. Then,*
- *if $\dim X \leq 4$, X has a Chow–Künneth decomposition which satisfies Murre's conjecture (B);*
- *if $\dim X \leq 4$ and $\dim Y \leq 3$, X satisfies Murre's conjecture (D);*
- *if $\dim X \leq 3$, X is finite-dimensional;*
- *if $\dim X \leq 3$ and $\dim Y \leq 2$, X satisfies Murre's conjecture (C).*

Proof. By Theorem 3.11, there is a decomposition $\mathfrak{h}(X) = Q \oplus R(1)$, where Q is isomorphic to a direct summand of $\mathfrak{h}(Y) \oplus \mathfrak{h}(Y)(\dim X - \dim Y)$ and where $R(1)$ is isomorphic to a direct summand of $\mathfrak{h}(Z)(1)$ for some variety Z of dimension at most two. By assumption $\mathfrak{h}(Y)$ is of abelian type, so that Q is also of abelian type and hence has a Chow–Künneth decomposition. The motive $R(1)$ has a Chow–Künneth decomposition which satisfies Murre's conjectures (B) and (D) by [20, Theorem 3.5]. Hence X has a Chow–Künneth decomposition. It only remains to show that a motive P which is of abelian type and which is isomorphic to the direct summand of the motive of a variety of dimension d, satisfies Murre's conjecture (B) if $d \leq 4$, satisfies Murre's conjecture (D) if $d \leq 3$, and satisfies Murre's conjecture (C) if $d \leq 2$. This is contained, respectively, in Theorem 4.5, Theorem 4.8 and Proposition 3.1 of [22]. □

Let X be a smooth projective variety over k. Assume that there exist smooth projective varieties $X_0, \ldots, X_{N-1}, X_N = X$ such that, for all $n \leq N$, X_n and X_{n-1} satisfy one of the following properties.

(1) There is a dominant rational map $\varphi_n : X_{n-1} \dashrightarrow X_n$;

(2) There is a dominant morphism $\psi_n : X_n \to X_{n-1}$ whose generic fiber has trivial Chow group of zero-cycles after base-change to a universal domain.

Proposition 3.13. *Let X be as above. Then there is a correspondence $\Gamma \in \mathrm{CH}^{\dim X}(X_0 \times X)$ such that $(\Gamma_\Omega)_* : \mathrm{CH}_0((X_0)_\Omega) \to \mathrm{CH}_0(X_\Omega)$ is surjective.*

Proof. Let Y and Z be two smooth projective varieties over k. On the one hand, a dominant rational map $\varphi : Y \dashrightarrow Z$ induces a surjection $\varphi_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(Z)$. On the other hand, given a dominant morphism $\psi : Z \to Y$ as in (2) and given an ample class $h \in \mathrm{CH}^1(Z)$, there is a surjection $h^{\dim Z - \dim Y} \circ \psi^* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(Z)$. Here $h^{\dim Z - \dim Y}$ denotes the $(\dim Z - \dim Y)$-fold intersection with h; see [20, Theorem 1.3]. This proves the proposition. □

An immediate consequence of Theorem 3.12 and Proposition 3.13 is the following theorem.

Theorem 3.14. *Let X be as above and assume X_0 is a product of curves. Then,*
- *if $\dim X \leq 4$, X has a Chow–Künneth decomposition which satisfies Murre's conjecture (B);*
- *if $\dim X \leq 3$, X is finite-dimensional and satisfies Murre's conjecture (D).* □

Example 3.15. Theorem 3.14 notably applies to any smooth projective variety over k which is rationally dominated by a product of curves. Thus a 3-fold rationally dominated by a product of curves is finite-dimensional and a 4-fold rationally dominated by a product of curves has a Chow–Künneth decomposition.

Example 3.16. Theorem 3.14 also applies to a smooth projective complex variety X whose tangent bundle T_X is nef, that is if the line-bundle $\mathcal{O}_{T_X}(1)$ on $\mathbb{P}(T_X)$ is nef. Indeed, a theorem of Demailly–Peternell–Schneider [7] says that there is an étale cover $X' \to X$ of X such that the Albanese morphism $X' \to \mathrm{Alb}_{X'}$ is a smooth morphism, whose fibres are smooth Fano varieties. Smooth Fano varieties are rationally connected and it follows that the generic fibre of $X' \to \mathrm{Alb}_{X'}$ is rationally connected and hence has trivial Chow group after base-change to a universal domain. Note that the case of a 3-fold with a nef tangent bundle was taken care of, by a different method relying on a stronger classification result due to Campana–Peternell, in [11].

