DECOMPOSITION OF THE DIAGONAL, INTERMEDIATE JACOBIANS, AND UNIVERSAL CODIMENSION-2 CYCLES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We consider the connections among algebraic cycles, abelian varieties, and stable rationality of smooth projective varieties in positive characteristic. Recently Voisin constructed two new obstructions to stable rationality for rationally connected complex projective threefolds by giving necessary and sufficient conditions for the existence of a cohomological decomposition of the diagonal. In this paper, we show how to extend these obstructions to rationally chain connected threefolds in positive characteristic via ℓ-adic cohomological decomposition of the diagonal. This requires extending results in Hodge theory regarding intermediate Jacobians and Abel–Jacobi maps to the setting of algebraic representatives. For instance, we show that the algebraic representative for codimension-two cycle classes on a geometrically stably rational threefold admits a canonical auto-duality, which in characteristic zero agrees with the principal polarization on the intermediate Jacobian coming from Hodge theory. As an application, we extend a result of Voisin, and show that in characteristic greater than two, a desingularization of a very general quartic double solid with seven nodes does not admit a universal codimension-two cycle class. In the process, we establish some results on the moduli space of nodal degree-four polarized K3 surfaces in positive characteristic.

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RÉSUMÉ. Nous examinons les relations entre cycles algébriques, variétés abéliennes, et la propriété de rationalité stable pour les variétés projectives et lisses en caractéristique positive. Récemment, Voisin a exhibé deux nouvelles obstructions à la rationalité stable pour les solides projectifs complexes rationnellement connexes en donnant des conditions nécessaires et suffisantes à l'existence d'une décomposition cohomologique de la diagonale. Dans cet article, nous montrons comment étendre ces obstructions aux solides projectifs rationnellement connexes par chaîne en caractéristique positive en utilisant la cohomologie ℓ-adique. Pour cela, nous étendons des résultats en théorie de Hodge concernant les jacobiennes intermédiaires et les applications d'Abel–Jacobi au contexte des représentants algébriques. Par exemple, nous établissons que le représentant algébrique pour les cycles de codimension deux sur un solide géométriquement rationnellement stable admet un isomorphisme canonique vers son dual qui coïncide en caractéristique nulle avec la polarisation principale sur la jacobienne intermédiaire provenant de la théorie de Hodge. Comme application, nous étendons un résultat de Voisin et montrons qu'en caractéristique positive différente de deux une désingularisation d'un solide quartique très général possédant sept nœuds n'admet pas de classe de cycle universel de codimension deux. En chemin, nous établissons des résultats concernant l'espace de modules des surfaces K3 nodales polarisées de degré quatre en caractéristique positive.

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In this paper we consider the connections among algebraic cycle classes, abelian varieties, and stable rationality of smooth projective varieties in positive characteristic. As motivation, recall that Clemens and Griffiths [CG72] have shown that if a complex projective rationally connected threefold X is rational, then the so-called minimal cohomology class

$$\frac{[\Theta_X]^{g-1}}{(g-1)!} \in H^{2g-2}(J^3(X), \mathbb{Z}) \tag{0.1}$$

is an *effective* algebraic cycle class, where $g = \dim J^3(X)$, and Θ_X is the canonical polarization on the intermediate Jacobian $J^3(X)$ induced by the cup product on $H^3(X, \mathbb{Z})$, which is principal as $h^{1,0}(X) = h^{3,0}(X) = 0$. For cubic threefolds, which are all unirational and therefore rationally connected, Clemens and Griffiths showed, rephrasing via the Matsusaka–Ran criterion, that $\frac{[\Theta_X]^{g-1}}{(g-1)!}$ is not an effective algebraic cycle class, and therefore that cubic threefolds are not rational.

Recently Voisin [Voi15] showed that if a complex projective rationally connected threefold X is stably rational, then the minimal cohomology class (0.1) is an algebraic cycle class (possibly *not* effective), and moreover, $J^3(X)$ admits a universal codimension-2 cycle class: there exists a cycle class $Z \in \mathrm{CH}^2(J^3(X) \times X)$, which is fiberwise algebraically trivial, such that the composition

$$\psi_Z : J^3(X) \longrightarrow A^2(X) \xrightarrow{AJ} J^3(X)$$

$$t \longmapsto Z_t \longmapsto AJ(Z_t)$$

is the identity. While it is not known whether there exist any principally polarized abelian varieties (A, Θ) where the class $\frac{[\Theta]^{g-1}}{(g-1)!}$ is not algebraic ($g = \dim A$), and thus it is unclear whether this test for stable irrationality via minimal cohomology classes can fail, the latter condition, on universal codimension-2 cycle classes, has so far been more tractable. For a smooth projective threefold X obtained as the desingularization of a very general quartic double solid with 7 nodes, Voisin showed [Voi15] that $J^3(X)$ does not admit a universal codimension-2 cycle class, and therefore, that such a unirational threefold is not stably rational. Recall that a nodal quartic double solid X is obtained as a double cover $X \to \mathbb{P}^3$ branched along a nodal quartic surface, and that any such variety is unirational.

A particularly interesting aspect of Voisin's example is that other standard tests of stable irrationality fail. More precisely, building on previous work of Artin–Mumford [AM72] and Bloch–Srinivas [BS83], Voisin showed [Voi15] that if a complex projective threefold X is stably rational, then:

(1) (Bloch–Srinivas) $H^1(X, \mathbb{Z}) = 0$;
(2) (Bloch–Srinivas) $H^{2i}(X, \mathbb{Z})$ is algebraic for all i;
(3) (Bloch–Srinivas) The Abel–Jacobi map $AJ : A^2(X) \to J^3(X)$ is surjective;
(4) (Artin–Mumford) Tors $H^\bullet(X, \mathbb{Z}) = 0$;
(5) (Voisin) $J^3(X)$ admits a universal codimension-2 cycle class;
(6) (Voisin) $\frac{[\Theta_X]^{g-1}}{(g-1)!} \in H^{2g-2}(J^3(X), \mathbb{Z})$ is an algebraic class, where $g = \dim J^3(X)$.

We have omitted the *a priori* weaker standard condition that $H^0(X, \Omega_X^i) = 0$ for $i > 0$ as it is implied by conditions (1)–(3). Also note that since the Albanese is a stable birational invariant, one gets (1) in characteristic 0 without using the Bloch–Srinivas arguments; however, for reference in the positive characteristic case, we prefer to call this a Bloch–Srinivas condition. We emphasize that any complex projective rationally connected threefold satisfies conditions (1)–(3); for (1) see *e.g.*, [BS83] or [Voi07b, Cor. 10.18], (2) is [Voi06, Thm. 2], and (3) is [BS83, Thm. 1(i)]. In other words,

(1)–(3) are obstructions to the rational connectivity of a threefold, while (4)–(6) are obstructions to the stable rationality of a rationally connected threefold.

Voisin's example is the first example of a unirational but stably irrational threefold satisfying (1)–(4) and (6) above, while only failing (5). Finding examples of unirational threefolds failing (4) has been the typical method of establishing stable irrationality. For instance, the first example of a unirational but stably irrational threefold was due to Artin and Mumford [AM72], who showed that there are threefolds X obtained as desingularizations of quartic double solids with 10 nodes in special position, such that $\mathrm{Tors}\, H^4(X, \mathbb{Z}) \neq 0$ (*i.e.*, (4) fails). In that example (5) and (6) hold trivially since $g = 0$. It is not known if there are examples of unirational varieties failing (6), as again, it is unknown if this condition fails for any principally polarized abelian variety.

The goal of this paper is to consider these types of questions in positive characteristic. The Clemens–Griffiths results on rationality have been investigated in this setting, for instance in [Mur73, Bea77] over algebraically closed fields, and more recently in [BW20, BW23] over arbitrary fields. In this paper we focus on the topic of stable rationality, with an emphasis on Voisin's conditions (5) and (6). As a brief digression, we recall that condition (4), as well as the condition that $H^0(X, \Omega^i_X) = 0$ for $i > 0$, have been studied extensively in the literature in positive characteristic in the context of stable rationality. In condition (4), one can for instance replace Betti cohomology with ℓ-adic cohomology, and Artin–Mumford [AM72] showed for example that over any algebraically closed field k of characteristic not equal to 2 there are threefolds X obtained as desingularizations of quartic double solids with 10 nodes in special position, such that $\mathrm{Tors}\, H^3(X, \mathbb{Z}_2) \neq 0$, *i.e.*, (4) fails, and therefore that these give examples of unirational stably irrational threefolds over k. Motivated by Voisin's degeneration techniques [Voi15] (see also [Kol96, Thm. V.5.14]), the condition on the Hodge numbers has been studied by Totaro [Tot16], who considered varieties X obtained as desingularizations of hypersurfaces in positive characteristic with the property that $H^0(X, \Omega^{\dim X - 1}_X) \neq 0$; by degeneration to positive characteristic, he gives examples of rationally connected but stably irrational hypersurfaces in characteristic 0.

Returning now to the focus of this paper, our goal is to show that over algebraically closed fields of positive characteristic, there are examples of unirational but stably irrational threefolds satisfying (1)–(4), and (6) above, while failing obstruction (5); *i.e.*, examples of unirational threefolds with no universal codimension-2 cycle class. We in fact study this question more generally over an arbitrary perfect field.

The first issue is to make sense of conditions (3), (5), and (6) over a perfect field K, since the conditions are defined in terms of the intermediate Jacobian and the Abel–Jacobi map, which are inherently transcendental. We take two approaches, one that works in characteristic 0, and one that works in arbitrary characteristic. In the former case, where we may take $K \subseteq \mathbb{C}$, we have shown [ACMV20] that $J^3_a(X^{\mathrm{an}})$, the image of the Abel–Jacobi map $AJ : \mathrm{A}^2(X^{\mathrm{an}}) \to J^3(X^{\mathrm{an}})$ on algebraically trivial cycle classes, descends to a distinguished model $J^3_{a, X/K}$ over K such that the Abel–Jacobi map is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. It is easy to see that if X^{an} is a rationally connected threefold (or more generally, has universally trivial *rational* Chow group of zero cycles), in which case $J^3_a(X^{\mathrm{an}}) = J^3(X^{\mathrm{an}})$, then the canonical principal polarization $\Theta_{X^{\mathrm{an}}}$ (see Remark 7.8) descends to a principal polarization Θ_X on $J^3_{a, X/K}$ (Theorem 12.12; see also [BW20, Prop. 2.5]).

Without the assumption that $\mathrm{char}(K) = 0$, there are two replacements for the Abel–Jacobi map in condition (3) that both play a crucial role in our treatment. To motivate this, we recall from [Mur85, Thm. 10.3] that $T_\ell AJ : T_\ell \mathrm{A}^2(X) \to T_\ell J^3_a(X)$ is an isomorphism, and that the Abel–Jacobi map is surjective if and only if the ℓ-adic Abel–Jacobi map $T_\ell AJ : T_\ell \mathrm{A}^2(X) \to T_\ell J^3(X)$, or equivalently $T_\ell AJ : T_\ell \mathrm{A}^2(X) \to H^3(X, \mathbb{Z}_\ell)_\tau$, is an isomorphism, where the subscript τ indicates the torsion-free quotient. For technical reasons we find this formulation to be easier to work with in

positive characteristic, and so we actually consider two replacements for the ℓ-adic Abel–Jacobi map.

The first, which is defined on torsion cycles and takes values in the odd cohomology with torsion coefficients, is the *Bloch map* [Blo79]; taking Tate modules defines the *ℓ-adic Bloch map* $T_\ell \lambda^2 : T_\ell \operatorname{CH}^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau$, with values in ℓ-adic cohomology modulo torsion. This map, which is in fact defined for cycles of any codimension, was first considered by Suwa [Suw88] in the case $\ell \neq p$ and by Gros–Suwa [GS88] in the case $\ell = p$. We will focus on the restriction of this map to algebraically trivial cycle classes:

$$T_\ell \lambda^2 : T_\ell \operatorname{A}^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau. \tag{0.2}$$

We recently studied the map further in [ACMV21], and will rely on the definitions and results presented there.

The second replacement for the Abel–Jacobi map, which is defined on algebraically trivial cycle classes of codimension-2 and takes values in an abstract abelian variety, is the *second algebraic representative* [Mur85] (see §1):

$$\phi^2_{X_{\overline{K}}} : \operatorname{A}^2(X_{\overline{K}}) \longrightarrow \operatorname{Ab}^2_{X_{\overline{K}}/\overline{K}}(\overline{K}). \tag{0.3}$$

Building on work of Murre [Mur85] over an algebraically closed field, we showed in [ACMV17] that the algebraic representative $\operatorname{Ab}^2_{X_{\overline{K}}/\overline{K}}$ over $\overline{K}$ admits a distinguished model $\operatorname{Ab}^2_{X/K}$ over K, distinguished by the fact that the universal regular homomorphism (0.3) is $\operatorname{Gal}(K)$-equivariant (see §1). Note that if $K \subseteq \mathbb{C}$, then $\operatorname{Ab}^2_{X/K} = J^3_{a,X/K}$; *i.e.*, the algebraic representative agrees with the distinguished model of the algebraic intermediate Jacobian [ACMV17]. Taking Tate modules in (0.3) yields a map

$$T_\ell \phi^2_{X_{\overline{K}}} : T_\ell \operatorname{A}^2(X_{\overline{K}}) \longrightarrow T_\ell \operatorname{Ab}^2_{X_{\overline{K}}/\overline{K}}. \tag{0.4}$$

While in characteristic zero both maps (0.2) and (0.4) identify canonically with the ℓ-adic Abel–Jacobi map ([Blo79, Prop. 3.7], [ACMV21]), it is not known if they agree in positive characteristic (*i.e.*, after making some identification of the cohomology of the abelian variety $\operatorname{Ab}^2_{X/K}$ with that of X). However, in parallel with the characteristic 0 case, for a smooth projective geometrically rationally chain connected variety X the maps (0.2) and (0.4) are known to be isomorphisms [BW20, Prop. 2.3]. We also point out here that in characteristic 0, condition (3) is simply equivalent to $H^3(X, \mathbb{Q})$ being supported on a divisor.

In positive characteristic, the replacement for condition (5) is given by the notion of a *universal codimension-2 cycle class* for X, which is a cycle class $Z \in \operatorname{CH}^2(\operatorname{Ab}^2_{X/K} \times_K X)$, viewed as a family of cycles on X parameterized by $\operatorname{Ab}^2_{X/K}$, which is fiberwise algebraically trivial, such that the composition

$$\psi_Z : \operatorname{Ab}^2_{X/K}(\overline{K}) \longrightarrow \operatorname{A}^2(X_{\overline{K}}) \longrightarrow \operatorname{Ab}^2_{X/K}(\overline{K})$$

$$t \longmapsto Z_t \longmapsto \phi^2_{X_{\overline{K}}}(Z_t)$$

is the identity.

Finally, we turn to condition (6); *i.e.*, to a replacement for the principal polarization Θ_X. Recall that in characteristic 0 we have $\operatorname{Ab}^2_{X/K} = J^3_{a,X/K}$, so that for a projective geometrically rationally connected threefold over a field of characteristic 0, the algebraic representative $\operatorname{Ab}^2_{X/K}$ comes equipped with the principal polarization obtained via intersection in cohomology, as explained above. While in positive characteristic we do not have a way to define a distinguished (principal) polarization on $\operatorname{Ab}^2_{X/K}$ for every rationally chain connected threefold X, we can do

something similar for a class of rationally chain connected threefolds that includes geometrically stably rational threefolds:

Theorem 1 (Auto-duality of the algebraic representative). *Let X be a smooth projective threefold over a perfect field K.*

(1) *If X is geometrically rationally chain connected, then there is a canonical purely inseparable symmetric K-isogeny*

$$\Theta_X : \mathrm{Ab}^2_{X/K} \longrightarrow \widehat{\mathrm{Ab}}^2_{X/K} . \tag{0.5}$$

(2) *If $V_\ell\lambda^2$ is an isomorphism for some prime $\ell \neq \mathrm{char}(K)$ (e.g., X is geometrically uniruled), and $X_{\overline{K}}$ admits a universal codimension-2 cycle class Z, then the homomorphism of abelian varieties induced by the cycle class $-({}^tZ \circ Z) \in \mathrm{CH}^1(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \times_{\overline{K}} \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}})$ descends to K to give a canonical symmetric K-isogeny*

$$\Theta_X : \mathrm{Ab}^2_{X/K} \longrightarrow \widehat{\mathrm{Ab}}^2_{X/K} , \tag{0.6}$$

which is independent of the choice of universal cycle class Z. If moreover X is geometrically rationally chain connected, then (0.6) agrees with (0.5).

(3) *If X is geometrically stably rational, then $X_{\overline{K}}$ admits a universal codimension-2 cycle class Z, and the purely inseparable symmetric K-isogeny Θ_X is an isomorphism.*

Theorem 1 is proven in Theorem 4.4 and Theorem 12.12. The idea of the proof is as follows. One chooses a miniversal codimension-2 cycle class $Z \in \mathrm{CH}^2(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ of degree N for some natural number N, *i.e.*, $\psi_Z : \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \to \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}$ is multiplication by N; such cycles are known to exist for any surjective regular homomorphism. The cycle class ${}^tZ \circ Z \in \mathrm{CH}^1(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \times_{\overline{K}} \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}})$ then defines, via the theory of Picard schemes, a symmetric homomorphism $\Lambda_Z : (\mathrm{Ab}^2_{X/K})_{\overline{K}} \to (\widehat{\mathrm{Ab}}^2_{X/K})_{\overline{K}}$. In Theorem 12.12 we show that Λ_Z is surjective, descends to K, and is independent of the choice of a miniversal cycle class of degree N (although it will depend on N). We then show that Λ_Z is divisible by N^2, giving a symmetric isogeny Λ_X, which, when X is geometrically rationally chain connected, we show is an isomorphism on Tate modules for all primes l, and is therefore a purely inseparable isogeny. For reasons having to do with positivity, we take $\Theta_X = -\Lambda_X$ in Theorem 1. When X is assumed to be geometrically stably rational, we give a different proof of these facts, and obtain the stronger result, that Λ_X is an isomorphism. More precisely, in Proposition 4.3, we show that there is a universal codimension-2 cycle class Z over $\overline{K}$. As before, the cycle class ${}^tZ \circ Z$ then defines a symmetric homomorphism $\Lambda_{X_{\overline{K}}} : (\mathrm{Ab}^2_{X/K})_{\overline{K}} \to (\widehat{\mathrm{Ab}}^2_{X/K})_{\overline{K}}$, which we show via a similar argument is surjective, descends to K, and is independent of the choice of a universal cycle class, giving a symmetric isogeny Λ_X. However, to show that Λ_X is an isomorphism, for which it suffices to show that $\Lambda_{X_{\overline{K}}}$ is an isomorphism, we use a crucial new ingredient: we show that for a stably rational threefold over an algebraically closed field, the second algebraic representative is induced by a cycle class (the meaning of this is made precise in §1.5 and Proposition 4.3).

Recall that a symmetric isogeny from an abelian variety over a field K to its dual is, after base change to the algebraic closure $\overline{K}$, induced by a symmetric line bundle, and is called a polarization if the line bundle is ample (see §A.2 for a review). We note that if $\mathrm{char}(K) = 0$, then Θ_X in Theorem 1 is the Hodge-theoretic principal polarization induced via the intersection product in cohomology, as described above. In fact, in positive characteristic as well, Θ_X is induced by the intersection product in the middle cohomology of X; the meaning of this is made precise in Definition 12.1. When X is a geometrically *rational* threefold, Benoist–Wittenberg [BW20,

Cor. 2.8] recently constructed a *principal polarization* Θ_X on $\mathrm{Ab}^2_{X/K}$, which agrees with Θ_X of Theorem 1; in fact, extending the result of Clemens–Griffiths, they show that if X is rational *over K* then $(\mathrm{Ab}^2_{X/K}, \Theta_X)$ is the product of principally polarized Jacobians of curves. For some rationally chain connected threefolds we can show that Θ_X is a polarization, so that Theorem 1 in fact provides a partial answer to a question of [BW20, p.6] (see §13.2 and Corollary 13.3).

While in the introduction we have so far discussed the results in the context of rationality, the results, as well as the techniques, are in fact most naturally explained in terms of decomposition of the diagonal (see §2). The basic implications we use are that stably rational (resp. rationally chain connected) implies universally trivial integral (resp. rational) Chow group of zero cycles, which implies strict integral (resp. rational) Chow decomposition of the diagonal (see Remarks 2.7 and 2.8). In order to study the question of universal codimension-2 cycle classes, however, we must consider the yet weaker notion of *cohomological* decomposition of the diagonal (see §6).

We can now state the following theorem regarding strict cohomological $\mathbb{Z}$-decomposition of the diagonal with respect to $H^\bullet(-, \mathbb{Z}_\ell)$ (see Definition 6.2), which generalizes [Voi15] to algebraically closed fields of positive characteristic. Recall that Voisin has shown that a smooth complex projective threefold admits a strict cohomological $\mathbb{Z}$-decomposition of the diagonal with respect to $H^\bullet(-, \mathbb{Z})$ if and only if conditions (1)–(6) above hold ([Voi15, Thm. 1.7] and [Voi17, Thm. 4.1]). (Note that it is assumed in [Voi17, Thm. 4.1] that the threefold be rationally connected, but this is used only to ensure conditions (1)–(3) hold, which we have explained above hold for any complex projective rationally connected threefold.) We give necessary and sufficient conditions over an algebraically closed field for the existence of a strict cohomological $\mathbb{Z}_\ell$-decomposition with respect to $H^\bullet(-, \mathbb{Z}_\ell)$ in Theorem 15.5, however here, we prefer to mention our result on strict cohomological $\mathbb{Z}$-decompositions:

Theorem 2 (Cohomological decomposition of the diagonal). *Let X be a smooth projective threefold over an algebraically closed field k, and fix a prime number $\ell \neq \mathrm{char}(k)$. If $\Delta_X \in \mathrm{CH}^3(X \times_k X)$ admits a strict cohomological $\mathbb{Z}$-decomposition with respect to $H^\bullet(-, \mathbb{Z}_\ell)$, then:*

   (1) *$H^1(X, \mathbb{Z}_\ell) = 0$;*
   (2) *$H^{2i}(X, \mathbb{Z}_\ell(i))$ is $\mathbb{Z}$-algebraic for all i;*
   (3) *The ℓ-adic Bloch map $T_\ell \lambda^2 : T_\ell \, \mathrm{A}^2(X) \to H^3(X, \mathbb{Z}_\ell(2))_\tau$ is an isomorphism;*
   (3') *The ℓ-adic map $T_\ell \phi^2_{X/k} : T_\ell \, \mathrm{A}^2(X) \to T_\ell \, \mathrm{Ab}^2_{X/k}$ is an isomorphism;*
   (4) *$\mathrm{Tors}\, H^\bullet(X, \mathbb{Z}_\ell) = 0$;*
   (5) *$\mathrm{Ab}^2_{X/k}$ admits a universal codimension-2 cycle class;*
   (6) *Assuming (3) and (5), and setting $\Theta_X : \mathrm{Ab}^2_{X/k} \to \widehat{\mathrm{Ab}}^2_{X/k}$ to be the symmetric isogeny of Theorem 1(2), we have that $T_\ell \Theta_X$ is an isomorphism, and $\frac{[\Theta_X]^{g-1}}{(g-1)!} \in H^{2g-2}(\mathrm{Ab}^2_{X/k}, \mathbb{Z}_\ell(g-1))$ is a $\mathbb{Z}$-algebraic class, where $g = \dim \mathrm{Ab}^2_{X/k}$ and $[\Theta_X]$ is the first Chern class of the line bundle associated to Θ_X.*

As a partial converse, if (1)–(6) (including (3')) hold, then $\Delta_X \in \mathrm{CH}^3(X \times_k X)$ admits a strict cohomological $\mathbb{Z}_\ell$-decomposition with respect to $H^\bullet(-, \mathbb{Z}_\ell)$.

Theorem 2 is proven in Theorem 15.1, which is in fact stronger, addressing for instance the case of perfect fields, as well as cohomological decompositions of the diagonal supported on curves, rather than points (see Remark 15.2). We emphasize that we have omitted an assertion about the vanishing of Hodge numbers. In characteristic 0, a strict cohomological $\mathbb{Q}$-decomposition of the diagonal implies that $H^0(X, \Omega^i_X) = 0$ for $i > 0$ (see *e.g.* [Voi13, Thm. 4.4(iii)]); however, in positive characteristic, we only know this holds under the stronger assumption of a strict *Chow* $\mathbb{Z}$-decomposition of the diagonal (see *e.g.*, [Tot16, Lem. 2.2], and also Remark 7.8). In addition,

while conditions (1)–(6) in Theorem 2 are sufficient for the existence of a cohomological $\mathbb{Z}_\ell$-decomposition of the diagonal, they are not necessary. This will follow from our Theorem 3 below (and [Voi15] over $\mathbb{C}$), which establishes that the standard desingularization of a very general quartic double solid with exactly 7 nodes does not admit a universal codimension-2 cycle class (*i.e.*, (5) fails); on the other hand, it is well-known that twice the class of the diagonal admits a strict Chow decomposition, and consequently the diagonal admits a strict cohomological $\mathbb{Z}_\ell$-decomposition for all $\ell \neq 2, \operatorname{char}(k)$ (see Remarks 17.5 and 17.6). We reiterate that over an algebraically closed field, we give necessary and sufficient conditions for a cohomological $\mathbb{Z}_\ell$-decomposition of the diagonal in Theorem 15.5.

Also, while we know that (1)–(3′) hold for all geometrically rationally connected threefolds in characteristic 0, for geometrically rationally chain connected threefolds in positive characteristic we only know that (1) holds (Corollary 7.7), and that (3) and (3′) hold [BW20, Prop. 2.3]. In fact, we expect (3′) may hold for all smooth projective varieties over any field, and for this reason have separated it from condition (3), although both (3) and (3′) replace the condition (3) in the complex setting, namely the surjectivity of the Abel–Jacobi map. In other words, in positive characteristic, (1), (3), and (3′) should still be viewed as an obstruction to a threefold being rationally chain connected, while in contrast to the characteristic zero case, (2) could potentially be an obstruction to the stable rationality of a rationally chain connected threefold. However, in Corollary 15.4 we show that if a geometrically rationally chain connected threefold lifts to characteristic 0 to a geometrically rationally connected threefold with no torsion in cohomology, then conditions (1)–(3′) hold (as well as condition (4)).

Regarding the proof of Theorem 2, under the assumption of the cohomological decomposition of the diagonal, (1), (2), and (4) are now standard in the literature: they follow from the techniques in [BS83] and [Voi15] (we recall the proof in our setting in §7.2, §7.3, and §7.1, respectively). In short, under the assumption of the cohomological decomposition of the diagonal, the main focus is on the conditions (3) and (3′), (5), and (6). Conditions (3) and (3′) were investigated recently in [BW20, Prop. 2.3] in the context of a *Chow* decomposition of the diagonal; in that setting (3′) is essentially a consequence of [BS83, Thm. 1(i)] and (3) is proven similarly. The key addition here, in the context of cohomological decompositions, is that we show that morphisms induced by families of cycle classes via universal regular homomorphisms depend only on the cohomology class of the family of cycles (see Proposition 8.1 and Corollary 8.2). As Abel–Jacobi maps enjoy this property, we view this as a significant improvement on the theory of algebraic representatives in positive characteristic. This addition also allows us to establish (5), which is an extension of a result of Voisin [Voi13, Thm. 4.4(iii)] [Voi17, Thm. 4.2] to the case of finite and algebraically closed fields (see Corollary 9.3).

Condition (6) follows Voisin's arguments in Hodge theory, as well as Mboro's work on cubic threefolds for $\operatorname{char}(k) \neq 2$, but we note that there are several significant additions needed in our work. First and foremost, one needs Theorem 1 to provide a replacement for the principal polarization, which in Voisin's case comes from Hodge theory, and in Mboro's case comes from the theory of Prym varieties and fibrations in quadrics, which rules out the case $\operatorname{char}(k) = 2$. Note that Θ_X in Theorem 1(1) and (2) (and therefore in Theorem 2) is not known to be a polarization, or even an isomorphism. This is an important point in the sense that, unlike the cases considered by Voisin and Mboro, one does not automatically have condition (6) when $\dim \operatorname{Ab}^2_{X/k} \leq 3$. We will see this subtle point come into play later. The second key addition is Proposition 11.6, which is a technical point relating regular homomorphisms and actions of correspondences, which generalizes a classical result regarding Abel–Jacobi maps (see [Voi07a, Thm. 12.17]), and plays a central role in Voisin's Hodge-theoretic arguments in [Voi13, Voi17]. These techniques are used also in [Mbo17], and for instance, Proposition 11.6 applied to cubic threefolds provides a proof of the

assertion [Mbo17, Lem. 3.3]. Along the way, we positively answer some cases of a conjecture of Gros–Suwa [GS88, Conj. III.4.1(iii)] (see Lemma 11.5).

In light of Theorem 2, we extend to positive characteristic Voisin's result that there exist unirational complex smooth projective varieties with no universal codimension-2 cycle class.

Theorem 3 (Quartic double solids). *Let k be an uncountable algebraically closed field with $\mathrm{char}(k) \neq 2$. Let $\widetilde{X}$ be the standard resolution of singularities of a very general quartic double solid X with exactly $n \leq 9$ nodes (and no other singularities). Then for $\ell = 2$:*

(A) *If $n \leq 6$, then (1)–(4) of Theorem 2 hold for $\widetilde{X}$, and one or both of (5) and (6) fail.*
(B) *If $7 \leq n \leq 9$, then (1)–(4), and (6), of Theorem 2 hold for $\widetilde{X}$, while (5) fails. In other words, $\widetilde{X}$ does not admit a universal codimension-2 cycle class.*

The precise notion of the meaning of a very general quartic double solid with $n \leq 9$ nodes is given in §17; essentially it is a quartic double solid obtained from a quartic surface with exactly n nodes, which corresponds to a very general point of the moduli of degree 4 polarized K3 surfaces with exactly n nodes, which we show is irreducible. Note that one can conclude from Theorems 2 and 3, that over an uncountable algebraically closed field k with $\mathrm{char}(k) \neq 2$, the standard resolution of singularities $\widetilde{X}$ of a very general quartic double solid X with at most 9 nodes is not stably rational.

Our proof of Theorem 3, which is given in §17.2, is similar to that in [Voi15], but involves several key additions. First, as mentioned above, for every $n \leq 9$ we show that in the moduli space of polarized K3 surfaces of degree 4, the discriminant locus corresponding to K3s with exactly $n \leq 9$ nodes is irreducible (Proposition 17.2), and that for a 10-nodal quartic K3, the nodes can be deformed independently (Lemma 17.1), so that the Artin–Mumford example is in the boundary of each of these components of the discriminant (Corollary 17.3). This result is slightly more general than what is proven in [Voi15] in characteristic 0 (see Remark 17.4), and for instance, even in characteristic 0, gives a clean statement of Theorem 3 for $n = 8, 9$ nodes (*cf.* [Voi15, p.210]). The point is that while the locus of n-nodal quartic surfaces in the *Hilbert scheme* of quartic surfaces with exactly $n = 6, 7, 8, 9$ nodes is known to be reducible (see [Voi15, Rem. 1.2]), and for $n = 6, 7$ Voisin picks out a distinguished component containing the Artin–Mumford example, the locus in the *moduli space of polarized K3 surfaces* is irreducible, and we are free to take very general points of of these irreducible components. Next we show that for $n \leq 9$ nodes, one can lift a nodal quartic surface, along with its nodes, to characteristic 0 (Lemma 17.1). From this, we can use specialization from characteristic 0 to show that conditions (1)–(4) hold (Corollary 15.4).

Having established that the Artin–Mumford examples are degenerations of our examples, the next step is to consider degenerations of decompositions of the diagonal. Since we must use singular quartic double solids, as well as their resolutions, this requires us to work with ℓ-adic *homological* decompositions of the diagonal. This is discussed in §16, where we show that existence of an ℓ-adic homological decomposition of the diagonal is stable under specialization from the very general fiber (Theorem 16.6), as well as under resolution of singularities of nodes (Proposition 16.3). From our degeneration to the Artin–Mumford example, we can then conclude that the standard resolution of singularities of the very general quartic double solid with at most 9 nodes does not admit a cohomological $\mathbb{Z}_2$-decomposition of the diagonal.

Therefore, from Theorem 2, we can conclude that condition (5) or (6) must fail. Turning now to condition (6), we assume that condition (5) holds, and let $\Theta_{\widetilde{X}}$ be the associated symmetric isogeny. We note that unlike the case of characteristic 0, where $\Theta_{\widetilde{X}}$ is known to be a principal polarization, in positive characteristic understanding the algebraicity of $[\Theta_{\widetilde{X}}]^{g-1}/(g-1)!$, even when $g \leq 3$, is more subtle. In addition, algebraic representatives need not be stable under specialization, so that even with a lift to characteristic 0, there is no guarantee that $\Theta_{\widetilde{X}}$ is a principal polarization. To get

around this issue, we first show (Corollary 13.3) that, due to the liftability of $\widetilde{X}$ to characteristic 0, the symmetric isogeny $\Theta_{\widetilde{X}}$ is a polarization (although not necessarily principal). From this it follows that the polarized abelian variety $(\mathrm{Ab}^2_{\widetilde{X}/k}, \Theta_{\widetilde{X}})$ admits an isogeny to a principally polarized abelian variety, which has the same dimension, namely $g = 10 - n$, and therefore, for dimension reasons, must be a Jacobian of a curve if $n = 7, 8, 9$. Consequently, provided $n = 7, 8, 9$, it follows that $[\Theta_{\widetilde{X}}]^{g-1}/(g-1)!$ is $\mathbb{Z}$-algebraic, being the pull back under the isogeny of the class of the Abel–Jacobi embedded curve in its Jacobian (see Proposition 14.3). Therefore, for $n = 7, 8, 9$, we must have had that condition (5) fails, since otherwise conditions (1)–(6) would hold and $\widetilde{X}$ would admit a cohomological $\mathbb{Z}_2$-decomposition of the diagonal, which we know is not the case.

We note that once one has established that the very general quartic double solid with at most 9 nodes degenerates to the Artin–Mumford example, then the conclusion regarding stable irrationality follows also from the degeneration and resolution of singularities results of [CTP16, Thm. 1.12], [HKT16, Prop. 8, Thm. 9], [Tot16, Thm. 2.3] (and for $\mathrm{char}(k) = 0$, from the degeneration results of [Voi15, Thm. 2.1], [NS19, Thm. 4.2.11], [KT19, Thm. 1]). The *irrationality* of desingularizations of *all* quartic double solids with exactly $n \leq 1$ nodes was established via the Clemens–Griffiths criterion in the case $n = 0$ (over $\mathbb{C}$) in [Voi88], and for $n = 1$ in [Bea77, Thm. 4.9]; recall that the Clemens–Griffiths criterion was extended to threefolds over an algebraically closed field in [Mur73, Thm. p.63] and [Bea77, Prop. 4.6], and to threefolds over arbitrary fields in [BW20, Thm. 2.7] and [BW23, Thm. C].

Outline. The paper is split into three parts. Part 1 focuses on applications of Chow decompositions of the diagonal to the second algebraic representative. There we start by reviewing the theory of algebraic representatives and fix the notation for decompositions of the diagonal, both of which will be used throughout the paper. We then proceed to prove Theorem 1 under the hypothesis of stable rationality, and draw some consequences. The main objective of Part 2 is the proof of Theorem 2. We proceed by first proving that the existence of a strict cohomological of the diagonal of a threefold implies conditions (1)–(6) of Theorem 2, and then conclude this part in §15 by establishing that, conversely, conditions (1)–(6) ensure the existence of a strict $\mathbb{Z}_\ell$-decomposition of the diagonal. (Where possible, we also study p-adic decompositions in positive characteristic p.) Along the way we complete the proof of Theorem 1 in §12. In Theorem 15.5 we give necessary and sufficient conditions for the existence of a strict cohomological $\mathbb{Z}_\ell$-decomposition of the diagonal. Finally, in Part 3 we prove Theorem 3.

0.1. **Conventions.** A *variety* over a field is a geometrically reduced separated scheme of finite type over that field. For a scheme X of finite type over a field K, we denote by $\mathrm{CH}^n(X)$ the Chow group of codimension-n cycle classes on X, and by $\mathrm{A}^n(X)$ the group of algebraically trivial cycle classes. Unless explicitly stated otherwise, $\mathcal{H}^\bullet$ denotes ℓ-adic cohomology with coefficient ring $R_{\mathcal{H}} = \mathbb{Z}_\ell$ for some $\ell \neq \mathrm{char}(K)$. For a smooth projective variety X over a field K, we denote the cycle class map by $[-] : \mathrm{CH}^n(X) \to \mathcal{H}^{2n}(X)$. For a commutative ring R, and a scheme X of finite type over a field K, we denote by $\mathrm{CH}^\bullet(X)_R := \mathrm{CH}^\bullet(X) \otimes_{\mathbb{Z}} R$ the Chow group with coefficients in R.

The symbol ℓ always denotes a rational prime (*i.e.*, a natural number that is a prime) invertible in the base field, while l is allowed to be *any* rational prime, including the characteristic of the base field.

If G is an abelian group, then G_τ denotes the quotient of G by its torsion subgroup, $G[\ell^\infty]$ denotes its ℓ-primary torsion and $G_{\mathbb{Q}}$ denotes $G \otimes_{\mathbb{Z}} \mathbb{Q}$.

Given a field K with algebraic closure $\overline{K}$ and separable closure denoted K^{sep}, together with an $\mathrm{Aut}(\overline{K}/K) = \mathrm{Gal}(K^{\mathrm{sep}}/K)$-module M, we denote $T_l M$ the Tate module $\varprojlim M \otimes_{\mathbb{Z}} \mathbb{Z}/l^n \mathbb{Z}$. As usual, we denote $\mathbb{Z}_\ell(1)$ the Tate module $\varprojlim \boldsymbol{\mu}_{\ell^n}$, where $\boldsymbol{\mu}_{\ell^n}$ is the group of ℓ^n-th roots of unity. Given a $\mathbb{Z}_\ell$-module M, we denote $M^\vee := \mathrm{Hom}(M, \mathbb{Z}_\ell)$ and $M(n) := M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)^{\otimes n}$ its n-th Tate twist, where for $n < 0$ we have $\mathbb{Z}_\ell(1)^{\otimes n} := (\mathbb{Z}_\ell(1)^\vee)^{\otimes -n}$.

If X is a smooth projective variety over a field K and if l is a prime, we will denote by

$$T_l \lambda^n : T_l \, \mathrm{CH}^n(X_{\overline{K}}) \to H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau$$

the l-adic Bloch map defined by Suwa [Suw88] in case l is invertible in K and by Gros–Suwa [GS88] otherwise; see also [ACMV21]. Abusing notation, we will also denote by $T_l \lambda^n : T_l \, \mathrm{A}^n(X_{\overline{K}}) \to H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau$ the restriction of the above map to $T_l \, \mathrm{A}^n(X_{\overline{K}})$.

For K of positive characteristic p, see [GS88, §I.3.1] for details on $H^j(X_{\overline{K}}, \mathbb{Z}_p)$. We let $\mathbb{W}(K)$ denote the ring of Witt vectors over K, and $\mathbb{B}(K)$ its field of fractions.

Let $\mathcal{H}^\bullet$ be a Weil cohomology theory. For X smooth projective and geometrically connected over a field K of pure dimension d, the intersection product $\mathcal{H}^k(X)_\tau \times \mathcal{H}^{2d-k}(X)_\tau \to \mathcal{H}^{2d}(X)$ provides a canonical identification

$$\mathcal{H}^i(X)_\tau(d) \overset{\cup}{=} \mathcal{H}^{2d-i}(X)_\tau^\vee.$$

Part 1. **Chow decomposition of the diagonal and algebraic representatives**

In this section we review the notion of an algebraic representative. In positive characteristic, this takes the role of the intermediate Jacobian.

1.1. **Galois-equivariant regular homomorphisms and algebraic representatives.** We start by reviewing the definition of a regular homomorphism and of an algebraic representative (*i.e.*, [Mur85, Def. 1.6.1] or [Sam60, 2.5]), as well as the notion of a Galois-equivariant algebraic representative ([ACMV17, Def. 4.2]).

Let X be a smooth projective variety over an algebraically closed field k and let n be a nonnegative integer. For a smooth separated scheme T of finite type over k, we denote

$$\mathscr{A}^n_{X/k}(T) := \{Z \in \mathrm{CH}^n(T \times_k X) \mid \forall t \in T(k), \text{ the Gysin fiber } Z_t \text{ is algebraically trivial}\}$$

and for all $Z \in \mathscr{A}^n_{X/K}(T)$ we denote by

$$w_Z : T(k) \to \mathrm{A}^n(X)$$

the map defined by $w_Z(t) = Z_t$. Given an abelian variety A/k, a *regular homomorphism* (in codimension n)

$$\phi : \mathrm{A}^n(X) \longrightarrow A(k)$$

is a homomorphism of groups such that for every $Z \in \mathscr{A}^n_{X/k}(T)$, the composition

$$T(k) \xrightarrow{w_Z} \mathrm{A}^n(X) \xrightarrow{\phi} A(k)$$

is induced by a morphism of varieties

$$\psi_Z : T \to A.$$

An *algebraic representative* (in codimension n) is a regular homomorphism

$$\phi^n_{X/k} : \mathrm{A}^n(X) \to \mathrm{Ab}^n_{X/k}(k)$$

that is initial among all regular homomorphisms (in codimension n); in particular if it exists then it is unique up to unique isomorphism. For $n = 1$, the algebraic representative is given by $(\mathrm{Pic}^0_{X/k})_{\mathrm{red}}$ together with the Abel–Jacobi map. For $n = d_X$, the algebraic representative is given by the Albanese variety and the Albanese map. For $n = 2$, it is a result of Murre [Mur85, Thm. A] that there exists an algebraic representative.

We now review the extension in [ACMV17] to the case of a smooth projective variety X over a perfect field K. Given an abelian variety A/K, we say that a regular homomorphism $\phi : \mathrm{A}^n(X_{\overline{K}}) \to A(\overline{K})$ is Galois-equivariant if it is equivariant with respect to the natural actions of $\mathrm{Gal}(K)$. We say an algebraic representative $\phi^n_{X_{\overline{K}}/\overline{K}} : \mathrm{A}^n(X_{\overline{K}}) \to \mathrm{Ab}^n_{X_{\overline{K}}/\overline{K}}(\overline{K})$ is Galois-equivariant if $\mathrm{Ab}^n_{X_{\overline{K}}/\overline{K}}$ descends to an abelian variety $\mathrm{Ab}^n_{X/K}$ defined over K in such a way that $\phi^n_{X_{\overline{K}}/\overline{K}}$ is a Galois-equivariant regular homomorphism. We show [ACMV17, Thm. 4.4] that if $X_{\overline{K}}$ admits an algebraic representative in codimension n, $(\mathrm{Ab}^n_{X_{\overline{K}}/\overline{K}}, \phi^n_{X_{\overline{K}}/\overline{K}})$, then $\mathrm{Ab}^n_{X_{\overline{K}}/\overline{K}}$ descends uniquely to an abelian variety, denoted $\mathrm{Ab}^n_{X/K}$, over K making $\phi^n_{X_{\overline{K}}/\overline{K}}$ Galois-equivariant. We also show that these are stable under Galois base change of field, as well as under algebraically closed base change of field.

Importantly for our purposes, we show that for any Galois-equivariant regular homomorphism $\phi : \mathrm{A}^n(X_{\overline{K}}) \to A(\overline{K})$, any smooth separated scheme T of finite type over K, and any cycle class $Z \in \mathrm{CH}^n(T \times_K X)$ such that for every $t \in T(\overline{K})$ the Gysin fiber Z_t is algebraically trivial, the induced map $\psi_{Z_{\overline{K}}} : T_{\overline{K}} \to A_{\overline{K}}$ descends to a morphism $\psi_Z : T \to A$ of K-schemes.

1.2. **The functorial approach.** In [ACMV23] we have translated the notion of a Galois-equivariant regular homomorphism into a functorial language, which greatly clarifies many of the arguments in [ACMV17], allowing us to extend some of those results, and also discuss algebraic representatives in families. As we believe this is the correct language to use going forward, we will use this notation in this paper. Here we briefly review the definition, referring the reader to [ACMV23] for details. Over a perfect field (which is the setting here), the functorial approach is entirely equivalent to the notion of a Galois-equivariant regular homomorphism, and the reader is free to simply interchange the notation throughout.

Fix a field K. We start by defining the category of spaces that provide parameter spaces for our cycles. Specifically, we define

$$\mathsf{Sm}/K$$

to be the category with *objects being smooth separated schemes of finite type over K*, and with *morphisms being morphisms of K-schemes*. Note that every morphism $t : T' \to T$ in Sm/K is lci in the sense of [Ful98, B.7.6] (see [Ful98, B.7.3]), so that there is a refined Gysin pull-back $t^!$ [Ful98, §6.6]. The *functor of codimension-n algebraically trivial cycle classes on X over K* is the contravariant functor

$$\mathscr{A}^n_{X/K} : \mathsf{Sm}/K \longrightarrow \mathsf{AbGp}$$

to the category of abelian groups AbGp given by families of algebraically trivial cycles on X/K. Precisely, given T in Sm/K, we take $\mathscr{A}^i_{X/K}(T)$ to be the group of cycle classes $Z \in \mathrm{CH}^i(T \times_K X)$ such that $Z_t \in \mathrm{CH}^i(X_{K^s})$ is algebraically trivial for some (equivalently, for any) separably closed point $t : \mathrm{Spec}\, K^s \to T$; see [ACMV23, §1.1]. The functor is defined on morphisms $t : T' \to T$ in Sm/K via the refined Gysin pullback $t^!$ for lci morphisms.

Let A/K be an abelian variety, viewed via Yoneda as the contravariant representable functor $\mathrm{Hom}(-, A) : \mathsf{Sm}/K \to \mathsf{AbGp}$. A *regular homomorphism in codimension n from $\mathscr{A}^n_{X/K}$ to A/K* is a natural transformation of functors

$$\Phi : \mathscr{A}^n_{X/K} \to A.$$

Here we parse the definition. Given T in Sm/K, we obtain $\Phi(T) : \mathscr{A}^n_{X/K}(T) \to A(T)$; in other words, given a cycle class $Z \in \mathscr{A}^n_{X/K}(T)$, *i.e.*, a family of algebraically trivial cycle classes on X parameterized by T, we obtain a K-morphism $\Phi(T)(Z) : T \to A$. The regular homomorphism Φ is said to be *surjective* if it is surjective on K^{sep}-points, *i.e.*, if

$$\phi := \Phi(K^{\mathrm{sep}}) : \mathscr{A}^n_{X/K}(K^{\mathrm{sep}}) = \mathrm{A}^n(X_{K^{\mathrm{sep}}}) \to A(K^{\mathrm{sep}})$$

is surjective. An *algebraic representative in codimension n* consists of an abelian variety $\mathrm{Ab}^n_{X/K}$ over K together with a natural transformation of functors

$$\Phi^n_{X/K} : \mathscr{A}^n_{X/K} \to \mathrm{Ab}^n_{X/K}$$

over Sm/K that is initial among all regular homomorphisms $\Phi : \mathscr{A}^n_{X/K} \to A$. An algebraic representative, if it exists, is a surjective regular homomorphism [ACMV23, Prop. 5.1] and it is unique up to unique isomorphism.

