# ON THE CHOW RING OF CYNK–HULEK CALABI–YAU VARIETIES AND SCHREIEDER VARIETIES

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Abstract. This note is about certain complete families of Calabi–Yau varieties constructed by Cynk and Hulek, and certain varieties constructed by Schreieder. We prove that the cycle class map on the Chow ring of powers of these varieties admits a section, and that these varieties admit a multiplicative self-dual Chow–Künneth decomposition. As a consequence of both results, we prove that the subring of the Chow ring generated by divisors, Chern classes, and intersections of two cycles of positive codimension injects into cohomology, via the cycle class map. We also prove that the small diagonal of Schreieder surfaces admits a decomposition similar to that of K3 surfaces. As a by-product, we verify a conjecture of Voisin concerning zero-cycles on the self-product of Cynk–Hulek Calabi–Yau varieties, and in the odd-dimensional case we verify a conjecture of Voevodsky concerning smash-equivalence. Finally, in positive characteristic, we study the Chow ring of supersingular Cynk–Hulek Calabi–Yau varieties.

## Introduction

In the course of a quest for Calabi–Yau varieties that are modular, Cynk and Hulek [7] constructed certain Calabi–Yau varieties $X$ of arbitrary dimension $n$ over $\mathbb{C}$. Their construction starts from a product of $n$ complex elliptic curves $E_1, \ldots, E_n$. The Calabi–Yau variety $X$ is obtained by considering

$$\begin{tikzcd} & E_1 \times \cdots \times E_n \arrow[d, "p"] \\ X \arrow[r, "f"] & (E_1 \times \cdots \times E_n)/G := \bar{X} \end{tikzcd}$$

where $G$ is a certain group of automorphisms (specifically $G \cong \mathbb{Z}_2^{n-1}$, or $G \cong \mathbb{Z}_3^{n-1}$ and $E_1 = \cdots = E_n$ is an elliptic curve with an order-3 automorphism, and $f$ is a crepant resolution of singularities). We refer to Theorems 1.1 and 1.2 below for explicit definitions, and to Propositions 4.3 and 4.4 together with the proof of Claim 4.8 for an explicit construction.

The difference between the two types of Cynk–Hulek varieties ($G = \mathbb{Z}_2^{n-1}$, resp. $G = \mathbb{Z}_3^{n-1}$) is illustrated by their Hodge diamond. In the first case (*i.e.* $G = \mathbb{Z}_2^{n-1}$), the Hodge diamond

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looks like

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & * & & & & \\ & & & & \vdots & & & & \\ 1 & * & \ldots & & \ldots & & \ldots & * & 1 \\ & & & & \vdots & & & & \\ & & & & * & & & & \\ & & & & 1 & & & & \end{array}$$

(where $*$ means some unspecified number, and all empty entries are 0), whereas for the second case (*i.e.* $G = \mathbb{Z}_3^{n-1}$), the Hodge diamond is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & * & & & & \\ & & & & \vdots & & & & \\ 1 & 0 & \ldots & 0 & * & 0 & \ldots & 0 & 1 \\ & & & & \vdots & & & & \\ & & & & * & & & & \\ & & & & 1 & & & & \end{array}$$

Recently, Stefan Schreieder [29] generalized the construction of Cynk–Hulek in order to solve some construction problems for Hodge numbers. His construction starts with the hyperelliptic curve $C$ which is the smooth projectivization of the affine curve $\{y^2 = x^{3^c} - 1\}$ equipped with the action of a primitive $3^c$-th root of unity $\zeta$ acting as $(x, y) \mapsto (\zeta \cdot x, y)$. The variety $X$ is an explicit smooth projective birational model of $C^n/G$, where $G$ is a certain subgroup of $(\mathbb{Z}_{3^c})^n$ isomorphic to $(\mathbb{Z}_{3^c})^{n-1}$ (see Proposition 4.5 and the proof of Claim 4.8). The Hodge diamond of a Schreieder variety looks like

$$\begin{array}{ccccccccccccccccc} & & & & & & & & 1 & & & & & & & & \\ & & & & & & & & * & & & & & & & & \\ & & & & & & & & \vdots & & & & & & & & \\ & & & & & & & & \vdots & & & & & & & & \\ 0 & \ldots & 0 & g & 0 & \ldots & 0 & & * & & 0 & \ldots & 0 & g & 0 & \ldots & 0 \\ & & & & & & & & \vdots & & & & & & & & \\ & & & & & & & & \vdots & & & & & & & & \\ & & & & & & & & * & & & & & & & & \\ & & & & & & & & 1 & & & & & & & & \end{array}$$

where $g = (3^c - 1)/2$ can occur at any desired place $h^{a,b}$ with $a + b = n$.

From an arithmetic perspective, the construction of Schreieder has been used by Flapan and Lang [11] to construct motives associated to certain algebraic Hecke characters, thereby generalizing the modularity result of Cynk and Hulek [7].

The varieties of Cynk–Hulek and of Schreieder are thus both very special from a Hodge-theoretic point of view and from an arithmetic point of view. The aim of this note is to confirm that these varieties are also very special from a cycle-theoretic point of view.

Let $\mathrm{CH}^i(X)$ denote the Chow groups with rational coefficients, let $\mathrm{CH}^i_{num}(X)$ denote the subgroup of homologically trivial cycles, and let $\overline{\mathrm{CH}}^i(X)$ denote the quotient. Our main result concerns the multiplicative structure of the Chow ring of $X$ :

**Theorem 1** (Theorem 4.1)**.** *Let $X$ be either a Cynk–Hulek Calabi–Yau variety as in Theorem 1.1 or 1.2, or a Schreieder variety as in Theorem 1.4. Then the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}^*(X^m) \to \overline{\mathrm{CH}}^*(X^m)$ admits a section for all positive integers $m$.*

*Moreover, assuming $n := \dim X \geq 2$, the graded subalgebra $R^*(X) \subseteq \mathrm{CH}^*(X)$ generated by divisors, Chern classes and by cycles that are the intersection of two cycles in $X$ of positive codimension injects into $\overline{\mathrm{CH}}^*(X)$. In particular, for any $k$ the image of the intersection map*

$$\mathrm{CH}^i(X) \otimes \mathrm{CH}^{k-i}(X) \to \mathrm{CH}^k(X) \qquad (0 < i < k)$$

*injects into cohomology.*

Theorem 4.1 is similar to results in the Chow ring of K3 surfaces [3], and is closely related to the conjectural "splitting property" of Beauville [4]. Presumably, the fact that $R^n(X) = \mathbb{Q}c_n(X)$ is true for *any* Calabi–Yau variety[1] ; for instance, this was established for Calabi–Yau complete intersections [37], [12]. On the other hand, the full statement of Theorem 4.1 is certainly *not* true for all Calabi–Yau varieties [4, Example 2.1.5] ; this behavior is peculiar to the Cynk–Hulek Calabi–Yau varieties.

Somewhat surprisingly, the Schreieder varieties give examples in any dimension, and with arbitrarily large geometric genus, for which the intersection product in the Chow ring is "as degenerate as possible". (This should be contrasted with the behaviour of the surfaces $S \subset \mathbb{P}^3$ exhibited in [27], for which the rank of $\mathrm{Im}\big(\mathrm{CH}^1(S)\otimes\mathrm{CH}^1(S) \to \mathrm{CH}^2(S)\big)$ gets arbitrarily large when the degree of $S$ grows.) Schreieder surfaces of genus 1 are K3 surfaces while Schreieder surfaces of higher genus are modular elliptic of Kodaira dimension 1 (see [10] and Remark 1.5). For those, we obtain as a corollary the existence of a decomposition of the small diagonal similar to that of K3 surfaces proved by Beauville and Voisin [3] :

**Corollary 1.** *Let $S$ be a Schreieder surface. Then there exists a point $p \in S$ such that*

$$(x,x,x)-(x,x,p)-(x,p,x)-(p,x,x)+(p,p,x)+(p,x,p)+(x,x,p) = 0 \quad \textit{in } \mathrm{CH}^4(S\times S\times S).$$

*Here $(x,x,x),(x,x,p),(x,p,p)$ are the classes of the images of $S$ into $S\times S\times S$ by the maps $x \mapsto (x,x,x)$, $x \mapsto (x,x,p)$, $x \mapsto (x,p,p)$.*

In order to show that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}^*(X^m) \to \overline{\mathrm{CH}}^*(X^m)$ of Theorem 4.1 admits a section, we prove that $X$ satisfies a certain condition $(\star)$ (*cf.* Definition 2.6) which was introduced in [14]. The "moreover" part of Theorem 4.1 is not a formal consequence of the existence of a section, and is obtained, via Proposition 2.10, by computing the motive of $X$ and by establishing yet another result related to the splitting property (see §2.1 for the notion of *multiplicative Chow–Künneth decomposition*) :

**Theorem 2** (Theorem 4.2)**.** *Let $X$ be either a Cynk–Hulek Calabi–Yau variety as in Theorem 1.1 or 1.2, or a Schreieder variety as in Theorem 1.4. Then $X$ admits a multiplicative self-dual Chow–Künneth decomposition, in the sense of* [31].

[1] This expectation is perhaps overly optimistic. Voisin [38, p. 101] writes more prudently : "It would be very interesting to understand the class of Calabi–Yau varieties satisfying conclusions analogous to" Calabi–Yau complete intersections. Bazhov [2] states (and proves in certain cases) a weaker version of this expectation, only considering 0-cycles that are intersections of divisors.

Other varieties admitting a multiplicative Chow–Künneth decomposition include abelian varieties, hyperelliptic curves, Hilbert schemes of points of K3 surfaces and of abelian surfaces [34], and generalized Kummer varieties [13]. Theorem 4.2 provides the first examples of Calabi–Yau varieties of dimension $> 2$ with a multiplicative Chow–Künneth decomposition, while Theorem 4.1 provides the first examples of Calabi–Yau varieties of dimension $> 2$ for which the subalgebra of the Chow ring generated by divisors injects into cohomology via the cycle class map.

Along the way, we compute (Corollary 3.7) the Chow motive of certain finite quotients of products of (hyper)elliptic curves (including the quotients considered in Theorems 1.1, 1.2 and 1.4) and establish the three following consequences:

**Theorem 3** (Theorem 5.3). *Let $X$ be a Cynk–Hulek Calabi–Yau variety of dimension $n$ as in Theorem 1.1 or 1.2. Then any $a, a' \in \mathrm{CH}^n_{num}(X)$ satisfy*

$$a \times a' = (-1)^n \, a' \times a \quad \text{in } \mathrm{CH}^{2n}(X \times X) \ .$$

According to an old conjecture of Voisin ([36], *cf.* also Section 5 below), the statement of Theorem 5.3 should hold for *any* Calabi–Yau variety. As far as we are aware, Theorem 5.3 gives the first examples of Calabi–Yau varieties of arbitrary dimension verifying Voisin's conjecture.

A second consequence concerns Voevodsky's conjecture on smash-equivalence; we refer to [35] and Section 6 below for the definition and background of smash-equivalence.

**Proposition** (Proposition 6.4). *Let $X$ be either a Cynk–Hulek Calabi–Yau variety as in Theorem 1.1 or 1.2, or a Schreieder variety as in Theorem 1.4. Assume that $X$ is odd-dimensional. Then smash-equivalence and numerical equivalence coincide for all $\mathrm{CH}^i(X)$.*

Finally, in a brief excursion to positive characteristic, we obtain the following:

**Proposition** (Proposition 7.1). *Let $k$ be an algebraically closed field of characteristic $\geq 5$. Let $X$ be a Cynk–Hulek Calabi–Yau variety over $k$ as in Theorem 1.1 or 1.2, where the elliptic curves are assumed to be supersingular, and $\dim X \geq 2$. Then the cycle class map to $\ell$-adic cohomology induces isomorphisms*

$$\mathrm{CH}^i(X)_{\mathbb{Q}_\ell} \ \xrightarrow{\cong} \ H^{2i}(X, \mathbb{Q}_\ell(i)) \quad \forall i$$

*(where $\ell$ is a prime different from $\mathrm{char}\, k$).*

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## 1. The varieties of Cynk–Hulek and Schreieder

We denote $\mathbb{Z}_n$ the cyclic group of order $n$. Given a smooth projective variety, we denote $H_{tr}(X)$ its transcendental cohomology; it is the orthogonal complement (with respect to the choice of a polarization) of the subspace spanned by algebraic classes.

### 1.1. The Cynk–Hulek construction.

**Theorem 1.1** (Cynk–Hulek [7]). *Let $E_1, \ldots, E_n$ be elliptic curves. For any $n \in \mathbb{N}$, let*

$$G = \{(m_1, \ldots, m_n) \ \in \mathbb{Z}_2^n : m_1 + \cdots + m_n = 0\} \cong \mathbb{Z}_2^{n-1}$$

*act on $E_1 \times \cdots \times E_n$, where the generator of $\mathbb{Z}_2$ acts on $E_i$ by the $[-1]$-involution. Then there exists a crepant resolution*

$$f\colon\ X \ \to\ \bar{X} := (E_1 \times \cdots \times E_n)/G\ ,$$

*and so $X$ is a Calabi–Yau variety. Moreover, such Calabi–Yau varieties form a complete family.*

*Proof.* This is [7, Corollary 2.3]. In fact, the crepant resolution $X$ can be constructed explicitly, inductively on the number of elliptic curves, *cf.* the proof of Proposition 4.3 below. That such Calabi–Yau varieties form a complete family can be seen as follows : since elliptic curves have a one-dimensional deformation family, $X$ clearly fits into an $n$-dimensional deformation family. On the other hand, $H^n(X)$ is isomorphic to $H^1(E_1) \otimes \cdots \otimes H^1(E_n)$ plus possibly some algebraic classes, and in particular $h^{1,n-1}(X) = n$ ; see [7, Lemma 2.4] or Corollary 3.7 below. By Serre duality $H^1(X, \mathcal{T}_X) \cong H^{n-1}(X, \Omega^1_X)$ so that $\dim H^1(X, \mathcal{T}_X) = n$. □

In the case of elliptic curves with extra endomorphisms (precisely, automorphisms of order 3), Cynk and Hulek construct examples of Calabi–Yau varieties with cohomology "as simple as possible".

