NORMAL FUNCTIONS FOR ALGEBRAICALLY TRIVIAL CYCLES ARE ALGEBRAIC FOR ARITHMETIC REASONS

JEFFREY D. ACHTER, SEBASTIAN CASALAINA-MARTIN, AND CHARLES VIAL

ABSTRACT. For families of smooth complex projective varieties we show that normal functions arising from algebraically trivial cycle classes are algebraic, and defined over the field of definition of the family. In particular, the zero loci of those functions are algebraic and defined over such a field of definition. This proves a conjecture of Charles.

1. INTRODUCTION

Let $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ be a smooth surjective projective morphism of complex algebraic manifolds, let n be an integer, and let $\mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B}) \to \mathsf{B}$ be the $(2n+1)$-st relative Griffiths intermediate Jacobian. If $\mathsf{Z} \in \mathrm{CH}^{n+1}(\mathsf{X})$ is an algebraic cycle class such that for every $\mathsf{b} \in \mathsf{B}$ the Gysin fiber Z_b is algebraically (resp. homologically) trivial, then there is an associated holomorphic function

$$\nu_\mathsf{Z} : \mathsf{B} \longrightarrow \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B}), \qquad \nu_\mathsf{Z}(\mathsf{b}) = \mathsf{AJ}_{\mathsf{X}_\mathsf{b}}(\mathsf{Z}_\mathsf{b}),$$

where $\mathsf{AJ}_{\mathsf{X}_\mathsf{b}} : \mathrm{CH}^{n+1}(\mathsf{X}_\mathsf{b})_{hom} \to \mathsf{J}^{2n+1}(\mathsf{X}_\mathsf{b})$ is the Abel–Jacobi map on homologically trivial cycles in the fiber X_b. Such a function is called an algebraically motivated (resp. motivated) normal function motivated by the cycle class Z.

More generally, let B be a complex manifold, and let $\mathcal{H}$ be a variation of pure negative weight integral Hodge structures over B. In [Sai96], Saito defines the notion of an admissible normal function as a holomorphic section $\nu : \mathsf{B} \to \mathsf{J}(\mathcal{H})$ of the associated family of generalized intermediate Jacobians $\mathsf{J}(\mathcal{H}) \to \mathsf{B}$ that satisfies a version of Griffiths horizontality and has controlled asymptotic behavior near the boundary (see *e.g.*, [BP13]). Despite the transcendental nature of the definition of admissible normal functions, there is the following conjecture due to Green and Griffiths (*e.g.*, [BP09, p.883], [Cha10, Conj. 1], [Sch12, Conj. 1.1], [BP13, p.1914]):

Conjecture 1 (Green–Griffiths). *The zero locus of an admissible normal function on a complex algebraic manifold is algebraic.*

Proofs of this conjecture were given in a series of papers: $\dim \mathsf{B} = 1$ [Sai08, Cor. 1], [BP09, Thm. 4.5], $\dim \mathsf{B} \geq 1$ [Sch12, Thm. C], [BP13, Cor. 1.3] (see also Sém. Bourbaki [Cha14]). In this paper, we are interested in algebraic and arithmetic questions concerning motivated normal functions. First, for algebraically motivated (resp. motivated) normal functions, if X, B, f, and Z are all defined over a subfield $F \subseteq \mathbb{C}$, we say that the normal function ν_Z is algebraically F-motivated (resp. F-motivated), and it is natural to ask whether the zero locus of ν_Z in B is also defined over F [Cha10, p.2284] (see also [KP11, Conj. 81]):

Conjecture 2 (Charles). *Let $F \subseteq \mathbb{C}$ be a subfield. The zero locus of an algebraically F-motivated (resp. F-motivated) normal function is algebraic and defined over F.*

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Several partial results are known. Regarding the F-motivated case of Conjecture 2, a special case of a result of Saito [Sai16, Cor. 1] on admissible normal functions implies that if an irreducible component of the zero locus of an F-motivated normal function contains a point of B that is defined over F, then the entire component of the zero locus is defined over F (see also [Cha10, Thm. 3], [KP11, Thm. 89]). Regarding the algebraically F-motivated case of Conjecture 2, Kerr and Pearlstein have shown in [KP11, Con. 81, $\widetilde{\mathfrak{ZL}}(D,1)_{alg}$, Thm. 88] that the zero locus of an algebraically F-motivated normal function is an algebraic subset of B defined over a finite extension of F. All of the aforementioned results take as a starting point the validity of Conjecture 1.

In this paper we directly prove Conjecture 2 in the algebraically F-motivated case. (In particular, we do not rely on earlier work on Conjecture 1.) In fact, we prove a stronger result, namely that algebraically F-motivated normal functions are themselves algebraic and defined over F:

Theorem 1. *Let $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ be a smooth surjective projective morphism of complex algebraic manifolds (not necessarily connected), let n be a nonnegative integer, let $\mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B}) \to \mathsf{B}$ be the $(2n+1)$-st relative Griffiths intermediate Jacobian. There is a relative algebraic complex subtorus $\mathsf{J}_a^{2n+1}(\mathsf{X}/\mathsf{B}) \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$ over B such that for very general $\mathsf{u} \in \mathsf{B}$ the fiber $\mathsf{J}_a^{2n+1}(\mathsf{X}/\mathsf{B})_\mathsf{u} \subseteq \mathsf{J}^{2n+1}(\mathsf{X}_\mathsf{u})$ is the image $\mathsf{J}_a^{2n+1}(\mathsf{X}_\mathsf{u})$ of the Abel–Jacobi map $\mathsf{AJ}_{\mathsf{X}_\mathsf{u}} : \mathrm{A}^{n+1}(\mathsf{X}_\mathsf{u}) \to \mathsf{J}^{2n+1}(\mathsf{X}_\mathsf{u})$, and for any algebraic cycle class $\mathsf{Z} \in \mathrm{CH}^{n+1}(\mathsf{X})$ such that for every $\mathsf{b} \in \mathsf{B}$ the Gysin fiber Z_b is algebraically trivial:*

1. *The normal function $\nu_\mathsf{Z} : \mathsf{B} \to \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$ has image contained in $\mathsf{J}_a^{2n+1}(\mathsf{X}/\mathsf{B})$ and is an algebraic map.*
2. *If, moreover, X, B, f, and Z are all defined over a field $F \subseteq \mathbb{C}$, then so are $\mathsf{J}_a^{2n+1}(\mathsf{X}/\mathsf{B})$ and the morphisms $\mathsf{J}_a^{2n+1}(\mathsf{X}/\mathsf{B}) \to \mathsf{B}$ and ν_Z.*

See Remark 5.2 for a caution about the notation $\mathsf{J}_a^{2n+1}(\mathsf{X}/\mathsf{B})$, and see the notation and conventions below for a reminder on very general points. Conjecture 1 in the algebraically motivated case follows immediately from Theorem 1(1), and in this way we obtain a short proof of this case of the conjecture. Conjecture 2 in the algebraically F-motivated case follows immediately from Theorem 1(2). In summary, we have:

Corollary 1. *Let $F \subseteq \mathbb{C}$ be a subfield. The zero locus of an algebraically F-motivated normal function is algebraic and defined over F.*

We review some special easy cases of Theorem 1, with an eye toward explaining why their generalization is not immediate. In the case of $n+1 = 1$, *i.e.*, of $\mathsf{Pic}^0_{\mathsf{X}/\mathsf{B}}$, and $n+1 = \dim_\mathsf{B} \mathsf{X}$, *i.e.*, of $\mathsf{Alb}_{\mathsf{X}/\mathsf{B}}$, it is well known that algebraically F-motivated normal functions are algebraic and defined over F (*e.g.*, [Gro62, Thm. VI.3.3], [Kle05, Def. 4.6 and Thm. 4.8]). In the case where B is quasiprojective, $\mathsf{X} = \mathsf{B} \times \mathsf{Y}$ for some smooth projective complex manifold Y, and $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ is the first projection, part (1) of the theorem is elementary by embedding B in a smooth complex projective manifold $\overline{\mathsf{B}}$, extending the cycle class Z to a cycle class $\overline{\mathsf{Z}}$ on $\overline{\mathsf{B}} \times \mathsf{Y}$, and obtaining a normal function $\nu_{\overline{\mathsf{Z}}} : \overline{\mathsf{B}} \to \overline{\mathsf{B}} \times \mathsf{J}_a^{2n+1}(\mathsf{Y})$ that is a holomorphic map between complex projective manifolds. From our work in [ACMV18], one can then easily deduce (2) of the theorem in this case, as well. The difficulty in using the same strategy to prove part (1) of the theorem in general is twofold. First, the family $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ may not extend to a smooth family over $\overline{\mathsf{B}}$, in which case it is difficult to know how to extend $\mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$ and ν_Z to the boundary. Second, even if one can extend $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ to a smooth family $\overline{\mathsf{f}} : \overline{\mathsf{X}} \to \overline{\mathsf{B}}$, the geometric coniveau of the family can jump along a countable union of algebraic subsets of $\overline{\mathsf{B}}$, and so there is no obvious algebraic target $\mathsf{J}_a^{2n+1}(\overline{\mathsf{X}}/\overline{\mathsf{B}})$ for an extended normal function.