Remark 1.1 (Connection with Galois-equivariant regular homomorphisms). If K is perfect, then regular homomorphisms and algebraic representatives in this sense are equivalent to Galois-equivariant regular homomorphisms and algebraic representatives over $\overline{K}$ (see [ACMV23]). One translates the notation as follows. Given a regular homomorphism $\Phi : \mathscr{A}^n_{X/K} \to A$, then $\phi = \Phi(\overline{K}) : \mathscr{A}^n_{X/K}(\overline{K}) = \mathrm{A}^n(X_{\overline{K}}) \to A(\overline{K})$ is a Galois-equivariant regular homomorphism. Conversely, given a Galois-equivariant regular homomorphism $\phi : \mathrm{A}^n(X_{\overline{K}}) \to A(\overline{K})$, then we define a regular homomorphism $\Phi : \mathscr{A}^n_{X/K} \to A$ as follows. For T in Sm/K and $Z \in \mathscr{A}^n_{X/K}(T)$, it is shown in [ACMV23] that there is a morphism $\Phi(T)(Z) = \psi_Z : T \to A$ of K-schemes determined by the map

of $\overline{K}$-points given by $t \mapsto \phi(Z_t)$. The assignment on morphisms $T' \to T$ is made in the obvious way.

Remark 1.2. We have shown in [ACMV23, Thm. 1] that algebraic representatives satisfy base change and descent along separable field extensions. More precisely, for a smooth projective variety X over a field K and a (not necessarily algebraic) separable field extension Ω/K, an algebraic representative $\Phi^i_{X_\Omega} : \mathscr{A}^i_{X_\Omega/\Omega} \to \mathrm{Ab}^i_{X_\Omega/\Omega}$ exists if and only if an algebraic representative $\Phi^i_X : \mathscr{A}^i_{X/K} \to \mathrm{Ab}^i_{X/K}$ exists. If this is the case, we have in addition that there is a canonical isomorphism $\mathrm{Ab}^i_{X_\Omega/\Omega} \xrightarrow{\sim} (\mathrm{Ab}^i_{X/K})_\Omega$, and $\Phi^i_{X_\Omega}(\Omega) : \mathrm{A}^i(X_\Omega) \to \mathrm{Ab}^i_{X_\Omega/\Omega}(\Omega)$ is $\mathrm{Aut}(\Omega/K)$-equivariant, relative to the above identification. We will typically use this in the case where Ω/K is an extension of a perfect field K by an algebraically closed field Ω.

1.3. **Miniversal and universal cycles.** Let X be a smooth projective variety over a field K and let $\Phi : \mathscr{A}^n_{X/K} \to A$ be a regular homomorphism. A *miniversal cycle class* for Φ is a cycle $Z \in \mathscr{A}^n_{X/K}(A)$ such that the homomorphism $\psi_Z := \Phi(A)(Z) : A \to A$ is given by multiplication by r for some natural number r, which we call the *degree* of Z. A miniversal cycle class is called *universal* if $\psi_Z := \Phi(A)(Z) : A \to A$ is the identity morphism, *i.e.*, if it is miniversal of degree one. In the case where Φ is an algebraic representative for codimension-n cycles on X, we call a universal cycle class for Φ a universal cycle in codimension-n for X (or for $\mathrm{Ab}^2_{X/K}$).

If K is algebraically closed, it is a classical and crucial fact [Mur85, 1.6.2 & 1.6.3] that a miniversal cycle class exists if and only if Φ is surjective ; this also holds without any restrictions on the field K by [ACMV23, Lem. 4.7]. In particular, since an algebraic representative is always a surjective regular homomorphism [ACMV23, Prop. 5.1], it always admits a miniversal cycle class. However, the existence of a universal cycle class is restrictive: Voisin [Voi15] established that the standard desingularization of the very general complex double quartic solid with 7 nodes does not admit a universal cycle class in codimension 2. One of the main results of this paper, Theorem 3, consists in extending Voisin's result to the positive characteristic case.

Nonetheless, recall [ACMV23, §7.1] that if X is a smooth projective variety over a field K, then its first algebraic representative exists and it coincides with the reduced Picard scheme $(\mathrm{Pic}^0_X)_{\mathrm{red}}$; in addition if X possesses a zero-cycle of degree-1 (*e.g.* if K is finite or separably closed), then X admits a universal cycle class in codimension 1.

1.4. **Regular homomorphisms and torsion.** The following lemma is crucial.

Lemma 1.3 ([Bea83a, Prop. 11, Lem. p.259]). *Let A be an abelian variety over K. The map $A(K^{\mathrm{sep}}) \to \mathrm{A}_0(A_{K^{\mathrm{sep}}})$, $a \mapsto [a] - [0]$ is an isomorphism on torsion. In particular, for any integer $N > 1$, it sends N-torsion to N-torsion.* $\qquad\square$

It admits the following consequence, which will be used in the proofs of Theorem 4.4 and Proposition 8.1.

Lemma 1.4. *Let X be a smooth projective variety over a field K. If $Z \in \mathscr{A}^n_{X/K}(B)$ is a family of algebraically trivial cycles on X parameterized by an abelian variety B over K with $Z_0 = 0 \in \mathrm{A}^n(X)$, then $w_Z : B(K^{\mathrm{sep}}) \to \mathrm{A}^n(X_{K^{\mathrm{sep}}}), b \mapsto Z_b$ is a homomorphism on torsion. In particular, if $\Phi : \mathscr{A}^n_{X/K} \to A$ is a regular homomorphism, then for any prime l we have $T_l \psi_Z = T_l \Phi(K^{\mathrm{sep}}) \circ T_l w_Z : T_l B \to T_l A$.*

Proof. That $w_Z : B(K^{\mathrm{sep}}) \to \mathrm{A}^n(X_{K^{\mathrm{sep}}}), b \mapsto Z_b$ is a homomorphism on torsion follows simply from the fact that it factors through $B(K^{\mathrm{sep}}) \to \mathrm{A}_0(B_{K^{\mathrm{sep}}})$, $b \mapsto [b] - [0]$ and from Lemma 1.3.

Note that given any $Z \in \mathscr{A}^n_{X/K}(T)$ for any smooth variety T over K, we do have $\Phi(T)(Z) =: \psi_Z = \Phi \circ w_Z$, where $w_Z : T \to \mathscr{A}^n_{X/K}$ is seen as a natural transformation. The assertion about Tate modules when T is an abelian variety uses the above-established fact that in that case $w_Z(K^{\mathrm{sep}}) : B(K^{\mathrm{sep}}) \to \mathscr{A}^n_{X/K}(K^{\mathrm{sep}})$ is a homomorphism on torsion. $\qquad\square$

Proposition 1.5. *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$, and let $\Phi^n_{X/K} : \mathscr{A}^n_{X/K} \to \mathrm{Ab}^n_{X/K}$ be the algebraic representative of X for $n = 1, 2$ or $\dim X$. Then $(\phi^n_X)_{\mathrm{tors}} : \mathrm{A}^n(X_{\overline{K}})_{\mathrm{tors}} \to \mathrm{Ab}^n_{X/K}(\overline{K})_{\mathrm{tors}}$ is an isomorphism. In particular $T_\ell \phi^n_X : T_\ell \mathrm{A}^n(X_{\overline{K}}) \to T_\ell \mathrm{Ab}^n_{X/K}$ is an isomorphism for all primes ℓ.*

Proof. For $n = 1$ this is a result of Bloch and for $n = \dim X$ this is a result of Roitman (see *e.g.*, [ACMV21] for references). For $n = 2$, this is a direct result of [Mur85] over $\mathbb{C}$ and [ACMV23] that algebraic representatives are stable under base change of field from K to an algebraically closed field Ω containing K. $\square$

In contrast, if $\mathrm{char}(K) > 0$, it is not known whether $(\Phi^2_{X/K})_{\mathrm{tors}} : \mathrm{A}^2(X_{K^{\mathrm{sep}}})_{\mathrm{tors}} \to \mathrm{Ab}^2_{X/K}(K^{\mathrm{sep}})_{\mathrm{tors}}$ is injective. The following proposition shows however that $(\Phi^2_{X/K})_{\mathrm{tors}}$ is surjective. In order to deduce that $T_\ell \Phi^2_{X/K}$ is surjective, one needs some more assumptions:

Proposition 1.6. *Let X be a smooth projective variety over a perfect field K and let $\Phi : \mathscr{A}^n_{X/K} \to A$ be a surjective regular homomorphism. Denote $\phi := \Phi(\overline{K}) : \mathrm{A}^n(X_{\overline{K}}) \to A(\overline{K})$. Then*

(1) *$\phi_{\mathrm{tors}} : \mathrm{A}^n(X_{\overline{K}})_{\mathrm{tors}} \to A(\overline{K})_{\mathrm{tors}}$ is surjective.*
(2) *$\phi[l^\infty] : \mathrm{A}^n(X_{\overline{K}})[l^\infty] \to A(\overline{K})[l^\infty]$ is surjective for all primes l.*
(3) *$T_l \phi : T_l \mathrm{A}^n(X_{\overline{K}}) \to T_l A(\overline{K})$ has finite cokernel for all primes l, and if $\Phi_{\overline{K}}$ admits a miniversal cycle class of degree r coprime to l, then $T_l \phi$ is surjective.*

Proof. Parts (1) and (2) are [ACMV20, Lem. 3.2, Rem. 3.3]; part (3) follows immediately from the definition of miniversality, and the fact that all surjective regular homomorphisms admit a miniversal cycle class of some degree. $\square$

1.5. **Regular homomorphisms induced by cycle classes.** One of the difficulties in working with regular homomorphisms is that it is hard to construct non-trivial examples. In this subsection we explain (§1.5.1) that given any abelian variety A/K and any cycle class $\widehat{Z} \in \mathrm{CH}^{d_X + 1 - n}(X \times_K \widehat{A})$ there is an induced a regular homomorphism

$$\widehat{Z}_* : \mathscr{A}^n_{X/K} \longrightarrow A. \tag{1.1}$$

While there is no guarantee that such a regular homomorphism will be nonzero (any such regular homomorphism would be zero in the case $n = 1$ if $H^1(X, \mathcal{O}_X) = 0$, since in this case the algebraic representative $(\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$ is trivial), we nevertheless find this to be quite helpful in constructing regular homomorphisms. Conversely, regular homomorphisms induced by cycle classes enjoy properties that we exploit in §4.

1.5.1. Let X and Y be smooth projective varieties over a field K. Then any cycle class

$$Z \in \mathrm{CH}^{d_X + 1 - n}(X \times_K Y)$$

induces a regular homomorphism

$$\Phi_Z = Z_* : \mathscr{A}^n_{X/K} \xrightarrow{\ Z_*\ } \mathscr{A}^1_{Y/K} \xrightarrow{\ AJ\ } (\mathrm{Pic}^0_{Y/K})_{\mathrm{red}}, \tag{1.2}$$

where $AJ : \mathscr{A}^1_{Y/K} \to (\mathrm{Pic}^0_{Y/K})_{\mathrm{red}}$ is the Abel–Jacobi map to the first algebraic representative, and $Z_* : \mathscr{A}^n_{X/K} \to \mathscr{A}^1_{Y/K}$ is the canonical natural transformation induced by the correspondence Z. In other words, given any $\Gamma \in \mathscr{A}^n_{X/K}(T)$, we have $Z \circ \Gamma \in \mathrm{CH}^1(T \times_K Y) = \mathrm{A}^1(T \times_K Y)$, and the theory of regular homomorphisms (*i.e.*, the Abel–Jacobi map) provides a morphism $T \to (\mathrm{Pic}^0_{Y/K})_{\mathrm{red}}$. Note that on points, this sends a point t of T to the point corresponding to the line bundle associated to the divisor $(Z \circ \Gamma)_t = Z_*(\Gamma_t)$ on Y. We observe that our notation overloads the use of Z_*, since it has two meanings in (1.2); some motivation for this is that the Abel–Jacobi map is an

isomorphism when evaluated on $\overline{K}$-points, and the meaning of Z_* should always be clear from the context. For $Z \in \mathrm{CH}^{d_X+1-n}(Y \times_K X)$, we denote by Z^* the regular homomorphism $({}^t Z)_*$ induced by the transpose ${}^t Z$.

If we apply this construction to the case $Y = \widehat{A}$ for an abelian variety A/K (and switch notation $Z = \widehat{Z}$), we obtain the regular homomorphism (1.1).

Question 1.7. *Given a regular homomorphism* $\Phi : \mathscr{A}^n_{X/K} \to A$, *when is* Φ *induced by a cycle? In other words, when is there a cycle class* $\widehat{Z} \in \mathrm{CH}^{d_X+1-n}(X \times_K \widehat{A})$ *with* $\Phi = \widehat{Z}_*$?

If X admits a zero-cycle of degree 1, the question has a positive answer for $n = \dim X$:

Lemma 1.8. *Let* X *be a smooth projective variety over a field* K *admitting a zero-cycle of degree* 1. *The algebraic representative* $\Phi^{d_X}_{X/K} : \mathscr{A}^{d_X}_{X/K} \to \mathrm{Alb}_{X/K}$ *is induced by the universal codimension-1 cycle class* $Z \in \mathrm{CH}^1((\mathrm{Pic}^0_{X/K})_{\mathrm{red}} \times_K X)$; *i.e.,* $\Phi^{d_X}_{X/K} = Z^* := {}^t Z_*$.

Proof. This is explained in [ACMV23, Thm. 7.9(i) and Rem. 7.4]. $\qquad \square$

We will see in Proposition 4.3 that the question also has a positive answer for codimension-2 cycles on stably rational threefolds over an algebraically closed field; this is a crucial step towards establishing Theorem 1.

1.5.2. The main point of the following discussion is to prove Lemma 1.9, regarding regular homomorphisms induced by correspondences. We start by recalling some basic facts concerning the Picard scheme. Let X and Y be smooth projective varieties over a field K admitting K-points x_0 and y_0 respectively. Then there is a canonical isomorphism

$$\frac{\mathrm{CH}^1(X \times_K Y)}{p_X^* \mathrm{CH}^1(X) + p_Y^* \mathrm{CH}^1(Y)} = \mathrm{Hom}(\mathrm{Alb}_{X/K}, \mathrm{Pic}^0_{Y/K}) \ \ (= \mathrm{Hom}(\mathrm{Alb}_{Y/K}, \mathrm{Pic}^0_{X/K})), \qquad (1.3)$$

which is induced from the homomorphism sending a line-bundle $\mathcal{L}$ on $X \times_K Y$ with trivial restriction on $\{x_0\} \times_K Y$ to the unique homomorphism $\mathrm{Alb}_{X/K} \to \mathrm{Pic}^0_{Y/K}$ induced by the morphism $X \to \mathrm{Pic}^0_{Y/K}$ given on points by $x \mapsto \mathcal{L}|_{\{x\} \times_K Y}$. Note that the homomorphism $\mathrm{Alb}_{X/K} \to \mathrm{Pic}^0_{Y/K}$ above is the one induced from the regular homomorphism $\mathscr{A}^{d_X}_{X/K} \to \mathrm{Pic}^0_{Y/K}$ by the universal property of the albanese map, considered as an algebraic representative for zero-cycles on X. Note also that the second equality of (1.3) is simply given by taking the dual homomorphism. Finally, we note that $\mathrm{Hom}(\mathrm{Alb}_{X/K}, \mathrm{Pic}^0_{Y/K}) = \mathrm{Hom}(\mathrm{Alb}_{X/K}, (\mathrm{Pic}^0_{Y/K})_{\mathrm{red}})$, since $\mathrm{Alb}_{X/K}$, being an abelian variety, is reduced.

Now consider the case where $X = B$ is an abelian variety of dimension g. If

$$Z \in \mathrm{CH}^1(B \times_K Y),$$

then identifying $B = \mathrm{Alb}_{B/K}$, the homomorphism $B \to (\mathrm{Pic}^0_{Y/K})_{\mathrm{red}}$ induced by (1.3) coincides with the homomorphism $\Phi(B)(\Delta_B - (B \times \{0\})) : B \to (\mathrm{Pic}^0_{Y/K})_{\mathrm{red}}$ where Φ is the regular homomorphism

$$\Phi = Z_* : \mathscr{A}^g_{B/K} \to \mathscr{A}^1_{Y/K} \to (\mathrm{Pic}^0_Y)_{\mathrm{red}}$$

of (1.2) with $n = g$, and

$$\Delta_B \in \mathrm{CH}^g(B \times_K B)$$

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is the family of dimension-0 cycles given by the diagonal. Indeed, it is enough to check that these two homomorphisms agree on K^{sep}-points. We have a commutative diagram

$$
\begin{array}{ccccc}
B(K^{\text{sep}}) & \longrightarrow & A_0(B_{K^{\text{sep}}}) & \xrightarrow{Z_*} & A^1(Y_{K^{\text{sep}}}) \\
& \searrow{\scriptstyle\text{id}} & \downarrow{\scriptstyle\text{alb}} & & \downarrow{\scriptstyle AJ} \\
& & B(K^{\text{sep}}) & \longrightarrow & \text{Pic}^0_Y(K^{\text{sep}})
\end{array}
$$

where the left triangle commutes because $\text{alb}([x] - [0]) = x$ by definition of the albanese map, and where the bottom horizontal arrow is the homomorphism induced by the fact that alb is an algebraic representative for $\mathscr{A}^g_{B/K}$. The latter coincides, by construction, with the homomorphism induced by the canonical isomorphism (1.3).

A special case we will use often is:

Lemma 1.9. *Let X be a smooth projective variety over a field K, and let A and B be abelian varieties over K. Given correspondences $Z \in \text{CH}^n(B \times_K X)$ and $\widehat{Z} \in \text{CH}^{d_X+1-n}(X \times_K \widehat{A})$, the regular homomorphism*

$$
\Phi : \mathscr{A}^{d_B}_{B/K} \xrightarrow{Z_*} \mathscr{A}^n_{X/K} \xrightarrow{\widehat{Z}_*} A
$$

when evaluated at the abelian variety B and the cycle $\Delta_B - (B \times_K \{0\}) \in \mathscr{A}^{d_B}_{B/K}(B)$ gives a homomorphism

$$
\Phi(B)(\Delta_B - (B \times_K \{0\})) : B \longrightarrow A, \tag{1.4}
$$

which agrees with the homomorphism induced from the correspondence $\widehat{Z} \circ Z \in \text{CH}^1(B \times_K A)$ via (1.3), and which on $\overline{K}$ points factors as

$$
B(\overline{K}) \xrightarrow{w_Z} A^n(X_{\overline{K}}) \xrightarrow{\widehat{Z}_*} A(\overline{K}). \tag{1.5}
$$

Moreover, the dual homomorphism $\widehat{A} \to \widehat{B}$ to (1.4) is induced by ${}^tZ \circ {}^t\widehat{Z}$.

Proof. Everything except the assertion (1.5) follows from the discussion above. For (1.5), we simply note that from the definitions, there is a factorization

$$
\begin{array}{ccc}
B(\overline{K}) & \xrightarrow{w_Z} A^n(X_{\overline{K}}) \xrightarrow{\widehat{Z}_*} & A(\overline{K}). \\
& \searrow{\scriptstyle w_{\Delta_B - (B \times_K \{0\})}} \quad \uparrow{\scriptstyle Z_*} & \\
& A^{d_B}(B_{\overline{K}}) &
\end{array}
$$

$\square$

2. Preliminaries on Chow decomposition of the diagonal

In this section we recall the definition of a Chow decomposition of the diagonal, and the connection with universal CH_0-triviality. The purpose is primarily to fix terminology, since the terminology is somewhat fluid in the literature. In particular, we consider here decompositions of the diagonal that are slightly more general than those equivalent to universal CH_0-triviality, and we wish, as well, to keep track of the exact multiple of the diagonal that may admit a decomposition.

Recall that for a scheme X of finite type over a field K, we say that a cycle class $Z \in \text{CH}^n(X)$ is supported on a closed subscheme $W \subseteq X$ if it is in the kernel of the restriction map $\text{CH}^n(X) \to \text{CH}^n(X \smallsetminus W)$. Equivalently, from the exact sequence

$$
\text{CH}^n(W) \to \text{CH}^n(X) \to \text{CH}^n(X \smallsetminus W) \to 0, \tag{2.1}
$$

we can say that Z is *supported on W* if and only if it is the push forward of a cycle class on W.

Definition 2.1 (Chow decomposition of a cycle class). Let R be a commutative ring. Let X be a smooth projective variety over a field K, and let

$$j_i : W_i \overset{\neq}{\hookrightarrow} X , \quad i = 1, 2$$

be reduced closed subschemes not containing any component of X. An *R-decomposition of type* (W_1, W_2) *of a cycle class* $Z \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is an equality

$$Z = Z_1 + Z_2 \in \mathrm{CH}^{d_X}(X \times_K X)_R \tag{2.2}$$

where $Z_1 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is supported on $W_1 \times_K X$ and $Z_2 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is supported on $X \times_K W_2$. When $R = \mathbb{Z}$, we call this a *decomposition of type* (W_1, W_2).

We say that $Z \in \mathrm{CH}^{d_X}(X \times_K X)_R$ has an *R-decomposition of type* (d_1, d_2) if it admits an R-decomposition of type (W_1, W_2) with $\dim W_1 \le d_1$ and $\dim W_2 \le d_2$.

Remark 2.2. We note here for convenience that if X is smooth and projective, a decomposition of a multiple $N\Delta_X$ of the diagonal must have $d_1 + d_2 \ge d_X - 1$. Indeed, if $d_1 + d_2 < d_X - 1$, then a short argument using (3.4), below, would imply that $N \cdot \mathcal{H}^2(X) = 0$, giving a contradiction as the class of any ample line bundle on X is not torsion.

In what follows, let

$$\mathrm{pr}_i : X \times_K X \to X, \quad i = 1, 2$$

be the respective projection maps.

Example 2.3. For projective space $\mathbb{P}_K^r$, the diagonal class is $\Delta_{\mathbb{P}_K^r} = \sum_{i=0}^{r-1} \mathrm{pr}_1^*[H]^i \times \mathrm{pr}_2^*[H]^{r-i} \in \mathrm{CH}^r(\mathbb{P}_K^r \times_K \mathbb{P}_K^r)$, where $[H]$ is the class of a hyperplane in $\mathbb{P}_K^r$. Thus for any non-negative integers d_1, d_2 with $d_1 + d_2 = r - 1$, the cycle class $\Delta_{\mathbb{P}_K^r} \in \mathrm{CH}^r(\mathbb{P}_K^r \times_K \mathbb{P}_K^r)$ has a decomposition of type (W_1, W_2) with $W_i \subseteq \mathbb{P}_K^n$ a linear space of dimension d_i, $i = 1, 2$.

In many situations, we will want to specify a more restricted type of decomposition, which is common in the literature due to its connection with universal CH_0-triviality (see Remark 2.6 below).

Definition 2.4 (Strict decomposition of a cycle class). A *strict R-decomposition of a cycle class* $Z \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is an R-decomposition of type $(d_X - 1, 0)$. In other words, it is an equality $Z = Z_1 + Z_2 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ as in (2.2) where Z_1 is supported on $D \times_K X$ for some codimension-1 subvariety $D \subseteq X$ and Z_2 is supported on $X \times_K W_2$ for some 0-dimensional subvariety $W_2 \subseteq X$. When $R = \mathbb{Z}$, we call this a *strict decomposition*.

Remark 2.5 (Strict decomposition of the diagonal). If for some integer $N > 0$ we have $N\Delta_X = Z_1 + Z_2 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is a strict R-decomposition of N times the diagonal, then the image of the degree map $\mathrm{CH}_0(X)_R \to R$ contains NR and $Z_2 = \mathrm{pr}_2^* \alpha$ for any zero-cycle $\alpha \in \mathrm{CH}_0(X)$ of degree N. Indeed, by definition of a strict decomposition, we must have $Z_2 = \mathrm{pr}_2^* \alpha$ for some zero-cycle $\alpha \in \mathrm{CH}_0(X)_R$. Letting Δ_X act on zero-cycles on X, and since Z_1 acts trivially on zero-cycles on X, we find that for all $\beta \in \mathrm{CH}_0(X)_R$ we have $N\beta = (\Delta_X)_* \beta = (\mathrm{pr}_2^* \alpha)_* \beta = \deg(\beta)\alpha$. It follows that $\deg(\alpha) = N$ and that any zero-cycle of degree N is rationally equivalent to α. In particular, in the situation where $R = \mathbb{Z}$ and $X(K) \ne \emptyset$, if Δ_X admits a strict decomposition, then $Z_2 = X \times_K x$ for any K-point $x \in X(K)$.

Remark 2.6 (Universal CH_0-triviality). A proper variety X over a field K is said to be *universally* $(\mathrm{CH}_0)_R$-*trivial* if, for any field extension L/K, the degree map $\mathrm{CH}_0(X_L)_R \to R$ is an isomorphism. When $R = \mathbb{Z}$, we simply say *universally* CH_0-*trivial*. It follows classically from [BS83] that a

smooth proper variety X over a field K is universally $(\mathrm{CH}_0)_R$-trivial if and only if the class of the diagonal admits a strict R-decomposition.

Remark 2.7 (Stably rational varieties and decomposition of the diagonal). It is well-known (see, *e.g.*, [Ful98, Ex. 16.1.11]) that universal CH_0-triviality is a stable birational invariant for smooth proper varieties. Thus a stably rational proper variety is universally CH_0-trivial. As a consequence, for a stably rational smooth projective variety X, we have that $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict decomposition.

Remark 2.8 (Rationally chain connected varieties and decomposition of the diagonal). Let X/K be a smooth projective rationally chain connected variety over a field K. From say [Kol96, Thm. IV.3.13], we have that X_Ω is CH_0-trivial for every algebraically closed field Ω/K, and therefore that X is universally $(\mathrm{CH}_0)_\mathbb{Q}$-trivial. It follows that some nonzero integer multiple of the diagonal $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict decomposition.

Remark 2.9 (Unirational varieties and decomposition of the diagonal). Let X be a smooth projective unirational variety over a field K. As X is rationally chain connected, it follows that some nonzero integer multiple of the diagonal $N\Delta_X \in \mathrm{CH}^{\dim X}(X \times_K X)$ admits a strict decomposition. In fact, if $\mathbb{P}^n \dashrightarrow X$ is a dominant rational map of degree N, then $\mathrm{CH}_0(X)_{\mathbb{Z}[1/N]}$ is universally trivial, so that $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict $\mathbb{Z}[\frac{1}{N}]$-decomposition (Remark 2.6). However, if either $\mathrm{char}(K) = 0$, or K is perfect and $d_X \leq 3$, then we obtain the stronger result that $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict decomposition; this is well-known, and we direct the reader to the *proof* of [Mbo17, Prop. 2.2].

Remark 2.10 (Uniruled varieties and decomposition of the diagonal). Let X be a smooth projective geometrically uniruled variety over a field K. It is clear that $\mathrm{CH}_0(X)_\mathbb{Q}$ is universally supported on a subvariety W_2 of dimension $d_X - 1$, in the sense that the push-forward map $\mathrm{CH}_0((W_2)_L)_\mathbb{Q} \to \mathrm{CH}_0(X_L)_\mathbb{Q}$ is surjective for all field extensions L/K. By [BS83], we see that $\Delta_X \in \mathrm{CH}^{\dim X}(X \times_K X)_\mathbb{Q}$ admits a $\mathbb{Q}$-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_X - 1$. Clearing denominators, there is some nonzero integer multiple $N\Delta_X \in \mathrm{CH}^{\dim X}(X \times_K X)$ that admits a decomposition of type (W_1, W_2).

3. FACTORING CORRESPONDENCES WITH GIVEN SUPPORT

A key tool we will use in what follows is the fact that a decomposition of the diagonal, viewed from the perspective of correspondences, gives a factorization of the identity into maps involving lower dimensional varieties. Here we consider the situation in the various cohomology theories. The key point is that correspondences require intersection theory, and therefore we prefer to work on smooth spaces.

3.1. **A factorization lemma for correspondences.** Let X be a scheme of finite type over a field K. We say that X has dimension $\leq d$ if all irreducible components of X have dimension $\leq d$, while we say X has pure dimension d if all irreducible components of X have dimension d.

Lemma 3.1. *Let X and Y be connected smooth proper varieties over a field K of characteristic exponent p, and let $j : W \hookrightarrow X$ be a closed subscheme of dimension $\leq n$. If $Z \in \mathrm{CH}_c(X \times_K Y)$ is a cycle class of dimension c supported on $W \times_K Y$, then Z, seen as a correspondence from X to Y with $\mathbb{Z}[\frac{1}{p}]$-coefficients, factors through a scheme $\widetilde{W}$ which is smooth proper and of finite type over a finite purely inseparable extension of K and of pure dimension n. More precisely, there exist a nonnegative integer e and correspondences $\widetilde{Z} \in \mathrm{CH}_c(\widetilde{W} \times_K Y)$ and $\tilde{\gamma} \in \mathrm{CH}^{\dim X}(X \times_K \widetilde{W})$ such that*

$$p^e Z = \widetilde{Z} \circ \tilde{\gamma} \quad \text{in } \mathrm{CH}_c(X \times_K Y).$$

In addition, assuming K is perfect and that resolution of singularities holds in dimensions $\leq n$ over K, the correspondence Z factors as above with $e = 0$ and with $\widetilde{W}$ smooth proper of finite type over K of pure dimension n.

Proof. It is clearly sufficient to establish the lemma for a cycle class Z that is the class of an integral closed subscheme of $X \times_K Y$ that, by abuse of notation, we still denote Z. Replacing W with the (scheme-theoretic) image of Z inside X via the first projection, we may assume that W is integral and that Z dominates W. Let us consider an alteration $\tau : W' \to W$ (*i.e.,* τ proper and generically finite) of W with W' smooth and proper over a finite purely inseparable extension L of K. By [Tem17, Thm. 4.3.1] we may in fact choose τ of degree p^e for some nonnegative integer e, while we can choose τ to be of degree 1 if K is perfect and resolution of singularities holds in dimensions $\leq n$ (which is the case for $n = 3$ in positive characteristic [CP09, Thm. p.1893] and for any n in zero characteristic). Since the push-pull along the field extension L/K is multiplication by $[L : K]$, which in our case is a power of p, we may by pulling back Z to $X_L \times_L Y_L = (X \times_K Y)_L$ assume that $L = K$. We now define $\tilde{j} := j \circ \tau \circ p_{W'} : \widetilde{W} \to X$, where $p_{W'} : \widetilde{W} := \mathbb{P}_L^{n-\dim W} \times_L W' \to W'$ is the natural projection, and we set $\tilde{\gamma}$ to be the transpose of the class of the graph of $\tilde{j}$. We claim that there is a cycle class $\widetilde{Z} \in \mathrm{CH}_c(\widetilde{W} \times_K Y)$ such that

$$(\tilde{j} \times \mathrm{Id}_Y)_* \widetilde{Z} = p^e \, Z. \tag{3.1}$$

Together with the identity $(\tilde{j} \times \mathrm{Id}_Y)_* \widetilde{Z} = \widetilde{Z} \circ \tilde{\gamma}$ (see [Ful98, 16.1.1]), we obtain the sought-after cycles $\tilde{\gamma}$ and $\widetilde{Z}$. To establish the claim, consider the fibered product diagram

$$
\begin{array}{ccc}
W' \times_K Y & \xrightarrow{\tau \times \mathrm{Id}_Y} & W \times_K Y \\
\uparrow & & \uparrow \\
W'^\circ \times_K Y & \xrightarrow{\tau \times \mathrm{Id}_Y} & W^\circ \times_K Y
\end{array}
$$

where W° is the smooth locus of W, and W'° is the pre-image of W^0 under τ. Since W°, W'°, and W' are all smooth, all of the morphisms in the diagram admit lci factorizations in the sense of Fulton (see [Ful98, Note, p.439]); thus all the morphisms admit refined Gysin pull-backs. We further restrict W° (and W'°) so that $W'^\circ \to W^\circ$ is finite and flat (it is generically finite and generically étale). With this set-up, we set Z' to be the closure in $W' \times_K Y$ of the flat pull back of Z along the composition $W'^\circ \times_K Y \to W^\circ \times_K Y \hookrightarrow W \times_K Y$, where Z is considered as a cycle on $W \times_K Y$; the assertion $(\tilde{j} \times \mathrm{Id}_Y)_* Z' = p^e Z$ then follows from functoriality of pull-back, and the fact that the push-forward of a pull-back along a finite flat morphism is multiplication by the degree. Indeed, we are assuming that Z dominates W. Since $W'^\circ \times_K Y \to W^\circ \times_K Y \hookrightarrow W \times_K Y$ is a composition of flat morphisms, we may use flat pull-back. Taking the closure in $W' \times_K Y$ we obtain a class in $W' \times_K Y$. Now by definition of the push-forward, and the fact that Z dominates W, we can compute this on $W'^\circ \times_K Y \to W^\circ \times_K Y$ giving the result.

Finally, we define $\widetilde{Z} := \{0\} \times Z' \in \mathrm{CH}_c((\mathbb{P}_L^{n-\dim W} \times_L W') \times_K Y)$ and we clearly have $(p_{W'} \times \mathrm{Id}_Y)_* \widetilde{Z} = Z'$ and hence $(\tilde{j} \times \mathrm{Id}_Y)_* \widetilde{Z} = p^e Z$. $\qquad\square$

3.2. **Factorizations of morphisms induced by correspondences.** With the set-up of Lemma 3.1 and its proof, we can factor the action of the correspondence $p^e Z$ in various settings as follows.

3.2.1. *Chow groups.* The correspondence $p^e Z^* : \mathrm{CH}^n(X) \to \mathrm{CH}^n(X)$, $\alpha \mapsto p^e (\mathrm{pr}_1)_*(\mathrm{pr}_2^* \alpha \cdot Z)$, and the correspondence $p^e Z_* : \mathrm{CH}^n(X) \to \mathrm{CH}^n(X)$, $\alpha \mapsto p^e(\mathrm{pr}_2)_*(\mathrm{pr}_1^* \alpha \cdot Z)$, factor, respectively, as:

$$
\begin{array}{cc}
\begin{array}{c}
\mathrm{CH}^{d_{\widetilde{W}} - d_X + n}(\widetilde{W}) \\
\nearrow_{\widetilde{Z}^*} \qquad \searrow_{\tilde{j}_*} \\
\mathrm{CH}^n(X) \xrightarrow{\;p^e Z^*\;} \mathrm{CH}^n(X)
\end{array}
&
\begin{array}{c}
\mathrm{CH}^n(X) \xrightarrow{\;p^e Z_*\;} \mathrm{CH}^n(X) \\
\searrow_{\widetilde{Z}_*} \qquad \nearrow_{\tilde{j}^*} \\
\mathrm{CH}^n(\widetilde{W})
\end{array}
\end{array}
\qquad (3.2)
$$

3.2.2. *Algebraic representatives.* We obtain factorizations similar to (3.2) if we consider algebraically trivial cycle classes. This induces diagrams of algebraic representatives, if they exist:
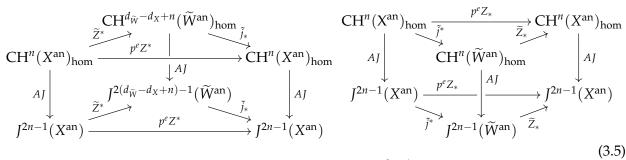

$$(3.3)$$

where the dashed arrows are induced by the universal property of the algebraic representative.

3.2.3. *Cohomology groups.* In the same situation at the start of §3, fix a Weil cohomology theory $\mathcal{H}^\bullet$ with coefficient ring $R_{\mathcal{H}}$, and a ring homomorphism $R \to R_{\mathcal{H}}$. Then the correspondences $p^e Z^* : \mathcal{H}^n(X) \to \mathcal{H}^n(X)$ and $p^e Z_* : \mathcal{H}^n(X) \to \mathcal{H}^n(X)$ factor, respectively, as:

$$
\begin{array}{cc}
\begin{array}{c}
\mathcal{H}^{2(d_{\widetilde{W}} - d_X) + n}(\widetilde{W})(d_{\widetilde{W}} - d_X) \\
\nearrow_{\widetilde{Z}^*} \qquad \searrow_{\tilde{j}_*} \\
\mathcal{H}^n(X) \xrightarrow{\;p^e Z^*\;} \mathcal{H}^n(X)
\end{array}
&
\begin{array}{c}
\mathcal{H}^n(X) \xrightarrow{\;p^e Z_*\;} \mathcal{H}^n(X) \\
\searrow_{\tilde{j}^*} \qquad \nearrow_{\widetilde{Z}_*} \\
\mathcal{H}^n(\widetilde{W})
\end{array}
\end{array}
$$

$$(3.4)$$

Note that in the case where n is even, these diagrams are compatible with the cycle class maps and the diagrams (3.2).

3.2.4. *Abel–Jacobi maps.* Consider now the case where $K \subseteq \mathbb{C}$. Since correspondences induce morphisms of integral Hodge structures, we obtain factorizations similar to (3.3). Here $\mathrm{CH}^*(-)_{\mathrm{hom}}$ denotes the kernel of the cycle class map $\mathrm{CH}^*(-) \to H^{2*}(-, \mathbb{Z}(*))$.

$$(3.5)$$

In [ACMV20], we have defined a distinguished model $J_{a,X/K}^{2n-1}$ of the image of the Abel–Jacobi map on algebraically trivial cycle classes $AJ : \mathrm{A}^n(X^{\mathrm{an}}) \to J^{2n-1}(X^{\mathrm{an}})$. From the functoriality statement of [ACMV20, Prop. 5.1], we obtain commutative diagrams:

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$$
\begin{array}{c}
\text{(3.6)}
\end{array}
$$

Left diagram:
$$
\begin{array}{ccc}
& \mathscr{A}^{d_{\widetilde{W}}-d_X+n}_{\widetilde{W}/K} & \\
\mathscr{A}^n_{X/K} \xrightarrow{\ \widetilde{Z}^*\ } \quad \xrightarrow{\ p^e Z^*\ } & \downarrow AJ & \mathscr{A}^n_{X/K} \xleftarrow{\ \widetilde{j}_*\ } \\
\downarrow AJ & J^{2(d_{\widetilde{W}}-d_X+n)-1}_{a,\widetilde{W}/K} & \downarrow AJ \\
J^{2n-1}_{a,X/K} \xrightarrow{\ \widetilde{Z}^*\ } \xrightarrow{\ p^e Z^*\ } & \xrightarrow{\ \widetilde{j}_*\ } & J^{2n-1}_{a,X/K}
\end{array}
$$

Right diagram:
$$
\begin{array}{ccc}
\mathscr{A}^n_{X/K} \xrightarrow{\ p^e Z_*\ } & & \mathscr{A}^n_{X/K} \\
\downarrow AJ \quad {}^{\widetilde{j}^*}\searrow & \mathscr{A}^n_{\widetilde{W}/K} & {}^{\widetilde{Z}_*}\nearrow \quad \downarrow AJ \\
J^{2n-1}_{a,X/K} \xrightarrow{\ p^e Z_*\ } & \downarrow AJ & J^{2n-1}_{a,X/K} \\
& J^{2n-1}_{a,\widetilde{W}/K} &
\end{array}
$$

Note that the morphisms of complex tori in (3.5) and (3.6) are induced by the morphisms in cohomology (3.4) with $\mathcal{H}^\bullet(-) = H^\bullet((-)^{\mathrm{an}}, \mathbb{Z})$.

3.3. **Chow decomposition of the diagonal and existence of algebraic representatives.** As already hinted at in [ACMV17, §6.2], the existence of certain Chow decompositions of the diagonal implies the existence of algebraic representatives.

Proposition 3.2 (Existence of algebraic representatives). *Let X be a smooth projective variety over a perfect field K and let n be a positive integer. Assume that $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)_{\mathbb{Q}}$ admits a decomposition of type (d_1, d_2) with $d_1 \leq d_X - (n-1)$ and $d_2 \leq n-1$. Then there is an algebraic representative $\Phi^n_{X/K} : \mathscr{A}^n_{X/K} \to \mathrm{Ab}^n_{X/K}$.*

Proof. By Lemma 3.1, we may write $\Delta_X = \widetilde{Z}_1 \circ \tilde{\gamma}_1 + {}^t\tilde{\gamma}_2 \circ {}^t\widetilde{Z}_2$, with $\widetilde{Z}_i \in \mathrm{CH}_{d_X}(\widetilde{W}_i \times_K X)_{\mathbb{Q}}$ and $\tilde{\gamma}_i \in \mathrm{CH}^{d_X}(X \times_K \widetilde{W}_i)_{\mathbb{Q}}$ for some smooth projective varieties $\widetilde{W}_i$ of pure dimension d_i with $d_1 = d_X - n + 1$ and $d_2 = n - 1$. By (3.2) with rational coefficients, this decomposition provides a surjective homomorphism $A^1((\widetilde{W}_1)_{\overline{K}}) \to A^n(X_{\overline{K}})$ induced by a correspondence, and we conclude the existence of the algebraic representative by Saito's criterion (*e.g.*, [ACMV23, Prop. 5.3]), as in the proof of [ACMV17, Prop. 6.7]. $\qquad\square$

4. CHOW DECOMPOSITION AND SELF-DUALITY OF THE ALGEBRAIC REPRESENTATIVE

The aim of this section is to prove the auto-duality statement of Theorem 1 in the case where the threefold is assumed to be geometrically stably rational; this is Theorem 4.4(2). We start with a motivic result (Theorem 4.2), which allows us to show that threefolds admitting a strict decomposition of the diagonal have universal regular homomorphisms for codimension-2 cycle classes that are themselves induced by a cycle class (Proposition 4.3). With this we prove Theorem 4.4.

4.1. **A motivic statement.** To start with, we consider a commutative unital ring R and we consider the category $\mathfrak{M}_{K,R}$ of pure Chow motives over K with R-coefficients as described in [And04, §4]. Denote by $\mathfrak{h}(X)_R$ the Chow motive of X with R-coefficients. Here is a general proposition, which is a version of [Via15, Thm. 2.1] with R-coefficients.

Proposition 4.1. *Let X be a smooth projective variety of pure dimension d over a perfect field K, and let $p \in \mathrm{Hom}_{\mathfrak{M}_{K,R}}(\mathfrak{h}(X)_R, \mathfrak{h}(X)_R) := \mathrm{CH}^d(X \times_K X)_R$ be an idempotent correspondence with the property that $p_* \mathrm{CH}_0(X_L)_R = 0$ for all field extensions L/K. Assume either that the exponential characteristic of K is invertible in R or that resolution of singularities holds in dimensions $\leq d - 1$. Then there exist a smooth projective variety Y over K of dimension $d - 1$ and an idempotent $q \in \mathrm{CH}^{d-1}(Y \times_K Y)_R$ such that $(X, p, 0) \simeq (Y, q, -1)$ in $\mathfrak{M}_{K,R}$.*

Proof. Let us denote X_i the connected components of X and $p_{ij} \in \mathrm{Hom}_{\mathfrak{M}_{K,R}}(\mathfrak{h}(X_i)_R, \mathfrak{h}(X_j)_R)$ the (i,j)-component of p. By assumption, if η_i denotes the generic point of X_i, then we have $p_*[\eta_i] = 0$

and hence $(p_{ij})_*[\eta_i] = 0$ for all j. But $(p_{ij})_*[\eta_i]$ is the restriction of $p_{ij} \in \mathrm{CH}^d(X_i \times_K X_j)_R$ to $\varinjlim \mathrm{CH}_d(U \times_K X_j)_R = \mathrm{CH}_0((X_j)_{k(\eta_i)})_R$, where the limit is taken over all open subsets U of X_i. Therefore, by the localization exact sequence for Chow groups, there exist for all j a proper closed subset $D_{ij} \subset X_i$ and a correspondence $\gamma_{ij} \in \mathrm{CH}_d(D_{ij} \times_K X_j)_R$ such that γ_{ij} maps to p_{ij} via the inclusion $D_{ij} \times_K X_j \to X_i \times_K X_j$. In other words, p_{ij} is supported on $D_{ij} \times_K X_j$. By Lemma 3.1, we get a factorization $p_{ij} = r_{ij} \circ s_{ij}$, where $r_{ij} \in \mathrm{CH}_d(Y_{ij} \times_K X_j)_R$ and $s_{ij} \in \mathrm{CH}^d(X_i \times_K Y_{ij})_R$ and where Y_{ij} is smooth projective over K of pure dimension $d - 1$. If we now consider the correspondences $r = \sqcup r_{ij}$ and $s = \sqcup s_{ij}$, we have $p = r \circ s$. Moreover, the correspondence $q := s \circ r \circ s \circ r = s \circ p \circ r \in \mathrm{CH}^{d-1}(Y \times_K Y)_R$ is an idempotent, and $p \circ r \circ q \circ s \circ p = p$ as well as $q \circ s \circ p \circ r \circ q = q$. These last two equalities exactly mean that $p \circ r \circ q$, seen as a morphism of Chow motives with R-coefficients from $(Y, q, -1)$ to $(X, p, 0)$, is an isomorphism, with inverse $q \circ s \circ p$. □

Theorem 4.2. *Let X be a (geometrically) connected smooth projective threefold over a perfect field K. Assume that $\mathrm{CH}_0(X)_R$ is universally spanned by a degree-1 zero-cycle $x \in \mathrm{CH}_0(X)_R$. Set*

$$\Gamma := \Delta_X - x \times_K X - X \times_K x \quad \in \mathrm{CH}^3(X \times_K X)_R.$$

Then there exist a smooth projective curve C over K, an idempotent correspondence $p \in \mathrm{CH}^1(C \times_K C)_R$ and an isomorphism of Chow motives with R-coefficients

$$(X, \Gamma) \simeq (C, p, -1).$$

Concretely, there exist a smooth projective curve C over K and correspondences $\alpha \in \mathrm{CH}^2(X \times_K C)_R$ and $\beta \in \mathrm{CH}^2(C \times_K X)_R$ such that $\Gamma = \beta \circ \alpha$.

Proof. First note that Γ does not depend on the choice of the degree-1 zero-cycle x. That X admits a decomposition of the diagonal implies not only that $\Gamma_* \mathrm{CH}_0(X_L)_R = 0$ but also that $\Gamma^* \mathrm{CH}_0(X_L)_R = 0$ for all field extensions L/K. Since resolution of singularities holds for surfaces over perfect fields, we may apply Proposition 4.1 and obtain a smooth projective surface S together with an idempotent $q \in \mathrm{CH}^2(S \times_K S)_R$ such that the Chow motive with R-coefficients $(S, q, -1)$ is isomorphic to $(X, \Gamma, 0)$. Since $\Gamma^* \mathrm{CH}_0(X_L)_R = 0$ for all field extensions L/K, we find that $q^* \mathrm{CH}_0(S_L)_R = 0$ for all field extensions L/K. Applying Proposition 4.1 to the motive $(S, {}^t q, 0)$, we obtain a smooth projective curve C and an idempotent $p \in \mathrm{CH}^1(C \times_K C)_R$ such that $(S, {}^t q, 0)$ is isomorphic to $(C, {}^t p, -1)$. Dualizing we get that $(S, q, -1)$ is isomorphic to $(C, p, -1)$, thereby concluding the proof. □

4.2. **Proof of auto-duality in Theorem 1.** As a first consequence of Theorem 4.2, one obtains information on the second algebraic representative for threefolds admitting a decomposition of the diagonal. The key feature of the following proposition is that for such threefolds the second algebraic representative is induced by an algebraic cycle.

