DERIVED EQUIVALENT THREEFOLDS, ALGEBRAIC REPRESENTATIVES, AND THE CONIVEAU FILTRATION

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ABSTRACT. A conjecture of Orlov predicts that derived equivalent smooth projective varieties have isomorphic Chow motives. We show a result in that direction : two derived equivalent threefolds over a field of characteristic zero have isogenous intermediate Jacobians. For threefolds over an arbitrary perfect field, we show that the isogeny class of the algebraic representative (for algebraically trivial cycles of codimension two) is a derived invariant.

INTRODUCTION

A fundamental invariant of a smooth projective variety X is given by $D^b(X)$, the bounded derived category of coherent sheaves on X ; see [Huy06] for a survey. Two smooth projective varieties over a field K are said to be *derived equivalent* if there exists a K-linear exact equivalence of categories between their bounded derived categories of coherent sheaves. The following conjecture states that $D^b(X)$ is a finer invariant than the Chow motive of X with rational coefficients.

Conjecture (Orlov [Orl05]). *If two smooth projective varieties X and Y defined over a field K are derived equivalent, then the Chow motives of X and Y with $\mathbb{Q}$-coefficients are isomorphic.*

The conjecture is true for varieties X and Y with ample canonical or anti-canonical bundle, since then a fundamental theorem of Bondal–Orlov [BO01] states that X and Y are isomorphic. It has recently been observed by Huybrechts that the conjecture is true for surfaces [Huy17]. A direct consequence of the conjecture is the following weaker conjecture : if X and Y are smooth projective varieties defined over a subfield $K \subseteq \mathbb{C}$ that are derived equivalent, then the Hodge structures $H^i(X_{\mathbb{C}}, \mathbb{Q})$ and $H^i(Y_{\mathbb{C}}, \mathbb{Q})$ are isomorphic for all i. The case $i = 0$ is true and obvious. The first nontrivial result in this direction is due to Popa and Schnell, building on work of Rouquier [Rou11], who secure the case $i = 1$ with $K = \mathbb{C}$. In fact, together with Honigs, we have established that their result is valid over an arbitrary field [ACHV17] :

Theorem 1 (Popa–Schnell [PS11]). *Assume that X and Y are two derived equivalent smooth projective varieties over a field K. Then the abelian varieties $\mathrm{Pic}^0(X)_{\mathrm{red}}$ and $\mathrm{Pic}^0(Y)_{\mathrm{red}}$ are isogenous.*

In particular, for all $\ell \neq \mathrm{char}(K)$, the $\mathrm{Gal}(K)$-representations $H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ and $H^1(Y_{\overline{K}}, \mathbb{Q}_\ell)$ are isomorphic. Our goal is to extend these results to other Galois representations and abelian varieties attached to smooth projective varieties.

Indeed, in addition to Hodge theoretic considerations, Orlov's motivic conjecture gives impetus for studying the arithmetic of derived equivalent varieties over an arbitrary field. On one hand, recent work of Antieau, Krashen and Ward [AKW17] gives examples of varieties X and Y over a field K that are derived equivalent, but are nontrivial twists of each other. For instance, they describe distinct K-isomorphism classes of genus one curves that are derived equivalent. These pairs

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of curves are twists of the same elliptic curve, and so have isomorphic Jacobians, in accordance with Theorem 1. On the other hand, Honigs recently proved that two derived equivalent surfaces [Hon15] or threefolds [Hon17] over a finite field share the same zeta function.

In this paper, we observe that in fact the ℓ-adic étale cohomology groups $H^i(-,\mathbb{Q}_\ell)$ of two derived equivalent threefolds over an arbitrary field K (of characteristic $\neq \ell$) are isomorphic as $\mathrm{Gal}(K)$-modules. Moreover, we show in some cases that these isomorphisms can be chosen to be compatible with the coniveau filtration (as reviewed in §1), and when K is a perfect field we show that they share a common motivic invariant, namely, the isogeny classes of their *algebraic representatives* $\mathrm{Ab}^i(-)$. (The algebraic representative $\mathrm{Ab}^i(X)$ is an abelian variety over K which is universal for algebraically trivial cycles on X of codimension i ; see §3 for details.) Our main result is :

Theorem 2. *Let X and Y be two smooth projective varieties of dimension* 3 *over a perfect field K. Assume that X and Y are derived equivalent.*

