

A COMPLETE ANSWER TO ALBANESE BASE CHANGE FOR INCOMPLETE VARIETIES

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ABSTRACT. Albanese varieties provide a standard tool in algebraic geometry for converting questions about varieties in general, to questions about Abelian varieties. A result of Serre provides the existence of an Albanese variety for any geometrically connected and geometrically reduced scheme of finite type over a field, and a result of Grothendieck–Conrad establishes that Albanese varieties are stable under base change of field provided the scheme is, in addition, proper. A result of Raynaud shows that base change can fail for Albanese varieties without this properness hypothesis. In this paper we show that Albanese varieties of geometrically connected and geometrically reduced schemes of finite type over a field are stable under separable field extensions. We also show that the failure of base change in general is explained by the L/K -image for purely inseparable extensions L/K .

1. INTRODUCTION

Consider a K -pointed scheme (V, v) of finite type over a field K . A *pointed Albanese variety* for this object, if it exists, consists of an abelian variety $\mathrm{Alb}_{V/K}$ over K and a pointed K -morphism

$$a_v : V \longrightarrow \mathrm{Alb}_{V/K}$$

taking v to $0_{\mathrm{Alb}_{V/K}}$, which is initial for pointed K -morphisms from V to abelian varieties. More generally, if V is a scheme of finite type over a field K , an *Albanese torsor* for V , if it exists, is a morphism $a : V \rightarrow \mathrm{Alb}_{V/K}^1$ to a torsor under an abelian variety $\mathrm{Alb}_{V/K}$, the *Albanese variety* of V , which is initial for morphisms to torsors under abelian varieties. For complex projective manifolds, Albanese varieties were a classical, and central, tool in algebraic geometry; they provide a method of converting geometric questions about a variety into related questions about abelian varieties, and extend the techniques used for studying smooth projective curves via the Jacobian and the Abel map to varieties of higher dimension. In 1960, Serre showed such an Albanese variety exists for any geometrically connected and geometrically reduced scheme of finite type over any field [Ser60] (see Theorem 2.2 and Remark 3.3), thereby allowing for the extension of these classical techniques to this setting. Other treatments were considered at about the same time [NS52, Che60].

After existence, perhaps the most important structural question is base change. In the special case where V is in addition assumed to be proper and geometrically normal over K , Grothendieck [Gro62] identified $\mathrm{Alb}_{V/K}$ with $((\mathrm{Pic}_{V/K}^0)_{\mathrm{red}})^\vee$, the dual abelian variety of the reduction of $\mathrm{Pic}_{V/K}^0$. In this setting, since the formation of the Picard scheme is compatible with base change, it follows that if L/K is any field extension, then the canonical map $\beta_{V,L/K} : \mathrm{Alb}_{V/L} \rightarrow (\mathrm{Alb}_{V/K})_L$ is an isomorphism. However, without the hypothesis that V be proper, it is known that base change can fail:

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Example 1.1 (Raynaud: Albanese are not stable under base change). Let L/K be a finite purely inseparable field extension, and let A/L be an abelian variety. Let $G = R_{L/K}A$ be the Weil restriction, which is a smooth connected commutative algebraic K -group with $\dim G = [L : K] \cdot \dim A$ [DG70, Exp. XVII, App. II, Prop. 5.1]. There is a short exact sequence of L -groups [DG70, Exp. XVII, App. II, Prop. 5.1]

$$1 \longrightarrow U \longrightarrow G_L \xrightarrow{u} A \longrightarrow 0,$$

where U is a smooth connected and unipotent linear algebraic L -group, exhibiting A as the Albanese of G_L [Bri17, Ex. 4.2.7]. As we will see in Theorem 5.6, $\text{Alb}_{G/K} = \text{im}_{L/K} A$, the L/K -image of A (see §5). If A is defined over K , then $(\text{Alb}_{G/K})_L = (\text{im}_{L/K} A)_L = A = \text{Alb}_{G_L/L}$. However, if A is *not* defined over K , then G is not an extension of an abelian variety over K by a smooth algebraic K -group [DG70, Exp. XVII, App. II, Cor.(ii) to Prop. 5.1]¹; Brion uses this to show that in this case $(\text{Alb}_{G/K})_L \neq A = \text{Alb}_{G_L/L}$ [Bri17, Ex. 4.2.7], providing an example where base change fails.

Despite this failure of base change, there are a few striking features of this example. First, the field extension is purely inseparable, and second, it happens that $\text{Alb}_{G/K} \cong \text{im}_{L/K} \text{Alb}_{G_L/L}$. The main result of this paper shows that these observations about the Raynaud example represent the general situation. In other words, it is the inseparability of L/K , and not the improperness of G , that drives the failure of base change in the Raynaud example. Indeed, for a purely inseparable extension, the Albanese is the L/K -image of the Albanese of the base change:

Theorem. *Let V be a geometrically connected and geometrically reduced scheme of finite type over a field K , and let L/K be an extension of fields.*

(A) (Theorem 5.1) *If L/K is separable, then $\text{Alb}_{V_L/L} \cong (\text{Alb}_{V/K})_L$.*

(B) (Theorem 5.6) *If L/K is a purely inseparable extension, then $\text{Alb}_{V/K} \cong \text{im}_{L/K} \text{Alb}_{V_L/L}$.*

Recall that any field extension L/K factors as $L/L'/K$ with L'/K separable and L/L' purely inseparable (see, e.g., §4.1), so that the theorem above completely describes base change for arbitrary field extensions. We note that our proof of Theorem (A) relies on Theorem (B), due to our use of de Jong’s regular alterations, and the fact that these regular alterations are only smooth over a purely inseparable extension of the base field. Theorem (A) generalizes the abelian (as opposed to semi-abelian) case of [Wit08, Cor. A.5], which requires V to be an open subset of a smooth proper geometrically integral scheme over K ; it also generalizes [Moc12, Prop. A.3(i)] to the case of non-perfect base fields, and non-algebraic field extensions. Theorem (B) completely explains the behavior studied in [Bri17, Ex. 4.2.7]. Theorem (B) implies the weaker statement that, for a purely inseparable extension L/K , $\text{Alb}_{V_L/L}$ and $(\text{Alb}_{V/K})_L$ differ by a purely inseparable isogeny. This, as well as Theorem (A), has also been secured by Schröer [Sch23, Thm. 6.1] for separated schemes, under a hypothesis on the ring of global functions.

We note that the geometrically connected hypothesis for V in the theorem is necessary, as it is necessary for the existence of an Albanese (Corollary 2.5). In contrast, the geometric reducedness hypothesis in the theorem is more subtle (Example 2.7, Example 2.9, Example 3.4), although one can at least say that the reducedness of $\Gamma(V, \mathcal{O}_V)$ is necessary in Theorem (A), and the geometric reducedness of $\Gamma(V, \mathcal{O}_V)$ is necessary in Theorem (B), as the reducedness (but *not* geometric reducedness) of $\Gamma(V, \mathcal{O}_V)$ is necessary for the existence of an Albanese (Corollary 2.8, Example 2.11). See also Proposition 2.12(i) where we summarize some necessary conditions for the existence of an Albanese.

¹Note that the statement of [DG70, Exp. XVII, App. II, Cor.(ii) to Prop. 5.1] has the implicit hypothesis that A not be defined over K : the proof uses this hypothesis to conclude that the linear algebraic subgroup of G in the proof is strictly larger than U when base changed to L ; moreover, the hypothesis that A not be defined over K is required in the statement of the corollary, as this example shows.

In light of the modern treatment of Albanese following Grothendieck, our definition of the Albanese, and our subsequent focus on base change of field may seem slightly archaic. Indeed, Grothendieck would require that the Albanese of (V, v) satisfy the stronger condition that for any morphism of schemes $S \rightarrow K$, and any pointed S -morphism $f : V_S \rightarrow A$ to an abelian scheme A/S sending v_S to $\mathbf{0}_A$, there exists a unique S -homomorphism $g : (\text{Alb}_{V/K})_S \rightarrow A$ such that $g \circ a_S = f$. Such an Albanese would automatically satisfy arbitrary base change. Grothendieck and Conrad show that for a pointed *proper* geometrically connected and geometrically reduced scheme V of finite type over K , the Albanese as defined here satisfies this stronger condition. Because of Theorem (B), the best possible result along these lines without the properness hypothesis is:

Theorem (C). (Theorem 6.3) *Let (V, v) be a K -pointed (geometrically) connected and geometrically reduced scheme of finite type over a field K . Then for any (inverse limit of) smooth morphism of schemes $S \rightarrow \text{Spec } K$, and any pointed S -morphism $f : V_S \rightarrow A$ to an abelian scheme A/S sending v_S to $\mathbf{0}_A$, there exists a unique S -homomorphism $g : (\text{Alb}_{V/K})_S \rightarrow A$ such that $g \circ a_S = f$.*

We refer to Theorem 6.3 for the Albanese torsor version of Theorem (C) valid for (not necessarily K -pointed) geometrically connected and geometrically reduced scheme of finite type over a field K .

We originally worked out these arguments as part of our development of a functorial approach to regular homomorphisms [ACMV23]. Indeed, Theorem (A) originally appeared as an appendix to *op. cit.*. However, since it seemed that these results on Albanese varieties might be useful to a wider audience, we decided to make them available in a separate document. Since that preprint originally appeared, Laurent and Schröer have studied the existence of a relative Albanese for proper families [LS24]. Moreover, in the context of schemes over a field Schröer, using different techniques, has extended some of our results under the further hypothesis that the scheme be separated [Sch23]. Combining our Proposition 2.12(i) with [Sch23, Thm. p.2] provides necessary and sufficient conditions for the existence of an Albanese for a *separated* scheme of finite type over a field (Proposition 2.12(ii)). A formulation of Schröer's base change result [Sch23, Thm. p.4] can be found in Proposition 5.7.

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2. ALBANESE VARIETIES

2.1. Serre's existence theorem. Let V be a scheme of finite type over a field K . Recall that an *Albanese datum* for V consists of a triple

$$(\text{Alb}_{V/K}, \text{Alb}_{V/K}^1, a_{V/K} : V \rightarrow \text{Alb}_{V/K}^1) \quad (2.1)$$

with $\text{Alb}_{V/K}$ an abelian variety over K , $\text{Alb}_{V/K}^1$ a torsor under $\text{Alb}_{V/K}$ over K , and $a : V \rightarrow \text{Alb}_{V/K}^1$ a morphism of K -schemes which is initial, meaning that given any triple $(A, P, f : V \rightarrow P)$ with A an abelian variety over K , P a torsor under A over K , and $f : V \rightarrow P$ a morphism of K -schemes, there is a unique K -morphism $g : \text{Alb}_{V/K}^1 \rightarrow P$ making the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{a_{V/K}} & \text{Alb}_{V/K}^1 \\ & \searrow f & \downarrow \exists! g \\ & & P \end{array}$$

We will respectively call the three objects in this datum the *Albanese variety*, the *Albanese torsor*, and the *Albanese morphism* of V/K (although of course the torsor is itself a variety, too).

Remark 2.1. Recall that if A is an abelian variety over K and P is a torsor under A over K , then there is a natural isomorphism $A^\vee \xrightarrow{\cong} \text{Pic}_{P/K}^0$ (e.g., [Ols08, §2.1]). Moreover, if A and A' are abelian varieties over K , and P and P' are torsors under A and A' , respectively, then for any K -morphism $g : P \rightarrow P'$, there is a unique K -homomorphism $\phi : A \rightarrow A'$ making g equivariant, and moreover, $g(P)$ is a torsor under $\phi(A)$; more precisely, ϕ is the composition $A \xrightarrow{\cong} (\text{Pic}_{P/K}^0)^\vee \xrightarrow{(g^*)^\vee} (\text{Pic}_{P'/K}^0)^\vee \xrightarrow{\cong} A'$. In particular, in the definition of the Albanese data above, there is a unique K -homomorphism $\text{Alb}_{V/K} \rightarrow A$ making $g : \text{Alb}_{V/K}^1 \rightarrow P$ equivariant.

When V/K is equipped with a K -point $v : \text{Spec } K \rightarrow V$ over K , then one can define a *pointed Albanese variety and morphism*, by requiring all the maps in the previous paragraph to be pointed. This reduces to the following situation: a *pointed Albanese datum* for (V, v) is a pair

$$(\text{Alb}_{V/K}, a_{V/K, v} : (V, v) \rightarrow (\text{Alb}_{V/K}, \mathbf{0})) \quad (2.2)$$

where $\text{Alb}_{V/K}$ is an abelian variety, and $a_{V/K, v} : V \rightarrow \text{Alb}_{V/K}$ is a morphism of K -schemes taking v to the zero section $\mathbf{0} = \mathbf{0}_{\text{Alb}_{V/K}}$, which is initial, meaning that given any K -morphism $f : V \rightarrow A$ to an abelian variety A/K , taking v to the zero section $\mathbf{0}_A$, there is a unique K -homomorphism $g : \text{Alb}_{V/K} \rightarrow A$ making the following diagram of pointed K -morphisms commute:

$$\begin{array}{ccc} (V, v) & \xrightarrow{a_{V/K, v}} & (\text{Alb}_{V/K}, \mathbf{0}_{\text{Alb}_{V/K}}) \\ & \searrow f & \downarrow \exists! g \\ & & (A, \mathbf{0}_A) \end{array}$$

As we noted in the introduction, the existence of Albanese data in the case of complex projective manifolds is classical, while in the more general setting goes back essentially to Serre [Ser60]. We direct the reader to [Wit08, Thm. A.1 and p.836] for an exposition valid over an arbitrary field; the assertion there is made for V/K a geometrically integral scheme² of finite type over a field K , although the argument holds under the slightly weaker hypothesis that V is a geometrically connected and geometrically reduced scheme of finite type over K :

Theorem 2.2 (Serre). *Let V be a geometrically connected and geometrically reduced scheme of finite type over a field K . Then V admits Albanese data, and if V admits a K -point, then V admits pointed Albanese data.*

2.2. Necessity of geometric connectedness and geometric reducedness. We now briefly discuss the hypotheses in Theorem 2.2 that V be geometrically connected and geometrically reduced. In short, the geometric connectedness of V is necessary for V to admit Albanese data (Corollary 2.5), while the geometric reducedness of V is not (Example 2.7). The situation is summarized in Proposition 2.12(i).