One faces similar difficulties in trying to prove Conjecture 1, and the approach taken in [Sai08, BP09, Sch12] overcomes these complications by constructing Néron models for the relative intermediate Jacobians (see also [GGK10]) that provide manageable targets for extending admissible normal functions. In the special case where $\dim \mathsf{B} = 1$, Schnell and Kerr independently communicated to us arguments using these techniques to prove part (1) of the theorem, up to replacing the normal function ν_{Z} with $M \cdot \nu_{\mathsf{Z}}$ for some integer M, depending on Z. It appears however that it would be difficult to extend these arguments to the case where $\dim \mathsf{B} \geq 2$. It also appears it would be difficult to use these techniques to prove part (2) of the theorem regarding the field of definition, even in the case where $\dim \mathsf{B} = 1$.

The starting point of our proof consists in showing that, for a smooth projective variety X defined over a subfield $K \subseteq \mathbb{C}$, the kernel of the Abel–Jacobi map restricted to algebraically trivial cycles defined over K is independent of the choice of field embedding $K \subseteq \mathbb{C}$. This is embodied in Corollary 3.3 ; in fact, a stronger result is proved in Proposition 3.1 where it is shown that the distinguished model of [ACMV18] does not depend on a choice of field embedding. The proof uses in an essential way the fact proven in [ACMV19] that algebraically trivial cycles defined over K are parameterized by abelian varieties, and builds on our previous work [ACMV18]. Consequently, the relevant material of [ACMV18] is reviewed in §2. An important consequence of Proposition 3.1 is that an algebraically F-motivated normal function vanishes at a very general point if and only if it vanishes on a Zariski open subset, and is therefore identically zero ; see Example 3.6 and Remarks 3.7 and 3.8.

In fact, the initial step of our strategy is to consider a very general fiber $\mathsf{X_u}$ and, thanks to Proposition 3.1, to descend the image of the Abel–Jacobi map $\mathsf{J}^{2n+1}_a(\mathsf{X_u})$ for this fiber to an abelian variety over the generic point of B, which admits a natural section related to the normal function. We then spread this abelian variety and section to a Zariski open subset of B, all defined over F. In Theorem 4.1, we compare this abelian scheme together with the induced section to the analytic normal function. This is achieved through comparing the related variations of Hodge structures via an algebraic correspondence defined over K, provided by Theorem 2.1. There is a technical point here, that the correspondence only identifies the integral Hodge structures, as well as our algebraic section and the analytic normal function, up to an integer multiple M ; in Theorem 4.1 we show that the image of the morphism of abelian varieties induced by the correspondence, inside the Griffiths intermediate Jacobian, is in fact the spread of our distinguished model, and that the algebraic section and analytic normal function are identified. The final step is to extend this to all of B (§5). We extend the relative algebraic torus over the generic points of codimension-1 boundary loci by using the good reduction of X and the Néron–Ogg–Shafarevich criterion, and then extend over codimension-2 loci using the Faltings–Chai Extension Theorem. The normal function is handled separately at each step. In short, rather than having to worry about extending admissible normal functions to projective compactifications in order to obtain algebraicity as a consequence of Chow's theorem, we extend algebraic maps defined over F on a Zariski open subset of B to all of B, and in this way also manage to maintain control over the field of definition.

In forthcoming work [ACMV] we will study the notion of regular homomorphisms in the relative setting ; Theorem 1 shows that the Abel–Jacobi map provides such a relative regular homomorphism.

Finally, although our results are algebraic in nature and only concern algebraically trivial cycle classes, there are important instances of families of varieties for which homological and algebraic equivalence of cycles in certain codimensions agree and for which the corresponding intermediate Jacobians are algebraic [BS83]. For example, a direct application of Theorem 1 concerns codimension-2 cycles on uniruled threefolds :

Corollary 2. *Let* $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ *be a smooth projective family of uniruled threefolds, defined over a field* $F \subseteq \mathbb{C}$*, and let* $\mathsf{Z} \in \mathrm{CH}^2(\mathsf{X})$ *be a cycle class defined over* F *that is fiber-wise homologically trivial. Then the analytic normal function* ν_{Z} *is algebraic, and defined over* F*. In particular, its zero-locus is an algebraic sub-variety of* B *defined over* F*.*

In the opposite direction, as a specific example of a case where it is not clear how to apply our results to the F-motivated case of Conjecture 2, one can consider families of Calabi–Yau threefolds ; specifically for very general quintic threefolds, Voisin [Voi00, Thm. 2,3] has shown that the image of the Abel–Jacobi map on *algebraically* trivial codimension-2 cycle classes is trivial, while the image of the Abel–Jacobi map on *homologically* trivial codimension-2 cycle classes is a countable abelian group of infinite rank. In general, it seems to us that it would be interesting to construct a canonical arithmetic structure on the image of all homologically trivial cycles under the Abel–Jacobi map. Such a construction is a necessary first step for extending our methods to normal functions motivated by homologically trivial cycles.

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Notation and conventions. Let X be a scheme of finite type over a field $F \subseteq \mathbb{C}$. We denote by $X(F)$ the set of F-morphisms $\operatorname{Spec} F \to X$. We denote by $\mathsf{X} = X_{an}$ the associated complex analytic space. After identifying the sets

$$\begin{aligned} \mathsf{X} &= \{\mathfrak{p} \in |X_{\mathbb{C}}| : \mathfrak{p} \text{ is closed in the underlying topological space } |X_{\mathbb{C}}|\} \\ &= \{x \in X_{\mathbb{C}}(\mathbb{C}) : x(\operatorname{Spec}\mathbb{C}) \text{ is closed in } |X_{\mathbb{C}}|\}, \end{aligned}$$

we say that $\mathsf{x} \in \mathsf{X}$ is *F-very general* if the corresponding morphism $x : \operatorname{Spec}\mathbb{C} \to X$ has image a generic point of $|X|$. If F is countable, then so is the collection of all closed algebraic subsets of X that are defined over F and not equal to X ; any F-very general point x is in the complement of the union of these closed algebraic subsets. If Y is merely a complex algebraic variety, then a very general point of Y is an F-very general point for some field of definition F of Y which is of finite type over $\mathbb{Q}$.

A *variety* over a field is a geometrically reduced separated scheme of finite type over that field. Given a smooth projective variety X over a field $F \subseteq \mathbb{C}$ we denote by $\mathrm{CH}^i(X)$ (resp. $\mathrm{CH}^i(\mathsf{X})$) the Chow group of codimension-i algebraic cycle classes on X (resp. X), and by $\mathrm{A}^i(X)$ (resp. $\mathrm{A}^i(\mathsf{X})$) the subgroup of algebraically trivial algebraic cycle classes on X (resp. X).

For the remainder of the paper, the domain of the Abel–Jacobi map $\mathsf{AJ} : \mathrm{A}^{n+1}(\mathsf{X}) \to \mathsf{J}^{2n+1}(\mathsf{X})$ is the group of algebraically trivial algebraic cycle classes. We denote by $\mathsf{J}^{2n+1}_a(\mathsf{X})$ the image of this Abel–Jacobi map, and by $\mathsf{i}^{2n+1}_{a,\mathsf{X}} : \mathsf{J}^{2n+1}_a(\mathsf{X}) \to \mathsf{J}^{2n+1}(\mathsf{X})$ the natural inclusion.

2. Distinguished models of intermediate Jacobians and distinguished normal functions

In this section we recall some results from [ACMV17, ACMV18] regarding descending intermediate Jacobians to a field of definition.

2.1. **Distinguished models of intermediate Jacobians.**

We start by recalling the main result of [ACMV18]. We note that while in §2.1, 2.2 we work over a field $K \subseteq \mathbb{C}$, starting from §3.2 we will implement these results in the case where the field K is the residue field of the generic point of the integral base B of a smooth projective family $f : X \to B$, all defined over a field $F \subseteq \mathbb{C}$.