**Theorem 1.2** (Cynk–Hulek [7]). *Let $E$ be an elliptic curve with an order 3 automorphism $\nu$. For any $n \in \mathbb{N}$, let*

$$G = \{(m_1, \ldots, m_n) \ \in \mathbb{Z}_3^n : m_1 + \cdots + m_n = 0\} \cong \mathbb{Z}_3^{n-1}$$

*act on $E^n$ by $\nu^{m_i}$ on the $i$-th factor. There exists a crepant resolution*

$$f\colon\ X \ \to\ \bar{X} := E^n/G\ ,$$

*and so $X$ is a Calabi–Yau variety. Moreover, for $n > 2$, such Calabi–Yau varieties are rigid, and their transcendental cohomology has Hodge numbers $h^{p,q}_{tr} = 1$ if $\{p, q\} = \{n, 0\}$, and $h^{p,q}_{tr} = 0$ otherwise.*

*Proof.* This is [7, Theorem 3.3] (the construction of $X$ is also explained in [16, Section 5.3]). In fact, the crepant resolution $X$ can be constructed explicitly, inductively on the number of elliptic curves, *cf.* the proof of Proposition 4.4 below.

Arguing as in the proof of Theorem 1.1, we see that such $X$ is rigid because $h^{1,n-1}(X) = 0$ ; see [7, Theorem 3.3] or Corollary 3.7 below. □

**Remark 1.3.** The Cynk–Hulek varieties $X$ of Theorem 1.2 are $N^1$–maximal, in the sense of [6] ; this means that $\dim H^n_{tr}(X, \mathbb{Q}) = 2$.

### 1.2. The Schreieder construction.

By using iterated resolutions of $\mathbb{Z}_3$-quotient singularities, Schreieder generalizes (see however Remark 1.6) the Cynk–Hulek construction of Theorem 1.2 and proves the following theorem.

**Theorem 1.4** (Schreieder [29]). *Let $c$ be a positive integer, and let $\zeta$ be a primitive $3^c$-th roof of unity. Let $C$ be the smooth projective hyperelliptic curve obtained as the smooth* *projectivization of the affine curve* $\{y^2 = x^{3^c} + 1\}$*. Endow* $C$ *with the action of* $\mathbb{Z}_{3^c}$ *given by* $(x, y) \mapsto (\zeta \cdot x, y)$*. For any* $n \in \mathbb{N}$ *and any integers* $a, b \geq 0$ *such that* $a > b$ *and* $a + b = n$*, let*

$$G_{a,b} = \{(m_1, \dots, m_n) \ \in \mathbb{Z}_{3^c}^n : m_1 + \cdots + m_a - m_{a+1} - m_{a+b} = 0\} \cong \mathbb{Z}_3^{n-1}$$

*act on* $C^n$ *by* $\zeta^{m_i}$ *on the* $i$*-th factor. Then* $C^n/G_{a,b}$ *admits a smooth projective model* $X$ *whose transcendental cohomology has Hodge numbers* $h_{tr}^{p,q} = (3^c - 1)/2$ *if* $\{p, q\} = \{a, b\}$*, and* $h_{tr}^{p,q} = 0$ *otherwise.*

Schreieder provides in [29, §8] an explicit construction of $X$. The construction is inductive on the number of factors $C$, and is recalled in §4.3. When referring to the "Schreieder varieties", we will mean those explicit models.

**Remark 1.5.** A Schreieder variety of dimension 2 is a K3 surface when $c = 1$ (these K3 surfaces have been intensively studied by Shioda–Inose [32]), and is an elliptic modular surface of Kodaira dimension 1 for all $c > 1$ [10, Theorems 3.2 & 9.2]. These surfaces are very special: they are $\rho$–maximal (in the sense of [5]) and have Mordell–Weil rank 0 [10, Corollary 6.1].

**Remark 1.6.** In case $c = 1$ and $b = 0$, the Schreieder variety $X_S$ (given by Theorem 1.4) and the Cynk–Hulek variety $X_{CH}$ (given by Theorem 1.2) are both resolutions of the same singular variety $C^n/G_{n,0}$. They share the same Hodge numbers $h^{p,q}$ for $p \neq q$, but they are (*a priori*) different; indeed, $X_{CH}$ is Calabi–Yau, whereas $X_S$ is only "numerically Calabi–Yau". The difference in the construction of $X_S$ and $X_{CH}$ is outlined in Remark 4.7.

## 2. Multiplicative Chow–Künneth decompositions and distinguished cycles

The aim of this section is to recall briefly the notions of *multiplicative Chow–Künneth decomposition*, and of *distinguished cycle* on varieties with motive of abelian type. Combining both notions, we reduce the proof of the main Theorem 4.1 to showing that the transcendental cohomology $H^i_{tr}(X)$ is concentrated in degree $i = \dim X$, and that the motive of $X$ satisfies a certain condition $(\star)$ (Definition 2.6); *cf.* Proposition 2.10 and the final Remark 2.11.

### 2.1. Multiplicative Chow–Künneth decompositions.

**Definition 2.1** (Murre [26])**.** Let $X$ be a smooth projective variety of dimension $n$. We say that $X$ has a *Chow–Künneth decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \quad \text{in } \mathrm{CH}^n(X \times X) \ ,$$

such that the $\pi_X^i$ are mutually orthogonal idempotents and $(\pi_X^i)_* H^*(X) = H^i(X)$. Given a Chow–Künneth decomposition for $X$, we set

$$\mathrm{CH}^i(X)_{(j)} := (\pi_X^{2i-j})_* \,\mathrm{CH}^i(X).$$

The Chow–Künneth decomposition is said to be *self-dual* if

$$\pi_X^i = {}^t\pi_X^{2n-i} \quad \text{in } \mathrm{CH}^n(X \times X) \quad \forall i \ .$$

(Here ${}^t\pi$ denotes the transpose of a cycle $\pi$.)

**Remark 2.2.** The existence of a Chow–Künneth decomposition for any smooth projective variety is part of Murre's conjectures [26]. It is expected that for any $X$ with a Chow–Künneth decomposition, one has

$$\mathrm{CH}^i(X)_{(j)} \stackrel{??}{=} 0 \quad \text{for } j < 0 \ , \quad \mathrm{CH}^i(X)_{(0)} \cap \mathrm{CH}^i_{num}(X) \stackrel{??}{=} 0.$$

These are Murre's conjectures B and D, respectively.

**Definition 2.3** (Definition 8.1 in [30])**.** Let $X$ be a smooth projective variety of dimension $n$. Let $\delta_X \in \mathrm{CH}^{2n}(X\times X\times X)$ be the class of the small diagonal

$$\delta_X := \big\{ (x,x,x) : x\in X\big\} \ \subset\ X\times X\times X\ .$$

A Chow–Künneth decomposition $\{\pi^i_X\}$ of $X$ is *multiplicative* if it satisfies

$$\pi^k_X\circ \delta_X\circ (\pi^i_X\otimes \pi^j_X)=0\quad \text{in}\ \ \mathrm{CH}^{2n}(X\times X\times X)\quad \text{for all}\ i+j\not= k\ .$$

In that case,

$$\mathrm{CH}^i(X)_{(j)}:= (\pi_X^{2i-j})_\ast \mathrm{CH}^i(X)$$

defines a bigraded ring structure on the Chow ring ; that is, the intersection product has the property that

$$\mathrm{Im} \Big(\mathrm{CH}^i(X)_{(j)}\otimes \mathrm{CH}^{i^\prime}(X)_{(j^\prime)} \xrightarrow{\cdot} \mathrm{CH}^{i+i^\prime}(X)\Big)\ \subseteq\ \mathrm{CH}^{i+i^\prime}(X)_{(j+j^\prime)}\ .$$

The property of having a multiplicative Chow–Künneth decomposition is severely restrictive, and is closely related to Beauville's "(weak) splitting property" [4]. For more ample discussion, and examples of varieties admitting a multiplicative Chow–Künneth decomposition, we refer to [30, Chapter 8], as well as [34], [31], [13].

2.2. **Distinguished cycles on varieties with motive of abelian type.** The following crucial notion was introduced by O'Sullivan [28].

**Definition 2.4** (Symmetrically distinguished cycles on abelian varieties [28])**.** Let $A$ be an abelian variety and $\alpha\in \mathrm{CH}^*(A)$. For each integer $m\geq 0$, denote by $V_m(\alpha)$ the $\mathbb{Q}$-vector subspace of $\mathrm{CH}^*(A^m)$ generated by elements of the form

$$p_*(\alpha^{r_1}\times \alpha^{r_2}\times\cdots\times \alpha^{r_n}),$$

where $n\leq m$, $r_j\geq 0$ are integers, and $p : A^n\to A^m$ is a closed immersion with each component $A^n\to A$ being either a projection or the composite of a projection with $[-1] : A\to A$. Then $\alpha$ is *symmetrically distinguished* if for every $m$ the restriction of the projection $\mathrm{CH}^*(A^m)\to \overline{\mathrm{CH}}^*(A^m)$ to $V_m(\alpha)$ is injective.

The main result of [28] is :

**Theorem 2.5** (O'Sullivan [28])**.** *Let $A$ be an abelian variety. Then $\mathrm{DCH}^*(A)$, the symmetrically distinguished cycles in $\mathrm{CH}^*(A)$, form a graded sub-$\mathbb{Q}$-algebra that contains symmetric divisors and that is stable under pull-backs and push-forwards along homomorphisms of abelian varieties. Moreover the composition*

$$\mathrm{DCH}^*(A)\hookrightarrow \mathrm{CH}^*(A)\twoheadrightarrow \overline{\mathrm{CH}}^*(A)$$

*is an isomorphism of $\mathbb{Q}$-algebras.*

Let $X$ be a smooth projective variety such that its Chow motive $\mathfrak{h}(X)$ belongs to the strictly full and thick subcategory of Chow motives generated by the motives of abelian varieties. We say that $X$ has motive *of abelian type*. A *marking* for $X$ is an isomorphism $\phi : \mathfrak{h}(X)\xrightarrow{\simeq} M$ of Chow motives with $M$ a direct summand of a Chow motive of the form $\oplus_i\mathfrak{h}(A_i)(n_i)$ cut out by an idempotent matrix $P\in\mathrm{End}(\oplus_i\mathfrak{h}(A_i)(n_i))$ whose entries are symmetrically distinguished cycles, where $A_i$ is an abelian variety and $n_i$ is an integer (the Tate twist). We refer to [14, Definition 3.1] for a precise definition. Given a marking $\phi : \mathfrak{h}(X)\xrightarrow{\simeq} M$, we define the subgroup of *distinguished cycles* of $X$, denoted $\mathrm{DCH}^*_\phi(X)$ to be the pre-image of

$\mathrm{DCH}^*(M) := P_* \bigoplus_i \mathrm{DCH}^{*-n_i}(A_i)$ via the induced isomorphism $\phi_* : \mathrm{CH}^*(X) \xrightarrow{\simeq} \mathrm{CH}^*(M)$. Given another smooth projective variety $Y$ with a marking $\psi : \mathfrak{h}(Y) \to N$, the tensor product $\phi \otimes \psi : \mathfrak{h}(X \times Y) \to M \otimes N$ defines naturally a marking for $X \times Y$. A morphism $f : X \to Y$ will be said to be a *distinguished morphism* if its graph is distinguished with respect to the product marking $\phi \otimes \psi$.

The composition

$$\mathrm{DCH}^*_\phi(X) \hookrightarrow \mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$$

is clearly bijective. In other words, $\phi$ provides a section (as graded vector spaces) of the natural projection $\mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$. In [14], we found sufficient conditions on the marking $\phi$ for $\mathrm{DCH}^*_\phi(X)$ to define a $\mathbb{Q}$-subalgebra of $\mathrm{CH}^*(X)$:

**Definition 2.6** (Definition 3.7 in [14])**.** We say that the marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ satisfies the condition $(\star)$ if the following two conditions are satisfied:

($\star_{\mathrm{Mult}}$) the small diagonal $\delta_X$ belongs to $\mathrm{DCH}^*_{\phi^{\otimes 3}}(X^3)$; that is, under the induced isomorphism $\phi_*^{\otimes 3} : \mathrm{CH}^*(X^3) \xrightarrow{\simeq} \mathrm{CH}^*(M^{\otimes 3})$, the image of $\delta_X$ is symmetrically distinguished, *i.e.* in $\mathrm{DCH}^*(M^{\otimes 3})$.

($\star_{\mathrm{Chern}}$) all Chern classes $c_i(X)$ belong to $\mathrm{DCH}^*_\phi(X)$;

If in addition $X$ is equipped with the action of a finite group $G$, we say that the marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ satisfies $(\star_G)$ if:

($\star_G$) the graph $g_X$ of $g : X \to X$ belongs to $\mathrm{DCH}^*_{\phi^{\otimes 2}}(X^2)$ for all $g \in G$.