The basic starting point is the following existence result, which states that given an abelian variety and a zero-dimensional scheme, there is a second abelian variety containing the zero dimensional scheme as a closed subscheme, and which admits no non-trivial homomorphisms from the first abelian variety:

Proposition 2.3. *Given an abelian variety A/K and a finite dimensional K -algebra R with each residue field a simple extension of K that is either separable or purely inseparable, there exists an abelian variety A'/K such that there is a closed embedding of K -schemes $\text{Spec } R \hookrightarrow A'$ and such that $\text{Hom}(A_{\overline{K}}, A'_{\overline{K}}) = 0$.*

²Note that Wittenberg uses the term variety for a scheme of finite type over a field [Wit08, p.807].

The proof of Proposition 2.3 is somewhat lengthy, and so to maintain the flow of the ideas in this subsection, we postpone the proof until §2.3. As an immediate consequence of Proposition 2.3, we have the following:

Theorem 2.4. *Let V be a scheme of finite type over a field K , and suppose there exists a nontrivial finite dimensional K -algebra R with $R \subseteq \Gamma(V, \mathcal{O}_V)$. Then given an abelian variety A/K , a torsor P/K under A , and a K -morphism $f : V \rightarrow P$, there exists a torsor P'/K under an abelian variety A'/K and a K -morphism $f' : V \rightarrow P'$ that does not factor through f . In particular, V does not admit an Albanese datum.*

Proof. The first claim is that R contains a nontrivial finite dimensional K -subalgebra $R' \subseteq R$ with residue fields that are separable extensions of K or simple purely inseparable extensions of K . Indeed, R being Artinian is a direct sum $R = \bigoplus_{i=1}^c R_i$ of finite local K -algebras $(R_i, \mathfrak{m}_i)$. For each i , we may and do choose a sub-algebra R'_i such that, if the residue field $\kappa_i := R_i/\mathfrak{m}_i$ is a nontrivial extension of K , then $\kappa'_i := R'_i/(\mathfrak{m}_i \cap R'_i)$ is a nontrivial extension which is either separable or simple and purely inseparable. Indeed, if κ_i is purely inseparable, let $K_i \subseteq \kappa_i$ be a sub- K -extension of degree $\text{char}(K_i)$, and otherwise let K_i be the separable closure of K in κ_i . In either case, let $R_i = \omega_i^{-1}(K_i)$, where $\omega_i : R_i \rightarrow \kappa_i$ is the projection. If $\kappa_i = K$ then we simply set $R'_i = R_i$. Finally we let $R' = \bigoplus_{i=1}^c R'_i$.

By virtue of Proposition 2.3, let A'/K be an abelian variety such that $\text{Hom}(A, A') = 0$, and such that there is a closed immersion $\text{Spec } R' \hookrightarrow A'$. Since $V \rightarrow \text{Spec } \Gamma(V, \mathcal{O}_V)$ is scheme-theoretically surjective (for any ring S we have $\text{Hom}(V, \text{Spec } S) = \text{Hom}(\text{Spec } \Gamma(V, \mathcal{O}_V), \text{Spec } S)$) and since the inclusion $R' \subseteq \Gamma(V, \mathcal{O}_V)$ induces a scheme-theoretic surjection (for affine schemes the scheme-theoretic image is determined by the factorization of a ring homomorphism into a surjection followed by an inclusion), we have that $V \rightarrow \text{Spec } R'$ is scheme-theoretically surjective.

Now let f' be the composition $f' : V \twoheadrightarrow \text{Spec } R' \hookrightarrow A'$. We obtain a diagram

$$\begin{array}{ccc} V & \twoheadrightarrow & \text{Spec } R' \\ f \downarrow & \searrow f' & \downarrow \\ P & \xrightarrow{\quad g \quad} & A', \end{array} \tag{2.3}$$

so that if we had a factorization $f' = g \circ f$, as indicated by the dashed arrow, then the morphism g would be an equivariant morphism over a K -homomorphism $\phi : A \rightarrow A'$ of abelian varieties. The hypothesis that $\text{Hom}(A, A') = 0$, would force ϕ to be the trivial map, so that g would be constant, with image a K -point of A' . The commutativity of the diagram would then imply $\text{Spec } R' \cong \text{Spec } K$, which we have assumed is not the case. $\square$

Corollary 2.5 (Geometric connectedness is necessary). *Suppose that V is a scheme of finite type over a field K , and V fails to be geometrically connected. Then V does not admit an Albanese datum.*

Proof. By Theorem 2.4, it suffices to show that $\Gamma(V, \mathcal{O}_V)$ contains a nontrivial finite K -algebra. Write $V = \coprod_{i=1}^c V_i$ as a disjoint union of connected K -schemes. Then there are idempotents $e_i \in \Gamma(V, \mathcal{O}_V)$ such that $\Gamma(V_i, \mathcal{O}_{V_i}) = \Gamma(V_i, \mathcal{O}_V) = e_i \Gamma(V, \mathcal{O}_V)$. Therefore, if V is disconnected, then $c > 1$ and $R := \bigoplus_{i=1}^c K e_i \cong K^{\oplus c} \subseteq \Gamma(V, \mathcal{O}_V)$ is a nontrivial finite K -algebra.

Otherwise, assume V is connected, but geometrically disconnected. Let L/K be a finite Galois extension such that V_L is disconnected (e.g., [GW20, Prop. 5.53]). As before, write $V_L = \coprod_{i=1}^c W_i$ as a disjoint union of $c > 1$ connected L -schemes, and let $e_1, \dots, e_c \in \Gamma(V_L, \mathcal{O}_{V_L})$ be the corresponding idempotents. Then $\text{Gal}(L/K)$ permutes the e_i , and acts transitively because V itself is connected. As before, we have $\bigoplus_{i=1}^c L e_i \cong L^{\oplus c} \subseteq \Gamma(V_L, \mathcal{O}_{V_L})$. In fact, letting $H \subseteq \text{Gal}(L/K)$ be the stabilizer of e_1 , then $c = |\text{Gal}(L/K)|/|H|$, and we can enumerate the components of V_L by the cosets $g_1 H, \dots, g_c H$ for some elements $g_1, \dots, g_c \in \text{Gal}(L/K)$. In this notation, we can take

$e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc., and then the action of $\text{Gal}(L/K)$ on $L^{\oplus c}$ is given by

$$g \cdot (\ell_{g_1 H}, \ell_{g_2 H}, \dots, \ell_{g_n H}) = (g \cdot \ell_{g^{-1} g_1 H}, g \cdot \ell_{g^{-1} g_2 H}, \dots, g \cdot \ell_{g^{-1} g_n H});$$

in other words, $\text{Gal}(L/K)$ permutes the components according to its action on the cosets of H , and then acts on the entries according to the action of the Galois group on L .

Now, because $\text{Gal}(L/K)$ does not fix e_1 , it follows that H is a proper subgroup of $\text{Gal}(L/K)$, and its fixed field L^H satisfies $[L^H : K] = |\text{Gal}(L/K)|/|H| > 1$. From the description of the action of $\text{Gal}(L/K)$ on $L^{\oplus c}$, it follows that there is a copy of L^H in $\bigoplus^c L \subseteq \Gamma(V_L, \mathcal{O}_{V_L})$ given by

$$\ell \mapsto (g_1 \cdot \ell, g_2 \cdot \ell, \dots, g_n \cdot \ell),$$

which is, by construction, invariant under the action of $\text{Gal}(L/K)$. Thus $\Gamma(V, \mathcal{O}_V)$, being the $\text{Gal}(L/K)$ -invariants of $\Gamma(V_L, \mathcal{O}_{V_L})$, contains a ring isomorphic to the finite nontrivial K -algebra L^H . $\square$

Remark 2.6 (Geometric connectedness of $\Gamma(V, \mathcal{O}_V)$ is necessary). We note that for a scheme V of finite type over a field K , since V is disconnected if and only if $\text{Spec } \Gamma(V, \mathcal{O}_V)$ is disconnected, we have that V fails to be geometrically connected if and only if $\text{Spec } \Gamma(V, \mathcal{O}_V)$ fails to be geometrically connected.

We now turn our attention to the geometric reducedness hypothesis in Theorem 2.2, which is more subtle. We first observe that since there are non-reduced schemes V of finite type over a field K such that $\Gamma(V, \mathcal{O}_V)$ does not admit any non-trivial finite K -subalgebra R (see e.g., Example 2.7), the proof of Corollary 2.5 cannot be used to rule out the existence of an Albanese in the case where V is non-reduced. In fact, there are non-reduced schemes that admit Albaneses:

Example 2.7 (Non-reduced scheme with an Albanese). Let K be a field, $H \subseteq \mathbb{P}_K^2$ be a line, and take $V = 2H \subseteq \mathbb{P}_K^2$. Then $\text{Alb}_{V/K} = \text{Spec } K$, $\text{Alb}_{V/K}^1 = \text{Spec } K$, and $a : V \rightarrow \text{Alb}_{V/K}^1$ is the structure map (of V as a K -scheme). Indeed, observe first that taking the long exact sequence in cohomology associated to $0 \rightarrow \mathcal{O}_{\mathbb{P}_K^2}(-2H) \rightarrow \mathcal{O}_{\mathbb{P}_K^2} \rightarrow \mathcal{O}_V \rightarrow 0$, one has that $\Gamma(V, \mathcal{O}_V) = K$. Therefore, as under the standard identification $\text{Hom}(V, \text{Spec } R) = \text{Hom}(R, \Gamma(V, \mathcal{O}_V))$ for a ring R , every morphism $V \rightarrow \text{Spec } R$ factors through the natural morphism $V \rightarrow \text{Spec } \Gamma(V, \mathcal{O}_V)$, then for any scheme-theoretically surjective morphism $V \rightarrow \text{Spec } R$, we have $R = K$. Now, since $V_{\text{red}} = \mathbb{P}_K^1$, then given any K -morphism $V \rightarrow P$ to a torsor P under an abelian variety A/K , the composition $\mathbb{P}_K^1 = V_{\text{red}} \hookrightarrow V \rightarrow P$ has set-theoretic image a K -point of P . Thus the scheme-theoretic image of V in P is an affine scheme $Z = \text{Spec } R$ where R is a finite K -algebra. Thus $Z = \text{Spec } K$ and we are done.

Nevertheless, Theorem 2.4 does give examples of non-reduced schemes that do not admit Albaneses:

Corollary 2.8 (Reducedness of $\Gamma(V, \mathcal{O}_V)$ is necessary). *Suppose that V is a scheme of finite type over a field K , and $\text{Spec } \Gamma(V, \mathcal{O}_V)$ fails to be reduced. Then V does not admit an Albanese datum.*

Proof. If $\text{Spec } \Gamma(V, \mathcal{O}_V)$ is non-reduced, then there exists a nilpotent element $r \in \Gamma(V_L, \mathcal{O}_{V_L})$ such that $r^n \neq 0$ and $r^{n+1} = 0$ for some natural number n . Then consider the subring $K[x]/(x^{n+1}) \cong R := K[r] \subseteq \Gamma(V, \mathcal{O}_V)$. We conclude using Theorem 2.4. $\square$

Example 2.9 (Non-reduced scheme with no Albanese). Let K be a field, let L/K be a nontrivial finite purely inseparable field extension, let Y be any scheme of finite type over L , and let $V = Y \times_K L$. Then $\Gamma(V, \mathcal{O}_V)$ contains $L \otimes_K L$ and thus has nontrivial nilpotents; by Corollary 2.8, V/L does not admit an Albanese datum. Similarly, if Y is any scheme of finite type over K and $V = Y_{K[\epsilon]/(\epsilon^2)}$, then V/K does not admit an Albanese datum. As a consequence, in contrast with

Example 2.7 where we saw that the nonreduced scheme $V = 2H \subseteq \mathbb{P}_K^2$ admits an Albanese datum over K , we have that the nonreduced scheme $\mathbb{P}_{K[\epsilon]/(\epsilon^2)}^1$ does *not* admit an Albanese datum over K , even as the reductions of both schemes are isomorphic to $\mathbb{P}_K^1$, assuming H is chosen with a K -point.

Example 2.10 (Reduced but geometrically non-reduced scheme with no Albanese). Let K be a field, let L/K be a nontrivial finite purely inseparable field extension, and let Y/L be a smooth irreducible variety. As a K -scheme, Y is reduced but not geometrically reduced; and the presence of the nontrivial finite K -algebra L in $\Gamma(Y, \mathcal{O}_Y)$ prevents Y from admitting an Albanese datum.

Allowing Y to be affine in Example 2.10 raises the possibility that the geometric reducedness of $\text{Spec } \Gamma(V, \mathcal{O}_V)$ is necessary. However, we have:

Example 2.11 (Geometric reducedness of $\Gamma(V, \mathcal{O}_V)$ is *not* necessary). We use a well-known example due to MacLane [ML39, p.384], which seems to be used frequently as an example of a geometrically nonreduced variety with interesting properties. Let $K = \mathbb{F}_p(t_1, t_2)$, let $S = K[x_1, x_2]/(t_1 x_1^p + t_2 x_2^p - 1)$, and define $V := \text{Spec } S$. Then V is integral and geometrically connected, but not geometrically reduced. Indeed, setting $L = \mathbb{F}_p(t_1^{1/p}, t_2^{1/p})$, we have $V_L = \text{Spec } L[x_1, x_2]/((t_1^{1/p} x_1 + t_2^{1/p} x_2 - 1)^p)$. One can check that K is algebraically closed in S , so that S , being reduced, admits no non-trivial finite dimensional K -subalgebras $R \subseteq S$. Differently put, one cannot use Theorem 2.4 to try to show that V does not admit an Albanese datum. In fact, we claim that the structure morphism $V \rightarrow \text{Spec } K$ is an Albanese datum. In other words, there are no nontrivial morphisms $V \rightarrow P$ to a torsor under an abelian variety over K . This follows from the fact that the reduction of V_L is a rational curve. More precisely, given a morphism $V \rightarrow P$ to a torsor under an abelian variety over K , if the image were zero dimensional, then since S admits no non-trivial finite dimensional K -subalgebras $R \subseteq S$, the image of V in P would have to be isomorphic to $\text{Spec } K$. If the image of V were 1-dimensional, then after base change to an algebraic closure $\bar{K}$, and considering the reduction of $V_{\bar{K}}$, one would have a non-trivial map from a rational curve to an abelian variety, which is not possible. Since $\dim V = 1$, we are done.