(a) *For each nonnegative integer i, the Galois representations $H^i(X_{\overline{K}},\mathbb{Q}_\ell)$ and $H^i(Y_{\overline{K}},\mathbb{Q}_\ell)$ are isomorphic. If $K \subseteq \mathbb{C}$, then the rational Hodge structures $H^i(X_{\mathbb{C}},\mathbb{Q})$ and $H^i(Y_{\mathbb{C}},\mathbb{Q})$ are isomorphic.*

(b) *If $K \subseteq \mathbb{C}$, then there exist isomorphisms in (a) compatible with the coniveau filtration (for both the ℓ-adic and Betti cohomology).*

(c) *For each nonnegative integer i, the algebraic representatives $\mathrm{Ab}^i(X)$ and $\mathrm{Ab}^i(Y)$ are isogenous over K.*

Insofar as $\mathrm{Ab}^1(X) \simeq \mathrm{Pic}^0(X)_{\mathrm{red}}$, part (c) is a natural extension of Theorem 1 in the special case of threefolds. In fact, the case $i = 3$ in Theorem 2(c) also follows directly from Theorem 1.

To a smooth projective variety X over a subfield of $\mathbb{C}$ one may associate the total image of the Abel–Jacobi map restricted to algebraically trivial cycles, $J_a(X_{\mathbb{C}})$, an abelian variety defined over K (§2, [ACV16]) that sits inside the total intermediate Jacobian. We show (Proposition 2.1) that if X and Y are derived equivalent smooth projective varieties of arbitrary dimension over a subfield of $\mathbb{C}$, then $J_a(X_{\mathbb{C}})$ and $J_a(Y_{\mathbb{C}})$ are isogenous over K ; over $\mathbb{C}$ this provides a short proof of a special case of a result of [BT16], which also considers semi-orthogonal decompositions.

Notation. Given a field K, we denote by $\overline{K}$ an algebraic closure and by $\mathrm{Gal}(K) = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. For a smooth projective variety X over K, $\mathrm{CH}(X)$ denotes the Chow group of X, and $H^i(X)(j)$ denotes one of the following Weil cohomology theories :

- For ℓ a prime $\neq \mathrm{char}(K)$, $H^i(X)(j) = H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_\ell(j))$ viewed as a $\mathbb{Q}_\ell[\mathrm{Gal}(K)]$-module.
- For $K \subseteq \mathbb{C}$, $H^i(X)(j) = H^i(X_{\mathbb{C}},\mathbb{Q}(j))$ viewed as a pure rational Hodge structure.

The notation is intentionally ambiguous so that we may give statements and proofs for both cohomology theories simultaneously ; the meaning will be clear from the context. Maps between $\mathbb{Q}_\ell[\mathrm{Gal}(K)]$-modules are always assumed to be morphisms. Likewise, maps between rational Hodge structures are always assumed to be morphisms.

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1. Derived equivalence and the coniveau filtration

If X and Y are two derived equivalent smooth projective varieties over a field K, then we have (ungraded) isomorphisms (e.g., [Huy06, Prop. 5.33], [LO15, §2], [Hon15, Lem. 3.1])

$$\bigoplus_i H^{2i}(X)(i) \simeq \bigoplus_i H^{2i}(Y)(i) \quad \text{and} \quad \bigoplus_i H^{2i+1}(X)(i) \simeq \bigoplus_i H^{2i+1}(Y)(i). \tag{1}$$

Let $\mathrm{N}^\bullet$ denote the (geometric) coniveau filtration, i.e.,

$$\mathrm{N}^j H^i(X) := \sum \ker(H^i(X) \to H^i(X\backslash Z)),$$

where the sum runs through all closed K-subschemes $Z \subseteq X$ of codimension $\geq j$. The isomorphisms (1) can be upgraded to take into account the coniveau filtration :

Proposition 1.1. *Let X and Y be smooth projective varieties over a field K. Assume that X and Y are derived equivalent. Then there exist for all integers j (ungraded) isomorphisms*

$$\bigoplus_i \mathrm{N}^{i-j} H^{2i}(X)(i) \simeq \bigoplus_i \mathrm{N}^{i-j} H^{2i}(Y)(i) \text{ and } \bigoplus_i \mathrm{N}^{i-j} H^{2i+1}(X)(i) \simeq \bigoplus_i \mathrm{N}^{i-j} H^{2i+1}(Y)(i). \tag{2}$$