Proposition 4.3. *Let X be a smooth projective threefold over a field K that is either finite or algebraically closed. Assume that there exists a natural number N such that $N\Delta_X$ admits a strict decomposition. Then the second algebraic representative*

$$\Phi^2_{X/K} : \mathscr{A}^2_{X/K} \to \mathrm{Ab}^2_{X/K}$$

admits a degree-N miniversal cycle $Z \in \mathrm{A}^2(\mathrm{Ab}^2_X \times_K X)$. Moreover, $\phi^2_{X_{\overline{K}}} := \Phi^2_{X/K}(\overline{K}) : \mathrm{A}^2(X_{\overline{K}}) \to \mathrm{Ab}^2_{X/K}(\overline{K})$ is an isomorphism and there is a nonnegative integer d such that $N^d \Phi^2_{X/K}$ is induced by a cycle $\widehat{Z} \in \mathrm{CH}^2(\widehat{\mathrm{Ab}}^2_X \times_K X)$, i.e., with the notation of §1.5

$$N^d \Phi^2_{X/K} = \widehat{Z}^* : \mathscr{A}^2_{X/K} \to \mathrm{Ab}^2_{X/K}.$$

Proof. First, we note that more generally, $\phi^2_{X_{\overline{K}}}$ is an isomorphism under the weaker hypothesis that X is a smooth proper variety of any dimension whose diagonal $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X) \otimes \mathbb{Q}$ admits

a Q-decomposition of type $(d_X - 1, 1)$; see Proposition 10.1. Likewise, $\Phi^2_{X/K}$ admits a degree-N miniversal cycle under the weaker hypothesis that X is a smooth proper variety of dimension ≤ 4 such that $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a decomposition of type $(d_X - 1, 1)$; see Theorem 9.1. In fact, in the aforementioned two results, it is enough to assume the existence of a *cohomological* decomposition (we discuss this notion in §6). Hence the key feature of Proposition 4.3, *i.e.*, requiring the hypothesis of a strict integral (Chow) decomposition of N times the diagonal, is that we establish that $N^d\Phi^2_{X/K}$ is induced by a cycle $\widehat{Z}$ for some nonnegative integer d.

It remains to prove that there is a nonnegative integer d such that $N^d\Phi^2_{X/K}$ is induced by a cycle. Via Theorem 4.2 with $R = \mathbb{Z}[1/N]$, the proof reduces to the case of codimension-1 cycles on curves. Indeed, with the notation of Theorem 4.2 and after clearing out denominators, we have a commutative diagram

$$
\begin{array}{ccccc}
\mathscr{A}^2_{X/K} & \xrightarrow{\alpha} & \mathscr{A}^1_{C/K} & \xrightarrow{\beta} & \mathscr{A}^2_{X/K} \\
\downarrow{\scriptstyle \Phi^2_{X/K}} & & \simeq \downarrow{\scriptstyle AJ} & & \downarrow{\scriptstyle \Phi^2_{X/K}} \\
\mathrm{Ab}^2_{X/K} & \xrightarrow{f} & \mathrm{Pic}^0_{C/K} & \xrightarrow{g} & \mathrm{Ab}^2_{X/K}
\end{array}
$$

where $\alpha \in \mathrm{CH}^2(X \times_K C)$ and $\beta \in \mathrm{CH}^2(C \times_K X)$ are *integral* correspondences such that $\alpha \circ \beta = N^d p$ and $\beta \circ \alpha = N^d \Gamma$ for some nonnegative integer d, and where f and g are the (unique) homomorphisms induced by the universal property of the algebraic representatives. Since the integral correspondences $N^d(x \times_K X)$ and $N^d(X \times_K x)$ act as zero on $\mathscr{A}^2_{X/K}$, the integral correspondence $N^d\Gamma$ acts as multiplication by N^d on $\mathscr{A}^2_{X/K}$. Thus, if $\mathrm{alb}_C : C \to \mathrm{Pic}^0_C$ denotes the K-morphism $c \mapsto \mathcal{O}_C(c - c_0)$ for any choice of zero-cycle c_0 of degree 1 on C (which exists if K is finite or algebraically closed) and if $\mathcal{P}_{\mathrm{Ab}^2_X}$ denotes the universal line-bundle on $\mathrm{Ab}^2_{X/K} \times_K \widehat{\mathrm{Ab}}^2_{X/K}$, then, by Lemma 1.8, $N^d\Phi^2_{X/K}$ is induced by the correspondence $\mathcal{P}_{\mathrm{Ab}^2_X} \circ g \circ \mathrm{alb}_C \circ \alpha$. $\qquad\square$

We are now in a position to prove the auto-duality statement of Theorem 1, which we formulate in the more precise form of Theorem 4.4, below. The basic observation is that when X is a threefold, then given any miniversal cycle $Z \in \mathscr{A}^2(\mathrm{Ab}^2_{X/K} \times_K X)$, the symmetric correspondence ${}^tZ \circ Z \in \mathrm{CH}^1(\mathrm{Ab}^2_{X/K} \times_K \mathrm{Ab}^2_{X/K})$ induces a morphism $\mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$ that is symmetric. We return to investigating such morphisms later, in §12, in more generality. Here we focus on the case of threefolds under the assumption that the diagonal admits a strict (Chow) decomposition, so that we can use Proposition 4.3, which implies the stronger conclusion that the given morphism is an isomorphism.

Theorem 4.4 (Auto-duality). *Let X be a smooth projective threefold over a perfect field K and let Ω/K be an algebraically closed field extension.*

(1) *Let $Z \in \mathrm{CH}^2(\mathrm{Ab}^2_{X_\Omega/\Omega} \times_\Omega X_\Omega)$ be a miniversal cycle over Ω, of degree, say, r. Let N be a natural number. Assume that $N\Phi^2_{X_\Omega/\Omega}$ is induced by a cycle $\widehat{Z} \in \mathrm{CH}^2(\widehat{\mathrm{Ab}}^2_{X_\Omega} \times_\Omega X_\Omega)$. Then the symmetric Ω-homomorphism*

$$
\mathrm{Ab}^2_{X_\Omega/\Omega} \longrightarrow \widehat{\mathrm{Ab}}^2_{X_\Omega/\Omega}
$$

induced by ${}^tZ \circ Z \in \mathrm{CH}^1(\mathrm{Ab}^2_{X_\Omega/\Omega} \times_\Omega \mathrm{Ab}^2_{X_\Omega/\Omega})$ is an isogeny with kernel contained in the torsion subscheme $\mathrm{Ab}^2_{X_\Omega/\Omega}[Nr^2]$. In particular, the degree of this isogeny divides $(Nr^2)^{2g}$ where g is the dimension of $\mathrm{Ab}^2_{X_\Omega/\Omega}$.

(2) *Assume that Δ_{X_Ω} admits a strict decomposition. Let $Z \in \mathrm{CH}^2(\mathrm{Ab}^2_{X_\Omega/\Omega} \times_\Omega X_\Omega)$ be a universal cycle over Ω, the existence of which is provided by Proposition 4.3. Then the symmetric Ω-homomorphism*

$$\mathrm{Ab}^2_{X_\Omega/\Omega} \longrightarrow \widehat{\mathrm{Ab}}^2_{X_\Omega/\Omega}$$

induced by ${}^t Z \circ Z \in \mathrm{CH}^1(\mathrm{Ab}^2_{X_\Omega/\Omega} \times_\Omega \mathrm{Ab}^2_{X_\Omega/\Omega})$ is an isomorphism that descends to a symmetric K-isomorphism

$$\Lambda_X : \ \mathrm{Ab}^2_{X/K} \longrightarrow \widehat{\mathrm{Ab}}^2_{X/K}$$

independent of the choice of the universal cycle Z.

Proof. From Chow rigidity, and descent for regular homomorphisms along separable field extensions [ACMV23, Thm. 1] (see Remark 1.2), we may immediately reduce to the case $\Omega = \overline{K}$.

(1) We have on $\overline{K}$-points a commutative diagram

$$(4.1)$$

where $N\phi^2_{X_{\overline{K}}} = \widehat{Z}^*$ and where $F : \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \to \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}}$ is the homomorphism induced by the universal property of $\Phi^2_{X/K}$. The bottom horizontal arrow is by definition $\psi_Z := \phi^2_{X_{\overline{K}}} \circ w_Z$ and is multiplication by r because Z is a degree-r miniversal cycle for $\Phi^2_{X/K}$. The top horizontal arrow $Z^* \circ w_{\widehat{Z}}$ is multiplication by Nr. Indeed, by Lemma 1.9, it is induced by the cycle ${}^t Z \circ \widehat{Z}$ and is the dual of the homomorphism $\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \to \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}$ induced by the transpose ${}^t \widehat{Z} \circ Z$, which in turn is nothing but $N\psi_Z$, as it coincides with $\widehat{Z}^* \circ w_Z = N\phi^2_{X_{\overline{K}}} \circ w_Z$. In particular, applying Lemma 1.9 to the composition $Z^* \circ w_Z$, and using the bottom horizontal arrow, we see that rF is induced by ${}^t Z \circ Z$.

Now, since $F \circ \psi_{\widehat{Z}} = Nr \cdot \mathrm{id} : \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}} \to \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}'}$ and since an abelian variety and its dual have the same dimension, it is clear that F, and hence rF, is an isogeny. In addition, we find that the kernel of F is contained in the torsion subscheme $\mathrm{Ab}^2_{X_\Omega/\Omega}[Nr]$ and hence that the kernel of rF is contained in the torsion subscheme $\mathrm{Ab}^2_{X_\Omega/\Omega}[Nr^2]$.

(2) By Proposition 4.3, $\Phi^2_{X_{\overline{K}}/\overline{K}}$ admits a universal cycle $Z \in \mathrm{CH}^2(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ and is induced by a cycle $\widehat{Z} \in \mathrm{CH}^2(\widehat{\mathrm{Ab}}^2_{X_{\overline{K}}/\overline{K}} \times_{\overline{K}} X_{\overline{K}})$. By point (1), the symmetric $\overline{K}$-homomorphism $\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \longrightarrow \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}/\overline{K}}$ induced by ${}^t Z \circ Z$ is an isomorphism and it coincides with the $\overline{K}$-homomorphism F of (4.1). We now proceed to show that the isomorphism F descends to a homomorphism Λ_X over K and is independent of the choice of a universal cycle Z for $\Phi^2_{X/K}$. The starting point is that $\Phi^2_{X/K} : \mathrm{A}^2(X_{\overline{K}}) \to \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}(\overline{K})$ is an isomorphism that is $\mathrm{Gal}(K)$-equivariant by [ACMV17], so that its inverse w_Z is also $\mathrm{Gal}(K)$-equivariant. In order to show that $Z^* \circ w_Z$ is $\mathrm{Gal}(K)$-equivariant and independent of Z, it suffices to show that the induced map on Tate modules $Z^* \circ w_Z : T_\ell \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \to T_\ell \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}/\overline{K}}$ is $\mathrm{Gal}(K)$-equivariant and independent of Z, for some prime $\ell \neq \mathrm{char}\, K$. However, the isomorphism $Z^* : T_\ell \mathrm{A}^2(X_{\overline{K}}) \to T_\ell \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}}$ is the dual of the $\mathrm{Gal}(K)$-equivariant isomorphism

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$(T_\ell \Phi^2_{X/K})^{-1} : T_\ell \mathrm{Ab}^2_{X_{\overline{K}}/K} \to T_\ell \, \mathrm{A}^2(X_{\overline{K}})$, where $T_\ell \, \mathrm{A}^2(X_{\overline{K}})$ is identified with its dual via the $\mathrm{Gal}(K)$-equivariant isomorphism $T_\ell \lambda^2 : T_\ell \, \mathrm{A}^2(X_{\overline{K}}) \to H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))$ provided by [ACMV21, Prop. 5.2] (see also Proposition 7.12 below), and the $\mathrm{Gal}(K)$-equivariant perfect pairing given by the intersection product $H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2)) \times H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2)) \to \mathbb{Z}_\ell(1)$, via the following commutative diagram

$$
\begin{array}{ccccccc}
& \overset{w_Z = (\Phi^2_{X/K})^{-1}}{\overbrace{\hspace{10cm}}} & & & & & \\
T_\ell \, \mathrm{Ab}^2_{X/K} & \longrightarrow & T_\ell \, \mathrm{A}_0(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}) & \xrightarrow{Z_*} & T_\ell \, \mathrm{A}^2(X_{\overline{K}}) & \xrightarrow{Z^*} & T_\ell \, \mathrm{A}^1(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}) \\
& \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle T_\ell \lambda^0} \simeq & & \downarrow{\scriptstyle T_\ell \lambda^2} \simeq & & \downarrow{\scriptstyle T_\ell \lambda^1} \simeq \\
& & T_{\overline{\ell}} \, \mathrm{Ab}^2_{X/K} & \xrightarrow{Z_*} & H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2)) & \xrightarrow{Z^*} & T_\ell \widehat{\mathrm{Ab}}^2_{X/K}
\end{array}
\tag{4.2}
$$

where the left triangle commutes thanks to §1.5.2 together with Lemma 1.4 and the fact that the Bloch map λ^0 coincides with the albanese map on ℓ-primary torsion [Blo79, Prop. 3.9]. □

Remark 4.5. In Theorem 4.4(2), note that although the homomorphism $\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \to \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}/\overline{K}}$ descends to K, the cycle $Z \in \mathrm{A}^2(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ might not be defined over K.

Remark 4.6. Note that the proof of Theorem 4.4(2) realizes $\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}$ as a direct factor of $\mathrm{Pic}^0_{C/\overline{K}}$ for some smooth projective curve C over $\overline{K}$. Although $\mathrm{Pic}^0_{C/\overline{K}}$ is principally polarizable, this is not enough to conclude that $\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}$ is principally polarizable and hence is isomorphic to its dual. Indeed, any abelian variety is the direct summand of a principally polarizable abelian variety; this can be seen for instance by Zarhin's trick which states that, for any abelian variety A, the abelian variety $(A \times_K \widehat{A})^4$ is principally polarizable, while A need not be principally polarizable.

As an application of Theorem 4.4(2), we can refine [ACMV21, Thm. 15 & 6.4] in the case of stably rational threefolds:

Corollary 4.7. *Let X be a smooth projective stably rational threefold over a field K that is either finite or algebraically closed. Then there exist correspondences $Z \in \mathrm{CH}^2(\mathrm{Ab}^2_X \times_K X)$ and $Z' \in \mathrm{CH}_2(X \times_K \mathrm{Ab}^2_X)$ inducing for all primes l inverse isomorphisms*

$$Z_* : T_l \, \mathrm{Ab}^2_X \xrightarrow{\; \cong \;} H^3(X_{\overline{K}}, \mathbb{Z}_l(2)) \quad and \quad Z'_* : H^3(X_{\overline{K}}, \mathbb{Z}_l(2)) \xrightarrow{\; \cong \;} T_l \, \mathrm{Ab}^2_X$$

of $\mathrm{Gal}(K)$-modules.

Proof. By Proposition 4.3, let $Z \in \mathrm{CH}^2(\mathrm{Ab}^2_X \times_K X)$ be a universal codimension-2 cycle for X. By [ACMV21, Thm. 6.4] and the fact that resolution of singularities holds for surfaces over perfect fields, Z induces for all primes l an isomorphism $Z_* : T_l \, \mathrm{Ab}^2_X \xrightarrow{\cong} H^3(X_{\overline{K}}, \mathbb{Z}_l(2))$. With $\Theta : \mathrm{Ab}^2_X \to \widehat{\mathrm{Ab}}^2_X$ the canonical K-isomorphism induced by $Z^* \circ Z_*$ provided by Theorem 4.4, we get that $Z'_* := \Theta^{-1} \circ Z^*$ provides the inverse to Z_*. □

5. SPECIALIZATION AND POLARIZATION ON THE ALGEBRAIC REPRESENTATIVE

Given an abelian variety A over a field K, recall that a symmetric isogeny $\Theta : A \to \widehat{A}$ is called a *polarization* if there exists an ample symmetric line-bundle L on $A_{\overline{K}}$ such that $\Theta_{\overline{K}} : A_{\overline{K}} \to \widehat{A}_{\overline{K}}$ is given by $a \mapsto t_a^* L \otimes L^{-1}$, where $t_a : A_{\overline{K}} \to A_{\overline{K}}$ is the translation-by-a morphism; see also §A.2. In characteristic zero, by Hodge theory, the isomorphism $\Theta_X = -\Lambda_X$ in Theorem 4.4 agrees with the canonical principal polarization induced by the cup-product on $H^3(X_{\mathbb{C}}, \mathbb{Z})$ (Theorem 12.12 and Remark 12.13). Although there are abelian varieties A that are isomorphic to their dual but are not

principally polarizable (we thank Bas Edixhoven for explaining an example to us), we are led to make the following conjecture:

Conjecture 5.1 (Canonical polarization). *Let X be a smooth projective threefold over a perfect field K and let Ω/K be an algebraically closed field extension. Assume that Δ_{X_Ω} admits a strict decomposition (i.e., X_Ω is universally CH_0-trivial). Then the canonical symmetric K-isomorphism $\Theta_X : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$, where $\Theta_X = -\Lambda_X$ (Theorem 4.4) is a principal polarization.*

With this as motivation, we now establish some results regarding specialization and Chow decomposition. For that purpose, let $\mathcal{X} \to S$ be a smooth projective morphism, where S is the spectrum of a DVR with generic point η and closed point s. We denote $\bar{\eta}$ and $\bar{s}$ algebraic closures of η and s, respectively, and we denote $\mathcal{X}_\eta$ and $\mathcal{X}_s$ the generic fiber and the closed fiber of $\mathcal{X} \to S$, respectively.

We start with the following basic result about polarizations:

Lemma 5.2 (Polarizations and specializations for algebraic representatives). *If $\Theta_\eta : \mathrm{Ab}^2_{\mathcal{X}_\eta/\eta} \to \widehat{\mathrm{Ab}}^2_{\mathcal{X}_\eta/\eta}$ is a degree-d isogeny, then $\mathrm{Ab}^2_{\mathcal{X}_\eta/\eta}$ and $\Theta_{\mathcal{X}_\eta}$ extend to a degree-d isogeny $\Theta_\mathcal{X} : \mathrm{Ab}^2_{\mathcal{X}/S} \to \widehat{\mathrm{Ab}}^2_{\mathcal{X}/S}$ of abelian S-schemes, and the following are equivalent: $\Theta_\mathcal{X}$ is a polarization, $\Theta_{\mathcal{X}_\eta}$ is a polarization, $\Theta_\mathcal{X}|_s$ is a polarization.*

Proof. The fact that $\mathrm{Ab}^2_{\mathcal{X}_\eta/\eta}$ extends to an abelian scheme $\mathrm{Ab}^2_{\mathcal{X}/S}$ is [ACMV23, Thm. 8.3], and essentially follows directly from the Ogg–Néron–Shafarevich criterion, and the fact that the second Bloch map is injective ([ACMV23, Thm. 8.3] makes the stronger assertion that this extension is the relative algebraic representative for $\mathcal{X}/S$, which we do not use here). The fact that $\Theta_{\mathcal{X}_\eta}$ then extends to an isogeny (resp. isomorphism) $\Theta_\mathcal{X}$ over S is standard (see *e.g.*, [ACMV23, Prop. 4.5] and [MFK94, Prop. 6.1]). Regarding polarizations, recall that by [MFK94, p.121], we have that $2\Theta_\mathcal{X}$ is induced by a line bundle $\mathcal{L}$ on $\mathrm{Ab}^2_{\mathcal{X}/S}$. Thus, for questions of polarizations, if suffices to establish the ampleness of the fibers of $\mathcal{L}$. On the one hand, since relative ampleness is an open property over the base, we have $\mathcal{L}$ is ample if and only if $\mathcal{L}|_s$ is ample. On the other hand, assuming $\mathcal{L}_\eta$ is ample, it suffices to show that $\mathcal{L}|_s$ is ample. In general, relative ampleness is not a closed condition, however, for abelian varieties, a nondegenerate line bundle is ample if and only if it is effective [Mum70]. Therefore, taking D_η to be an effective divisor realizing the line bundle $\mathcal{L}_\eta$, we may take the closure of D_η to obtain an effective divisor D over S realizing $\mathcal{L}$. Thus $\mathcal{L}_s$ is effective and nondegenerate, and therefore ample. $\square$

Proposition 5.3. *Assume both the generic point η and the closed point s of S have perfect residue fields and assume that the diagonal $\Delta_{\mathcal{X}_{\bar{\eta}}}$ has a strict decomposition (e.g., $\mathcal{X}_\eta$ is geometrically stably rational). (By specialization, $\Delta_{\mathcal{X}_{\bar{s}}}$ also has a strict decomposition [Voi15, Thm. 2.1], [CTP16, Thm. 1.12].) Denote by $\Theta_{\mathcal{X}_s} : \mathrm{Ab}^2_{\mathcal{X}_s/s} \to \widehat{\mathrm{Ab}}^2_{\mathcal{X}_s/s}$ and $\Theta_{\mathcal{X}_\eta} : \mathrm{Ab}^2_{\mathcal{X}_\eta/\eta} \to \widehat{\mathrm{Ab}}^2_{\mathcal{X}_\eta/\eta}$ the negatives of the canonical isomorphisms provided by Theorem 4.4(2).*

Then $\mathrm{Ab}^2_{\mathcal{X}_\eta/\eta}$ and $\Theta_{\mathcal{X}_\eta}$ extend to an isomorphism $\Theta_\mathcal{X} : \mathrm{Ab}^2_{\mathcal{X}/S} \to \widehat{\mathrm{Ab}}^2_{\mathcal{X}/S}$ of abelian S-schemes, and there is a canonical isomorphism

$$(\mathrm{Ab}^2_{\mathcal{X}_s/s}, \Theta_{\mathcal{X}_s}) \cong (\mathrm{Ab}^2_{\mathcal{X}/S}|_s, \Theta_\mathcal{X}|_s). \tag{5.1}$$
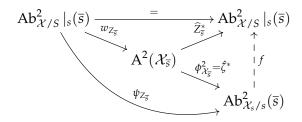
In particular, $\Theta_{\mathcal{X}_\eta}$ is a polarization (and therefore a principal polarization) if and only if $\Theta_{\mathcal{X}_s}$ is.

Proof. From Lemma 5.2, it suffices to establish the canonical isomorphism (5.1).

By Proposition 4.3, $\phi^2_{\mathcal{X}_{\bar{\eta}}}$ and $\phi^2_{\mathcal{X}_{\bar{s}}}$ are isomorphisms, and, by [ACMV21, Prop. 5.2] (see also Proposition 7.12 below), the maps $T_\ell \lambda^2_{\mathcal{X}_{\bar{\eta}}} : T_\ell \mathrm{A}^2(\mathcal{X}_{\bar{\eta}}) \to H^3(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_\ell(2))$ and $T_\ell \lambda^2_{\mathcal{X}_{\bar{s}}} : T_\ell \mathrm{A}^2(\mathcal{X}_{\bar{s}}) \to$

$H^3(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(2))$ are isomorphisms for all primes ℓ. Choose a prime ℓ invertible in the function fields $\kappa(s)$ and $\kappa(\eta)$. On the one hand, we obtain a Galois-equivariant isomorphism $T_\ell \lambda^2_{\mathcal{X}_{\bar{\eta}}} \circ (T_\ell \phi^2_{\mathcal{X}_\eta})^{-1}$: $T_\ell \mathrm{Ab}^2_{\mathcal{X}_\eta/\eta} \to H^3(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_\ell(2))$, showing that $\mathrm{Ab}^2_{\mathcal{X}_\eta/\eta}$ has good reduction. On the other hand, since the specialization map $H^3(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_\ell(2)) \to H^3(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(2))$ is an isomorphism (by smooth proper base change), it follows that $\dim \mathrm{Ab}^2_{\mathcal{X}_s/s} = \dim \mathrm{Ab}^2_{\mathcal{X}_\eta/\eta}$.

Let $Z \in \mathrm{A}^2(\mathrm{Ab}^2_{\mathcal{X}_{\bar{\eta}}/\bar{\eta}} \times_{\bar{\eta}} \mathcal{X}_{\bar{\eta}})$ be a universal cycle for $\mathrm{Ab}^2_{\mathcal{X}_{\bar{\eta}}/\bar{\eta}}$ and $\widehat{Z}$ be a cycle inducing $\Phi^2_{\mathcal{X}_{\bar{\eta}}/\bar{\eta}}$. We denote $Z_{\bar{s}}$ and $\widehat{Z}_{\bar{s}}$ their specializations. Let also ζ be a universal cycle for $\mathrm{Ab}^2_{\mathcal{X}_{\bar{s}}}$ and $\hat{\zeta}$ be a cycle inducing $\Phi^2_{\mathcal{X}_{\bar{s}}}$. The key point is that since $\phi_{\bar{\eta}}$ is induced by a cycle $\widehat{Z}$, its specialization defines a regular homomorphism, namely the one induced by the specialization of the cycle $\widehat{Z}$. Therefore, we have a commutative diagram

where the upper triangle is the specialization of the corresponding triangle over η, where f is the homomorphism induced by the universal property of $\Phi^2_{\mathcal{X}_s}$, and where $\psi_{Z_{\bar{s}}} : \mathrm{Ab}^2_{\mathcal{X}/S}|_{\bar{s}} \to \mathrm{Ab}^2_{\mathcal{X}_{\bar{s}}/\bar{s}}$ is the homomorphism induced by the cycle $Z_{\bar{s}}$. Hence $f : \mathrm{Ab}^2_{\mathcal{X}_s/s} \to \mathrm{Ab}^2_{\mathcal{X}/S}|_s$ is surjective. We already established that $\dim \mathrm{Ab}^2_{\mathcal{X}_s/s} = \dim \mathrm{Ab}^2_{\mathcal{X}/S}$, and we can thus conclude that f is an isomorphism (with inverse $\psi_{Z_{\bar{s}}}$).

We now check that the isomorphism $\psi_{Z_{\bar{s}}}$ is canonical, *i.e.*, it does not depend on the choice of a universal cycle Z for $\mathrm{Ab}^2_{\mathcal{X}_{\bar{\eta}}/\bar{\eta}}$. Choose a prime ℓ invertible in the function fields $\kappa(s)$ and $\kappa(\eta)$. It is enough to check that $T_\ell \psi_{Z_{\bar{s}}}$ is canonical and, since on $\bar{s}$-points we have $\psi_{Z_{\bar{s}}} = \phi^2_{\mathcal{X}_s} \circ w_{Z_{\bar{s}}}$, by Lemma 1.4, it suffices to check that $T_\ell w_{Z_{\bar{s}}}$ is canonical. We have a commutative diagram

$$
\begin{array}{ccccc}
T_\ell \mathrm{Ab}^2_{\mathcal{X}_\eta/\eta} & \xrightarrow{T_\ell w_Z} & T_\ell \mathrm{A}^2(\mathcal{X}_{\bar{\eta}}) & \xrightarrow[\cong]{T_\ell \lambda^2} & H^3(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_\ell(2)) \\
{\scriptstyle \mathrm{sp}} \downarrow {\scriptstyle \cong} & & {\scriptstyle \mathrm{sp}} \downarrow & & {\scriptstyle \mathrm{sp}} \downarrow {\scriptstyle \cong} \\
T_\ell \mathrm{Ab}^2_{\mathcal{X}_s/s} & \xrightarrow{T_\ell w_{Z_{\bar{s}}}} & T_\ell \mathrm{A}^2(\mathcal{X}_{\bar{s}}) & \xrightarrow[\cong]{T_\ell \lambda^2} & H^3(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(2)).
\end{array}
\tag{5.2}
$$

The commutativity of the right square shows that the middle specialization map is an isomorphism. We can conclude that $T_\ell w_{Z_{\bar{s}}}$ is canonical by noting that $T_\ell w_Z = (T_\ell \phi^2_{\mathcal{X}_\eta})^{-1}$ is canonical.

Finally, the isomorphism $f : \mathrm{Ab}^2_{\mathcal{X}_s/s} \xrightarrow{\sim} \mathrm{Ab}^2_{\mathcal{X}/S}|_s$ satisfies $\Theta_{\mathcal{X}_s} = f^\vee \circ \Theta_{\mathcal{X}_\eta}|_s \circ f$; *i.e.*, it is in fact an isomorphism $f : (\mathrm{Ab}^2_{\mathcal{X}_s}, \Theta_{\mathcal{X}_s}) \xrightarrow{\sim} (\mathrm{Ab}^2_{\mathcal{X}/S}|_s, \Theta_{\mathcal{X}_S}|_s)$. This follows from the commutativity of the outer square of the diagram

$$
\begin{array}{ccc}
\mathrm{Ab}^2_{\mathcal{X}/S}|_s(\bar{s}) & \xrightarrow{\ \ -\Theta_{\mathcal{X}_S}|_s\ \ } & \widehat{\mathrm{Ab}}^{\,2}_{\mathcal{X}/S}|_s(\bar{s})
\end{array}
$$

with intermediate object $A^2(\mathcal{X}_{\bar{s}})$, maps $w_{Z_{\bar{s}}}$, $Z^*_{\bar{s}}$, w_ζ, ζ^*, vertical maps f, $\psi_{Z_{\bar{s}}}$, $\psi^\vee_{Z_{\bar{s}}}$, $f^\vee$, and bottom row

$$
\mathrm{Ab}^2_{\mathcal{X}_s/s}(\bar{s}) \xrightarrow{\ \ -\Theta_{\mathcal{X}_s}\ \ } \widehat{\mathrm{Ab}}^{\,2}_{\mathcal{X}_s/s}(\bar{s})
$$

This whole diagram is in fact commutative: by duality it suffices to check that $-\Theta_{\mathcal{X}_s} \circ \psi_{Z_{\bar{s}}}$ is induced by ${}^t\zeta \circ Z_{\bar{s}} \in \mathrm{CH}^1(\mathrm{Ab}^2_{\mathcal{X}/S}|_{\bar{s}} \times_{\bar{s}} \mathrm{Ab}^2_{\mathcal{X}_{\bar{s}}/\bar{s}})$. By construction, $-\Theta_{\mathcal{X}_s}$ is such that the regular homomorphism $\zeta^* : \mathscr{A}^2_{\mathcal{X}_{\bar{s}}/\bar{s}} \to \widehat{\mathrm{Ab}}^{\,2}_{\mathcal{X}_{\bar{s}}}$ is equal to $(-\Theta_{\mathcal{X}_s})_{\bar{s}} \circ \phi^2_{\mathcal{X}_{\bar{s}}}$, and it follows that the homomorphism induced by ${}^t\zeta \circ Z_{\bar{s}}$ coincides with the homomorphism $(-\Theta_{\mathcal{X}_s})_{\bar{s}} \circ \phi^2_{\mathcal{X}_{\bar{s}}}(\mathrm{Ab}^2_{\mathcal{X}/S}|_{\bar{s}})(Z_{\bar{s}}) = (-\Theta_{\mathcal{X}_s})_{\bar{s}} \circ \psi_{Z_{\bar{s}}}$. $\hspace{1cm}\square$

Remark 5.4. As already mentioned, in characteristic 0, the isomorphism $\Theta_X = -\Lambda_X$ in Theorem 4.4 agrees with the canonical principal polarization induced by the cup-product on $H^3(X_{\mathbf{C}}, \mathbb{Z})$ (Theorem 12.12 and Remark 12.13). In particular, Proposition 5.3 implies that for a smooth projective geometrically stably rational threefold over a perfect field K that lifts to a smooth projective geometrically stably rational threefold in characteristic zero, the isomorphism Θ_X is a principal polarization. Later, we will strengthen this to show that we only need to assume that X lifts to a geometrically *rationally chain connected* threefold (Corollary 13.3). We point out, however, that while we have a stronger hypothesis in Proposition 5.3, we also get the stronger conclusion that $(\mathrm{Ab}^2_{\mathcal{X}/S})_s \cong \mathrm{Ab}^2_{\mathcal{X}_s/s}$, and moreover, that the principal polarization on the generic fiber extends to a principal polarization on the special fiber, which agrees with the canonical auto-duality of Theorem 4.4. In Corollary 13.3 we will only obtain that $(\mathrm{Ab}^2_{\mathcal{X}/S})_s$ is isogenous to $\mathrm{Ab}^2_{\mathcal{X}_s/s}$.

Part 2. Cohomological decomposition of the diagonal and algebraic representatives

6. PRELIMINARIES ON COHOMOLOGICAL DECOMPOSITION OF THE DIAGONAL

In this section, we deviate slightly from our Conventions 0.1 and fix an arbitrary Weil cohomology theory $\mathcal{H}^\bullet$ and denote $R_\mathcal{H}$ its coefficient ring. If X is a smooth projective variety over a field K, the cohomology class of a cycle class $Z \in \mathrm{CH}^i(X)$ will be denoted $[Z] \in \mathcal{H}^{2i}(X)(i)$. The following definition parallels the definition of Chow decomposition of the diagonal.

Definition 6.1 (Cohomological decomposition of a cycle class). Let $R \to R_\mathcal{H}$ be a homomorphism of commutative rings. Let X be a smooth projective variety over a field K, and let

$$j_i : W_i \overset{\neq}{\hookrightarrow} X \,, \quad i = 1,2$$

be reduced closed subschemes not equal to X. A *cohomological R-decomposition of type (W_1, W_2) of a cycle class $Z \in \mathrm{CH}^{d_X}(X \times_K X)_R$ with respect to $\mathcal{H}^\bullet$* is an equality

$$[Z] = [Z_1] + [Z_2] \in \mathcal{H}^{2d_X}(X \times_K X)(d_X), \tag{6.1}$$

where $Z_1 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is supported on $W_1 \times_K X$ and $Z_2 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is supported on $X \times_K W_2$ (see (2.1) for the support of a cycle). When $R = \mathbb{Z}$, we call this a *cohomological decomposition of type (W_1, W_2) with respect to $\mathcal{H}^\bullet$*. We say that $Z \in \mathrm{CH}^{d_X}(X \times_K X)_R$ has a *cohomological R-decomposition of type (d_1, d_2) with respect to $\mathcal{H}^\bullet$* if it admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_1 \le d_1$ and $\dim W_2 \le d_2$.

Beware that these definitions depend *a priori* on the choice of the Weil cohomology theory $\mathcal{H}^\bullet$. We emphasize that the cycle classes Z_1, Z_2 in the definition are cycle classes with R-coefficients, and the statement about support of Z_1, Z_2 is in terms of the Chow group (not the cohomology group). Clearly, by applying the cycle class map, we see that if a cycle class Z has an R-decomposition of type (W_1, W_2), then it has a cohomological R-decomposition of type (W_1, W_2) (with respect to any Weil cohomology theory). We will primarily be interested in the case where $R = \mathbb{Z}$ and where $Z = N\Delta_X$ is a multiple of the diagonal in $X \times_K X$.

Note that with the notations of Lemma 3.1, if a cycle $Z \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is cohomologically equivalent to a cycle supported on $W \times_K X$, then the maps $p^e Z^* : \mathcal{H}^n(X) \to \mathcal{H}^n(X)$ and $p^e Z_* : \mathcal{H}^n(X) \to \mathcal{H}^n(X)$ factor, respectively, as:

$$
\begin{array}{ccc}
& \mathcal{H}^{2(d_{\widetilde{W}} - d_X) + n}(\widetilde{W})(d_{\widetilde{W}} - d_X) & \\
{\scriptstyle \widetilde{Z}^*}\nearrow & & \searrow{\scriptstyle \widetilde{j}_*} \\
\mathcal{H}^n(X) \xrightarrow{\hspace{2cm} p^e Z^* \hspace{2cm}} & & \mathcal{H}^n(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{H}^n(X) \xrightarrow{\hspace{1.5cm} p^e Z_* \hspace{1.5cm}} & & \mathcal{H}^n(X) \\
{\scriptstyle \widetilde{j}^*}\searrow & & \nearrow{\scriptstyle \widetilde{Z}_*} \\
& \mathcal{H}^n(\widetilde{W}) &
\end{array}
$$

$$\tag{6.2}$$

Definition 6.2 (Strict cohomological decomposition of a cycle class). A *strict cohomological R-decomposition of a cycle class $Z \in \mathrm{CH}^{d_X}(X \times_K X)_R$* is a cohomological R-decomposition of type $(d_X - 1, 0)$. In other words, it is an equality as in (6.1) such that $Z_1 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ is supported on $D \times_K X$ for some codimension-1 subvariety $D \subseteq X$, and $Z_2 \in \mathrm{CH}^{d_X}(X \times_K X)_R$ for some zero-cycle class $\alpha \in \mathrm{CH}_0(X)_R$. When $R = \mathbb{Z}$, we call this a *strict cohomological decomposition*.

Remark 6.3 (Strict cohomological decomposition of the diagonal). As in Remark 2.5, if $N[\Delta_X] = [Z_1] + [Z_2] \in \mathcal{H}^{2d_X}(X \times_K X)(d_X)$ is a strict cohomological R-decomposition, then $[Z_2] = \mathrm{pr}_2^*[\alpha]$ for any zero-cycle $\alpha \in \mathrm{CH}_0(X)$ of degree N. In particular, in the situation where $R = \mathbb{Z}$ and $X(K) \ne \varnothing$, if Δ_X admits a strict cohomological decomposition, then $[Z_2] = [X \times_K x]$ for any K-point $x \in X(K)$.

Remark 6.4. In the case where $K \subseteq \mathbb{C}$ and $R = \mathbb{Z}$ (resp. $\mathbb{Q}$), the comparison isomorphisms in cohomology imply that if $Z \in \mathrm{CH}^d(X_1 \times_K X_2)_R$ has a cohomological R-decomposition of type (W_1, W_2) (resp. a strict cohomological R-decomposition) with respect to $H^\bullet((-)^{an}, \mathbb{Z})$ (resp. $H^\bullet((-)^{an}, \mathbb{Q})$) then for each prime number ℓ it has a cohomological R-decomposition of type (W_1, W_2) (resp. a strict R-decomposition) with respect to $H^\bullet((-)_{\overline{K}}, \mathbb{Z}_\ell)$ (resp. $H^\bullet((-)_{\overline{K}}, \mathbb{Q}_\ell)$).

7. COHOMOLOGICAL DECOMPOSITION, TORSION, ALGEBRAICITY, AND THE BLOCH MAP

We now proceed to recall some of the basic results concerning decomposition of the diagonal, which essentially go back to Bloch–Srinivas, and have recently been strengthened by Voisin. The main addition in this section is to explain how to modify these results to hold in the case of varieties over arbitrary perfect fields.

A projective variety W over a perfect field K admits an alteration $\widetilde{W} \to W$ of degree M invertible in $R_{\mathcal{H}}$ with $\widetilde{W}$ smooth projective over K in the following situations:

- If $\mathcal{H}^\bullet$ is ℓ-adic étale cohomology $\mathcal{H}^\bullet(X) = H^\bullet(X_{\overline{K}}, \mathbb{Z}_\ell)$ with ℓ invertible in K. This is Gabber's ℓ'-alteration theorem. A strengthening, due to Temkin [Tem17], shows that a projective variety W over K admits an alteration $\widetilde{W} \to W$ as above of degree some power of the characteristic exponent of K.
- If W admits a resolution of singularities (in which case M can be chosen to be equal to 1). This holds unconditionally if $\mathrm{char}(K) = 0$ or if $\dim W \leq 3$ [CP09].

We fix a ring homomorphism $R \to R_{\mathcal{H}}$.

7.1. Cohomological decomposition of the diagonal and torsion in cohomology.
Each of Betti, ℓ-adic and crystalline cohomology has the property that $\mathcal{H}^0$, $\mathcal{H}^1$ and $\mathcal{H}^{2\dim X}$ are torsion-free for proper smooth varieties X, and vanish in degrees greater than $2 \dim X$. (In degree 1 this follows, for instance, from the identification of $\mathcal{H}^1(X)$ with $\mathcal{H}^1(\mathrm{Alb}_{X/K})$, and the known calculation for abelian varieties. See [Ill79, II.3.11.2] for the case of crystalline cohomology.)

Proposition 7.1 ([Voi13, Thm. 4.4(i)]). *Let K be a perfect field, and assume $\mathcal{H}^i$ is torsion-free for $i = 0, 1$. Let X/K be a smooth projective variety. Assume that for some N in R the multiple $N\Delta_{X_{\overline{K}}} \in \mathrm{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})_R$ admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_X - 1$ and $\dim W_2 \leq 1$. Let $\widetilde{W}_1 \to W_1$ be an alteration of degree M invertible in $R_{\mathcal{H}}$ with $\widetilde{W}_1$ smooth projective over $\overline{K}$.*

(1) *If $\mathcal{H}^{i-2}(\widetilde{W}_1)$ is torsion-free (e.g., $i \leq 3$), then torsion in $\mathcal{H}^i(X)$ is killed by multiplication by N.*
(2) *If $\mathcal{H}^i(\widetilde{W}_1)$ is torsion-free (e.g., $i > 2d_X - 3$), then torsion in $\mathcal{H}^i(X)$ is killed by multiplication by N.*

Proof. This follows directly from the proof [Voi13, Thm. 4.4(i)]; *i.e.*, from the factorization of correspondences in cohomology (6.2). □

As before, if N is a natural number whose image is zero in the coefficient ring of the cohomology theory, the conclusions of Proposition 7.1 are trivial. At the opposite extreme, if the image of N is a unit in $R_{\mathcal{H}}$, then under the hypotheses we may conclude that $\mathcal{H}^i(X)$ is torsion-free.

Corollary 7.2. *Let K be a perfect field, and assume $\mathcal{H}^i(-)$ is torsion-free for $i = 0, 1$. Let X/K be a smooth projective variety. If $\Delta_{X_{\overline{K}}} \in \mathrm{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_X - 1$ and $\dim W_2 \leq 1$, and with W_1 admitting a resolution of singularities, then $\mathcal{H}^i(X)$ is torsion-free for $i \leq 3$ and $i \geq 2d_X - 2$.*

In particular, if X/K is a smooth projective threefold such that $\Delta_{X_{\overline{K}}} \in \mathrm{CH}^3(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_2 \leq 1$, then $\mathcal{H}^i(X)$ is torsion-free for all i.

Proof. This is immediate from the proposition, using the fact that there is resolution of singularities for surfaces over perfect fields. $\qquad \square$

Remark 7.3. For a smooth projective threefold X over a field K, and $\mathcal{H}^\bullet$ denoting Betti or ℓ-adic cohomology ($\ell \neq \operatorname{char}(K)$) recall that $\mathcal{H}^0(X)$, $\mathcal{H}^1(X)$, and $\mathcal{H}^6(X)$ are torsion-free. Moreover, $\operatorname{Tors} \mathcal{H}^2(X) \cong \operatorname{Tors} \mathcal{H}^5(X)$ and $\operatorname{Tors} \mathcal{H}^3(X) \cong \operatorname{Tors} \mathcal{H}^4(X)$. If X is unirational over $\mathbb{C}$, a result of Serre [Ser59] implies that the fundamental group of X is trivial, so that $\mathcal{H}_1(X) = \mathcal{H}^5(X) = 0$.

Corollary 7.4. *Let X be a smooth projective threefold over a perfect field K. If the class of the diagonal $\Delta_{X_{\overline{K}}} \in \operatorname{CH}^3(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ admits a cohomological $\mathbb{W}(K)$-decomposition of type (W_1, W_2) with respect to crystalline cohomology $H^\bullet_{\mathrm{cris}}(-/\mathbb{W}(K))$, with $\dim W_1 \leq d_X - 1$ and $\dim W_2 \leq 1$, then the Picard scheme $\operatorname{Pic}^0_{X/K}$ is reduced.*

Proof. Note that W_1, like any surface over a perfect field, admits a resolution of singularities. By Corollary 7.2, we have that $H^2_{\mathrm{cris}}(X/\mathbb{W}(K))$ is torsion-free, and so (*e.g.*, [Ill79, Prop. II.5.16]) $\operatorname{Pic}^0_{X/K}$ is reduced. $\qquad \square$

For threefolds, it is often convenient to also consider the following situation:

Proposition 7.5. *Let X be a smooth projective threefold over a perfect field K, and let $\mathcal{H}^\bullet$ denote Betti, crystalline or ℓ-adic cohomology (with $\ell \neq \operatorname{char} K$). Assume that for some $N \in R$ the multiple $N\Delta_{X_{\overline{K}}} \in \operatorname{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})_R$ admits a cohomological R-decomposition of type (W_1, W_2). Let $\widetilde{W}_2 \to W_2$ be a resolution of singularities.*

  (1) *For $i \leq 3$, if $\mathcal{H}^i(\widetilde{W}_2)$ is torsion-free (e.g., $\widetilde{W}_2$ is rational), then torsion in $\mathcal{H}^i(X)$ is killed by multiplication by N.*
  (2) *For $i > 3$, if $\mathcal{H}^{i-2}(\widetilde{W}_2)$ is torsion-free (e.g., $\widetilde{W}_2$ is rational), then $\mathcal{H}^i(X)$ is killed by multiplication by N.*

Proof. The arguments are the same as for Proposition 7.1. $\qquad \square$

7.2. Cohomological decomposition of the diagonal and vanishing cohomology.
We now consider some results on vanishing of cohomology. The main take-away for our applications is Corollary 7.7 for threefolds.

Proposition 7.6. *Let X be a smooth projective variety over a perfect field K. Assume that for some $N \in R$ the multiple $N\Delta_{X_{\overline{K}}} \in \operatorname{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})_R$ admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_X - 1$ and $\dim W_2 = 0$. Let $\widetilde{W}_1 \to W_1$ be an alteration of degree M invertible in R with $\widetilde{W}_1$ smooth projective over $\overline{K}$.*

  (1) *If $i \geq 1$ and $\mathcal{H}^{i-2}(\widetilde{W}_1) = 0$ (e.g., $i = 1$), then $\mathcal{H}^i(X)$ is killed by multiplication by N.*
  (2) *If $i \neq 2d_X$ and $\mathcal{H}^i(\widetilde{W}_1) = 0$ (e.g., $i = 2d_X - 1$), then $\mathcal{H}^i(X)$ is killed by multiplication by N.*

Proof. This follows directly from the proof [Voi13, Thm. 4.4(i)]; *i.e.*, from the factorization of correspondences in cohomology (6.2). $\qquad \square$

Note that if the image of N is zero in the coefficient ring of the cohomology theory, the conclusions of Proposition 7.1 are trivial. At the opposite extreme, if the image of N is a unit in $R_\mathcal{H}$, then under the hypotheses we may conclude that $\mathcal{H}^i(X) = 0$.