3.6. **Application to smash-nilpotent 1-cycles.** R. Sebastian [19] proved that every 1-dimensional cycle on a product of curves which is numerically trivial is smash-nilpotent. Theorem 3.11 makes it possible to extend Sebastian's result.

Theorem 3.17. *Let C_i be smooth projective curves and let $f : \mathfrak{h}(\prod_i C_i) \to M$ be a morphism of effective motives over k. Assume that $(f_\Omega)_* : \mathrm{CH}_0(\prod_i C_{i,\Omega}) \to \mathrm{CH}_0(M_\Omega)$ is surjective. Then smash-nilpotence equivalence agrees with numerical equivalence on 1-cycles on M.*

Proof. By Theorem 3.11, there is a splitting $M \cong Q \oplus R(1)$, where Q is a direct summand of $\mathfrak{h}(\prod_i C_i)$ and where R is effective. Therefore, $\mathrm{CH}_1(M)_{\mathrm{num}}$ is spanned via the action of correspondences by $\mathrm{CH}_1(\prod_i C_i)_{\mathrm{num}} \oplus \mathrm{CH}_0(R)_{\mathrm{num}}$. Numerically trivial cycles in $\mathrm{CH}_0(R)$ are clearly smash-nilpotent and, by Sebastian's theorem, numerically trivial cycles in $\mathrm{CH}_1(\prod_i C_i)$ are smash-nilpotent. The theorem is thus proved. □

3.7. **A splitting result without finite-dimensionality.** Let $f : N \to M$ be a morphism of motives over k such that $(f_\Omega)_* : \mathrm{CH}_i(N_\Omega) \to \mathrm{CH}_i(M_\Omega)$ is surjective for all i and assume that N is finite-dimensional. An immediate corollary to Theorem 3.11 (or rather its proof) is that f has a right-inverse. In this paragraph, we show that, in the situation above, the assumption that N be finite-dimensional can be dropped.

Theorem 3.18. *Let $f : N \to M$ be a morphism of motives over k such that $(f_\Omega)_* : \mathrm{CH}_*(N_\Omega) \to \mathrm{CH}_*(M_\Omega)$ is surjective. Then f has a right-inverse.*

Proof. Up to replacing k with a field of definition of f which is finitely generated, we may assume that Ω is a universal domain over k. Let's write $M = (X, p, n)$ and $N = (Y, q, m)$. Let Z be an irreducible variety over k and let's define, as in [10, proof of Lemma 1], an action $f \otimes Z : \mathrm{CH}_*(Y \times Z) \to \mathrm{CH}_*(X \times Z)$ by $(f \otimes Z)\alpha := (p_{X,Z})_*(p_{Y,Z}^*\alpha \cdot_{p_{Y,X}} f)$. Here, $p_{X,Y}$, $p_{Y,Z}$ and $p_{X,Z}$ are the natural projections, and the product "$\cdot_{p_{Y,X}}$" is Fulton's refined intersection [9, §8] with respect to the projection $p_{Y,X} : Y \times X \times Z \to Y \times X$. By [9, Corollary 8.1.2], it can be checked that, when Z is smooth (projective), we have $(f \otimes Z)\alpha = (f, \mathrm{id}_Z)_*\alpha$.

We are going to prove by induction on $\dim Z$ that $f \otimes Z : \mathrm{CH}_*(Y \times Z) \to \mathrm{CH}_*(X \times Z)$ has same image as $p \otimes Z : \mathrm{CH}_*(X \times Z) \to \mathrm{CH}_*(X \times Z)$. Let K be the function field of Z over k. We have a diagram with exact rows

$$\begin{tikzcd} \bigoplus \mathrm{CH}_*(Y \times D) \arrow[r] \arrow[d, "\bigoplus f\otimes D"] & \mathrm{CH}_*(Y \times Z) \arrow[r] \arrow[d, "f\otimes Z"] & \mathrm{CH}_*(Y_K) \arrow[r] \arrow[d, "f\otimes K"] & 0 \\ \bigoplus \mathrm{CH}_*(X \times D) \arrow[r] & \mathrm{CH}_*(X \times Z) \arrow[r] & \mathrm{CH}_*(X_K) \arrow[r] & 0. \end{tikzcd}$$

