PROJECTORS ON THE INTERMEDIATE ALGEBRAIC JACOBIANS

CHARLES VIAL

Abstract. Let X be a complex smooth projective variety of dimension d. Under some assumption on the cohomology of X, we construct mutually orthogonal idempotents in $\mathrm{CH}_d(X \times X) \otimes \mathbb{Q}$ whose action on algebraically trivial cycles coincides with the Abel–Jacobi map. Such a construction generalizes Murre's construction of the Albanese and Picard idempotents and makes it possible to give new examples of varieties admitting a self-dual Chow–Künneth decomposition as well as new examples of varieties having a Kimura finite dimensional Chow motive. For instance, we prove that fourfolds with Chow group of zero-cycles supported on a curve (e.g. rationally connected fourfolds) have a self-dual Chow–Künneth decomposition. We also prove that hypersurfaces of very low degree are Kimura finite dimensional.

Introduction

Let X be a smooth projective variety of dimension d over an algebraically closed field $k \subset \mathbb{C}$. The Chow group $\mathrm{CH}_i(X)$ of cycles of dimension i on X is the $\mathbb{Q}$-vector space generated by i-cycles on X modulo rational equivalence. Given $\sim$ an equivalence relation on cycles, $\mathrm{CH}_i(X)_\sim$ denotes those cycles which are ~ 0. In this paper, $\sim$ will either be algebraic, homological or numerical equivalence. All three equivalence relations agree on zero-cycles and are spanned by the zero cycles of degree zero.

Being able to exhibit cycles in $\mathrm{CH}_d(X \times X)$ with appropriate action on the homology of X is essential to Grothendieck's theory of pure motives. As discussed for instance in [11], being able to exhibit cycles in $\mathrm{CH}_d(X \times X)$ which are idempotents is a prerequisite to the understanding of Chow groups as part of the framework of the Bloch–Beilinson–Murre philosophy. Roughly speaking, such a framework predicts that the Chow groups of X should be controlled by the cohomology of X. In this paper we address a question of a slightly different nature as whether the Chow groups of X dictate its Chow motive. Of course, we do not answer such a question in generality. However, we completely answer this question in the case when the Chow groups of X are generated by the Chow groups of zero-cycles of curves. For this purpose we construct appropriate idempotents in $\mathrm{CH}_d(X \times X)$. In the spirit of the BBM philosophy, work of Esnault and Levine [6] (and Jannsen [11] in the case of points) shows that if the Chow groups of X are generated by the Chow groups of curves, then the cohomology of X is generated by the cohomology of curves. Here, as a consequence of the construction of appropriate idempotents, we show that if the Chow groups of X are generated by the Chow groups of curves, then not only is the cohomology of X generated by the cohomology of curves but the Chow motive of X is generated by the Chow motives of curves (see Theorem 4 below). In particular, this complements Esnault and Levine's theorem by showing that the Chow motive of X is finite dimensional in the sense of Kimura [14]. The basic properties of pure motives are explained in [20] and the (covariant) notations we use are those of [13].

Murre [17] constructed mutually orthogonal idempotents Π_1 and Π_{2d-1} in $\mathrm{CH}_d(X \times X)$ called respectively the Albanese projector and the Picard projector. Such idempotents satisfy the following properties.

Date: October 25, 2013.
2010 *Mathematics Subject Classification.* 14C15, 14C25.

- $\Pi_1 = {}^t\Pi_{2d-1}$.
- $(\Pi_1)_*H_*(X) = H_1(X)$ and $(\Pi_{2d-1})_*H_*(X) = H_{2d-1}(X)$.
- $(\Pi_1)_*\mathrm{CH}_*(X) = (\Pi_1)_*\mathrm{CH}_0(X) \cong \mathrm{Alb}_X(k) \otimes \mathbb{Q}$.
- $(\Pi_{2d-1})_*\mathrm{CH}_*(X) = \mathrm{CH}_{d-1}(X)_{\mathrm{hom}} \cong \mathrm{Pic}^0_X(k) \otimes \mathbb{Q}$.

Scholl [20] then showed that it is possible to modify slightly the construction of these idempotents in order to, in addition, have a Lefschetz isomorphism :

- The map $\iota_*\iota^* : (X, \Pi_{2d-1}, 0) \to (X, \Pi_1, d-1)$ is an isomorphism of Chow motives. Here $\iota : C \to X$ is a smooth linear section of dimension one of X.