Proof. By Orlov [Orl03], the equivalence $D^b(X) \simeq D^b(Y)$ is induced by a Fourier–Mukai functor. Denote by $\mathcal{E} \in D^b(X \times_K Y)$ its kernel, and by $\mathcal{F} \in D^b(Y \times_K X)$ the kernel of its inverse. To $\mathcal{E}$, one can associate the Mukai vector

$$v(\mathcal{E}) := p_X^* \sqrt{\mathrm{td}_X} \cdot \mathrm{ch}(\mathcal{E}) \cdot p_Y^* \sqrt{\mathrm{td}_Y} \in \mathrm{CH}(X \times_K Y) \otimes \mathbb{Q},$$

where $p_X : X \times_K Y \to X$ and $p_Y : X \times_K Y \to Y$ are the natural projections, and where td_X and td_Y are the Todd classes of X and Y, respectively. A square root is taken formally. Likewise, we may consider the Mukai vector $v(\mathcal{F})$, and we have (see e.g., [Huy06, Prop. 5.10, Lem. 5.32])

$$v(\mathcal{F}) \circ v(\mathcal{E}) = v(\mathcal{O}_{\Delta_X}) = \Delta_X \in \mathrm{CH}(X \times_K X) \otimes \mathbb{Q},$$
$$v(\mathcal{E}) \circ v(\mathcal{F}) = v(\mathcal{O}_{\Delta_Y}) = \Delta_Y \in \mathrm{CH}(Y \times_K Y) \otimes \mathbb{Q}.$$

It is then apparent that the action of $v(\mathcal{E})$ induces the isomorphisms in (1). The isomorphisms in (2) follow readily from the functoriality of the coniveau filtration with respect to the action of correspondences (which itself is a consequence of the formalism of cohomology with support). □

Remark 1.2. Proposition 1.1 also holds if one replaces the geometric coniveau filtration with the Hodge coniveau filtration or the Tate coniveau filtration ; see e.g. [ACV, §1.2] for definitions. As a consequence, given two derived equivalent smooth projective varieties X and Y over a field K, if $K = \mathbb{C}$ and the Hodge conjecture (or the generalized Hodge conjecture) holds for X, or if K is finitely generated over its prime subfield and the Tate conjecture (or the generalized Tate conjecture) holds for X, then the corresponding conjecture also holds for Y.

The conjecture of Orlov predicts that the isomorphisms (2) can be chosen to be i-graded. We verify this prediction in the case where X and Y are derived equivalent threefolds. Note that it was already observed by Honigs [Hon17] that the theorem of Popa–Schnell implies in this case that $H^3(X) \simeq H^3(Y)$ (it was previously observed in [PS11] that X and Y have the same Betti numbers).

Proposition 1.3. *Let X and Y be two smooth projective varieties of dimension 3 over a field K. Assume that X and Y are derived equivalent. Then for all odd i there are isomorphisms $H^i(X) \simeq H^i(Y)$ that are compatible with the coniveau filtration. If $K \subseteq \mathbb{C}$, then for all i there are isomorphisms $H^i(X) \simeq H^i(Y)$ that are compatible with the coniveau filtration.*

Proof. Note that for a 3-fold V, we have

$$0 = \mathrm{N}^1 H^0(V) \subseteq \mathrm{N}^0 H^0(V) = H^0(V)$$
$$0 = \mathrm{N}^1 H^1(V) \subseteq \mathrm{N}^0 H^1(V) = H^1(V)$$
$$0 = \mathrm{N}^2 H^2(V) \subseteq \mathrm{N}^1 H^2(V) \subseteq \mathrm{N}^0 H^2(V) = H^2(V)$$
$$0 = \mathrm{N}^2 H^3(V) \subseteq \mathrm{N}^1 H^3(V) \subseteq \mathrm{N}^0 H^3(V) = H^3(V)$$
$$0 = \mathrm{N}^3 H^4(V) \subseteq \mathrm{N}^2 H^4(V) \subseteq \mathrm{N}^1 H^4(V) = H^4(V)$$

$$0 = \mathrm{N}^3 H^5(V) \subseteq \mathrm{N}^2 H^5(V) = H^5(V)$$
$$0 = \mathrm{N}^4 H^6(V) \subseteq \mathrm{N}^3 H^6(V) = H^6(V).$$

Recall that the category of polarizable rational Hodge structures is semi-simple, and that the category of $\mathbb{Q}_\ell[\mathrm{Gal}(K)]$-modules that are finite-dimensional as $\mathbb{Q}_\ell$-vector spaces is a Krull–Schmidt category. This means that any finite-dimensional $\mathbb{Q}_\ell[\mathrm{Gal}(K)]$-module can be written in an essentially unique way as a direct sum of indecomposable objects ; see e.g., [CR90, Thm. 6.12, Cor. 6.15]. In particular, in both categories, we have the following cancellation property : if we have an isomorphism $A \oplus B \simeq A \oplus B'$, then $B \simeq B'$.