We summarize the situation in the following corollary, including the relation to Schröer [Sch23, Thm. p.2], which has the additional separated hypothesis:

Proposition 2.12. *Let V be a scheme of finite type over a field K .*

- (i) *If V admits an Albanese datum, then*
 - (a) *V is geometrically connected,*
 - (b) *$\text{Spec } \Gamma(V, \mathcal{O}_V)$ is geometrically connected,*
 - (c) *$\text{Spec } \Gamma(V, \mathcal{O}_V)$ is reduced, and*
 - (d) *K is algebraically closed in $\Gamma(V, \mathcal{O}_V)$.*
- (ii) (Schröer) *If, moreover, V is separated, then the converse holds. More precisely, for a separated scheme V of finite type over a field K , one has that V admits an Albanese datum if and only if $\text{Spec } \Gamma(V, \mathcal{O}_V)$ is connected and reduced, and K is algebraically closed in $\Gamma(V, \mathcal{O}_V)$.*

Remark 2.13. Note that if (d) holds in Proposition 2.12(i), then V (resp. $\text{Spec } \Gamma(V, \mathcal{O}_V)$) connected implies V (resp. $\text{Spec } \Gamma(V, \mathcal{O}_V)$) is geometrically connected.

Proof. Assuming V admits an Albanese datum, (a) and (b) are Corollary 2.5 and Remark 2.6. (c) is Corollary 2.8. The assertion (d) follows immediately from Theorem 2.4, since if K is not algebraically closed in $\Gamma(V, \mathcal{O}_V)$, then $\Gamma(V, \mathcal{O}_V)$ contains a finite nontrivial extension field of K .

Conversely, assume that V is a separated scheme V of finite type over a field K , $\text{Spec } \Gamma(V, \mathcal{O}_V)$ is connected and reduced, and K is algebraically closed in $\Gamma(V, \mathcal{O}_V)$. The conclusion that V admits an Albanese datum is then due to Schröer [Sch23, Thm. p.2], after one observes that for a

separated scheme V of finite type over a field K , with $\text{Spec } \Gamma(V, \mathcal{O}_V)$ connected and reduced, then V is naturally endowed with a scheme structure over the essential field of constants K' [Sch23, p.19], which is by construction finite over K ; i.e., there is a factorization $V \rightarrow \text{Spec } \Gamma(V, \mathcal{O}_V) \rightarrow \text{Spec } K' \rightarrow \text{Spec } K$. Therefore, if K is algebraically closed in $\Gamma(V, \mathcal{O}_V)$, then the essential field of constants for V is K . The assertion is then exactly the statement of [Sch23, Thm. p.2]. $\square$

2.3. Embedding zero-dimensional schemes in abelian varieties. While Proposition 2.3 is well suited to proving Theorem 2.4, the following stronger existence result seems easier to verify:

Proposition 2.14. *Let L/K be a finite simple extension.*

- (a) *If L/K is separable, then there exists a collection of abelian varieties $\{A_i/K\}$ of unbounded dimension such that, for each i ,*
 - A_i is absolutely simple;
 - A_i has a closed point with residue field L ; and
 - $\#A_i(K) \geq 2$.
- (b) *If L/K is purely inseparable, then there exist a collection of abelian varieties $\{A_i/K\}$ of unbounded dimension and a collection of abelian varieties $\{B_i/K\}$ such that, for each i ,*
 - A_i is absolutely simple;
 - $(B_i)_{\bar{K}} \cong (A_i \times_K A_i)_{\bar{K}}$ if $\text{char}(K) \neq 2$, and $(B_i)_{\bar{K}} \cong (A_i \times_K A_i \times_K A_i)_{\bar{K}}$ if $\text{char}(K) = 2$;
 - B_i has a closed point with residue field L ; and
 - $\#B_i(K) \geq 2$.

Before proving Proposition 2.14, we explain how it implies Proposition 2.3:

Proof of Proposition 2.3 (using Proposition 2.14). First assume that R is local, and set $Z = \text{Spec } R$ for simplicity of notation. By assumption $Z \subseteq \text{Spec } K[x_1, \dots, x_n]$ for any sufficiently large n , and in particular we may assume $n > 3 \dim A$. Take A' to be an abelian variety from Proposition 2.14 with $\dim A' \geq n$, and replacing n with $\dim A'$, we can and will assume that $n = \dim A'$. Since $(A')_{\bar{K}}$ is a product of at most 3 simple abelian varieties each of which, from our assumptions on n , must have dimension greater than $\dim A$, we have that $\text{Hom}(A_{\bar{K}}, A'_{\bar{K}}) = 0$. Therefore, we only need to show that $Z \subseteq A'$.

For this, using the definition of smoothness, we have a commutative fibered product diagram Zariski locally on A' :

$$\begin{array}{ccc} Z' & \hookrightarrow & (A', a') \\ \text{ét} \downarrow & & \downarrow \text{ét} \\ (Z, z) & \hookrightarrow & (A_K^n, a) \end{array}$$

where we have marked each scheme with its respective L -point, having residue field L . The L -points, and the commutativity of the diagram give an L -point we will call z' on Z' . Let Z'' be the component of Z' containing z' , and consider the pointed scheme (Z'', z') . Note that the residue field of z' must also be equal to L . Since all the morphisms above induce isomorphisms on the complete local rings (they are étale and induce isomorphisms on residue fields [Gro67, Prop. 17.6.3]), and since (Z'', z') and (Z, z) are affine pointed schemes associated to finite K -algebras (which are therefore products of complete local K -algebras), we have that Z'' and Z are isomorphic. This completes the proof in the case where R is local.

In general, R , being Artinian, is a product of finitely many local rings. Now use the nontrivial K -points and a product construction. (In more detail, if $R \cong \prod_{j=1}^r R_j$ is a product of local Artin algebras, using the previous paragraph, let A_j/K be an absolutely simple abelian variety equipped with an embedding $\alpha_j : \text{Spec } R_j \rightarrow A_j$ whose image is *not* supported at the identity element. Let

$A' = \prod A_j$, and let $\iota_j : A_j \rightarrow A'$ be the natural embedding. Define a morphism $\alpha : \operatorname{Spec} R \rightarrow A$ by $\alpha|_{\operatorname{Spec}(R_j)} = \iota_j \circ \alpha_j$; then α is a closed embedding. $\square$

Proof of Proposition 2.14 when K is infinite and L/K is a finite separable extension. We suppose $\operatorname{char}(K) \neq 2$, and indicate the necessary changes for even characteristic at the end. Using the separability hypothesis, choose a polynomial $f(s) \in K[s]$ such that $L \cong K[s]/f(s)$; note that f is squarefree.

For each i , let $h_i(x) \in K[x]$ be a polynomial of degree i which factors completely over K , and such that $xh_i(x)$ is squarefree. Let t be a parameter on $\mathbb{A}_K^1$, and let $\mathcal{C}_i \rightarrow \mathbb{A}_K^1$ be the family of curves whose fiber over t has affine model $y^2 = f(x)xh_i(x)(x-t)$. Fix $\ell > 3$ invertible in K . The geometric mod ℓ monodromy of this family is $\operatorname{Sp}_{2g}(\mathbb{Z}/\ell)$ [Hal08].

Let $K_0 \subseteq K$ be a subfield, finitely generated over the prime field, such that $\mathcal{C}_i \rightarrow \mathbb{A}_K^1$ descends to a model over K_0 . Since K_0 is finitely generated over $\mathbb{Q}$ (if $\operatorname{char}(K) = 0$) or $\mathbb{F}_p(s)$ (if $\operatorname{char}(K) = p > 0$), K_0 is Hilbertian. By Hilbert's irreducibility theorem, there exists some $t_0 \in \mathbb{A}^1(K_0) \subset \mathbb{A}^1(K)$ such that, for $\mathcal{C}_i := \mathcal{C}_{i,t_0}$, the image of $\operatorname{Gal}(K_0)$ acting on $H^1(\mathcal{C}_{i,\bar{K}_0}, \mathbb{Z}/\ell)$ contains $\operatorname{Sp}_{2g}(\mathbb{Z}/\ell)$. In particular, let $A_i = \operatorname{Jac}(\mathcal{C}_{i,t_0})/K$; a standard argument then shows $\operatorname{End}(A_{i,\bar{K}}) \cong \mathbb{Z}$. (Briefly, for group-theoretic reasons, since $\ell > 3$ and since the image of $\operatorname{Gal}(K_0)$ in $\operatorname{Aut}(H^1(\mathcal{C}_{i,\bar{K}_0}, \mathbb{Z}/\ell))$ is an ℓ -adically closed group which contains a subgroup surjecting onto $\operatorname{Sp}_{2g}(\mathbb{Z}/\ell)$, it contains all of $\operatorname{Sp}_{2g}(\mathbb{Z}/\ell)$. Replacing K_0 with a finite extension K'_0 replaces the image of $\operatorname{Gal}(K_0)$ with a subgroup of finite index, but it is still Zariski dense in $\operatorname{Sp}_{2g, \mathbb{Q}_\ell}$. Then $\operatorname{End}_{K'_0}(A_i) \otimes \mathbb{Q}_\ell$, being contained in the commutant of $\operatorname{Sp}_{2g, \mathbb{Q}_\ell}$ in $\operatorname{Aut}(H^1(\mathcal{C}_{i,\bar{K}_0}, \mathbb{Q}_\ell))$, is just $\mathbb{Q}_\ell$, and thus $\operatorname{End}_{K'_0}(A_i) \cong \mathbb{Z}$.) We use the base point $(0,0)$ to embed $\mathcal{C}_i$ in A_i .

Let $Z_i \subseteq \mathcal{C}_i$ be the vanishing locus of the function y . Then Z_i is the spectrum of

$$R_i := \frac{K[x, y]}{(y^2 - xh_i(x)(x - t_0), y)} \cong \frac{K[x]}{f(x)} \oplus K^{\oplus i+2} \cong L \oplus K^{\oplus i+2},$$

and we have $Z_i \hookrightarrow \mathcal{C}_i \hookrightarrow A_i$. In particular, A_i contains a subscheme isomorphic to $\operatorname{Spec} L$, and $\#A_i(K) \geq i + 2 \geq 2$.

In fact, the same argument works if we replace $y^2 = xh_i(x)(x - t)$ with $y^r = xh_i(x)(x - t)$ for any prime r [Kat19, §2]. Briefly, every K -rational fiber contains $\operatorname{Spec} L \oplus \operatorname{Spec} K$ as a closed subscheme; the monodromy group of the family contains a special unitary group; Hilbert irreducibility and an ℓ -adic calculation show that the absolute endomorphism ring of the Jacobian of any fiber outside a thin set is $\mathbb{Z}[\zeta_r]$; and such a Jacobian is an absolutely simple abelian variety. $\square$

We now move to the case of a finite simple purely inseparable extension L/K , which we can take to be of the form $L = K[x]/(x^{p^r} - a)$ for some choice of $a \in K$. A natural approach (for $\operatorname{char} K \neq 2$) would then be to consider the family of hyperelliptic curves $-y + y^2 + xy + (x^{p^r} - a)(x^p - a_1)(x^p - a_2) \cdots (x^p - a_s)(x^p - t)$, or the family $-ty + y^2 + xy + (x^{p^r} - a)(x^p - a_1)(x^p - a_2) \cdots (x^p - a_s)$, both of which give smooth affine curves containing $\operatorname{Spec} L$ as a closed subscheme. We note that completing the square of the second family gives the family $y^2 - \frac{1}{4}(x - t)^2 + (x^{p^r} - a)(x^p - a_1)(x^p - a_2) \cdots (x^p - a_s)$. The complication in this approach is to determine if the general member of the family has large mod ℓ monodromy; i.e., whether the general curve in the family is absolutely simple. To avoid this issue, we use an argument we learned from David Grant:

Proof of Proposition 2.14 when K is infinite and L/K is a finite simple purely inseparable extension. As in the previous proof, we reduce to the case where K is finitely generated over the prime field, and again, we explain the case where $p := \operatorname{char}(K) > 2$ first. To begin, we fix for each i the curve $\mathcal{C}_i$ over K from the previous proof, which is a smooth projective hyperelliptic curve of appropriately large genus admitting a number of K -points, and which has absolutely simple Jacobian $\operatorname{Jac}(\mathcal{C}_i)$,

which we denote by A_i/K . Since L/K is assumed to be simple and purely inseparable, we can take L to be of the form $L = K[x]/(x^{p^r} - a)$ for some choice of $a \in K$. In particular, we have $\text{Spec } L \subseteq \mathbb{P}_K^1$, and by changing C_i (moving the branch points) we may and will assume that $\text{Spec } L$ has support disjoint from the branch locus of the structure map $C_i \rightarrow \mathbb{P}_K^1$. The pre-image of $\text{Spec } L$ in C_i is a closed subscheme $\text{Spec } N \subseteq C_i$, either consisting of two distinct L -points, or consisting of a single point, in which case N/L is a degree 2 extension of fields, separable since $\text{char}(K) \neq 2$. In the former case we can simply take $B_i = A_i \times_K A_i$.

In the latter case, taking the separable closure of K in N , we obtain another subfield M/K of N , necessarily of degree 2 over K , giving us a diagram of fields

$$\begin{array}{ccccc} & & N & & \\ & \text{sep} \downarrow & \swarrow \text{insep} & & \\ & L & & M & \\ & \swarrow p^r & & \downarrow 2 \text{ | sep} & \\ & & K & & \end{array}$$

Using that $[L : K]$ and $[M : K]$ are coprime, we have that $N = LM$, and L and M are linearly disjoint over K ($N = L \otimes_K M$).

Now since C_i is a smooth projective curve with a K -point, and therefore embeds in its Jacobian, we have that A_i admits $\text{Spec } N$ as a closed subscheme (and has a number of K -points). Then $B_i := \text{Res}_{M/K}((A_i)_M)$ has the property that $(B_i)_{\bar{K}} \cong (A_i \times_K A_i)_{\bar{K}}$ (e.g., [FT15, Lem. 5]), and we claim that B_i admits $\text{Spec } L$ as a closed subscheme (as well as a K -point for each K -point of A_i). This latter assertion follows from the fact that $\text{Res}_{M/K} \text{Spec } N$ contains $\text{Spec } L$ as a closed subscheme, and the fact that closed immersions are preserved by the Weil restriction [BLR90, §7.6, Prop. 2, p.192].