In this paper, we wish to generalize Murre's construction in the following sense : we wish to construct mutually orthogonal idempotents $\Pi_{2i+1,i}$ in $\mathrm{CH}_d(X \times X)$ which, in homology, define projectors on the largest sub-Hodge structure of $H_{2i+1}(X)$ generated by the H_1's of curves. Here $H_k(X) := H_k(X(\mathbb{C}), \mathbb{Q})$ which is isomorphic to $H^{2d-k}(X(\mathbb{C}), \mathbb{Q})$. We offer two different constructions.

The first construction is explained in the first section. It is defined for all smooth projective varieties X but we cannot show that the idempotents constructed there act appropriately in homology without making some assumptions on X. In some sense the idempotents constructed there lift the largest sub-motive of a curve contained in the numerical motive of X. What is needed is Jannsen's semi-simplicity theorem [10] in order to produce idempotents modulo numerical equivalence, and then a lifting property from numerical equivalence up to rational equivalence (Proposition 1.1).

The second construction, which is much more precise, gives the required idempotents but depends on an assumption on the cohomology of X which we describe below. Let us define $N^iH_{2i}(X)$ to be the image of the rational cycle class map $cl_i : \mathrm{CH}_i(X) \to H_{2i}(X)$ and

$$N^iH_{2i+1}(X) := \sum \mathrm{im}\left(\Gamma_* : H_1(C) \to H_{2i+1}(X)\right),$$

where the sum runs through all smooth projective curves C and through all correspondences $\Gamma \in \mathrm{CH}_{i+1}(C \times X)$. The use of the notation $N^iH_{2i+1}(X)$ is not arbitrary since it can be shown that this sub-group of $H_{2i+1}(X)$ is spanned by those classes that vanish in the open complement of some sub-variety of X of dimension $i+1$. The group $N^iH_{2i+1}(X)$ is thus the last step of the coniveau filtration on $H_{2i+1}(X)$.

Given an integer i, the assumption we need on X in order to construct the idempotent that we will denote $\Pi_{i,\lfloor i/2 \rfloor}$ is that the cup product pairing $H_{2d-i}(X) \times H_i(X) \to \mathbb{Q}$ restricts to a non degenerate pairing on $N^{\lfloor (2d-i)/2 \rfloor}H_{2d-i}(X) \times N^{\lfloor i/2 \rfloor}H_i(X)$. We begin the second section by showing in Lemma 2.1 that such pairings are non degenerate for a large class of varieties. Lemma 2.1 also shows that these pairings are expected to be non degenerate for all smooth projective varieties if one believes in Grothendieck's standard conjectures.

The construction of the projectors $\Pi_{2i,i}$ is unsurprising and is usually used to extract the Néron–Severi group $NS_i(X)$ out of $\mathrm{CH}_i(X)$:

Theorem 1. *Let i be an integer. Assume that the pairing $N^{d-i}H_{2d-2i}(X) \times N^iH_{2i}(X) \to \mathbb{Q}$ is non degenerate. Then there exist idempotents $\Pi_{2i,i}$ and $\Pi_{2d-2i,d-i}$ in $\mathrm{CH}_d(X \times X)$ such that*

- $\Pi_{2i,i} = {}^t\Pi_{2d-2i,d-i}$.
- $(\Pi_{2i,i})_*H_*(X) = N^iH_{2i}(X)$.
- $\mathrm{CH}_i(X)_{\mathrm{hom}} = \ker\left(\Pi_{2i,i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\right)$.
- *The Chow motive $(X, \Pi_{2i,i}, 0)$ is isomorphic to $(\mathbb{L}^{\otimes i})^{\oplus d_i}$ where $d_i = \dim N^iH_{2i}(X)$.*
- *If $2i \geq d$ there is a Lefschetz isomorphism of Chow motives $(X, \Pi_{2i,i}, 0) \to (X, \Pi_{2d-2i,d-i}, 2i-d)$ given by intersecting $2i-d$ times with a smooth hyperplane section of X.*

We now turn to the construction of the projectors $\Pi_{2i+1,i}$. In particular, our construction gives a motivic interpretation of the Abel–Jacobi map to the algebraic part of the intermediate Jacobians. Write $J_i^a(X)$ for the image of the Abel–Jacobi map $AJ_i : \mathrm{CH}_i(X)^{\mathbb{Z}}_{\mathrm{alg}} \to J_i(X(\mathbb{C}))$, it is an algebraic torus defined over k.