That $H^0(X) \simeq H^0(Y)$ and $H^6(X) \simeq H^6(Y)$ is obvious. Theorem 1 asserts that $H^1(X) \simeq H^1(Y)$ and duality then implies that $H^5(X) \simeq H^5(Y)$. By (1) and cancellation, we obtain $H^3(X) \simeq H^3(Y)$. We also obtain from (2) and from cancellation that $\mathrm{N}^1 H^3(X) \simeq \mathrm{N}^1 H^3(Y)$.

By (1), and by cancellation, we obtain that $H^2(X)(1) \oplus H^4(X)(2) \simeq H^2(Y)(1) \oplus H^4(Y)(2)$. By Poincaré duality, and by semi-simplicity of the category of polarizable $\mathbb{Q}$-Hodge structures in the case where H is Betti cohomology or by the Krull–Schmidt theorem [CR90, 6.12] in the case where H is ℓ-adic cohomology, we obtain that $H^2(X) \simeq H^2(Y)$, and then by duality that $H^4(X) \simeq H^4(Y)$.

We now assume that $K \subseteq \mathbb{C}$. Recall that in that case the coniveau filtration on $H^i(X)$ is split ; see e.g. [ACV16, Cor. 4.4]. Therefore, in order to prove isomorphisms of $\mathbb{Q}_\ell[\mathrm{Gal}(K)]$-modules or of rational Hodge structures that are compatible with the coniveau filtration, it suffices to prove that $\mathrm{N}^j H^i(X) \simeq \mathrm{N}^j H^i(Y)$ for all i and j. It only remains to show that $\mathrm{N}^1 H^2(X) \simeq \mathrm{N}^1 H^2(Y)$ and $\mathrm{N}^2 H^4(X) \simeq \mathrm{N}^2 H^4(Y)$. Note that by the comparison isomorphisms (see e.g., [ACV, (1.3)]), intersecting with a smooth hyperplane section of X defined over K induces an isomorphism $\mathrm{N}^1 H^2(X)(1) \simeq \mathrm{N}^2 H^4(X)(2)$, and similarly for Y. Therefore, by semi-simplicity or by the Krull–Schmidt theorem, we obtain the required isomorphisms for $\mathrm{N}^1 H^2$ and $\mathrm{N}^2 H^4$, which concludes the proof. $\square$

Remark 1.4. The same arguments show that if X and Y are derived equivalent smooth projective varieties of dimension 4 over a field K, then there are isomorphisms $H^3(X) \simeq H^3(Y)$ and $H^5(X) \simeq H^5(Y)$. If in addition the monomorphism $\mathrm{N}^1 H^3(X) \hookrightarrow \mathrm{N}^2 H^5(X)$ induced by cupping with an ample divisor is surjective (e.g., if the generalized Hodge (or Tate) conjecture holds for X, or if the standard conjectures hold for X), then there are isomorphisms $\mathrm{N}^1 H^3(X) \simeq \mathrm{N}^1 H^3(Y)$ and $\mathrm{N}^2 H^5(X) \simeq \mathrm{N}^2 H^5(Y)$ compatible with the aforementioned isomorphisms.

2. Derived equivalence and total intermediate Jacobians

If $K \subseteq \mathbb{C}$, we see from the second isomorphism of (1) that the isogeny class of the total intermediate Jacobian

$$J(X_{\mathbb{C}}) := \bigoplus_i J^{2i-1}(X_{\mathbb{C}})$$

is a derived invariant of smooth projective varieties. Here, $J^{2i-1}(X_{\mathbb{C}})$ is Griffiths' intermediate Jacobian ; as a complex torus it is defined as

$$J^{2i-1}(X) := F^i H^{2i-1}(X, \mathbb{C}) \backslash H^{2i-1}(X, \mathbb{C}) / H^{2i-1}(X, \mathbb{Z}),$$