The direct sums are taken over all closed irreducible codimension-1 subschemes of Z and the left horizontal arrows are induced by the natural inclusions. Moreover, the diagram is commutative. The left-hand square is commutative by the projection formula [9, Proposition 8.1.1.(c)] and the right-hand square is commutative by [9, Theorem 6.4] and by [9, Proposition 8.1.1.(d)] which is valid for "regular imbedding" replaced by "l.c.i morphism" thanks to [9, Proposition 6.6.(c)]. By assumption and by Lemma 3.1, for all $\alpha \in \mathrm{CH}_*(X_K)$, there is a $\beta \in \mathrm{CH}_*(Y_K)$ such that $(p \otimes K)\alpha = (f \otimes K)\beta$. By induction, there is also, for all irreducible divisors D in Z and all $\gamma \in \mathrm{CH}_*(X \times D)$, a cycle $\delta \in \mathrm{CH}_*(Y \times D)$ such that $(p \otimes D)\gamma = (f \otimes D)\delta$. Therefore, by a simple diagram-chase and by commutativity of the similar diagram involving $\bigoplus p \otimes D$, $p \otimes Z$ and $p \otimes K$, we see that the middle vertical map has image coinciding with the image of $p \otimes Z$.

Thus, $(f \times \mathrm{id}_Z)_* : \mathrm{CH}_*(N \otimes \mathfrak{h}(Z)) \to \mathrm{CH}_*(M \otimes \mathfrak{h}(Z))$ is surjective for all smooth projective varieties Z over k. By Kimura [15, Lemma 6.8], it follows that f has a right-inverse. □

3.8. **Homological motives and splittings.** Theorem 3.18 can be thought of as an analogue modulo rational equivalence of Theorem 3.22 below which is concerned with motives modulo homological equivalence. The arguments used in this section go back to André [1] and improve slightly on the results of Arapura [3]. The difference with [3] is that we are able to ignore the middle cohomology of X; see Proposition 3.19.

Recall [16] that, in characteristic zero, for a smooth projective variety, the standard conjectures reduce to the Lefschetz standard conjecture. Given a smooth projective variety X of dimension d over k, we write $H^i(X)$ for the Betti cohomology of X with rational coefficients and we write $H_i(X)$ for $H^{2d-i}(X)$.

Proposition 3.19. *Assume that* $\mathrm{char}\, k = 0$. *Let X and Y be smooth projective varieties over k such that there exists a morphism $f : \bigoplus_{m,n} \mathfrak{h}(Y)^{\otimes n}(m) \to \mathfrak{h}(X)$ that induces a surjection on $H_i(X)$ for all $i > \dim X$. Assume that Y satisfies the standard conjectures. Then X also satisfies the standard conjectures.*

Proof. If Y satisfies the standard conjectures, then so do all of its powers. For varieties satisfying the standard conjectures, the Lefschetz involution $*_L$ on cohomology is induced by a correspondence; see [1] and [16]. Let's fix a polarization on Y and let's endow Y^n with the product polarization. Let $f_{m,n}$ be the restriction of f to $\mathfrak{h}(Y)^{\otimes n}(m)$ and, for $i > \dim X$, consider the composite map

$$\varphi_{m,n} : H^i(X) \xrightarrow{f^*_{m,n}} H^{i-2m}(Y^n) \xrightarrow{L} H_{i-2m}(Y^n) \xrightarrow{*_L} H_{i-2m}(Y^n) \xrightarrow{(f_{m,n})_*} H_i(X).$$

Here the map L is the Lefschetz isomorphism with respect to the polarization on Y^n. It is induced by a correspondence by assumption. Since $H_{i-2m}(Y^n)$ is polarized with respect to the bilinear form $\langle L^{-1}-, *_L- \rangle$, we obtain that $\varphi_{m,n}$ has same image as $(f_{m,n})_*$. Thus, there exists $h : \mathfrak{h}(X)(-d) \to \bigoplus_{m,n} \mathfrak{h}(Y)^{\otimes n}(m)$ such that $f \circ h$ induces a surjective map $H^i(X) \to H_i(X)$. It is then classical to deduce, thanks to the theorem of Cayley–Hamilton, that the inverse to the Lefschetz isomorphism $H_i(X) \to H^i(X)$ is induced by a correspondence. Thus, X satisfies the Lefschetz standard conjecture and hence the standard conjectures. □