Corollary 7.7. *Let X be a smooth projective variety over a perfect field K. Assume that that for some $N \in R$ the multiple $N\Delta_X \in \operatorname{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})_R$ admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_X - 1$ and $\dim W_2 = 0$, with W_1 admitting a resolution of singularities. Then $\mathcal{H}^1(X)$ and $\mathcal{H}^{2d_X-1}(X)$ are killed by multiplication by N. If $N = 1$, then $\mathcal{H}^1(X) = \mathcal{H}^{2d_X-1}(X) = 0$.*

In particular, if X/K is a smooth projective threefold such that $N\Delta_{X_{\overline{K}}} \in \mathrm{CH}^3(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})_R$ admits a strict cohomological R-decomposition (e.g., X is geometrically rationally chain connected), then $\mathcal{H}^1(X) = 0$ and $\mathcal{H}^5(X)$ is killed by multiplication by N. If $N = 1$ (e.g., X is geometrically stably rational), then $\mathcal{H}^1(X) = \mathcal{H}^5(X) = 0$.

Proof. This is immediate from the proposition. The key point is that $\mathcal{H}^1(X)$ and $\mathcal{H}^{2d_X-1}(X)$ model the cohomology of the Abelian varieties $(\mathrm{Pic}^0_{X_K/K})_{\mathrm{red}}$ and $\mathrm{Alb}_{X_K/K}$, respectively, and that $\mathcal{H}^1(X)$ is torsion while torsion in $\mathcal{H}^{2d_X-1}(X)$ is killed by multiplication by N under our assumption on the decomposition of the diagonal by virtue of Corollary 7.2). The assertion for threefolds follows as there is resolution of singularities for surfaces over perfect fields. For the case where X is assumed to be geometrically rationally chain connected, see Remark 2.8. $\square$

Remark 7.8 (Vanishing Hodge numbers). Let X be a smooth projective variety over a field K. If X is geometrically rationally chain connected and K is perfect, the previous corollary implies that $\mathcal{H}^1(X) = 0$. In characteristic 0, this implies the vanishing of $H^0(X, \Omega_X)$ and $H^1(X, \mathcal{O}_X)$; here we recall some further results on vanishing Hodge numbers. First, if X is geometrically separably rationally connected over a field K, then $H^0(X, \Omega_X^{\otimes i}) = 0$ for all $i > 0$ [Kol96, Cor. IV.3.8], and $H^1(X, \mathcal{O}_X) = 0$ [Gou14, Thm. p.872]. More generally, if $\Delta_{X_{\overline{K}}}$ admits a strict *Chow* decomposition, then [Tot16, Lem. 2.2] implies that $H^0(X, \Omega_X^i) = 0$ for $i > 0$. If $K \subseteq \mathbb{C}$, the *proof* of [Voi14, Thm. 3.13], or equivalently of [Voi07b, Thm. 10.17, p.294], shows that if for some natural number N the multiple $N\Delta_{X_{\mathbb{C}}} \in \mathrm{CH}^{d_X}(X_{\mathbb{C}} \times_{\mathbb{C}} X_{\mathbb{C}})$ admits a cohomological decomposition of type (d_1, d_2) with respect to $H^\bullet((-)^{an}, \mathbb{Z})$, then $H^0(X, \Omega_X^i) = 0$ for all $i > d_2$.

7.3. Cohomological decomposition of the diagonal and algebraic cycle classes.

Definition 7.9 (Algebraic cohomology classes). Let X be a smooth projective variety over a perfect field K. Let $R \to R_{\mathcal{H}}$ be a ring homomorphism. We say that a class $\alpha \in \mathcal{H}^{2i}(X)(i)$ is *R-algebraic* if it is in the image of the cycle class map

$$[-] : \mathrm{CH}^i(X)_R \to \mathcal{H}^{2i}(X)(i).$$

We say that $\mathcal{H}^{2i}(X)$ is *R-algebraic* if the map above is surjective. We say that $\alpha \in \mathcal{H}^{2i}(X)(i)$ is *algebraic* (resp. $\mathcal{H}^{2i}(X)$ is *algebraic*) if it is $R_{\mathcal{H}}$-algebraic. In other words, $\mathcal{H}^{2i}(X)$ is algebraic if $\mathcal{H}^{2i}(X)(i)$ is spanned over $R_{\mathcal{H}}$ by the image of the cycle class map $[-] : \mathrm{CH}^i(X) \to \mathcal{H}^{2i}(X)(i)$.

Note that we require algebraic cohomology classes to be the classes of algebraic cycles on X, rather than on $X_{\overline{K}}$. This leads to some subtleties. For instance, even for $i = 0$, one may have $\mathcal{H}^0(X)$ is not algebraic: consider X connected but not geometrically connected. Likewise, even for $i = d_X$, one may have $\mathcal{H}^{2d_X}(X)$ is not algebraic: taking X to be a smooth conic over $\mathbb{Q}$ with no $\mathbb{Q}$-points, $H^2(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_2)$ is not spanned by zero-cycles on X, as there is no zero-cycle of odd degree defined over $\mathbb{Q}$. Note however that, if K is a finite field, then any variety over K admits a zero-cycle of degree-1 and hence $\mathcal{H}^{2d_X}(X)$ is indeed algebraic in that case.

Proposition 7.10. *Let X be a smooth projective variety over a perfect field K. Assume that for some natural number N the multiple $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)_R$ admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_X - 1$ and $\dim W_2 \leq 1$. Let $\widetilde{W}_1 \to W_1$ be an alteration of degree M invertible in $R_{\mathcal{H}}$ with $\widetilde{W}_1$ smooth projective over K.*

(1) If $\mathcal{H}^{2i-2}(\widetilde{W}_1)$ is R-algebraic (e.g., $2i \leq 2$), then $N \cdot \mathcal{H}^{2i}(X)$ is R-algebraic.

(2) If $\mathcal{H}^{2i}(\widetilde{W}_1)$ is R-algebraic (e.g., $2i \geq 2d_X - 2$), then $N \cdot \mathcal{H}^{2i}(X)$ is R-algebraic.

Proof. This follows directly from the proof of [BS83, Thm. 1(iv)] or [Voi13, Thm. 4.4(ii)]; *i.e.*, from the factorization of correspondences in cohomology (3.4). $\square$

Note that if the image of N is zero in the coefficient ring of the cohomology theory, the conclusions of Proposition 7.10 are trivial.

Corollary 7.11. *Let X be a smooth projective variety over an algebraically closed field K. If $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a cohomological R-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_X - 1$ and $\dim W_2 \leq 1$, and with W_1 admitting a resolution of singularities, then $\mathcal{H}^{2i}(X)$ is R-algebraic for $2i \leq 2$ and $2i \geq 2d_X - 2$.*

In particular, for any smooth projective threefold *X over an algebraically closed field K such that $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a cohomological R-decomposition of type $(2, 1)$, we have $\mathcal{H}^{2i}(X)$ is R-algebraic for all i.*

Proof. This is immediate from Proposition 7.10. $\qquad\square$

7.4. **Cohomological decomposition of the diagonal and the Bloch map.** We observe here that the assumption of [ACMV21, Prop. 5.2] involving a Chow decomposition can be weakened to a *cohomological* decomposition:

Proposition 7.12. *Let X be a smooth projective variety over a perfect field K of characteristic exponent p. Fix a natural number N and a prime ℓ. Either let $R_\mathcal{H} = \mathbb{Z}_\ell$ and assume $\ell \nmid pN$, or let $R_\mathcal{H} = \mathbb{Q}_\ell$ and assume $\ell \neq p$. Fix a ring homomorphism $R \to R_\mathcal{H}$. Assume that $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits an R-cohomological decomposition of type $(d_X - 1, 2)$ with respect to $H^\bullet(-, R_\mathcal{H})$.*

Then the inclusion $V_\ell \mathrm{A}^2(X_{\overline{K}}) \hookrightarrow V_\ell \mathrm{CH}^2(X_{\overline{K}})$ is an isomorphism of $\mathrm{Gal}(K)$-modules, and the second ℓ-adic Bloch map

$$V_\ell \lambda^2 : V_\ell \mathrm{CH}^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Q}_\ell(2))$$

is an isomorphism of Galois modules.

Moreover, if $R_\mathcal{H} = \mathbb{Z}_\ell$, then

$$T_\ell \lambda^2 : T_\ell \mathrm{CH}^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau$$

is an isomorphism of $\mathrm{Gal}(K)$-modules; and if if $\dim W_1 \leq d_X - 1$ and $\dim W_2 \leq 1$, then $H^3(X_{\overline{K}}, \mathbb{Z}_\ell)$ is torsion-free.

Proof. The proof is exactly the same as that of [ACMV21, Prop. 5.2], where the assertion is made only for a Chow decomposition of the diagonal. For convenience, we include the proof here for a cohomological R-decomposition with respect to $H^\bullet(-, \mathbb{Z}_\ell)$; the case of $H^\bullet(-, \mathbb{Q}_\ell)$ is identical. Let ℓ be a prime. That $T_\ell \lambda^2$ is an injective morphism of $\mathrm{Gal}(K)$-modules was reviewed in [ACMV21]. Using Lemma 3.1, one decomposes $Np^e \Delta_X^* = p^e Z_1^* + p^e Z_2^*$, with factorizations $p^e Z_1^* = (\tilde{j}_1)_* \circ \widetilde{Z}_1^*$ and $p^e Z_2^* = (\widetilde{Z}_2)_* \circ \tilde{j}_2^*$. (For use in Lemma 7.13 below, we note that we can set $e = 0$ in case $\dim X \leq 4$ due to resolution of singularities [CP09] in dimensions ≤ 3.) We obtain by the naturality of the ℓ-adic Bloch map a commutative diagram

$$
\begin{array}{ccccc}
T_\ell \mathrm{CH}^2(X_{\overline{K}}) & \xrightarrow{\widetilde{Z}_1^* \oplus \tilde{j}_2^*} & T_\ell \mathrm{CH}^1((\widetilde{W}_1)_{\overline{K}}) \oplus T_\ell \mathrm{CH}^2((\widetilde{W}_2)_{\overline{K}}) & \xrightarrow{(\tilde{j}_1)_* + (\widetilde{Z}_2)_*} & T_\ell \mathrm{CH}^2(X_{\overline{K}}) \\
\downarrow{\scriptstyle T_\ell \lambda^2} & & \simeq \downarrow{\scriptstyle T_\ell \lambda^1_{\widetilde{W}_1} \oplus T_\ell \lambda^2_{\widetilde{W}_2}} & & \downarrow{\scriptstyle T_\ell \lambda^2} \\
H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau & \xrightarrow{\widetilde{Z}_1^* \oplus \tilde{j}_2^*} & H^1((\widetilde{W}_1)_{\overline{K}}, \mathbb{Z}_\ell(1)) \oplus H^3((\widetilde{W}_2)_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau & \xrightarrow{(\tilde{j}_1)_* + (\widetilde{Z}_2)_*} & H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau.
\end{array}
$$

(7.1)

The middle vertical arrow is an isomorphism by Kummer theory and Rojtman's theorem (see [ACMV21]), while the composition of the (bottom) horizontal arrows is multiplication by Np^e. A diagram chase then establishes the surjectivity of $T_\ell \lambda^2$.

Finally, in case $\dim W_2 \leq 1$, that $H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))$ is torsion-free follows simply from the factorization of the multiplication by Np^e map

$$H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2)) \xrightarrow{\widetilde{Z}_1^*} H^1((\widetilde{W}_1)_{\overline{K}}, \mathbb{Z}_\ell(1)) \xrightarrow{(\tilde{j}_1)_*} H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))$$

and the fact that $H^1((\widetilde{W}_1)_{\overline{K}}, \mathbb{Z}_\ell(1))$ is torsion-free. $\qquad\square$

For the sake of completeness, we record an analogous statement about p-torsion in Chow. Note that in positive characteristic p, crystalline cohomology, unlike $H^\bullet(-, \mathbb{Q}_p)$, is a Weil cohomology.

Lemma 7.13. *Let X be a smooth projective variety over a perfect field K of characteristic $p > 0$. Let $R_{\mathcal{H}}$ be either $\mathbb{W}(K)$ or $\mathbb{B}(K)$, and fix a ring homomorphism $R \to R_{\mathcal{H}}$. Assume that for some natural number N, we have that $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits an R-cohomological decomposition of type $(d_X - 1, 2)$ with respect to $H^\bullet_{\mathrm{cris}}(-/R_{\mathcal{H}})$.*

Then the inclusion $V_p \mathrm{A}^2(X_{\overline{K}}) \hookrightarrow V_p \mathrm{CH}^2(X_{\overline{K}})$ is an isomorphism of $\mathrm{Gal}(K)$-modules, and the second p-adic Bloch map

$$V_p\lambda^2 : V_p \mathrm{CH}^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Q}_p(2))$$

is an isomorphism of $\mathrm{Gal}(K)$-modules.

Moreover, if $R_{\mathcal{H}} = \mathbb{W}(K)$, if $p \nmid N$, and if resolution of singularities holds in dimension $< d_X$, then $T_p \mathrm{A}^2(X_{\overline{K}}) \to T_p \mathrm{CH}^2(X_{\overline{K}})$ is an isomorphism, and the second p-adic Bloch map

$$T_p\lambda^2 : T_p \mathrm{CH}^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_p(2))_\tau$$

is an isomorphism of $\mathrm{Gal}(K)$-modules. If in addition we have $\dim W_1 \leq d_X - 1$ and $\dim W_2 \leq 1$, then $H^3_{\mathrm{cris}}(X/\mathbb{W}(K))$ and $H^3(X_{\overline{K}}, \mathbb{Z}_p(2))$ are torsion-free.

Proof. The proof strategy is identical to that of Proposition 7.12, bearing in mind that a decomposition in crystalline cohomology induces maps on p-adic cohomology, and that the necessary properties of the p-adic Bloch map are secured by Gros and Suwa [GS88]. $\qquad\square$

We can similarly give a small improvement on [BW20, Prop. 2.3(ii)] regarding the Bloch map, which shows that under the stronger assumption that the decomposition is of type $(d_X - 1, 1)$, one gets the stronger conclusion that the ℓ-adic Bloch map is an isomorphism at all primes.

Proposition 7.14 ([BW20, Prop. 2.3(ii)]). *Let X be a smooth projective variety over a perfect field K of characteristic exponent p. Fix a natural number N and a prime l; let $\mathcal{H}^\bullet(-)$ denote $H^\bullet(-, \mathbb{Z}_l)$ if $l \neq p$ and $H^\bullet_{\mathrm{cris}}(-/\mathbb{W}(K))$ if $l = p$.*

Suppose that $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a $\mathbb{Z}$-cohomological decomposition of type $(d_X - 1, 1)$ with respect to $\mathcal{H}^\bullet$. Then the second Bloch map

$$\lambda^2 : \mathrm{A}^2(X_{\overline{K}})[l^\infty] \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$$

is an isomorphism of $\mathrm{Gal}(K)$-modules. Taking Tate modules, the second l-adic Bloch map

$$T_l\lambda^2 : T_l \mathrm{A}^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_l(2))_\tau$$

is also an isomorphism of $\mathrm{Gal}(K)$-modules.

Proof. We adapt the proof of [BW20, Prop. 2.3(ii)] to the setting of cohomological decomposition of the diagonal, and verify in the process that argument of Benoist–Wittenberg works for p-adic

cohomology, as well. For X smooth and projective, we have a diagram with exact row (see *e.g.*, [ACMV21, A.16] and [GS88, (3.33)])

$$\mathrm{CH}^2(X_{\overline{K}})[l^\infty]$$

$$0 \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow H^3(X_{\overline{K}}, \mathbb{Q}_l/\mathbb{Z}_l(2)) \longrightarrow H^4(X_{\overline{K}}, \mathbb{Z}_l(2))$$

with vertical map λ_l^2 from $\mathrm{CH}^2(X_{\overline{K}})[l^\infty]$ and a dashed diagonal arrow.

where the dashed arrow is, up to sign, the cycle class map ([CTSS83, Cor. 4], [GS88, Prop. III.1.16 and Prop. III.1.21]). Since algebraically trivial cycles are homologically trivial, it follows that the image of $\mathrm{A}^2(X_{\overline{K}})[l^\infty]$ under λ_l^2 is contained in $H^3(X_{\overline{K}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \subseteq H^3(X_{\overline{K}}, \mathbb{Q}_l/\mathbb{Z}_l)$. In particular, the cokernel of $\lambda_l^2 : \mathrm{A}^2(X_{\overline{K}})[l^\infty] \to H^3(X_{\overline{K}}, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$ is divisible.

Now suppose $N\Delta_X$ admits a cohomological decomposition of type $(d_X - 1, 1)$. Arguing as in the proof of Proposition 7.12 and Lemma 7.13, coker λ_l^2 is annihilated by Np^e. Therefore, this cokernel is trivial, and $\lambda_l^2 : \mathrm{A}^2(X_{\overline{K}})[l^\infty] \to H^3(X_{\overline{K}}, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$ is an isomorphism. Taking the Tate module of this isomorphism gives the assertion for the l-adic Bloch map, since $H^3(X_{\overline{K}}, \mathbb{Z}_l)$ is a finitely generated $\mathbb{Z}_l$-module, and so it is elementary to check that $T_l(H^3(X_{\overline{K}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l) \cong H^3(X_{\overline{K}}, \mathbb{Z}_l)_\tau$. $\square$

8. Cohomological correspondences and algebraic representatives

Recall that for a complex projective manifold X, given a cycle class $Z \in \mathrm{CH}^n(T \times X)$ parameterized by a complex projective manifold T with fiber-wise homologically trivial cycle classes, the Abel–Jacobi map induces a holomorphic map $T \to J^{2n-1}(X)$, $t \mapsto AJ(Z_t)$, which, by construction, only depends on the cohomology class of Z in $H^{2n}(T \times X, \mathbb{Z})$. The main goal of this section is to show (Corollary 8.2) that the same holds for morphisms induced by algebraic representatives. In fact, this follows from Proposition 8.1, which will allow us in many situations to "lift" to rational equivalence the action of a cohomological decomposition on algebraic representatives. Recall that, given a smooth projective variety X over a field K, an algebraic representative $\Phi_X^i : \mathscr{A}_{X/K}^i \to \mathrm{Ab}_{X/K}^i$ exists for $i \in \{1, 2, d_X\}$; see [ACMV23].

Proposition 8.1. *Let X and Y be smooth proper varieties of dimension d_X and d_Y respectively over a field K and let $\gamma \in \mathrm{CH}^{d_X+j-i}(X \times_K Y)$ be a correspondence. Assume that $j \in \{1, 2, d_Y\}$ and that an algebraic representative $\Phi_X^i : \mathscr{A}_{X/K}^i \to \mathrm{Ab}_{X/K}^i$ exists. Let $f : \mathrm{Ab}_{X/K}^i \to \mathrm{Ab}_{Y/K}^j$ be the morphism induced by γ and the universal property of the algebraic representative via the commutative diagram*

$$\begin{array}{ccc} \mathscr{A}_{X/K}^i & \xrightarrow{\gamma_*} & \mathscr{A}_{Y/K}^j \\ \Phi_X^i \downarrow & & \downarrow \Phi_Y^j \\ \mathrm{Ab}_{X/K}^i & \xrightarrow{f} & \mathrm{Ab}_{Y/K}^j \end{array} \tag{8.1}$$

If $[\gamma] = 0 \in H^{2(d_X+j-i)}(X_{K^{\mathrm{sep}}} \times_{K^{\mathrm{sep}}} Y_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell(d_X + j - i))$ for some $\ell \neq \mathrm{char}(K)$, then $f = 0$. In other words, f depends only on the cohomology class of γ.

Moreover, if $[\gamma] = 0 \in H^{2(d_X+j-i)}(X_{K^{\mathrm{sep}}} \times_{K^{\mathrm{sep}}} Y_{K^{\mathrm{sep}}}, \mathbb{Z}/\ell^{v+1}\mathbb{Z}(d_X + j - i))$ for some $\ell \neq \mathrm{char}(K)$ and some natural number v, then $f[\ell^v] : \mathrm{Ab}_{X/K}^i[\ell^v] \to \mathrm{Ab}_{Y/K}^j[\ell^v]$ is the zero map.

Proof. First note that, since $\Phi_Y^j \circ \gamma_* : \mathscr{A}_{X/K}^i \to \mathrm{Ab}_{Y/K}^j$ defines a regular homomorphism, the universal property of the algebraic representative $\mathrm{Ab}_{X/K}^i$ provides a K-homomorphism $f : \mathrm{Ab}_{X/K}^i \to \mathrm{Ab}_{Y/K}^j$ giving the commutative diagram (8.1).

Let Z be a miniversal cycle (see §1.3) for Φ_X^i and let us denote r the natural number such that the K-homomorphism $\Phi_X^i(\mathrm{Ab}_{X/K}^i)(Z)$ is given by multiplication by r. Recalling that $\Phi_X^i(\mathrm{Ab}_{X/K}^i)(Z)$ is simply given by $\phi_{X_{\overline{K}}}^i \circ w_Z$, we then have

$$\phi_{Y_{\overline{K}}}^j \circ \gamma_* \circ w_Z = f \circ (\phi_{X_{\overline{K}}}^i \circ w_Z) = r\, f. \tag{8.2}$$

Now, if $[\gamma] = 0 \in H^{2(d_X+j-i)}(X_{K^{\mathrm{sep}}} \times_{K^{\mathrm{sep}}} Y_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell(d_X + j - i))$, clearly $[\gamma \circ Z]$ also vanishes. By naturality of the ℓ-adic Bloch maps, we hence get a commutative diagram

$$
\begin{array}{ccc}
& \xrightarrow{\quad \gamma_* \circ w_Z \quad} & \\
V_\ell \mathrm{Ab}_{X/K}^i \longrightarrow V_\ell \mathrm{A}_0((\mathrm{Ab}_{X/K}^i)_{K^{\mathrm{sep}}}) & \xrightarrow{\gamma_* \circ Z_*} & V_\ell \mathrm{A}^j(Y_{K^{\mathrm{sep}}}) \\
\Big\downarrow {\scriptstyle V_\ell \lambda^0} & & \Big\downarrow {\scriptstyle V_\ell \lambda^j} \\
{\scriptstyle \mathrm{id}} \quad V_\ell \mathrm{Ab}_{X/K}^i & \xrightarrow{\gamma_* \circ Z_* = 0} & H^{2j-1}(Y_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell(j)).
\end{array}
$$

Note that the left triangle commutes thanks to §1.5.2 together with Lemma 1.4 and the fact that the Bloch map λ^0 coincides with the albanese map on ℓ-primary torsion [Blo79, Prop. 3.9]. Recalling that $V_\ell \lambda^j$ is injective for $j = 1, 2, d_Y$, we conclude that $\gamma_* \circ w_Z : V_\ell \mathrm{Ab}_{X/K}^i \to V_\ell \mathrm{A}^j(Y_{K^{\mathrm{sep}}})$ is zero, and hence in view of (8.2) that $r \cdot V_\ell f = 0$; i.e., $f = 0$.

The same argument works in the case $[\gamma] = 0 \in H^{2(d_X+j-i)}(X_{K^{\mathrm{sep}}} \times_{K^{\mathrm{sep}}} Y_{K^{\mathrm{sep}}}, \mathbb{Z}/\ell^{\nu+1}\mathbb{Z}(d_X + j - i))$, using that the finite level Bloch maps $\lambda^j[\ell^\nu]$ are injective for $j = 1, 2, d_Y$ (see [ACMV21, App. A, Prop. A.27]), and replacing the Tate modules with ℓ^ν-torsion in the diagram above. $\square$

From Proposition 8.1 we can show that morphisms induced by projective families of cycles and universal regular homomorphisms depend only on the cohomology class of the family of cycles.

Corollary 8.2. *Let X and T be a smooth projective varieties over a field K, with $T(K) \neq \varnothing$, and let $Z \in \mathscr{A}_{X/K}^i(T)$ be a family of algebraically trivial cycle classes. Assume that there exists an algebraic representative $\Phi_{X/K}^i : \mathscr{A}_{X/K}^i \to \mathrm{Ab}_{X/K}^i$ (e.g., $i \in \{1, 2, d_X\}$), and let*

$$\psi_Z : T \longrightarrow \mathrm{Ab}_{X/K}^i$$

be the associated morphism. If $[Z] = 0 \in H^{2i}((T \times_K X)_{\overline{K}}, \mathbb{Q}_\ell(i))$, then $\psi_Z = 0$; in other words, ψ_Z depends only on the cohomology class of Z.

Proof. Fix $t_0 \in T(K)$. Set $\Gamma = \Delta_T - (T \times_K t_0)$ and $Z' = Z + (T \times_K Z_{t_0})$. We have the commutative diagram

$$
\begin{array}{ccccc}
T & \xrightarrow[w_\Gamma]{t \mapsto t - t_0} & \mathrm{A}^{d_T}(T_{\overline{K}}) & \xrightarrow{Z'_*} & \mathrm{A}^i(X_{\overline{K}}) \\
\Big\downarrow {\scriptstyle \mathrm{alb}_{t_0}} & & \Big\downarrow {\scriptstyle \phi_{T_{\overline{K}}}^{d_T}} {\scriptstyle \psi_Z} & & \Big\downarrow {\scriptstyle \phi_{X_{\overline{K}}}^i} \\
\mathrm{Alb}_{T/K} & = = = & \mathrm{Ab}_{X/K}^{d_T} & \xrightarrow{f} & \mathrm{Ab}_{X/K}^i
\end{array}
$$

By Proposition 8.1, f depends only on the cohomology class $[Z']$, and therefore, by commutativity, we see that ψ_Z depends only on the cohomology class $[Z']$. Now since the cohomology class $[Z_{t_0}]$ depends only on the cohomology class $[Z]$, we see that the cohomology class $[Z']$ depends only on the cohomology class $[Z]$. $\square$

Remark 8.3. In both Proposition 8.1 and Corollary 8.2, if $K \subseteq \mathbb{C}$, then by the comparison theorems in cohomology, one is free to replace ℓ-adic cohomology with Betti cohomology.

9. Cohomological decomposition and universal codimension-2 cycle classes

This section contains our new results regarding cohomological decomposition of the diagonal and universal codimension-2 cycle classes.

For a surjective regular homomorphism $\Phi : \mathscr{A}^n_{X/K} \to A$, a *universal* (resp. *miniversal*) *codimension-n cycle class* (for Φ) is a cycle class $Z \in \mathscr{A}^n_{X/K}(A)$ such that the induced morphism $\psi_Z = \Phi(A)(Z) : A \to A$ is the identity (resp. N times the identity for some natural number N). It is a basic fact that there always exist miniversal cycle classes [ACMV17, Lem. 4.9]. Theorem 9.1 relates the number N in the case of algebraic representatives to cohomological decompositions of N times the diagonal.

We start by recalling the well-known story for $n = 1$, coming from the identification $\mathrm{Ab}^1_{X/K} = (\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$. For brevity, we will call a universal codimension-1 cycle class for $\mathrm{Ab}^1_{X/K}$ a universal divisor. There exists a universal divisor if each component of X admits a K-point (*e.g.*, if K is separably closed). In general, there need not exist a universal divisor; in fact, one can exhibit curves over fields of characteristic 0 with no universal divisor. However, any smooth projective variety X over a finite field K (with or without a K-point) does admit a universal divisor. We refer the reader to [ACMV23, §7.1] for more details.

The albanese morphism provides an algebraic representative in codimension $n = \dim X$, and we have the identification $\mathrm{Ab}^n_{X/K} = \mathrm{Alb}_{X/K}$. Voisin [Voi23, Cor. 0.14] showed that a universal zero-cycle does not necessarily exist even for K algebraically closed, thereby answering the question raised in [ACMV23, Rem. 7.11].

We now investigate the connection between universal codimension-2 cycle classes and cohomological decompositions of the diagonal.

Theorem 9.1 (Miniversal cycle classes on algebraic representatives). *Let X be a smooth projective variety over a perfect field K of characteristic exponent p. Fix a positive integer n and a prime number $\ell \neq p$. Assume:*

- *For some natural number N (resp. some natural number N coprime to ℓ) the multiple $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ of the diagonal admits a cohomological decomposition (resp. cohomological $\mathbb{Z}_\ell$-decomposition) of type (W_1, W_2) with $d_{W_1} \leq d_X - (n-1)$ and $d_{W_2} \leq n-1$;*
- *W_1 admits a smooth projective alteration of degree p^e admitting a universal divisor (e.g., K is finite or algebraically closed).*

If there exists an algebraic representative $\Phi^n_{X/K} : \mathscr{A}^n_{X/K} \to \mathrm{Ab}^n_{X/K}$ (e.g., $n \in \{1, 2, d_X\}$), then there exists a family of algebraically trivial cycle classes $Z \in \mathscr{A}^n_{X/K}(\mathrm{Ab}^n_{X/K})$ such that the induced morphism of abelian varieties over K,

$$\psi_Z = \Phi^n_{X/K}(\mathrm{Ab}^n_{X/K})(Z) : \mathrm{Ab}^n_{X/K} \to \mathrm{Ab}^n_{X/K},$$

is multiplication by Np^e (resp. multiplication by r for some natural number r coprime to ℓ).

Remark 9.2. In Theorem 9.1, the constraints on d_{W_1} and d_{W_2} are perhaps more restrictive than they first appear. In fact, we can have $d_{W_1} = d_X - (n-1)$ and $d_{W_2} = n-1$ or $n-2$, or we can have $d_{W_1} = d_X - n$ and $d_{W_2} = n-1$, since from Remark 2.2, we know that $d_{W_1} + d_{W_2} \geq d_X - 1$. At the same time, since $0 \leq d_{W_1}, d_{W_2} \leq d_X - 1$, this also puts the constraints $2 \leq n \leq d_X$. Moreover, in the case $d_{W_1} = d_X - n$, it will follow from the proof of Theorem 9.1 that $\Phi^n_{X/K} = 0$.

Proof. Assuming a cohomological decomposition, the proof of [Voi13, Thm. 4.4(iii)] applies here; *i.e.*, this follows from the factorization of correspondences (3.3) and Proposition 8.1. More precisely, using Lemma 3.1, one decomposes $Np^e\Delta^*_X = p^e Z^*_1 + p^e Z^*_2$, with factorizations $p^e Z^*_1 = (\tilde{j}_1)_* \circ \widetilde{Z}^*_1$ and $p^e Z^*_2 = (\widetilde{Z}_2)_* \circ \tilde{j}^*_2$. Since $n > \dim \widetilde{W}_2$, it follows that $\tilde{j}^*_2 = 0$ on Chow groups, and therefore, by diagram (3.3), this also holds for the induced morphism of abelian varieties. We

therefore obtain a commutative diagram

$$
\begin{array}{ccccc}
\mathscr{A}^n_{X/K} & \xrightarrow{\ \widetilde{Z}^*_1\ } & \mathscr{A}^1_{\widetilde{W}_1/K} & \xrightarrow{\ (\tilde{j}_1)_*\ } & \mathscr{A}^n_{X/K} \\
\downarrow{\scriptstyle \Phi^n_X} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \Phi^1_{\widetilde{W}_1}} & & \downarrow{\scriptstyle \Phi^n_X} \\
\mathrm{Ab}^n_X & \xrightarrow{\ \widetilde{Z}^*_1\ } & (\mathrm{Pic}^0_{\widetilde{W}_1/K})_{\mathrm{red}} & \xrightarrow{\ (\tilde{j}_1)_*\ } & \mathrm{Ab}^n_X,
\end{array}
\tag{9.1}
$$

where the composition of the horizontal arrows is given by multiplication by Np^e and where we have denoted $\widetilde{Z}^*_1 : \mathrm{Ab}^n_{X/K} \to (\mathrm{Pic}^0_{\widetilde{W}_1})_{\mathrm{red}}$ and $(\tilde{j}_1)_* : (\mathrm{Pic}^0_{\widetilde{W}_1})_{\mathrm{red}} \to \mathrm{Ab}^n_{X/K}$ the K-homomorphisms induced by the correspondences $\widetilde{Z}^*_1$ and $(\tilde{j}_1)_*$. Here we are using Proposition 8.1.

We have assumed the existence of a universal divisor $\widetilde{\mathcal{D}} \in \mathscr{A}^1_{\widetilde{W}_1/K}((\mathrm{Pic}^0_{\widetilde{W}_1/K})_{\mathrm{red}})$, meaning that the induced morphism:

$$
\psi_{\widetilde{\mathcal{D}}} = \Phi^1_{\widetilde{W}_1/K}((\mathrm{Pic}^0_{\widetilde{W}_1/K})_{\mathrm{red}})(\widetilde{\mathcal{D}}) : (\mathrm{Pic}^0_{\widetilde{W}_1/K})_{\mathrm{red}} \to (\mathrm{Pic}^0_{\widetilde{W}_1/K})_{\mathrm{red}}
$$

is the identity. Consider the cycle class

$$
Z := (\tilde{j}_1)_* \circ \widetilde{\mathcal{D}} \circ \widetilde{Z}^*_1 \quad \in \mathscr{A}^n_{X/K}(\mathrm{Ab}^n_{X/K}).
$$

It is then clear from (9.1) that the associated homomorphism $\psi_Z = \Phi^n_{X/K}(\mathrm{Ab}^n_{X/K})(Z) : \mathrm{Ab}^n_{X/K} \to \mathrm{Ab}^n_{X/K}$ is given by $Np^e \mathrm{Id}_{\mathrm{Ab}^n_{X/K}}$.

The case where one assumes a cohomological $\mathbb{Z}_\ell$-decomposition is similar. For simplicity, since we are assuming that $\ell \nmid N$, N is invertible in $\mathbb{Z}_\ell$, and so we may and do assume $N = 1$. We assume that we have Z_1, Z_2, as in the definition of the $\mathbb{Z}_\ell$-decomposition of the diagonal, observing that they are given as cycles with $\mathbb{Z}_\ell$-coefficients. We have by definition that $[\Delta_X] = [Z_1] + [Z_2]$ in $H^{2d_X}((X \times_K X)_{\overline{K}}, \mathbb{Z}_\ell(d_X))$. Now let Z'_1 and Z'_2 be integral cycles which ℓ-adically approximate Z_1 and Z_2. (Concretely, if $Z_i = \sum a_{ij} Z_j$ with $a_{ij} \in \mathbb{Z}_\ell$, choose $a'_{ij} \in \mathbb{Z}$ with $a'_{ij} \equiv a_{ij} \bmod \ell^2$, and let $Z'_i = \sum a'_{ij} Z_i$.) We therefore have the equality $[\Delta_X] = [Z'_1] + [Z'_2]$ in $H^{2d_X}((X \times_K X)_{\overline{K}}, \mathbb{Z}/\ell^2\mathbb{Z}(d_X))$. The rest of the proof goes through identically, so that we find a cycle Z such that that the associated homomorphism $\psi_Z[\ell] = \Phi^n_{X/K}(\mathrm{Ab}^n_{X/K})(Z)[\ell] : \mathrm{Ab}^n_{X/K}[\ell] \to \mathrm{Ab}^n_{X/K}[\ell]$ is given by $Np^e \mathrm{Id}_{\mathrm{Ab}^n_{X/K}}$. Note that the cycle Z depended on our choice to approximate to order ℓ^2, as well as our choice of lift; truncating at higher order would also work, although it typically yields a different cycle Z. In any case, since ψ_Z is an injection on ℓ-torsion, it is an injection on ℓ-power torsion, and therefore an isomorphism on ℓ-power torsion.

It follows that ψ_Z is an isogeny, of degree coprime to ℓ. Arguing as in [ACMV17, Lem. 4.9] or [ACMV23, Lem. 4.7], one can find a cycle $Z' \in \mathscr{A}^n_{X/K}(\mathrm{Ab}^n_{X/K})$ so that $\psi_{Z'}$ is multiplication by r for some natural number r with $\ell \nmid r$. $\qquad\square$

Corollary 9.3 (Universal codimension-2 cycle classes). *Suppose X is a smooth projective variety of dimension ≤ 4 over a field K that is either finite or algebraically closed. If Δ_X admits a cohomological decomposition (resp. cohomological $\mathbb{Z}_\ell$-decomposition) of type $(d_X - 1, 1)$, then there exists a universal codimension-2 cycle class (resp. a miniversal codimension-2 cycle class of degree coprime to ℓ).*

Proof. This follows immediately from Theorem 9.1 and the fact [CP09] that resolution of singularities holds for varieties of dimension ≤ 3 over perfect fields. $\qquad\square$

We now consider the case where X is a smooth projective variety over a field $K \subseteq \mathbb{C}$, and we consider the surjective regular homomorphism $AJ : \mathscr{A}^n_{X/K} \to J^{2n+1}_{a,X/K}$ to the distinguished model, given by the Abel–Jacobi map, as defined in [ACMV20].

Theorem 9.4 (Mini- and universal cycle classes on distinguished models). *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$, and fix a positive integer n. Assume further that for some natural number N the multiple $N\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a cohomological decomposition of type (W_1, W_2) with $d_{W_2} \leq n - 1$, $d_{W_1} \leq d_X + 1 - n$, and that W_1 admits a resolution of singularities with each component admitting a K-point. Then there exists a family of algebraically trivial cycle classes $Z \in \mathscr{A}^n_{X/K}(J^{2n-1}_{a,X/K})$ such that the induced morphism of abelian varieties over K,*

$$\psi_Z : J^{2n-1}_{a,X/K} \to J^{2n-1}_{a,X/K},$$

is multiplication by N.

Proof. The proof of Theorem 9.4 is identical to that of Theorem 9.1, except one uses diagram (3.6) rather than (3.3) and the fact that the action of a correspondence on intermediate Jacobians only depends on its cohomology class. □

Remark 9.5. In the cases $n = 1, 2$ or d_X, we have $J^{2n-1}_{a,X/K} = \mathrm{Ab}^n_{X/K}$, and the Abel–Jacobi map is the universal regular homomorphism $\Phi^n_{X/K}$, so that in those cases Theorems 9.4 and 9.1 are closely related. The benefit of Theorem 9.4 is that in characteristic 0, we always have the distinguished model of the image of the Abel–Jacobi map for any n, whereas algebraic representatives, in any characteristic, are known to exist in general only for $n = 1, 2, d_X$.

10. THE STANDARD ASSUMPTION

We now discuss an isomorphism, (10.5) below, that plays a central role in our presentation moving forward (Theorem 12.6 and consequently Theorem 1(1)–(2)). It makes it possible, in positive characteristic, to relate the Tate module of the second algebraic representative of a smooth projective variety X to the third cohomology group of X. Since we expect such an isomorphism to hold true in general (and it does in characteristic zero and in any characteristic for geometrically rationally chain connected varieties; see Proposition 10.3), we call this the standard assumption (Definition 10.5). In order to slightly streamline the presentation, we recall our convention that ℓ always denotes a rational prime (*i.e.*, a prime number in $\mathbb{Z}$) invertible in the base field (which we denote by K), while l is allowed to be the (positive) characteristic of the base field.

10.1. **Cohomological decomposition of the diagonal and Bloch–Srinivas' result on algebraic representatives.** A fundamental result due to Bloch–Srinivas [BS83, Thm. 1(i)] states that over an algebraically closed field of characteristic 0, a decomposition of the diagonal implies the algebraic representative in codimension-2 is isomorphic to the group of algebraically trivial codimension-2 cycle classes. Their argument carries over to smooth projective varieties over perfect fields (this was also recently observed in [BW20, Prop. 2.3]). Here we improve their result by only assuming the existence of a *cohomological* decomposition of the diagonal.

Proposition 10.1. *Let X be a smooth projective variety over a perfect field K of characteristic exponent p and let ℓ be a prime $\neq p$. Suppose that, for some natural number N, the multiple $N\Delta_{X_{\overline{K}}} \in \mathrm{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ admits a cohomological decomposition (resp. cohomological $\mathbb{Z}_\ell$-decomposition) of type $(d_X - 1, 1)$ with respect to $H^\bullet(-, \mathbb{Z}_\ell)$. Then the universal regular homomorphism $\Phi^2_{X/K} : \mathscr{A}^2_{X/K} \to \mathrm{Ab}^2_{X/K}$ is an isomorphism of $\mathrm{Gal}(K)$-modules on $\overline{K}$-points; i.e.,*

$$\Phi^2_{X/K}(\overline{K}) : \mathrm{A}^2(X_{\overline{K}}) \xrightarrow{\sim} \mathrm{Ab}^2_{X/K}(\overline{K})$$

(resp. is an isomorphism on ℓ-power torsion if $\ell \nmid N$; i.e., $\Phi^2_{X/K}(\overline{K})[\ell^\infty]$ is an isomorphism).

Proof. We combine the proof in [BS83, Thm. 1(i)], where it is assumed that $N\Delta_{X_{\overline{K}}} \in \mathrm{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ admits a decomposition of type $(d_X - 1, 1)$, with Proposition 8.1.

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First, it suffices to prove the result after base change to the algebraic closure $\overline{K}$, so we assume that $K = \overline{K}$ to make the notation more streamlined. We only need to show that $\phi_X^2 = \Phi_{X/K}^2(K) : A^2(X) \to \mathrm{Ab}_{X/K}^2(K)$ is injective, since the map to the algebraic representative is always surjective on points over an algebraically closed field.

We are given that

$$N[\Delta_X] = [Z_1] + [Z_2] \in H^{2d_X}(X \times_K X, \mathbb{Z}_\ell(d_X))$$

with $Z_1 \in \mathrm{CH}^{d_X}(X \times_K X)$ supported on $W_1 \times_K X$ where W_1 is a divisor, and $Z_2 \in \mathrm{CH}^{d_X}(X \times_K X)$ is supported on $X \times_K W_2$ with $\dim W_2 \leq 1$. Since $K = \overline{K}$ is perfect, we have a smooth projective alteration of W_1 of some degree a power of p, say p^e. We then consider the correspondence $Np^e\Delta_X^*$. By Proposition 8.1, $Np^e\Delta_X^*$ acts as $p^eZ_1^* + p^eZ_2^*$ on $\mathrm{Ab}_{X/K}^2$. Now we use our factorization (3.3). As $\dim W_2 \leq 1$, we have $p^eZ_2^* = 0$ on $A^2(X)$, so that $Np^e\Delta_X^* = p^eZ_1^*$ on $\mathrm{Ab}_{X/K}^2$. Under the identification $A^1(\widetilde{W}_1) = (\mathrm{Pic}^0(\widetilde{W}_1))_{\mathrm{red}}(K)$ we obtain a map $r := (\tilde{j}_1)_* \circ \widetilde{Z}_1^* : \mathrm{Ab}_{X/K}^2(K) \to A^2(X)$, which is a homomorphism (via a diagram chase) with kernel contained in the Np^e-torsion. The inverse of the isomorphism $\mathrm{Ab}_{X/K}^2(K)/(\ker r) \xrightarrow{\sim} A^2(X)$ is a regular homomorphism (use that for any smooth variety T we have $A^2(T \times_K X)$ is divisible), and then a diagram chase using the universal property of the algebraic representative shows that $\Phi_{X/K}^2$ is injective, completing the proof.

In the case of a $\mathbb{Z}_\ell$-cohomological decomposition, we make the same modifications as in Theorem 9.1. From Proposition 1.6, we know that $\Phi_{X/K}^2(\overline{K})[\ell^\infty]$ is surjective, so we only need to show it is injective. As before, we may set $N = 1$. We assume that we have Z_1, Z_2, as in the definition of the $\mathbb{Z}_\ell$-decomposition of the diagonal, observing that they are given as cycles with $\mathbb{Z}_\ell$-coefficients. We have by definition that $[\Delta_X] = [Z_1] + [Z_2]$ in $H^{2d_X}((X \times_K X)_{\overline{K}}, \mathbb{Z}_\ell(d_X))$. As in the proof of Theorem 9.1, choose integral cycles Z_1' and Z_2' which respectively approximate Z_1 and Z_2 modulo ℓ^2. We have the equality $[\Delta_X] = [Z_1'] + [Z_2']$ in $H^{2d_X}((X \times_K X)_{\overline{K}}, \mathbb{Z}/\ell^2\mathbb{Z}(d_X))$. The rest of the proof goes through identically, so that we find that $\Phi_{X/K}^2(\overline{K})$ is an injection on ℓ-torsion. This implies it is an injection on ℓ-power torsion, and we are done. $\qquad\square$

Remark 10.2. In Proposition 10.1, the natural transformation $\Phi_{X/K}^2 : \mathscr{A}_{X/K}^2 \to \mathrm{Ab}_{X/K}^2$ need not be an isomorphism of functors [ACMV23, Rem. 5.2].

We have the following application:

Proposition 10.3. *Let X be a smooth projective variety over a perfect field K. Assume one of the following:*

(1) $\mathrm{char}(K) = 0$, *or,*

(2) *for some natural number N, the multiple $N\Delta_{X_{\overline{K}}} \in \mathrm{CH}^{d_X}(X_{\overline{K}} \times_{\overline{K}} X_{\overline{K}})$ admits a cohomological decomposition (resp. cohomological $\mathbb{Z}_\ell$-decomposition) of type $(d_X - 1, 1)$ with respect to $H^\bullet(-, \mathbb{Z}_\ell)$ for some prime $\ell \neq \mathrm{char}(K)$.*

Then for all primes l (resp. for $l = \ell$), the morphisms

$$\phi_{X_{\overline{K}}}^2[l^\infty] : A^2(X_{\overline{K}})[l^\infty] \xrightarrow{\sim} \mathrm{Ab}_{X/K}^2(\overline{K})[l^\infty] \tag{10.1}$$

$$T_l\phi_{X_{\overline{K}}}^2 : T_l\, A^2(X_{\overline{K}}) \xrightarrow{\sim} T_l\, \mathrm{Ab}_{X/K}^2 \tag{10.2}$$

are isomorphisms of $\mathrm{Gal}(K)$-*modules.*

Proof. Under the hypothesis (1), applying T_l to both sides of (10.1), we see that the morphism (10.2) follows from (10.1). For (10.1), using Lecomte's rigidity theorem [Lec86], we immediately reduce to the case $K = \mathbb{C}$, in which case this is [Mur85, Thm. 10.3]. We note that there is a small gap in Murre's proof; it is not obvious that for a surjective regular homomorphism, the induced

morphism on ℓ-torsion is surjective. This uses Lemma 1.3 and the existence of miniversal cycles, and is explained in [ACMV20, Lem. 3.2(b) & Rem. 3.3].

Under hypothesis (2), the conclusions follow immediately from Proposition 10.1. $\square$

Remark 10.4. If in Proposition 10.3 the decomposition is of type $(d_X - 1, 2)$, and $\ell \nmid Np$, then (10.1) and (10.2) are isomorphisms. This follows by combining Proposition 8.1 with the arguments for [ACMV21, Prop. 3.8(3)] (which addresses the case of a Chow decomposition).