where $F^\bullet$ denotes the Hodge filtration. The intermediate Jacobian is of interest because it receives cohomologically trivial cycles under the Abel–Jacobi map $AJ : \mathrm{CH}^i(X_{\mathbb{C}})_{\mathrm{hom}} \to J^{2i-1}(X_{\mathbb{C}})$. In this paper we will be interested in the Abel–Jacobi map

$$AJ : \mathrm{A}^i(X_{\mathbb{C}}) \longrightarrow J^{2i-1}(X_{\mathbb{C}}) \tag{3}$$

obtained by restricting to the subgroup of algebraically trivial cycles $\mathrm{A}^i(X_{\mathbb{C}}) \subseteq \mathrm{CH}^i(X_{\mathbb{C}})_{\mathrm{hom}}$. We denote by $J_a^{2i-1}(X_{\mathbb{C}})$ the subtorus that is the image of $\mathrm{A}^i(X_{\mathbb{C}})$ under the Abel–Jacobi map (e.g., [Mur85, Lem. 1.6.2]). In terms of the coniveau filtration, we have

$$H^1(J_a^{2i-1}(X), \mathbb{Q}) \simeq \mathrm{N}^i\, H^{2i-1}(X, \mathbb{Q}(i)).$$

A choice of polarization on X endows $J_a^{2i-1}(X_{\mathbb{C}})$ with a polarization, thereby turning it into a complex abelian variety. Likewise, we see from the second isomorphism of (2) that the isogeny class of the total image of the Abel–Jacobi map (3),

$$J_a(X_{\mathbb{C}}) := \bigoplus_i J_a^{2i-1}(X_{\mathbb{C}}),$$

is a derived invariant of smooth projective varieties.

We would like to upgrade this observation to an isogeny of abelian varieties defined over K. Recall from [ACV16] that $J_a^{2i-1}(X_{\mathbb{C}})$ descends canonically to an abelian variety $J_a^{2i-1}(X)$ over K. Precisely, $J_a^{2i-1}(X)$ is the abelian variety over K such that the Abel–Jacobi map $\mathrm{A}^i(X_{\mathbb{C}}) \to J_a^{2i-1}(X)_{\mathbb{C}}$ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant.

Proposition 2.1. *Let X and Y be smooth projective varieties over a subfield K of $\mathbb{C}$. Assume that X and Y are derived equivalent. Then $J_a(X) := \bigoplus_i J_a^{2i-1}(X)$ and $J_a(Y) := \bigoplus_i J_a^{2i-1}(Y)$ are isogenous abelian varieties over K.*

Proof. Recall that the Abel–Jacobi map is functorial with respect to the action of correspondences. With the notations of the proof of Proposition 1.1, the Mukai vector $v(\mathcal{E})$ thus induces an isogeny $v(\mathcal{E})_* : J_a(X_{\mathbb{C}}) \to J_a(Y_{\mathbb{C}})$. Then it follows from [ACV16, Prop. 2.5] that this isogeny descends to an isogeny over K. □

As mentioned earlier, over $\mathbb{C}$ this provides a short proof of a special case of a result of [BT16], which also considers semi-orthogonal decompositions.

For threefolds, as a corollary to Proposition 2.1 (Proposition 1.3 would suffice if $K = \mathbb{C}$) and Theorem 1, we may single out the second intermediate Jacobian (we will give an alternate proof of this corollary in Theorem 3.1) :

Corollary 2.2. *Let X and Y be two smooth projective varieties of dimension 3 over a field $K \subseteq \mathbb{C}$. Assume that X and Y are derived equivalent. Then the intermediate Jacobians $J^3(X_{\mathbb{C}})$ and $J^3(Y_{\mathbb{C}})$ are isogenous complex tori, and $J_a^3(X)$ and $J_a^3(Y)$ are K-isogenous abelian varieties.* □

Remark 2.3. Following Remarks 1.2 and 1.4, we may in fact improve Corollary 2.2. Suppose that X and Y are two derived-equivalent smooth projective varieties of dimension 4 over a field $K \subseteq \mathbb{C}$; then the intermediate Jacobians $J^{2i-1}(X_{\mathbb{C}})$ and $J^{2i-1}(Y_{\mathbb{C}})$ are isogenous complex tori for all i. If in addition the monomorphism $\mathrm{N}^1 H^3(X) \hookrightarrow \mathrm{N}^2 H^5(X)$ induced by cupping with an ample divisor is surjective, then $J_a^{2i-1}(X)$ and $J_a^{2i-1}(Y)$ are K-isogenous abelian varieties for all i.