Theorem 2.1 (Distinguished models [ACMV18, Thm. 1])**.** *Suppose X is a smooth projective variety over a field $K \subseteq \mathbb{C}$, with associated complex analytic space X, and let n be a nonnegative integer. Then $\mathsf{J}^{2n+1}_a(\mathsf{X})$, the algebraic complex torus that is the image of the Abel–Jacobi map $\mathsf{AJ} : \mathrm{A}^{n+1}(\mathsf{X}) \to \mathsf{J}^{2n+1}(\mathsf{X})$,* *admits a distinguished model $J^{2n+1}_{a,X/K}$ over K such that the Abel–Jacobi map is* $\mathrm{Aut}(\mathbb{C}/K)$*-equivariant. Moreover, there exist a correspondence* $\Gamma \in \mathrm{CH}^{\dim(J^{2n+1}_{a,X/K})+n}(J^{2n+1}_{a,X/K} \times_K X)$ *and a positive integer M such that the induced morphism* $\Gamma_* : \mathsf{J}^{2n+1}_a(\mathsf{X}) \to \mathsf{J}^{2n+1}(\mathsf{X})$ *is* $M \cdot \mathsf{i}^{2n+1}_{a,\mathsf{X}}$; i.e., *M times the natural inclusion.*

Remark 2.2 (Uniqueness of the distinguished model). By Chow's rigidity theorem, an abelian variety $A/\mathbb{C}$ descends to at most one model defined over $\overline{K}$. On the other hand, an abelian variety $A/\overline{K}$ may descend to more than one model defined over K. Nevertheless, since $\mathsf{AJ} : \mathsf{A}^{n+1}(\mathsf{X}) \to \mathsf{J}^{2n+1}_a(\mathsf{X})$ is surjective, the algebraic complex torus $\mathsf{J}^{2n+1}_a(\mathsf{X})$ admits at most one structure of a variety over K such that AJ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. More precisely, setting $J^{2n+1}_a(X_{\mathbb{C}})$ to be the abelian variety associated to the algebraic complex torus $\mathsf{J}^{2n+1}_a(\mathsf{X})$, there an abelian variety $J^{2n+1}_{a,X/K}$ which is unique up to unique isomorphism, such that there is an isomorphism $(J^{2n+1}_{a,X/K})_{\mathbb{C}} \to J^{2n+1}_a(X_{\mathbb{C}})$ such that the induced action of $\mathrm{Aut}(\mathbb{C}/K)$ on $\mathsf{J}^{2n+1}_a(\mathsf{X})$ makes the Abel–Jacobi map $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. This is the sense in which $\mathsf{J}^{2n+1}_a(\mathsf{X})$ admits a *distinguished model* over K.

In [ACMV18, Thm. 1] we show that the correspondence Γ in Theorem 2.1 induces a morphism of complex tori $\Gamma_* : \mathsf{J}^{2n+1}_a(\mathsf{X}) \to \mathsf{J}^{2n+1}(\mathsf{X})$ with image $\mathsf{J}^{2n+1}_a(\mathsf{X})$. In fact, this morphism respects K-structures :

Lemma 2.3. *In the situation of Theorem 2.1, the morphism* $\Gamma_* : \mathsf{J}^{2n+1}_a(\mathsf{X}) \to \mathsf{J}^{2n+1}_a(\mathsf{X})$ *is induced by an isogeny* $\psi : J^{2n+1}_{a,X/K} \to J^{2n+1}_{a,X/K}$ *over K.*

Proof. It suffices to show that Γ_* is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant on torsion. This is achieved by identifying the map on N-torsion with the map $\Gamma_* : H^1(J^{2n+1}_a(X_{\mathbb{C}}), \boldsymbol{\mu}_N) \to H^1(J^{2n+1}_a(X_{\mathbb{C}}), \boldsymbol{\mu}_N) \subseteq H^{2n+1}(X_{\mathbb{C}}, \boldsymbol{\mu}_N^{\otimes(n+1)})$ (similar to [ACMV18, (2.3)]). Let M be the exponent of ψ, and let $\tilde{\psi} : J^{2n+1}_{a,X/K} \to J^{2n+1}_{a,X/K}$ be such that $\tilde{\psi} \circ \psi = M$. With $\Gamma_{\tilde{\psi}}$ the correspondence associated to the morphism $\tilde{\psi}$, let $\Gamma' = \Gamma_{\tilde{\psi}} \circ \Gamma$. It follows that the induced morphism $\Gamma'_* : \mathsf{J}^{2n+1}_a(\mathsf{X}) \to \mathsf{J}^{2n+1}(\mathsf{X})$ has image $\mathsf{J}^{2n+1}_a(\mathsf{X})$, and is given as $M \cdot \mathsf{i}^{2n+1}_{a,\mathsf{X}}$; *i.e.*, M times the natural inclusion. ☐

Remark 2.4 (Extensions $K \subseteq L \subseteq \mathbb{C}$). In the notation of the theorem, suppose we have an intermediate field extension $K \subseteq L \subseteq \mathbb{C}$. Then the base change $(J^{2n+1}_{a,X/K})_L$ is the distinguished model for X_L. Indeed, the distinguished model over L is determined uniquely by the fact that the Abel–Jacobi map for X is $\mathrm{Aut}(\mathbb{C}/L)$-equivariant ; but if the Abel–Jacobi map is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant with respect to the K-structure on $J^{2n+1}_{a,X/K}$, then it is $\mathrm{Aut}(\mathbb{C}/L)$-equivariant for the L-structure on $(J^{2n+1}_{a,X/K})_L$.

2.2. **Distinguished normal functions.** In [ACMV17] we established some results regarding equivariant regular homomorphisms. In this section we recall the consequences of that work in the context of normal functions and the distinguished model.

Let X be smooth projective variety over $K \subseteq \mathbb{C}$. Given $Z \in \mathrm{CH}^{n+1}(X)$ with the base change $Z_{\mathbb{C}}$ algebraically trivial, and $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, we showed in [ACMV17] that the following diagram commutes :

$$\begin{array}{ccccc}
(\operatorname{Spec}\mathbb{C})_{\mathbb{C}}(\mathbb{C}) & \xrightarrow{w_{Z_{\mathbb{C}}}} & \mathrm{A}^{n+1}(X_{\mathbb{C}}) & \xrightarrow{AJ} & J^{2n+1}_a(X_{\mathbb{C}})(\mathbb{C}) \\
\| & & \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma(\mathbb{C})} \\
(\operatorname{Spec}\mathbb{C})_{\mathbb{C}}(\mathbb{C}) & \xrightarrow{w_{Z_{\mathbb{C}}}} & \mathrm{A}^{n+1}(X_{\mathbb{C}}) & \xrightarrow{AJ} & J^{2n+1}_a(X_{\mathbb{C}})(\mathbb{C})
\end{array}$$

where $(\operatorname{Spec}\mathbb{C})_{\mathbb{C}}(\mathbb{C}) = \{\mathrm{Id}_{\mathbb{C}}\}$, and $w_{Z_{\mathbb{C}}}(\mathrm{Id}_{\mathbb{C}}) = Z_{\mathbb{C}}$. Indeed the right hand side is the precise meaning of the statement in Theorem 2.1 that the Abel–Jacobi map is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant, while the left hand side is elementary (see [ACMV17, Rem. 4.3] for more on this). This corresponds to the commutativity of the diagram of sets :

$$\begin{array}{ccccc} (\operatorname{Spec}\mathbb{C})_{an} & \xrightarrow{w_{\mathsf{Z}}} & \mathsf{A}^{n+1}(\mathsf{X}) & \xrightarrow{\mathsf{AJ}} & \mathsf{J}_a^{2n+1}(\mathsf{X}) \\ \| & & \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma} \\ (\operatorname{Spec}\mathbb{C})_{an} & \xrightarrow{w_{\mathsf{Z}}} & \mathsf{A}^{n+1}(\mathsf{X}) & \xrightarrow{\mathsf{AJ}} & \mathsf{J}_a^{2n+1}(\mathsf{X}) \end{array}$$

where $w_{\mathsf{Z}}((\operatorname{Spec}\mathbb{C})_{an}) = \mathsf{Z}$, the complex analytic cycle class associated to Z, and $\sigma : \mathsf{J}_a^{2n+1}(\mathsf{X}) \to \mathsf{J}_a^{2n+1}(\mathsf{X})$ is the map of sets induced by $\sigma(\mathbb{C})$ in the previous diagram. Note that $\mathsf{AJ} \circ w_{\mathsf{Z}} = \nu_{\mathsf{Z}}$, the normal function associated to Z. As mentioned in [ACMV17, Rem. 4.3], the commutativity of the diagrams above implies that $AJ \circ w_{Z_{\mathbb{C}}}$, and hence ν_{Z}, descend to K to give a morphism

$$\delta_Z : \operatorname{Spec} K \to J^{2n+1}_{a,X/K}. \tag{2.1}$$

We call this the *distinguished normal function* associated to Z.