10.2. **The standard assumption.** To start our discussion, we observe that given a regular homomorphism

$$\Phi : \mathscr{A}^n_{X/K} \longrightarrow A ,$$

if the natural map

$$T_l \Phi(\overline{K}) : T_l \, \mathrm{A}^n(X_{\overline{K}}) \xrightarrow{\simeq} T_l A \tag{10.3}$$

is an isomorphism, then there is a canonical morphism ι of $\mathrm{Gal}(K)$-modules:

$$T_l A(\overline{K}) \xrightarrow{T_l \Phi(\overline{K})^{-1}} T_l \, \mathrm{A}^n(X_{\overline{K}}) \lhook\joinrel\longrightarrow T_l \, \mathrm{CH}^n(X_{\overline{K}}) \xrightarrow{T_l \lambda^n} H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau. \tag{10.4}$$

$$\iota$$

When $K = \mathbb{C}$, n is $1, 2$, or d_X, and $A = \mathrm{Ab}^n_{X/K}$, this agrees with the canonical inclusion coming from Hodge theory (as the Bloch map agrees with the Abel–Jacobi map on torsion [Blo79, Prop. 3.7]).

An isomorphism of the type (10.3) in the case $n = 2$ turns out to be quite central to our treatment and deserves to be singled out:

Definition 10.5 (The standard assumption). We say a smooth projective variety X over a field K and a prime l satisfy the *standard assumption*, or that X satisfies the standard assumption at l, if the homomorphism

$$\phi^2_{X_{\overline{K}}}[l^\infty] : \mathrm{A}^2(X_{\overline{K}})[l^\infty] \longrightarrow \mathrm{Ab}^2_{X/K}[l^\infty] \tag{10.5}$$

is an isomorphism.

If the standard assumption holds, then, by taking Tate modules,

$$T_l \phi^2_{X_{\overline{K}}} : T_l \, \mathrm{A}^2(X_{\overline{K}}) \longrightarrow T_l \, \mathrm{Ab}^2_{X/K} \tag{10.6}$$

is an isomorphism, as well, in which case we will denote

$$\iota : \; T_l \, \mathrm{Ab}^2_{X/K} \xrightarrow[\sim]{(T_l \phi^2_{X_{\overline{K}}})^{-1}} T_l \, \mathrm{A}^2(X_{\overline{K}}) \xrightarrow{T_l \lambda^2} H^3(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau \tag{10.7}$$

the composition, and similarly with $\mathbb{Q}_l$-coefficients.

Remark 10.6. As explained in Proposition 10.3, the standard assumption holds unconditionally if $\mathrm{char}(K) = 0$, and holds in positive characteristic for those smooth projective varieties X whose diagonal admit a positive multiple with a cohomological decomposition of type $(d_X - 1, 1)$; *e.g.*, geometrically rationally chain connected varieties. We in fact expect that the standard assumption holds unconditionally; to establish this for almost all primes, it would suffice to show the standard assumption holds for varieties over finite fields (see [ACMV21, Lem. 4.3]).

It is a basic fact in Hodge theory that for a family of homologically trivial cycle classes, the normal function defined via the Abel–Jacobi map induces on cohomology the same morphism as the family of cycle classes viewed as a correspondence (see (11.1)). As this fact is quite useful in characteristic 0, the purpose of this section is to explain how to interpret this fact in positive characteristic, which we do in terms of the commutativity of a certain diagram (see (11.2)). The main results are Proposition 11.6, as well as Proposition 11.6 together with its consequence, Corollary 11.8, regarding the case of codimension-2 cycle classes, which essentially says that the diagram is commutative for geometrically rationally chain connected varieties (see Remark 11.9).

11.1. **Defining the commutative diagram.** For motivation, consider the situation where X and T are complex projective manifolds, and $Z \in \mathrm{CH}^n(T \times X)$ is a cycle class that is fiberwise algebraically trivial, *i.e.*, $Z \in \mathscr{A}_{X/\mathbb{C}}(T)$. Via the Abel–Jacobi map, we obtain a morphism

$$\psi_Z : T \longrightarrow J^{2n-1}(X).$$

It is a standard fact (see *e.g.*, [Voi07a, Thm. 12.17]) that $(\psi_Z)_*$ and the correspondence Z_* agree on $H^{2d_T-1}(T,\mathbb{Z})$ in the sense that there is a commutative diagram

$$
\begin{array}{ccc}
H_1(T,\mathbb{Z}) \xrightarrow{\ (\psi_Z)_*\ } & H_1(J^{2n-1}(X),\mathbb{Z}) =\!=\!=\!=\!= & H^{2n-1}(X,\mathbb{Z}(n))_\tau \\
\ \ \searrow{\scriptstyle Z_*} & & \nearrow \\
& H^{2n-1}(X,\mathbb{Z}(n)) &
\end{array}
\tag{11.1}
$$

We note that ψ_Z has image contained in the image $J_a^{2n-1}(X)$ of the restriction of the Abel–Jacobi map to algebraically trivial cycles $AJ : \mathrm{A}^n(X) \to J^{2n-1}(X)$. If X admits $K \subseteq \mathbb{C}$ as a field of definition, the subtorus $J_a^{2n-1}(X) \subseteq J^{2n-1}(X)$ was shown in [ACMV20] to admit a distinguished model $J_{a,X/K}^{2n-1}$ over K, in such a way that $AJ : \mathrm{A}^n(X) \to J_a^{2n-1}(X)$ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. Since the Bloch map agrees with the Abel–Jacobi map on torsion [Blo79, Prop. 3.7], we obtain for smooth projective varieties T, X over a field $K \subseteq \mathbb{C}$ and $Z \in \mathscr{A}_{X/K}^n(T)$ a commutative diagram:

$$
\begin{array}{ccc}
& T_\ell\, \mathrm{A}^n(X_{\overline{K}}) & \\
& {\scriptstyle T_\ell AJ}\ \swarrow \qquad \searrow\ {\scriptstyle T_\ell \lambda^n} & \\
H^{2d_T-1}(T_{\overline{K}},\mathbb{Z}_\ell(d_T)) \xrightarrow{\ (\psi_Z)_*\ } T_\ell J_{a,X/K}^{2n-1} \hookleftarrow \qquad \longrightarrow & H^{2n-1}(X_{\overline{K}},\mathbb{Z}_\ell(n))_\tau \\
\ \searrow{\scriptstyle Z_*} & \nearrow & \\
H^{2n-1}(X_{\overline{K}},\mathbb{Z}_\ell(n)) &
\end{array}
$$

As the Abel–Jacobi map is complex in nature, it is not immediately clear how to generalize this statement to varieties over fields of positive characteristic. However, as algebraic representatives provide a replacement for the Abel–Jacobi maps in positive characteristic, let us assume that there exists an algebraic representative $\Phi_{X/K}^n : \mathscr{A}_{X/K}^n \to \mathrm{Ab}_{X/K}^n$, which is always the case if $n = 1, 2, d_X$. While in general it is unclear whether there is a canonical map $T_l\,\mathrm{Ab}_{X/K}^n \to H^{2n-1}(X_{\overline{K}},\mathbb{Z}_l(n))_\tau$, if we assume that

$$T_l\phi_{X_{\overline{K}}}^n : T_l\,\mathrm{A}^n(X_{\overline{K}}) \xrightarrow{\ \sim\ } T_l\,\mathrm{Ab}_{X/K}^n$$

is an isomorphism, then we obtain a diagram

$$H^{2d_T-1}(T_{\overline{K}}, \mathbb{Z}_l(d_T)) \xrightarrow{(\psi_Z)_*} T_l \mathrm{Ab}^n_{X/K} \xrightarrow[\simeq]{(T_l\phi^n_X)^{-1}} T_l \mathrm{A}^n(X_{\overline{K}}) \xrightarrow{T_l\lambda^n} H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau$$

$$\searrow_{Z_*} \qquad H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))$$

(11.2)

Note that if $n = 1$ or $n = d_X$, then $T_l\phi^n_{X_{\overline{K}}}$ is an isomorphism for all primes l (for $n = 1$ this is Kummer theory, while for $n = d_X$ this is Rojtman's theorem; see [Blo79, GS88, Mil82], and [ACMV21, Appendix] for a review). Recall also from Remark 10.6 that $T_l\phi^n_{X_{\overline{K}}}$ is an isomorphism for $n = 2$ in characteristic zero and for geometrically rationally chain connected varieties in positive characteristic.

The question is then the following:

Question 11.1. *Assuming* $T_l\phi^n_{X_{\overline{K}}} : T_l \mathrm{A}^n(X_{\overline{K}}) \xrightarrow{\sim} T_l \mathrm{Ab}^n_{X/K}$ *is an isomorphism, under what conditions is diagram* (11.2) *commutative?*

In Lemma 11.4 below, we establish the commutativity of (11.2) in the case $n = 1$ and in Corollary 11.7 below, we establish the commutativity of (11.2) in the case $n = d_X$. If $n = 2$ and $K \subseteq \mathbb{C}$, the algebraic representative $\Phi^2_{X/K}$ coincides with the Abel–Jacobi map after base-change to $\mathbb{C}$. As such, under the above conditions, the diagram (11.2) commutes. The commutativity of (11.2) for $n = 2$ and for perfect fields K of positive characteristic will be established in Corollary 11.8 below, assuming that $T_l\phi^2_{X_{\overline{K}}}$ is an isomorphism and that $V_l\lambda^2$ is surjective.

Remark 11.2. Note that since $H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau$ is by definition torsion-free, it suffices to show that (11.2) is commutative with $\mathbb{Q}_l$-coefficients. Indeed, suppose M and N are $\mathbb{Z}_l$-modules, and $f, g : M \to N$ are $\mathbb{Z}_l$-module homomorphisms that agree after tensoring with $\mathbb{Q}_l$. If N is torsion-free, then $f = g$: given $m \in M$, we have that $r(f(m) - g(m)) = 0$ for some $r \in \mathbb{Z}_l$; but since N is torsion-free, this implies that $f(m) = g(m)$.

11.2. **Diagram** (11.2) **in the case** $n = 1$. In case $\mathrm{char}(K) = 0$, as explained in §11.1, the diagram (11.2) with $n = 1$ commutes. We now observe that it also commutes with $n = 1$ for any perfect field K. We start by recalling the following fact:

Proposition 11.3. *Let X be a smooth projective variety over a perfect field K, and let n be a natural number. Then there exist a smooth projective, geometrically integral, curve C over K admitting a K-point and a correspondence $\gamma \in \mathrm{CH}^n(X \times_K C)$ such that for all primes $\ell \neq \mathrm{char}(K)$*

$$\mathrm{N}^{n-1} H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) = \mathrm{Im}\left(\gamma_* : H^1(C_{\overline{K}}, \mathbb{Q}_\ell(1)) \longrightarrow H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))\right).$$

Here, $\mathrm{N}^\bullet$ is the geometric coniveau filtration.

Proof. This is [ACMV20, Prop. 1.1] which actually holds over any perfect field K. □

Lemma 11.4 (Diagram (11.2) with $n = 1$). *Let X be a smooth projective variety over a perfect field K and let l be a prime number. The map $T_l\phi^1_{X_{\overline{K}}} : T_l \mathrm{A}^1(X_{\overline{K}}) \xrightarrow{\sim} T_l \mathrm{Ab}^1_{X/K}$ is an isomorphism, and for any smooth projective variety T and any $Z \in \mathscr{A}^1_{X/K}(T)$, the diagram (11.2) commutes in the case $n = 1$. In particular, if $T = B$ is an abelian variety, then $T_l\lambda^1 \circ T_l w_Z = Z_* : T_l B \to H^1(X_{\overline{K}}, \mathbb{Z}_l(1))$.*

Proof. Using the fact that $\mathrm{Ab}^1_{X/K} = (\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$ (e.g., [ACMV23, Rem. 7.2]), it follows from the definition of the Picard functor that $\phi^1_{X_{\overline{K}}} : \mathrm{A}^1(X_{\overline{K}}) \xrightarrow{\sim} \mathrm{Ab}^1_{X/K}(\overline{K})$ is an isomorphism. Taking Tate modules gives that the map $T_l\phi^1_{X_{\overline{K}}} : T_l \mathrm{A}^1(X_{\overline{K}}) \xrightarrow{\sim} T_l \mathrm{Ab}^1_{X/K}$ is an isomorphism.

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We now proceed to establish the commutativity of (11.2) in the case $n = 1$. We first consider the case where T is a curve. The key point is then to use the identification, valid even when $l = \mathrm{char}(K)$ (*e.g.*, [GS88, Cor. 3.3]), that $H^1(X_{\overline{K}}, \boldsymbol{\mu}_{l^\nu}) = \mathrm{Pic}^0_{X_{\overline{K}}/\overline{K}}(\overline{K})[l^\nu]$, obtained via identifying $\boldsymbol{\mu}_{l^\nu}$-torsors over X with étale covers, and then with torsion line bundles (and similarly for T). Using that $d_T = 1$ (so that $H^{2d_T - 1}(T_{\overline{K}}, \mathbb{Z}_l(d_T)) = H^1(T_{\overline{K}}, \mathbb{Z}_l(1))$), the commutativity follows from the definitions of the maps.

For the general case, it is slightly more convenient to use Remark 11.2, and prove commutativity with $\mathbb{Q}_l$-coefficients.

Initially, suppose $l = \ell \neq \mathrm{char}(K)$. To start, we use Proposition 11.3: there exist a smooth projective curve C over K and a correspondence $\gamma \in \mathrm{CH}^1(C \times_K T)$ such that $H^{2d_T - 1}(T_{\overline{K}}, \mathbb{Q}_\ell(d_T)) = \gamma_* H^1(C_{\overline{K}}, \mathbb{Q}_\ell(1))$. We then consider the diagram

$$
H^1(C_{\overline{K}})(1) \xrightarrow{\gamma_*} H^{2d_T - 1}(T_{\overline{K}})(d_T) \xrightarrow{(\psi_Z)_*} V_l \mathrm{Ab}^1_{X/K} \xrightarrow[\simeq]{(V_l \phi^1_X)^{-1}} V_l \, \mathrm{A}^1(X_{\overline{K}}) \xrightarrow{\lambda^1_X} H^1(X_{\overline{K}})(1) \quad (11.3)
$$

with the upper arc $(\psi_{Z \circ \gamma})_*$, the lower arc $(Z \circ \gamma)_*$, and $Z_* : H^{2d_T-1}(T_{\overline{K}})(d_T) \to H^1(X_{\overline{K}})(1)$,

where we are using $\mathbb{Q}_\ell$-coefficients. A diagram chase then completes the proof of commutativity.

We now consider the case where $l = p = \mathrm{char}(K) > 0$. The same γ and C used before yield a surjection of F-isocrystals $\gamma_* : H^1_{\mathrm{cris}}(C/\mathbb{B}(\overline{K}))(1) \to H^{2d_T - 1}_{\mathrm{cris}}(T/\mathbb{B}(\overline{K}))(d_T)$. Upon taking Frobenius invariants, we obtain a surjection $\gamma_{*,p} : H^1(C_{\overline{K}}, \mathbb{Q}_p(1)) \to H^{2d_T - 1}(X_{\overline{K}}, \mathbb{Q}_p(d_T))$. The same chase of (11.3) establishes the commutativity.

Finally, that $T_l \lambda^1 \circ T_l w_Z = Z_*$ in case T is an abelian variety follows from the commutativity of (11.2) and Lemma 1.4. $\qquad\square$

11.3. Diagram (11.2) in the case $n > 1$.

We start with the following observation which answers positively the conjecture [GS88, Conj. III.4.1(iii)] in case $2n - 1 \leq d_X$ (and in case $2n - 1 > d_X$ provided X satisfies the Lefschetz standard conjecture):

Lemma 11.5 ([ACMV21, Prop. 6.1 & Rem. 6.2]). *Let X be a smooth projective variety over a perfect field K. Let n be a natural number and let $\ell_0 \neq \mathrm{char}(K)$ be a prime. Suppose that*

- $V_{\ell_0} \lambda^m : V_{\ell_0} \mathrm{A}^m(X_{\overline{K}}) \longrightarrow H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m))$ *is surjective for $m := \min\{n, d_X - n + 1\}$.*

Then $V_l \lambda^n : V_l \mathrm{A}^n(X_{\overline{K}}) \longrightarrow H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n))$ is surjective for all primes l.

In addition, there exist an abelian variety A over K, and correspondences $\Gamma \in \mathrm{CH}^{d_X + 1 - n}(X \times_K \widehat{A})$ and $\Gamma' \in \mathrm{CH}^n(A \times_K X)$, which induce for all primes l isomorphisms of $\mathrm{Gal}(K)$-modules

$$
\Gamma_* : H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n)) \xrightarrow{\sim} V_l A \qquad \textit{and} \qquad \Gamma'_* : V_l A \xrightarrow{\sim} H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)). \qquad (11.4)
$$

Proof. This is essentially [ACMV21, Prop. 6.1] and we provide a proof here for convenience. We start with the given $\ell_0 \neq \mathrm{char}(K)$. We can of course assume $0 \leq n \leq d$, so that $0 \leq m \leq d/2$. Since $V_{\ell_0} \lambda^m$ is surjective, and since $\mathrm{im}(V_{\ell_0} \lambda^m) = \mathrm{N}^{m-1} H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m))$ (see [Suw88, Prop. 5.2] or [ACMV21, Prop. 2.1]), there exist by Proposition 11.3 a smooth projective, geometrically integral, curve C over K admitting a K-point and a correspondence $\gamma \in \mathrm{CH}^m(X \times_K C)$ inducing a surjection

$$
H^1(C_{\overline{K}}, \mathbb{Q}_{\ell_0}(1)) \xrightarrow{\gamma^*} H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m)).
$$

Taking a hyperplane section H_X and dualizing the surjection above we obtain an injection

$$H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m)) \xrightarrow[\sim]{H_X^{d-2m+1}} H^{2d-2m+1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(d-m+1)) \xhookrightarrow{\gamma_*} H^1(C_{\overline{K}}, \mathbb{Q}_{\ell_0}(1)).$$

The correspondence $\gamma_* \circ H_X^{d-2m+1} \circ \gamma^* \in \mathrm{CH}^1(C \times_K C)_{\mathbb{Q}}$ induces, via the choice of a K-point on C, an element of $f \in \mathrm{End}(\mathrm{Alb}_C) \otimes \mathbb{Q}$. Using the semi-simplicity of the category of abelian varieties over K up to isogeny, taking A to be the image of f, and clearing denominators, we obtain correspondences $\Gamma \in \mathrm{CH}^{d_X+1-m}(X \times_K \widehat{A})$ and $\Gamma' \in \mathrm{CH}^m(A \times_K X)$ such that $\Gamma \circ \Gamma' \in \mathrm{CH}^1(A \times \widehat{A})$ induces an isogeny $A \to A$ and such that

$$\Gamma_* : H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m)) \xrightarrow{\sim} V_{\ell_0} A \qquad \text{and} \qquad \Gamma'_* : V_{\ell_0} A \xrightarrow{\sim} H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m))$$

are isomorphisms. In case $m = d - n + 1$, by dualizing, we also obtain isomorphisms

$${}^t\Gamma'_* : H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(n)) \xrightarrow{\sim} V_{\ell_0} \widehat{A} \qquad \text{and} \qquad {}^t\Gamma_* : V_{\ell_0} \widehat{A} \xrightarrow{\sim} H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(n)).$$

We conclude that the homomorphisms (11.4) are isomorphisms for all primes $\ell \neq p$ using the invariance of the ℓ-adic Betti numbers for all $\ell \neq p$. Since crystalline and ℓ-adic Betti numbers coincide, Γ induces an isomorphism of isocrystals $H_{\mathrm{cris}}^{2n-1}(X/\mathbb{B}(K))(n) \cong H_{\mathrm{cris}}^1(A/\mathbb{B}(K))$; taking Frobenius invariants yields (11.4) with $l = p$.

Finally, from the right-hand side isomorphism of (11.4), we see that for all primes l we have $H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n)) = \mathrm{N}^{n-1}H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n))$. We conclude, by [ACMV21, Prop. 2.1] again, that $V_l \lambda^n$ is surjective. $\qquad\square$

The main result of this section is the following proposition providing an answer to Question 11.1 under certain conditions.

Proposition 11.6. *Let X be a smooth projective variety over a perfect field K. Suppose that X admits an algebraic representative $\Phi_X^n : \mathscr{A}_{X/K}^n \to \mathrm{Ab}_{X/K}^n$ in codimension n and suppose that there exists a prime $\ell_0 \neq \mathrm{char}(K)$ such that $V_{\ell_0} \lambda^m : V_{\ell_0} \mathrm{A}^m(X_{\overline{K}}) \longrightarrow H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m))$ is surjective for $m := \min\{n, d - n + 1\}$.*

Let l be any prime such that $T_l \phi_{X_{\overline{K}}}^n : T_l \mathrm{A}^n(X_{\overline{K}}) \xrightarrow{\sim} T_l \mathrm{Ab}_{X/K}^n$ is an isomorphism. Then for any smooth projective variety T and any $Z \in \mathscr{A}_{X/K}^n(T)$, the diagram (11.2) commutes.

Proof. As mentioned in Remark 11.2, it suffices to prove the commutativity of (11.2) with $\mathbb{Q}_l$-coefficients. The proof consists in reducing to the case of codimension-1 cycles by showing that the diagram (11.2) with $\mathbb{Q}_l$-coefficients is the direct summand of a similar diagram with Ab^1 in place of Ab^n, in which case the commutativity is proven in Lemma 11.4.

Using the given ℓ_0, choose A, Γ and Γ' as in Lemma 11.5; recall that these objects induce the isomorphisms (11.4) for *all* l. By the universal property of the algebraic representatives, the correspondence Γ induces a K-morphism $f : \mathrm{Ab}_X^n \to \mathrm{Ab}_A^1 = A^\vee$ and the correspondence Γ' induces a K-morphism $g : \mathrm{Ab}_A^1 \to \mathrm{Ab}_X^n$, making the following diagrams commute:

$$
\begin{array}{ccc}
\mathrm{A}^n(X_{\overline{K}}) & \xrightarrow{\phi_{X_{\overline{K}}}^n} & \mathrm{Ab}_X^n(\overline{K}) \\
\downarrow{\scriptstyle\Gamma_*} & & \downarrow{\scriptstyle f} \\
\mathrm{A}^1(A_{\overline{K}}) & \xrightarrow{\phi_{A_{\overline{K}}}^1} & \mathrm{Ab}_A^1(\overline{K})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{A}^1(A_{\overline{K}}) & \xrightarrow{\phi_{A_{\overline{K}}}^1} & \mathrm{Ab}_A^1(\overline{K}) \\
\downarrow{\scriptstyle\Gamma'_*} & & \downarrow{\scriptstyle g} \\
\mathrm{A}^n(X_{\overline{K}}) & \xrightarrow{\phi_{X_{\overline{K}}}^n} & \mathrm{Ab}_X^n(\overline{K}).
\end{array}
$$

Given a regular homomorphism $\Phi : \mathscr{A}^i_{X/K} \to D$ and a smooth separated variety T over K, a correspondence $\Theta \in \mathscr{A}^i_{X/K}(T)$ induces a K-morphism $\psi_\Theta : T \to D$, which itself induces a morphism $(\psi_\Theta)_* : H^{2d_T-1}(T_{\overline{K}}, \mathbb{Z}_l(d_T)) \to T_l D$. We obtain a commutative diagram

$$
\begin{array}{ccccccc}
H^{2d_T-1}(T_{\overline{K}}, \mathbb{Q}_l(d_T)) & \xrightarrow{\psi_{Z,*}} & V_l\, \mathrm{Ab}^n_{X/K} & \xrightarrow[\simeq]{(V_l\phi^n_{X_{\overline{K}}})^{-1}} & V_l\, \mathrm{A}^n(X_{\overline{K}}) & \xrightarrow{\lambda^n_X} & H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n)) \qquad (11.5)\\
& {}_{\psi_{\Gamma\circ Z,*}}\searrow & \downarrow{\scriptstyle V_l f} & & \downarrow{\scriptstyle \Gamma_*} & & \downarrow{\scriptstyle \Gamma_*}\\
& & V_l\, \mathrm{Ab}^1_{A/K} & \xrightarrow[\simeq]{(V_l\phi^1_{A_{\overline{K}}})^{-1}} & V_l\, \mathrm{A}^1(A_{\overline{K}}) & \xrightarrow[\simeq]{\lambda^1_A} & H^1(A_{\overline{K}}, \mathbb{Q}_l(1))\\
& {}_{\psi_{\Gamma'\circ\Gamma\circ Z,*}}\searrow & \downarrow{\scriptstyle V_l g} & & \downarrow{\scriptstyle \Gamma'_*} & & \downarrow{\scriptstyle \Gamma'_*}\\
& & V_l\, \mathrm{Ab}^n_{X/K} & \xrightarrow[\simeq]{(V_l\phi^n_{X_{\overline{K}}})^{-1}} & V_l\, \mathrm{A}^n(X_{\overline{K}}) & \xrightarrow{\lambda^n_X} & H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n)).
\end{array}
$$

The right squares commute thanks to the naturality of the Bloch map (see [GS88] for the case $l = p$), the middle squares commute by construction of f and g and the left part of the diagram commutes by the definition of regular homomorphisms. The commutativity of (11.5) yields $\Gamma'_* \circ \Gamma_* \circ \lambda^n_X \circ (V_l\phi^2_X)^{-1} \circ \psi_{Z_*} = \Gamma'_* \circ \lambda^1_A \circ (V_l\phi^1_A)^{-1} \circ \psi_{\Gamma\circ Z,*} = \Gamma'_* \circ \Gamma_* \circ Z_*$, *i.e.*, that the diagram (11.2) commutes after composing with $\Gamma'_* \circ \Gamma_*$. Since $\Gamma'_* \circ \Gamma_* : H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n)) \to H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n))$ is an isomorphism thanks to Lemma 11.5, we conclude the diagram (11.2) commutes with $\mathbb{Q}_l$-coefficients. $\qquad\square$

We obtain unconditionally the commutativity of the diagram (11.2) in case $n = \dim X$:

Corollary 11.7 (Diagram (11.2) with $n = \dim X$). *Let X be a smooth projective variety of dimension d over a perfect field K. Then, for any smooth projective variety T, any $Z \in \mathscr{A}^d_{X/K}(T)$ and any prime l, the diagram (11.2) with $n = d$ commutes.*

Proof. Recall that an algebraic representative for codimension-d cycles on $X_{\overline{K}}$ is given by the albanese morphism. The latter is an isomorphism on torsion by Rojtman's theorem and so $T_l\phi^d_{X_{\overline{K}}}$ is an isomorphism. On the other hand $T_l\lambda^1$ is an isomorphism by Kummer theory. The assumptions of Proposition 11.6 are met and we can conclude. $\qquad\square$

Finally, since this will be important to us, we state explicitly Proposition 11.6 in the case $n = 2$:

Corollary 11.8 (Diagram (11.2) with $n = 2$). *Let X be a smooth projective variety over a perfect field K. If $\mathrm{char}(K) > 0$, suppose that there exists a prime $\ell_0 \neq \mathrm{char}(K)$ such that $V_{\ell_0}\lambda^2 : V_{\ell_0}\, \mathrm{A}^2(X_{\overline{K}}) \to H^3(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(2))$ is surjective.*
If X satisfies the standard assumption at l (Definition 10.5), then for any smooth projective variety T and for any $Z \in \mathscr{A}^2_{X/K}(T)$, the diagram (11.2) with $n = 2$ commutes.

Proof. The case where $\mathrm{char}(K) = 0$ was explained in §11.1 (and the standard assumption is satisfied for all l). The case where $\mathrm{char}(K) > 0$ is Proposition 11.6 in case $d_X > 2$ and Corollary 11.7 in case $d_X = 2$ (in which case the assumptions on $T_\ell\phi^2_{X_{\overline{K}}}$ and $V_\ell\lambda^2$ are superfluous). $\qquad\square$

Remark 11.9. Note that, due to Proposition 7.12 and Proposition 10.1, the assumptions of Corollary 11.8 are met for smooth projective varieties X over a perfect field K whose $\mathrm{CH}_0(X)_\mathbb{Q}$ is universally supported on a curve, *e.g.*, for smooth projective geometrically rationally chain connected varieties, or for smooth projective varieties with geometric MRC quotient of dimension ≤ 1.

11.4. **Commutativity in the case of an abelian variety.** In case the parameter space $T = B$ is an abelian variety, we can rephrase the commutativity of (11.2) under less restrictive assumptions.

The point is that in this case (11.2) becomes

$$
\begin{array}{c}
\xymatrix{
& & T_\ell w_Z & & \\
T_l B \ar@/^2pc/[urr] \ar@<0.5ex>[r]^{(\psi_Z)_*} \ar[dr]_{Z_*} & T_l \mathrm{Ab}^n_{X/K} \ar[rr]^{(T_l \phi^n_X)^{-1}}_{\simeq} & & T_l\, \mathrm{A}^n(X_{\overline{K}}) \ar[r]^{T_l \lambda^n} & H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau \\
& H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n)) \ar@/_1pc/[urr] & & &
}
\end{array}
$$

and therefore, ignoring the existence of the algebraic representative, and whether $T_l \phi^n_X$ is an isomorphism if the algebraic representative exists, we can rephrase commutativity as:

Proposition 11.10. *Let X be a smooth projective variety over a perfect field K and let n be a natural number. Suppose there exists a prime $\ell_0 \neq \mathrm{char}(K)$ such that:*

- *$V_{\ell_0} \lambda^m : V_{\ell_0}\, \mathrm{A}^m(X_{\overline{K}}) \longrightarrow H^{2m-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(m))$ is surjective for $m := \min\{n, d - n + 1\}$.*

Then, for any abelian variety B over K, any $Z \in \mathscr{A}^n_{X/K}(B)$ and any prime l, the diagram

$$
\begin{array}{c}
\xymatrix{
T_l B \ar[r]^{w_Z} \ar[dr]_{Z_*} & T_l\, \mathrm{A}^n(X_{\overline{K}}) \ar[d]^{T_l \lambda^n} \\
& H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau
}
\end{array}
\tag{11.6}
$$

is commutative.

Proof. Recall from Lemma 1.4 that in our situation w_Z induces a map on l-adic Tate modules. Since $T_l B$ and $H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_l(n))_\tau$ are torsion-free, it suffices to prove commutativity of (11.2) with $\mathbb{Q}_l$-coefficients. The proof consists then in using Lemma 11.5 to reduce to the case of codimension-1 cycles, in which case the commutativity is proven in Lemma 11.4.

Using ℓ_0, choose A, Γ and Γ' as in Lemma 11.5. We obtain a commutative diagram

$$
\begin{array}{c}
\xymatrix{
T_l B \ar[r]^{w_Z} \ar[dr]_{w_{\Gamma \circ Z}} & T_l\, \mathrm{A}^n(X_{\overline{K}}) \ar[r]^{T_l \lambda^n_X} \ar[d]^{\Gamma_*} & H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n))_\tau \ar[d]^{\Gamma_*} \\
& T_l\, \mathrm{A}^1(A_{\overline{K}}) \ar[r]^{T_l \lambda^1_A}_{\simeq} \ar[d]^{\Gamma'_*} & H^1(A_{\overline{K}}, \mathbb{Q}_l(1)) \ar[d]^{\Gamma'_*} \\
& T_l\, \mathrm{A}^n(X_{\overline{K}}) \ar[r]^{T_l \lambda^n_X} & H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n))_\tau.
}
\end{array}
\tag{11.7}
$$

The squares commute thanks to the naturality of the Bloch map and the left part of the diagram commutes by the definition of the maps w_Z and $w_{\Gamma \circ Z}$. Therefore $\Gamma'_* \circ \Gamma_* \circ T_l \lambda^n_X \circ w_Z = \Gamma'_* \circ T_l \lambda^1_A \circ w_{\Gamma \circ Z} = \Gamma'_* \circ (\Gamma \circ Z)_*$, where the second equality follows from Lemma 11.4. Thus the diagram (11.6) commutes after composing with $\Gamma'_* \circ \Gamma_*$. Since $\Gamma'_* \circ \Gamma_* : H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n)) \to H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_l(n))$ is an isomorphism, we conclude the diagram (11.6) commutes with $\mathbb{Q}_l$-coefficients. $\square$

12. COHOMOLOGICAL DECOMPOSITION AND SELF-DUALITY OF THE ALGEBRAIC REPRESENTATIVE

Let X be a smooth projective threefold over a field K and assume that $X_{\overline{K}}$ admits a universal codimension-2 cycle Z. The aim of this section is to study the symmetric $\overline{K}$-homomorphism

$$
\Theta_X : \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \to \widehat{\mathrm{Ab}}^2_{X_{\overline{K}}/\overline{K}}
$$

induced (see §1.5) by the cycle $-({}^t Z \circ Z) \in \mathrm{CH}^1(\mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}} \times_{\overline{K}} \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}})$. Under the assumption that $V_\ell \lambda^2$ is surjective for some prime $\ell \neq \mathrm{char}(K)$, we show in Theorem 12.6 that Θ_X is an isogeny that descends to K and is independent of the choice of the universal cycle Z. Moreover, in characteristic 0, in the case where X is geometrically rationally connected, we show that Θ_X is the Hodge-theoretic polarization induced via the intersection pairing in cohomology (see Remark 12.7), and that a similar statement holds in positive characteristic for the symmetric K-isogeny Θ_X induced by $-({}^t Z \circ Z)$ (the precise meaning of this is explained in Definition 12.1). In particular, this shows that the isomorphism in Theorem 4.4, in characteristic 0, agrees with the Hodge-theoretic polarization induced via the intersection pairing in cohomology. In addition, let us already mention that, in positive characteristic, we will show in Proposition 13.1 that Θ_X is a polarization under some liftability conditions to characteristic zero. For instance, in Corollary 13.3, we will show that if X is a geometrically stably rational threefold, and if X lifts to characteristic 0 as a geometrically rationally connected threefold, then Θ_X is a principal polarization.

12.1. **Motivation from Hodge theory: morphisms of abelian varieties induced by cup product in cohomology.**

Let X be a complex projective manifold, let H be an ample divisor on X, let n be a nonnegative integer with $1 \leq 2n - 1 \leq d_X$, and assume $\mathrm{N}^{n-1} H^{2n-1}(X, \mathbb{Q}) = H^{2n-1}(X, \mathbb{Q})$, which implies

$$H^{2n-1}(X, \mathbb{C}) = H^{n+1,n}(X) \oplus H^{n,n-1}(X).$$

Then the Hodge–Riemann bilinear pairing $(\alpha, \beta) \mapsto i(-1)^n \int_X \alpha \wedge \overline{\beta} \wedge [H]^{d_X - 2n - 1}$ gives a Hermitian form h on $H^{n,n+1}(X)$ so that $-\operatorname{Im} 2h$ is the cup product in cohomology (up to a sign):

$$H^{2n-1}(X, \mathbb{Z})_\tau \otimes H^{2n-1}(X, \mathbb{Z})_\tau \longrightarrow H^{2d_X}(X, \mathbb{Z}) = \mathbb{Z}$$

$$(\alpha, \beta) \mapsto (-1)^{n-1} \alpha \cup \beta \cup [H]^{d_X - 2n - 1}$$

The associated linear map $2h : H^{n-1,n}(X) \to \overline{H^{n-1,n}(X)}$ therefore induces a symmetric isogeny

$$\Theta_X : J^{2n-1}(X) \to \widehat{J}^{2n-1}(X)$$

on the intermediate Jacobian. Note that under the assumption $\mathrm{N}^{n-1} H^{2n-1}(X, \mathbb{Q}) = H^{2n-1}(X, \mathbb{Q})$, we have that $J^{2n-1}(X)$ is equal to $J^{2n-1}_a(X)$, *i.e.*, the image of the Abel–Jacobi map on algebraically trivial cycle classes, which is an abelian variety.

The above discussion can be rephrased as saying that Θ_X induces a commutative diagram

$$
\begin{array}{ccc}
H_1(J^{2n-1}(X), \mathbb{Z}) & \xrightarrow{\Theta_X} & H_1(\widehat{J}^{2n-1}(X), \mathbb{Z}) \\
{\scriptstyle =}\downarrow & & \uparrow{\scriptstyle =} \\
H^{2n-1}(X, \mathbb{Z})_\tau & \xrightarrow{\cup(-1)^{n-1}[H]^{d_X - 2n+1}} & H^{2d_X - 2n+1}(X, \mathbb{Z})_\tau\,.
\end{array}
$$

The left vertical arrow is the canonical identification coming from the construction of the intermediate Jacobian, while the right vertical arrow is the dual identification, where we identify $H_1(J^{2n-1}(X), \mathbb{Z})^\vee = H_1(\widehat{J}^{2n-1}(X), \mathbb{Z})$ via the Weil pairing, and we identify $H^{2n-1}(X, \mathbb{Z})^\vee_\tau = H^{2d_X - 2n+1}(X, \mathbb{Z})_\tau$ via the cup product. We review this standard Hodge theory in §A.4.

Taking the Tate module of the Abel–Jacobi map gives us two equivalent maps [Blo79, Prop. 3.7], namely the maps $T_\ell AJ : T_\ell \mathrm{A}^n(X) \to T_\ell J^{2n-1}_a(X)$ and $T_\ell \lambda^n : T_\ell \mathrm{A}^n(X) \to H^{2n-1}(X, \mathbb{Z}_\ell)_\tau$, and we

can rephrase the diagram above ℓ-adically as saying the following diagram commutes:

$$
\begin{array}{ccc}
H_1(J^{2n-1}(X), \mathbb{Z}_\ell) & \xrightarrow{\;T_\ell \Theta_X\;} & H_1(\widehat{J}^{2n-1}(X), \mathbb{Z}_\ell) \\
\;\;\uparrow{\scriptstyle T_\ell AJ} & & \downarrow{\scriptstyle (T_\ell AJ)^\vee} \\
T_\ell \, \mathrm{A}^n(X) & & T_\ell \, \mathrm{A}^n(X)^\vee \\
\;\;\downarrow{\scriptstyle T_\ell \lambda^n} & & \uparrow{\scriptstyle (T_\ell \lambda^n)^\vee} \\
H^{2n-1}(X, \mathbb{Z}_\ell)_\tau & \xrightarrow{\;\cup (-1)^{n-1}[H]^{d_X - 2n+1}\;} & H^{2d_X - 2n+1}(X, \mathbb{Z}_\ell)_\tau
\end{array}
$$

The isogeny Θ_X is in fact the only morphism $\Theta : J^{2n-1}(X) \to \widehat{J}^{2n-1}(X)$ making the above diagram commute (for any ℓ). Indeed, since for abelian varieties A, B, the natural map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(V_\ell A, V_\ell B)$ is an inclusion, it suffices to show that $V_\ell AJ : V_\ell \, \mathrm{A}^n(X) \to V_\ell J^{2n-1}(X)$ is surjective; this follows from the Proposition 1.6 using the fact that the Abel–Jacobi map is a surjective regular homomorphism ($T_\ell AJ$ is in fact an isomorphism for $n = 1, 2, d_X$).

Note that if the Hodge coniveau filtration satisfies $\mathrm{N}_H^{i-1} H^{2i-1}(X, \mathbb{Q}) = 0$ for all $i < n$; i.e., the middle two terms of the Hodge decomposition are zero for all odd cohomology in degree less than $2n - 1$, which holds for instance if $n = 1$, or if $d_X \geq 3$, $n = 2$, and $h^{1,0} = 0$, then the Hodge–Riemann bilinear pairing is positive definite, so that Θ_X is a polarization.

12.2. Distinguished homomorphisms.
The discussion above motivates the following definition:

Definition 12.1 (Distinguished homomorphism). Let X be a smooth projective variety over a perfect field K, let $H \in \mathrm{CH}^1(X)$ be the class of an ample divisor, let n be a natural number such that $1 \leq 2n - 1 \leq d_X$, let Ω / K be an algebraically closed field extension, let l be a prime, and let $\Phi : \mathscr{A}_{X/K}^n \to A$ be a *surjective* regular homomorphism. We say that a homomorphism

$$
\Lambda : A_\Omega \longrightarrow \widehat{A}_\Omega
$$

is *l-distinguished* (with respect to H) if it is induced by cup product in cohomology in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
T_l A_\Omega & \xrightarrow{\;\;\;\;\;\;T_l \Lambda\;\;\;\;\;\;} & T_l \widehat{A}_\Omega \\
\;\;\uparrow{\scriptstyle T_l \phi} & & \downarrow{\scriptstyle (T_l \phi)^\vee} \\
T_l \, \mathrm{A}^n(X_\Omega) & & T_l \, \mathrm{A}^n(X_\Omega)^\vee(1) \\
\;\;\downarrow{\scriptstyle T_l \lambda^n} & & \uparrow{\scriptstyle (T_l \lambda^n)^\vee} \\
H^{2n-1}(X_\Omega, \mathbb{Z}_l(n))_\tau & \xrightarrow{\;\cup [H]^{d_X - 2n+1}\;} & H^{2d_X - 2n+1}(X_\Omega, \mathbb{Z}_l(d_X - n + 1))_\tau
\end{array}
\tag{12.1}
$$

Here $(T_l A_\Omega)^\vee(1)$ is identified with $T_l \widehat{A}_\Omega$ via the Weil pairing (see A.1), and $H^{2n-1}(X_\Omega, \mathbb{Z})_\tau^\vee$ is identified with $H^{2d_X - 2n+1}(X_\Omega, \mathbb{Z})_\tau$ via the cup product. We say that Λ is *distinguished* if it is distinguished for all primes l.

Remark 12.2. Note that in comparison to the motivation in §12.1, we have removed the factor of $(-1)^{n-1}$ in the bottom row of (12.1) to simplify some of the diagrams, and to make the presentation more clearly motivic. When we are interested in questions of positivity, we will always replace the (ℓ-)distinguished homomorphism Λ with the homomorphism $\Theta = (-1)^{n-1}\Lambda$.

Lemma 12.3 (Uniqueness and descent). *Let $\ell \neq \mathrm{char}(K)$ be prime. In the notation of Definition 12.1, there is at most one ℓ-distinguished homomorphism $\Lambda : A_\Omega \to \widehat{A}_\Omega$, and this morphism descends to a K-homomorphism $\underline{\Lambda} : A \to \widehat{A}$. Moreover, $\Theta = (-1)^{n-1}\Lambda$ is a (principal) polarization if and only if $\underline{\Theta} = (-1)^{n-1}\underline{\Lambda}$ is a (principal) polarization.*

Proof. As in the case of the Abel–Jacobi map (§12.1), by virtue of the fact that the natural map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(V_\ell A, V_\ell B)$ is an inclusion, and the fact that $V_\ell \phi$ is surjective for all ℓ (Proposition 1.6(3)), a diagram chase in (12.1) shows that an ℓ-distinguished homomorphism, if it exists, is unique. To show an ℓ-distinguished homomorphism Λ descends to K it suffices to show that it is $\mathrm{Aut}(\Omega/K)$-equivariant on Tate modules. In fact, it suffices to show that $V_\ell \Lambda$ is $\mathrm{Aut}(\Omega/K)$-equivariant; since $V_\ell \phi$ is surjective, this follows again from a diagram chase in (12.1). Finally, it is standard that $\Theta = (-1)^{n-1}\Lambda$ is a (principal) polarization if and only if $\underline{\Theta} = (-1)^{n-1}\underline{\Lambda}$ is a (principal) polarization (see §A.2). $\qquad\square$

In light of the uniqueness and descent of Lemma 12.3, if $\Phi^n_{X/K} : \mathscr{A}^n_{X/K} \to \mathrm{Ab}^n_{X/K}$ is an algebraic representative, we use the notation

$$\Lambda_X : \mathrm{Ab}^n_{X/K} \to \widehat{\mathrm{Ab}}^n_{X/K}$$

for a distinguished symmetric K-isogeny, if it exists. Note that unless $d = 2n - 1$, we have that Λ_X depends *a priori* on H as well as X.

Remark 12.4. Note that since the inclusion $\mathrm{Hom}(A, B) \to \mathrm{Hom}_{\mathrm{Gal}(K)}(T_\ell A, T_\ell B)$ is bijective if and only if $\mathrm{Hom}(A, B) = 0$, the existence of an ℓ-distinguished morphism Λ is not formal from the rest of the diagram (12.1).

In summary, with Hodge theory as our inspiration, our goal is to investigate when algebraic representatives admit distinguished polarizations. Obviously, motivated by the case of characteristic 0, we have the following examples (recall that the distinguished model of the algebraic intermediate Jacobian agrees with the algebraic representative in characteristic 0 for $n = 1, 2, d_X$):

Example 12.5 (Distinguished polarizations for distinguished models in characteristic 0)**.** As in §12.1, let X be a complex projective manifold, let H be an ample divisor on X, let n be a nonnegative integer with $1 \le 2n - 1 \le d_X$, and assume $\mathrm{N}^{n-1} H^{2n-1}(X, \mathbb{Q}) = H^{2n-1}(X, \mathbb{Q})$, which implies

$$H^{2n-1}(X, \mathbb{C}) = H^{n+1,n}(X) \oplus H^{n,n-1}(X).$$

Then the symmetric isogeny $\Lambda_X := (-1)^{n-1}\Theta_X : J^{2n-1}(X) \to \widehat{J}^{2n-1}(X)$ induced by H and the cup product in cohomology is ℓ-distinguished, and therefore descends to symmetric K-isogeny Λ_X on $J^{2n-1}_{a,X/K}$. If the Hodge coniveau filtration satisfies $\mathrm{N}^{i-1}_H H^{2i-1}(X, \mathbb{Q}) = 0$ for all $i < n$, then $\Theta_X = (-1)^{n-1}\Lambda_X$ is a polarization on $J^{2n-1}_{a,X/K}$.

In positive characteristic, we will use miniversal cycles to construct distinguished homomorphisms.

12.3. **Distinguished homomorphisms and miniversal cycles.** Let X be a smooth projective variety over a perfect field K, let $H \in \mathrm{CH}^1(X)$ be the class of an ample divisor, and let n be a natural number such that $1 \le 2n - 1 \le d_X$. Let A be an abelian variety over K, and let Ω/K be an algebraically closed field extension. Then for any cycle $Z \in \mathrm{CH}^n(X_\Omega \times_\Omega A_\Omega)$, the cycle $^tZ \circ [\cup H^{d_X - 2n+1}] \circ Z \in \mathrm{CH}^1(A_\Omega \times_\Omega A_\Omega)$ induces, via §1.5, a symmetric Ω-homomorphism

$$\Lambda_Z : A_\Omega \longrightarrow \widehat{A}_\Omega. \tag{12.2}$$

Our goal is to investigate when this construction gives a distinguished homomorphism on A, in the sense of Definition 12.1.