**Proposition 2.7** (Proposition 3.12 in [14])**.** *If the marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ satisfies the condition $(\star)$, then there is a section, as* graded algebras*, for the natural surjective morphism $\mathrm{CH}^*(X) \to \overline{\mathrm{CH}}^*(X)$ such that all Chern classes of $X$ are in the image of this section.*

*In other words, under $(\star)$, we have a graded $\mathbb{Q}$-sub-algebra $\mathrm{DCH}^*_\phi(X)$ of the Chow ring $\mathrm{CH}^*(X)$, which contains all the Chern classes of $X$ and is mapped isomorphically to $\overline{\mathrm{CH}}^*(X)$. Elements of $\mathrm{DCH}^*_\phi(X)$ are called* distinguished cycles*.*

We refer to [14] for example of varieties satisfying $(\star)$; for our purpose here, we mention that these include abelian varieties, hyperelliptic curves (see Proposition 3.3), and varieties with trivial Chow groups[2]. The property $(\star)$ is very flexible; in [14, Section 4], it is shown that this property is stable under product, projectivization of vector bundles, and blow-ups, under certain conditions on some Chern classes. Those will be utilized in the proof of our main theorems where the smooth models will be obtained by blowing up subvarieties with trivial Chow groups inside a product of hyperelliptic curves, taking finite quotients and iterating; see the arguments in Section 4. For the record, let us write down explicitly one of the results of [14]:

**Proposition 2.8** (Propositions 4.5 and 4.8 in [14])**.** *Let $X$ be a smooth projective variety and let $i : Y \hookrightarrow X$ be a closed smooth subvariety. Let $\tilde{X}$ be the blow-up of $X$ along $Y$ and let $E$*

[2]A smooth projective variety $X$ is said to have *trivial Chow groups* if the Chow groups of $X$ base-changed to a universal domain are finite-dimensional $\mathbb{Q}$-vector spaces.

*be the exceptional divisor, so that we have a commutative diagram*

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"'] & \tilde{X} \arrow[d, "\tau"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd}$$

*If we have markings satisfying the condition* $(\star)$ *for* $X$ *and* $Y$ *such that* $i : Y \hookrightarrow X$ *is distinguished, then* $E$ *and* $\tilde{X}$ *have natural markings that satisfy* $(\star)$ *and are such that the morphisms* $i, j, \tau$ *and* $p$ *are all distinguished.*

*If in addition* $X$ *is equipped with the action of a finite group* $G$ *such that* $G \cdot Y = Y$ *and such that the markings of* $X$ *and* $Y$ *satisfy* $(\star_G)$*, then the natural markings of* $E$ *and* $\tilde{X}$ *also satisfy* $(\star_G)$*.* □

Theorem 4.2 and Theorem 4.1 are related by:

**Proposition 2.9** (Proposition 6.1 in [14]). *Let* $X$ *be a smooth projective variety with a marking* $\phi$ *that satisfies* $(\star_{\mathrm{Mult}})$*. Then* $X$ *has a self-dual multiplicative Chow–Künneth decomposition, consisting of distinguished cycles in* $\mathrm{CH}^*(X \times X)$*, with the property that*

$$\mathrm{DCH}^*_\phi(X) \subseteq \mathrm{CH}^*(X)_{(0)}$$

*(and equality holds for* $* = 0, 1, \dim X - 1, \dim X$*).* □

Thus if we have a smooth projective variety $X$ with a marking $\phi$ that satisfies $(\star_{\mathrm{Mult}})$, we have a chain of homomorphisms

$$\mathrm{DCH}^*_\phi(X) \hookrightarrow \mathrm{CH}^*(X)_{(0)} \hookrightarrow \mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X),$$

whose composition is an isomorphism, and where the left inclusion arrow is conjecturally an isomorphism (by Murre's conjecture D).

## 2.3. A crucial proposition.

The following proposition is crucial to the proof of the second part of Theorem 4.1.

**Proposition 2.10.** *Let* $X$ *be a smooth projective variety of dimension* $n \geq 2$ *that admits a marking satisfying the condition* $(\star)$ *of Definition 2.6. Assume that the cohomology of* $X$ *is spanned by algebraic classes in degree* $\neq n$*. Then the graded subalgebra* $R^*(X) \subseteq \mathrm{CH}^*(X)$ *generated by divisors, Chern classes and by cycles that are the intersection of two cycles in* $X$ *of positive codimension injects into* $\overline{\mathrm{CH}}^*(X)$*.*

*Proof.* Fix a marking $\phi : \mathfrak{h}(X) \to M$ that satisfies $(\star)$; in particular, $X$ has motive of abelian type. First we exploit the condition on the cohomology of $X$. This condition means that $H^{2i+1}(X) = 0$ for $2i + 1 \neq n$, and that the cycle class map $\mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X)$ is surjective for $2i \neq n$. By using the nondegenerate cup-product pairing between $\mathrm{H}^{2i}(X)$ and $\mathrm{H}^{2n-2i}(X)$, we obtain a Künneth decomposition of the diagonal $\Delta_X = p_X^n + \sum_{2i \neq n} p_X^{2i} \in \mathrm{H}^{2n}(X \times X)$, where the classes $p_X^j$ are algebraic and such that the homological motives $(X, p_X^{2i})$ are isomorphic to a direct sum of copies of the Lefschetz motive $\mathbb{1}(-i)$ for $2i \neq n$, and where $(p_X^n)_* \mathrm{H}^*(X) = \mathrm{H}^n(X)$. By finite-dimensionality of the motive of $X$, this decomposition lifts to a decomposition of the Chow motive of $X$:

$$\mathfrak{h}(X) \simeq \mathfrak{h}^n(X) \oplus \bigoplus \mathbb{1}(*), \tag{1}$$

where $\mathrm{H}^*(\mathfrak{h}^n(X)) = \mathrm{H}^n(X)$.

By Proposition 2.7, it is enough to show that $R^*(X) \subseteq \mathrm{DCH}^*_\phi(X)$. Since we already know that $\mathrm{DCH}^*_\phi(X)$ is a subalgebra of $\mathrm{CH}^*(X)$ that contains the Chern classes of $X$, it is enough to show that $\mathrm{DCH}^1_\phi(X) = \mathrm{CH}^1(X)$ and that the intersection of any two cycles of positive codimension belongs to $\mathrm{DCH}^*_\phi(X)$. That $\mathrm{DCH}^1_\phi(X) = \mathrm{CH}^1(X)$ is clear from the description of the motive of $X$ given in (1). By Proposition 2.9, $X$ has a multiplicative self-dual Chow–Künneth decomposition $\{\pi^i_X : 0 \le i \le 2n\}$ that induces a bigrading on the Chow ring of $X$ with the property that $\mathrm{DCH}^*_\phi(X) \subseteq \mathrm{CH}^*(X)_{(0)}$. By finite-dimensionality of $X$, any two Chow–Künneth decompositions of the motive of $X$ are isomorphic; therefore, by (1), the bigrading on $\mathrm{CH}^*(X)$ has the form:

$$\mathrm{CH}^i(X) = \mathrm{CH}^i(X)_{(0)} \oplus \mathrm{CH}^i(X)_{(2i-n)}. \tag{2}$$

By multiplicativity we have $\mathrm{Im}\left(\mathrm{CH}^i(X)_{(s)} \otimes \mathrm{CH}^j(X)_{(t)} \to \mathrm{CH}^{i+j}(X)\right) \subseteq \mathrm{CH}^{i+j}(X)_{(s+t)}$. Combining the multiplicativity with (2), we find that, for $0 < i, j < n$, we have

$$\mathrm{Im}\left(\mathrm{CH}^i(X) \otimes \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X)\right) = \mathrm{Im}\left(\mathrm{CH}^i(X)_{(0)} \otimes \mathrm{CH}^j(X)_0 \to \mathrm{CH}^{i+j}(X)\right) \subseteq \mathrm{CH}^{i+j}(X)_{(0)}.$$

Now the motive $(X, \pi^{2i}_X)$ is isomorphic to a direct sum of Lefschetz motives $\mathbb{1}(-i)$ for all $i \neq \frac{n}{2}$, so that $\mathrm{CH}^i(X)_{(0)}$ maps isomorphically to $\overline{\mathrm{CH}}^i(X)$ for all $i \neq \frac{n}{2}$. Since $\mathrm{DCH}^i_\phi(X)$ maps isomorphically to $\overline{\mathrm{CH}}^i(X)$ for all $i$, we see that the inclusion $\mathrm{DCH}^i_\phi(X) \subseteq \mathrm{CH}^i(X)_{(0)}$ is an equality for all $i \neq \frac{n}{2}$. This establishes that the intersection of any two cycles of positive codimension belongs to $\mathrm{DCH}^*_\phi(X)$, and thereby finishes the proof of Proposition 2.10. ☐

**Remark 2.11.** In view of Proposition 2.7 (together with the fact that the property ($\star$) is stable under product by [14, Prop. 4.1]), Propositions 2.9 and 2.10, the proof of the main Theorems 4.1 and 4.2 reduces to showing that the relevant varieties $X$ satisfy the following two properties:

(1) The cohomology of $X$ is spanned by algebraic classes in degree $\neq n$;
(2) $X$ admits a marking that satisfies the condition ($\star$).

## 3. The motive of $\bar{X}$

3.1. **Hyperelliptic curves.** Let $C$ be a smooth projective hyperelliptic curve of genus $g \ge 0$, that is, $C$ comes equipped with a 2-to-1 morphism $\pi : C \to \mathbb{P}^1$. This morphism induces an involution on $C$ which we call the hyperelliptic involution. By definition, the *Weierstraß points* of $C$ are the $2g+2$ ramification points of the morphism $\pi : C \to \mathbb{P}^1$, that is, the $2g+2$ fixed points of the involution. An elliptic curve will be seen as a hyperelliptic curve via its $[-1]$-involution. We have the following basic lemma.

**Lemma 3.1.** *The fixed points for the hyperelliptic involution are pairwise rationally equivalent,* i.e., *define the same class in* $\mathrm{CH}_0(C)_\mathbb{Q}$.

*Proof.* Since $\pi$ is flat of degree 2, we see that any two Weierstraß points $P$ and $Q$ of $C$ satisfy $2[P] = 2[Q] \in \mathrm{CH}_0(C)$. Therefore the Weierstraß points on $C$ define the same class in $\mathrm{CH}_0(C)_\mathbb{Q}$. ☐

3.2. **The hyperelliptic curves $y^2 = x^{2g+1} + D$.** Let $g$ be a natural number and let $D$ be a non-zero rational number, and let $C_{g,D}$ be the smooth projective model of the affine curve $Y = \{y^2 = x^{2g+1} + D\}$. When $g > 1$, the projective closure $X$ of $Y$ has a cusp at the point $\infty$, and the hyperelliptic curve $C_{g,D}$ is its normalization. In particular $C_{g,D}$ is obtained from $Y$ by adding one additional point at $\infty$.

The curve $C_{g,D}$ is endowed with the hyperelliptic involution $\sigma$ which on the open $Y$ is given by $(x,y) \mapsto (x,-y)$. The fixed points for that action are called the Weierstraß points, and are explicitly given by the $2g+2$ points $\{(\zeta \cdot |D|^{\frac{1}{2g+1}}, 0) : \zeta \in \mu_{2g+1}\} \cup \{\infty\}$.

The curve is also endowed with an action of $\mu_{2g+1}$, which on the open $Y$ is given by $\zeta \cdot (x,y) = (\zeta \cdot x, y)$. Its fixed points are the points $(0,\sqrt{D})$, $(0,-\sqrt{D})$, and $\infty$. Note that these points are defined over $\mathbb{Q}$ if $D=1$.

**Lemma 3.2.** *The fixed points for the hyperelliptic involution and for the $\mu_{2g+1}$-action are pairwise rationally equivalent,* i.e., *define the same class in* $\mathrm{CH}_0(C_{g,D})_{\mathbb{Q}}$.

*Proof.* By Lemma 3.1, the fixed points for the hyperelliptic involution on $C_{g,D}$ (which include the $\infty$ point) define the same class in $\mathrm{CH}_0(C_{g,D})_{\mathbb{Q}}$.

Consider the lines $\{y=\sqrt{D}\}, \{y=-\sqrt{D}\}$ and $\{x=0\}$ in $\mathbb{A}^2$. These lines intersect the curve $Y$ in $(2g+1)[(0,\sqrt{D})]$, $(2g+1)[(0,-\sqrt{D})]$ and $[(0,\sqrt{D})]+[(0,-\sqrt{D})]$, respectively. We deduce that these 3 cycles are rationally equivalent on $Y$. By excision, we see that the points $(0,\sqrt{D})$, $(0,-\sqrt{D})$ and $\infty$ define the same class in $\mathrm{CH}_0(C_{g,D})_{\mathbb{Q}}$. (Note that in the case $g=1$, *i.e.*, in the case where $(C_{g,D},\infty)$ is an elliptic curve, the points $(0,\sqrt{D})$ and $(0,-\sqrt{D})$ are 3-torsion points.) □

## 3.3. **Key proposition.**

**Proposition 3.3.** *Let $C$ be a smooth projective curve equipped with the action of a finite group $H$. Assume that $(C,H)$ is either:*

*(i) a hyperelliptic curve equipped with its hyperelliptic involution;*
*(ii) a hyperelliptic curve $C_{g,D}$ as in subsection 3.2 equipped with the action of $\mu_{2g+1}$.*

*Then $C$ has a marking that satisfies the condition $(\star)$ and $(\star_H)$, with the additional property that if $P$ is a fixed point of $H$, then the embedding $P \hookrightarrow C$ is distinguished.*

*Proof.* By [14, Corollary 5.4], the embedding of a hyperelliptic curve inside its Jacobian $\mathrm{AJ} : C \to J(C), x \mapsto O_C(x-q)$, where $q$ is a Weierstraß point, provides a marking for $C$ that satisfies $(\star)$. Moreover the embedding $q \hookrightarrow C$ is distinguished by construction. Since by Lemma 3.2 all fixed points of $H$ and all Weierstraß points are rationally equivalent, we see that the embedding $P \hookrightarrow C$ is distinguished for any choice of fixed point $P$ of $H$.