To see that $\text{Res}_{M/K} \text{Spec } N$ contains $\text{Spec } L$ as a closed subscheme one can argue as follows. We have $N = M[x_1, \dots, x_n]/(f_1, \dots, f_m)$. As for any affine scheme and any finite extension of fields, we can write $\text{Res}_{M/K} \text{Spec } N$ as $\text{Spec } K[y_{i,j}]/(g_{l,r})$, where $y_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq s$) are new variables, and $g_{l,r}$ ($1 \leq l \leq m, 1 \leq r \leq s$) are polynomials in $y_{i,j}$ given by taking a basis $e_1, \dots, e_s$ of M over K and setting $x_i = y_{i,1}e_1 + \dots + y_{i,s}e_s$ and $f_t = g_{t,1}e_1 + \dots + g_{t,s}e_s$. In our case, $s = 2$. Now since L and M are linearly disjoint and $[N : L] = 2$, we have that e_1, e_2 form a basis of N over L . So, if we write α_i for the class of x_i in N , then we can write $\alpha_i = a_{i,1}e_1 + a_{i,2}e_2$ for some elements $a_{i,j} \in L$. Therefore, by definition, taking $y_{i,j} = a_{i,j}$, we obtain an L -point of the Weil restriction. Since not all of the $a_{i,j}$ can be in K (otherwise $L = M$), we have in fact an L -point of the Weil restriction with residue field L . This completes the proof in the case where $\text{char}(K) \neq 2$.

In the case where $\text{char}(K) = 2$, we replace the family of curves $y^2 = xh_i(x)(x - t)$ with the family $y^3 = xh_i(x)(x - t)$; the rest of the proof goes through identically. $\square$

We now take up the task of dealing with finite fields.

Proof of Proposition 2.14 when K is finite. Let $K = \mathbb{F}_q$ and $L = \mathbb{F}_{q^r}$. There exist absolutely simple abelian varieties over K of every dimension, and most of them (in particular, at least one in every dimension) have at least two K -rational points [HZ02]. It thus suffices to assume that $r > 1$ and show that if A/K is a simple abelian variety then, with finitely many exceptions, A has a closed point with residue field L . (In the case $(q, q^r) = (2, 4)$, we will prove a slightly weaker statement which is still adequate for our purpose.)

Let X/K be any geometrically irreducible variety. If X does not contain a closed subscheme isomorphic to $\text{Spec } L$, then every L -rational point $P \in X(L)$ is actually defined over some subfield K' , where $K \subseteq K' \subsetneq L$. It suffices to consider points defined over *maximal* proper subfields

of L . Crudely estimating, we have the criterion that if

$$\#X(\mathbb{F}_{q^r}) > \sum_{\ell|r} \#X(\mathbb{F}_{q^{r/\ell}})$$

(where ℓ ranges over prime divisors of r), then X has a closed subscheme isomorphic to $\text{Spec } L$.

Let A/K be an abelian variety of dimension g . Weil's theorem on the eigenvalues of Frobenius of an abelian variety easily yields, for any extension $\mathbb{F}_{q^d}$ of $\mathbb{F}_q$, that

$$(q^d + 1 - 2\sqrt{q^d})^g \leq \#A(\mathbb{F}_{q^d}) \leq (q^d + 1 + 2\sqrt{q^d})^g.$$

(In fact, each occurrence of $2\sqrt{q^d}$ can be replaced by $\lfloor 2\sqrt{q^d} \rfloor$ [AHL13, §1].) In particular, let ℓ_0 be the smallest prime divisor of r . We have the coarse estimate

$$\begin{aligned} \frac{\#\bigcup_{K' \subsetneq L} A(K')}{\#A(L)} &\leq \frac{\sum_{\ell|r} \#A(\mathbb{F}_{q^{r/\ell}})}{\#A(\mathbb{F}_{q^r})} \\ &\leq \frac{(\#\{\ell : \ell|r\}) (q^{r/\ell_0} + 1 + 2q^{r/2\ell_0})^g}{(q^r + 1 - 2\sqrt{q^r})^g} \\ &\leq \log(r) \left(\frac{q^{r/\ell_0} + 1 + 2q^{r/2\ell_0}}{q^r + 1 - 2\sqrt{q^r}} \right)^g. \end{aligned}$$

For sufficiently large g , this quantity is less than one, *unless*

$$(q^{r/\ell_0}, q^r) \in \{(2, 4), (3, 9), (4, 16), (2, 8)\}.$$

Consider one of these remaining cases. Then $\mathbb{F}_{q^{r/\ell_0}}$ is the unique maximal proper subfield of $\mathbb{F}_{q^r}$. If the abelian variety $A/\mathbb{F}_q$ fails to have a closed subscheme isomorphic to $\text{Spec } \mathbb{F}_{q^r}$, then $A(\mathbb{F}_{q^{r/\ell_0}}) = A(\mathbb{F}_{q^r})$. Except for the case $(q^{r/\ell_0}, q^r) = (2, 4)$, this cannot happen if A is simple of dimension at least three. Indeed, the case $(q^{r/\ell_0}, q^r) = (2, 8)$ literally follows from [Ked22, Lem. 3.1], while the other two cases follow from its proof and [Ked22, Lem. 2.1(b)].

We now address the remaining case $K = \mathbb{F}_2$ and $L = \mathbb{F}_4$ by adapting Kedlaya's argument to our needs. Assume that A/K is an absolutely simple ordinary abelian variety of dimension g with $\#A(K) \geq 2$. (Again, this is possible by [HZ02].) Let A' be its nontrivial quadratic twist; it, too, is absolutely simple. We will show that if $A(K) = A(L)$, then $A'(K) \subsetneq A'(L)$. Let $B = R_{L/K}(A_L)$; then B is isogenous to $A \times_K A'$. Since $B(K) = A(L)$, if $A(K) = A(L)$, then $\#A'(K) = 1$. For g in the complement of a thin set of natural numbers – in particular, for infinitely many g – this uniquely determines the isogeny class of A' [Ked22, Lem. 2.1(c)]. If A' *also* has the property that $A'(K) = A'(L)$, then the quadratic twist A'' of A' also satisfies $A''(K) = 1$. Since $A'' \cong A$, we find in particular that A and A' are isogenous; but this is impossible for a simple ordinary abelian variety (e.g., [AC15, Ex. 1.7]). Consequently, at least one of A and A' admits a subscheme isomorphic to $\text{Spec } L$. $\square$

As is clear from the proofs above, given a simple extension of fields L/K one can quickly write down an abelian variety with $\text{Spec } L$ as a closed subscheme. The difficulty is finding such an abelian variety that is absolutely simple (or, whose base change to the algebraic closure is a product of simple abelian varieties of sufficiently large dimension). In order to shorten the proofs above, one might hope that given any abelian variety A/K , and any simple extension of fields L/K , there exists a closed subscheme of A isomorphic to $\text{Spec } L$. The following example shows that this is not the case:

Example 2.15. Let E be the elliptic curve over $\mathbb{F}_2$ with affine model $y^2 + y = x^3 + x^2$. One can check that $\#E(\mathbb{F}_2) = \#E(\mathbb{F}_4) = 5$. In particular, there is no point of E with residue field $\mathbb{F}_4$.

3. BASE CHANGE FOR ALBANESE VARIETIES

Let V be a geometrically connected and geometrically reduced scheme of finite type over a field K , and let $(\text{Alb}_{V/K}, \text{Alb}_{V/K}^1, a_{V/K})$ be Albanese data for V (2.1). Recall that this includes the Albanese morphism

$$V \xrightarrow{a_{V/K}} \text{Alb}_{V/K}^1$$

to the Albanese torsor. If L/K is any field extension, then after base change we obtain a diagram

$$\begin{array}{ccc} V_L & \xrightarrow{a_{V_L/L}} & \text{Alb}_{V_L/L}^1 \\ \parallel & & \downarrow \beta_{V,L/K}^1 \\ V_L & \xrightarrow{(a_{V/K})_L} & (\text{Alb}_{V/K}^1)_L \end{array} \quad (3.1)$$

where $\beta_{V,L/K}^1$ is induced by the the universal property of the Albanese. Via the dual of the pull-back morphism on line bundles (see Remark 2.1), this is equivariant with respect to a base change morphism of abelian varieties

$$\beta_{V,L/K} : \text{Alb}_{V_L/L} \longrightarrow (\text{Alb}_{V/K})_L. \quad (3.2)$$

A diagram similar to (3.1) holds in the pointed case, as well.

We say that the Albanese data of V is *stable under (separable) base change of field* if the Albanese data exists and $\beta_{V,L/K}$ and $\beta_{V,L/K}^1$ are isomorphisms for all (separable) field extensions L/K . Note that in particular, this means that $((\text{Alb}_{V/K})_L, (\text{Alb}_{V/K}^1)_L, (a_{V/K})_L)$ gives Albanese data for V_L .

There is an analogous notion for pointed Albanese data (2.2) to be *stable under (separable) base change of field*. A before, when this holds, the suitable base change of pointed Albanese data is again pointed Albanese data.

Initially, we remark that in the unpointed case, $\beta_{V,L/K}$ is an isomorphism if and only if $\beta_{V,L/K}^1$ is an isomorphism:

Lemma 3.1. *Let V/K be a geometrically connected and geometrically reduced scheme of finite type over a field K , and let L/K be an extension of fields. Then $\beta_{V,L/K}$ is an isomorphism if and only if $\beta_{V,L/K}^1$ is an isomorphism.*

Proof. On one hand, let T be a torsor under an abelian variety A over K ; then $A \cong (\text{Pic}_{T/K}^0)^\vee$ (Remark 2.1). Consequently, if $\text{Alb}_{V_L/L}^1$ and $(\text{Alb}_{V/K}^1)_L$ are isomorphic via $\beta_{V,L/K}^1$, then so are $\text{Alb}_{V_L/L}$ and $(\text{Alb}_{V/K})_L$ via $\beta_{V,L/K}$.

On the other hand, suppose $\beta_{V,L/K}$ is an isomorphism. Then $\beta_{V,L/K}^1$ is a nontrivial map of torsors over an isomorphism of abelian varieties. Since $\beta_{V,L/K}$ and $\beta_{V,L/K}^1$ agree up to translation after base change to the algebraic closure of L , $\beta_{V,L/K}^1$ is an isomorphism. $\square$

The Raynaud example (Example 1.1) shows that, in general, Albanese varieties are not stable under base change of field. There are two possible issues to focus on in this example. First, the variety G/K is not proper, and second, the extension L/K is not separable. Regarding the former, it has been understood that if one assumes V is proper, then the Albanese variety is stable under base change:

Theorem 3.2 (Grothendieck–Conrad). *Let V be a proper geometrically connected and geometrically reduced scheme over a field K . Then Albanese data for V (2.1) is stable under base change of field, and if V admits a K -point v , then pointed Albanese data for (V, v) (2.2) is stable under base change of field.* $\square$

Remark 3.3 (References for Theorem 3.2). Recall that Grothendieck provides an Albanese torsor (resp. pointed Albanese variety) for any proper geometrically connected and geometrically normal scheme V (resp. pointed proper geometrically connected and geometrically normal scheme (V, v)) over a field K in the following way. As V/K is proper and geometrically normal, one has that $\text{Pic}_{V/K}^0$ is proper [Gro62, Thm. VI.2.1(ii)]; then by [Gro62, Prop. VI.2.1], one has that $(\text{Pic}_{V/K}^0)_{\text{red}}$ is a group scheme (*i.e.*, without the usual hypothesis that K be perfect and the group scheme be smooth). It then follows from [Gro62, Thm. VI.3.3(iii)] that $((\text{Pic}_{V/K}^0)_{\text{red}})^\vee$ is an Albanese variety, and using that V/K is geometrically connected, that there exists an Albanese torsor. Conrad has generalized Grothendieck's argument to show that any proper geometrically connected and geometrically reduced scheme V over a field K admits an Albanese torsor, and a pointed Albanese variety if V admits a K -point. For lack of a better reference, we direct the reader to [Con17, Thm.]. His argument is to show that the Albanese variety is the dual abelian variety to the maximal abelian subvariety of the (possibly non-reduced and non-proper) Picard scheme $\text{Pic}_{V/K}$. Grothendieck's theorem can then be summarized in this context by saying that his additional hypothesis that V be geometrically normal implies that the maximal abelian subvariety of $\text{Pic}_{V/K}$ is $\text{Pic}_{V/K}^0$. That Grothendieck's and Conrad's Albanese varieties are stable under arbitrary field extension is [Gro62, Thm. VI.3.3(iii)] and [Con17, Prop.], respectively. In fact, the Albanese variety enjoys an even stronger universal property; see §6 below.

While the hypothesis in the theorem that V be geometrically connected is necessary (Corollary 2.5), we point out here that it is possible for geometrically non-reduced schemes to admit Albanese data that is stable under base change of field:

Example 3.4 (Albanese base change for a non-reduced scheme). Let V be the non-reduced scheme defined in Example 2.7. Then the Albanese torsor (and the pointed Albanese variety) of V is stable under base change of field.

The second potential difficulty in Example 1.1, namely, the inseparability of the field extension L/K , shows that in the absence of properness, something like the separability hypothesis in Theorem (A) is necessary.

4. EXTENSIONS OF FIELDS

We briefly detour from our development of the Albanese to gather some results on separable and primary extensions of fields.

4.1. Separable extensions. The following elementary results on separable extensions will ultimately be used to extend the standard Lemma 5.4 below to arbitrary separable extensions (as opposed to separable algebraic extensions).

For clarity with the terminology, we recall that a (not necessarily algebraic) field extension L/K is separable if for every extension of fields M/K , one has that $M \otimes_K L$ is reduced. Setting p to be the characteristic exponent, this is equivalent to the condition that L^p and K be linearly disjoint over K^p ; *i.e.*, that the natural map $L^p \otimes_{K^p} K \rightarrow L^p K$ be injective [Bou81, Rem., p.V.119]. We say that a field K is separably closed if it admits no separable *algebraic* field extensions. We say an extension of fields L/K is purely inseparable if for every $x \in L$, there is an integer n such that $x^{p^n} \in K$, or equivalently [Bou81, Prop. 13, p.V.42], if it is an algebraic extension and there are no nontrivial separable subextensions.