12.3.1. The first observation is that, for each prime l, we have by construction a commutative diagram

$$
\begin{array}{ccc}
T_l A & \xrightarrow{\ \ T_l \Lambda_Z\ \ } & T_l \widehat{A} \\
\downarrow{\scriptstyle Z_*} & & \uparrow{\scriptstyle Z^*} \\
H^{2n-1}(X_\Omega, \mathbb{Z}_l(n))_\tau & \xrightarrow{\ \cup [H]^{d_X-2n+1}\ } & H^{2d_X-2n+1}(X_\Omega, \mathbb{Z}_l(d_X-n+1))_\tau
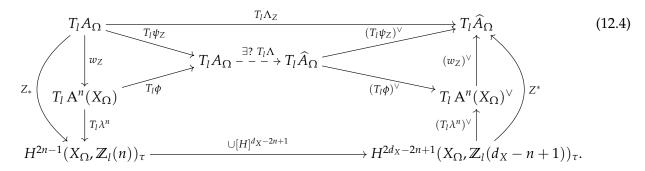\end{array}
\tag{12.3}
$$

where $(T_l A)^\vee(1)$ is identified with $T_l \widehat{A}$ via the Weil pairing, and we identify $T_l A_\Omega = T_l A$ by rigidity. When $l = \ell \neq \mathrm{char}(K)$, (12.3) follows by taking the ℓ-adic realization of an equality of cycle classes. If $\mathrm{char}(K) = p > 0$, then taking cycle classes in crystalline cohomology yields a diagram

$$
\begin{array}{ccc}
H_1^{\mathrm{cris}}(A/\mathbb{W}(K)) & \xrightarrow{\ \ \Lambda_{Z,\mathrm{cris}}\ \ } & H_1^{\mathrm{cris}}(\widehat{A}/\mathbb{W}(K)) \\
\downarrow{\scriptstyle Z_{*,\mathrm{cris}}} & & \uparrow{\scriptstyle Z^*_{\mathrm{cris}}} \\
H_{\mathrm{cris}}^{2n-1}(X/\mathbb{W}(K)(n))_\tau & \xrightarrow{\ [H]^{d_X-2n+1}\ } & H_{\mathrm{cris}}^{2d-2n+1}(X/\mathbb{W}(K)(d-n+1))_\tau
\end{array}
$$

and then taking F-invariants gives (12.3) with $l = p$.

12.3.2. Suppose now that $\Phi : \mathscr{A}^n_{X/K} \to A$ is a surjective regular homomorphism and assume that $V_\ell \lambda^n : V_\ell \, \mathrm{A}^n(X_{\overline{K}}) \longrightarrow H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ is surjective for some prime $\ell \neq \mathrm{char}\,K$. Then, by combining (12.1) and (12.3) together with Lemma 11.5 and Proposition 11.10, we obtain for all primes l a commutative diagram

$$
\tag{12.4}
$$

and we are asking whether there exists an Ω-homomorphism $\Lambda : A_\Omega \to \widehat{A}_\Omega$ making the diagram commute for all l. Note from Lemma 12.3 that Λ, if it exists, is unique and descends to K, and note also that the homomorphism $\Lambda_Z : A_\Omega \to \widehat{A}_\Omega$ is uniquely determined (since $\mathrm{Hom}(A, B) \to \mathrm{Hom}(T_\ell A, T_\ell B)$ is injective for $\ell \neq \mathrm{char}\,K$).

We fix a miniversal cycle $Z \in \mathscr{A}^n_{X_\Omega/\Omega}(A_\Omega)$ of degree, say, r, meaning that $\psi_Z : A \to A$ is multiplication by r. By considering the diagram (12.4) at a prime $\ell \neq \mathrm{char}(K)$ and arguing as in the proof of Lemma 12.3, we see that Λ_Z descends to K and only depends on r (i.e., does not depend on the choice of a miniversal cycle of degree r).

If $V_\ell \lambda^n$ is bijective, then the symmetric K-homomorphism Λ_Z is an isogeny (for instance, replace T_ℓ by V_ℓ in the above diagram (12.4) and use that r is invertible in $\mathbb{Q}_\ell$).

In case $Z \in \mathscr{A}^n_{X_\Omega/\Omega}(A_\Omega)$ is a universal cycle (i.e., $\psi_Z = \mathrm{id}_A$), then there exists a distinguished homomorphism Λ, namely, $\Lambda = \Lambda_Z$, since $\psi_Z = \mathrm{id}_A$.

The following theorem summarizes the above discussion:

Theorem 12.6 (Distinguished morphisms motivically). *Let X be a smooth projective variety over a perfect field K, let H be an ample divisor on X, let n be a natural number such that $1 \leq 2n - 1 \leq d_X$, and let Ω/K be an algebraically closed field extension. Further, let $\Phi : \mathscr{A}^n_{X/K} \to A$ be a surjective regular homomorphism and let $Z \in \mathscr{A}^n_{X_\Omega/\Omega}(A_\Omega)$ be a miniversal cycle of degree r. We denote $\Lambda_Z : A_\Omega \to \widehat{A}_\Omega$ the symmetric Ω-homomorphism induced by the cycle ${}^tZ \circ H^{d_X - 2n + 1} \circ Z \in \mathrm{CH}^1(A_\Omega \times_\Omega A_\Omega)$.*

Assume that for some prime $\ell_0 \neq \mathrm{char}(K)$:

- *$V_{\ell_0}\lambda^n : V_{\ell_0} \mathrm{A}^n(X_{\overline{K}}) \longrightarrow H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_{\ell_0}(n))$ is surjective.*

Then the symmetric Ω-homomorphism $\Lambda_Z : A_\Omega \to \widehat{A}_\Omega$ descends to K and depends only on H and r (and not on the choice of the miniversal cycle Z of degree r). Moreover,

(1) *If Z is universal (i.e., $r = 1$), then Λ_Z is a distinguished symmetric K-homomorphism.*

(2) *If $T_{l_i}\phi$ are isomorphisms for a given set of primes $\{l_i\}_{i \in I}$,*

   then there exists a symmetric K-homomorphism $\Lambda' : A \to \widehat{A}$ such that

$$\Lambda_Z = \Big(\prod_{l \in \{l_i\}_{i \in I}} l^{v_l(r)} \Big)^2 \Lambda'. \tag{12.5}$$

(3) *If $T_l\phi$ is an isomorphism at the primes l dividing r, then there exists a distinguished symmetric K-homomorphism Λ, which makes (12.4) commute at all primes l, and $\Lambda_Z = r^2\Lambda$.*

Finally, if in addition $V_{\ell_0}\lambda^n$ is bijective, then Λ_Z is a symmetric K-isogeny (and hence so are Λ' and Λ in (2) and (3), respectively).

Proof. Everything except (2) and (3) follows immediately from the commutativity of (12.4) and from the discussion above. Clearly (2) $\implies$ (3). Concerning (2), at the primes l_i for which $T_{l_i}\phi$ are isomorphisms, we obtain from (12.4) a Galois-equivariant homomorphism $\alpha_i : T_{l_i}A \to T_{l_i}\widehat{A}$ such that $r^2\alpha_i = T_{l_i}\Lambda_Z$. We next recall the following elementary fact: Given a free $\mathbb{Z}$-module M (of finite rank), an element $m \in M$ and a prime l, denoting by $\pi : M \hookrightarrow M \otimes \mathbb{Z}_l$ the canonical map, if $\pi(m)$ is divisible by an integer N (*i.e.*, in the image of the multiplication by N map), then m is divisible by $l^{v_l(N)}$. Applying this to the abelian group of homomorphisms from A_Ω to $\widehat{A}_\Omega$, this implies that Λ_Z is divisible by $l_i^{2v_{l_i}(r)}$. Hence, there exists a symmetric K-homomorphism $\Lambda' : A \to \widehat{A}$ such that (12.5) holds. In particular, we see that Λ_Z becomes divisible by r^2 after inverting the primes l dividing r but distinct from the l_i. $\qquad\square$

Remark 12.7 (Characteristic 0). Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$, as in Example 12.5. Assuming that the bullet point condition in Theorem 12.6 holds, and that there is a universal codimension-n cycle class for $X_\mathbb{C}$ (resp. $T_\ell\phi$ and $T_\ell\lambda^n$ are isomorphisms for all primes ℓ), then the symmetric K-isogeny $\Theta = (-1)^{n-1}\Lambda$ of Theorem 12.6(1) (resp. (3)) agrees with the polarization Θ_X on $J^{2n-1}_{a,X/K}$ from Example 12.5. Indeed, both are ℓ-distinguished, and so we may employ Lemma 12.3.

Remark 12.8 ($\Theta_Z = (-1)^{n-1}\Lambda_Z$). We emphasize that in light of the Hodge theory, it is $\Theta_Z := (-1)^{n-1}\Lambda_Z$ in Theorem 12.6 that one might hope is a polarization on A. We discuss this further in §13.

Corollary 12.9 (Distinguished homomorphisms for codimension-1 cycles). *Let $\Phi^1_{X/K} : \mathscr{A}^1_{X/K} \to (\mathrm{Pic}^0_X)_{\mathrm{red}} = \mathrm{Ab}^1_{X/K}$ be the Abel–Jacobi map (the first algebraic representative). Then there exists a distinguished the symmetric K-isogeny $\Lambda_X : (\mathrm{Pic}^0_X)_{\mathrm{red}} \to (\mathrm{Pic}^0_X)^\vee_{\mathrm{red}}$, which, in case $K \subseteq \mathbb{C}$, agrees with the polarization Θ_X induced by Hodge theory (see Example 12.5).*

Proof. Note that $V_l\lambda^1$ is an isomorphism for all primes l and that the Abel–Jacobi map $\Phi : \mathscr{A}^1_{X/K} \to (\mathrm{Pic}^0_X)_{\mathrm{red}}$ always admits a universal cycle $Z \in \mathscr{A}^n_{X_\Omega/\Omega}(A_\Omega)$ so that Theorem 12.6(1) applies. The

agreement of Λ_X with the polarization Θ_X induced by Hodge theory comes from the fact that both are distinguished homomorphisms. □

Remark 12.10 (Curves). In Corollary 12.9, if X is a curve, then Λ_X is independent of the choice of H, and agrees with the canonical principal polarization on the Jacobian of the curve, since the canonical principal polarization is known to be distinguished.

Corollary 12.11 (Distinguished homomorphisms for codimension-2 cycles). *Let $\Phi^2_{X/K} : \mathscr{A}^2_{X/K} \to \mathrm{Ab}^2_{X/K}$ be the second algebraic representative and let N be a natural number. Suppose that, for a prime $\ell \neq \mathrm{char}(K)$, we have that $N\Delta_{X_\Omega}$ admits a cohomological $\mathbb{Z}_\ell$-decomposition of type $(d_X - 1, 1)$ with respect to $H^\bullet(-, \mathbb{Z}_\ell)$, e.g., X is geometrically rationally connected. Then*

(1) *There is a distinguished symmetric K-isogeny $\Lambda_X : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$;*
(2) *If in addition $\dim X \leq 4$, then X admits a codimension-2 miniversal cycle $Z \in \mathscr{A}^n_{X_\Omega/\Omega}(A_\Omega)$ of degree N (Corollary 9.3) and $\Lambda_Z = N^2 \Lambda_X$;*
(3) *If further $N = 1$, e.g., if X is geometrically stably rational of dimension ≤ 4, then X admits a codimension-2 universal cycle Z and $\Lambda_Z = \Lambda_X$.*

Moreover, in case $K \subseteq \mathbb{C}$, the symmetric K-isogeny $-\Lambda_X$ agrees with the polarization Θ_X induced by Hodge theory (see Example 12.5).

Proof. The assumption on $N\Delta_{X_\Omega}$ implies that $V_\ell\lambda^2$ is bijective (Proposition 7.12) and that the $T_l\phi$ are isomorphisms for all primes l (Proposition 10.3). Items (1)–(3) then follow from Theorem 12.6. In case $K \subseteq \mathbb{C}$, the agreement of $-\Lambda_X$ with the polarization Θ_X induced by Hodge theory comes from the fact that both Λ_X and $-\Theta_X$ are distinguished homomorphisms. □

As another consequence of Theorem 12.6 we obtain a cohomological analogue to Theorem 4.4; this result will in fact show that the isomorphism Θ_X in Theorem 4.4 is distinguished, and therefore, via Lemma 12.3, in characteristic 0, agrees with the principal polarization coming from Hodge theory:

Theorem 12.12 (Threefolds). *Let X be a smooth projective threefold over a perfect field K, and let Ω/K be an algebraically closed field extension. Let $Z \in \mathscr{A}^2_{X_\Omega/\Omega}(\mathrm{Ab}^2_{X_\Omega/\Omega})$ be a miniversal cycle class.*

(1) *If $V_\ell\lambda^2$ is an isomorphism for some prime $\ell \neq \mathrm{char}(K)$ (e.g., if X is geometrically uniruled), then the Ω-homomorphism $\Lambda_Z : \mathrm{Ab}^2_{X_\Omega/\Omega} \to \widehat{\mathrm{Ab}}^2_{X_\Omega/\Omega}$ induced by the cycle class ${}^t Z \circ Z \in \mathrm{CH}^1(\mathrm{Ab}^2_{X_\Omega/\Omega} \times_\Omega \mathrm{Ab}^2_{X_\Omega/\Omega})$ descends to a symmetric K-isogeny*

$$\Lambda_Z : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$$

depending only on the degree of Z as a miniversal cycle. Moreover, if Z is universal, then Λ_Z is distinguished and we denote Λ_Z by Λ_X.
(2) *If $\mathrm{CH}_0(X_{\overline{K}}) \otimes \mathbb{Z}[\frac{1}{N}]$ is universally trivial for some natural number N (e.g., X is geometrically rationally chain connected), then there exists a distinguished purely inseparable symmetric K-isogeny $\Lambda_X : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$. Moreover, if $\mathrm{char}(K) \nmid N$, then Λ_X is an isomorphism.*
(3) *If $\mathrm{CH}_0(X_{\overline{K}})$ is universally trivial (e.g., X is geometrically stably rational), then the distinguished symmetric K-isogeny Λ_X is an isomorphism.*

Moreover, if $K \subseteq \mathbb{C}$, then the symmetric K-isogeny $-\Lambda_X$ of (2) and (3) agrees with the principal polarization Θ_X induced by Hodge theory (see Example 12.5).

Proof. Item (1) follows immediately from Theorem 12.6. In cases (2) and (3), the diagonal of $X_{\overline{K}}$ admits in particular a Chow $\mathbb{Q}$-decomposition of type $(2, 2)$ and it follows from Proposition 7.14 that $T_l\lambda^2$ (and hence $V_l\lambda^2$) is bijective for all primes l. Concerning (2), the diagonal of $X_{\overline{K}}$ admits

in particular a Chow $\mathbb{Q}$-decomposition of type $(2,1)$, and hence $T_l\phi^2$ is an isomorphism for all primes l by Proposition 10.3. Combined with the fact that $T_l\lambda^2$ is bijective for all primes l, there exists from Theorem 12.6(3) a distinguished symmetric K-isogeny Λ_X and it follows from diagram (12.4) that $T_l\Lambda_X$ is an isomorphism for all primes l and hence that Λ_X is a purely inseparable isogeny. On the other hand, by Theorem 4.4(1), the universal triviality of $\mathrm{CH}_0(X_{\overline{K}})\otimes\mathbb{Z}[\frac{1}{N}]$ yields that the degree of the isogeny Λ_X divides a power of N; it follows that, if $\mathrm{char}(K)\nmid N$, Λ_X is an isomorphism. For (3), if $\mathrm{CH}_0(X_{\overline{K}})$ is universally trivial, then by Corollary 9.3 $X_{\overline{K}}$ admits a universal codimension-2 cycle Z and Λ_Z is the distinguished symmetric K-isogeny (*i.e.*, it coincides with Λ_X) by Theorem 12.6(1). Now, the fact that Λ_Z is an isomorphism follows by noting that by construction Λ_Z coincides with the symmetric K-isomorphism Λ_X of Theorem 4.4(2). $\qquad\square$

Remark 12.13. Note that by construction, the symmetric isogeny Λ_X in Theorem 12.12 agrees with that in Theorem 4.4 when there is a (Chow) decomposition of the diagonal, so that the symmetric K-isomorphism Λ_X of Theorem 4.4(2) is distinguished.

13. Specialization and polarization on the algebraic representative

Regarding whether the symmetric K-isogeny $\Theta_X = (-1)^{n-1}\Lambda_X$ of Theorem 12.6(2) is a polarization, in this section we present results for threefolds that essentially say that if a geometrically stably rational threefold can be lifted to a geometrically rationally chain connected threefold in characteristic 0, then Θ_X is a principal polarization. While many of the examples we have in mind are liftable to characteristic 0 (see *e.g.*, [Har10, Thms. 22.1, 22.3]), recall, of course, that there are smooth, projective, even rational varieties, over perfect fields of characteristic $p > 0$ that do not lift (see *e.g.*, [AZ17]).

13.1. **Inducing polarizations on the algebraic representative via specialization.**

Proposition 13.1 (Polarizations and specialization). *Suppose that S is the spectrum of a DVR with generic point η and special point s both with perfect residue fields. Suppose $f : \mathcal{X} \to S$ is a smooth projective morphism, and let H be a relatively ample divisor. Fix a natural number n such that $2n-1 \le d = \dim_S \mathcal{X}$, and let ℓ be a prime invertible in $\kappa(s)$.*
  Assume that:
  - *There exist algebraic representatives $\Phi^n_{\mathcal{X}_\eta/\eta} : \mathscr{A}^n_{\mathcal{X}_\eta/\eta} \to \mathrm{Ab}^n_{\mathcal{X}_\eta/\eta}$ and $\Phi^n_{\mathcal{X}_s/s} : \mathscr{A}^n_{\mathcal{X}_s/s} \to \mathrm{Ab}^n_{\mathcal{X}_s/s}$;*
  - *$T_\ell\phi^n_{\mathcal{X}_\eta} : T_\ell \mathrm{A}^n(\mathcal{X}_{\overline{\eta}}) \to T_\ell\,\mathrm{Ab}^n_{\mathcal{X}_\eta}$ and $T_\ell\phi^n_{\mathcal{X}_s} : T_\ell\,\mathrm{A}^n(\mathcal{X}_{\overline{s}}) \to T_\ell\,\mathrm{Ab}^n_{\mathcal{X}_s}$ are isomorphisms;*
  - *$V_\ell\lambda^n_{\mathcal{X}_{\overline{\eta}}} : V_\ell\,\mathrm{A}^n(\mathcal{X}_{\overline{\eta}}) \to H^{2n-1}(\mathcal{X}_{\overline{\eta}},\mathbb{Q}_\ell(n))$ and $V_\ell\lambda^n_{\mathcal{X}_{\overline{s}}} : V_\ell\,\mathrm{A}^n(\mathcal{X}_{\overline{s}}) \to H^{2n-1}(\mathcal{X}_{\overline{s}},\mathbb{Q}_\ell(n))$ are isomorphisms.*

Let $Z_\eta \in \mathscr{A}^n_{\mathcal{X}_\eta/\eta}(\mathrm{Ab}^n_{\mathcal{X}_\eta/\eta})$ be a miniversal cycle of degree r_η, and let $\zeta \in \mathscr{A}^n_{\mathcal{X}_s/s}(\mathrm{Ab}^n_{\mathcal{X}_s/s})$ be a miniversal cycle of degree r_s. Let Λ_{Z_η} and Λ_ζ be the symmetric K-isogenies of Theorem 12.6.
  Then $(\mathrm{Ab}^n_{\mathcal{X}_\eta}, \Lambda_{Z_\eta})$ extends to an abelian scheme $(\mathrm{Ab}^n_{\mathcal{X}/S}, \Lambda)$ over S, and Z_η induces an isogeny

$$h : (\mathrm{Ab}^n_{\mathcal{X}/S}|_s, r^2_s\Lambda|_s) \to (\mathrm{Ab}^n_{\mathcal{X}_s/s}, \Lambda_\zeta);\qquad\qquad (13.1)$$

the notation above using pairs, consisting of an abelian variety and a morphism to the dual abelian variety, indicates that the indicated extensions and morphisms make the associated diagrams commute. In particular, $\Theta_{Z_\eta} := (-1)^{n-1}\Lambda_{Z_\eta}$ is a polarization if and only if $\Theta_\zeta := (-1)^{n-1}\Lambda_\zeta$ is.
  Moreover, if $\ell \nmid r$, $d = 2n-1$, and $T_\ell\lambda^n_{\mathcal{X}_{\overline{\eta}}}$ and $T_\ell\lambda^n_{\mathcal{X}_{\overline{s}}}$ are isomorphisms, then $T_\ell h$ is an isomorphism.

Remark 13.2. With the view to lifting to characteristic 0, we will want to employ Proposition 13.1 in the case where $\mathrm{Ab}^n_{\mathcal{X}_\eta/\eta}$ admits in addition an ℓ-distinguished symmetric K-isogeny Λ'_η. In that case, Λ'_η extends to a symmetric K-isogeny Λ' on $\mathrm{Ab}^n_{\mathcal{X}/S}$, and we have $\Lambda' = r^2_\eta\Lambda$, so that we have in that case an isogeny $h : (\mathrm{Ab}^n_{\mathcal{X}/S}|_s, r^2_s r^2_\eta\Lambda'|_s) \to (\mathrm{Ab}^n_{\mathcal{X}_s/s}, \Lambda_\zeta)$. In particular, $\Theta'_\eta := (-1)^{n-1}\Lambda'_\eta$

is a polarization if and only if $\Theta_\zeta := (-1)^{n-1}\Lambda_\zeta$ is a polarization. If $\mathrm{Ab}^n_{\mathcal{X}/s}$ also admits an ℓ-distinguished symmetric K-isogeny Λ_s, then we have the isogeny $h : (\mathrm{Ab}^n_{\mathcal{X}/S}|_s, r^2_\eta \Lambda'|_s) \to (\mathrm{Ab}^n_{\mathcal{X}_s/s}, \Lambda_s)$.

Proof. First, from the bullet point assumptions, we have a Galois-equivariant isomorphism $V_\ell \lambda^n_{\mathcal{X}_{\overline{\eta}}} \circ (V_\ell \phi^n_{\mathcal{X}_\eta})^{-1} : V_\ell \, \mathrm{Ab}^n_{\mathcal{X}_\eta/\eta} \to H^{2n-1}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell(n))$, showing by the Ogg–Néron–Shafarevich criterion that $\mathrm{Ab}^n_{\mathcal{X}_\eta/\eta}$ extends to an abelian scheme $\mathrm{Ab}^n_{\mathcal{X}/S}$ over S. The fact that Λ_{Z_η} then extends to a morphism Θ over S is standard (see *e.g.*, [ACMV23, Prop. 4.5] and [MFK94, Prop. 6.1]). Just as in Lemma 5.2, we have that the following are equivalent: Θ is a polarization; Θ_η is a polarization; $\Theta|_s$ is a polarization. Thus we have reduced to showing the existence of the isogeny h in the statement in the theorem, as well as the assertion about $T_\ell h$.

Let now $Z_\eta \in \mathrm{CH}^n(\mathrm{Ab}^n_{\mathcal{X}_\eta/\eta} \times_\eta \mathcal{X}_\eta)$ and $\zeta \in \mathrm{CH}^n(\mathrm{Ab}^n_{\mathcal{X}_s/s} \times_s \mathcal{X}_s)$ be miniversal codimension-n cycle classes. From our assumptions we have the commutative diagrams (12.4):

$$(13.2)$$

whose arrows are all isomorphisms after tensoring with $\mathbb{Q}_\ell$ and where $\iota := T_\ell \lambda^n \circ (T_\ell \phi^n)^{-1}$, with ϕ^n denoting the universal regular homomorphism. We have omitted the Tate twists in the diagram above for space. Note that while Bloch maps are stable under specialization, regular homomorphisms need not be, and so we may not simply take the specialization of the bottom half of the diagram on the left, and expect to obtain the bottom of the diagram on the right. In any case, we can already draw the conclusion that $T_\ell \Lambda_{Z_\eta} = r^2_\eta \alpha_\eta$ and $T_\ell \Lambda_\zeta = r^2_s \alpha_s$.

Let us now consider $Z_s \in \mathrm{A}^n(\mathrm{Ab}^n_{\mathcal{X}_S/S}|_s \times_s \mathcal{X}_s)$, the specialization of the cycle Z_η. This cycle induces a homomorphism $h = \psi_{Z_s} : \mathrm{Ab}^n_{\mathcal{X}_S/S}|_s \to \mathrm{Ab}^n_{\mathcal{X}_s/s}$, and we consider the diagram

$$(13.3)$$

Thanks to Proposition 11.6, the vertical arrows form commutative diagrams. We observe that the outer rectangle is nothing but the specialization of the outer rectangle on the left-hand side of diagram (13.2) and is hence commutative. In addition, the bottom square, being the right-hand side of (13.2), is commutative. Thus the diagram (13.3) is commutative.

Tensoring (13.3) with $\mathbb{Q}_\ell$, the morphisms ι and $(Z_s)_*$ are isomorphisms, and therefore $V_\ell h$ is an isomorphism, implying that h is an isogeny. On the other hand, the commutativity of the top square of diagram (13.3) implies that $T_\ell \Lambda_{Z_s} = T_\ell \hat{h} \circ \alpha_s \circ T_\ell h$. Thus $T_\ell(r^2_s \Lambda_{Z_s}) = T_\ell \hat{h} \circ r^2_s \alpha_s \circ T_\ell h = T_\ell \hat{h} \circ T_\ell \Lambda_\zeta \circ T_\ell h = T_\ell(\hat{h} \circ \Lambda_\zeta \circ h)$, establishing (13.1).

Finally, if we assume that $T_\ell \lambda^n_{\mathcal{X}_{\bar{\eta}}} : T_\ell \, \mathrm{A}^n(\mathcal{X}_{\bar{\eta}}) \to H^{2n-1}(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_\ell(n))$ and $T_\ell \lambda^n_{\mathcal{X}_{\bar{s}}} : T_\ell \, \mathrm{A}^n(\mathcal{X}_{\bar{s}}) \to H^{2n-1}(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(n))$ are isomorphisms, that $\ell \nmid r$, and that $d = 2n - 1$, then all of the morphisms in (13.3) are isomorphisms, and therefore the isogeny h induces an isomorphism $T_\ell h : T_\ell \, \mathrm{Ab}^n_{\mathcal{X}/S}|_s \to T_\ell \, \mathrm{Ab}^n_{\mathcal{X}_s}$. $\hfill\square$

We can apply this to the case of a threefold liftable to characteristic 0, which essentially says that for a smooth projective geometrically rationally chain connected threefold X over a perfect field K that lifts to a geometrically rationally connected threefold in characteristic 0, then Θ_X is a principal polarization.

Corollary 13.3 (Threefolds). *Suppose that $\mathcal{X}/S$ is a smooth projective threefold over the spectrum S of a DVR, such that the fraction field $\kappa(\eta)$ has characteristic 0 and the residue field $\kappa(s)$ is perfect. With $\mathcal{X}_{\mathbb{C}}$ the base change of the generic fiber $\mathcal{X}_\eta$ to $\mathbb{C}$, assume that $H^1(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}) = 0$ and $AJ : \mathrm{A}^2(\mathcal{X}_{\mathbb{C}}) \to J^3(\mathcal{X}_{\mathbb{C}})$ is surjective (e.g., $\mathcal{X}_{\mathbb{C}}$ is rationally connected).*

If for some natural number N, the multiple $N\Delta_{\mathcal{X}_{\bar{s}}} \in \mathrm{CH}^3(\mathcal{X}_{\bar{s}} \times_{\bar{s}} \mathcal{X}_{\bar{s}})$ admits a cohomological $\mathbb{Z}$-decomposition of type $(2, 1)$ with respect to $H^\bullet(-, \mathbb{Z}_{\ell_s})$ for all primes $\ell \neq \mathrm{char}(\kappa(s))$ (e.g., $\mathcal{X}_s$ is geometrically rationally chain connected), then $\Theta_{\mathcal{X}_s}$, the negative of the ℓ-distinguished symmetric isogeny $\Lambda_{\mathcal{X}_s}$ of Theorem 12.12, is a polarization on $\mathrm{Ab}^2_{\mathcal{X}_s/s}$.

If moreover, $\Delta_{\mathcal{X}_{\bar{s}}} \in \mathrm{CH}^3(\mathcal{X}_{\bar{s}} \times_{\bar{s}} \mathcal{X}_{\bar{s}})_{\mathbb{Z}[\frac{1}{N}]}$ admits a strict Chow decomposition for some natural number N coprime to $\mathrm{char}(K)$ (e.g., $\mathcal{X}_s$ is geometrically stably rational), then $\Theta_{\mathcal{X}_s}$ is a principal polarization.

Proof. On the generic fiber, we know that in characteristic 0 both $T_\ell \phi^2$ and $T_\ell \lambda^2$ are isomorphisms so long as the Abel–Jacobi map is surjective (*e.g.*, $\mathcal{X}_{\mathbb{C}}$ is uniruled [Voi07a, Thm. 12.22]). Thus the last two bullet points of Proposition 13.1 are satisfied for the generic fiber. Moreover, in characteristic 0, the Hodge-theoretic polarization $\Theta_{\mathcal{X}_\eta}$ induces a distinguished symmetric isogeny $\Lambda_\eta = -\Theta_{\mathcal{X}_\eta}$ on $\mathrm{Ab}^2_{X_\eta/\eta}$, since we have assumed that $H^1(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}) = 0$ (Theorem 12.6(2) and Remark 12.7), which holds if we assume $\mathcal{X}_{\mathbb{C}}$ is rationally connected.

For the special fiber, the cohomological decomposition of a multiple of the diagonal implies that $\mathrm{Ab}^2_{\mathcal{X}_s/s}$ admits a distinguished symmetric K-isogeny (Corollary 12.11), and that $T_\ell \phi^2$ and $T_\ell \lambda^2$ are isomorphisms (Propositions 10.1 and 7.14), where ϕ^2 indicates the universal regular homomorphism. Thus the last two bullet points of Proposition 13.1 are satisfied for the special fiber.

Thus we can conclude from Proposition 13.1 that $\Theta_{\mathcal{X}_s}$ is a polarization. $\hfill\square$

13.2. **Rationally chain connected threefolds and the work of Benoist–Wittenberg.** Let X be a smooth projective threefold over a perfect field K, and let $\Lambda : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$ be a symmetric K-isogeny. Benoist–Wittenberg introduced the following terminology:

   (i) Λ satisfies [BW20, Property 2.4(i)] if Λ is distinguished.
   (ii) Λ satisfies [BW20, Property 2.4(ii)] if $\Theta = -\Lambda$ is a principal polarization.

Benoist–Wittenberg give an equivalent formulation via the first Chern class $[\Lambda]$ (see §A.2). However we prefer to work in the setting of symmetric isogenies. They ask [BW20, p.6]:

Question 13.4 ([BW20]). *Let X be a smooth projective threefold over a perfect field K of positive characteristic, with $\mathrm{CH}_0(X_{\overline{K}})_{\mathbb{Q}}$ universally trivial. Does there exist a symmetric K-isogeny Λ_X on $\mathrm{Ab}^2_{X/K}$ such that:*

   (1) *Λ_X is distinguished (i.e., satisfies [BW20, Property 2.4(i)])?*
   (2) *Λ_X is distinguished and principal (i.e., Λ_X is an isomorphism)?*
   (3) *Λ_X is distinguished and $\Theta_X = -\Lambda_X$ is a polarization?*
   (4) *Λ_X is distinguished and $\Theta_X = -\Lambda_X$ is a principal polarization (i.e., satisfies [BW20, Property 2.4(i) and (ii)])?*

Recall that if the characteristic of K is zero, then the Hodge theoretic principal polarization Θ_X has the property that $\Lambda_X = -\Theta_X$ is distinguished (*i.e.*, Λ_X satisfies [BW20, Property 2.4(i) and (ii)]).

When X is a geometrically *rational* threefold over a perfect field K, Benoist–Wittenberg provide an affirmative answer to all parts of their question by constructing a principal polarization Θ_X on $\mathrm{Ab}^2_{X/K}$ such that $\Lambda_X = -\Theta_X$ is distinguished [BW20, Cor. 2.8]. In fact, they extend the Clemens–Griffiths condition over $\mathbb{C}$, showing that if X is rational *over* K, then $(\mathrm{Ab}^2_{X/K}, \Theta_X)$ is the product of principally polarized Jacobians of curves. Note that since their symmetric isogeny Λ_X is distinguished, their result shows that in this case the symmetric K-isomorphism $\Theta_X = -\Lambda_X : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$ of Theorem 4.4(2) and Theorem 12.12 is a principal polarization.

Our results provide an affirmative answer to part (1) of their question, and provide some further evidence for an affirmative answer to the other parts of their question. More precisely, given a smooth projective threefold X over a perfect field K of positive characteristic, with $\mathrm{CH}_0(X_{\overline{K}})_{\mathbb{Q}}$ universally trivial, there exists a purely inseparable symmetric K-isogeny Λ_X on $\mathrm{Ab}^2_{X/K}$ such that:

(1) Λ_X is distinguished (*i.e.*, satisfies [BW20, Property 2.4(i)]);
(2) Λ_X is distinguished and principal if $\mathrm{CH}_0(X_{\overline{K}})_{\mathbb{Z}[\frac{1}{N}]}$ is universally trivial for some natural number N coprime to $\mathrm{char}(K)$.
(3) Λ_X is distinguished and $\Theta_X = -\Lambda_X$ is a polarization if X lifts to characteristic 0 to a geometrically universally $(\mathrm{CH}_0)_{\mathbb{Q}}$-trivial smooth projective threefold.
(4) Λ_X is distinguished and $\Theta_X = -\Lambda_X$ is a principal polarization (*i.e.*, satisfies [BW20, Property 2.4(i) and (ii)]) if $\mathrm{CH}_0(X_{\overline{K}})_{\mathbb{Z}[\frac{1}{N}]}$ is universally trivial for some natural number N coprime to $\mathrm{char}(K)$, and X lifts to characteristic 0 to a geometrically universally $(\mathrm{CH}_0)_{\mathbb{Q}}$-trivial smooth projective threefold.

The existence of the purely inseparable symmetric K-isogeny Λ_X, as well as (1) and (2), are shown in Theorem 12.12. (3) and (4) are shown in Corollary 13.3. Note that in the language here, Conjecture 5.1 asserts that if we assume further that $\mathrm{CH}_0(X_{\overline{K}})$ is universally trivial, then the distinguished K-isomorphism Λ_X of (2) above should have $\Theta_X = -\Lambda_X$ being a (principal) polarization; *i.e.*, it should also satisfy condition (4), without requiring the further hypothesis of lifting to characteristic 0.

14. Cohomological decomposition of the diagonal, algebraic representatives, and minimal cohomology classes

In this section, we relate cohomological decomposition of the diagonal to minimal cohomology classes. This follows Voisin's work, with the key addition being that we must use replacements for the canonical principal polarization coming from Hodge theory, as well as the canonical identification of the cohomology of a rationally connected threefold with the first homology of the intermediate Jacobian. The starting point is a technical condition $(*_R)$, which is central to the discussion.

14.1. **The technical condition $(*_R)$ for a cohomological decomposition of the diagonal.** The following technical theorem has a number of interesting applications. The key point is the condition $(*_R)$ for a smooth projective variety X over a field K, with respect to a fixed Weil cohomology

theory $\mathcal{H}$ and a ring homomorphism $R \to R_{\mathcal{H}}$:

There exist finitely many (not necessarily distinct) smooth projective varieties Y_i over K of dimension $d_X - 2$, and correspondences $\Gamma_i \in \mathrm{CH}^{d_X - 1}(Y_i \times_K X)_R$, such that for any $\alpha, \beta \in \mathcal{H}^{d_X}(X)$,

$$\langle \alpha, \beta \rangle_X = \sum_i \langle \Gamma_i^* \alpha, \Gamma_i^* \beta \rangle_{Y_i}. \qquad (*_R)$$

This technical condition was introduced by Voisin [Voi17, (35)] in the case $K = \mathbb{C}$, $R = \mathbb{Z}$, and $\mathcal{H}^\bullet$ is Betti cohomology. We will see in §14.3, following Voisin, that this condition can be related to universal cycle classes and to minimal cohomology classes on abelian varieties.

Remark 14.1. If K is either separably closed or finite and if $d_X > 2$, it is equivalent in $(*_R)$ to require the correspondences Γ_i to belong to $\mathscr{A}_{X/K}^{d_X - 1}(Y_i)_R$. To see this one replaces Γ_i with $\Gamma_i - Y_i \times \Gamma_i|_{y_{i,0}}$ for any choice of zero-cycle $y_{i,0} \in \mathrm{CH}_0(Y_i)$ of degree 1, and one notes that $(Y_i \times \Gamma_i|_{y_{i,0}})^* \alpha = 0$ for all $\alpha \in \mathcal{H}^{d_X}(X)$ for dimension reasons.

Theorem 14.2 ([Voi17, Thm. 3.1], [Mbo17, Thm. 3.1]). *Let X be a smooth projective variety over a field K, and let $R \to R_{\mathcal{H}}$ be a ring homomorphism.*

(A) *If $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict cohomological R-decomposition such that $D \subseteq X$ (in the notation of Definition 6.2) admits an embedded resolution to a normal crossing divisor (e.g., K is perfect and $d_X = 3$ [CP09]), then $(*_R)$ is satisfied.*

(B) *As a partial converse, if $(*_R)$ is satisfied, and the additional criteria are met:*
   (a) *$\mathcal{H}^\bullet(X)$ has no torsion,*
   (b) *$\mathcal{H}^{2i}(X)(i)$ is R-algebraic for $2i \neq d_X$,*
   (c) *$\mathcal{H}^{2i+1}(X) = 0$ for $2i + 1 \neq d_X$,*
   then $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict cohomological $R_{\mathcal{H}}$-decomposition.

Proof. The proof of [Voi17, Thm. 3.1] carries over directly to this case. The case at hand here, where $K = \overline{K}$ and $\mathcal{H}^\bullet$ is ℓ-adic cohomology ($\ell \neq \mathrm{char}(K)$), is [Mbo17, Thm. 3.1]. $\qquad \square$

14.2. **A first result concerning minimal cohomology classes.** For use in the proof of Theorem 3, we record the following consequence of Corollary 13.3 regarding minimal cohomology classes:

Proposition 14.3 (Minimal cohomology classes). *With the same notation and assumptions as in Corollary 13.3, if $g = \dim \mathrm{Ab}_{\mathcal{X}_{\overline{\eta}}/\overline{\eta}}^2 = \dim \mathrm{Ab}_{\mathcal{X}_{\overline{s}}/\overline{s}}^2 \leq 3$, then $[\Theta_{\mathcal{X}_s}]^{g-1}/(g-1)!$ is $\mathbb{Z}$-algebraic.*

Proof. We know from Corollary 13.3 that $\Theta_{\mathcal{X}_s}$ is a polarization. Thus there is an isogeny of polarized abelian varieties

$$h : (\mathrm{Ab}_{\mathcal{X}_s/s}^2, \Theta_{\mathcal{X}_s}) \to (A, \Theta)$$

with target a principally polarized abelian variety of dimension g (see *e.g.*, [EvdGM, Prop. 11.25, Cor. 11.26], or [Mum70, Cor. 1, p.234]). For dimension reasons, (A, Θ) is the Jacobian of a (possibly reducible) curve C, and therefore $[\Theta]^{g-1}/(g-1)!$ is the class of the Abel–Jacobi embedded curve $[C]$. Consequently, $h^*[C] = h^*[\Theta]^{g-1}/(g-1)! = [\Theta_{\mathcal{X}_s}]^{g-1}/(g-1)!$ is $\mathbb{Z}$-algebraic. $\qquad \square$

14.3. **Cohomological decompositions, universal codimension-2 cycles, and minimal cohomology classes.** We now revisit the technical Theorem 14.2, and convert the condition $(*_R)$ to a condition on universal codimension-2 cycles and minimal cohomology classes.

Proposition 14.4. *Let X be a smooth projective threefold over a perfect field K and let $\ell \neq \mathrm{char}(K)$ be a prime. Assume that*

- *$H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))$ is torsion-free;*

- $T_\ell \phi_X^2 : T_\ell A^2(X_{\overline{K}}) \longrightarrow T_\ell \operatorname{Ab}_{X/K}^2$ is an isomorphism (standard assumption at ℓ);
- $T_\ell \lambda^2 : T_\ell A^2(X_{\overline{K}}) \longrightarrow H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))$ is an isomorphism.

Assume further (e.g., K is finite or algebraically closed and X admits a universal codimension-2 cycle class; see Theorem 12.6) that there exists an ℓ-distinguished symmetric K-isogeny $\Lambda_X : \operatorname{Ab}_{X/K}^2 \to \widehat{\operatorname{Ab}}_{X/K}^2$ (Definition 12.1) such that $T_\ell \Lambda_X$ is an isomorphism, and set $[\Lambda_X]$ to be the first Chern class (§A.2).

(A) *If $(*_R)$ holds, and if K is either finite or algebraically closed, then*

$$\frac{[\Lambda_X]^{g-1}}{(g-1)!} \in H^{2g-2}((\operatorname{Ab}_{X/K}^2)_{\overline{K}}, \mathbb{Z}_\ell(g-1))$$

is R-algebraic, where $g := \dim \operatorname{Ab}_{X/K}^2$.

(B) *If the class $\frac{[\Lambda_X]^{g-1}}{(g-1)!} \in H^{2g-2}((\operatorname{Ab}_{X/K}^2)_{\overline{K}}, \mathbb{Z}_\ell(g-1))$ is R-algebraic, and $\operatorname{Ab}_{X/K}^2$ admits a universal codimension-2 cycle class, then $(*_R)$ holds.*

Remark 14.5. The R-algebraicity of $[\Lambda_X]^{g-1}/(g-1)!$ is a tautology if $(g-1)!$ is a unit in the coefficient ring R.

Proof. A version where $K = \mathbb{C}$, $R = \mathbb{Z}$, and one uses Betti cohomology is [Voi17, Pf. of Thm. 4.1] in case (A) and [Voi13, Pf. of Thm. 4.9] in case (B). The case where X is a cubic threefold, $K = \overline{K}$, $\operatorname{char}(K) \neq 2$, $\ell = 2$, and $R = \mathbb{Z}_2$ is [Mbo17, Thm. 3.2]. To be precise, we note that our assumptions are slightly different from those of Voisin or Mboro. In Voisin's version, the bullet point conditions in the statement of the proposition are replaced with the canonical identification $H^3(X, \mathbb{Z}) = H_1(J^3(X), \mathbb{Z}) \cong H^1(J^3(X), \mathbb{Z})$ induced via the principal polarization $\Theta_X = -\Lambda_X$ coming from the intersection product on $H^3(X, \mathbb{Z})$. Similarly, Mboro uses the principally polarized Prym variety (P, Ξ) of the cubic threefold as a replacement for the intermediate Jacobian, which Beauville has shown is in fact the algebraic representative and is isomorphic to the group of algebraically trivial codimension-2 cycle classes, as well as Beauville's identification $H^3(X, \mathbb{Z}_\ell) = H^1(P, \mathbb{Z}_\ell)$ (see [Bea77]).

While our conditions essentially reduce to Voisin's and Mboro's in these special cases, our proposition applies more generally. Nevertheless, essentially the same argument as in [Voi17, Pf. of Thm. 4.1] and [Mbo17, Thm. 3.2] carries over to this situation: a key point is that one can replace [Mbo17, Lem 3.3] with Corollary 11.8. Note also that in the work of both Voisin and Mboro, Θ_X is a principal polarization, but the arguments in the setting of $H^\bullet(-, \mathbb{Z}_\ell)$ only require that it be an ℓ-distinguished symmetric isogeny that induces an isomorphism on ℓ-adic Tate modules.

Since our assumptions are more general, we provide a proof. For brevity, we denote $\mathcal{H}^\bullet(-) = H^\bullet(-, \mathbb{Z}_\ell)$. First, by our assumptions that Λ_X is ℓ-distinguished and $T_\ell \Lambda_X$ is an isomorphism, then by Lemma A.1, we have a commutative diagram (see (12.1))

$$
\begin{array}{ccc}
T_\ell \operatorname{Ab}_{X/K}^2 & \xleftarrow{\ \cup \frac{[\Lambda_X]^{g-1}}{(g-1)!}\ } & \mathcal{H}^1(\operatorname{Ab}_{X/K}^2)(1) \\
\simeq \downarrow{\scriptstyle \iota} & & {\scriptstyle \iota^\vee}\uparrow \simeq \\
\mathcal{H}^3(X)(2) & =\!=\!=\!=\!= & \mathcal{H}^3(X)(2)
\end{array}
$$

where $\iota = T_\ell \lambda^2 \circ (T_\ell \phi^2)^{-1}$. Since $\mathcal{H}^3(X)(2)$ is identified with $\mathcal{H}^3(X)^\vee(-1)$ via the intersection pairing, we get

$$\langle \alpha, \beta \rangle_X = \left\langle \iota^\vee(\alpha), \iota^\vee(\beta) \cup \frac{[\Lambda_X]^{g-1}}{(g-1)!} \right\rangle_{\operatorname{Ab}_{X/K}^2} \qquad \text{for all } \alpha, \beta \in \mathcal{H}^3(X)(2). \tag{14.1}$$

We proceed to prove (A). Assume that $(*_R)$ holds. Since we are assuming K finite or algebraically closed, we may assume by Remark 14.1 that the cycles $\Gamma_i \in \mathrm{CH}^{d_X-1}(Y_i \times_K X)_R$ in fact sit in $\mathscr{A}_{X/K}^{d_X-1}(Y_i)_R$. We can thus consider the K-morphisms

$$\gamma_i := \Phi_{X/K}^2(Y_i)(\Gamma_i) : Y_i \to \mathrm{Ab}_{X/K}^2.$$

Corollary 11.8 implies that

$$(\Gamma_i)_* = \iota \circ (\gamma_i)_* : \mathcal{H}_1(Y_i) \to \mathcal{H}^3(X)(2).$$

By dualizing we get

$$\Gamma_i^* = \gamma_i^* \circ \iota^\vee : \mathcal{H}^3(X)(2) \to \mathcal{H}^1(Y_i)(1).$$

By combining $(*_R)$ with (14.1), we get for all $\alpha, \beta \in \mathcal{H}^3(X)(2)$

$$\left\langle \iota^\vee(\alpha), \iota^\vee(\beta) \cup \frac{[\Lambda_X]^{g-1}}{(g-1)!} \right\rangle_{\mathrm{Ab}_{X/K}^2} = \sum_i n_i \langle \Gamma_i^*\alpha, \Gamma_i^*\beta \rangle_{Y_i}$$

$$= \sum_i n_i \gamma_i^* \langle \iota^\vee(\alpha), \iota^\vee(\beta) \rangle_{\mathrm{Ab}_{X/K}^2}.$$

It follows that

$$\frac{[\Lambda_X]^{g-1}}{(g-1)!} = \sum_i n_i (\gamma_i)_* \gamma_i^*[\mathrm{Ab}_{X/K}^2] = \sum_i n_i (\gamma_i)_*[Y_i],$$

thereby establishing the R-algebraicity of $\frac{[\Lambda_X]^{g-1}}{(g-1)!}$.