It remains to show that for any $h \in H$, the graph $\Gamma_h \in \mathrm{CH}^1(C\times C)$ is distinguished with respect to the product marking $\mathrm{AJ}\otimes\mathrm{AJ}$. Let $P$ be a fixed point of $H$ (which by Lemma 3.2 is rationally equivalent to any Weierstraß point) and consider the following Chow–Künneth decomposition

$$\pi_C^0 = P\times C, \quad \pi_C^2 = C\times P, \quad \text{and} \quad \pi_C^1 = \Delta_C - \pi_C^0 - \pi_C^2.$$

These are distinguished cycles in $C\times C$ and by Proposition 2.9 they define a multiplicative Chow–Künneth decomposition such that $\mathrm{DCH}^*(C\times C) \subseteq \mathrm{CH}^*(C\times C)_{(0)}$. Since in codimension 1 the previous inclusion is an equality (see Proposition 2.9), we are reduced to showing that $\Gamma_h$ belongs to $\mathrm{CH}^1(C\times C)_{(0)}$ with respect to the product Chow–Künneth decomposition on the product $C\times C$. That is, we are reduced to showing that

$$(\pi_C^0\otimes\pi_C^1 + \pi_C^1\otimes\pi_C^0)_*\Gamma_h = 0 \quad \text{in } \mathrm{CH}^1(C\times C).$$

By orthogonality and symmetry, we are reduced to showing that

$$\pi_C^1\circ\Gamma_h\circ\pi_C^2 = 0 \quad \text{in } \mathrm{CH}^1(C\times C).$$

But $\Gamma_h \circ \pi_C^2 = C \times h(P) = C \times P = \pi_C^2$ (since $P$ is a fixed point of $H$). We can then conclude by orthogonality of $\pi_C^1$ and $\pi_C^2$. $\square$

3.4. **The motive of $\bar{X}$.** In this subsection we consider a projective variety of the form

$$\bar{X} = (C_1 \times \cdots \times C_n)/G,$$

where the $C_i$ are hyperelliptic curves and $G$ is a certain finite subgroup of automorphisms of $C_1 \times \cdots \times C_n$. Specifically, we assume one of the following:

(a) Each $C_i$ is a hyperelliptic curve equipped with the action of $H \cong \mathbb{Z}_2$ induced by its hyperelliptic involution, and

$$G = \{(h_1, \ldots, h_n) \in H^n : h_1 + \cdots + h_n = 0\}.$$

(b) Let $g$ be a positive integer and let $a, b \geq 0$ be integers such that $n = a + b$ and $a > b$. Each $C_i$ is a hyperelliptic curve of genus $g$ as in Subsection 3.2 equipped with the action of $H = \mu_{2g+1}$ given by $(x, y) \mapsto (\zeta \cdot x, y)$, and

$$G = G_{a,b} = \{(h_1, \ldots, h_n) \in H^n : h_1 + \cdots + h_a - h_{a+1} - \cdots - h_{a+b} = 0\}.$$

Note that in case (a) if the curves are chosen to be elliptic curves endowed with the $[-1]$-involution, then $\bar{X}$ is the variety considered in Theorem 1.1, while in case (b) if the curves are chosen to be elliptic curves with an order 3 automorphism, and one takes $b = 0$, then $\bar{X}$ is the variety considered in Theorem 1.2.

In this subsection, we determine the motive of $\bar{X}$; this will be used later on in Section 4. We also show how the formalism of distinguished cycles (and multiplicative Chow–Künneth decomposition) works for $\bar{X}$. This is done for the reader's benefit, and is not necessary for the results in Section 4.

In what follows, a hyperelliptic curve $C$ is always endowed with the Chow–Künneth decomposition given by

$$\pi_C^0 := P \times C, \quad \pi_C^2 = C \times P, \quad \pi_C^1 = \Delta_C - \pi_C^0 - \pi_C^2,$$

where $P$ is the class of a Weierstraß point. A product of hyperelliptic curves $C_1 \times \cdots \times C_n$ is endowed with the product Chow–Künneth decomposition

$$\pi^k := \sum_{k = k_1 + \cdots + k_n} \pi_{C_1}^{k_1} \otimes \cdots \otimes \pi_{C_n}^{k_n}.$$

In the case where $C$ is an elliptic curve endowed with the $[-1]$-involution, note that the 0 element is a Weierstraß point, so that the above Chow–Künneth decomposition is the Deninger–Murre decomposition [8]. By unicity of the Deninger–Murre decomposition [8, Theorem 3.1], the above product Chow–Künneth decomposition for a product of elliptic curves is the one of Deninger–Murre [8].

Since the variety $\bar{X}$ is obtained as the quotient of the smooth projective variety $C_1 \times \cdots \times C_n$ by a finite group $G$, and since we are only concerned with algebraic cycles with rational coefficients, the motive of $\bar{X}$ identifies with the $G$-invariant part of the motive of $C_1 \times \cdots \times C_n$, as algebra objects. In particular, the notion of multiplicative Chow–Künneth decomposition and the condition $(\star_{\mathrm{Mult}})$ make sense for $\bar{X}$, so that Proposition 2.10 holds with the Chern classes omitted. These are established for $\bar{X}$ via the following proposition, which is the main result of this section:

**Proposition 3.4.** *Let $\bar{X} = (C_1 \times \cdots \times C_n)/G$ with $C_1, \ldots, C_n$ and $G$ as in (a) or (b) above. Then the $\mathbb{Q}$-subalgebra of $\mathrm{CH}^*(\bar{X})$ generated by $\mathrm{CH}^1(\bar{X})$ and by the images of the intersection products*

$$\mathrm{CH}^{k-l}(\bar{X}) \otimes \mathrm{CH}^l(\bar{X}) \ \to \ \mathrm{CH}^k(\bar{X}) \quad (0 < l < k)$$

*injects into $\overline{\mathrm{CH}}^*(\bar{X})$.*

In the next section, this statement, together with the existence of a multiplicative Chow–Künneth decomposition, will be extended from $\bar{X}$ to the crepant resolution $X$ for Calabi–Yau varieties as in Theorems 1.1 and 1.2, and to the Schreieder resolution $X$ for varieties as in Theorem 1.4.

Note that Proposition 3.3 (together with [14, Remark 4.3 and Proposition 4.12]) establishes the existence of a marking for $\bar{X}$ that satisfies $(\star_{\mathrm{Mult}})$. Therefore the proof of Proposition 3.4 reduces, thanks to Proposition 2.10 and Remark 2.11, to an explicit computation of the Chow motive of $\bar{X}$. This is achieved in Corollary 3.7. These computations will also be used in Section 5 in order to establish Theorem 5.3. First, we start with a general lemma and a general proposition.

**Lemma 3.5.** *Let $C$ be a smooth projective curve endowed with the action of a finite group $H$ such that $C/H$ is rational. Then, choosing a degree-1 zero-cycle $\alpha$ on $C$ that is $H$-invariant (e.g. $\alpha = \frac{1}{|H|}\sum_{h\in H} h^*[P]$ for any choice of point $P \in C$), and denoting $\pi^0_C := \alpha \times C$, $\pi^2_C := C \times \alpha$ and $\pi^1_C := \Delta_C - \pi^0_C - \pi^2_C$, we have*

$$\sum_{h\in H} \Gamma_h \circ \pi^1_C = 0 \quad \text{in } \mathrm{CH}^1(C\times C), \tag{3}$$

*whereas*

$$\Gamma_h \circ \pi^j_C = \pi^j_C \quad \text{in } \mathrm{CH}^1(C\times C), \quad \text{for } j=0 \text{ or } 2, \text{ and for all } h \in H. \tag{4}$$

*In particular, if $E$ is an elliptic curve and $H$ is a non-trivial subgroup of the group of automorphisms, then $\sum_{h\in H}\Gamma_h\circ\pi^1_E = 0$, where $\pi^1_E$ is the Chow–Künneth projector of Deninger–Murre.*

*Proof.* That $\Gamma_h \circ \pi^0_C = \pi^0_C$ and $\Gamma_h \circ \pi^2_C = \pi^2_C$ for all $h \in H$ is clear. Let $p : C \to C/H$ be the projection morphism. On the one hand, we have

$${}^t\Gamma_p \circ \Gamma_p = \sum_{h\in H} \Gamma_h.$$

On the other hand, since $C/H$ is rational we have $\Delta_{C/H} = \pi^0_{C/H} + \pi^2_{C/H}$ with $\pi^0_{C/H} = \beta \times C/H$ and $\pi^2_{C/H} = C/H \times \beta$ for any choice of degree-1 zero cycle $\beta$ on $C/H$. We also have

$${}^t\Gamma_p \circ \Gamma_p = {}^t\Gamma_p \circ (\pi^0_{C/H} + \pi^2_{C/H}) \circ \Gamma_p = (p\times p)^*(\pi^0_{C/H} + \pi^2_{C/H}) = |H|(\pi^0_C + \pi^2_C).$$

We conclude by orthogonality of the Chow–Künneth projectors $\pi^i_C$. □

**Proposition 3.6.** *Let $C_1, \ldots, C_n$ be smooth projective curves endowed with the action of finite abelian group $H$ such that each $C_i/H$ is rational. For integers $a, b \geq 0$ such that $a+b = n$ and $a > b$, consider the group*

$$G = G_{a,b} = \{(h_1, \ldots, h_n) \ \in H^n \ : \ h_1 + \cdots + h_a - h_{a+1} - \cdots - h_{a+b} = 0_H\}$$

*together with its natural action on the product $C_1 \times \cdots \times C_n$ and with the induced quotient morphism $p : C_1 \times \cdots \times C_n \to (C_1 \times \cdots \times C_n)/G$. Then we have the implication*

$$0 < |\{j : i_j = 1\}| < n \quad \Rightarrow \quad \Gamma_p \circ (\pi^{i_1}_{C_1} \otimes \cdots \otimes \pi^{i_n}_{C_n}) \circ {}^t\Gamma_p = 0.$$

*In particular, the Chow motive of $(C_1 \times \cdots \times C_n)/G$ decomposes into a direct sum of Lefschetz motives and one copy of the motive $T := (C_1 \times \cdots \times C_n, \frac{1}{|G|}\sum_{g\in G} \Gamma_g \circ (\pi^1_{C_1} \otimes \cdots \otimes \pi^1_{C_n}))$.*

*Proof.* Let us write $\Pi$ for $\pi^{i_1}_{C_1} \otimes \cdots \otimes \pi^{i_n}_{C_n}$. The action of $G$ commutes with $\Pi$, therefore $\frac{1}{|G|}\Gamma_p \circ \Pi \circ {}^t\Gamma_p$ is an idempotent, and it is zero if and only if

$$\sum_{g\in G} \Gamma_g \circ \Pi = 0.$$

Assume that $0 < |\{j : i_j = 1\}| < n$. By symmetry, we may assume without loss of generality that $\Pi = \pi^1_{C_1} \otimes \Pi' \otimes \pi_{C_n}$, where $\pi_{C_n} = \pi^0_{C_n}$ or $\pi^2_{C_n}$, and $\Pi' = \pi^{i_2}_{C_2} \otimes \cdots \otimes \pi^{i_{n-1}}_{C_{n-1}}$. Then, partitioning $G$ by the first entry of its elements, we have

$$\sum_{g\in G} \Gamma_g \circ \Pi = \sum_{g' := (h_2,\ldots,h_{n-1})\in H^{n-2}} \left( \sum_{h\in H} (\Gamma_h \circ \pi^1_{C_1}) \otimes (\Gamma'_g \circ \Pi') \otimes \pi_{C_n} \right) = 0.$$

The first equality follows from (3) and the second equality follows from (4) of Lemma 3.5.

Now assume $|\{j : i_j = 1\}| = 0$, *i.e.*, $\Pi = \pi^{i_1}_{C_1} \otimes \cdots \otimes \pi^{i_n}_{C_n}$ with $\{i_1, \ldots, i_n\} \subseteq \{0, 2\}$. In that case, we also have for all $g \in G$ that $\Gamma_g \circ \Pi = \Pi$, and thus $\sum_{g\in G} \Gamma_g \circ \Pi \neq 0$. Moreover the motive $((C_1 \times \cdots \times C_n)/G, \frac{1}{G}\Gamma_p \circ \Pi \circ {}^t\Gamma_p)$ is isomorphic to the Lefschetz motive $\mathbb{1}(-k)$, where $k = |\{j : i_j = 2\}|$.

Finally, when considering $|\{j : i_j = 1\}| = n$, one is left with the motive

$$((C_1 \times \cdots \times C_n)/G, \frac{1}{|G|}\Gamma_p \circ (\pi^1_{C_1} \times \cdots \times \pi^1_{C_n}) \circ {}^t\Gamma_p) \ ,$$

which (under $\Gamma_p$) is isomorphic to

$$T := (C_1 \times \cdots \times C_n, \frac{1}{|G|} \sum_{g\in G} \Gamma_g \circ (\pi^1_{C_1} \otimes \cdots \otimes \pi^1_{C_n})) \ .$$

□

**Corollary 3.7.** *Let $\bar{X} = (C_1 \times \cdots \times C_n)/G$ with $C_1, \ldots, C_n$ and $G$ as in (a) or (b) above. Then the Chow motive of $\bar{X}$ decomposes into a direct sum of Lefschetz motives and one copy of the motive*

$$T := (C_1 \times \cdots \times C_n, \frac{1}{|G|} \sum_{g\in G} \Gamma_g \circ (\pi^1_{C_1} \otimes \cdots \otimes \pi^1_{C_n})) \quad \in \mathcal{M}_{\mathrm{rat}} \ .$$

*In case (a), $T := (C_1 \times \cdots \times C_n, \pi^1_{C_1} \otimes \cdots \otimes \pi^1_{C_n})$, and in case (b) the motive $T$ is such that $H^j(T) = 0$ for $j \neq n$, and its transcendental cohomology has Hodge numbers*

$$h^{p,n-p}_{tr}(T) = \begin{cases} g & \text{for } p \in \{a, b\}\,; \\ 0 & \text{for } p \neq \frac{n}{2} \ . \end{cases}$$

*Proof.* By Proposition 3.6, we only need to compute $T$.