Remark 2.5 (Uniqueness of the distinguished normal function). The distinguished normal function is unique in the sense that given the distinguished model $J^{2n+1}_{a,X/K}$ (unique up to unique isomorphism by Remark 2.2), there is a unique morphism $\delta_Z : \operatorname{Spec} K \to J^{2n+1}_{a,X/K}$ such that $(\delta_Z)_{an} : (\operatorname{Spec}\mathbb{C})_{an} \to \mathsf{J}_a^{2n+1}(\mathsf{X})$ is equal to the analytic normal function ν_{Z}.

Remark 2.6 (Extensions $K \subseteq L \subseteq \mathbb{C}$). In the notation of Theorem 2.1, suppose we have an intermediate field extension $K \subseteq L \subseteq \mathbb{C}$. In light of Remark 2.4, we have a fibered product diagram

$$\begin{array}{ccc} J^{2n+1}_{a,X_L/L} & \longrightarrow & J^{2n+1}_{a,X/K} \\ {\scriptstyle \delta_{Z_L}}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle \delta_Z} \\ \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K \end{array}$$

2.3. **Review of the construction of the distinguished model.** Because it will be relevant later, we recall the construction of the distinguished model $J^{2n+1}_{a,X/K}$ from [ACMV18]. The starting point is [ACMV18, Prop. 1.1], which provides a smooth projective geometrically integral curve C/K (admitting a K-point) and a correspondence $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)$ such that the induced morphism of complex tori $\gamma_* : \mathsf{J}(\mathsf{C}) \to \mathsf{J}^{2n+1}(\mathsf{X})$ has image $\mathsf{J}_a^{2n+1}(\mathsf{X})$. We thus obtain a short exact sequence of algebraic compact complex analytic groups $0 \to \mathsf{P} \to \mathsf{J}(\mathsf{C}) \to \mathsf{J}_a^{2n+1}(\mathsf{X}) \to 0$, where P is defined to be the kernel.

The next step is to show that P descends to K. For this it suffices to show that for every natural number N, the N-torsion $\mathsf{P}[N]$ is preserved by $\operatorname{Aut}(\mathbb{C}/K)$ (since torsion is dense in any sub-group scheme of an abelian variety ; see *e.g.*, [ACMV18, Lem. 2.3]). For this one shows that $\mathsf{P}[N]$ is equal to the kernel of the morphism $\gamma_* : H^1(C_{\mathbb{C}}, \boldsymbol{\mu}_N) \to H^{2n+1}(X_{\mathbb{C}}, \boldsymbol{\mu}_N^{\otimes(n+1)})$ (see [ACMV18, (2.3)]), which is equivariant as it is induced by a correspondence defined over K. Thus $\mathsf{P}[N]$ is preserved by $\operatorname{Aut}(\mathbb{C}/K)$, so that P descends to a group scheme P/K, and consequently $\mathsf{J}_a^{2n+1}(\mathsf{X})$ descends to a model $J^{2n+1}_{a,X/K}$ over K, as well. This is the distinguished model.

Remark 2.7. We reiterate here that the distinguished model is unique up to unique isomorphism (see Remark 2.2), so that $J^{2n+1}_{a,X/K}$ is in fact independent of the curve C and the correspondence γ used in the construction. In other words, given any smooth projective geometrically integral curve C'/K (admitting a K-point) and a correspondence $\gamma' \in \mathrm{CH}^{n+1}(C' \times_K X)$ such that the induced morphism of complex tori $\gamma'_* : \mathsf{J}(\mathsf{C}') \to \mathsf{J}^{2n+1}(\mathsf{X})$ defines a short exact sequence of algebraic compact complex analytic groups $0 \to \mathsf{P}' \to \mathsf{J}(\mathsf{C}') \to \mathsf{J}_a^{2n+1}(\mathsf{X}) \to 0$, the descent datum on $C'_{\mathbb{C}}$ defines $J^{2n+1}_{a,X/K}$.

3. CHANGING THE EMBEDDING $K \subseteq \mathbb{C}$

In this section we show that if K is a field of finite transcendence degree over $\mathbb{Q}$, then up to isomorphism, the distinguished model and distinguished normal function do not depend on the embedding $K \subseteq \mathbb{C}$. An important consequence is that the normal function associated to a fiber-wise algebraically trivial cycle defined over a field F of finite transcendence degree over $\mathbb{Q}$ (*i.e.*, an algebraically F-motivated normal function in our terminology) vanishes at an F-very general point if and only if it vanishes at all F-very general points ; see Example 3.6 and Remark 3.8.

3.1. **The distinguished normal function is independent of the field embedding.** The following proposition complements, in particular, Theorem 2.1 by showing that the distinguished model does not depend on the choice of an embedding $K \subseteq \mathbb{C}$.

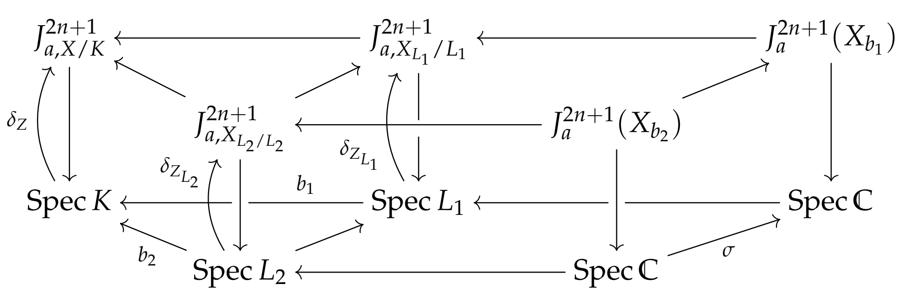
Proposition 3.1. *Let X be a smooth projective variety over a field K of finite transcendence degree over $\mathbb{Q}$, let n be a nonnegative integer, and let $Z \in \mathrm{A}^{n+1}(X)$ be an algebraically trivial cycle class. Let $b_1, b_2 : K \hookrightarrow \mathbb{C}$ be two inclusions of fields, let $L_i = b_i(K)$, and denote by $\sigma : \mathbb{C} \to \mathbb{C}$ an automorphism inducing a commutative diagram of field homomorphisms*

$$\begin{array}{ccccc} & \overset{b_1}{\nearrow} & L_1 & \hookrightarrow & \mathbb{C} \\ K & & \downarrow{\scriptstyle b_2 b_1^{-1}}\ \wr & & \downarrow{\scriptstyle \sigma} \\ & \underset{b_2}{\searrow} & L_2 & \hookrightarrow & \mathbb{C} \end{array}$$

which exists due to the assumption that K is of finite transcendence degree over $\mathbb{Q}$.

For $i = 1, 2$ let X_{b_i} be the base change of X over $b_i : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} K$, with associated complex analytic space $\mathsf{X}_{\mathsf{b}_i}$, let $J^{2n+1}_{a,X_{L_i}/L_i}$ be the distinguished model of $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{b}_i})$ over L_i, and let $\delta_{Z_{L_i}} : \operatorname{Spec} L_i \to J^{2n+1}_{a,X_{L_i}/L_i}$ be the distinguished normal function. Let $J^{2n+1}_a(X_{b_i})$ be the complex abelian variety associated to $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{b}_i})$.

Then $J^{2n+1}_{a,X_{L_2}/L_2}$ is the pullback of $J^{2n+1}_{a,X_{L_1}/L_1}$ by $b_1^{-1}b_2 : \operatorname{Spec} L_2 \to \operatorname{Spec} L_1$, and there is a commutative fibered product diagram



(3.1)

where $J^{2n+1}_{a,X/K}$ and $\delta_Z : \operatorname{Spec} K \to J^{2n+1}_{a,X/K}$ are defined from the rest of the diagram via fibered product.

Remark 3.2. Proposition 3.1 allows one to define *the distinguished model* $J^{2n+1}_{a,X/K}$ of the image of the Abel–Jacobi map, and *the distinguished normal function* δ_Z associated to a cycle class $Z \in \mathrm{CH}^{n+1}(X)$, without first needing to specify a particular inclusion $K \hookrightarrow \mathbb{C}$.