Remark 3.20. In some cases, Proposition 3.19 can be slightly improved. Indeed, if Y is a 3-fold, although Y might not be known to satisfy the Lefschetz standard conjecture, it is known that the Lefschetz involution $*_L$ on $H^3(Y)$ is algebraic. It is thus possible to prove the following statement. Let X be a smooth projective variety of dimension d defined over a subfield of $\mathbf{C}$ and let $i > d$. Assume that $H_i(X) = \widetilde{N}^{\lfloor i/2 \rfloor -1} H_i(X)$. Here $\widetilde{N}$ denotes the niveau filtration defined in [22]. In other words assume that

- if i is odd, then there exist a threefold Y_i and a correspondence $\Gamma_i \in \mathrm{CH}_{(i+3)/2}(Y_i \times X)$ such that $(\Gamma_i)_* : H_3(Y_i) \to H_i(X)$ is surjective, and
- if i is even, then there exist a surface Z_i and a correspondence $\Gamma_i \in \mathrm{CH}_{(i+2)/2}(Z_i \times X)$ such that $(\Gamma_i)_* : H_2(Z_i) \to H_i(X)$ is surjective.

Then the inverse to the Lefschetz isomorphism $L^{d-i} : H_i(X) \to H^i(X)$ is induced by an algebraic correspondence. □

A combination of Proposition 3.5 and Proposition 3.19 gives a criterion on $\mathrm{CH}_0(X)$ for a complex fourfold X to satisfy the Lefschetz standard conjecture.

Proposition 3.21. *Let X be a smooth projective variety of dimension $d \leq 4$ over $\mathbf{C}$. Assume that there exist a smooth projective variety Y which satisfies Grothendieck's Lefschetz standard* *conjecture, as well as a correspondence* $\Gamma \in \mathrm{CH}_{\dim Y}(Y \times X)$ *such that* $\Gamma_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ *is surjective. Then* X *satisfies the Lefschetz standard conjecture.*

Proof. By Lemma 3.2 and Proposition 3.5, we get that

$$\Delta_X = \Gamma \circ \Gamma_1 + \Gamma_2 \in \mathrm{CH}_d(X \times X),$$

for some correspondence $\Gamma_1 \in \mathrm{CH}_d(X \times Y)$ and some correspondence $\Gamma_2 \in \mathrm{CH}_d(X \times X)$ that factors as $\Gamma_2 = \alpha \circ \beta$ for some smooth projective variety Z and for some $\beta \in \mathrm{CH}^d(X \times Z)$ and some $\alpha \in \mathrm{CH}_d(Z \times X)$. By looking at the action of Δ_X on $H^i(X)$, we see that

$$H^i(X) = \Gamma_1^* H^i(Y) + \beta^* H^{i-2}(Z).$$

This settles the proposition thanks to Proposition 3.19. □

Theorem 3.22. *Assume that* $\mathrm{char}\, k = 0$*. Let* X *and* Y *be smooth projective varieties over* k *such that there exists a morphism* $f : \bigoplus_{m,n} \mathfrak{h}(Y)^{\otimes n}(m) \to \mathfrak{h}(X)$ *that induces a surjection on* $H_i(X)$ *for all* i*. Assume that* Y *satisfies the standard conjectures. Then* f *has a right-inverse modulo homological equivalence. If, moreover,* $\mathfrak{h}(X)$ *is finite-dimensional, then* f *has a right-inverse.*

Proof. By Proposition 3.19, X satisfies the standard conjectures. By André [1, §4], the full, thick and rigid sub-category of homological motives spanned by the motives of X and Y is semi-simple. Therefore, f has a right-inverse modulo homological equivalence. The last statement follows from Lemma 3.6. □

Proof of Theorem 8. This is a direct consequence of Theorem 3.22 because a motive of abelian type is a direct factor of the motive of a product of curves and because a product of curves satisfies the standard conjectures. □

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