3. Derived equivalent 3-folds and algebraic representatives

The aim of this section is to extend Corollary 2.2 to varieties defined over a perfect field K. The abelian variety that plays the role of the second intermediate Jacobian over algebraically closed fields of positive characteristic is called an *algebraic representative* (for codimension-2 cycles).

Let X/K be a smooth, projective variety, and consider the group $\mathrm{A}^i(X_{\overline{K}})$ of algebraically trivial codimension-i cycles up to rational equivalence on $X_{\overline{K}}$. The *algebraic representative*, if it exists, is an abelian variety $\mathrm{Ab}^i(X_{\overline{K}})$ equipped with an Abel–Jacobi map $\phi^i : \mathrm{A}^i(X_{\overline{K}}) \to \mathrm{Ab}^i(X_{\overline{K}})(\overline{K})$. Its defining universal property is that ϕ^i is initial among all *regular homomorphisms* $\mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$, that is, among all homomorphisms of groups $\psi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ to the $\overline{K}$-points of an abelian variety A over $\overline{K}$ such that for every pair $((T,t_0),Z)$ with (T,t_0) a pointed smooth integral variety over $\overline{K}$, and $Z \in \mathrm{CH}^i(T \times X)$, the composition $T(\overline{K}) \to \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K}), t \mapsto \psi(Z_t - Z_{t_0})$ is induced by a morphism of varieties $\psi_Z : T \to A$ over $\overline{K}$. The Abel–Jacobi map to the algebraic representative is unique (up to a unique automorphism) and surjective. In particular, $\mathrm{Ab}^1(X_{\overline{K}}) \simeq \mathrm{Pic}^0(X_{\overline{K}})_{\mathrm{red}}$, while $\mathrm{Ab}^{\dim X}(X_{\overline{K}}) \simeq \mathrm{Alb}(X_{\overline{K}})$. The existence of $\mathrm{Ab}^2(X_{\overline{K}})$ was established by Murre [Mur85].

In [ACV], we extended Murre's theorem by showing that an initial regular homomorphism $\phi^i : \mathrm{A}^i(X_{\overline{K}}) \to \mathrm{Ab}^i(X_{\overline{K}})(\overline{K})$ can be made Galois-equivariant, thus providing a canonical descent datum on $\mathrm{Ab}^i(X_{\overline{K}})$ and hence, by descent, a distinguished model $\mathrm{Ab}^i(X)$ over K of the abelian variety $\mathrm{Ab}^i(X_{\overline{K}})$. The corresponding statements for the Picard and Albanese varieties are well-known.

Our main result extends Theorem 1 to $\mathrm{Ab}^2(X)$ when $\dim X \leq 3$ and when K is a perfect field.

Theorem 3.1. *Let X and Y be two smooth projective varieties of dimension 3 over a perfect field K. Assume that X and Y are derived equivalent. Then the abelian varieties $\mathrm{Ab}^2(X)$ and $\mathrm{Ab}^2(Y)$ are K-isogenous.*

Proof. By Orlov [Orl03], the equivalence $D^b(X) \simeq D^b(Y)$ is induced by a Fourier–Mukai functor. Denote $\mathcal{E} \in D^b(X \times_K Y)$ its kernel. As above, to $\mathcal{E}$ one can associate the Mukai vector

$$\gamma = v(\mathcal{E}) := p_X^* \sqrt{\mathrm{td}_X} \cdot \mathrm{ch}(\mathcal{E}) \cdot p_Y^* \sqrt{\mathrm{td}_Y} \in \mathrm{CH}^*(X \times_K Y) \otimes \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}].$$

By Riemann–Roch we obtain, after inverting 2, 3 and 5, an isomorphism induced by the action of the correspondence γ

$$\gamma_* : \mathrm{A}^*(X_{\overline{K}}) \otimes \mathbb{Z}[1/30] \xrightarrow{\simeq} \mathrm{A}^*(Y_{\overline{K}}) \otimes \mathbb{Z}[1/30].$$

In particular we obtain, for all positive integers N relatively prime to 30, isomorphisms

$$\gamma_* : \mathrm{A}^*(X_{\overline{K}})[N] \xrightarrow{\simeq} \mathrm{A}^*(Y_{\overline{K}})[N].$$