Theorem 2. *Let i be an integer. Assume that the pairing $N^{d-i-1}H_{2d-2i-1}(X) \times N^i H_{2i+1}(X) \to \mathbb{Q}$ is non degenerate. Then there exist idempotents $\Pi_{2i+1,i}$ and $\Pi_{2d-2i-1,d-i-1}$ in $\mathrm{CH}_d(X \times X)$ such that*

- $\Pi_{2i+1,i} = {}^t\Pi_{2d-2i-1,d-i-1}$.
- $(\Pi_{2i+1,i})_* H_*(X) = N^i H_{2i+1}(X)$.
- $\ker\big(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i^a(X)(k) \otimes \mathbb{Q}\big) = \ker\big(\Pi_{2i+1,i} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\big)$.
- *The Chow motive $(X, \Pi_{2i+1,i}, 0)$ is isomorphic to $\mathfrak{h}_1(J_i^a(X))(i)$.*
- *If $2i+1 \geq d$ there is a Lefschetz isomorphism of Chow motives $(X, \Pi_{2i+1,i}, 0) \to (X, \Pi_{2d-2i-1,d-i-1}, 2i+1-d)$ given by intersecting $2i+1-d$ times with a smooth hyperplane section of X.*

These generalize Murre's construction of the Albanese and Picard projectors ($\Pi_{1,0}$ and $\Pi_{2d-1,d-1}$ respectively) because in the cases $i = 0$ or $i = d-1$ we have $N^0H_1(X) = H_1(X)$ and $N^{d-1}H_{2d-1}(X) = H_{2d-1}(X)$. The pairing $N^{d-i-1}H_{2d-2i-1}(X) \times N^i H_{2i+1}(X) \to \mathbb{Q}$ is thus just the cup product pairing between $H_{2d-1}(X)$ and $H_1(X)$ and is always non degenerate.

Finally Lemma 2.1 shows that the above pairings are all non degenerate for curves, surfaces, abelian varieties, complete intersections, uniruled threefolds, rationally connected fourfolds and any smooth hyperplane section, product and finite quotient thereof. For those varieties X for which those idempotents can be constructed for all i we can show, thanks to the Gram–Schmidt process of Lemma 2.12, that it is possible to choose the idempotents of Theorems 1 and 2 to be pairwise orthogonal :

Theorem 3. *If the pairings $N^{\lfloor (2d-i)/2 \rfloor}H_{2d-i}(X) \times N^{\lfloor i/2 \rfloor}H_i(X) \to \mathbb{Q}$ are non degenerate for all i then the idempotents of Theorems 1 and 2 can be chosen to be pairwise orthogonal.*

The second section is then devoted to the proof of these theorems.

In the third section, we compute the Chow motive of those varieties whose Chow groups are all representable. We say that $\mathrm{CH}_i(X)_{\mathrm{alg}}$ is *representable* if there exists a curve C and a correspondence $\Gamma \in \mathrm{CH}_{i+1}(C \times X)$ such that $\mathrm{CH}_i(X)_{\mathrm{alg}} = \Gamma_*\mathrm{CH}_0(C)_{\mathrm{alg}}$. We show that if X is a variety whose Chow groups are all representable then the pairings $N^{\lfloor (2d-i)/2 \rfloor}H_{2d-i}(X) \times N^{\lfloor i/2 \rfloor}H_i(X) \to \mathbb{Q}$ are non degenerate for all i. We then use Theorems 1, 2 and 3 to compute the Chow motive of varieties having representable Chow groups. Informally, Esnault and Levine [6] showed that if the Chow groups of a variety X are all representable then the cohomology of X is generated by the cohomology of curves. Conversely Kimura [14] proved that if the cohomology groups of X are generated by the cohomology of curves and if the Chow motive of X is finite dimensional (See [14] for a definition) then the Chow groups of X are representable. Here we prove a stronger statement:

Theorem 4. *Let $k \subseteq \mathbb{C}$ be an algebraically closed field. Let X be a smooth projective variety of dimension d over k. Write $X_{\mathbb{C}} := X \times_{\mathrm{Spec} k} \mathrm{Spec}\, \mathbb{C}$ and $b_j := \dim H_j(X)$. The following statements are equivalent.*