Note that if $\text{char}(K) = 0$ then any field extension of K is separable [Bou81, Thm. 1, p.V.117]. In general, any field extension L/K factors as $L/L'/K$ with L'/K separable and L/L' purely inseparable; indeed, taking any transcendence basis T for L/K [Bou81, Thm. 1, p.V.105], one has $K(T)/K$ is separable [Bou81, Prop. 6, p.V.116], and then the algebraic extension $L/K(T)$ factors

as a separable extension $L'/K(T)$ followed by a purely inseparable extension L/L' . Here we are using that the composite of two separable extensions is separable [Bou81, Prop. 9, p.V.117].

Lemma 4.1. *Let Ω/k be an extension of separably closed fields. Then Ω/k is separable if and only if $\Omega^{\text{Aut}(\Omega/k)} = k$.*

Proof. Without any assumptions on K , if Ω/K is any field extension, then Ω is separable over $\Omega^{\text{Aut}(\Omega/K)}$; see e.g. [Bou81, §15.3, Prop. 7]. In particular, if $\Omega^{\text{Aut}(\Omega/K)} = K$, then Ω/K is separable.

Conversely, assume that Ω/k is a separable extension of separably closed fields. Since k is separably closed and since any sub-extension $\Omega/K/k$ satisfies K/k separable [Bou81, Prop. 8, p.V.116], in order to show that $\Omega^{\text{Aut}(\Omega/k)} = k$, it is enough to show that $\Omega^{\text{Aut}(\Omega/k)}/k$ is algebraic. Let $\alpha \in \Omega$ be a transcendental element over k . Since α extends to a transcendence basis of Ω/k , the map $\alpha \mapsto \alpha + 1$ extends to an automorphism of Ω which fixes k . Consequently, no element of Ω transcendental over k is fixed by all of $\text{Aut}(\Omega/k)$, and $\Omega^{\text{Aut}(\Omega/k)}/k$ is algebraic, as desired. $\square$

Recall that if $L'/L/K$ is a tower of field extensions, then on the one hand, if L'/L is separable and L/K is separable, then L'/K is separable [Bou81, Prop. 9, p.V.117]. On the other hand, if L'/K is separable, then L/K is separable [Bou81, Prop. 8, p.V.116], but L'/L may not be separable (e.g., $\mathbb{F}_p(T)/\mathbb{F}_p(T^p)/\mathbb{F}_p$). Nevertheless, we have:

Lemma 4.2. *Suppose that L/K is a separable extension of fields. Then $L^{\text{sep}}/K^{\text{sep}}$ is separable.*

Proof. In characteristic 0 there is nothing to show. So let $p = \text{char } K > 0$. We start with a small observation [Bou81, Exe. 4 p.V.165]: If $F/E/K$ is a tower of field extensions, with F/K separable, then if $E^p K = E$, then F/E is separable. To prove this, it suffices to show that the natural map $F^p \otimes_{E^p} E \rightarrow F^p E$ is injective. By the assumption $E = E^p K$, we therefore must show $F^p \otimes_{E^p} E^p K \rightarrow F^p(E^p K)$ is injective. Since E^p/K is separable [Bou81, Prop. 8, p.V.116], we have that $E^p \otimes_{K^p} K \hookrightarrow E^p K$ is injective. Since field extensions are (faithfully) flat, tensoring by $F^p \otimes_{E^p} (-)$ we obtain

$$F^p \otimes_{E^p} (E^p \otimes_{K^p} K) \hookrightarrow F^p \otimes_{E^p} E^p K \rightarrow F^p(E^p K). \quad (4.1)$$

The composition is identified with the map $F^p \otimes_{K^p} K \rightarrow F^p K \subseteq F^p(E^p K)$, which is injective since F/K is assumed to be separable. However, since $E^p K$ is the field of fractions of $E^p \otimes_{K^p} K$ under the inclusion $E^p \otimes_{K^p} K \hookrightarrow E^p K$, we see that the right hand map $F^p \otimes_{E^p} E^p K \rightarrow F^p(E^p K)$ in (4.1) is injective, as claimed, since it is obtained from the composition $F^p \otimes_{E^p} (E^p \otimes_{K^p} K) \rightarrow F^p(E^p K)$ in (4.1) by localization.

To prove the lemma, we apply the observation in the previous paragraph with $F = L^{\text{sep}}$ and $E = K^{\text{sep}}$. Thus we just need to show that $(K^{\text{sep}})^p K = K^{\text{sep}}$. Thus we have reduced to the following: if E/K is a separable algebraic extension, then $E^p K = E$. Indeed, we have a tower of extensions $E/E^p K/K$. The extension $E/E^p K$ is purely inseparable (the p -th power of every element of E belongs to $E^p K$) while the extension E/K is separable. This implies $E = E^p K$. $\square$

4.2. L/K -images. Let L/K be a primary extension of fields, i.e., the algebraic closure of K in L is purely inseparable over K , or equivalently, K equals its separable closure in L . Suppose A/L is an abelian variety. The L/K -image of A is a pair $(\text{im}_{L/K}(A), \lambda)$ consisting of an abelian variety $\text{im}_{L/K}(A)$ over K and a homomorphism $\lambda : A \rightarrow (\text{im}_{L/K}(A))_L$ of abelian varieties over L that is initial for pairs (B, f) consisting of an abelian variety B over K and a homomorphism $f : A \rightarrow B_L$:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & (\text{im}_{L/K}(A))_L \\ & \searrow f & \downarrow \exists! \\ & & B_L \end{array} \quad (4.2)$$

The idea of such an image (and the complementary notion of the trace, which is final for pairs (B, f) consisting of an abelian variety B over K and a homomorphism $f : B_L \rightarrow A$) goes back to Chow, but we appeal to [Con06] as a modern and comprehensive reference. The existence of $\text{im}_{L/K}(A)$ is proven in [Con06, Thm. 4.1].

For later reference, given a separable extension M/K , and an algebraically disjoint extension L/K ([Bou81, Def. 5, p.V.108]), we have that LM/L is separable [Bou81, Prop. 5, p.V.131]. Similarly, given a purely inseparable extension L/K and an arbitrary extension M/K , we have that LM/M is purely inseparable. In other words, if M/K is separable, and L/K is purely inseparable (and therefore algebraic, so that L is algebraically disjoint from M), then we have a tower:

$$\begin{array}{ccc} & LM & \text{insep} \\ \text{sep} \downarrow & & \swarrow \\ & M & \\ & \downarrow \text{sep} & \\ & K & \text{insep} \\ & \swarrow & \\ L & & \end{array} \quad (4.3)$$

where “insep” means “purely inseparable”.

One fact we will use later is that formation of the image is insensitive to separable field extensions. Indeed, a special case of [Con06, Thm. 5.4] states:

Lemma 4.3. *If L/K is purely inseparable, if M/K is separable, and A/L is an abelian variety, then*

$$\text{im}_{LM/M}(A_{LM}) \cong (\text{im}_{L/K}(A))_M. \quad \square$$

In fact, we will want a small strengthening of this lemma (Proposition 4.8). To obtain this strengthening, we will first need a few more small results. First, we will need a slight variation on Mumford’s Rigidity Lemma [MFK94, Prop. 6.1(1)]. If V is a scheme, we use $|V|$ to denote the underlying topological space.

Lemma 4.4 (Rigidity Lemma). *Given a diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & & S \end{array} \quad \begin{array}{c} \epsilon \\ \nearrow \end{array}$$

where S is a Noetherian scheme and:

- (a) $p_* \mathcal{O}_X \cong \mathcal{O}_S$;
- (b) ϵ is a section of p , and $|S|$ consists of a single point, s ; and
- (c) the set-theoretic image $f(|X_s|)$ is a single point of $|Y|$;

Then there exists a section $\eta : S \rightarrow Y$ of q such that $f = \eta \circ p$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & & S \end{array} \quad \begin{array}{c} \epsilon \\ \nearrow \\ \eta \end{array}$$

Proof. This is almost verbatim [MFK94, Prop. 6.1(1)]. Indeed, Mumford’s hypotheses in [MFK94, Prop. 6.1(1)] are the same, except that our assumption (a) is replaced in [MFK94, Prop. 6.1(1)] by the assumption that p be flat and that $H^0(X_s, \mathcal{O}_{X_s}) \cong \kappa(s)$. However, these two hypotheses are only used in the proof of [MFK94, Prop. 6.1(1)] at the top of [MFK94, p.116], where the reader is invited to verify that these conditions imply that $p_* \mathcal{O}_X \cong \mathcal{O}_S$; but this is our hypothesis (a). $\square$

Chow's rigidity theorem for abelian varieties (e.g., [Con06, Thm. 3.19]) implies that a morphism of abelian varieties which is defined after a purely inseparable extension is already defined over the base field. Here, we use Mumford's rigidity lemma to prove an analogous statement when the source of the morphism is an arbitrary geometrically integral variety.

Proposition 4.5. *Let L/K be a purely inseparable extension of fields. Let $(U, u)/K$ be a pointed geometrically integral separated scheme of finite type, let A/K be an abelian variety, and suppose that $g : (U_L, u_L) \rightarrow (A_L, \mathbf{0}_{A_L})$ is a pointed L -morphism. Then g descends to K .*

Proof. Since U and A are of finite type over K , so is g . Therefore, there is a subextension $L_0 \subseteq L$, finite over K , over which g is defined. Replacing L by L_0 if necessary, we may and do assume L/K is finite and purely inseparable. Then $\mathrm{Spec}(L \otimes_K L)$ is a Noetherian scheme with a single point. This point has residue field L ; let $s : \mathrm{Spec} L \hookrightarrow \mathrm{Spec}(L \otimes_K L)$ be its inclusion.

Let $p_i : \mathrm{Spec}(L \otimes_K L) \rightarrow \mathrm{Spec} L$ be the two projections. As usual, since U_L is the base change of a K -scheme, there is a canonical isomorphism $p_1^*(U_L) \cong p_2^*(U_L)$, and we simply call this object $U_{L \otimes_K L}$. We similarly define the pullback of u , A and $\mathbf{0}_A$ to $L \otimes_K L$. We want to use fpqc descent to show that $g : U_L \rightarrow A_L$ descends to K ; for this we need to show an equality of morphisms

$$p_1^*g \stackrel{?}{=} p_2^*g : U_{L \otimes_K L} \longrightarrow A_{L \otimes_K L}. \quad (4.4)$$

This equality will follow from the Rigidity Lemma (Lemma 4.4), as we will see. At the moment, however, we have a diagram

$$\begin{array}{ccc} U_{L \otimes_K L} & \xrightarrow{p_1^*g - p_2^*g} & A_{L \otimes_K L} \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec} L \otimes_K L & \end{array} \quad (4.5)$$

$u_{L \otimes_K L}$ is indicated by a curved arrow from $U_{L \otimes_K L}$ to $\mathrm{Spec} L \otimes_K L$.

If U/K were proper, one could easily check that the hypotheses of the Rigidity Lemma held for the diagram (4.5) (see the proof below), and then it would follow quickly from the Rigidity Lemma that equality holds in (4.4) (again, see the proof below). But, since we are not assuming that U/K is proper, we must do a little work first to get around this issue.

To begin, let $\omega : X \rightarrow \mathrm{Spec} K$ be a (Nagata) compactification of U [Con07]. Since X is proper over K and contains the geometrically integral scheme U as an open dense set, X is geometrically integral (e.g., [GW20, Prop. 5.51(iii)]) and so (e.g., [Sta22, Lem. 0FD2]) $\omega_*\mathcal{O}_X \cong \mathcal{O}_{\mathrm{Spec} K}$. We now base change to $\mathrm{Spec} L$. Using [Lüt93, Lem. 2.2] or [Con07, Rem. 2.5], there is a U_L -admissible blowup

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\omega'} & X_L \\ & \searrow \tilde{\omega} \quad \swarrow \omega_L & \\ & \mathrm{Spec} L & \end{array}$$

such that $g : U_L \rightarrow A_L$ extends to a morphism

$$\tilde{g} : \tilde{X} \rightarrow A_L.$$

Moreover, using the same argument as before, i.e., that $\tilde{X}$ is proper over L and contains the geometrically integral scheme U_L as an dense open subset, we have that $\tilde{X}$ is geometrically integral so that $\tilde{\omega}_*\mathcal{O}_{\tilde{X}} \cong \mathcal{O}_{\mathrm{Spec} L}$.

We now base change to $L \otimes_K L$ and obtain a diagram

$$\begin{array}{ccc} \tilde{X}_{L \otimes_K L} & \xrightarrow{\omega'_{L \otimes_K L}} & X_{L \otimes_K L} \\ & \searrow \tilde{\omega}_{L \otimes_K L} \quad \swarrow \omega_{L \otimes_K L} & \\ & \text{Spec } L \otimes_K L & \end{array}$$

Via cohomology and base change for flat base change, and using that $\tilde{\omega}_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_{\text{Spec } L}$, we have that $(\tilde{\omega}_{L \otimes_K L})_* \mathcal{O}_{\tilde{X}_{L \otimes_K L}} \cong \mathcal{O}_{\text{Spec } L \otimes_K L}$.