Proof. From the diagram (3.1) it is clear that it suffices to establish that there is a commutative fibered product diagram

$$\begin{array}{ccc} J^{2n+1}_{a,X_{L_2}/L_2} & \longrightarrow & J^{2n+1}_{a,X_{L_1}/L_1} \\ {\scriptstyle \delta_{Z_{L_2}}}\uparrow\ \downarrow & & \downarrow\ \uparrow{\scriptstyle \delta_{Z_{L_1}}} \\ \operatorname{Spec} L_2 & \xrightarrow{b_1^{-1}b_2} & \operatorname{Spec} L_1 \end{array} \tag{3.2}$$

i.e., it is enough to focus on the sub-diagram that is the left hand face of the cube in diagram (3.1). We break the proof into two parts. First we establish the result for the distinguished models, and second for the distinguished normal functions.

Step 1: The distinguished models. Let C be a geometrically integral curve over K (admitting a K-point) and let $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)$ be a correspondence such that, for $i = 1, 2$, the induced morphisms of complex tori $\gamma_{i*} : \mathsf{J}(\mathsf{C}_i) \to \mathsf{J}^{2n+1}(\mathsf{X}_i)$ have respective images equal to $J_a^{2n+1}(\mathsf{X}_i)$ ([ACMV18, Prop. 1.1]). Here, X_i and C_i are the complex analytic spaces associated to the pull backs of X and C to $\operatorname{Spec}\mathbb{C}$ via the given inclusions $b_i : K \hookrightarrow \mathbb{C}$. Thus, for $i = 1, 2$, we obtain short exact sequences

$$0 \longrightarrow \mathsf{P}_i \longrightarrow \mathsf{J}(\mathsf{C}_i) \longrightarrow \mathsf{J}_a^{2n+1}(\mathsf{X}_i) \longrightarrow 0$$

where P_i is defined to be the kernel of the morphism of complex tori induced by γ. Moreover, we have seen in Remark 2.7 that P_i descends to an abelian scheme P_i over L_i. This gives short exact sequences

$$0 \longrightarrow P_i \longrightarrow J(C_{L_i}) \longrightarrow J^{2n+1}_{a,X_{L_i}/L_i} \longrightarrow 0$$

defining the distinguished models $J^{2n+1}_{a,X_{L_i}/L_i}$. We want to show that the distinguished models differ by base change over $b_1^{-1}b_2 : \operatorname{Spec} L_2 \to \operatorname{Spec} L_1$. We will do this by showing that P_1 and P_2 differ by base change over $b_1^{-1}b_2$.

Let $P_{1,L_2} \subseteq J(C_{L_2})$ be the base change of P_1 to L_2. To show that $P_{1,L_2} = P_2$, it suffices to show that for all natural numbers N, the N-torsion of P_{1,L_2} and P_2 are equal; *i.e.*, $P_{1,L_2}[N] = P_2[N]$. But the N-torsion $P_i[N]$ is equal to the kernel of the morphism $\gamma_{i*} : H^1(C_{b_i}, \boldsymbol{\mu}_N) \to H^{2n+1}(X_{b_i}, \boldsymbol{\mu}_N^{\otimes(n+1)})$; see [ACMV18, (2.3)]. These are related by the diagram

$$\begin{array}{ccc} H^1(C_{b_1}, \boldsymbol{\mu}_N) & \xrightarrow{(\gamma_1)_*} & H^{2n+1}(X_{b_1}, \boldsymbol{\mu}_N^{\otimes(n+1)}) \\ \simeq\downarrow\sigma^* & & \simeq\downarrow\sigma^* \\ H^1(C_{b_2}, \boldsymbol{\mu}_N) & \xrightarrow{(\gamma_2)_*} & H^{2n+1}(X_{b_2}, \boldsymbol{\mu}_N^{\otimes(n+1)}) \end{array}$$

implying that $P_{1,L_2}[N] = P_2[N]$, completing the proof.

Step 2: The distinguished normal functions. We now show that the distinguished normal functions $\delta_{Z_{L_1}}$ and $\delta_{Z_{L_2}}$ fit into the fibered product diagram (3.2); *i.e.*, that they agree under base change. To begin, recall that the distinguished normal function $\delta_{Z_{L_i}}$ is characterized by the condition that $(\delta_{Z_{L_i}})_{an} = \nu_{(Z_{b_i})_{an}}$; *i.e.*, the analytic map induced by δ_{Z_L} agrees with the analytic normal function (Remark 2.5). Algebraically, this is the condition that $(\delta_{Z_{L_i}})_{b_i} = AJ \circ w_{Z_{b_i}}$ (see §2.2). In other words, to complete the proof of the theorem, it suffices to show that $(\delta_{Z_{L_1}})_{b_2} = (\delta_{Z_{L_2}})_{b_2}$. Put differently, by virtue of the fact that $J_a^{2n+1}(X_{b_1})$ and $J_a^{2n+1}(X_{b_2})$ have been identified in Step 1 via base change over $\sigma : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$, it suffices to show that the outer rectangle of the diagram

$$\begin{array}{ccccc} (\operatorname{Spec}\mathbb{C})_{\mathbb{C}}(\mathbb{C}) & \xrightarrow{w_{Z_{b_1}}} & \mathrm{A}^{n+1}(X_{b_1}) & \xrightarrow{AJ} & J_a^{2n+1}(X_{b_1})(\mathbb{C}) \\ \downarrow\sigma & & \downarrow\sigma^* & & \downarrow\sigma(\mathbb{C}) \\ (\operatorname{Spec}\mathbb{C})_{\mathbb{C}}(\mathbb{C}) & \xrightarrow{w_{Z_{b_2}}} & \mathrm{A}^{n+1}(X_{b_2}) & \xrightarrow{AJ} & J_a^{2n+1}(X_{b_2})(\mathbb{C}) \end{array} \tag{3.3}$$

is commutative. In other words, it suffices to show that $\sigma AJ(Z_{b_1}) = AJ((Z_{b_2}))$.

To do this, we first show the commutativity of the right hand side of (3.3) on torsion (see also Remark 3.4). For this one considers the diagram

$$\begin{array}{ccccc} \mathrm{A}^{n+1}(X_{b_1})[N] & \xrightarrow{AJ} & J_a^{2n+1}(X_{b_1})[N] & \hookrightarrow & H^{2n+1}(X_{b_1}, \boldsymbol{\mu}_N^{\otimes(n+1)}) \\ \downarrow{\scriptstyle \sigma^*[N]} & & \downarrow{\scriptstyle \sigma(\mathbb{C})[N]} & & \downarrow{\scriptstyle \sigma^*} \\ \mathrm{A}^{n+1}(X_{b_2})[N] & \xrightarrow{AJ} & J_a^{2n+1}(X_{b_2})[N] & \hookrightarrow & H^{2n+1}(X_{b_2}, \boldsymbol{\mu}_N^{\otimes(n+1)}) \end{array}$$

which is commutative for all integers $N > 1$, due to the fact that by construction the right-hand square is commutative, and the fact that the outer square is commutative because the composition of horizontal arrows is the Bloch map, which is functorial with respect to automorphisms of the field (see *e.g.*, [ACMV18, §2.3]).

Now, since Z is defined over K, there exist by [ACMV19, Thm. 1] an abelian variety A over K, a K-point $p \in A(K)$, and a correspondence $\Xi \in \mathrm{CH}^{n+1}(A \times_K X)$ such that $Z = \Xi_p - \Xi_0$. Since the Abel–Jacobi map is a regular homomorphism, the base change of Ξ along b_i induces a homomorphism $\psi_{\Xi,i} : A_{b_i} \to J_a^{2n+1}(X_{b_i})$ with $\psi_{\Xi,i}(q) = AJ(\Xi_q - \Xi_0)$, in particular $(\psi_\Xi)_i(p) = AJ(Z_{b_i})$. We have then a not *a priori* commutative diagram

$$\begin{array}{ccc} A_{b_1} & \xrightarrow{\psi_{\Xi,1}} & J_a^{2n+1}(X_{b_1}) \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma} \\ A_{b_2} & \xrightarrow{\psi_{\Xi,2}} & J_a^{2n+1}(X_{b_2}). \end{array}$$

However, since the right hand side of (3.3) is commutative on torsion, this diagram is also commutative on torsion (note that $\Xi_q - \Xi_0$ is torsion in $\mathrm{A}^{n+1}(X_{b_i})$ whenever q is a torsion point in A_{b_i}; *e.g.* [ACMV18, Lem. 3.2]) and since torsion points are dense, the diagram is in fact commutative, thereby establishing the desired identity $\sigma AJ(Z_{b_1}) = AJ(Z_{b_2})$. □