(1) $\mathfrak{h}(X) = \mathbb{1} \oplus \mathfrak{h}_1(\mathrm{Alb}_X) \oplus \mathbb{L}^{\oplus b_2} \oplus \mathfrak{h}_1(J_1^a(X))(1) \oplus (\mathbb{L}^{\otimes 2})^{\oplus b_4} \oplus \cdots \oplus \mathfrak{h}_1(J_{d-1}^a(X))(d-1) \oplus \mathbb{L}^{\otimes d}$.
(2) The cycle class maps $cl_i : \mathrm{CH}_i(X) \to H_{2i}(X)$ and the rational Deligne cycle class maps $cl_i^{\mathcal{D}} : \mathrm{CH}_i(X_{\mathbb{C}}) \to H_{2i}^{\mathcal{D}}(X, \mathbb{Q}(i))$ are surjective for all i and $\mathfrak{h}(X)$ is finite dimensional.
(3) The rational Deligne cycle class maps $cl_i^{\mathcal{D}} : \mathrm{CH}_i(X_{\mathbb{C}}) \to H_{2i}^{\mathcal{D}}(X, \mathbb{Q}(i))$ are injective for all i.
(4) The Chow groups $\mathrm{CH}_i(X_{\mathbb{C}})_{\mathrm{alg}}$ are representable for all i.

Esnault and Levine [6] proved that if the total Deligne cycle class map of X is injective then it is surjective. Theorem 4 gives thus a better insight to the link between injectivity and surjectivity of cycle class maps.

The proof of the theorem goes as follows. The first statement is certainly the strongest, i.e. it implies the three others. The equivalence $(3 \Leftrightarrow 4)$ is certainly known but we couldn't find a reference for $(4 \Rightarrow 3)$, so we include a proof in §3.2. The proof relies on a generalized decomposition of the diagonal and was essentially written in [6]. The implication $(2 \Rightarrow 4)$ is due to Kimura and appears in [14, Theorem 7.10]. Our main input is then a proof of $(4 \Rightarrow 1)$ which settles the theorem and which we give in §3.3. For sake of completeness we also give a direct proof of $(2 \Rightarrow 1)$ in §3.1 using our projectors $\Pi_{2i,i}$ and $\Pi_{2i+1,i}$.

As an immediate corollary we get the following theorem, which is a generalization of a result by Jannsen [11, Th. 3.6] who proved that if the total cycle class map of X is injective then it is surjective. Theorem 5 was also proved by Kimura [15].

Theorem 5. *Let $k \subseteq \mathbb{C}$ be an algebraically closed field. Let X be a smooth projective variety of dimension d over k. The following statements are equivalent.*

1. $\mathfrak{h}(X) = \bigoplus_{i=0}^{d} (\mathbb{L}^{\otimes i})^{\oplus b_{2i}}$.
2. *The rational cycle class map $cl : \mathrm{CH}_*(X_{\mathbb{C}}) \to H_*(X)$ is surjective and $\mathfrak{h}(X)$ is finite dimensional.*
3. *The rational cycle class maps $cl : \mathrm{CH}_*(X_{\mathbb{C}}) \to H_*(X)$ is injective.*
4. *The Chow groups $\mathrm{CH}_i(X_{\mathbb{C}})$ are finite dimensional $\mathbb{Q}$-vector spaces for all i.*

Again this theorem makes more precise the link between injectivity and surjectivity of cycle class maps.

In the fourth and last section we are interested in using our construction of idempotents to give new examples of varieties for which we can compute explicitly a Chow–Künneth decomposition of the diagonal. Such examples include 3-folds X satisfying $H^2(X, \Omega^1_X) = 0$ (e.g. Calabi–Yau 3-folds), rationally connected 4-folds, and 4-folds admitting a rational map to a curve with rationally connected general fiber.

We are also interested in giving new examples of varieties whose Chow motives are finite dimensional in the sense of Kimura [14]. These will be given by smooth hyperplane sections of hypersurfaces covered by a family of linear projective varieties of dimension $\lfloor n/2 \rfloor$ in $\mathbf{P}^{n+1}$. These were considered by Esnault, Levine and Viehweg [7] and also subsequently by Otwinowska [18] and include hypersurfaces of very small degree, e.g. cubic 5-folds, 5-folds which are the smooth intersection of a cubic and a quadric and 7-folds which are the smooth intersection of two quadrics. Other examples are given by rationally connected threefolds, a case which was treated by Gorchinskiy and Guletskii in [8].