(a) Denote $\sigma_i$ the non-trivial hyperelliptic involution of $C_i$. By Lemma 3.5, we have

$$\sigma_i \circ \pi^1_{C_i} = -\pi^1_{C_i}.$$

Writing $\Pi = \pi^1_{C_1} \otimes \cdots \otimes \pi^1_{C_n}$, we therefore have for all $g \in G$ that $g \circ \Pi = \Pi$, and thus $\sum_{g\in G} g \circ \Pi = |G|\Pi \neq 0$. In particular, we see that the motive $((C_1 \times \cdots \times C_n)/G, \frac{1}{|G|}\Gamma_p \circ (\pi^1_{C_1} \otimes \cdots \otimes \pi^1_{C_n}) \circ {}^t\Gamma_p)$ is isomorphic to the motive $(C_1 \times \cdots \times C_n, \pi^1_{C_1} \otimes \cdots \otimes \pi^1_{C_n})$.

(b) In case $g = 1$ and $b = 0$ (which is the set–up of Theorem 1.2), it is proven in [7, Theorem 3.3] that $H^n(T)$ has dimension 2 when $n$ is odd, and that $H^n_{tr}(T)$ has dimension 2 when $n$ is even.

For the case $g > 1$, this is essentially done in Schreieder [29]. The proof goes as follows in the case where each curve $C_i$ is the curve $C_{g,1}$. A basis of $H^{1,0}(C_{g,1})$ is given by the differential forms $\omega_i = \frac{x^{i-1}}{y}\mathrm{d}x$, $1 \leq i \leq g$, and for $\zeta \in \mu_{2g+1}$ we have $\zeta^*\omega_i = \zeta^i\omega_i$. The proof then consists in understanding the invariants in $H^{0,1}(C_{g,1})^{\otimes d} \otimes H^{1,0}(C_{g,1})^{\otimes(n-d)}$. Since this result will not be used in this paper, let us only mention that this can be done combinatorially and was essentially carried out by Schreieder in [29, Lemma 8]. ☐

*Proof of Proposition 3.4.* In view of Proposition 3.3 and Corollary 3.7, this is an immediate consequence of Proposition 2.10 (with the Chern classes omitted). ☐

## 4. The motive of $X$

This section contains the proof of the main result of this note:

**Theorem 4.1.** *Let $X$ be either a Calabi–Yau variety as in Theorem 1.1 or 1.2, or the Schreieder variety of Theorem 1.4. Then for all integers $m \geq 1$ the $\mathbb{Q}$-algebra epimorphisms $\mathrm{CH}^*(X^m) \to \overline{\mathrm{CH}}^*(X^m)$ admit a section. Moreover, assuming $\dim X \geq 2$, the graded subalgebra $R^*(X) \subseteq \mathrm{CH}^*(X)$ generated by divisors, Chern classes and by cycles that are the intersection of two cycles in $X$ of positive codimension injects into $\overline{\mathrm{CH}}^*(X)$.*

We also establish the following:

**Theorem 4.2.** *Let $X$ be either a Cynk–Hulek Calabi–Yau variety of dimension $n$ as in Theorem 1.1 or 1.2, or a Schreieder variety as in Theorem 1.4. Then $X$ admits a multiplicative self-dual Chow–Künneth decomposition, in the sense of* [31].

Before giving the proof of Theorems 4.1 and 4.2, we detail the inductive constructions of Cynk–Hulek [7] and Schreieder [29]. This will allow us to prove the theorems, by applying the reduction argument outlined in Remark 2.11. That is, the proof will consist in checking that each step of the construction only changes algebraic classes in cohomology, and preserves the condition $(\star)$ of Definition 2.6.

### 4.1. $\mathbb{Z}_2$-actions.

**Proposition 4.3.** *Let $X_i$, $i = 1, 2$, be smooth projective varieties endowed with an action of $H_i = \mathbb{Z}_2$. Assume that, for $i = 1, 2$, $(X_i, H_i)$ enjoys the following properties:*

*(i) $X_i$ has a marking that satisfies $(\star)$ and $(\star_{H_i})$;*
*(ii) the quotients $X_i/H_i$ are smooth;*
*(iii) $B_i := \mathrm{Fix}_{X_i}(H_i)$ is a smooth divisor;*
*(iv) $B_i$ has trivial Chow groups (in particular, $B_i$ has a marking that satisfies $(\star)$);*

*(v) The inclusion morphism $B_i \hookrightarrow X_i$ is distinguished with respect to the above markings.*

*Let $Z$ be the blow-up of $X_1 \times X_2$ along $B_1 \times B_2$, and let $\tilde{B}$ be the exceptional divisor; the action of $H_1 \times H_2$ on $X_1 \times X_2$ naturally endows $Z$ with an action of $H_1 \times H_2$. Let*

$$G := \{(h_1, h_2) \in H_1 \times H_2 : h_1 + h_2 = 0\}.$$

*Then the quotient variety $X := Z/G$ is smooth and the pair $(X, H) := (Z/G, (H_1 \times H_2)/G)$ enjoys properties (i)–(v), with $\tilde{B} = \mathrm{Fix}_X(H)$.*

*Proof.* This is our take on the inductive construction of Cynk–Hulek [7, Propositions 2.1 and 2.2] (where the $X_i$ are in addition assumed to be Calabi–Yau, and it is proven that the resulting variety $X := Z/G$ is again Calabi–Yau). As in *loc. cit.*, the various varieties fit into a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X_1 \times X_2 \arrow[d] \\ X := Z/G \arrow[r] \arrow[d] & (X_1 \times X_2)/G \arrow[d] \\ Y := Z/(H_1 \times H_2) \arrow[r] & X_1/H_1 \times X_2/H_2 =: Y_1 \times Y_2 \end{tikzcd}$$

(where we adhere to the notation of [7]). Here, horizontal arrows are blow-ups, and vertical arrows are 2-to-1 morphisms. This (doubly !) explains why $X$ is smooth. On the one hand, $X$ is the blow-up of the quotient $(X_1 \times X_2)/G$ along the singular locus (which is isomorphic to $B_1 \times B_2$) consisting of $A_1$ singularities. On the other hand, $X$ is the double cover of the smooth variety $Y$ branched along the smooth divisor obtained by blowing up the smooth image of $B_1 \times B_2$ in $Y_1 \times Y_2$. This also shows that the fixed loci $\mathrm{Fix}_Z(H_1 \times H_2)$ and $\mathrm{Fix}_Z(G)$ (which coincide with the branch loci of the covers $Z \to Y$, resp. $Z \to X$) are isomorphic to the exceptional divisor $\tilde{B}$ [7, Proof of Proposition 2.1].

Let us endow $X_1 \times X_2$ and $B_1 \times B_2$ with the product markings; these satisfy $(\star)$ by [14, Prop. 4.1], the inclusion morphism $B_1 \times B_2 \hookrightarrow X_1 \times X_2$ is distinguished by [14, Prop. 3.5], and the pushforwards and pullbacks along the projection morphisms $X_1 \times X_2 \to X_i$ and $B_1 \times B_2 \to B_i$ are distinguished. Moreover, the pair $(X_1 \times X_2, H_1 \times H_2)$, where $\mathrm{Fix}_{X_1 \times X_2}(H_1 \times H_2) = B_1 \times B_2$, satisfies properties (i) and (v) by [14, Proposition 4.1] (and also [14, Remark 4.3]). Since $\tilde{B}$ is a $\mathbb{P}^1$–bundle over $B_1 \times B_2$, property (iv) is satisfied.

Since $\tilde{B} = \mathrm{Fix}_Z(H_1 \times H_2) = \mathrm{Fix}_Z(G)$, it follows that $(Z, H_1 \times H_2)$ satisfies *(iii)*. Now this is enough to ensure that $(Z, H_1 \times H_2)$ satisfies properties (i)–(v) by [14, Proposition 4.8].

Consider now the quotient morphism $Z \to X = Z/G$; it is a $\mathbb{Z}_2$-covering branched along the smooth divisor $\tilde{B}$ (which we view as a divisor on $Z$ and $X$ via the quotient morphism). We have already seen that $\tilde{B}$ satisfies $(\star)$; and $X$ satisfies $(\star)$ by [14, Prop. 4.12]. That the inclusion morphism $\tilde{B} \to X$ is distinguished follows from the fact that the inclusion morphism $\tilde{B} \to Z$ is distinguished and the fact that the quotient morphism $Z \to X$ is distinguished [*loc. cit.*]. In order to conclude, it remains to see that $X$ satisfies $(\star_H)$. But then this again follows from the fact that the quotient morphism $Z \to X$ is distinguished, together with the fact that $Z$ satisfies $(\star_{H_1 \times H_2})$. $\square$

4.2. $\mathbb{Z}_3$**-actions.** We take care of the inductive approach in order to treat the case of the Cynk–Hulek Calabi–Yau varieties of Theorem 1.2. This is very similar to the arguments in the next subsection, but we include detailed arguments here for the sake of readability.

**Proposition 4.4.** *Let $X_i$, $i = 1, 2$, be smooth projective Calabi–Yau varieties endowed with an action of $H_i = \mathbb{Z}_3$. Assume the following properties:*

*(i) The action of $H_i$ on $X_i$ does not preserve the canonical form of $X_i$;*
*(ii) $X_i$ has a marking that satisfies $(\star)$ and $(\star_{H_i})$;*
*(iii) $B_1 := \mathrm{Fix}_{X_1}(H_1)$ is a smooth divisor, whereas $B_2 := \mathrm{Fix}_{X_2}(H_2)$ is the disjoint union of a smooth divisor $B_{2,1}$ and a smooth codimension 2 subvariety $B_{2,2}$;*
*(iv) $B_i$ has trivial Chow groups (in particular, $B_i$ has a marking that satisfies $(\star)$);*
*(v) The inclusion morphism $B_i \hookrightarrow X_i$ is distinguished with respect to the above markings.*

*Let*

$$G := \{(h_1, h_2) \in H_1 \times H_2 : h_1 + h_2 = 0\} \ .$$

*Then there exists a crepant resolution of singularities*

$$X \ \to \ (X_1 \times X_2)/G \ ,$$

*and an action of $H = (H_1 \times H_2)/G \cong \mathbb{Z}_3$ on $X$ (induced by the action of* $\mathrm{id} \times H_2$ *on $X_1 \times X_2$), such that the pair $(X, H)$ satisfies the same assumptions as $(X_2, H_2)$.*

*Proof.* This is essentially the inductive argument of [7, Proposition 3.1], on which we have additionally grafted condition $(\star)$. We briefly resume the construction of $X$ given in [7, Proposition 3.1] (retaining the notation of *loc. cit.*).

The quotient $(X_1 \times X_2)/G$ has $A_2$–singularities along a codimension 2 stratum $W_1$ (isomorphic to $B_1 \times B_{2,1}$), plus other singularities along a codimension 3 stratum $W_2$ (isomorphic to $B_1 \times B_{2,2}$). A crepant resolution

$$X \ \to \ (X_1 \times X_2)/G$$

is explicitly described in local coordinates in [7, Proof of Proposition 3.1]. Moreover, it is checked in [7, Proof of Proposition 3.1] that $(X, H)$ satisfies conditions (i) and (iii) (just as $(X_2, H_2)$). Therefore it only remains to check that $X$ also satisfies conditions (ii), (iv) and (v).

As explained in *loc. cit.*, the variety $X$ can also be obtained as follows: Let $Z_1$ be the blow-up of $X_1 \times X_2$ along $B_1 \times B_2$. The action of

$$G := \{(h_1, h_2) \in H_1 \times H_2 : h_1 + h_2 = 0\}.$$

on $X_1 \times X_2$ naturally endows $Z$ with an action of $G$. Let $Z_2 \to Z_1$ be the blow–up with center the codimension 2 part of $\mathrm{Fix}_{Z_1}(G)$ (this center consists of two disjoint copies of $W_1$, as can be seen from [29, Lemma 18]). The action of $G$ lifts to $Z_2$, and we define

$$Z := Z_2/G.$$

The crepant resolution $X$ is now attained by performing a blow–down

$$b : Z \ \to \ X,$$

where the exceptional divisor of $b$ in $Z$ corresponds to the strict transform of the exceptional divisor of the first blow–up $Z_1 \to X_1 \times X_2$. The exceptional locus $V \subset X$ of $b$ is an isomorphic copy of $W_1$ (this exceptional locus $V \cong W_1$ corresponds to the intersection of the 2 irreducible components of the exceptional divisor in $X$ lying over the stratum $W_1$).

Once again, we endow $X_1 \times X_2$ and $B_1 \times B_2$ with the product markings. These markings satisfy ($\star$) by [14, Prop. 4.1], the inclusion morphism $B_1 \times B_2 \hookrightarrow X_1 \times X_2$ is distinguished by [14, Prop. 3.5], and the pushforwards and pullbacks along the projection morphisms $X_1 \times X_2 \to X_i$ and $B_1 \times B_2 \to B_i$ are distinguished. Moreover, the pair $(X_1 \times X_2, H_1 \times H_2)$, where $\mathrm{Fix}_{X_1\times X_2}(H_1 \times H_2) = B_1 \times B_2$, satisfies properties (ii), (iv) and (v) by [14, Prop. 4.1] (and also [14, Rem 4.3], plus the fact that condition (iv) is stable under taking products). In view of [14, Prop. 4.8], this implies that $(Z_1, H_1 \times H_2)$ and $(Z_1, G)$ satisfy (ii).

The codimension 2 part of $\mathrm{Fix}_{Z_1}(G)$ consists of 2 disjoint copies of $W_1 \cong B_1 \times B_{2,1}$, and so it has a marking satisfying ($\star$). Let $E_1 \subset Z_1$ denote the exceptional divisor. The inclusion of the 2 copies of $W_1$ in $Z_1$ is distinguished, because the inclusion morphism $W_1 \to E_1$ is distinguished (indeed, both $W_1$ and $E_1$ have trivial Chow groups), and the inclusion morphism $E_1 \to Z_1$ is also distinguished. Again applying [14, Prop. 4.8], and reasoning as before, this implies that $(Z_2, H_1 \times H_2)$ and $(Z_2, G)$ in turn satisfy (ii), (iv) and (v).