We now prove (B). By assumption, there are finitely many connected curves C_i in $\mathrm{Ab}_{X/K}^2$ and constants $n_i \in R$ such that $\frac{[\Lambda_X]^{g-1}}{(g-1)!} = \sum_i n_i[C_i]$. Let $\widetilde{C}_i \to C_i$ be the normalization morphisms and denote $j_i : \widetilde{C}_i \to \mathrm{Ab}_{X/K}^2$ the natural morphism. The map $\cup \frac{[\Lambda_X]^{g-1}}{(g-1)!} : H^1(\mathrm{Ab}_{X/K}^2)(1) \to T_\ell \mathrm{Ab}_{X/K}^2$ is then given by $\sum_i n_i (j_i)_* j_i^*$. Let now $Z \in \mathscr{A}_{X/K}^2(\mathrm{Ab}_{X/K}^2)$ be a universal cycle. By Corollary 11.8, ι coincides with Z_* and $\iota^\vee$ coincides with Z^*; see (12.4). We get

$$\langle \alpha, \beta \rangle_X = \left\langle Z^*\alpha, Z^*\beta \cup \frac{[\Lambda_X]^{g-1}}{(g-1)!} \right\rangle_{\mathrm{Ab}_{X/K}^2}$$

$$= \langle \alpha, \sum_i n_i (j_i)_* j_i^* Z^*\beta \rangle_{\mathrm{Ab}_{X/K}^2}$$

$$= \sum_i \langle j_i^* Z^*\alpha, j_i^* Z^*\beta \rangle_{\mathrm{Ab}_{X/K}^2},$$

where the first equality is (14.1) and the last equality is obtained using the projection formula. This establishes $(*_R)$. $\qquad\square$

We next translate this into a statement about cohomological decompositions of the diagonal:

Corollary 14.6. *With the assumptions of Proposition 14.4, we have :*

(A) *If K is finite or algebraically closed and if $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict cohomological R-decomposition, then the cohomology class $\frac{[\Lambda_X]^{g-1}}{(g-1)!} \in H^{2g-2}((\mathrm{Ab}_{X/K}^2)_{\overline{K}}, \mathbb{Z}_\ell(g-1))$ is R-algebraic, where $g := \dim \mathrm{Ab}_{X/K}^2$.*

Assume further (to the assumptions of Proposition 14.4) that :

- *$H^\bullet(X_{\overline{K}}, \mathbb{Z}_\ell)$ has no torsion,*
- *$H^{2i}(X_{\overline{K}}, \mathbb{Z}_\ell(i))$ are $\mathbb{Z}_\ell$-algebraic for all i,*
- *$H^1(X_{\overline{K}}, \mathbb{Z}_\ell) = 0$.*

Then we have :

(B) *If the class $\frac{[\Lambda_X]^{g-1}}{(g-1)!} \in H^{2g-2}((\mathrm{Ab}^2_{X/K})_{\overline{K}}, \mathbb{Z}_\ell(g-1))$ is $\mathbb{Z}_\ell$-algebraic, and $\mathrm{Ab}^2_{X/K}$ admits a universal codimension-2 cycle, then $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ admits a strict cohomological $\mathbb{Z}_\ell$-decomposition.*

Proof. This is immediate from Theorem 14.2 and Proposition 14.4. $\qquad\square$

15. PROOF OF THEOREM 2

We are now in a position to prove Theorem 2. The main result of this section is Theorem 15.1, which is more general. We also provide necessary and sufficient conditions for a threefold over an algebraically closed field to admit a cohomological $\mathbb{Z}_\ell$-decomposition in Theorem 15.5.

15.1. **Proof of Theorem 2.**

Theorem 15.1. *Let X be a smooth projective threefold over a perfect field K.*

If $K = \overline{K}$ is algebraically closed and the diagonal $\Delta_X \in \mathrm{CH}^3(X \times_K X)$ admits a strict cohomological $\mathbb{Z}$-decomposition with respect to $H^\bullet(-, \mathbb{Z}_\ell)$, then for all prime numbers $\ell \neq \mathrm{char}(K)$:

(1) $H^1(X_{\overline{K}}, \mathbb{Z}_\ell) = 0$;

(2) $H^{2i}(X_{\overline{K}}, \mathbb{Z}_\ell(i))$ *is $\mathbb{Z}$-algebraic for all i;*

(3) *The ℓ-adic Bloch map $T_\ell \lambda^2 : T_\ell \, \mathrm{A}^2(X_{\overline{K}}) \to H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau$ is an isomorphism;*

(3') *The ℓ-adic map $T_\ell \phi^2_{X/K} : T_\ell \, \mathrm{A}^2(X_{\overline{K}}) \to T_\ell \, \mathrm{Ab}^2_{X/K}$ is an isomorphism;*

(4) $\mathrm{Tors}\, H^\bullet(X_{\overline{K}}, \mathbb{Z}_\ell) = 0$;

(5) $\mathrm{Ab}^2_{X/K}$ *admits a universal codimension-2 cycle class;*

(6) *Assuming (3) and (5), and setting $\Theta_X : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}\,^2_{X/K}$ to be the symmetric isogeny of Theorem 1(2), we have that $T_\ell \Theta_X$ is an isomorphism, and the class $\frac{[\Theta_X]^{g-1}}{(g-1)!} \in H^{2g-2}((\mathrm{Ab}^2_{X/K})_{\overline{K}}, \mathbb{Z}_\ell(g-1))$ is a $\mathbb{Z}$-algebraic class, where $g = \dim \mathrm{Ab}^2_{X/K}$ and $[\Theta_X]$ is the first Chern class of the line bundle associated to Θ_X.*

As a partial converse, let K be any perfect field and assume that (1)–(6) hold (including (3')) for some prime number $\ell \neq \mathrm{char}(K)$, where in (6) we define $[\Theta_X]$ to be the first Chern class of the line bundle associated to $(\Theta_X)_{\overline{K}}$ (§A.3). Then the diagonal $\Delta_X \in \mathrm{CH}^3(X \times_K X)$ admits a strict cohomological $\mathbb{Z}_\ell$-decomposition.

Remark 15.2. In the case where $K = \overline{K}$, if we just assume that $\Delta_X \in \mathrm{CH}^3(X \times_K X)$ admits a cohomological $\mathbb{Z}$-decomposition of type $(2,1)$, then (2)–(5) hold.

Proof. Assume $K = \overline{K}$ and that the diagonal $\Delta_X \in \mathrm{CH}^3(X \times_K X)$ admits a strict cohomological $\mathbb{Z}$-decomposition. Item (1) is Corollary 7.7 (and in fact we get this using just a strict $\mathbb{Z}_\ell$-decomposition). Item (2) is Corollary 7.11 (and in fact we get $\mathbb{Z}_\ell$-algebraicity using just a strict cohomological $\mathbb{Z}_\ell$-decomposition of type $(2,1)$, and using just a cohomological $\mathbb{Z}$-decomposition of type $(2,1)$ we get $\mathbb{Z}$-*algebraicity*). Note that this is the only place where we use that the field is algebraically closed; if one could prove Corollary 7.11 over finite fields, then Theorem 15.1 would hold over finite fields, as well. Item (3) is Proposition 7.12 (and in fact we get this using just a $\mathbb{Z}_\ell$-decomposition of type $(2,2)$). Item (3') is Proposition 10.1 (and in fact we get this using just a $\mathbb{Z}_\ell$-decomposition of type $(2,1)$). Item (4) is Corollary 7.2 (and in fact we get this using just a $\mathbb{Z}_\ell$-decomposition of type $(2,1)$). Item (5) is Corollary 9.3 (and in fact we get this using just a $\mathbb{Z}$-decomposition of type $(2,1)$). Item (6) is Corollary 14.6(A).

The converse statement is Corollary 14.6(B). As this is the easier direction of Corollary 14.6, for convenience, we provide a brief proof in the notation of Theorem 15.1. Working component-wise, we may and do assume X is connected. By assumption (4), the intersection pairing $\mathcal{H}^2(X) \times \mathcal{H}^4(X) \to R_{\mathcal{H}}(-3)$ is perfect. By assumption (1) in case $i = 1$ and $i = 2$, it follows that the

Künneth projectors π_X^2 and π_X^4 belong to the image of $\mathrm{CH}^3(X \times_K)_{R_{\mathcal{H}}} \to \mathcal{H}^6(X \times_K X)(3)$. Moreover, assumption (1) in case $i = 3$ provides a zero-cycle $x \in \mathrm{CH}_0(X)_{R_{\mathcal{H}}}$ of degree 1. We then consider the cycle

$$\pi_X^3 := \Delta_X - x \times_K X - \pi_X^2 - \pi_X^4 - X \times_K x \quad \in \mathrm{CH}^3(X \times_K X)_{R_{\mathcal{H}}},$$

whose cohomology class defines, by assumptions (1) and (4), the Künneth projector on $\mathcal{H}^3(X)$. We aim to show that $[\pi_X^3]$ is supported on $D \times_K X$ for some divisor D. By assumptions (3), (3'), and (5), together with Lemma A.1 and Corollary 11.8, we have a commutative diagram (where as usual $\Lambda_X = -\Theta_X$):

$$
\begin{array}{ccc}
T_\ell \, \mathrm{Ab}_{X/K}^2 & \xleftarrow{\cup \frac{[\Lambda_X]^{g-1}}{(g-1)!}} & \mathcal{H}^1(\mathrm{Ab}_{X/K}^2)(1) \\
\simeq \downarrow Z_* & & Z^* \uparrow \simeq \\
\mathcal{H}^3(X)(2) & =\!=\!=\!= & \mathcal{H}^3(X)(2)
\end{array}
$$

where $Z \in \mathscr{A}_{X/K}^2(\mathrm{Ab}_{X/K}^2)$ is any universal codimension-2 cycle. Therefore,

$$[\pi_X^3] = [\pi_X^3] \circ Z_* \circ \left(-\cup \frac{[\Lambda_X]^{g-1}}{(g-1)!} \right) \circ Z^* \circ [\pi_X^3].$$

By assumption (6), there are finitely many connected curves C_i in $\mathrm{Ab}_{X/K}^2$ and constants $n_i \in R_{\mathcal{H}}$ such that $\frac{[\Lambda_X]^{g-1}}{(g-1)!} = \sum_i n_i [C_i]$. Let $\widetilde{C}_i \to C_i$ be the normalizations morphisms and denote $j_i : \widetilde{C}_i \to \mathrm{Ab}_{X/K}^2$ the natural morphisms. The map $\cup \frac{[\Lambda_X]^{g-1}}{(g-1)!} : H^1(\mathrm{Ab}_{X/K}^2)(1) \to T_\ell \, \mathrm{Ab}_{X/K}^2$ is then given by $\sum_i n_i (j_i)_* j_i^*$. Therefore, the cohomological correspondence $[\pi_X^3]$ factors through $\bigoplus_i \mathcal{H}^1(\widetilde{C}_i)(-1)$, thereby establishing it is supported on $D \times_K X$ for some divisor D in X. $\qquad\square$

15.2. Regarding conditions (1)–(4) of Theorem 15.1 in positive characteristic.
As explained in the introduction, for any complex projective rationally connected threefold, conditions (1)–(3') of Theorem 15.1 hold. We show here that any smooth projective geometrically rationally chain connected threefold that lifts to a smooth projective geometrically rationally connected threefold in characteristic zero with no torsion in cohomology also satisfies conditions (1)–(3'), as well as (4) (Corollary 15.4).

We start with a preliminary result that follows directly from a result of Voisin [Voi06, Thm. 2]:

Theorem 15.3 (Voisin). *Suppose that $\mathcal{X}/S$ is a smooth projective threefold over the spectrum S of a DVR R, such that the fraction field $K := \kappa(\eta) \subseteq \mathbb{C}$ is characteristic 0, and the residue field $k := \kappa(s)$ is algebraically closed. Let $\mathcal{X}_{\mathbb{C}}$ be the base change of $\mathcal{X}_\eta$, and fix a prime $\ell \neq \mathrm{char}\,\kappa(s)$.*

(1) If $H^\bullet(\mathcal{X}_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z})$ has no ℓ-torsion, then $H^\bullet(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell)$ has no torsion.

(2) If in addition $\mathcal{X}_{\mathbb{C}}$ is rationally connected or satisfies $H^2(\mathcal{X}_{\mathbb{C}}, \mathcal{O}_{\mathcal{X}_{\mathbb{C}}}) = 0$ (resp. $\mathcal{X}_{\mathbb{C}}$ is uniruled or satisfies $K_{\mathcal{X}_{\mathbb{C}}} \cong \mathcal{O}_{\mathcal{X}_{\mathbb{C}}}$ and $H^2(\mathcal{X}_{\mathbb{C}}, \mathcal{O}_{\mathcal{X}_{\mathbb{C}}}) = 0$), then $H^2(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(1))$ (resp. $H^4(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(2))$) is algebraic.

(3) If $R = \mathbb{W}(k)$, the ring of Witt vectors of k, and if $p \geq 5$, then under the hypothesis that $H^\bullet(\mathcal{X}_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z})$ has no torsion, the crystalline cohomology $H^\bullet_{\mathrm{cris}}(\mathcal{X}_{\bar{s}}/\mathbb{W}(k))$ has no torsion; and under the hypotheses of (2), $H^2_{\mathrm{cris}}(\mathcal{X}_{\bar{s}}/\mathbb{W}(k))$ and $H^4_{\mathrm{cris}}(\mathcal{X}_{\bar{s}}/\mathbb{W}(k))$, respectively, are algebraic.

Proof. If $H^\bullet(\mathcal{X}_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z})$ has no ℓ-torsion, then $H^\bullet(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}_\ell) = H^\bullet(\mathcal{X}_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}_\ell) = H^\bullet(\mathcal{X}_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, which also has no torsion. By proper base change, we also have that $H^\bullet(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}_\ell) = H^\bullet(\mathcal{X}_{\bar{K}}, \mathbb{Z}_\ell) = H^\bullet(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell)$, and so we have completed the argument for (1).

For claim (2) on algebraicity, we argue as follows. First, as $H^2(\mathcal{X}_{\mathbb{C}}, \mathcal{O}_{\mathcal{X}_{\mathbb{C}}}) = 0$, either by assumption or else by using the assumption that $\mathcal{X}_{\mathbb{C}}$ is rationally connected, then we can conclude that

$H^2(\mathcal{X}^{an}_{\mathbb{C}}, \mathbb{C}) = H^{1,1}(\mathcal{X}^{an}_{\mathbb{C}})$, so that algebraicity of $H^2(\mathcal{X}^{an}_{\mathbb{C}}, \mathbb{Z})$ follows from the Lefschetz-$(1,1)$ theorem. In the case where $\mathcal{X}_{\mathbb{C}}$ is uniruled, or $K_{\mathcal{X}_{\mathbb{C}}} \cong \mathcal{O}_{\mathcal{X}_{\mathbb{C}}}$ and $H^2(\mathcal{X}_{\mathbb{C}}, \mathcal{O}_{\mathcal{X}_{\mathbb{C}}}) = 0$, the algebraicity of $H^4(\mathcal{X}^{an}_{\mathbb{C}}, \mathbb{Z})$ is Voisin's result [Voi06, Thm. 2].

Since we are assuming (1), we have $H^{\bullet}(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}_{\ell}) = H^{\bullet}(\mathcal{X}^{an}_{\mathbb{C}}, \mathbb{Z}_{\ell}) = H^{\bullet}(\mathcal{X}^{an}_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$ and it follows from Chow's theorem that $H^{2i}(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}_{\ell}(i))$ is algebraic for $i = 1, 2$ so long as $H^{2i}(\mathcal{X}^{an}_{\mathbb{C}}, \mathbb{Z}(i))$ is. We next claim that this implies $H^{2i}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_{\ell}(i))$ is algebraic. Indeed, given any $\alpha \in H^{2i}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_{\ell}(i))$, by proper base change, α is identified with a class in $H^{2i}(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}_{\ell}(i))$. By algebraicity, we can write $\alpha = \sum a_i [Z_i]$ for some cycles Z_i on $\mathcal{X}_{\mathbb{C}}$, and some coefficients $a_i \in \mathbb{Z}_{\ell}$. Each cycle Z_i lies on some component of the Hilbert scheme for $\mathcal{X}_{\mathbb{C}}$, which is the base change to $\mathbb{C}$ of the Hilbert scheme for $\mathcal{X}_{\overline{K}}$. Since the cohomology class of a cycle is the same for any cycle in the same component of the Hilbert scheme (we can use resolution of singularities, for instance, to get a smooth curve interpolating), we can replace Z_i with a cycle Z'_i defined over $\overline{K}$ (corresponding to any $\overline{K}$-point of the corresponding Hilbert scheme), and we have $\alpha = \sum a_i [Z_i] = \sum a_i [Z'_i]$. Thus $H^{2i}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_{\ell}(i))$ is algebraic. Now using the identification $H^{2i}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_{\ell}(i)) = H^{2i}(\mathcal{X}_{\overline{s}}, \mathbb{Z}_{\ell}(i))$ via proper base change, and the fact that specialization of cycles in Chow respects the cycle class map [Ful98, Exa. 20.3.5] (e.g., (16.5)), we have that $H^{2i}(\mathcal{X}_{\overline{s}}, \mathbb{Z}_{\ell}(i))$ is algebraic.

For (3), the hypothesis that $\mathcal{X}_{\overline{s}}$ lifts to an unramified mixed characteristic DVR guarantees that the freeness of $H^{\bullet}(\mathcal{X}_{\mathbb{C}}, \mathbb{Z})$, and thus that of $H^{\bullet}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$, implies that of $H^{\bullet}_{cris}(\mathcal{X}_{\overline{s}}/\mathbb{W}(k))$ [Car08, Thm. 1.1]. Moreover, this freeness allows one to define an integral crystalline cycle class map. Now suppose that $H^{2i}(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}(i))$ is algebraic. The comparison isomorphism between Betti and de Rham cohomology is compatible with cycle class maps, and thus $H^{2i}_{dR}(\mathcal{X}^{an}_{\mathbb{C}})$ is algebraic. Spreading and specializing shows that there is a finite extension $L/\mathbb{B}(k)$ such that $H^{2i}_{dR}(\mathcal{X}_L)$ is algebraic. (Note that the residue field of L is again k.) The comparison isomorphism between the de Rham cohomology of $\mathcal{X}_L$ and the crystalline cohomology of $\mathcal{X}_{\overline{s}}$ is compatible with cycle class maps [GM87, App. B], and thus $H^{2i}_{cris}(\mathcal{X}_{\overline{s}}/\mathbb{W}(k))$ is algebraic as well. $\qquad\square$

We can now show that conditions (1)–(4) of Theorem 15.1 hold for all threefolds liftable to rationally connected threefolds in characteristic 0 with no torsion in cohomology.

Corollary 15.4. *Suppose that $\mathcal{X}/S$ is a smooth projective threefold over the spectrum S of a DVR R, such that the fraction field $\kappa(\eta) \subseteq \mathbb{C}$ is characteristic 0, and the residue field $\kappa(s)$ is algebraically closed. If $\mathcal{X}_s$ and $\mathcal{X}_{\eta}$ are geometrically rationally chain connected, and $H^{\bullet}(\mathcal{X}_{\overline{\eta}}, \mathbb{Z}_{\ell})$ is torsion-free, then $\mathcal{X}_s$ satisfies conditions (1)–(4) of Theorem 15.1.*

Proof. Since $\mathcal{X}_s$ is rationally chain connected, a multiple of the diagonal admits a strict decomposition. Then condition (1) is Corollary 7.7. (2) and (4) follow from Theorem 15.3. (3) and (3′) are [ACMV21, Cor. 7.4]. $\qquad\square$

15.3. **Necessary and sufficient conditions for a cohomological $\mathbb{Z}_{\ell}$-decomposition.** We will see below, in Remark 17.6, that there are examples of smooth projective threefolds over an algebraically closed field $k = K = \overline{K}$ such that the diagonal admits a cohomological $\mathbb{Z}_{\ell}$-decomposition for some prime $\ell \neq \text{char}(k)$, but where condition (5) in Theorem 15.1 fails. In other words, while conditions (1)–(6) are sufficient for a cohomological $\mathbb{Z}_{\ell}$-decomposition of the diagonal, they are not necessary. The following theorem gives necessary and sufficient conditions for a cohomological $\mathbb{Z}_{\ell}$-decomposition:

Theorem 15.5 (Cohomological $\mathbb{Z}_{\ell}$-decomposition of the diagonal). *Let X be a smooth projective threefold over an algebraically closed field k, and fix a prime number $\ell \neq \text{char}(k)$. Then $\Delta_X \in \text{CH}^3(X \times_k X)$ admits a strict cohomological $\mathbb{Z}_{\ell}$-decomposition with respect to $H^{\bullet}(-, \mathbb{Z}_{\ell})$ if and only if*

(1) $H^1(X, \mathbb{Z}_{\ell}) = 0$;

(2_ℓ) $H^{2i}(X, \mathbb{Z}_\ell(i))$ is $\mathbb{Z}_\ell$-algebraic for all i;

(3) The ℓ-adic Bloch map $T_\ell\lambda^2 : T_\ell A^2(X) \to H^3(X, \mathbb{Z}_\ell(2))_\tau$ is an isomorphism;

(3′) The ℓ-adic map $T_\ell\phi^2_{X/k} : T_\ell A^2(X) \to T_\ell \mathrm{Ab}^2_{X/k}$ is an isomorphism;

(4) Tors $H^\bullet(X, \mathbb{Z}_\ell) = 0$;

(5_ℓ) $\mathrm{Ab}^2_{X/k}$ admits a miniversal codimension-2 cycle class Z of degree coprime to ℓ;

(6_ℓ) Assuming (3) and (5_ℓ), and setting $\Theta_Z : \mathrm{Ab}^2_{X/k} \to \widehat{\mathrm{Ab}}^2_{X/k}$ to be the morphism induced by the cycle class $-({}^tZ \circ Z)$ (see (12.2) and Theorem 12.6), we have that Θ_Z is a symmetric isogeny of degree coprime to ℓ, and $\frac{[\Theta_Z]^{g-1}}{(g-1)!} \in H^{2g-2}(\mathrm{Ab}^2_{X/k}, \mathbb{Z}_\ell(g-1))$ is a $\mathbb{Z}_\ell$-algebraic class, where $g = \dim \mathrm{Ab}^2_{X/k}$ and $[\Theta_Z]$ is the first Chern class of the line bundle associated to Θ_Z.

Proof. The necessity of the conditions above is given in the proof of Theorem 15.1: the results cited in that proof show that the weaker conditions in Theorem 15.5 hold under the weaker hypothesis of a cohomological $\mathbb{Z}_\ell$-decomposition. Note that in going from (3) and (5_ℓ) to the symmetric isogeny Θ_X of (6_ℓ), we are using Theorem 12.6 in the case of a miniversal cycle, rather than the case of a universal cycle.

The proof of sufficiency is again Corollary 14.6(B). Technically, if Z is a miniversal codimension-2 cycle class of degree r coprime to ℓ, then the symmetric isogeny $\Lambda_Z = -\Theta_Z$ of Theorem 12.6 is not ℓ-distinguished, but rather, satisfies $T_\ell\Lambda_Z = r^2 \circ \iota^\vee \circ \iota$, where $\iota = T_\ell\lambda^2 \circ (T_\ell\phi^2)^{-1}$. It is easy to see that Proposition 14.4 holds under this hypothesis, and therefore that Corollary 14.6(B) does, as well. One notationally easy way to express that is to say that $\frac{1}{r^2}\Lambda_Z \in \mathrm{Hom}(\mathrm{Ab}^2_{X/K}, \widehat{\mathrm{Ab}}^2_{X/})_\mathbb{Q}$ is ℓ-distinguished, in the sense that $\frac{1}{r^2}T_\ell\Lambda_Z = \iota^\vee \circ \iota$, and one can check that Proposition 14.4 and Corollary 14.6(B) hold for symmetric K-isogenies $\Lambda : \mathrm{Ab}^2_{X/K} \to \widehat{\mathrm{Ab}}^2_{X/K}$ such that $T_\ell\Lambda$ is an isomorphism, and such that there is an integer N invertible in R, such that $\frac{1}{N}\Lambda$ is ℓ-distinguished. $\square$

Remark 15.6. Since conditions (1)–(4) of Theorem 15.1 imply conditions (1)–(4) of Theorem 15.5, one can apply Corollary 15.4 to Theorem 15.5, as well.

Part 3. **Stable rationality and quartic double solids in positive characteristic**

16. HOMOLOGICAL DECOMPOSITION OF THE DIAGONAL, RESOLUTION OF SINGULARITIES,
AND SPECIALIZATION

The goal of this section is to show that cohomological decomposition of the diagonal is stable under specialization, and under resolution of singularities for nodal projective varieties. The results generalize Voisin's results over $\mathbb{C}$. Since those arguments are made with cycle classes in Betti homology, the central point is to convert elements of the arguments to work in Borel–Moore homology, or more precisely, in the algebraic setting of ℓ-adic homology. For this we need a few basic results on ℓ-adic homology, which unfortunately we could not find in the literature.

In this section, for uniformity, we fix a coefficient ring $\Lambda = \mathbb{Z}$ or $\mathbb{Z}_\ell$. For an algebraically closed field k and a scheme $\pi : X \to \operatorname{Spec} k$ of finite type over k, we denote by $\omega_X := R\pi^! \Lambda$ the dualizing sheaf in the derived category of constructible Λ-sheaves on X. For a scheme $\pi : X \to \operatorname{Spec} K$ of finite type over a field K, we denote by $\mathcal{H}_i(X) := \mathbb{H}^{-i}(X_{\overline{K}}, \omega_{X_{\overline{K}}})$ the ℓ-adic homology, or alternatively, the Borel–Moore homology, when working over $K = \mathbb{C}$ and with the analytic topology; we refer the reader to [Lau76] for details on ℓ-adic homology, and to [BM60] for details on the Borel–Moore homology.

16.1. **Homological decomposition of the diagonal.** As before, let $R_\mathcal{H}$ denote the coefficient ring of $\mathcal{H}_\bullet$; *i.e.*, $\mathbb{Z}_\ell$ for ℓ-adic homology, and $\mathbb{Z}$ for Borel–Moore homology. We adapt Definition 6.1 to the setting of homology:

Definition 16.1 (Homological decomposition of a cycle class). Let $R \to R_\mathcal{H}$ be a homomorphism of commutative rings. Let X be a scheme of finite type over a field K, and let

$$j_i : W_i \overset{\neq}{\hookrightarrow} X \, , \quad i = 1, 2$$

be reduced closed subschemes not equal to X. A *homological R-decomposition of type* (W_1, W_2) *of a cycle class* $Z \in \operatorname{CH}^{d_X}(X \times_K X)_R$ (with respect to $\mathcal{H}_\bullet$) is an equality

$$[Z] = [Z_1] + [Z_2] \in \mathcal{H}_{2d_X}(X \times_K X)(-d_X), \tag{16.1}$$

where $Z_1 \in \operatorname{CH}^{d_X}(X \times_K X)_R$ is supported on $W_1 \times_K X$ and $Z_2 \in \operatorname{CH}^{d_X}(X \times_K X)_R$ is supported on $X \times_K W_2$ (see (2.1) for the support of a cycle). When $R = \mathbb{Z}$, we call this a *homological decomposition of type* (W_1, W_2) (with respect to $\mathcal{H}_\bullet$). We say that $Z \in \operatorname{CH}^{d_X}(X \times_K X)_R$ has a *homological R-decomposition of type* (d_1, d_2) (with respect to $\mathcal{H}_\bullet$) if it admits a homological R-decomposition of type (W_1, W_2) with $\dim W_1 \leq d_1$ and $\dim W_2 \leq d_2$.

Remark 16.2. Recall that in the case where X is smooth projective and equidimensional, a homological decomposition is the same as a cohomological decomposition. Indeed, setting $\mathcal{H}^\bullet$ to be ℓ-adic cohomology with $\mathbb{Z}_\ell$-coefficients in the case of ℓ-adic homology, or Betti cohomology in the case of Borel–Moore homology, the cap product with the fundamental class of X induces for all i (*e.g.*, [Lau76, p.173]) an isomorphism $- \cap [X] : \mathcal{H}^{2d_X - i}(X)(d_X) \overset{\sim}{\to} \mathcal{H}_i(X)$, and the cycle class maps are compatible [Lau76, Rem. 6.4].

16.2. **Homological decomposition of the diagonal and resolution of singularities.**

16.2.1. *Long exact sequences in ℓ-adic homology.* In this section we recall some long exact sequences in ℓ-adic homology. These are standard in Borel–Moore homology, going back to the original paper [BM60]. Unfortunately, these do not seem to appear in the literature in the analogous theory of ℓ-adic homology. Here we present the arguments of [BM60] in the modern language of the 6-functor formalism, establishing the results in either setting.

First, assume that $i : Z \subseteq X$ is a closed subvariety, and let $U = X - Z$ be the complement, with inclusion $j : U \subseteq X$. Then there is a long exact sequence (*e.g.*, [BM60, Thm. 3.8] in Borel–Moore homology)

$$\cdots \longrightarrow \mathcal{H}_i(Z) \xrightarrow{i_*} \mathcal{H}_i(X) \xrightarrow{j^*} \mathcal{H}_i(U) \longrightarrow \cdots \tag{16.2}$$

The key observation to establish this is that there is an exact triangle in the derived category of constructible sheaves

$$i_*\omega_Z \to \omega_X \to j_*\omega_U \to i_*\omega_Z[1].$$

Taking the long exact sequence in hyper-cohomology gives (16.2). To obtain this exact triangle, we replace ω_X with a quasi-isomorphic complex of injectives, $I^\bullet$, and then consider the short exact sequence (*e.g.*, [Har77, Exe. II.1.20])

$$0 \to \mathcal{H}_Z^0(I^\bullet) \to I^\bullet \to j_*(I^\bullet|_U) \to 0,$$

which is exact on the right since injectives are flasque. This gives the exact triangle

$$\mathcal{H}_Z^0(\omega_X) \to \omega_X \to j_*(\omega_X|_U) \to \mathcal{H}_Z^0(\omega_X)[1].$$

Now we recall that $\omega_X := R\pi^!\Lambda$, where $\pi : X \to \operatorname{Spec} K$ is the structure morphism. For closed immersions, we have $i_*Ri^! = \mathcal{H}_Z^0$ [Lau76, (0.3.2)(a)], so that on the left we have $\mathcal{H}_Z^0(\omega_X) = i_*i^!\omega_X = i_*i^!\pi^!\Lambda = i_*\omega_Z$. For open immersions, we have $Rj^! = j^*$ so that $\omega_X|_U = j^*\omega_X = Rj^!R\pi^!\Lambda = \omega_U$.

Similarly, suppose that we have $X = X_1 \cup X_2$ a decomposition of a variety into two irreducible components. Let $i_j : X_j \hookrightarrow X$, $j = 1, 2$, be the closed immersions of the components, and let $i_{12} : X_1 \cap X_2 \hookrightarrow X$ be the closed immersion of the intersection. Then we have a long exact sequence (*e.g.*, [BM60, Thm. 3.10] in Borel–Moore homology)

$$\cdots \longrightarrow \mathcal{H}_i(X_1 \cap X_2) \longrightarrow \mathcal{H}_i(X_1) \oplus \mathcal{H}_i(X_2) \longrightarrow \mathcal{H}_i(X) \longrightarrow \cdots \tag{16.3}$$

where the maps are the obvious inclusion and difference maps. Again, the point is that we have an exact triangle

$$i_{12*}\omega_{X_1 \cap X_2} \to i_{1*}\omega_{X_1} \oplus i_{2*}\omega_{X_2} \to \omega_X \to i_{12*}\omega_{X_1 \cap X_2}[1].$$

The argument to obtain this exact triangle is similar. We replace ω_X with a quasi-isomorphic complex of injectives, $I^\bullet$, and then consider the short exact sequence

$$0 \to \mathcal{H}_{X_1 \cap X_2}^0(I^\bullet) \to \mathcal{H}_{X_1}^0(I^\bullet) \oplus \mathcal{H}_{X_2}^0(I^\bullet) \to I^\bullet \to 0,$$

where again we get exactness on the right using that injectives are flasque. This gives the exact triangle

$$\mathcal{H}_{X_1 \cap X_2}^0(\omega_X) \to \mathcal{H}_{X_1}^0(\omega_X) \to \omega_X \to \mathcal{H}_{X_1 \cap X_2}^0(\omega_X)[1].$$

Now we use again the description of the extraordinary pull back for closed immersions to conclude $\mathcal{H}_{X_1 \cap X_2}^0(\omega_X) = i_{12*}\omega_{X_1 \cap X_2}$ and $\mathcal{H}_{X_j}^0(\omega_X) = i_{j*}\omega_{X_j}$.

16.2.2. *Applications to decomposition of the diagonal and resolution of singularities.* Recall that we say that a variety X over a perfect field K has at worst ordinary double point singularities if it is smooth over K, or has isolated singular points, each of which is a K-point with a projective tangent cone that is a smooth quadric over K.

Proposition 16.3. *Let X be a projective variety over an algebraically closed field k, having only ordinary double point singularities, let $\epsilon : \widetilde{X} \to X$ be the standard resolution obtained by blowing up the singular points of X, and assume that the even degree cohomology of $\widetilde{X}$ is algebraic and without torsion. Then $\Delta_X \in \operatorname{CH}^{d_X}(X \times_k X)$ admits a strict homological R-decomposition if and and only if $\Delta_{\widetilde{X}} \in \operatorname{CH}^{d_X}(\widetilde{X} \times_k \widetilde{X})$ admits a strict homological R-decomposition.*

Remark 16.4. A similar result holds for Chow groups [Voi15, Thm. 2.1] [CTP16, Thm. 1.14] [HKT16, Prop. 8]).

Proof. The case where $K = \mathbb{C}$ and cycles are taken in Betti homology is [Voi15, Thm. 2.1]. Since we have (16.2) and (16.3) in ℓ-adic (and Borel–Moore) homology, and the even degree cohomology of a smooth quadric over an algebraically closed field is algebraic [SGA73, Exp. XII,Thm. 3.3], Voisin's proof carries over essentially without change. For convenience, we include the proof here.

We start with the easy direction. Assume that $\Delta_{\widetilde{X}} \in \mathrm{CH}^{d_X}(\widetilde{X} \times_k \widetilde{X})$ has a strict R-decomposition, $[\Delta_{\widetilde{X}}] = [\widetilde{Z}_1] + [\widetilde{Z}_2]$ with $\widetilde{Z}_1$ supported on $\widetilde{D} \times_k \widetilde{X}$ and $\widetilde{Z}_2 = \tilde{\mathrm{pr}}_2^*[\tilde{\alpha}]$ for some zero-cycle class $\tilde{\alpha} \in \mathrm{CH}_0(\widetilde{X})$. Now since ϵ is proper, we have push forward in homology that is compatible with the cycle class maps [Lau76, §6]. Then we simply push-forward via the proper map $\epsilon \times \epsilon$. More precisely, we have $\Delta_X = (\epsilon \times \epsilon)_* \Delta_{\widetilde{X}}$, $Z_1 := (\epsilon \times \epsilon)_* \widetilde{Z}_1$ supported on $D \times_k X$ where D is the image of $\widetilde{D}$, and $Z_2 := (\epsilon \times \epsilon)_* \widetilde{Z}_2 = (\epsilon \times \epsilon)_* \tilde{\mathrm{pr}}_2^* \tilde{\alpha} = \mathrm{pr}_2^* \epsilon_* \tilde{\alpha}$. This provides the strict decomposition for Δ_X.

Conversely, assume we have a strict R-decomposition of the diagonal for X:

$$[\Delta_X] - [Z_1] - [Z_2] = 0 \in \mathcal{H}_{2d_X}(X \times_k X)(-d_X), \tag{16.4}$$

where $Z_2 = \mathrm{pr}_2^* \alpha$ for some zero-cycle class $\alpha \in \mathrm{CH}_0(X)_R$. For space, we will leave off all of the Tate twists in what follows.

We denote by x_i the singular points of X, x the union of the singular points, Q_i the exceptional divisors (smooth quadrics) for the resolution $\epsilon : \widetilde{X} \to X$, and Q the union of these quadrics.

There is a diagram of maps of homology groups:

$$\begin{array}{ccc}
\mathcal{H}_{2d_X}(X \times_k X) & \longrightarrow & \mathcal{H}_{2d_X}(X \times_k X - (X \times_k x \cup X \times_k x)) \\
& & \| \\
\mathcal{H}_{2d_X}(\widetilde{X} \times_k \widetilde{X}) & \longrightarrow & \mathcal{H}_{2d_X}(\widetilde{X} \times_k \widetilde{X} - (\widetilde{X} \times_k Q \cup Q \times_k \widetilde{X}))
\end{array}$$

Since $\epsilon : \widetilde{X} \to X$ is lci ($\widetilde{X}$ is smooth), we may take the Gysin pull back $\widetilde{Z}_1$ of $\widetilde{Z}$, and set $\widetilde{Z}_2 = \tilde{\mathrm{pr}}_2^* \epsilon^* \alpha$. We find that the classes

$$[\Delta_X] - [Z_1] - [Z_2] \quad (= 0 \in \mathcal{H}_{2d_X}(X \times_k X))$$

$$[\Delta_{\widetilde{X}}] - [\widetilde{Z}_1] - [\widetilde{Z}_2] \quad \left(\in \mathcal{H}_{2d_X}(\widetilde{X} \times_k \widetilde{X}) \right)$$

have the same image in $\mathcal{H}_{2d_X}(X \times_k X - (X \times_k x \cup X \times_k x)) = \mathcal{H}_{2d_X}(\widetilde{X} \times_k \widetilde{X} - (\widetilde{X} \times_k Q \cup Q \times_k \widetilde{X}))$.

From the long exact sequence (16.2), one obtains that

$$[\Delta_{\widetilde{X}}] - [Z_1] - [Z_2] \in \mathcal{H}_{2d_X}(\widetilde{X} \times_k \widetilde{X})$$

comes from a homology class

$$\beta \in \mathcal{H}_{2d_X}(\widetilde{X} \times_k Q \cup Q \times_k \widetilde{X}).$$

The next observation is that the closed subset $\widetilde{X} \times_k Q \cup Q \times_k \widetilde{X} \subseteq \widetilde{X} \times_k \widetilde{X}$ is the union of $\widetilde{X} \times_k Q$ and $Q \times_k \widetilde{X}$ glued along $Q \times_k Q$, so that we have from (16.3)

$$\cdots \to \mathcal{H}_{2d_X}(\widetilde{X} \times_k Q) \oplus \mathcal{H}_{2d_X}(Q \times_k \widetilde{X}) \to \mathcal{H}_{2d_X}(\widetilde{X} \times_k Q \cup Q \times_k \widetilde{X}) \to \mathcal{H}_{2d_X-1}(Q \times_k Q) \to \cdots$$

As $Q \times Q = \coprod_{i,j} Q_i \times Q_j$, and $Q_i \times Q_j$ has trivial homology in odd degree [SGA73, Exp. XII, Thm. 3.3], we conclude that $\mathcal{H}_{2d_X-1}(Q \times_k Q) = 0$, so that β comes from a homology class

$$\gamma = (\gamma_1, \gamma_2) \in \mathcal{H}_{2d_X}(\widetilde{X} \times_k Q) \oplus \mathcal{H}_{2d_X}(Q \times_k \widetilde{X}).$$

We now use the assumption made on $\widetilde{X}$, namely that its even degree cohomology is algebraic. As the cohomology of Q has no torsion and is algebraic [SGA73, Exp. XII,Thm. 3.3] (this is the only place we are using that k is algebraically closed), we get by the Künneth decomposition that

$$\mathcal{H}_{2d_X}(Q \times_k \widetilde{X}) = \mathcal{H}^{2d_X-2}(Q \times_k \widetilde{X}) = \bigoplus_{0 \leq 2i \leq 2d_X-2} \mathcal{H}^{2i}(Q) \otimes \mathcal{H}^{2d_X-2-2i}(\widetilde{X})$$

is generated by classes of algebraic cycles $z_j \times_k z_j' \subseteq Q \times_k \widetilde{X}$, and similarly for $\widetilde{X} \times_k Q$.

Putting everything together, we get an equality

$$\Delta_{\widetilde{X}} - [Z_1] - [Z_2] = \sum n_j [z_j \times_k z_j'] + \sum n_j' [z_j' \times_k z_j] \in \mathcal{H}_{2d_X}(\widetilde{X} \times_k \widetilde{X}).$$

This provides us with an integral cohomological decomposition of the diagonal. Indeed, all the cycle classes of the form $[\widetilde{\mathcal{X}_{\bar{s}}} \times_k pt]$ are cohomologous and they have to sum-up to zero, while all the other terms $[z_k' \times_k z_k]$ with $\dim z_k' < n$ are supported on $D \times_k \widetilde{X}$ for some closed algebraic subset $D \subsetneq \widetilde{X}$. □

16.3. Homological decomposition of the diagonal and specialization.

16.3.1. *Specialization and cycle class maps in ℓ-adic homology.* Let B be a smooth variety of dimension 1 over a field K, let $f : \mathcal{X} \to B$ be a smooth morphism. Setting $X_{\bar{\eta}}$ to be the geometric generic fiber, and X_b to be the fiber over a $\overline{K}$-point $b \in B(\overline{K})$, one has a commutative diagram [Ful98, Exa. 20.3.5] (see [Ful75, p.65] and [BGI71, SGA6, Exp. X, 7.13-7.16])

$$\begin{array}{ccc}
\mathrm{CH}_n(\mathcal{X}_{\bar{\eta}}) & \xrightarrow{\mathrm{sp}} & \mathrm{CH}_n(X_b) \\
\downarrow{[-]} & & \downarrow{[-]} \\
\mathcal{H}^{2n}(\mathcal{X}_{\bar{\eta}}) & =\!=\!=\!= & \mathcal{H}^{2n}(X_b)
\end{array} \tag{16.5}$$

where the top arrow is the specialization map of [Ful98, §20.3], and the bottom equality comes from proper base change. If $K = \mathbb{C}$, and we identify $\overline{\kappa(\eta)} = \mathbb{C}$, then we have the same result in Borel–Moore homology.

If more generally we want to consider a morphism $f : \mathcal{X} \to B$ of finite type, then there is a specialization map in ℓ-adic homology making the following diagram commute:

$$\begin{array}{ccc}
\mathrm{CH}_n(\mathcal{X}_{\bar{\eta}}) & \xrightarrow{\mathrm{sp}} & \mathrm{CH}_n(X_b) \\
\downarrow{[-]} & & \downarrow{[-]} \\
\mathcal{H}_{2n}(\mathcal{X}_{\bar{\eta}})' & \xrightarrow{\mathrm{sp}} & \mathcal{H}_{2n}(X_b)
\end{array} \tag{16.6}$$

where $\mathcal{H}_{2n}(X_{\bar{\eta}})' \subseteq \mathcal{H}_{2n}(X_{\bar{\eta}})$ is the image of the cycle class map. Again, if $K = \mathbb{C}$, and we identify $\overline{\kappa(\eta)} = \mathbb{C}$, then we have the same result in Borel–Moore homology. Since there does not appear to be a reference in the literature, we explain this now. Setting $B^\circ = B - \{b\}$ and $f^\circ : \mathcal{X}^\circ = B^\circ \times_B \mathcal{X} \to B^\circ$ to be the restriction, we obtain a commutative diagram

$$\begin{array}{ccccccc}
 & & & & & & \mathrm{CH}_n(X_b) \\
 & & & & & \nearrow^{i^!} & \downarrow \\
\mathrm{CH}_n(X_b) & \xrightarrow{i_*} & \mathrm{CH}_n(\mathcal{X}/B) & \xrightarrow{j^*} & \mathrm{CH}_n(\mathcal{X}^\circ/B^\circ) & \xrightarrow{\mathrm{sp}} & 0 \\
\downarrow{[-]} & & \downarrow{[-]} & & \downarrow{[-]} & & \downarrow{[-]} \\
 & & & & \xrightarrow{i^!} & & \mathcal{H}_{2n}(X_b) \\
\mathcal{H}_{2n}(X_b) & \xrightarrow{i_*} & \mathcal{H}_{2n}(\mathcal{X}) & \xrightarrow{j^*} & \mathcal{H}_{2n}(\mathcal{X}^\circ)
\end{array}$$

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where the horizontal sequences are exact ([Ful98, Prop. 1.8] and (16.2)). Since $i^! i_* = 0$ for Chow groups [Ful98, §20.3], the top of this diagram gives the definition of the specialization map [Ful98, §20.3]. The compatibility of the cycle class maps with proper push forward and flat pull back is standard [Lau76, §6].

On the images of the cycle class maps in homology, by commutativity, we have $i^! i_* = 0$, so that from the diagram above, we can define the specialization map in homology:

$$\begin{array}{ccc}
\mathrm{CH}_n(\mathcal{X}^\circ / B^\circ) & \xrightarrow{\mathrm{sp}} & \mathrm{CH}_n(X_b) \\
\downarrow{[-]} & & \downarrow{[-]} \\
\mathcal{H}_{2n}(\mathcal{X}^\circ)' & \xrightarrow{\mathrm{sp}} & \mathcal{H}_{2n}(X_b)
\end{array}$$

We obtain (16.6) by spreading cycle classes after finite base changes, as in [Ful98, Exa. 20.3.8].

Remark 16.5. Let $f : \mathcal{X} \to B$ be a smooth morphism of smooth varieties of finite type over a field K, and let $B' \subseteq B$ be a closed regular embedding of codimension 1 with trivial normal bundle. Let η (resp. η') be the generic point of B (resp. B'). The arguments above generalize to this setting (see [Ful98, Exa. 20.3.8]) to give a specialization map

$$\begin{array}{ccc}
\mathrm{CH}_n(\mathcal{X}_{\bar\eta}) & \xrightarrow{\mathrm{sp}} & \mathrm{CH}_n(\mathcal{X}_{\bar\eta'}) \\
\downarrow{[-]} & & {[-]}\downarrow \\
\mathcal{H}_{2n}(\mathcal{X}_{\bar\eta})' & \xrightarrow{\mathrm{sp}} & \mathcal{H}_{2n}(\mathcal{X}_{\bar\eta'})'
\end{array}$$

For any $b \in B(\overline{K})$, if we iteratively take smooth subvarieties $b \in B' \subseteq B$, and restrict to Zariski open subsets to trivialize the normal bundle, we obtain a restriction map (16.6) even if $\dim B > 1$.

16.3.2. *Application: strict decomposition of the diagonal and specialization.*

Theorem 16.6. *Let B be a smooth integral variety over a field K, let $\pi : \mathcal{X} \to B$ be a flat projective morphism of relative dimension d. If there exists a $\overline{K}$-point $b \in B(\overline{K})$ such that for the fiber X_b the class of the diagonal $\Delta_{X_b} \in \mathrm{CH}^d(X_b \times_{\overline{K}} X_b)$ does not admit a strict homological R-decomposition, then the same is true for the geometric generic fiber $X_{\bar\eta}$.*

Remark 16.7. A similar result holds for Chow groups [Voi15, Thm. 2.1] [CTP16, Thm. 1.14] [HKT16, Thm. 9].

Proof. Using Remark 16.5, it suffices to prove the theorem for $\dim B = 1$. The case where $K = \mathbb{C}$ is a special case of [Voi15, Thm. 2.1]. We proceed to give a similar proof in the setting of ℓ-adic homology.