Our goal now is to show that

$$p_1^* \tilde{g} \stackrel{?}{=} p_2^* \tilde{g} : \tilde{X}_{L \otimes_K L} \longrightarrow A_{L \otimes_K L}, \quad (4.6)$$

as this will establish (4.4), and we will be done. For this, we will want to apply the Rigidity Lemma (Lemma 4.4) to the diagram

$$\begin{array}{ccc} \tilde{X}_{L \otimes_K L} & \xrightarrow{p_1^* \tilde{g} - p_2^* \tilde{g}} & A_{L \otimes_K L} \\ & \searrow \tilde{\omega}_{L \otimes_K L} \quad \swarrow \omega_{L \otimes_K L} & \\ & \text{Spec } L \otimes_K L & \\ & \nwarrow u_{L \otimes_K L} & \end{array} \quad (4.7)$$

However, to apply the Rigidity Lemma we still need to check that $(p_1^* \tilde{g} - p_2^* \tilde{g})(|(\tilde{X}_{L \otimes_K L})_s|)$ is set-theoretically a single point of $|(\mathcal{O}_{A_{L \otimes_K L}})_s|$. To see this, we start with the observation that the fiber $(U_{L \otimes_K L})_s = s^* U_{L \otimes_K L}$ is canonically isomorphic to U_L , and $p_i^* g|_{(U_{L \otimes_K L})_s} = g$. Similarly, $|(\mathcal{O}_{A_{L \otimes_K L}})_s| = |A_L|$. Our next claim is that $|(\mathcal{O}_{A_{L \otimes_K L}})_s|$ is dense in $|(\tilde{X}_{L \otimes_K L})_s|$, but this just follows since U_L is dense in $\tilde{X}$ by construction. Now, moving forward, we know that $(p_1^* g - p_2^* g)(|(\mathcal{O}_{A_{L \otimes_K L}})_s|) = |\mathbf{0}_{(\mathcal{O}_{A_{L \otimes_K L}})_s}|$, where here we are denoting by $|\mathbf{0}_{(\mathcal{O}_{A_{L \otimes_K L}})_s}|$ the support of the image of $\mathbf{0}_{(\mathcal{O}_{A_{L \otimes_K L}})_s} : (\text{Spec } L \otimes_K L)_s \rightarrow (\mathcal{O}_{A_{L \otimes_K L}})_s$. This is only an equality on $|(\mathcal{O}_{A_{L \otimes_K L}})_s|$. However, we know that $s^*(p_1^* \tilde{g} - p_2^* \tilde{g})$, as a continuous map $|(\tilde{X}_{L \otimes_K L})_s| \rightarrow |(\mathcal{O}_{A_{L \otimes_K L}})_s|$, must take the closure of $|(\mathcal{O}_{A_{L \otimes_K L}})_s|$ to the closure of the point $|\mathbf{0}_{(\mathcal{O}_{A_{L \otimes_K L}})_s}|$. But, since $|\mathbf{0}_{(\mathcal{O}_{A_{L \otimes_K L}})_s}|$ is a closed point of $|(\mathcal{O}_{A_{L \otimes_K L}})_s|$ and $|(\tilde{X}_{L \otimes_K L})_s|$ is the closure of $|(\mathcal{O}_{A_{L \otimes_K L}})_s|$, we see that $s^*(p_1^* \tilde{g} - p_2^* \tilde{g})(|(\tilde{X}_{L \otimes_K L})_s|) = \mathbf{0}_{(\mathcal{O}_{A_{L \otimes_K L}})_s} \in |(\mathcal{O}_{A_{L \otimes_K L}})_s|$ is a single point.

Consequently, we can apply the Rigidity Lemma to diagram (4.7), and we find that $p_1^* \tilde{g}$ and $p_2^* \tilde{g}$ differ by a section η of $A_{L \otimes_K L}$ over $\text{Spec } L \otimes_K L$. It remains to show that this section η coincides with $\mathbf{0}_{A_{L \otimes_K L}}$. For this it suffices to show that $p_1^* \tilde{g}$ and $p_2^* \tilde{g}$ are equal along a section of $\tilde{X}_{L \otimes_K L}$ over $\text{Spec } L \otimes_K L$, and of course, it therefore suffices to check equality along a section of $U_{L \otimes_K L} \subseteq \tilde{X}_{L \otimes_K L}$ over $\text{Spec } L \otimes_K L$; we will use the section $u_{L \otimes_K L}$. Since g takes u_L to $\mathbf{0}_{A_L}$, we have that $p_1^* g$ and $p_2^* g$ both take the section $p_1^* u_L = p_2^* u_L = u_{L \otimes_K L}$ of $U_{L \otimes_K L}$ to $\mathbf{0}_{A_{L \otimes_K L}}$, and thus $\eta = \mathbf{0}_{A_{L \otimes_K L}}$. Note that here we have used that u is defined over K , to identify $p_1^* u_L = p_2^* u_L = u_{L \otimes_K L}$. $\square$

To implement this descent result in the setting we want to use it, we need one more result, which states that rational maps that extend to a morphism after base change of field, extend to a morphism over the ground field, as well.

Lemma 4.6. *Let V be a reduced scheme of finite type over a field K , let T/K be a separated scheme of finite type, let $U \subseteq V$ be a dense open subset, and let L/K be an arbitrary extension of fields. Given a morphism $f : U \rightarrow T$ over K , such that $f_L : U_L \rightarrow T_L$ extends to a morphism $V_L \rightarrow T_L$, we have that $f : U \rightarrow T$ extends to a morphism $V \rightarrow T$ over K .*

Proof. To fix some notation, write $\Gamma_g : X \rightarrow X \times_K Y$ for the graph of a K -morphism $g : X \rightarrow Y$, which is a closed embedding if Y is separated, and denote by $\overline{g}(X)$ the scheme-theoretic image of X under g . Now, considering the graph $\Gamma_f : U \rightarrow U \times_K T$ and the inclusion $\iota : U \hookrightarrow V$, we wish to show that the first projection $\overline{(\iota \times 1_T)}(\overline{\Gamma_f}(U)) \rightarrow V$ is an isomorphism, so that the composition $V \xrightarrow{\cong} \overline{(\iota \times 1_T)}(\overline{\Gamma_f}(U)) \rightarrow T$ gives an extension of $f : U \rightarrow T$. We are given that f_L extends to $\tilde{f}_L : V_L \rightarrow T_L$. Recalling that the scheme-theoretic image is stable under flat base change (e.g., [EH00, Prop. V-8, p.217]), we have $\overline{(\iota \times 1)}((\overline{\Gamma_f}(U))_L) = \overline{(\iota \times 1)}(\overline{\Gamma_{f_L}}(U_L)) = \overline{\Gamma_{\tilde{f}_L}}(V_L)$, where the last equality holds since V_L is reduced. Finally, since $\overline{\Gamma_{\tilde{f}_L}}(V_L) \rightarrow V_L$ is an isomorphism, we can deduce that $\overline{(\iota \times 1_T)}(\overline{\Gamma_f}(U)) \rightarrow V$ is an isomorphism, since isomorphisms satisfy fpqc descent (e.g., [GW20, p.583]). $\square$

Lemma 4.7. *Suppose L/K is a purely inseparable extension of fields, A/L is an abelian variety, $(V, v)/K$ is a K -pointed (geometrically) connected and geometrically reduced scheme of finite type over K , and $f : (V_L, v_L) \rightarrow (A, \mathbf{0})$ is a pointed K -morphism. Then the composition $\lambda \circ f$ of pointed L -morphisms*

$$(V_L, v_L) \xrightarrow{f} (A, \mathbf{0}) \xrightarrow{\lambda} ((\mathrm{im}_{L/K} A)_L, \mathbf{0}) \quad (4.8)$$

is initial for compositions of pointed L -morphisms $(V_L, v_L) \rightarrow (A, \mathbf{0}) \rightarrow (B_L, \mathbf{0})$, where B/K is an abelian variety over K .

If, moreover, V admits an open cover $\{(U_i, u_i)\}$ by separated (geometrically) connected and geometrically reduced schemes U_i of finite type over K , with each irreducible component of the U_i being geometrically integral and admitting a smooth K -point u_i , then the composition $\lambda \circ f$, descends to a unique pointed K -morphism

$$\underline{f} : (V, v) \longrightarrow (\mathrm{im}_{L/K} A, \mathbf{0}).$$

Proof. The universal property of (4.8) follows from the definition of the L/K -image. All that is left is to show the descent. It suffices to show descent on restriction to each of the U_i . Let U be any of the U_i .

In fact, it suffices to show descent on the normalization U^ν of U . Indeed, since U is geometrically reduced, it is generically smooth (smoothness may be verified fpqc locally on the base, and the base change to the algebraic closure is generically smooth), and therefore, there is a dense open subset $U' \subseteq U$ that is normal, so that the normalization $\nu : U^\nu \rightarrow U$ is an isomorphism over U' . If we show that the morphism from $(U^\nu)_L$ descends, then the morphism from U'_L descends. Then we use Lemma 4.6.

So we can and will assume that U is normal. We can then focus on one irreducible component at a time, and we can assume that U is integral, and therefore geometrically integral from our assumptions. Since each irreducible component of U was assumed to have a smooth K -point, this gives a K -point on each of the irreducible components of the normalizations. Now use Proposition 4.5. $\square$

Proposition 4.8. *In the situation of Lemma 4.7, if in addition M/K is a separable field extension, then the pointed LM-morphisms f_{LM} and λ_{LM} obtained by base change of (4.8) along LM/ L factor as*

$$(V_{LM}, v_{LM}) \xrightarrow{f_{LM}} (A_{LM}, \mathbf{0}) \xrightarrow{\quad \lambda_{LM} \quad} ((\mathrm{im}_{LM/M} A_{LM})_{LM}, \mathbf{0}) \longrightarrow ((\mathrm{im}_{L/K} A)_{LM}, \mathbf{0}). \quad (4.9)$$

If, moreover, V admits an open cover $\{(U_i, u_i)\}$ by separated (geometrically) connected and geometrically reduced schemes U_i of finite type over K , with each irreducible component of the U_i being geometrically

integral and admitting a smooth K -point u_i , then, excluding morphisms with source or target A_{LM} , the morphisms in (4.9) descend uniquely to M to give pointed M -morphisms

$$(V_M, v_M) \xrightarrow{\quad \quad \quad \underline{f}_M \quad \quad \quad} (\mathrm{im}_{LM/M} A_{LM}, \mathbf{0}) \xrightarrow{\cong} ((\mathrm{im}_{L/K} A)_M, \mathbf{0}) \quad (4.10)$$

where the morphism on the right in (4.10) is the isomorphism in Lemma 4.3.

Proof. The factorization (4.9) follows from the universal property in the first part of Lemma 4.7 applied to the composition $(V_{LM}, v_{LM}) \xrightarrow{f_{LM}} (A_{LM}, \mathbf{0}) \longrightarrow ((\mathrm{im}_{LM/M} A_{LM})_{LM}, \mathbf{0})$, and the observation that $(\mathrm{im}_{L/K} A)_{LM} = ((\mathrm{im}_{L/K} A)_L)_{LM} = ((\mathrm{im}_{L/K} A)_M)_{LM}$ is obtained by pull back of an abelian variety over M .

The descent in (4.10) comes from applying Lemma 4.7 to (4.9). The fact that the composition in (4.10) is identified with $\underline{f}_M$ comes from the fact that the pull back of the composition in (4.10) to LM is by definition $\lambda_{LM} \circ f_{LM}$, and, also by definition, we have $\underline{f}_{LM} = (\underline{f}_L)_M = (\lambda \circ f)_{LM} = \lambda_{LM} \circ f_{LM}$, so that the uniqueness of the descent shows that $\underline{f}_M$ is the composition in (4.10).

That the second morphism in (4.10) is the isomorphism in Lemma 4.3 follows from the fact that this is the same descended morphism constructed by Conrad [Con06, Thm. 5.4]. $\square$

5. PROOF OF THEOREM (A)

We state a more precise version of Theorem (A) here:

Theorem 5.1 (Separable base change). *Let V be a geometrically connected and geometrically reduced scheme of finite type over a field K . Then the Albanese data $(\mathrm{Alb}_{V/K}, \mathrm{Alb}_{V/K}^1, a_{V/K})$ for V (2.1) is stable under separable base change of field (§3), and if V admits a K -point $v \in V(K)$, then pointed Albanese data $(\mathrm{Alb}_{V/K}, a_{V/K, v})$ for (V, v) (2.2) is stable under separable base change of field (§3).*

For finite separable extensions, an easy argument shows:

Lemma 5.2. *Let V/K be a geometrically connected and geometrically reduced scheme of finite type over a field K . If L/K is finite and separable, then the base change morphisms $\beta_{V,L/K}^1$ (3.1) and $\beta_{V,L/K}$ (3.2) are isomorphisms.*

Proof. By virtue of Lemma 3.1, it suffices to show that $\beta_{V,L/K}^1$ is an isomorphism, and therefore, by the universal property, it suffices to show that if A/L is any abelian variety, T is a torsor under A , and $\alpha : V_L \rightarrow T$ is a morphism, then α factors through $a_L : V_L \rightarrow (\mathrm{Alb}_{V/K}^1)_L$.

Since L/K is finite and separable, the Weil restriction $R_{L/K}(A)$ is an abelian variety (e.g., [Mil72, §1]) and $R_{L/K}(T)$ is a torsor under $R_{L/K}(A)$. Since $\mathrm{Hom}_K(V, R_{L/K}(T)) = \mathrm{Hom}_L(V_L, T)$ (i.e., the adjoint property of the Weil restriction, e.g., [BLR90, p. 191, Lem. 1]), there is associated to α a K -morphism $V \rightarrow R_{L/K}(T)$. By the universal property of $\mathrm{Alb}_{V/K}$, this factors over K as

$$\begin{array}{ccc} V & \xrightarrow{\quad} & \mathrm{Alb}_{V/K}^1 \\ & \searrow & \downarrow \\ & & R_{L/K}(T) \end{array}$$

Again by the adjoint property of $R_{L/K}$, this induces a diagram over L

$$\begin{array}{ccc} V_L & \longrightarrow & (\text{Alb}_{V/K}^1)_L \\ & \searrow \alpha & \downarrow \\ & & T \end{array}$$

and so $(\text{Alb}_{V/K}^1)_L$ is the universal torsor receiving a map from V_L . $\square$

For clarity, recall that a normal Noetherian scheme is irreducible if and only if it is connected. In particular, if a scheme over K is smooth, then it is geometrically reduced and geometrically connected if and only if it is geometrically integral.

Lemma 5.3. *Let V/K be a smooth geometrically integral scheme over a field K . If $\iota : U \hookrightarrow V$ is an open immersion, then the universal morphism indicated with the dashed arrow in the diagram below:*

$$\begin{array}{ccc} U & \longrightarrow & \text{Alb}_{U/K}^1 \\ \iota \downarrow & & \downarrow \cong \\ V & \longrightarrow & \text{Alb}_{V/K}^1 \end{array}$$

is an isomorphism, equivariant with respect to a canonical isomorphism

$$\text{Alb}_{U/K} \cong \text{Alb}_{V/K}.$$

In particular, pre-composing with the inclusion $U \hookrightarrow V$ converts Albanese data (resp. pointed Albanese data) for V into Albanese (resp. pointed Albanese data) for U .