The next step is to take the quotient $Z_2 \to Z := Z_2/G$. Here, [14, Prop. 4.12] ensures that $Z$ has a marking satisfying ($\star$) and that the quotient morphism $Z_2 \to Z$ is distinguished. This last fact, combined with the fact that $(Z_2, H_1 \times H_2)$ verifies (ii), ensures that $(Z, H)$ also verifies (ii). The fact that $(Z_2, H_1 \times H_2)$ verifies (v), plus the fact that the quotient morphism $Z_2 \to Z$ is distinguished, ensures that $(Z, H)$ also verifies (v). Condition (iv) is satisfied for $(Z, H)$ since the fixed locus is dominated by the fixed locus of $(Z_2, G)$ which satisfied (iv).

The final step in the inductive process is the blow–down $b$ from $Z$ to $X$. Here, we know that the exceptional divisor $E \subset Z$ of $b$ has a marking that verifies ($\star$) and is such that the inclusion is distinguished. Also, we know that the exceptional locus $V \subset X$ (is isomorphic to $W_1$ and so) has trivial Chow groups, and thus verifies ($\star$). We remark that the correspondence

$$ {}^t\Gamma_b \circ \Gamma_b \quad \in \ \mathrm{CH}^n(Z \times Z) $$

is supported on $\Delta_Z \cup E \times_V E$ (by refined intersection). The fiber product $E \times_V E$ is a $\mathbb{P}^1 \times \mathbb{P}^1$–bundle over $V$; as such, it is smooth irreducible of dimension $n$ and has trivial Chow groups. The inclusion $E \times_V E \subset E \times E$ is distinguished (both sides have trivial Chow groups), and the inclusion $E \times E \subset Z \times Z$ is distinguished (as $E \subset V$ is distinguished). Therefore, $E \times_V E \subset Z \times Z$ is distinguished, and so we may conclude that ${}^t\Gamma_b \circ \Gamma_b$ is distinguished in $\mathrm{CH}^n(Z \times Z)$.

Now, applying [14, Prop. 4.9], it follows that $X$ has a marking that verifies ($\star_{\mathrm{Mult}}$), and such that the blow–up morphism $b$ is distinguished. To show that the marking of $X$ also verifies ($\star_{\mathrm{Chern}}$), one can reason as in the technical [31, Lemma 6.4], with $\mathrm{DCH}^*(-)$ instead of $\mathrm{CH}^*(-)_{(0)}$ (*cf.* also [14, Rem. 4.15], which deals with the same situation). Alternatively, one can argue as follows: according to Porteous' formula [15, Theorem 15.4], the difference

$$ d := c_i(Z) - b^*c_i(X) \quad \in \ \mathrm{CH}^i(Z) $$

can be expressed in terms of (push–forwards to $Z$ of pullbacks to $E$ of) Chern classes of $V$ and Chern classes of the normal bundle of $V$ in $X$. But any cycle on $E$ is distinguished (since $E$ has trivial Chow groups), and the inclusion morphism $E \to Z$ is distinguished, and hence this difference $d$ is distinguished. As the Chern classes $c_i(Z)$ are distinguished, this implies that $b^*c_i(X)$ is distinguished. Since the morphism $b$ is distinguished, this implies that

$$ c_i(X) = b_*b^*c_i(X) \quad \in \ \mathrm{DCH}^i(X), $$

*i.e.*, condition ($\star_{\mathrm{Chern}}$) (and hence condition ($\star$)) is verified for $X$.

To finish the proof, we observe that the inclusion of (each copy of) $W_1$ in the exceptional divisor of $Z_1 \to X_1 \times X_2$ is distinguished (since both $W_1$ and the exceptional divisor have trivial Chow groups). This implies that the same is true for the inclusion of (each copy of) $W_1$ in the strict transform of this exceptional divisor in $Z_2$. Since the inclusion of the exceptional divisor in $Z_2$ is distinguished, this implies that the inclusion of (each copy of) $W_1$ in $Z_2$ is distinguished. Since the quotient morphism $Z_2 \to Z$ and the blow–up $b$ are distinguished, it follows that the inclusion of $V \cong W_1$ in $X$ is distinguished.

The fact that $(Z, H)$ verifies (ii), plus the fact that $b$ is distinguished, guarantees that $(X, H)$ verifies (ii). The fixed locus $\mathrm{Fix}_X(H)$ is the disjoint union of the codimension 2 component $V$, and a divisor (which is the isomorphic image in $X$ of the exceptional divisor in $Z_2$ lying over the codimension 3 stratum $W_2 \subset X_1 \times X_2$). In view of the above, this implies that $(X, H)$ verifies the conditions (ii), (iv) and (v), and so we are done. $\square$

4.3. $\mathbb{Z}_{3^c}$**-actions.** In this section, we want to show that the inductive approach of Schreieder in [29, §8.2] can be strengthened to take into account the motivic structure and to keep track of the condition $(\star)$. For clarity, we follow the notations of [29].

Precisely, for natural numbers $a \neq b$ and $c \geq 0$, let $S_c^{a,b}$ denote the family of pairs $(X, \phi)$, consisting of a smooth projective complex variety $X$ of dimension $a+b$ and an automorphism $\phi \in \mathrm{Aut}(X)$ of order $3^c$, such that properties (i)–(v) below hold. Here $\zeta$ denotes a fixed primitive $3^c$-th root of unity and $g := (3^c - 1)/2$.

(i) The decomposition $\mathfrak{h}(X) = T \oplus \mathfrak{h}(X)^{\langle\phi\rangle}$ is such that $h^{a,b}(T) = h^{b,a}(T) = g$ and $h^{p,q}(T) = 0$ for all other $p \neq q$, and such that the summand $\mathfrak{h}(X)^{\langle\phi\rangle}$ (which is the $\phi$–invariant part of the motive of $X$) is isomorphic to a direct sum of Lefschetz motives $\bigoplus \mathbb{1}(*)$.
(ii) The action of $\phi$ on $H^{a,b}(X)$ has eigenvalues $\zeta, \ldots, \zeta^g$.
(iii) The set $\mathrm{Fix}_X(\phi^{3^{c-1}})$ can be covered by local holomorphic charts such that $\phi$ acts on each coordinate function by multiplication with some power of $\zeta$.
(iv) For $0 \leq l \leq c-1$, the motive of $\mathrm{Fix}_X(\phi^{3^l})$ is isomorphic to a sum of Lefschetz motives and the action of $\phi$ on that motive is the identity.
(v) $X$ has a marking that satisfies the condition $(\star)$ and the condition $(\star_{\langle\phi\rangle})$. Moreover, the inclusion morphism $\mathrm{Fix}_X(\phi^{3^l}) \hookrightarrow X$ is distinguished for $0 \leq l \leq c-1$.

In condition (v), note that it makes sense to say that the inclusion morphism is distinguished : by (iv) the motive of $\mathrm{Fix}_X(\phi^{3^l})$ is isomorphic to a sum of Lefschetz motives, and in particular it admits a marking that satisfies $(\star)$ (*cf.* [14, Prop. 5.2]).

Our condition (i) (resp. (iv)) is a motivic version of conditions (1) and (3) (resp. (5)) of Schreieder. Our conditions (ii) and (iii) are exactly the conditions (2) and (4) of Schreieder. The new feature is our condition (v).

As in [29, §8.2], we note that it follows from (iii) that $\mathrm{Fix}_X(\phi^{3^l})$ is smooth for all $0 \leq l \leq c-1$.

With this strengthened definition of $S_c^{a,b}$ (compared to that of [29]), we have the exact same statement as [29, Proposition 19] :

**Proposition 4.5.** *Let* $(X_1, \phi_1) \in S_c^{a_1,b_1}$ *and* $(X_2, \phi_2) \in S_c^{a_2,b_2}$. *Then*

$$(X_1 \times X_2)/\langle \phi_1 \times \phi_2 \rangle$$

*admits a smooth model* $X$ *such that the automorphism* $\mathrm{id} \times \phi_2$ *on* $X_1 \times X_2$ *induces an automorphism* $\phi \in \mathrm{Aut}(X)$ *with* $(X, \phi) \in S_c^{a,b}$, *where* $a = a_1 + a_2$ *and* $b = b_1 + b_2$.

Precisely, the variety $X$ is constructed inductively as follows. Consider the subgroup of $\mathrm{Aut}(X)$ given by

$$G := \langle \phi_1 \times \mathrm{id}, \mathrm{id} \times \phi_2 \rangle.$$

For each $1 \leq i \leq c$, consider the element of order $3^i$ in $G$ given by $\eta_i := (\phi_1 \times \phi_2)^{3^{c-i}}$, generating a cyclic subgroup $G_i := \langle \eta_i \rangle \subseteq G$. We obtain a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_c = \langle \phi_1 \times \phi_2 \rangle,$$

such that each quotient $G_i/G_{i-1}$ is cyclic of order 3, generated by the image of $\eta_i$. We set

$$Y_0 := X_1 \times X_2$$

equipped with the natural action of $G$. We define inductively

$$\begin{aligned} Y_i &= Y''_{i-1}/\langle \eta_i \rangle, \\ Y'_i &= \text{Blow up of } Y_i \text{ along } \mathrm{Fix}_{Y_i}(\eta_{i+1}), \\ Y''_i &= \text{Blow up of } Y'_i \text{ along } \mathrm{Fix}_{Y'_i}(\eta_{i+1}). \end{aligned}$$

Here the action of the group $G$ carries at each step. Schreieder shows that each $Y_i$ is a smooth model of $Y_0/G_i$, so that the variety $X$ of Proposition 4.5 is nothing but $Y_c$ equipped with the action of $G/G_c$. To summarize, we have the following diagram :

$$\begin{tikzcd}[column sep=small] & Y''_{c-1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Y''_1 \arrow[dl] \arrow[dr] & & Y''_0 \arrow[dl] \arrow[dr] & \\ Y_c & & Y_{c-1} & & Y_2 & & Y_1 & & Y_0. \end{tikzcd} \tag{5}$$

Each arrow to the right corresponds to the composition $Y''_i \to Y'_i \to Y_i$ of two blow-ups along fix loci (which turn out to be smooth), and each arrow to the left corresponds to a $3-1$ cover, branched along a smooth divisor ; see [29].

*Proof of Proposition 4.5.* Since there is no point in repeating Schreieder's arguments in full, we only indicate how to adapt his proof to show that the motivic statements and the condition (v) carry through.

First, since our conditions (i)–(v) imply Schreieder's conditions (1)–(5), we only need to prove that $X$ satisfies conditions (i), (iv) and (v). Concerning the latter two, they are contained in the following strengthening of [29, Lemma 20] :

**Lemma 4.6.** *Let $\Gamma \subseteq G$ be a subgroup which is not contained in $G_i$. Then $\mathrm{Fix}_{Y_i}(\Gamma)$, $\mathrm{Fix}_{Y'_i}(\Gamma)$ and $\mathrm{Fix}_{Y''_i}(\Gamma)$ are smooth, their motives are isomorphic to direct sums of Lefschetz motives, their $G$-actions restrict to actions on each irreducible component and the $G_c$-fixed part of their motive is also fixed by $G$. Moreover, $Y_i$, $Y'_i$ and $Y''_i$ are naturally equipped with markings that satisfy $(\star)$ and $(\star_G)$ with the additional property that the embeddings $\mathrm{Fix}_{Y_i}(\Gamma) \hookrightarrow Y_i$, $\mathrm{Fix}_{Y'_i}(\Gamma) \hookrightarrow Y'_i$ and $\mathrm{Fix}_{Y''_i}(\Gamma) \hookrightarrow Y''_i$ are distinguished.*

*Proof.* We follow word-for-word the proof of [29, Lemma 20], which is by induction on $i$. The motivic statement is obtained from Schreieder's arguments simply by noting the following :

(a) the fixed locus of $\Gamma$ on $Y_0$ is described as the product of fixed loci on $X_1$ and $X_2$ ;

(b) the irreducible components of the fixed locus of $\Gamma$ on $Y_i'$ are described either as projective bundles over irreducible components of a fixed locus on $Y_i$, or as strict transforms of irreducible component of some fixed locus, which are themselves blow-ups of irreducible components of some fixed locus along the irreducible component of some other fixed locus;

(c) the irreducible components of the fixed locus of $\Gamma$ on $Y_i''$ are described similarly as for $Y_i'$;

(d) the fixed locus of $\Gamma$ on $Y_{i+1}$ is described either as the isomorphic image of a fixed locus on $Y_i''$, or its irreducible components are quotients by $\langle \eta_i \rangle$ of irreducible components of fixed loci on $Y_i''$.

In all aforementioned descriptions, the property that the Chow groups are trivial is preserved. We then note that the motive of a variety is a direct sum of Lefschetz motives if and only if its Chow groups are finite-dimensional vector spaces if and only if the total cycle class map $\mathrm{CH}^*(X) \to \mathrm{H}^*(X)$ is an isomorphism; see [17, 21, 33]. In particular, assuming $Z$ is a smooth projective variety with trivial Chow groups, if the $G_c$-invariant part of the cohomology of $Z$ is spanned by $G$-invariant algebraic cycles, then the $G_c$-invariant part of the motive of this variety $Z$ is isomorphic to a direct sum of $G$-invariant Lefschetz motives. Together with [29, Lemma 20], this establishes the first part of the lemma.

Concerning the moreover part, we first recall that any smooth projective variety $Z$ whose motive is isomorphic to a direct sum of Lefschetz motives satisfies condition $(\star)$ (*cf.* [14, Prop. 5.2]). In addition, since for any choice of marking we have $\mathrm{DCH}^*(Z\times Z) = \mathrm{CH}^*(Z\times Z)$, any action of a finite group $G$ on $Z$ satisfies the condition $(\star_G)$.