According to the Bloch–Beilinson philosophy (see *e.g.* [Gre14, Lecture 3]), if X is a smooth projective variety defined over a subfield $K \subseteq \mathbb{C}$, then the kernel of the Abel–Jacobi map $AJ : \mathrm{CH}^{n+1}(X)_{\mathrm{hom}} \to \mathrm{CH}^{n+1}(X_{\mathbb{C}})_{\mathrm{hom}} \to J^{2n+1}(X_{\mathbb{C}})$ defined on homologically trivial cycles defined over K should be independent of the choice of embedding $K \hookrightarrow \mathbb{C}$, after tensoring with $\mathbb{Q}$. Even such a concrete consequence of the Bloch and Beilinson conjectures remains wide open. As a noteworthy consequence of Proposition 3.1, we can establish this, with integral coefficients, for algebraically trivial cycle classes:

Corollary 3.3. *Let X be a smooth projective variety over a field K of finite transcendence degree over $\mathbb{Q}$, and let $Z \in \mathrm{A}^{n+1}(X)$ be an algebraically trivial cycle class. Let $b : K \hookrightarrow \mathbb{C}$ be an inclusion of fields. Then $AJ(Z_b) = 0$ for one such embedding if and only if $AJ(Z_b) = 0$ for all such embeddings.* □

Remark 3.4. In fact, one can also show that the right hand side of (3.3) is commutative on all cycle classes (*i.e.*, not just the ones defined over K). In particular, the kernel of the Abel–Jacobi map restricted to algebraically trivial cycles (not necessarily defined over K) is independent of the choice of an embedding $K \hookrightarrow \mathbb{C}$, in the sense that if $Z \in \mathrm{A}^{n+1}(X_{\mathbb{C}})$ is such that $AJ(Z) = 0$, then $AJ(Z^\sigma) = 0$ for all automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$. For brevity, we have omitted the proof.

3.2. **Distinguished models and distinguished normal functions of very general fibers.** We now focus on the main case of interest in this paper. Let $F \subseteq \mathbb{C}$ be a field of finite transcendence degree over $\mathbb{Q}$. Let $f : X \to B$ be a smooth surjective projective morphism of smooth integral schemes of finite type over F, and let n be a nonnegative integer. Let $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ be the associated map of complex manifolds. Let η be the generic point of B with residue field K, which is also of finite transcendence degree over $\mathbb{Q}$.

Example 3.5 (Distinguished model of a very general fiber)**.** In the notation above, fix an inclusion $K \subseteq \mathbb{C}$, let X_η be the generic fiber, and let $J^{2n+1}_{a,X_\eta/K}$ be the corresponding distinguished model associated to $(X_{\mathbb{C}})_{an}$ (see also Proposition 3.1). Now let $\mathsf{u} \in \mathsf{B}$ be an F-very general point; *i.e.*, u corresponds to a closed $\mathbb{C}$-point $u : \operatorname{Spec}\mathbb{C} \to B_{\mathbb{C}}$, which is itself a morphism of $\mathbb{C}$-schemes, so that the composition $u : \operatorname{Spec}\mathbb{C} \to B_{\mathbb{C}} \to B$ has image the generic point of B, given by a second inclusion $i : K \hookrightarrow \mathbb{C}$. From Proposition 3.1, the distinguished model of $\mathsf{J}^{2n+1}(\mathsf{X}_\mathsf{u})$ is (after pull back to K) isomorphic over K to $J^{2n+1}_{a,X_\eta/K}$. In fact, the distinguished models of all F-very general fibers agree up to isomorphism over K. Put another way, for any F-very general point $\mathsf{u} \in \mathsf{B}$, corresponding to a point $u : \operatorname{Spec}\mathbb{C} \to B_{\mathbb{C}}$,

$$((J^{2n+1}_{a,X_\eta/K})_u)_{an} = \mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{u}).$$

Example 3.6 (Distinguished normal function of a very general fiber)**.** In the same situation as Example 3.5, let $Z \in \mathrm{CH}^{n+1}(X)$ be such that every Gysin fiber is algebraically trivial. Let $\delta_Z : \operatorname{Spec}K \to J^{2n+1}_{a,X_\eta/K}$ be the distinguished normal function (2.1), which *a priori* depends on the inclusion $K \subseteq \mathbb{C}$. However, from Proposition 3.1, the distinguished normal function associated to any F-very general point $\mathsf{u} \in \mathsf{B}$ (corresponding to an inclusion $i : K \hookrightarrow \mathbb{C}$) agrees (after pull back to K) with δ_Z. Put another way, for any F-very general point $\mathsf{u} \in \mathsf{B}$, corresponding to a point $u : \operatorname{Spec}\mathbb{C} \to B_{\mathbb{C}}$,

$$((\delta_Z)_u)_{an} = (\nu_\mathsf{Z})_\mathsf{u} : \mathsf{u} \to \mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{u})$$

where $\nu_\mathsf{Z} : \mathsf{B} \to \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$ is the analytic normal function associated to Z.

Remark 3.7 (The geometric generic fiber). Another way to frame the relationship between the distinguished models and distinguished normal functions associated to different F-very general points of B is to observe that the $\mathbb{C}$-scheme X_u associated to an F-very general fiber X_u of $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ is isomorphic as a K-scheme to a geometric generic fiber $X_{\overline{\mathbb{C}(B)}}$. In fact, after choosing a K-isomorphism $\alpha : \mathbb{C} \to \overline{\mathbb{C}(B)}$, we have that X_u and $X_{\overline{\mathbb{C}(B)}}$ are isomorphic over α (see *e.g.*, [Via13, Lem. 2.1]).

Remark 3.8 (First consequence for zero loci of normal functions: Conjecture 2 in the algebraically F-motivated case). Already, we obtain a quick proof of much of Corollary 1, *i.e.*, that that the zero-locus of ν_Z is a countable union of algebraic subsets of B defined over F. Indeed, from Example 3.6 we have that if $(\nu_\mathsf{Z})_\mathsf{u}$ is zero for one F-very general point $\mathsf{u} \in \mathsf{B}$, then it is zero for every F-very general point $\mathsf{u}' \in \mathsf{B}$ (see also Corollary 3.3). By continuity, if ν_Z is not identically zero, then the zero locus of ν_Z is contained in the complement of the F-very general points, which is a countable union of algebraic subsets of B defined over F, and not equal to B. Restricting ν_Z to an F-desingularization of each irreducible component and arguing recursively on each component, one obtains the claim. Note that together with Conjecture 1, one obtains a proof of Corollary 1. We will, however, give below a direct proof of Theorem 1, and hence of Corollary 1, that does not rely on the validity of Conjecture 1.

4. Spreading the distinguished model and distinguished normal function

We now consider a family of smooth complex projective varieties, and descend the image of the Abel–Jacobi map of a very general fiber to the generic point of the base of the family. We then spread this to an open subset of the base. The following theorem collects some properties of this spread. (Note that the caveat of Remark 5.2 already applies to the abelian scheme $\mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{U}/\mathsf{U})$ constructed below.)

Theorem 4.1. *Let $F \subseteq \mathbb{C}$ be a subfield of finite transcendence degree over $\mathbb{Q}$, let $f : X \to B$ be a smooth surjective projective morphism of smooth integral varieties of finite type over F, and let n be a nonnegative integer. Let $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ be the associated morphism of complex analytic spaces.*

Let η be the generic point of B with residue field K, and fix an inclusion $K \hookrightarrow \mathbb{C}$. Let X_η be the generic fiber of X over K, let $J^{2n+1}_{a,X_\eta/K}$ be the distinguished model of $\mathsf{J}^{2n+1}_a((X_{\mathbb{C}})_{an})$ over K and let

$$\Gamma \in \mathrm{CH}^{\dim(J^{2n+1}_{a,X_\eta/K})+n}(J^{2n+1}_{a,X/K} \times_K X_\eta) \quad \text{and} \quad M \in \mathbb{Z}_{>0} \tag{4.1}$$

be the correspondence and integer, respectively, from Theorem 2.1 (see also Proposition 3.1). Spread this data to a Zariski open subset $U \subseteq B$. More precisely, let $U \subseteq B$ be a Zariski open subset over which there is an abelian scheme

$$g : J^{2n+1}_{a,X_U/U} \to U$$

with generic fiber isomorphic to $J^{2n+1}_{a,X_\eta/\eta}$, and a cycle

$$\Gamma_U \in \mathrm{CH}^{\dim(J^{2n+1}_{a,X_\eta/K})+n}(J^{2n+1}_{a,X_U/U} \times_U X_U)$$

with $(\Gamma_U)_\eta = \Gamma$. Let $\mathsf{U} \subseteq \mathsf{B}$ be the Zariski open subset corresponding to $U \subseteq B$, and let Γ_{U} be the corresponding complex analytic correspondence.