Here, for an abelian group G, we denote by $G[N]$ the subgroup consisting of N-torsion elements. In fact there exists a positive integer n, whose only prime divisors are 2, 3 and 5, such that $n\gamma$ is an integral cycle, *i.e.*, an element of $\mathrm{CH}(X \times_K Y)$. For all positive integers $N > 1$ relatively prime to 30, we also have isomorphisms

$$n\gamma_* : \mathrm{A}^*(X_{\overline{K}})[N] \xrightarrow{\simeq} \mathrm{A}^*(Y_{\overline{K}})[N].$$

One can check that the homomorphism $(\phi_Y^3 \oplus \phi_Y^2 \oplus \phi_Y^1) \circ n\gamma_*$ is a regular homomorphism. Thus, for X and Y as in the statement of the theorem and by the universal property of the algebraic representatives, we have a diagram

$$\begin{array}{ccc} \mathrm{A}^*(X_{\overline{K}}) & \xrightarrow{\phi_X^3 \oplus \phi_X^2 \oplus \phi_X^1} & \mathrm{Alb}(X)(\overline{K}) \oplus \mathrm{Ab}^2(X)(\overline{K}) \oplus \mathrm{Pic}^0(X)_{\mathrm{red}}(\overline{K}) \\ {\scriptstyle n\gamma_*}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathrm{A}^*(Y_{\overline{K}}) & \xrightarrow{\phi_Y^3 \oplus \phi_Y^2 \oplus \phi_Y^1} & \mathrm{Alb}(Y)(\overline{K}) \oplus \mathrm{Ab}^2(Y)(\overline{K}) \oplus \mathrm{Pic}^0(Y)_{\mathrm{red}}(\overline{K}) \end{array} \tag{4}$$

where φ is induced by a $\overline{K}$-homomorphism

$$\mathrm{Alb}(X)_{\overline{K}} \times \mathrm{Ab}^2(X)_{\overline{K}} \times \mathrm{Pic}^0(X)_{\mathrm{red},\overline{K}} \to \mathrm{Alb}(Y)_{\overline{K}} \times \mathrm{Ab}^2(Y)_{\overline{K}} \times \mathrm{Pic}^0(Y)_{\mathrm{red},\overline{K}}.$$

In fact, since $n\gamma$ is a correspondence defined over K, it follows from [ACV, Thm. 4.4] that this $\overline{K}$-homomorphism descends to a K-homomorphism

$$\Phi : \mathrm{Alb}(X) \times_K \mathrm{Ab}^2(X) \times_K \mathrm{Pic}^0(X)_{\mathrm{red}} \to \mathrm{Alb}(Y) \times_K \mathrm{Ab}^2(Y) \times_K \mathrm{Pic}^0(Y)_{\mathrm{red}}.$$

Together with the fact that $n\gamma_*$ is an isomorphism when restricted to N-torsion for all integers $N > 1$ coprime to 2, 3 and 5, a simple diagram chase yields that Φ is surjective when restricted to N-torsion for all integers $N > 1$ coprime to 30.

Likewise, by considering the inverse equivalence, we see that there is a K-homomorphism

$$\Psi : \mathrm{Alb}(Y) \times_K \mathrm{Ab}^2(Y) \times_K \mathrm{Pic}^0(Y)_{\mathrm{red}} \to \mathrm{Alb}(X) \times_K \mathrm{Ab}^2(X) \times_K \mathrm{Pic}^0(X)_{\mathrm{red}}$$

which is surjective when restricted to N-torsion for all integers $N > 1$ coprime to 30. Therefore the abelian varieties $\mathrm{Alb}(X) \times_K \mathrm{Ab}^2(X) \times_K \mathrm{Pic}^0(X)_{\mathrm{red}}$ and $\mathrm{Alb}(Y) \times_K \mathrm{Ab}^2(Y) \times_K \mathrm{Pic}^0(Y)_{\mathrm{red}}$ are K-isogenous.

By Theorem 1, if X and Y are two derived equivalent smooth projective varieties over a field K, then the abelian varieties $\mathrm{Pic}^0(X)_{\mathrm{red}}$ and $\mathrm{Pic}^0(Y)_{\mathrm{red}}$ are K-isogenous ; by duality, the abelian varieties $\mathrm{Alb}(X)$ and $\mathrm{Alb}(Y)$ are K-isogenous. Therefore, by Poincaré reducibility, we find that $\mathrm{Ab}^2(X)$ and $\mathrm{Ab}^2(Y)$ are K-isogenous. $\square$

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