By induction, assuming that $\mathrm{Fix}_{Y_i}(\Gamma)$ and $Y_i$ have a marking satisfying $(\star)$ and $(\star_G)$ such that $\mathrm{Fix}_{Y_i}(\Gamma) \hookrightarrow Y_i$ is distinguished, it only remains to show that the graphs of the embeddings $\mathrm{Fix}_{Y_i'}(\Gamma) \hookrightarrow Y_i'$, $\mathrm{Fix}_{Y_i''}(\Gamma) \hookrightarrow Y_i''$ and $\mathrm{Fix}_{Y_{i+1}}(\Gamma) \hookrightarrow Y_{i+1}$ are distinguished for suitable choices of markings satisfying $(\star)$ and $(\star_G)$. In fact, we only need to show this component-wise for the irreducible components of the fixed loci of $\Gamma$.

In case (a), which is the initial case, this is obvious (see [14, Prop. 3.5]).

In case (b) (and also case (c) which is similar), we have the following more precise description ([29, pp. 326–27]) of the irreducible components of the fixed locus of $\Gamma$ on $Y_i'$. Let $P$ be an irreducible component of $\mathrm{Fix}_{Y_i'}(\Gamma)$, and let $Z$ be the image of $P$ inside $Y_i$. Then, depending on whether $Z$ is contained in $\mathrm{Fix}_{Y_i}(\langle \Gamma, \eta_{i+1}\rangle)$ or not, one of the following occurs:

- $Z$ is an irreducible component of $\mathrm{Fix}_{Y_i}(\langle \Gamma, \eta_{i+1}\rangle)$ and $P \to Z$ is a projective subbundle of the projective bundle $E_i'|_Z \to Z$, where $E_i'$ is the exceptional divisor of the blow-up $Y_i' \to Y_i$.
- $Z$ is an irreducible component of $\mathrm{Fix}_{Y_i}(\Gamma)$ and $P$ is the strict transform of $Z$ in $Y_i'$; in particular $P \to Z$ is the blow-up along $\mathrm{Fix}_Z(\eta_i)$.

In the first case, we have the composition of inclusions $P \hookrightarrow E_i' \hookrightarrow Y_i'$. The left inclusion is distinguished because as we saw, both $P$ and $E_i'$ have trivial Chow groups. As for the right inclusion, $Y_i'$ is the blow-up of $Y_i$ along $\mathrm{Fix}_{Y_i}(\Gamma)$, by induction $Y_i$ satisfies $(\star)$ and $(\star_G)$, and $\mathrm{Fix}_{Y_i}(\Gamma)$ has trivial Chow groups and has a suitable marking making the inclusion $\mathrm{Fix}_{Y_i}(\Gamma) \hookrightarrow Y_i$ distinguished; therefore by [14, Prop. 4.8] $E_i'$ and $Y_i'$ have markings that satisfy $(\star)$ and $(\star_G)$ such that $E_i' \hookrightarrow Y_i'$ is distinguished. In the second case, by arguing as in the proof of [31, Prop. 3.4] (with $\mathrm{CH}^*(-)_{(0)}$ replaced with $\mathrm{DCH}^*(-)$) and using the fact that $P$ has trivial Chow groups, one can show that $P \hookrightarrow Y_i'$ is distinguished.

In case (d), finally, we have that $\pi : Y_i'' \to Y_{i+1}$ is a $\mathbb{Z}_3$-cyclic covering branched along the smooth divisor $\mathrm{Fix}_{Y_i''}(\eta_i)$. That $Y_i''$ satisfies $(\star)$ and $(\star_G)$, together with the fact proved above (case (c)) that $\mathrm{Fix}_{Y_i''}(\eta_i) \hookrightarrow Y_i''$ is distinguished, is enough to conclude that the quotient $Y_{i+1}$ satisfies $(\star)$ and $(\star_G)$ ; see [14, Prop. 4.12]. It remains to show that $\mathrm{Fix}_{Y_{i+1}}(\eta_i) \hookrightarrow Y_{i+1}$ is distinguished. By the projection formula, any generically finite morphism $f : Z_1 \to Z_2$ of degree $d$ between smooth projective varieties induces a surjective morphism $f_* : \mathfrak{h}(Z_1) \to \mathfrak{h}(Z_2)$ with a section $\frac{1}{d}f^* : \mathfrak{h}(Z_2) \to \mathfrak{h}(Z_1)$. If in addition $Z_1$ has a marking, then $f$ is distinguished for the marking on $Z_2$ induced by that of $Z_1$. Let $P$ be an irreducible component of $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$. We know that there is an irreducible component $Z$ of some fixed locus in $Y_i''$ such that $\pi$ restricts to either an isomorphism or a 3-to-1 morphism $Z \to P$. By induction, $Z$ has a marking such that the inclusion $Z \hookrightarrow Y_i''$ is distinguished. Endow $P$ with the marking induced by that of $Z$ ; in particular $f$ is distinguished. Then the inclusion $P \hookrightarrow Y_{i+1}$ is distinguished because it is the composite of $\pi$, the inclusion $Z \hookrightarrow Y_i''$, and $\frac{1}{\deg f}f^*$, all of which are distinguished.

The proof of Lemma 4.6 is complete. □

We have now established properties (ii)–(v) for $X$. With Lemma 4.6, we have in fact showed that

$$\mathfrak{h}(X) \simeq \mathfrak{h}(Y_0)^{G_c} \oplus \bigoplus \mathbb{1}(*),$$

where the right hand side summand is fixed by the action of $G$. Since the motive of $Y_0$ is of abelian type (and hence finite-dimensional in the sense of Kimura), in order to establish (i), it thus suffices to see that the $G_c$-invariant cohomology of $Y_0$ is spanned by $G$-invariant algebraic classes, by $g$ linearly independent $(a, b)$-forms and their conjugates. This follows from conditions (i) and (ii) for $(X_1, \phi_1)$ and for $(X_2, \phi_2)$, as in [29, pp. 329–30].

The proof of Proposition 4.5 is now complete. □

**Remark 4.7.** As mentioned before (Remark 1.6), the Cynk–Hulek variety $X_{CH}$ (given by Theorem 1.2) and the Schreieder variety $X_S$ (given by the $c = 1, b = 0$ case of Theorem 1.4) are not the same. The difference in their construction is clear from comparison of subsections 4.2 and 4.3 : in the construction of $X_{CH}$, there is at each step the blow–down $b\colon Z \to X$ (in order to have a *crepant* resolution), whereas in the construction of $X_S$ the varieties $Z$ and $X$ coincide.

4.4. **Proof of the main Theorem 4.1.** In view of Remark 2.11, Theorem 4.1 follows from the following two claims :

**Claim 4.8.** *Let $X$ be the $n$-dimensional Calabi–Yau variety of Theorem 1.1 or 1.2, or a Schreieder variety as in Theorem 1.4. Then $X$ admits a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ that satisfies $(\star)$.*

**Claim 4.9.** *Let $X$ be the $n$-dimensional Calabi–Yau variety of Theorem 1.1 or 1.2, or a Schreieder variety as in Theorem 1.4. Then there is a decomposition*

$$\mathfrak{h}(X) = T \oplus \bigoplus \mathbb{1}(*) \quad \text{in } \mathcal{M}_{\mathrm{rat}} \ , \tag{6}$$

*where $T$ is such that $H^j(T) = 0$ for $j \neq n$, and $T$ is isomorphic to a direct summand of the Chow motive of $E_1 \times \cdots \times E_n$ (if $X$ is as in Theorem 1.1), resp. of $E^n$ (for $X$ as in Theorem 1.2), resp. of $C^n$ (for $X$ as in Theorem 1.4).*

*Proof of Claim 4.8.* The Cynk–Hulek varieties of Theorem 1.1 (resp. Theorem 1.2) are constructed inductively using the process of Proposition 4.3 (resp. Proposition 4.4), by adding an elliptic curve with $[-1]$-involution (resp. an elliptic curve with non-trivial $\mathbb{Z}_3$–action), at each step. Repeatedly applying Proposition 4.3 (resp. Proposition 4.4), we find that they admit a marking satisfying ($\star$).

Likewise, the Schreieder varieties are obtained inductively using Proposition 4.5, by adding the hyperelliptic curve $C_{g,1}$ at each step. A repeated application of Proposition 4.5 establishes Claim 4.8 for the Schreieder varieties. □

*Proof of Claim 4.9.* The Cynk–Hulek varieties of Theorem 1.1 are constructed inductively using the process of Proposition 4.3, by adding at each step an elliptic curve $E$ with $[-1]$-involution. Note that the quotient $E/[-1]$ is isomorphic to $\mathbb{P}^1$ and hence has Chow motive isomorphic to $\mathbb{1}\oplus\mathbb{1}(-1)$. In particular, the Chow motive of $E$ is isomorphic to $T\oplus(\mathbb{1}\oplus\mathbb{1}(-1))$, where $[-1]$ acts trivially on the right-hand side summand. Therefore, in order to prove Claim 4.9 for the Cynk–Hulek varieties of Theorem 1.1, it is enough to remark that Proposition 4.3 continues to hold if one adds the property

*(vi)* the decomposition $\mathfrak{h}(X_i)=T_i\oplus\mathfrak{h}(X_i)^{H_i}$ is such that $\mathfrak{h}(X_i)^{H_i}\simeq\bigoplus\mathbb{1}(*)$, and $H^j(T_i)=0$ if $j\neq\dim X_i$.

Recall that $X$ is the quotient by $G\simeq\mathbb{Z}_2$ of the blow-up $Z$ of $X_1\times X_2$ along $B_1\times B_2$, where $B_i$ is the fixed locus of $H_i$ acting on $X_i$ and is assumed to have motive isomorphic to a direct sum of Lefschetz motives. By the blow-up formula, we have

$$\mathfrak{h}(Z)\simeq\mathfrak{h}(X_1)\otimes\mathfrak{h}(X_2)\oplus\big(\mathfrak{h}(B_1)\otimes\mathfrak{h}(B_2)\oplus\mathfrak{h}(B_1)\otimes\mathfrak{h}(B_2)(-1)\big).$$

The right-hand side summand is fixed under the action of $H_1\times H_2$ and is isomorphic to a direct sum of Lefschetz motives. Thus in order to conclude it is enough to note that $(T_1\otimes\mathfrak{h}(X_2)^{H_2})^G=0$ (and similarly $\mathfrak{h}(X_1)^{H_1}\otimes T_2)^G=0$) and the $(H_1\times H_2)$-invariant part of the motive $\mathfrak{h}(X_1)\otimes\mathfrak{h}(X_2)$ is a direct sum of Lefschetz motives ; this follows at once from the assumption that $T_i^{H_i}=0$.

The proof of Claim 4.9 for the Schreieder varieties was already taken care of. Indeed, thanks to Proposition 4.5 we know that Schreieder varieties $X$ are in the class $S_c^{a,b}$; this entails in particular (by definition of $S_c^{a,b}$) that the motive of $X$ decomposes as

$$\mathfrak{h}(X)=T\oplus\bigoplus\mathbb{1}(*)\quad\text{in }\mathcal{M}_{\mathrm{rat}}\ ,$$

where $T$ is such that $H^j(T)=0$ for all $j\neq\dim X$. In fact, at the end of the proof of Proposition 4.5), we constructed $T$ as a direct summand of $\mathfrak{h}(Y_0)^{G_c}$, and so $T$ is indeed a direct summand of the Chow motive of $C^n$.

Finally, to prove Claim 4.9 for the Cynk–Hulek varieties $X_{CH}$ of Theorem 1.2 we argue as follows : the variety $X_{CH}$ is dominated by the Schreieder variety $X_S$ (of the same dimension, and where $c=1$ and $b=0$ in the Schreieder construction). Thus, the truth of Claim 4.9 for $X_S$ implies the truth of Claim 4.9 for $X_{CH}$. □

4.5. **Proof of Theorem 4.2.** This is immediate : in view of Proposition 2.9, Theorem 4.2 follows from Claim 4.8. □

**Remark 4.10.** Alternatively, we could have established the existence of a self-dual multiplicative Chow–Künneth decomposition for $X$ as in Theorem 4.2 directly (*i.e.*, without invoking Proposition 2.9 ([14, Prop. 6.1]) by using the results of [31] instead of those of [14].

4.6. **Proof of Corollary 1.** By Claim 4.8, we know that any Schreieder surface $S$ has a marking that satisfies $(\star)$. Since the cycle

$$(x,x,x)-(x,x,p)-(x,p,x)-(p,x,x)+(p,p,x)+(p,x,p)+(x,x,p)$$

is numerically trivial (this cycle is numerically trivial for any regular surface), it suffices to show that there exists a point $p$ in $S$ such that each summand is distinguished. By definition of $(\star_{\mathrm{Mult}})$, the cycle $(x,x,x)$ is distinguished. By [14, Lemma 3.8], the diagonal $\Delta_S=(x,x)$ is distinguished in $S\times S$; and it is clear that the fundamental class of $S$ is distinguished in $S$ (see also [14, Remark 3.4]). Therefore it suffices to exhibit a point $p$ that is distinguished in $S$. The variety $S$ is constructed using the diagram (5) starting from $Y_0$ the product of two hyperelliptic curves. Choose a fixed point $p_0$ of $G$ in $Y_0$. By Proposition 3.3 and the fact that $(\star)$ is stable under product, $Y_0$ has a marking that satisfies $(\star)$ such that $p_0$ is distinguished. Now by inspection of the proof of Proposition 4.5, we see, by defining inductively $p_i$ as the image of $p''_{i-1}$ under $Y''_{i-1}\to Y_i$, $p'_i$ as a point in the preimage of $p_i$ under $Y'_i\to Y_i$, and $p''_i$ as a point in the preimage of $p'_i$ under $Y''_i\to Y'_i$, that $p_c=p$ is a distinguished point in $S=Y_c$.