(1) *For every prime number ℓ the correspondence Γ_U induces a morphism of sheaves on U*

$$(\Gamma_U)_* : R^1 g_* \mathbb{Q}_\ell \longrightarrow R^{2n+1}(f|_U)_* \mathbb{Q}_\ell(n) \tag{4.2}$$

which at the geometric generic point $\overline{u} : \operatorname{Spec} \overline{K} \to \operatorname{Spec} K \to U$, induces an inclusion of $\mathrm{Gal}(\overline{K}/K)$-representations

$$(\Gamma_{U,\overline{u}})_* : H^1((J^{2n+1}_{a,X_\eta/K})_{\overline{K}}, \mathbb{Q}_\ell) \hookrightarrow H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) \tag{4.3}$$

(with image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$, where $\mathrm{N}^\bullet$ denotes the geometric coniveau filtration).

(2) *The correspondence Γ_{U} induces a morphism of variations of pure integral Hodge structures and thus a morphism of relative complex tori over U*

$$(\Gamma_{\mathsf{U}})_* : (J^{2n+1}_{a,X_U/U})_{an} \to \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_{\mathsf{U}}. \tag{4.4}$$

The image of (4.4) is an algebraic relative complex torus $\mathsf{A}_{\mathsf{U}} \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_{\mathsf{U}}$ over U, induced by an abelian scheme

$$A_U/U,$$

defined over F, with generic fiber $(A_U)_\eta$ isomorphic to $J^{2n+1}_{a,X_\eta/K}$ over K.

For F-very general $\mathsf{u} \in \mathsf{U}$, the morphism (4.4) restricts to a morphism

$$((\Gamma_{\mathsf{U}})_*)_{\mathsf{u}} : \mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{u}}) \to \mathsf{J}^{2n+1}(\mathsf{X}_{\mathsf{u}}) \tag{4.5}$$

that is given by $M \cdot \mathsf{i}^{2n+1}_{a,\mathsf{X}_{\mathsf{u}}}$, i.e., *$M$ times the natural inclusion (where M is defined in (4.1)). In particular, the image of (4.5),* i.e., *the fiber $\mathsf{A}_{\mathsf{U},\mathsf{u}}$, is equal to $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{u}})$.*

(3) *Let $Z \in \mathrm{CH}^{n+1}(X)$ be a cycle class with every Gysin fiber algebraically trivial, let Z_η be the restriction of Z to the generic fiber X_η, and let*

$$\delta_{Z_\eta} : \operatorname{Spec} K \to J^{2n+1}_{a,X_\eta/K}$$

be the associated distinguished normal function (see (2.1) and Proposition 3.1). After possibly replacing the Zariski open subset $U \subseteq B$ with a smaller Zariski open subset, let $\delta : U \to J^{2n+1}_{a,X_U/U}$ be the spread of the distinguished normal function, and let $\delta_{an} : \mathsf{U} \to (J^{2n+1}_{a,X_U/U})_{an}$ denote the associated *morphism of complex analytic spaces. We have the following formula relating the normal function ν_{Z}, the spread δ_{an} of the distinguished normal function, and the morphism (4.4):*

$$(\Gamma_{\mathsf{U}})_* \circ \delta_{an} = M \cdot \nu_{\mathsf{Z}}|_{\mathsf{U}}, \tag{4.6}$$

and $M \cdot \nu_{\mathsf{Z}}|_{\mathsf{U}}$ is algebraic, and defined over F.

Proof. (1) Correspondences induce morphisms of sheaves, giving (4.2). (4.3) is just a statement about fibers of correspondences, and follows from Theorem 2.1 (but see also [ACMV18, Thm. A]).

(2) Correspondences induce morphisms of Hodge structures, giving (4.4). To show that the image A_{U} of (4.4) is algebraic, one shows that the kernel of the morphism of relative complex tori is algebraic. For this it suffices to check that torsion is preserved by $\mathrm{Aut}(\mathbb{C}/F)$, and it is easy to see this holds from the fact that the morphism is induced by an algebraic cycle defined over F (as in Lemma 2.3) ; alternatively, it is dominated by the relative algebraic complex torus $(J^{2n+1}_{a,X_U/U})_{an}$.

The assertion (4.5) is just a statement about fibers of correspondences, and Theorem 2.1 and Example 3.5 provide the needed identification of fibers. One also uses the general observation that the image of the multiplication by M map is the same complex torus.

The final statement, that the generic fiber $(A_U)_\eta$ is isomorphic to $J^{2n+1}_{a,X_\eta/K}$ over K, can be established as follows. Let G be the kernel of the K-isogeny $J^{2n+1}_{a,X_\eta/K} \to (A_U)_\eta$. The isogeny, when pulled back to u, gives a morphism

$$\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{u}}) \twoheadrightarrow \mathsf{A}_{\mathsf{U},\mathsf{u}} \xrightarrow{\cong} \mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{u}})$$

with composition equal to the multiplication by M map. Thus G and $J^{2n+1}_{a,X_\eta/K}[M]$ are reduced K-subschemes of $J^{2n+1}_{a,X_\eta/K}$ with the same $\mathbb{C}$-points, and are therefore the same scheme. It follows that $(A_U)_\eta \cong J^{2n+1}_{a,X_\eta/K}/J^{2n+1}_{a,X_\eta/K}[M] = J^{2n+1}_{a,X_\eta/K}$.

(3) The only thing to show is (4.6). Since both sides of the equation are continuous functions, it suffices to prove the assertion for a dense subset of U, and in particular we can focus on F-very general points $\mathsf{u} \in \mathsf{U}$. The assertion then follows from (2) together with Example 3.6. ☐

We now use Theorem 4.1 to prove Theorem 1 over a Zariski dense open subset of the base :

Corollary 4.2. *Let $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ be a smooth surjective projective morphism of complex algebraic manifolds, let n be a nonnegative integer, let $\mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B}) \to \mathsf{B}$ be the $(2n+1)$-st relative Griffiths intermediate Jacobian. There is a Zariski open subset $\mathsf{U} \subseteq \mathsf{B}$, a relative algebraic complex subtorus $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{U}}/\mathsf{U}) \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_U$ over U such that for very general $\mathsf{u} \in \mathsf{U}$ the fiber $\mathsf{J}_a(\mathsf{X}_{\mathsf{U}}/\mathsf{U})_{\mathsf{u}} \subseteq \mathsf{J}^{2n+1}(\mathsf{X}_{\mathsf{u}})$ is the image $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{u}})$ of the Abel–Jacobi map $\mathsf{AJ}_{\mathsf{X}_{\mathsf{u}}} : \mathrm{A}^{n+1}(\mathsf{X}_{\mathsf{u}}) \to \mathsf{J}^{2n+1}(\mathsf{X}_{\mathsf{u}})$, and for any $\mathsf{Z} \in \mathrm{CH}^{n+1}(\mathsf{X})$ with every Gysin fiber algebraically trivial :*

1. *The restriction of the normal function $\nu_{\mathsf{Z}}|_{\mathsf{U}} : \mathsf{U} \to \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_{\mathsf{U}}$ has image contained in the relative algebraic complex torus $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{U}}/\mathsf{U})$ and is an algebraic map.*
2. *If, moreover, X, B, f, and Z are all defined over a subfield $F \subseteq \mathbb{C}$, then so are $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{U}}/\mathsf{U})$ and the morphisms $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{U}}/\mathsf{U}) \to \mathsf{U}$ and $\nu_{\mathsf{Z}}|_{\mathsf{U}}$.*

Proof. In case (1), since $\mathsf{f} : \mathsf{X} \to \mathsf{B}$ is defined over some field $F \subseteq \mathbb{C}$ that is finitely generated over $\mathbb{Q}$, we may as well make this assumption from the start. In case (2) we may take our field of definition F' to be contained in the given field F, and can base change to F at the end, if necessary, and so we may as well assume F is finitely generated over $\mathbb{Q}$ in case (2), as well. We are then in the situation of Theorem 4.1, and we will use the notation from that theorem moving forward.