Proof. It suffices to show that, if $f : U \rightarrow T$ is a morphism to a torsor under an abelian variety, then f extends to a morphism $\tilde{f} : V \rightarrow T$. If T is an abelian variety, this is a special case of [BLR90, §8.4, Cor. 6, p.234]. The general case then follows from this since if $f_L : U_L \rightarrow T_L$ extends to a morphism $V_L \rightarrow T_L$ for some field extension L/K , then $f : U \rightarrow T$ extends to a morphism $\tilde{f} : V \rightarrow T$ (Lemma 4.6), and so one reduces to the previous case by base change to a field L/K over which T_L admits an L -point. $\square$

The next two lemmas establish that for a variety V that admits a smooth alteration, formation of the Albanese torsor commutes with separable base change:

Lemma 5.4. *Let V/K be a geometrically connected and geometrically reduced scheme of finite type over a field K . Suppose that there is a diagram*

$$\begin{array}{ccc} U & \xrightarrow{\iota} & X \\ \downarrow \pi & & \\ V & & \end{array}$$

of K -schemes with π dominant, ι a dense open immersion, and X a smooth proper scheme over K each connected component of which is geometrically integral. Let L/K be any field extension such that $L^{\text{Aut}(L/K)} = K$. Then $\beta_{V,L/K}$ and $\beta_{V,L/K}^1$ are isomorphisms.

Proof. Let us write $\sqcup_i \iota_i : \sqcup_i U_i \rightarrow \sqcup_i X_i$ for $\iota : U \hookrightarrow X$, with the X_i being the connected components of X . Note that, by Lemma 5.3 and Theorem 3.2, $\beta_{U_i,L/K}^1$ and $\beta_{U_i,L/K}$ are isomorphisms for all i . By

the universal property, each Albanese morphism $a_i : U_i \rightarrow \text{Alb}_{U_i/K}^1$ induces a diagram

$$\begin{array}{ccc} U_i & \xrightarrow{a_i} & \text{Alb}_{U_i/K}^1 \\ \pi_i \downarrow & & \downarrow \delta_i^1 \\ V & \xrightarrow{a} & \text{Alb}_{V/K}^1 \end{array}$$

and each δ_i^1 is equivariant with respect to the induced morphism $\delta_i : \text{Alb}_{U_i/K} \rightarrow \text{Alb}_{V/K}$ of abelian varieties. Let $\delta : \prod_i \text{Alb}_{U_i/K} \rightarrow \text{Alb}_{V/K}$ be the homomorphism induced by the δ_i . We claim that δ is surjective and that, for any field extension L/K such that $L^{\text{Aut}(L/K)} = K$, we have that

$$\ker(\delta_L : \prod_i \text{Alb}_{(U_i)_L/L} \rightarrow \text{Alb}_{V_L/L})$$

is invariant under $\text{Aut}(L/K)$, so that $\ker \delta_L$ descends to $\ker \delta$.

The surjectivity of δ can be seen as follows. Choose a finite field extension M/L such that each $(U_i)_M$ acquires an M -point. Since the image of V_M in its Albanese variety $\text{Alb}_{V_M/M}$ generates $\text{Alb}_{V_M/M}$ and since the disjoint union $\bigsqcup_i (U_i)_M$ dominates V_M , we see that $\text{Alb}_{V_M/M}$ is generated by certain translates of the images of the induced homomorphisms $\text{Alb}_{(U_i)_M/M} \rightarrow \text{Alb}_{V_M/M}$ and it ensues that δ_M , and hence δ , is surjective (as surjective morphisms satisfy fpqc descent; *e.g.*, [GW20, p.584]).

For the $\text{Aut}(L/K)$ -invariance of $\ker \delta_L$, we argue as follows. Let $\sigma \in \text{Aut}(L/K)$, and for an L -scheme Y , denote by Y^σ the pull-back of Y along $\sigma : \text{Spec } L \rightarrow \text{Spec } L$. We want to show there is a canonical L -isomorphism $(\ker \delta_L)^\sigma = \ker \delta_L$. For this, consider the diagram

$$\begin{array}{ccccccc} V_L & \xrightarrow{a_L} & \text{Alb}_{V_L/L} & \xleftarrow{\delta_L} & \prod_i \text{Alb}_{(U_i)_L/L} & \xleftarrow{\quad} & \ker \delta_L \\ \parallel & & \downarrow & & \parallel & & \\ (V_L)^\sigma & \xrightarrow{(a_L)^\sigma} & (\text{Alb}_{V_L/L})^\sigma & \xleftarrow{(\delta_L)^\sigma} & \prod_i (\text{Alb}_{(U_i)_L/L})^\sigma & \xleftarrow{\quad} & (\ker \delta_L)^\sigma \end{array}$$

where the dashed arrow is induced by the universal property of the Albanese. One concludes from a diagram chase that there is a scheme-theoretic inclusion

$$\ker \delta_L \subseteq (\ker \delta_L)^\sigma.$$

Applying the same argument to σ^{-1} , we see that $\ker \delta_L \subseteq (\ker \delta_L)^{\sigma^{-1}}$, and then applying σ to both sides, we have $(\ker \delta_L)^\sigma \subseteq ((\ker \delta_L)^{\sigma^{-1}})^\sigma = \ker \delta_L$, so that $(\ker \delta_L)^\sigma = \ker \delta_L$, as claimed.

Now, since we have established that $(\ker \delta)_L = \ker \delta_L$, we have, for any field extension L/K such that $L^{\text{Aut}(L/K)} = K$, a commutative diagram:

$$\begin{array}{ccc} \text{Alb}_{V_L/L} & \xleftarrow{\cong} & (\prod_i \text{Alb}_{(U_i)_L/L}) / \ker \delta_L \\ \beta_{V,L/K} \downarrow & & \cong \downarrow \prod_i \beta_{U_i,L/K} \\ (\text{Alb}_{V_K/K})_L & \xleftarrow{\cong} & (\prod_i \text{Alb}_{U_i/K})_L / (\ker \delta)_L \end{array}$$

showing that $\beta_{V,L/K}$ is an isomorphism. By Lemma 3.1, $\beta_{V,L/K}^1$ is then also an isomorphism. $\square$

Lemma 5.5. *Suppose U , V , and X are as in Lemma 5.4. If L/K is any separable extension, then $\beta_{V,L/K}$ is an isomorphism.*

Proof. Let K^{sep} and L^{sep} be separable closures of, respectively, K and L , and consider the diagram of separable (thanks to Lemma 4.2) field extensions

$$\begin{array}{ccc} & L^{\text{sep}} & \\ & | & \searrow \\ & K^{\text{sep}} & \\ & \swarrow & \downarrow \\ & & L \\ & & | \\ & & K \end{array}$$

As we have $(K^{\text{sep}})^{\text{Aut}(K^{\text{sep}}/K)} = K$, $(L^{\text{sep}})^{\text{Aut}(L^{\text{sep}}/L)} = L$, and by Lemma 4.1 we also have the identification $(L^{\text{sep}})^{\text{Aut}(L^{\text{sep}}/K^{\text{sep}})} = K^{\text{sep}}$, we can apply Lemma 5.4 to all three extensions with solid segments in the diagram above. Together with the universal property of the Albanese morphism, we therefore obtain the diagram:

$$\begin{array}{ccc} \text{Alb}_{V_{L^{\text{sep}}}/L^{\text{sep}}} & \xrightarrow[\cong]{\beta_{V_{L^{\text{sep}}}/L^{\text{sep}}/L}} & (\text{Alb}_{V_L/L})_{L^{\text{sep}}} \\ \beta_{V_{L^{\text{sep}}}/K^{\text{sep}}} \downarrow \cong & & \downarrow (\beta_{V,L/K})_{L^{\text{sep}}} \\ (\text{Alb}_{V_{K^{\text{sep}}}/K^{\text{sep}}})_{L^{\text{sep}}} & & \\ (\beta_{V,K^{\text{sep}}/K})_{L^{\text{sep}}} \downarrow \cong & & \\ ((\text{Alb}_{V/K})_{K^{\text{sep}}})_{L^{\text{sep}}} & \equiv & ((\text{Alb}_{V/K})_L)_{L^{\text{sep}}} \end{array}$$

It follows that $\beta_{V,L/K}$ becomes an isomorphism after base-change to L^{sep} , and hence that it is an isomorphism. $\square$

In the case of a purely inseparable extension L/K , it turns out that the L/K image explains the Raynaud Example 1.1:

Theorem 5.6 (Theorem (B)). *Let V/K be a geometrically connected and geometrically reduced scheme of finite type over a field K . Suppose L/K is a purely inseparable extension. Then there is a commutative diagram*

$$\begin{array}{ccc} \text{Alb}_{V_L/L} & \xrightarrow{\beta_{V,L/K}} & (\text{Alb}_{V/K})_L \\ & \searrow \lambda & \downarrow \cong \\ & & (\text{im}_{L/K} \text{Alb}_{V_L/L})_L \end{array} \quad (5.1)$$

induced by an isomorphism $\text{Alb}_{V/K} \cong \text{im}_{L/K} \text{Alb}_{V_L/L}$, where λ is the universal morphism in the definition of the L/K -image (4.2).

If V admits a K -point v , then the composition of pointed L -morphisms

$$(V_L, v_L) \xrightarrow{a_{V_L/L}} (\text{Alb}_{V_L/L}, \mathbf{0}) \xrightarrow{\lambda} ((\text{im}_{L/K} \text{Alb}_{V_L/L})_L, \mathbf{0}) \quad (5.2)$$

is initial for pointed L -morphisms $(V_L, v_L) \rightarrow (A_L, \mathbf{0})$, where A/K is an abelian variety over K , and (5.2) descends to K to give a pointed K -morphism

$$(V, v) \xrightarrow{a_{V/L}} (\text{im}_{L/K} \text{Alb}_{V_L/L}, \mathbf{0}) \quad (5.3)$$

providing Albanese data for (V, v) ; i.e., $(\text{im}_{L/K} \text{Alb}_{V_L/L}, a_{V_L/L})$ is pointed Albanese data for (V, v) .

Proof. We wish to establish (5.1), first. To this end, let $V = \bigcup_{i=1}^n U_i$ be an affine open cover. Since V is geometrically reduced, each U_i admits a point over some finite separable extension M/K , which can be chosen to be independent of i . By Lemmas 4.3 and 5.2, it suffices to verify the lemma after base change to M . Thus, we may and do assume in particular that V admits a K -point, and consequently that the Albanese torsor and the Albanese abelian variety coincide. Moreover, each of the U_i is separated, being affine, and geometrically reduced, being contained in V . Moreover, we can take the U_i to be connected, and then, since we are allowed to take finite separable base changes, we may take the U_i to be geometrically connected, as well. In other words, we may assume that V admits an open cover $\{(U_i, u_i)\}$ by separated (geometrically) connected and geometrically reduced schemes U_i of finite type over K , with each irreducible component of the U_i being geometrically integral and admitting a smooth K -point u_i . Moreover, we have reduced to proving, under these hypotheses, the second assertion of the lemma, namely that (5.2) descends to (5.3), and that this gives pointed Albanese data.

Let a be the composite map (5.2)

$$a : V_L \xrightarrow{a_{V_L/L}} \text{Alb}_{V_L/L} \xrightarrow{\lambda} (\text{im}_{L/K} \text{Alb}_{V_L/L})_L.$$

From Lemma 4.7, it is initial for pointed maps from V_L to the base change to L of abelian varieties defined over K (establishing one of the claims of Theorem 5.6), and descends to a pointed K -morphism $\underline{a} : V \rightarrow \text{im}_{L/K} \text{Alb}_{V_L/L}$ over K . We claim that this implies that $\underline{a} : V \rightarrow \text{im}_{L/K} \text{Alb}_{V_L/L}$ is a pointed Albanese. Indeed, given a pointed morphism $V \rightarrow A$ to an abelian variety A , we obtain a unique morphism making the following diagram commute:

$$\begin{array}{ccc} V_L & \xrightarrow{a} & (\text{im}_{L/K} \text{Alb}_{V_L/L})_L \\ & \searrow & \downarrow \\ & & A_L \end{array}$$

Then, from Chow rigidity [Con06, Thm. 3.19], one has a unique morphism making the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{\underline{a}} & \text{im}_{L/K} \text{Alb}_{V_L/L} \\ & \searrow & \downarrow \\ & & A \end{array}$$

showing that $\underline{a} : V \rightarrow \text{im}_{L/K} \text{Alb}_{V_L/L}$ is the pointed Albanese. □

Finally, we can prove our main result.

Proof of Theorem 5.1. By Lemma 5.2, after possibly base-changing along a finite separable field extension, we may and do assume that the irreducible components of V are geometrically integral. We will identify a finite purely inseparable extension L/K such that V admits a smooth alteration relative to L , and chase Albanese varieties along the diagram of fields (4.3).

Let V_i be the irreducible components of V and for each i choose an open affine (and so separated) subset $V'_i \subseteq V_i$. Using Nagata compactification [Con07], embed $V'_i \hookrightarrow Y_i$ into a proper geometrically integral variety. Using [dj96, Thm. 4.1], there is a diagram

$$\begin{array}{ccc} U_i & \hookrightarrow & X_i \\ \downarrow & & \downarrow \\ V'_i & \hookrightarrow & Y_i \end{array}$$

in which the vertical arrows are alterations; moreover, there is a finite, purely inseparable extension L_i/K such that the structural morphism $X_i \rightarrow \operatorname{Spec} K$ factors through $\operatorname{Spec} L_i$, and $X_i \rightarrow \operatorname{Spec} L_i$ is smooth [dJ96, Rem. 4.2]. The composition $U_i \hookrightarrow X_i \rightarrow \operatorname{Spec} L_i$ together with the map $U_i \rightarrow V$ determine a unique morphism $U_i \rightarrow V_{L_i}$ over L_i , giving the following diagram over L_i :

$$\begin{array}{ccc} U_i & \hookrightarrow & X_i \\ \downarrow & & \\ V_{L_i} & & \end{array}$$

Letting L/K be the (purely inseparable) composite of the L_i (in some algebraically closed field containing the L_i), base changing to L , and then taking unions, i.e., $U := \bigsqcup_i U_i \times_{L_i} L$ and $X := \bigsqcup_i X_i \times_{L_i} L$, and we obtain a diagram over L :

$$\begin{array}{ccc} U & \hookrightarrow & X \\ \downarrow & & \\ V_L & & \end{array} \quad (5.4)$$

satisfying the hypotheses of Lemma 5.4 over L . Indeed, the only thing to check is that V_L is geometrically reduced over L , and that $U \rightarrow V_L$ is dominant. The former holds, as for any extension L'/L we have $(V_L) \times_L L' = (V \times_K L) \times_L L' = (V_K) \times_K L'$. For the latter, we started with $\bigsqcup_i U_i \rightarrow V$ dominant. It follows that the composition $U \rightarrow \bigsqcup_i U_i \rightarrow V$ is dominant. Moreover, this morphism factors through $V_L \rightarrow V$. From say [GW20, Prop. 4.35, p.111] one has that $V_L \rightarrow V$ is injective by virtue of the fact that L/K is purely inseparable, and one can conclude that $U \rightarrow V_L$ is dominant.