Alternately, Corollary 1 is a consequence of Theorem 4.2 combined with [30, Proposition 8.14] and with the fact that there exists a point $p$ in $S$ that is distinguished. □

4.7. **Final remarks.** We remark that Theorems 4.1 and 4.2 hold also in the following two situations. First in the Schreieder construction, we may add via Proposition 4.5 at each step, instead of the hyperelliptic curve $C_{g,1}$, more generally the hyperelliptic curve $C_{g,D}$ for any non-zero rational number $D$. Second in the construction of a smooth model of the Cynk–Hulek varieties of Theorem 1.1, one may add via Proposition 4.3 at each step a hyperelliptic curve equipped with its hyperelliptic involution instead of an elliptic curve equipped with its $[-1]$-involution.

## 5. Voisin's conjecture

Voisin [36] has formulated the following intriguing conjecture, which is a special instance of the Bloch–Beilinson conjectures.

**Conjecture 5.1** (Voisin [36]). *Let $X$ be a smooth projective variety of dimension $n$, with $p_g(X):=h^{n,0}(X)=1$ and $h^{j,0}(X)=0$ for $0<j<n$. Then any two zero–cycles $a,a'\in \mathrm{CH}^n_{num}(X)$ satisfy*

$$a\times a'=(-1)^n a'\times a \quad \textit{in } \mathrm{CH}^{2n}(X\times X)\ .$$

*(Here, $a\times a'$ is the exterior product $(p_1)^*(a)\cdot(p_2)^*(a')\in \mathrm{CH}^{2n}(X\times X)$, where $p_j$ is projection to the $j$-th factor.)*

For background and motivation for Conjecture 5.1, *cf.* [38, Section 4.3.5.2]. Conjecture 5.1 has been proven in some scattered special cases [36], [22], [23], [24], [6], but is still widely open for a general K3 surface.

**Remark 5.2.** Conjecture 5.1 can be thought of as a version of Bloch's conjecture for motives. Indeed, given $X$ as in Conjecture 5.1, consider the Chow motive $M$ defined as

$$M:=\begin{cases}\wedge^2\mathfrak{h}^n(X):=(X\times X,\frac12\sum_{\sigma\in\mathfrak{S}_2}\mathrm{sgn}(\sigma)\Gamma_\sigma\circ(\pi^n_X\times\pi^n_X),0) & \text{if } n \text{ is even },\\ \mathrm{Sym}^2\mathfrak{h}^n(X):=(X\times X,\frac12\sum_{\sigma\in\mathfrak{S}_2}\Gamma_\sigma\circ(\pi^n_X\times\pi^n_X),0) & \text{if } n \text{ is odd }.\end{cases}$$

(Here, for $\mathfrak{h}^n(X)$ to make sense, we need to assume that $X$ has a Chow–Künneth decomposition, in the sense of Definition 2.1). The condition on $p_g(X)$ implies that $h^{2n,0}(M)=0$, and so $M$ is a motive with
$$h^{j,0}(M)=0 \quad \forall\, j\ .$$
A motivic version of Bloch's conjecture would then imply that
$$\mathrm{CH}_0(M)=0\ .$$
On the other hand, the condition on $h^{j,0}(X)$ conjecturally implies that $\mathrm{CH}^n_{num}(X)=(\pi^n_X)_\ast\,\mathrm{CH}^n(X)$. It follows that given two zero-cycles $a,a'\in \mathrm{CH}^n_{num}(X)$, one conjecturally has
$$a\times a'-(-1)^n a'\times a=(\pi^n_X\times\pi^n_X)_\ast(a\times a')-(-1)^n\iota_\ast(\pi^n_X\times\pi^n_X)_\ast(a\times a')\ \text{ in }\ \mathrm{CH}_0(M)=0,$$
where $\iota$ is the non-trivial element of $\mathfrak{S}_2$. This heuristically explains Conjecture 5.1.

We now prove Voisin's conjecture for Cynk–Hulek Calabi–Yau varieties:

**Theorem 5.3.** *Let $X$ be a Calabi–Yau variety of dimension $n$ as in Theorem 1.1 or 1.2. Then Conjecture 5.1 is true for $X$: any $a,a'\in \mathrm{CH}^n_{num}(X)$ satisfy*
$$a\times a'=(-1)^n\, a'\times a \quad \textit{in}\ \mathrm{CH}^{2n}(X\times X)\ .$$

*Proof.* Consider morphisms
$$\begin{tikzcd} & E_1\times\cdots\times E_n \arrow[d, "p"] \\ X \arrow[r, "f"] & \bar{X} \end{tikzcd}$$
as in Theorem 1.1 or 1.2. The Chow group of 0–cycles is a birational invariant amongst varieties that are global quotients (this follows for instance from [15, Example 17.4.10]), and so $f^\ast\colon\ \mathrm{CH}^n(\bar{X})\to \mathrm{CH}^n(X)$ is an isomorphism. Consequently, it suffices to prove Conjecture 5.1 for $\bar{X}$. Using Corollary 3.7, we see that $\mathrm{CH}^n_{num}(\bar{X})$ is contained in $p_\ast\,\mathrm{CH}^n(E_1\times\cdots E_n)_{(n)}$. Therefore, we are reduced to proving that $a\times a'=(-1)^n a'\times a$ for all $a,a'\in \mathrm{CH}^n(E_1\times\cdots\times E_n)_{(n)}$; this is a special case of:

**Proposition 5.4** (Voisin, Example 4.40 in [38])**.** *Let $B$ be an abelian variety of dimension $n$. Let $a,a'\in \mathrm{CH}^n(B)_{(n)}$. Then*
$$a\times a'=(-1)^n\, a'\times a \quad \textit{in}\ \mathrm{CH}^{2n}(B\times B).$$

This concludes the proof of the theorem. $\square$

## 6. Voevodsky's conjecture

In this section, we give an application of our results to Voevodsky's conjecture on smash–equivalence.

**Definition 6.1** (Voevodsky [35])**.** Let $X$ be a smooth projective variety. A cycle $a\in \mathrm{CH}^i(X)$ is called *smash-nilpotent* if there exists $m\in\mathbb{N}$ such that
$$a^m:=\ \underbrace{a\times\cdots\times a}_{(m\ \text{times})}\ =0\ \text{ in }\ \mathrm{CH}^{mi}(X\times\cdots\times X)\ .$$

Two cycles $a,a'$ are called *smash-equivalent* if their difference $a-a'$ is smash-nilpotent. We will write $\mathrm{CH}^i_\otimes(X)\subseteq \mathrm{CH}^i(X)$ for the subgroup of smash-nilpotent cycles.

**Conjecture 6.2** (Voevodsky [35]). *Let $X$ be a smooth projective variety. Then*

$$\mathrm{CH}^i_{num}(X) \subseteq \mathrm{CH}^i_{\otimes}(X) \quad \textit{for all } i\ .$$

**Remark 6.3.** It is known [1, Théorème 3.33] that Conjecture 6.2 for all smooth projective varieties implies (and is strictly stronger than) Kimura's conjecture "all smooth projective varieties have finite-dimensional motive" [20].

Thanks to Claim 4.9, we can verify Voevodsky's conjecture for odd-dimensional Cynk–Hulek varieties and Schreieder varieties:

**Proposition 6.4.** *Let $X$ be a Cynk–Hulek Calabi–Yau variety as in Theorem 1.1 or 1.2, or a Schreieder variety as in Theorem 1.4. Suppose the dimension $n$ of $X$ is odd. Then*

$$\mathrm{CH}^i_{num}(X) \subseteq \mathrm{CH}^i_{\otimes}(X) \quad \textit{for all } i.$$

*Proof.* According to Claim 4.9, we have a decomposition

$$\mathfrak{h}(X) = T \oplus \bigoplus \mathbb{1}(*),$$

with $H^j(T) = 0$ for $j \neq n$, and $T$ isomorphic to a direct summand of $\mathfrak{h}(C_1 \times \cdots \times C_n)$. Here, the $C_i$ are elliptic curves in case $X$ is a Cynk–Hulek variety, and the hyperelliptic curves of §3.2 in case $X$ is a Schreieder variety.

By Kimura finite–dimensionality, $T$ is isomorphic to a direct summand of the motive $(C_1 \times \cdots \times C_n, \pi^n, 0)$, where $\pi^n$ is any Chow–Künneth projector on the degree-$n$ cohomology. But the Chow motive $(C_1 \times \cdots \times C_n, \pi^n, 0)$ is oddly finite-dimensional (in the sense of [20]). Hence, together with the fact that $\mathrm{CH}^i_{num}(X) = \mathrm{CH}^i_{num}(T)$, the corollary is implied by the following result (which is [20, Proposition 6.1], and which is also applied in [19]). ☐

**Proposition 6.5** (Kimura [20]). *Let $M$ be an oddly finite-dimensional Chow motive. Then*

$$\mathrm{CH}^i(M) \subseteq \mathrm{CH}^i_{\otimes}(M) \quad \textit{for all } i\ .$$

## 7. Supersingularity

The construction of the Cynk–Hulek Calabi–Yau varieties also makes sense in positive characteristic $\geq 5$. In this final section, we present supersingular Calabi–Yau varieties for which the motive behaves in stark contrast to the characteristic zero case:

**Proposition 7.1.** *Let $k$ be an algebraically closed field of characteristic $\geq 5$. Let $X$ be a Calabi–Yau variety over $k$ obtained as in Theorem 1.1 or 1.2, where the elliptic curves are assumed to be supersingular. Assume* $\dim X \geq 2$. *Then the cycle class map to $\ell$-adic cohomology induces isomorphisms*

$$\mathrm{CH}^i(X)_{\mathbb{Q}_\ell} \xrightarrow{\cong} H^{2i}(X, \mathbb{Q}_\ell(i)) \quad \forall i$$

*(where $\ell$ is a prime different from char$(k)$). Consequently, the Chow motive of $X$ is isomorphic to a direct sum of Lefschetz motives.*

*Proof.* First of all, we observe that the construction of the smooth projective Calabi–Yau varieties of Cynk–Hulek carries over to caracteristic $\geq 5$. Using Claim 4.9, we have a decomposition

$$\mathfrak{h}(X) = T \oplus \bigoplus \mathbb{1}(*),$$

with $H^j(T) = 0$ for $j \neq n$, and $T$ isomorphic to a direct summand of $\mathfrak{h}(E_1 \times \cdots \times E_n)$. Therefore one is reduced to proving that the cycle class map induces isomorphisms

$$\mathrm{CH}^i(T)_{\mathbb{Q}_\ell} \xrightarrow{\cong} H^{2i}(T, \mathbb{Q}_\ell(i)) \quad \forall i\ .$$

By Kimura finite–dimensionality, $T$ is isomorphic to a direct summand of the motive $(C_1 \times \cdots \times C_n, \pi^n, 0)$, where $\pi^n$ is any Chow–Künneth projector on the degree-$n$ cohomology.

Restricting to codimension 1 cycles (and noting that $n$ is at least 2), this implies in particular that

$$\mathrm{CH}^1(T)\ \subseteq\ (\pi^n)_* \mathrm{CH}^1(E_1 \times \cdots \times E_n)\ \subseteq\ \mathrm{CH}^1(E_1 \times \cdots \times E_n)_{(0)}\ . \tag{7}$$

A result of Fakhruddin [9, Proposition 1] describes the Chow ring of supersingular abelian varieties. The description is as follows: there exists a subring

$$B^*(E_1 \times \cdots \times E_n)\ \subset\ \mathrm{CH}^*(E_1 \times \cdots \times E_n)$$

(defined in terms of cycles generated by *abelian* subvarieties of $E_1 \times \cdots \times E_n$, *cf.* [9, Lemma 1]), such that $\mathrm{CH}^*(E_1 \times \cdots \times E_n)$ is generated as a module over $B^*(E_1 \times \cdots \times E_n)$ by 1 and $\mathrm{CH}^1(E_1 \times \cdots \times E_n)_{(1)}$. Fakhruddin's description, combined with property (7), implies that

$$(\pi^n)_* \mathrm{CH}^*(E_1 \times \cdots \times E_n)\ \subseteq\ B^*(E_1 \times \cdots \times E_n)\ .$$

However, the restriction of the cycle class map induces an isomorphism

$$B^*(E_1 \times \cdots \times E_n)_{\mathbb{Q}_\ell} \xrightarrow{\cong} H^{2*}(E_1 \times \cdots \times E_n, \mathbb{Q}_\ell(*)) \tag{8}$$

[9, Lemma 2], and so it follows that the cycle class map induces injections

$$\mathrm{CH}^i(T)\ \hookrightarrow\ H^{2i}(T, \mathbb{Q}_\ell(i)) \quad \forall i\ .$$

To see that $\mathrm{CH}^i(T)_{\mathbb{Q}_\ell} \to H^{2i}(T, \mathbb{Q}_\ell(i))$ is surjective, we observe that the isomorphism (8) implies in particular that the cycle class map induces a surjection

$$\mathrm{CH}^*(E_1 \times \cdots \times E_n)_{\mathbb{Q}_\ell}\ \twoheadrightarrow\ H^{2*}(E_1 \times \cdots \times E_n, \mathbb{Q}_\ell(*)) \tag{9}$$

Now let $p \in \mathrm{CH}^n((E_1 \times \cdots \times E_n) \times (E_1 \times \cdots \times E_n))$ denote the projector defining (the direct summand of $\mathfrak{h}(E_1 \times \cdots \times E_n)$ isomorphic to) the motive $T$. Letting $p$ act on both sides of the surjection (9), we obtain a surjection

$$\mathrm{CH}^i(T)_{\mathbb{Q}_\ell}\ \twoheadrightarrow\ H^{2i}(T, \mathbb{Q}_\ell(i))\ ,$$

and so we are done.

Finally, the above reasoning applies to any base extension $K \supset k$, and so we see that $\mathrm{CH}^*(T_K)$ is of finite rank. This implies that $T$ is isomorphic to a sum of Lefschetz motives [33], [21]. □

**Remark 7.2.** In dimension $n = 2$, Proposition 7.1 follows from the general result that supersingular K3 surfaces are unirational [25]. In case the ground field $k$ is finite, Proposition 7.1 is true without the assumption of supersingularity, as follows from [18].

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