We start with a few simplifying assumptions. Since the algebraic closure of the function field of B is isomorphic to the algebraic closure of the function field of $B_{\overline{K}}$, we may assume that $K = \overline{K}$. Also, since the geometric generic fiber does not change after finite base change, we are free to make finite base changes.

We will now prove the contrapositive of the theorem, and therefore we start by assuming $\Delta_{X_{\bar\eta}} \in \mathrm{CH}^d(X_{\bar\eta} \times_{\bar\eta} X_{\bar\eta})$ admits a strict homological R-decomposition. By the finite type hypotheses, there is some finite extension of the function field $\kappa(\eta)$ of B over which the decomposition is defined. Therefore, after spreading, we may assume that after a finite surjective base change $B' \to B$ there exist a divisor $\mathcal{D}' \subseteq \mathcal{X}' := B' \times_B \mathcal{X}$, which we may assume to contain no fiber of $\mathcal{X}' \to B'$, a cycle $\mathcal{Z}_1 \in \mathrm{CH}^d(\mathcal{X}' \times_{B'} \mathcal{X}')$ supported on $\mathcal{D}' \times_{B'} \mathcal{X}'$, and a zero cycle $\alpha' \in \mathrm{CH}_0(\mathcal{X}')$ of relative degree-1

such that setting $\mathcal{Z}_2 = \mathrm{pr}_2^* \alpha'$, we have

$$[\Delta_{X'_{\bar{\eta}}}] - [(\mathcal{Z}_1)_{\bar{\eta}}] - [(\mathcal{Z}_2)_{\bar{\eta}}] = 0 \in \mathcal{H}_{2d}(X'_{\bar{\eta}} \times_{\overline{\kappa(\eta)}} X'_{\bar{\eta}})(-d).$$

Since we are free to make finite base changes, to simplify the notation we relabel B' as B and continue the proof. We claim that at every $\overline{K}$-point $b \in B(\overline{K})$, we have

$$[\Delta_{X_b}] - [(\mathcal{Z}_1)_b] - [(\mathcal{Z}_2)_b] = 0 \in \mathcal{H}_{2d}(X_b \times_{\overline{K}} X_b)(-d).$$

But this follows from the compatibility of the cycle class map with specialization given in (16.6).

$\square$

17. QUARTIC DOUBLE SOLIDS AND THE PROOF OF THEOREM 3

We now use the results developed so far to show that there exist unirational threefolds in positive characteristic that have no universal codimension-2 cycle class.

The starting point is the result of Artin and Mumford, that over an algebraically closed field k of characteristic $\neq 2$ there exists a quartic double solid X with exactly 10 singular points, in special position, all of which are nodes; i.e, a double cover of $\mathbb{P}_k^3$ branched along a quartic with 10 nodes in special position, such that the standard resolution of singularities $\epsilon : \widetilde{X} \to X$ has non-trivial torsion in cohomology: for $\mathcal{H}^\bullet(-)$ given by Betti cohomology $H^\bullet(-, \mathbb{Z})$, or 2-adic cohomology $H^\bullet(-, \mathbb{Z}_2)$, we have $\mathrm{Tors}\, \mathcal{H}^4(\widetilde{X}) \neq 0$ [AM72, Prop. 3, §4]. It follows from Corollary 7.2 that the class of the diagonal $[\Delta_{\widetilde{X}}] \in \mathcal{H}^6(\widetilde{X} \times_k \widetilde{X})(3)$ does not admit an $R_{\mathcal{H}}$-decomposition.

We use this as a starting point for the investigation of desingularizations of quartic double solids with at most 9 nodes. In order to connect such threefolds to the Artin–Mumford example we take a brief excursion into the moduli spaces of (lattice-polarized) K3 surfaces in Section 17.1.

17.1. Nodal quartic surfaces.

Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$. Let $Y \subset \mathbb{P}_k^3$ be a quartic surface smooth away from rational double points $P_1, \ldots, P_n \in Y(k)$, with $n \geq 1$. Let $\omega : \widetilde{Y} \to Y$ be the minimal resolution of Y, obtained by blowing up the n nodes. Then $\widetilde{Y}$ is a smooth K3 surface. Let $\lambda = \omega^* \mathcal{O}_{\mathbb{P}^3}(1) \in \mathrm{Pic}(\widetilde{Y}) \cong \mathrm{NS}(\widetilde{Y})$, and for $1 \leq i \leq n$ let ϵ_i be the class of $\omega^{-1}(P_i)$ in $\mathrm{Pic}(\widetilde{Y})$. Under the intersection pairing, we have $(\lambda, \lambda) = 4$; $(\epsilon_i, \epsilon_i) = -2$; $(\lambda, \epsilon_i) = 0$; and, if $i \neq j$, then $(\epsilon_i, \epsilon_j) = 0$. Therefore, the $\mathbb{Z}$-span of $\{\lambda, \epsilon_1, \ldots, \epsilon_n\}$ is a primitive sub-lattice of $\mathrm{Pic}(\widetilde{Y})$ of rank $n + 1$. Moreover, for $N \gg 0$, $N\lambda - \sum \epsilon_i$ is (very) ample. In particular, the lattice spanned by $\{\lambda, \epsilon_1, \ldots, \epsilon_n\}$ contains the class of a polarization.

With this notation, it is not hard to show:

Lemma 17.1. *Let k be an algebraically closed field. Suppose that either $\mathrm{char}(k) \neq 2$ and $R = k[\![T]\!]$, or that $\mathrm{char}(k) = p > 2$ and that R is the ring of Witt vectors $\mathbb{W}(k)$. Let $B = \mathrm{Frac}(R)$.*

Let $Y \subset \mathbb{P}_k^3$ be a quartic surface which has n rational double points and is smooth elsewhere. Then for each $0 \leq m \leq n$, there exists a deformation $\mathcal{Y}/R$ of Y/k such that $\mathcal{Y}_B$ has exactly m nodes.

Proof. This follows from the deformation theory for K3 surfaces worked out in [Del81]; see, *e.g.*, [Ach20, Prop. 3.8] for details. We assume $\mathrm{char}(k) = p > 2$, since the deformation theory in [Del81] is built in analogy to the well-known classical case over the complex numbers. Let $R_{\mathrm{univ}} = \mathbb{W}(k)[\![T_1, \ldots, T_n]\!]$, and choose an isomorphism $\mathrm{Def}(\widetilde{Y}) \cong \mathrm{Spf}\, R_{\mathrm{univ}}$. We have seen that the collection $\{\lambda, \epsilon_1, \ldots, \epsilon_n\}$ is a linearly independent set of primitive elements of $\mathrm{Pic}(\widetilde{Y})$. Consequently, there exist $f_0, \ldots, f_n \in R_{\mathrm{univ}}$ such that, for $0 \leq m \leq n$, $R_{\mathrm{univ}}/(f_0, \ldots, f_m)$ is smooth over $\mathbb{W}(k)$ of relative dimension $19 - m$; and if A is an Artinian algebra, and if $\mu : R_{\mathrm{univ}} \to A$ is a deformation of $\widetilde{Y}$ to A, then ϵ_i (resp. λ) extends to $\widetilde{\mathcal{Y}}_A$ if and only if $\mu(f_i) = 0$ (resp. $\mu(f_0) = 0$). In particular, $\mathrm{Def}(\widetilde{Y}, \{\lambda, \epsilon_1, \ldots, \epsilon_m\}) \cong \mathrm{Spf}\, R_{\mathrm{univ}}/(f_0, \ldots, f_m)$.

So, let $\mathfrak{p}$ be the maximal ideal of R, and fix $0 \leq m \leq n$. Choose a compatible family of surjections $\mu_j : R_{\mathrm{univ}} \to R/(\mathfrak{p})^j$ such that $\mu_j(f_i) = 0$ if and only if $i \leq m$. We obtain a formal deformation of $\widetilde{Y}$ to Spf R. Moreover, since (for $N \gg 0$) $N\lambda - \sum_{i=1}^m \epsilon_i$ is ample on the generic fiber, the formal deformation algebraizes to yield an algebraic deformation $\widetilde{\mathscr{Y}}/R$ of $\widetilde{Y}$ over R. The only (-2)-curves on $\widetilde{\mathscr{Y}}_B$ are the curves representing $\epsilon_1, \ldots, \epsilon_m$; contracting these – equivalently, mapping $\widetilde{\mathscr{Y}}$ to $\mathbb{P}^3_R$ using the quasi-ample line bundle λ – gives the desired deformation of Y. $\qquad\square$

Already, this is adequate for producing examples of Theorem 3. To show that for $m \leq n$ an arbitrary m-nodal quartic surface degenerates to an arbitrary n-nodal quartic requires a brief detour into the moduli theory of K3 surfaces.

Let $\mathsf{R}_{4, \geq n} / \mathbb{Z}[1/2]$ be the moduli space of quartic surfaces with at least n rational double points. Our goal is to show:

Proposition 17.2. *If $n < 10$, then each fiber of $\mathsf{R}_{4, \geq n} \to \operatorname{Spec} \mathbb{Z}[1/2]$ is geometrically irreducible.*

Proof. Before proceeding, it may be worth recalling Madapusi Pera's strategy for showing that away from characteristics dividing $2d$, the moduli space R_{2d} of quasipolarized K3 surfaces of degree $2d$, is irreducible [MP15, Cor. 5.16]. The well-known period map for K3 surfaces realizes $\mathsf{R}_{2d}(\mathbb{C})$ as an arithmetic quotient of a Hermitian symmetric domain. This quotient admits a canonical model over $\mathbb{Z}[1/2d]$; the existence of a good arithmetic compactification for such an orthogonal Shimura variety implies, by Zariski's main theorem, that the space stays irreducible upon reduction modulo a prime. We adopt a similar strategy here, using the notion of a lattice-polarized K3 surface.

Initially, since the maximal number of nodes on a quartic surface is achieved by the 16 nodes of a Kummer surface, we merely assume that $1 \leq n \leq 16$. Let L_n be the free $\mathbb{Z}$-module generated by symbols $\ell, e_1, \ldots, e_n$, equipped with the pairing $(\ell, \ell) = 4$; $(e_i, e_i) = -2$; and all other pairings are zero. Then L_n is a lattice of rank $n + 1$ and signature $(1, n)$.

Now let Y/k be a quartic surface with n rational double points. A labeling of the these double points – equivalently, a labeling of n exceptional curves in the minimal resolution $\widetilde{Y} \to Y$ – induces a primitive embedding of lattices

$$L_n \xrightarrow{\ \alpha\ } \operatorname{Pic}(\widetilde{Y})$$

$$\ell \longmapsto \lambda$$

$$e_i \longmapsto \epsilon_i.$$

Moreover, λ is a quasi-polarization on $\widetilde{Y}$; and if Y is smooth away from the n double points, then $\alpha(L_n)$ contains the ample classes $N\lambda - \sum \epsilon_i$ for $N \gg 0$.

In short, $(\widetilde{Y}, \alpha)$ is an element of $\mathsf{R}_{L_n}(k)$, where R_{L_n} is the moduli space of K3 surfaces (quasi-) polarized by the lattice L_n ([Ach20, Dol96]). Let $\mathsf{R}_{4, \geq n}$ be the moduli space of quartic K3 surfaces with at least n rational double points. Contracting the classes represented by $\alpha(e_1), \ldots, \alpha(e_n)$ defines a morphism

$$\beta : \mathsf{R}_{L_n} \longrightarrow \mathsf{R}_{4, \geq n}$$

which restricts to an isomorphism

$$\mathsf{R}^\circ_{L_n} \longrightarrow \mathsf{R}_{4, =n},$$

where the source is the moduli space of K3 surfaces with an ample lattice polarization by L_n, and $\mathsf{R}_{4, =n}$ is the space of quartic surfaces in $\mathbb{P}^3$ with exactly n rational double points. By Lemma 17.1, $\mathsf{R}_{4, =n}$ is fiberwise (over $\mathbb{Z}[1/2]$) dense in $\mathsf{R}_{4, \geq n}$, and thus $\mathsf{R}^\circ_{L_n}$ is fiberwise dense in R_{L_n}.

It thus suffices to show that each fiber of R_{L_n} is geometrically irreducible; it is now that we start assuming $n \leq 9$. Then there is a unique primitive embedding of L_n into the standard K3 lattice

(*e.g.*, [Huy16, Thm. 14.1.12]). Consequently, over $\mathbb{C}$, there exist a Hermitian symmetric domain $\mathbb{D}^{L_n}$ of type IV, and an arithmetic group of automorphisms Γ^{L_n} of $\mathbb{D}^{L_n}$, such that the complex period map yields an isomorphism [DK07, Thm. 10.1]

$$\mathsf{R}_{L_n}(\mathbb{C}) \xrightarrow{\tau_{L_n,\mathbb{C}}} \Gamma^{L_n} \backslash \mathbb{D}^{L_n}.$$

In particular, $\mathsf{R}_{L_n,\mathbb{C}}$ is irreducible. The theory of integral canonical models of Shimura varieties provides a canonical stack Sh^{L_n} over $\mathbb{Z}[1/2]$ with $\mathsf{Sh}^{L_n}_{\mathbb{C}} = \Gamma^{L_n} \backslash \mathbb{D}^{L_n}$ [Kis10], and $\tau_{\mathbb{C}}$ is the complex fiber of a morphism

$$\mathsf{R}_{L_n} \xrightarrow{\tau_{L_n}} \mathsf{Sh}^{L_n}$$

of stacks over $\mathbb{Z}[1/2]$ [Ach20, Lem 6.4]. It is known that Sh^{L_n} is fiberwise geometrically irreducible [MP19, Cor. 4.1.11]. Because fibers of R_{L_n} and of Sh^{L_n} have the same dimension, it suffices to show that τ_{L_n} is an immersion.

Let p be an odd prime, and choose N so that $\mu := N\lambda - \sum \epsilon_i$ is ample and $d := \frac{1}{2}(\mu, \mu)$ is relatively prime to p. We have a morphism $\phi : \mathsf{R}_{L_n} \to \mathsf{R}_{2d}$ which, on S-points, is given by $(\widetilde{Y} \to S, \alpha) \mapsto (\widetilde{Y} \to S, \alpha(N\ell - \sum e_i))$.

As in [Ach20, §6], we have a commuting diagram of stacks over $\mathbb{Z}_{(p)}$:

$$
\begin{array}{ccc}
\mathsf{R}^{\circ}_{L_n} & \xrightarrow{\tau_{L_n}} & \mathsf{Sh}^{L_n} \\
\downarrow{\scriptstyle \phi} & & \downarrow \\
\mathsf{R}^{\circ}_{2d} & \xhookrightarrow{\tau_{2d}} & \mathsf{Sh}^{\langle 2d \rangle}
\end{array}.
$$

Since the minimal resolution of a K3 surface with exactly n nodes admits a unique L_n-polarization, ϕ is an immersion. Since τ_{2d} is an immersion [MP15, Cor. 5.15], τ_{L_n} is an immersion, too. $\qquad\square$

Corollary 17.3. *Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$. Suppose Y_1 and Y_2 are quartic surfaces with, respectively, exactly m and $n < 10$ nodes, with $m \leq \min(n, 9)$. Then there exist a twice-pointed curve (T, t_1, t_2) over k, and a relative quartic surface $\mathcal{Y} \to T$, such that $\mathcal{Y}_{t_1} \cong Y_1$ and $\mathcal{Y}_{t_2} \cong Y_2$.*

Proof. By Lemma 17.1, the closure of $\mathsf{R}_{4,=m}$ in $\mathsf{R}_{4,\geq m}$ contains $\mathsf{R}_{4,\geq n}$. Now use the fact (Proposition 17.2) that $\mathsf{R}_{4,\geq m,k}$ is irreducible. $\qquad\square$

Remark 17.4. While we show above that, for $n \leq 9$, the locus $\mathsf{R}_{4,\geq n}$ of degree-4 polarized K3 surfaces with greater than or equal to n nodes is irreducible, and therefore contains the Artin–Mumford example with $n = 10$ nodes (since one can deform the nodes independently for $n \leq 10$), the situation for the locus $\mathrm{Hilb}^4_{\mathbb{P}^3_k, \geq n}$ of quartic surfaces with greater than or equal to n nodes in the Hilbert scheme $\mathrm{Hilb}^4_{\mathbb{P}^3_k}$ of quartic surfaces is different. For contrast, we recall the situation over $\mathbb{C}$. It is known that for $n = 6, 7, 8, 9$, the locus $\mathrm{Hilb}^4_{\mathbb{P}^3_{\mathbb{C}}, \geq n}$ is reducible (see [Voi15, Rem. 1.2]), while for $n \leq 7$, there is a unique irreducible component of $\mathrm{Hilb}^4_{\mathbb{P}^3_{\mathbb{C}}, \geq n}$ dominating $(\mathbb{P}^3_{\mathbb{C}})^{(n)}$ by the map sending an m-nodal quartic to its set of nodes, and this component contains the Artin–Mumford examples [Voi15, §2]. For quartics with exactly $n = 8, 9$ nodes, it is a classical result that the nodes must be in special position in $\mathbb{P}^3_{\mathbb{C}}$ (there is no component of $\mathrm{Hilb}^4_{\mathbb{P}^3_{\mathbb{C}}, \geq n}$ dominating $(\mathbb{P}^3)^n$); we direct the reader to the MathSciNet review of [Voi15] for references.

17.2. Quartic double solids and the proof of Theorem 3. The previous section essentially says that quartic double solids are liftable to quartic double solids in characteristic 0, and that every quartic double solid with at most 9 nodes (and no other singularities) degenerates to the Artin–Mumford example with 10 nodes (since the nodes deform independently). We use this to give a proof of Theorem 3:

Proof of Theorem 3. In the case where char $k = 0$, this is [Voi15, Thm. 1.9]. We now proceed with the proof under the hypothesis that char $k > 2$.

To investigate conditions (1)–(4) of Theorem 2, we will first want to use that $\widetilde{X}$ lifts to characteristic 0. More precisely, let us fix notation, and suppose that Y is the n-nodal quartic surface over k defining $\widetilde{X}$. In other words, we have $\widetilde{X} \to X \to \mathbb{P}^3_k$, and the second morphism is the double cover of projective space branched over Y. For concreteness, suppose that Y is defined by $q(x_0, x_1, x_2, x_3) = 0$ for some homogeneous quartic polynomial $q(x_0, x_1, x_2, x_3)$, so that X is defined by $q(x_0, x_1, x_2, x_3) + x_4^2 = 0$ in the weighted projective space $\mathbb{P}^4_k(1,1,1,1,2)$. Now we may take the lift $\mathcal{Y}/S$ of Y to characteristic 0 of Lemma 17.1, and we define $\mathcal{X}/S$ in $\mathbb{P}^4_S(1,1,1,1,2)$ by taking the double cover of $\mathbb{P}^3_S$ branched along $\mathcal{Y}$. We then define $\widetilde{\mathcal{X}}/S$ by blowing up the locus in $\mathcal{Y}$ of nodes in the fibers; *i.e.*, the singular locus of the map $\mathcal{Y} \to S$. In other words, $\widetilde{X}$ lifts to characteristic 0 as the standard resolution of singularities of a quartic double solid with exactly n nodes.

With this lift $\widetilde{\mathcal{X}}/S$ of $\widetilde{X}$ to characteristic 0, then using the fact that in characteristic 0 the standard resolution of singularities of a quartic double solid with exactly n nodes is rationally connected and has no 2-torsion in cohomology for $n \leq 9$ [End99], we can conclude from Corollary 15.4 that $\widetilde{X}$ satisfies conditions (1)–(4) of Theorem 2 with $\ell = 2$ for $n \leq 9$.

Thus we can focus on conditions (5) and (6). For this we will use a new deformation of X, namely a deformation to the Artin–Mumford example. More precisely, with X and Y as above, according to Corollary 17.3, we can find a second family of quartic surfaces $\mathcal{Y}' \to S'$ over a k-curve S', with points $s_1', s_2' \in S'(k)$ such that $\mathcal{Y}'_{s_1'} \cong Y$ and $\mathcal{Y}'_{s_2'}$ is isomorphic to the quartic surface of the Artin–Mumford example. We take $\mathcal{X}'/S'$ to be the double cover of $\mathbb{P}^3_{S'}$ branched along $\mathcal{Y}'$. Then $\mathcal{X}'_{s_1'} \cong X$, and $\mathcal{X}'_{s_2'}$ is the Artin–Mumford quartic double solid. We denote by $\widetilde{\mathcal{X}}'_{s_2'} \to \mathcal{X}'_{s_2'}$ the standard resolution of singularities of the Artin–Mumford quartic double solid; *i.e.*, the Artin–Mumford example. As $\mathrm{Tors}\, H^4(\widetilde{\mathcal{X}}'_{s_2'}, \mathbb{Z}_2) \neq 0$ [AM72, Prop. 3, §4], we see that the Artin–Mumford example $\widetilde{\mathcal{X}}'_{s_2'}$ does not admit a strict cohomological $\mathbb{Z}_2$-decomposition of the diagonal (Corollary 7.2). Using Proposition 16.3 twice and Theorem 16.6 once, we can conclude that $\widetilde{X} = \widetilde{\mathcal{X}}'_{s_1'}$ does not admit a a strict cohomological $\mathbb{Z}_2$-decomposition of the diagonal.

Therefore, from Theorem 2, one of the conditions (5) or (6) must fail. Let us focus on (6). Using that (3) and (3′) hold for $\widetilde{X}$, it follows that $\dim \mathrm{Ab}^2_{\widetilde{X}/K} = \dim H^3(\widetilde{X}_{\overline{K}}, \mathbb{Q}_\ell)$. By proper base change, we know that $H^3(\widetilde{X}_{\overline{K}}, \mathbb{Q}_\ell) \cong H^3(\widetilde{\mathcal{X}}_{\overline{\eta}}, \mathbb{Q}_\ell)$, and it is well-known that $\dim H^3(\widetilde{\mathcal{X}}_{\overline{\eta}}, \mathbb{Q}_\ell) = 10 - n$ (for $n = 0$, this is the standard computation of the Betti numbers of a double cover, and for $n \geq 1$ this is [Bea77, 4.10.4]). Using again the lift $\widetilde{\mathcal{X}}/S$ of $\widetilde{X}$ to characteristic 0, then assuming (5) holds, we can use Proposition 14.3 to conclude that for $n = 7, 8, 9$, we have that $[\Theta_{\widetilde{X}}]^{g-1}/(g-1)!$ is $\mathbb{Z}$-algebraic, so that condition (6) holds. Therefore, for $n = 7, 8, 9$, we must have had that condition (5) fails, since otherwise conditions (1)–(6) would hold and $\widetilde{X}$ would admit a cohomological $\mathbb{Z}_2$-decomposition of the diagonal, which we know is not the case. $\qquad\square$

Remark 17.5. Here we recall that the standard desingularization $\widetilde{X}$ of a nodal quartic double solid is unirational. For the case of $n = 0$ nodes, we direct the reader to [Wel81, p.10], where a unirational parameterization of $\widetilde{X}$ is given; Welters works over $\mathbb{C}$ but the argument holds for char$(k) \neq 2$. For

$n \geq 1$, we have moreover that $\widetilde{X}$ is separably rationally connected, there is a degree 2 dominant rational map $\mathbb{P}^3_k \dashrightarrow \widetilde{X}$, and the class $2\Delta_{\widetilde{X}} \in \mathrm{CH}^3(\widetilde{X} \times_k \widetilde{X})$ admits a strict decomposition. This can be found in [Bea77, 4.5.4], [Bea83b, Exa. 3, p.25]. For convenience, we sketch the argument here. The key point is to show is that there is a degree-2 dominant rational map to $\widetilde{X}$ from a rational threefold. Projecting from a chosen node exhibits the blow-up X' of X at that node as a singular fibration in quadrics. The exceptional divisor $Q \subseteq X'$ is a quadric surface, which, under the structure map $X' \to \mathbb{P}^2_k$, gives a double cover of $\mathbb{P}^2_k$. The base change of X' to Q under this double cover admits a section, and therefore is rational (see *e.g.*, [Bea77, Prop. 4.1]), completing the proof. If $\mathbb{P}^3_k \dashrightarrow \widetilde{X}$ is the associated degree-2 dominant rational map from projective space, then since $\operatorname{char} k \neq 2$, we must have that this is a separable rational map, so that $\widetilde{X}$ is separably rationally connected [Kol96, Exa. IV.3.2.6.2]. Finally, the degree-2 dominant rational map $\mathbb{P}^3_k \dashrightarrow \widetilde{X}$ also implies that $2\Delta_X$ admits a strict decomposition (Remark 2.9).

Remark 17.6. In Theorem 3 we showed that the standard desingularization of a very general quartic double solid with exactly $n \leq 9$ nodes does not admit a cohomological $\mathbb{Z}_2$-decomposition of the diagonal. The previous remark implies that for $1 \leq n \leq 9$, the diagonal admits a cohomological $\mathbb{Z}_\ell$-decomposition for all $\ell \neq 2, \operatorname{char}(k)$, since 2 is invertible in $\mathbb{Z}_\ell$. This shows in particular that while conditions (1)–(6) in Theorem 2 are sufficient to imply the existence of a cohomological $\mathbb{Z}_\ell$-decomposition, they are not necessary. See Theorem 15.5 for necessary and sufficient conditions for a cohomological $\mathbb{Z}_\ell$-decomposition.

A.1. **The Weil pairing.** Let A/K be an abelian variety over a field, with dual abelian variety $\widehat{A}$. For any integer N, the group scheme $\widehat{A}[N]$ is canonically isomorphic to the Cartier dual $A[N]^D :=$ $\mathrm{Hom}(A[N], \mathbb{G}_M)$ of $A[N]$, and thus there is a canonical Weil pairing

$$A[N] \times \widehat{A}[N] \longrightarrow \boldsymbol{\mu}_N. \tag{A.1}$$

Let l be any prime. The l-adic Tate module of A is $T_l A := \varprojlim A[l^n](\overline{K})$. If $l = \ell \neq \mathrm{char}(K)$, then $T_\ell A$ is abstractly isomorphic to $\mathbb{Z}_\ell^{2 \dim A}$; but if $p = \mathrm{char}(K) > 0$, then $T_p A$ is free over $\mathbb{Z}_p$ of rank at most $\dim A$. For any l, we have a canonical isomorphism

$$T_l(\widehat{A}) = T_l(A)^\vee(1) \tag{A.2}$$

(the notation is recalled in our Conventions 0.1.) To see this in the case where $l = p = \mathrm{char}(K) = p > 0$, use (A.1) to obtain that $\widehat{A}[p^\infty]$ is isomorphic to the Serre dual $(A[p^\infty])^D = \mathrm{Hom}(A[p^\infty], \boldsymbol{\mu}_{p^\infty})$; then use the fact that $T_p A = T_p(A[p^\infty])$, and that for any p-divisible group G there are canonical isomorphisms $T_p(G^D) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G)(-1), \mathbb{Z}_p) = T_p(G)^\vee(1)$.

If $\ell \neq \mathrm{char}(K)$, then the intersection pairing, Poincaré duality and the Weil pairing yield canonical isomorphisms

$$H^{2g-1}(A, \mathbb{Z}_\ell(g)) \overset{\cup}{=} H^1(A, \mathbb{Z}_\ell)^\vee \overset{PD}{=} T_\ell A \overset{\mathrm{Weil}}{=} (T_\ell \widehat{A})^\vee(1) \overset{PD}{=} H^1(\widehat{A}, \mathbb{Z}_\ell(1)) \tag{A.3}$$

and

$$H^{2g-1}(A, \mathbb{Q}_\ell(g)) \overset{\cup}{=} H^1(A, \mathbb{Q}_\ell)^\vee \overset{PD}{=} V_\ell A \overset{\mathrm{Weil}}{=} (V_\ell \widehat{A})^\vee(1) \overset{PD}{=} H^1(\widehat{A}, \mathbb{Q}_\ell(1)).$$

If K is perfect of characteristic $p > 0$, there are canonical isomorphisms of F-crystals

$$H_{\mathrm{cris}}^{2g-1}(A/\mathbb{W})(g) = H_{\mathrm{cris}}^1(A/\mathbb{W})^\vee = H^1(\widehat{A}/\mathbb{W})(1);$$

taking invariants under F yields (A.3) at p.

A.2. **Line bundles and symmetric isogenies.** We review the link between symmetric isogenies $A \to \widehat{A}$ and line bundles on $A_{\overline{K}}$. We also prove Lemma A.1, which we will use later.

Let A be an abelian variety over a field K. Recall that, to an isomorphism class of a line bundle L on A, one associates the (symmetric, *i.e.*, self-dual) homomorphism $\varphi_L : A \to \widehat{A}$, which on points $\overline{a} \in A(\overline{K})$ is given by $\varphi_L(\overline{a}) = t_{\overline{a}}^* L \otimes L^{-1}$. When K is finite or algebraically closed, the assignment $L \mapsto \varphi_L$ surjects onto the set of symmetric isogenies (*e.g.*, [Con, Thm. 2.6] if K is finite and [Mum70, §20] if $K = \overline{K}$). We say that L is *nondegenerate* if φ_L is an isogeny. For nondegenerate line bundles L, L' on A, we have $\varphi_L = \varphi_{L'}$ if and only if L and L' are algebraically equivalent, *i.e.*, if and only if L and L' differ by translation.

A symmetric isogeny $\Lambda : A \to \widehat{A}$ is *principal* if it is an isomorphism. If L is a line bundle on $A_{\overline{K}}$ such that $\Lambda_{\overline{K}} = \varphi_L$, we have the equalities $L^g/g! = \chi(L) = \pm\sqrt{\deg \varphi_L}$, where $g = \dim A$. In other words, Λ is principal if and only if $\chi(L) = \pm 1$. We observe that if Ω/K is any algebraically closed field, then $\Lambda_\Omega : A_\Omega \to \widehat{A}_\Omega$ is principal if and only if Λ is.

Recall that a symmetric isogeny $\Lambda : A \to \widehat{A}$ is a polarization if $\Lambda_{\overline{K}} : A_{\overline{K}} \to \widehat{A}_{\overline{K}}$ is induced by an ample line bundle; *i.e.*, for any line bundle L on $A_{\overline{K}}$ such that $\Lambda_{\overline{K}} = \varphi_L$, we have that L is ample. Recall that a nondegenerate line bundle L is ample if and only if $h^0(A, L) > 0$ (*e.g.*, [Mum70, §17]), and so Λ is a polarization (resp. principal polarization) if and only if $h^0(A, L) > 0$ (resp. $h^0(A, L) = 1$). We observe that if Ω/K is any algebraically closed field, then $\Lambda_\Omega : A_\Omega \to \widehat{A}_\Omega$ is a (principal) polarization if and only if Λ is. Indeed, clearly if Λ is a polarization, then Λ_Ω is as well. Conversely, suppose that Λ_Ω is a polarization, and let L be a line bundle on $A_{\overline{K}}$ such that $\Lambda_{\overline{K}} = \varphi_L$. As L is by definition non-degenerate, and a non-degenerate line bundle on an abelian

variety is ample if and only if it is effective, it suffices to show that L is effective. For this, note that $\Lambda_\Omega = (\varphi_L)_\Omega = \varphi_{L_\Omega}$. Since Λ_Ω is a polarization, it follows that L_Ω is ample, and therefore effective. Using cohomology and base-change (over the affine base fields), one concludes that L is effective.

A.3. **First Chern class of a symmetric isogeny.** Given a symmetric isogeny $\Lambda : A \to \widehat{A}$, we denote by $[\Lambda] \in H^2(A_{\overline{K}}, \mathbb{Z}_\ell(1))$ the first Chern class of the unique line bundle (up to translation) on $A_{\overline{K}}$ inducing the base change of Λ to $\overline{K}$. In other words, if L is any line bundle on $A_{\overline{K}}$ such that $\Lambda_{\overline{K}} = \varphi_L : A_{\overline{K}} \to \widehat{A}_{\overline{K}}$, then we set $[\Lambda] := c_1(L)$. One can realize the first Chern class of Λ more directly in the following way. The symmetric isogeny Λ induces by Tensor-Hom adjunction a morphism $T_\ell\Lambda : T_\ell A \otimes T_\ell\widehat{A}^\vee \to \mathbb{Z}_\ell$. The Weil pairing $e : T_\ell A \times T_\ell\widehat{A} \to \mathbb{Z}_\ell(1)$ then provides an isomorphism $e : T_\ell\widehat{A}^\vee \xrightarrow{\sim} T_\ell A(-1)$, which all together gives the pairing

$$E^\Lambda : T_\ell A \times T_\ell A \to \mathbb{Z}_\ell(1), \quad (x, y) \mapsto e(x, T_\ell\Lambda(y)).$$

It is classical that the pairing E^Λ is alternating and that, seen as an element of $(\wedge^2 T_\ell A)^\vee(1) = H^2(A_{\overline{K}}, \mathbb{Z}_\ell(1))$, it coincides with $c_1(L)$ for any line bundle L such that $\Lambda = \varphi_L$; see *e.g.* [EvdGM, (11.23)].

In case a symmetric isogeny Λ induces an isomorphism $T_\ell\Lambda : T_\ell A \xrightarrow{\simeq} T_\ell\widehat{A}$, the inverse to $T_\ell\Lambda$ has an explicit description in terms of the first Chern class $[\Lambda]$:

Lemma A.1. *Let $\Lambda : A \to \widehat{A}$ be a symmetric isogeny of a g-dimensional abelian variety over a field K. If $T_\ell\Lambda : T_\ell A \xrightarrow{\simeq} T_\ell\widehat{A}$ is an isomorphism for some prime ℓ, then the inverse is given by the map*

$$(T_\ell\Lambda)^{-1} = \frac{[\Lambda]^{g-1}}{(g-1)!} \cup - : H^1(A_{\overline{K}}, \mathbb{Z}_\ell(1)) \to H^{2g-1}(A_{\overline{K}}, \mathbb{Z}_\ell(g)),$$

where we have identified $T_\ell A$ with $H^{2g-1}(A_{\overline{K}}, \mathbb{Z}_\ell(g))$ and $T_\ell\widehat{A}$ with $H^1(A_{\overline{K}}, \mathbb{Z}_\ell(1))$ as laid out in §A.1.

Proof. Identifying $[\Lambda]$ with the pairing E^Λ above, one concludes by using the fact that the cohomology algebra $H^\bullet(A_{\overline{K}}, \mathbb{Z}_\ell)$ identifies, via the intersection pairing, with the alternating algebra on $H^1(A_{\overline{K}}, \mathbb{Z}_\ell)$. $\square$

A.4. **Some Hodge theoretic conventions.** Suppose that X is a complex projective manifold of dimension d_X, with ample divisor H. Setting $\omega = c_1(H) \in H^2(X, \mathbb{Z})$, and using ω also for the associated 2-form in $H^2_{dR}(X, \mathbb{C})$ under the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) = H^2_{dR}(X, \mathbb{C})$, the Hodge–Riemann bilinear form on $H^{p,q}(X)$ is the Hermitian form given by

$$h(\alpha, \beta) = i^{p-q}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \overline{\beta} \wedge \omega^{n-k},$$

where $k = p + q$.

Fixing an integer n such that $1 \leq 2n - 1 \leq d_X$, assume that $\mathrm{N}^n H^{2n-1}(X, \mathbb{Q}) = H^{2n}(X, \mathbb{Q})$, which implies that

$$H^{2n-1}_{dR}(X, \mathbb{C}) = H^{n,n-1}(X) \oplus H^{n-1,n}(X).$$

Via the composition

$$H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}_{dR}(X, \mathbb{C}) \twoheadrightarrow H^{n-1,n}(X)$$

$$\alpha \mapsto \alpha^{n,n-1} + \alpha^{n-1,n} \mapsto \alpha^{n-1,n}$$

we obtain an inclusion $H^{2n+1}(X, \mathbb{Z})_\tau \subseteq H^{n,n+1}(X)$, where in the above we have $\alpha^{n,n-1} = \overline{\alpha^{n-1,n}}$. The intermediate Jacobian is the complex torus

$$J^{2n-1}(X) = H^{n-1,n}(X)/H^{2n-1}(X, \mathbb{Z})_\tau.$$

The Hodge–Riemann bilinear form on $H^{n-1,n}(X)$ is in this case

$$h(\alpha, \beta) = (i)^{(n-1)-n}(-1)^{(2n-1)(2n-2)/2} \int_X \alpha \wedge \overline{\beta} \wedge \omega^{d_X-2n+1} = i(-1)^n \int_X \alpha \wedge \overline{\beta} \wedge \omega^{d_X-2n+1}.$$

For any $\alpha, \beta \in H^{2n-1}(X, \mathbb{Z})$, after viewing them as classes in $H^{2n-1}_{dR}(X, \mathbb{C})$ under the natural map, we have that

$$\begin{aligned}
\alpha \cup \beta \cup [H]^{d_X-2n+1} &= \int_X \alpha \wedge \beta \wedge \omega^{d_X-2n+1} \\
&= \int_X (\overline{\alpha^{n-1,n}} + \alpha^{n-1,n}) \wedge (\overline{\beta^{n-1,n}} + \beta^{n-1,n}) \wedge \omega^{d_X-2n+1} \\
&= 2\operatorname{Re} \int_X \alpha^{n-1,n} \wedge \overline{\beta^{n-1,n}} \wedge \omega^{d_X-2n+1} \\
&= 2\operatorname{Re}(-i)(-1)^n h(\alpha^{n-1,n}, \beta^{n-1,n}) \\
&= 2(-1)^n \left(\operatorname{Im} h(\alpha^{n-1,n}, \beta^{n-1,n}) \right).
\end{aligned}$$

In other words, for the hermitian form $2h$ on $H^{n-1,n}(X)$, the associated alternating form

$$E = -\operatorname{Im} 2h : H^{n-1,n}(X) \times H^{n-1,n}(X) \to \mathbb{R}$$

when restricted to the image $H^{2n-1}(X, \mathbb{Z})_\tau \subseteq H^{n-1,n}(X)$ is given by $(-1)^{n-1}$ times the cup product in cohomology (via the morphism $H^{2n-1}(X, \mathbb{Z}) \to H^{n-1,n}(X)$). We note the consequence that E evaluated on $H^{2n-1}(X, \mathbb{Z})_\tau \subseteq H^{n-1,n}(X)$ takes integral values. Summarizing, we have the commutative diagram

$$\begin{array}{ccc}
H^{n-1,n}(X) \times H^{n-1,n}(X) & \xrightarrow{\quad E = -\operatorname{Im} 2h \quad} & \mathbb{R} \\
\Big\updownarrow & & \Big\updownarrow \\
H^{2n-1}(X, \mathbb{Z})_\tau \times H^{2n-1}(X, \mathbb{Z})_\tau & \xrightarrow{\quad E = -\operatorname{Im} 2h \quad} & \mathbb{Z} \\
\Big\uparrow & & \| \\
H^{2n-1}(X, \mathbb{Z}) \times H^{2n-1}(X, \mathbb{Z}) & \xrightarrow{(-1)^{n-1}\alpha \cup \beta \cup \omega^{d_X-2n+1}} & \mathbb{Z}
\end{array} \qquad \text{(A.4)}$$

As in [BL04, §2.2], since the hermitian form $2h$ has associated alternating form E taking integral values on the integral lattice, there is an induced line bundle Θ_X on $J^{2n-1}(X)$, such that

$$c_1(\Theta_X) = E = (-1)^{n-1} \int_X (-) \wedge (-) \wedge \omega^{d_X-2n-1}$$

under the identification $H^2(J^{2n-1}(X), \mathbb{Z}) = \bigwedge^2 H^1(J^{2n-1}(X), \mathbb{Z}) = \bigwedge^2 H^{2n-1}(X, \mathbb{Z})^\vee_\tau$. Note that in [BL04, §2.2], they associate to the hermitian form $2h$ the alternating form $\operatorname{Im} 2h$, and therefore their alternating form is the negative of $c_1(\Theta_X)$.

We now translate this discussion using Tensor-Hom adjunction. First, we see that $2h$ induces an isomorphism

$$H^{n-1,n}(X) \xrightarrow[\sim]{\alpha \mapsto 2h(\alpha, -)} \overline{H^{n-1,n}(X)}^\vee$$

Now, given a complex vector space V there is an identification

$$\operatorname{Hom}_{\mathbb{C}}(\overline{V}, \mathbb{C}) = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \qquad \text{(A.5)}$$

$$\ell \mapsto -\operatorname{Im} \ell.$$

The inverse map is given by $\phi \mapsto \phi(i(-)) - i\phi(-)$. Note that in [BL04, §2.2] the opposite identification is made by the assignment $\ell \mapsto \operatorname{Im} \ell$. Using the identification (A.5) above, we have from

(A.4) a commutative diagram

$$\begin{array}{ccc}
H^{n-1,n}(X) & \xrightarrow[\sim]{\alpha \mapsto 2h(\alpha,-)} & \overline{H^{n-1,n}(X)}^{\vee} \\
\| & & \| \\
H^{n-1,n}(X) & \xrightarrow[\sim]{\alpha \mapsto -\operatorname{Im} 2h(\alpha,-)} & \operatorname{Hom}_{\mathbb{R}}(H^{n-1,n}(X),\mathbb{R})
\end{array}$$

As a consequence of the discussion above, the Hodge–Rieman bilinear form induces a commutative diagram

$$\begin{array}{ccc}
H^{n-1,n}(X) & \xrightarrow[\sim]{\alpha \mapsto 2h(\alpha,-)} & \overline{H^{n-1,n}(X)}^{\vee} \\
\big\Uparrow & & \big\Uparrow \\
H^{2n-1}(X,\mathbb{Z})_\tau & \longrightarrow & (H^{2n-1}(X,\mathbb{Z})_\tau)\hat{}
\end{array}$$

where by definition

$$(H^{2n-1}(X,\mathbb{Z})_\tau)\hat{} = \{\varphi \in \overline{H^{n-1,n}(X)}^{\vee} : \varphi(\alpha) \in \mathbb{Z} \text{ for all } \alpha \in H^{2n-1}(X,\mathbb{Z})_\tau\} \tag{A.6}$$

$$= \{\phi \in \operatorname{Hom}_{\mathbb{R}}(H^{n-1,n}(X),\mathbb{R}) : \phi(\alpha) \in \mathbb{Z} \text{ for all } \alpha \in H^{2n-1}(X,\mathbb{Z})_\tau\} \tag{A.7}$$

This by definition induces an isogeny of complex tori

$$\Theta_X : J^{2n-1}(X) \longrightarrow \widehat{J}^{2n-1}(X).$$

As in [BL04, §2.2], one has that $\Theta_X = \varphi_{\Theta_X}$; note that despite our difference in conventions from [BL04, §2.2] regarding alternating forms, the map Θ_X is determined by its induced morphism on complex vector spaces, which is given by the hermitian form $2h$, which is the same in our conventions and those of [BL04, §2.2].

Taking the induced map in homology for Θ_X, and combining with the discussion above, we obtain a commutative diagram

$$\begin{array}{ccc}
H_1(J^{2n-1}(X),\mathbb{Z}) & \xrightarrow{\Theta_X} & H_1(\widehat{J}^{2n-1}(X),\mathbb{Z}) \\
= \big\downarrow & & \big\uparrow = \\
H^{2n-1}(X,\mathbb{Z})_\tau & \xrightarrow{\cup(-1)^{n-1}[H]^{d_X-2n+1}} & H^{2d_X-2n+1}(X,\mathbb{Z})_\tau \, .
\end{array}$$

The left vertical arrow is the canonical identification coming from the construction of the intermediate Jacobian, while the right vertical arrow is the dual identification, where we identify $H_1(\widehat{J}^{2n-1}(X),\mathbb{Z}) = H_1(J^{2n-1}(X),\mathbb{Z})^{\vee}$ via the Weil pairing, and we identify $H^{2n-1}(X,\mathbb{Z})_\tau^{\vee} = H^{2d_X-2n+1}(X,\mathbb{Z})_\tau$ via the cup product. Note that the Weil pairing is the composition of the identifications $H_1(\widehat{J}^{2n-1}(X),\mathbb{Z}) = (H^{2n-1}(X,\mathbb{Z})_\tau)\hat{} = H_1(J^{2n-1}(X),\mathbb{Z})^{\vee}$, where the first is the canonical identification from the definition, and the second comes from the evaluation pairing from the definition (A.7). This second identification uses the convention (A.5); using the opposite convention, *i.e.*, taking the imaginary part of a Hermitian form, rather than its negative, may lead one naturally in the analysis above to include an extra factor of (-1) in the bottom row of the diagram, in which case one would have to use the negative of the Weil pairing to make the diagram commute.

Finally, we note that taking Tate modules, this gives a commutative diagram

$$\begin{array}{ccc}
T_\ell J^{2n-1}(X) & \xrightarrow[c_1(\Theta_X)]{T_\ell \Theta_X} & T_\ell \widehat{J}^{2n-1}(X) \\
= \big\downarrow & & \big\uparrow = \\
H^{2n-1}(X,\mathbb{Z}_\ell)_\tau & \xrightarrow{\cup(-1)^{n-1}[H]^{d_X-2n+1}} & H^{2d_X-2n+1}(X,\mathbb{Z}_\ell)_\tau
\end{array}$$

where in the top row we have identified $T_\ell \Theta_X$ with $c_1(\Theta_X)$ as elements of $H^2(J^{2n-1}(X), \mathbb{Z}_\ell)$ as in §A.3.

A.4.1. *Polarizations.* If we assume that $N_H^m H^{2m-1}(X, \mathbb{Q}) = 0$ for all $1 \le m < n$, i.e., if $H^{m,m-1}(X) = 0$ for all $1 \le m < n$, then $H^{n-1,n}(X)$ is primitive, and the Hodge–Riemann bilinear form is positive definite on $H^{n-1,n}(X)$. In this case, Θ_X is ample, and gives a polarization on $J^{2n-1}(X)$.

Remark A.2 (Weight-1 Hodge structure)*.* The category of polarized abelian varieties is equivalent to the category of polarized weight-1 $\mathbb{Z}$-Hodge structures. In the case where $N_H^m H^{2m-1}(X, \mathbb{Q}) = 0$ for all $1 \le m < n$, the polarized abelian variety $(J^{2n-1}(X), \Theta_X)$ corresponds to the polarized weight-1 $\mathbb{Z}$-Hodge structure $(H^{2n-1}(X, \mathbb{Z})_\tau, Q)$, where Q is the alternating form

$$Q : H^{2n-1}(X, \mathbb{Z})_\tau \times H^{2n-1}(X, \mathbb{Z})_\tau \longrightarrow \mathbb{Z}$$

$$Q(\alpha, \beta) = (-1)^{n-1} \int_X \alpha \wedge \beta \wedge \omega^{d_X - 2n + 1}.$$

To be clear, the Hodge decomposition is given by $(H^{2n-1}(X, \mathbb{Z})_\tau \otimes_{\mathbb{Z}} \mathbb{C})^{1,0} = H^{n,n-1}(X)$. Note that under these identifications, the polarizations satisfy $Q = E = c_1(\Theta_X)$. The associated Hermitian forms also agree, as $i^{0-1}Q(\alpha, \bar\beta) = i^{-1}Q(\alpha, \bar\beta) = i(-1)^n \int_X \alpha \wedge \bar\beta \wedge \omega^{d_X - 2n + 1} = h(\alpha, \beta)$.

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