First, we can take $\mathsf{J}^{2n+1}_a(\mathsf{X}_{\mathsf{U}}/\mathsf{U}) = \mathsf{A}_{\mathsf{U}} \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_{\mathsf{U}}$, the image of (4.4). Let A_U be the corresponding abelian scheme over $U \subseteq B$. From Theorem 4.1(2) we have that the generic fiber $(A_U)_\eta$

is isomorphic over K to the distinguished model $J^{2n+1}_{a,X_\eta/K}$. With cycle class $Z \in \mathrm{CH}^{n+1}(X)$ as in the theorem, let

$$\delta_{Z_\eta} : \operatorname{Spec} K \to J^{2n+1}_{a,X_\eta/K} \cong (A_U)_\eta \tag{4.7}$$

be the distinguished normal function. This then spreads to an F-morphism $\delta : U \to A_U$, after possibly replacing U with a smaller Zariski open subset. The associated complex analytic map $\delta_{an} : \mathsf{U} \to \mathsf{A}_\mathsf{U} \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_\mathsf{U}$ has the property that for F-very general $\mathsf{u} \in \mathsf{U}$ we have $(\delta_{an})_\mathsf{u} : \mathsf{u} \to \mathsf{A}_{\mathsf{U},\mathsf{u}} = \mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{u})$ has image $\nu_\mathsf{Z}(\mathsf{u})$; *i.e.*, it agrees with the complex analytic normal function. This follows from (4.7) and Example 3.6. Therefore, since δ_{an} and $\nu_\mathsf{Z}|_\mathsf{U}$ are continuous and agree on the dense open subset of F-very general points of U, they agree on all of U. □

5. Extending the distinguished model and distinguished normal function

In light of Corollary 4.2, in order to prove Theorem 1, we only need to show that the distinguished model and distinguished normal function extend over the entire base B. We do this now.

Proof of Theorem 1. We use the notation from Theorem 1, and the partial result, Corollary 4.2. As mentioned above, we only need to show that the spread of the distinguished model $\mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{U}/\mathsf{U}) \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_\mathsf{U}$ and distinguished normal function $\delta_{an} : \mathsf{U} \to \mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{U}/\mathsf{U})$ extend over the entire base B, to give algebraic objects over F. To begin, we switch to the algebraic setting, and let $J^{2n+1}_{a,X_U/U}/U$ be the algebraic model of $\mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{U}/\mathsf{U})$, and let $\delta : U \to J^{2n+1}_{a,X_U/U}$ be the associated morphism of F-schemes.

First, we show that $J^{2n+1}_{a,X_U/U}$ extends to an abelian scheme $g : \tilde{J}^{2n+1}_{a,X/B} \to B$ over B. If $\dim B = 1$, we use the inclusion (4.3) and the Néron–Ogg–Shafarevich criterion as in [ACMV17, Lem. 6.1(a)]. If $\dim B \geq 2$, using the dimension 1 case we can extend over the generic points of divisors in the boundary $B - U$, and thus we can assume that $\operatorname{codim}_B(B-U) \geq 2$. The assertion now follows from the Faltings–Chai Extension Theorem [FC90, Cor. 6.8, p.185].

Next we show that the relative algebraic complex torus $\tilde{\mathsf{J}}^{2n+1}_a(\mathsf{X}/\mathsf{B}) := (\tilde{J}^{2n+1}_{a,X/B})_{an}$ induces an algebraic relative subtorus $\mathsf{J}^{2n+1}_a(\mathsf{X}/\mathsf{B}) \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$ extending $\mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{U}/\mathsf{U})$. For this we use the basic fact that any morphism of variations of Hodge structures extends over a locus of codimension at least 2. (Indeed, by purity, the natural map $\pi_1(\mathsf{U},\mathsf{u}) \to \pi_1(\mathsf{B},\mathsf{u})$ is an isomorphism ; now use [PS08, Thm. 10.11, p.243], or the proof of [Hai95, Lem. 6.3, p.117].) It now follows that the inclusion $\mathsf{J}^{2n+1}_a(\mathsf{X}_\mathsf{U}/\mathsf{U}) \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})|_\mathsf{U}$ extends to a morphism

$$\tilde{\mathsf{J}}^{2n+1}_a(\mathsf{X}/\mathsf{B}) \longrightarrow \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B}) \tag{5.1}$$

with finite kernel, which *a priori* may be nontrivial only over $\mathsf{B} - \mathsf{U}$. We define $\mathsf{J}^{2n+1}_a(\mathsf{X}/\mathsf{B}) \subseteq \mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$ to be the image of (5.1). Although it is not needed, we note that by Zariski's Main Theorem the morphism of relative algebraic complex tori $\tilde{\mathsf{J}}^{2n+1}_a(\mathsf{X}/\mathsf{B}) \to \mathsf{J}^{2n+1}_a(\mathsf{X}/\mathsf{B})$ is an isomorphism.

The normal function ν_Z has image contained in $\mathsf{J}^{2n+1}_a(\mathsf{X}/\mathsf{B})$, since $\mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$ is separated, and $\nu_\mathsf{Z}|_\mathsf{U}$ has image contained in $\mathsf{J}^{2n+1}_a(\mathsf{X}/\mathsf{B})$, which is closed in $\mathsf{J}^{2n+1}(\mathsf{X}/\mathsf{B})$. Finally, it is straight forward to check that ν_Z is algebraic, and defined over F, since $\nu_\mathsf{Z}|_\mathsf{U}$ is algebraic and defined over F. For completeness, we include this last assertion as Lemma 5.1 below. □

Lemma 5.1. *Let X, Y be schemes of finite type over $F \subseteq \mathbb{C}$, with X reduced and Y separated, let $U \subseteq X$ be a Zariski open subset, let $f_U : U \to Y$ be a morphism of F-schemes, and assume that the associated morphism of analytic spaces $(f_U)_{an} : \mathsf{U} \to \mathsf{Y}$ extends to a morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$. Then f_U extends to a morphism $f : X \to Y$ over F with $f_{an} = \mathsf{f}$.*

Proof. We may immediately reduce to the case with X integral, and Y reduced. Now consider the graph $\Gamma_{f_U} \subseteq U \times_F Y \subseteq X \times_F Y$, which is closed in $U \times_F Y$ since Y is separated. Let $\Gamma \subseteq X \times_F Y$

be the closure of Γ_{f_U}, which we observe is an integral subscheme. Now let $\Gamma = \Gamma_{an} \subseteq X \times Y$ be the associated complex analytic space. We have by assumption that $(\Gamma_{f_U})_{an} = (\Gamma_f)|_{U\times Y}$. Now Γ_f is the analytic closure of $(\Gamma_f)|_{U\times Y}$ in $X \times Y$. Since $(\Gamma_f)|_{U\times Y} = (\Gamma_{f_U})_{an} \subseteq \Gamma$, we have Γ_f is equal to the analytic closure of $(\Gamma_{f_U})_{an}$ in Γ. But Γ_{f_U} is a Zariski open subset of an integral scheme Γ of finite type over F, and so $\Gamma_f = \Gamma$. (In general, if T is a locally closed subset of a scheme $Z/\mathbb{C}$ which is locally of finite type, then T is dense in Z if and only if T is dense in Z [Gro71, Exposé XII, Cor. 2.3].)

Now we just need to conclude that Γ induces a morphism $X \to Y$. It suffices to show that the second projection $q_1 : \Gamma \to X$ is an isomorphism. But this follows from the fact that a morphism between the complex analytic spaces associated to two F-schemes descends to F if and only if it is $\mathrm{Aut}(\mathbb{C}/F)$-equivariant (apply, *e.g.*, [Voi13, §5.2] to its graph), and the fact that $q_1 : \Gamma \to X$ is an isomorphism. ☐

Remark 5.2. The notation $J_a^{2n+1}(X/B)$ may be slightly misleading, in the sense that formation of this object is not compatible with base change in B. While the very general fiber $J_a^{2n+1}(X/B)_u$ is equal to the image of the Abel–Jacobi map $J_a^{2n+1}(X_u)$, in some cases there is a countably infinite union of algebraic subsets of B over which the geometric coniveau of the fiber $N^n H^{2n+1}(X_b, \mathbb{Q})$ jumps. If this is the case, then over these points the fiber $J_a^{2n+1}(X/B)_b$ is strictly contained in $J_a^{2n+1}(X_b)$. Nonetheless, we feel that $J_a^{2n+1}(X/B)$ is good notation in the sense that this is the smallest relative algebraic subtorus of $J^{2n+1}(X/B)$ that interpolates between the very general $J_a^{2n+1}(X_u)$. Moreover, by part (1) of Theorem 1, it serves as a target for every algebraically motivated normal function.

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Colorado State University, Department of Mathematics, Fort Collins, CO 80523, USA
E-mail address: `j.achter@colostate.edu`

University of Colorado, Department of Mathematics, Boulder, CO 80309, USA
E-mail address: `casa@math.colorado.edu`

Universität Bielefeld, Fakultät für Mathematik, Germany
E-mail address: `vial@math.uni-bielefeld.de`