Now, let M/K be a separable extension of fields, and consider the tower of field extensions in (4.3). We then compute canonical isomorphisms:

$$\begin{array}{ccc} V_M & \xrightarrow{a_{V_M/M}} & \operatorname{Alb}_{V_M/M} \\ \parallel & & \parallel \\ V_M & \xrightarrow{a_{V_{LM}/LM}} & \operatorname{im}_{LM/M}(\operatorname{Alb}_{V_{LM}/LM}) & \text{(Theorem 5.6)} \\ \parallel & & \parallel \\ V_M & \xrightarrow{(a_{V_L/L})_{LM}} & \operatorname{im}_{LM/M}((\operatorname{Alb}_{V_L/L})_{LM}) & \text{(Lemma 5.5, (5.4))} \\ \parallel & & \parallel \\ V_M & \xrightarrow{(a_{V_L/L})_M} & (\operatorname{im}_{L/K} \operatorname{Alb}_{V_L/L})_M & \text{(Proposition 4.8)} \\ \parallel & & \parallel \\ V_M & \xrightarrow{(a_{V/K})_M} & (\operatorname{Alb}_{V/K})_M & \text{(Theorem 5.6)} \end{array}$$

completing the proof. $\square$

We note that under the separated hypothesis, combining Proposition 2.12 with [Sch23, Thm. p.4] one has the following base change result:

Proposition 5.7 (Schröer). *If V is a separated scheme of finite type over a field K that admits an Albanese datum (see Proposition 2.12(ii)), then the Albanese datum is stable under separable base change of field. For purely inseparable field extensions, if $\Gamma(V, \mathcal{O}_V)$ is in addition geometrically³ reduced then the base change morphism is a universal homeomorphism.*

³The hypothesis that $\Gamma(V, \mathcal{O}_V)$ be geometrically reduced is implicit in [Sch23, Thm. p.4] for field extensions that are not separable; the necessity of this assumption is made clear by Example 2.11.

Proof. Proposition 2.12(ii) implies that V satisfies the hypotheses of Schröer [Sch23, Thm. p.4]. $\square$

6. THE UNIVERSAL PROPERTY OF ALBANESE VARIETIES

In fact, Theorem 3.2 as stated above is weaker than what Grothendieck [Gro62, Thm. VI.3.3(iii)] and Conrad [Con17, Thm.] actually prove:

Theorem 6.1 (Grothendieck–Conrad). *Let V/K be a proper geometrically connected and geometrically reduced scheme over a field K . Then for any morphism of schemes $S \rightarrow \operatorname{Spec} K$, and any S -morphism $f : V_S \rightarrow P$ to a torsor under an abelian scheme A/S , there exists a unique S -morphism $g : (\operatorname{Alb}_{V/K}^1)_S \rightarrow P$ such that $g \circ a_S = f$. If V admits a K -point v , then for any morphism $S \rightarrow \operatorname{Spec} K$, and any pointed S -morphism $f : V_S \rightarrow A$ to an abelian scheme A/S taking v_S to $\mathbf{0}_A$, there exists a unique S -homomorphism $g : (\operatorname{Alb}_{V/K})_S \rightarrow A$ such that $g \circ (a_v)_S = f$.* $\square$

Remark 6.2. Recall (similarly to Remark 2.1) that if A and A' are abelian schemes over a scheme S , and P and P' are torsors under A and A' , respectively, then for any S -morphism $g : P \rightarrow P'$, there is a unique S -homomorphism $\phi : A \rightarrow A'$ making g equivariant, and moreover, $g(P)$ is a torsor under $\phi(A)$. In particular, in the theorem above, there is a unique S -homomorphism $(\operatorname{Alb}_{V/K})_S \rightarrow A$ making $g : (\operatorname{Alb}_{V/K}^1)_S \rightarrow P$ equivariant.

If one is willing to restrict to base change by smooth morphisms, one can derive a similar statement without a properness hypothesis.

Theorem 6.3 (Arbitrary separable base change). *Let V be a geometrically connected and geometrically reduced scheme of finite type over a field K . Then for any (inverse limit of) smooth morphism of schemes $S \rightarrow \operatorname{Spec} K$, and any S -morphism $f : V_S \rightarrow P$ to a torsor under an abelian scheme A/S , there exists a unique S -homomorphism $(\operatorname{Alb}_{V/K})_S \rightarrow A$ and a unique equivariant S -morphism $g : (\operatorname{Alb}_{V/K}^1)_S \rightarrow P$ such that $g \circ a_S = f$.*

If V admits a K -point v , then for any (inverse limit of) smooth morphism $S \rightarrow \operatorname{Spec} K$, and any pointed S -morphism $f : V_S \rightarrow A$ to an abelian scheme A/S taking v_S to $\mathbf{0}_A$, there exists a unique S -homomorphism $g : (\operatorname{Alb}_{V/K})_S \rightarrow A$ such that $g \circ (a_v)_S = f$.

Proof. We give the proof for the Albanese torsor; the case of the pointed Albanese variety is similar. It suffices to consider the case where S is irreducible. By assumption on the morphism $S \rightarrow \operatorname{Spec} K$, the extension $\kappa(S)/K$ is separable. Consider then the restriction f_{η_S} of $f : X_S \rightarrow P$ to the generic point η_S of S . By Theorem 5.1, f_{η_S} factors through $(\operatorname{alb}_{V/K}^1)_{\eta_S}$. This gives a canonical η_S -morphism of torsors $(\operatorname{Alb}_{V/K}^1)_{\eta_S} \rightarrow P_{\eta_S}$ over $(\operatorname{Alb}_{V/K})_{\eta_S}$. Let $U \subseteq S$ be an open dense subscheme to which these morphisms extend as $g^1 : (\operatorname{Alb}_{V/K}^1)_U \rightarrow P_U$ over $g : (\operatorname{Alb}_{V/K})_U \rightarrow A_U$. By Raynaud’s extension theorem [FC90, I.2.7], g extends to a morphism of abelian schemes over S . Let $S' \rightarrow S$ be an fpqc morphism such that $(\operatorname{Alb}_{V/K}^1)_{S'} \rightarrow S'$ and $T_{S'} \rightarrow S'$ admit sections. Then $(\operatorname{Alb}_{V/K}^1)_{S'}$ and $P_{S'}$ are trivial torsors under abelian schemes over S' , and so $g_{U \times_S S'}^1$ extends to a morphism $(g^1)' : (\operatorname{Alb}_{V/K}^1)_{S'} \rightarrow P_{S'}$. By fpqc descent (Lemma 6.4 below), $(g^1)'$ descends to a morphism $g^1 : (\operatorname{Alb}_{V/K}^1)_S \rightarrow T$, as desired. $\square$

Lemma 6.4. *Let S be a scheme and let X and Y be schemes over S , with Y/S separated. Let $U \subseteq S$ be an open dense subscheme, and let $S' \rightarrow S$ be faithfully flat and quasicompact. Suppose $f : X_U \rightarrow Y_U$ is a morphism of schemes over U . If $f_{S'} : X_U \times_S S' \rightarrow Y_U \times_S S'$ extends to a morphism $\tilde{f}' : X_{S'} \rightarrow Y_{S'}$, and $X_U \times_S S'$ is dense in $X_{S'}$, then $\tilde{f}'$ descends to a morphism $\tilde{f} : X \rightarrow Y$ over S , and $\tilde{f}|_U = f$.*

Proof. Let $S'' = S' \times_S S'$, equipped with the two projections $p_i : S'' \rightarrow S'$. Let $\Gamma_{\tilde{f}'} \subseteq X_{S'} \times_{S'} Y_{S'}$ be the graph of $\tilde{f}'$ (since the graph morphism $\Gamma_{\tilde{f}'} : X_{S'} \rightarrow X_{S'} \times_{S'} Y_{S'}$ is a closed embedding, as

Y/S is assumed to be separated, we are identifying the graph morphism with its scheme-theoretic image). By Grothendieck’s theory of fpqc descent (e.g., [BLR90, §6.1] or [Con06, Thm. 3.1]), it suffices to demonstrate an equality of closed subschemes $p_1^*(\Gamma_{\tilde{f}}) = p_2^*(\Gamma_{\tilde{f}})$. However, $p_i^*(\Gamma_{\tilde{f}})$ contains $p_i^*(\Gamma_{f_{S'}})$ as a dense set (here we are using that $X_U \times_S S'$ is assumed to be dense in $X_{S'}$ and that the scheme-theoretic image is stable under flat base change, e.g., [EH00, Prop. V-8, p.217]); and $p_1^*(\Gamma_{f_{S'}}) = p_2^*(\Gamma_{f_{S'}})$, because $f_{S'}$ descends to f . $\square$

REFERENCES

- [AC15] Jeffrey D. Achter and Clifton Cunningham, *A note on L-packets and abelian varieties over local fields*, Pacific J. Math. **273** (2015), no. 2, 395–412. MR 3317772
- [ACMV23] Jeffrey D. Achter, Sebastian Casalaina-Martin, and Charles Vial, *A functorial approach to regular homomorphisms*, Algebr. Geom. **10** (2023), no. 1, 87–129. MR 4537125
- [AHL13] Yves Aubry, Safia Haloui, and Gilles Lachaud, *On the number of points on abelian and Jacobian varieties over finite fields*, Acta Arith. **160** (2013), no. 3, 201–241. MR 3106095
- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR 1045822 (91i:14034)
- [Bou81] Nicolas Bourbaki, *Éléments de mathématique*, Masson, Paris, 1981, Algèbre. Chapitres 4 à 7. [Algebra. Chapters 4–7]. MR 643362
- [Bri17] Michel Brion, *Some structure theorems for algebraic groups*, Algebraic groups: structure and actions, Proc. Sympos. Pure Math., vol. 94, Amer. Math. Soc., Providence, RI, 2017, pp. 53–126. MR 3645068
- [Che60] C. Chevalley, *Sur la théorie de la variété de Picard*, Amer. J. Math. **82** (1960), 435–490. MR 118723
- [Con06] Brian Conrad, *Chow’s K/k -image and K/k -trace, and the Lang-Néron theorem*, Enseign. Math. (2) **52** (2006), no. 1-2, 37–108. MR 2255529 (2007e:14068)
- [Con07] ———, *Deligne’s notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257. MR 2356346
- [Con17] ———, *Albanese variety over non-perfect fields*, 2017, Available at <https://mathoverflow.net/questions/260982/albanese-variety-over-non-perfect-fields>, p. 1.
- [DG70] M. Demazure and A. Grothendieck, *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York, 1970. MR 0274459
- [dJ96] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93. MR 1423020
- [EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000. MR 1730819
- [FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford. MR 1083353
- [FT15] E. V. Flynn and D. Testa, *Finite Weil restriction of curves*, Monatsh. Math. **176** (2015), no. 2, 197–218. MR 3302155
- [Gro62] Alexander Grothendieck, *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*, Secrétariat mathématique, Paris, 1962. MR 0146040
- [Gro67] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 238860
- [GW20] Ulrich Görtz and Torsten Wedhorn, *Algebraic geometry I. Schemes—with examples and exercises*, Springer Studium Mathematik—Master, Springer Spektrum, Wiesbaden, [2020] ©2020, Second edition [of 2675155]. MR 4225278
- [Hal08] Chris Hall, *Big symplectic or orthogonal monodromy modulo l* , Duke Math. J. **141** (2008), no. 1, 179–203. MR 2372151
- [HZ02] Everett W. Howe and Hui June Zhu, *On the existence of absolutely simple abelian varieties of a given dimension over an arbitrary field*, J. Number Theory **92** (2002), no. 1, 139–163. MR 1880590
- [Kat19] Nicholas M. Katz, *A note on Galois representations with big image*, Enseign. Math. **65** (2019), no. 3-4, 271–301. MR 4113044
- [Ked22] Kiran S. Kedlaya, *The relative class number one problem for function fields, I*, Res. Number Theory **8** (2022), no. 4, Paper No. 79, 21. MR 4493405

- [LS24] Bruno Laurent and Stefan Schröer, *Para-abelian varieties and Albanese maps*, Bull. Braz. Math. Soc. (N.S.) **55** (2024), no. 1, Paper No. 4, 39. MR 4680514
- [Lüt93] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Math. **80** (1993), no. 1, 95–111. MR 1226600
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906
- [Mil72] J. S. Milne, *On the arithmetic of abelian varieties*, Invent. Math. **17** (1972), 177–190. MR 330174
- [ML39] Saunders Mac Lane, *Modular fields. I. Separating transcendence bases*, Duke Math. J. **5** (1939), no. 2, 372–393. MR 1546131
- [Moc12] Shinichi Mochizuki, *Topics in absolute anabelian geometry I: generalities*, J. Math. Sci. Univ. Tokyo **19** (2012), no. 2, 139–242. MR 2987306
- [NS52] André Néron and Pierre Samuel, *La variété de Picard d’une variété normale*, Ann. Inst. Fourier (Grenoble) **4** (1952), 1–30 (1954). MR 61858
- [Ols08] Martin C. Olsson, *Compactifying moduli spaces for abelian varieties*, Lecture Notes in Mathematics, vol. 1958, Springer-Verlag, Berlin, 2008. MR 2446415
- [Sch23] Stefan Schröer, *Albanese Maps for Open Algebraic Spaces*, International Mathematics Research Notices (2023), rrad187.
- [Ser60] J.-P. Serre, *Morphismes universels et variétés d’Albanese*, Séminaire C. Chevalley, 3ième année: 1958/59, vol. 4, École Normale Supérieure, Paris, 1960. MR 0157865
- [Sta22] The Stacks Project Authors, *Stacks Project*, <http://stacks.math.columbia.edu>, 2022.
- [Wit08] Olivier Wittenberg, *On Albanese torsors and the elementary obstruction*, Math. Ann. **340** (2008), no. 4, 805–838. MR 2372739

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