Let us mention that the construction given in the first section is used in [21] to prove a generalization of the implication $(4 \Rightarrow 1)$ in Theorem 4 to the case of Chow motives with representable Chow groups. The proof given there does not involve any cohomology theory, except implicitly through the use of Jannsen's semi-simplicity theorem whose proof requires the existence of a "good" cohomology theory. In [22], we prove Murre's conjectures for the varieties considered in §4. We also refer to [22, §2] for some statements that do not involve the cohomology (or the Chow groups) of X in all degrees.

Acknowledgements. Thanks to Burt Totaro for useful comments. This work was supported by a Nevile Research Fellowship at Magdalene College, Cambridge and an EPSRC Postdoctoral Fellowship under grant EP/H028870/1. I would like to thank both institutions for their support.

1. A first construction

1.1. **The coniveau filtration on numerical motives.** Let k be any field and X a smooth projective variety over k of dimension d. We refer to [13, §7.1] for the definition of pure motives in a covariant setting. Chow motives are pure motives with rational coefficients for rational equivalence and numerical motives are pure motives with rational coefficients for numerical equivalence. The category of Chow motives over k is denoted $\mathcal{M}$ and the category of numerical motives over k is denoted $\bar{\mathcal{M}}$. The reduction modulo numerical equivalence of a cycle $\gamma \in \mathrm{CH}_k(X)$ is denoted $\bar{\gamma}$. A fundamental result of Jannsen [10] states that the category of numerical motives is abelian semi-simple. In particular if $f : N \to M$ is a morphism of numerical motives with $M = (X, p, n)$, Jannsen's theorem gives the existence of a correspondence $\pi \in \mathrm{End}_k(M)$ such that $\mathrm{im}\, f \cong (X, \pi, n)$.

It is thus possible to define a coniveau filtration on numerical motives as in [1, §8] and [13, §7.7]: $N^j M := \sum \big(f : \bar{\mathfrak{h}}(Y)(j) \to M\big)$ where the sum runs through all smooth projective varieties Y and all morphisms $f \in \mathrm{Hom}_k(\bar{\mathfrak{h}}(Y)(j), M)$ and where $\bar{\mathfrak{h}}(Y)(j)$ denotes the numerical motive of Y tensored j times by the Lefschetz motive.

Let us imagine for a moment that Grothendieck's standard conjecture B (cf. [1, 5.2.4.1]) is true. Then [1, 5.4.2.1] each numerical motive M has a weight decomposition that we write $M = \bigoplus_i M_i$. Furthermore, for weight reasons $N^j M_i = \sum \big(f : \bar{\mathfrak{h}}_{i-2j}(Y)(j) \to M_i\big)$. Another consequence of Grothendieck's standard conjecture B is that if $i : Z \to Y$ is a smooth hyperplane section of dimension $i-2j$ of Y then $i_* : \bar{\mathfrak{h}}_{i-2j}(Z) \to \bar{\mathfrak{h}}_{i-2j}(Y)$ is surjective [1, 5.2.5.1]. Therefore we have $N^j M_i = \sum \mathrm{im}\,\big(f : \bar{\mathfrak{h}}_{i-2j}(Y)(j) \to M_i\big)$, where the sum runs through all smooth projective varieties Y with $\dim Y = i - 2j$ and through all morphisms $f \in \mathrm{Hom}_k(\bar{\mathfrak{h}}_{i-2j}(Y)(j), M_i)$.

1.2. **The idempotents $\bar{\pi}_{2j,j}$ and $\bar{\pi}_{2j+1,j}$.** Let us now forget about the standard conjectures. We know that points and curves have a weight decomposition [1, 4.3.2]; it is therefore natural for any integer j and for any numerical motive M to consider the following direct summands of M :

$$M_{2j,j} := \sum \mathrm{im}\,\big(f : \bar{\mathfrak{h}}_0(\mathrm{Spec}\, k)(j) \to M\big) \quad \text{and} \quad M_{2j+1,j} := \sum \mathrm{im}\,\big(f : \bar{\mathfrak{h}}_1(C)(j) \to M\big)$$

where the first sum runs through all morphisms $f \in \mathrm{Hom}_k(\bar{\mathfrak{h}}_0(\mathrm{Spec}\, k), M)$ and the second sum runs through all curves C and through all morphisms $f \in \mathrm{Hom}_k(\bar{\mathfrak{h}}_1(C), M)$. Thus in particular there exist for all integers j correspondences $\pi_{2j,j}$ and $\pi_{2j+1,j}$ in $\mathrm{CH}_d(X \times X)$ such that $M_{2j,j} = (X, \bar{\pi}_{2j,j}, 0)$ and $M_{2j+1,j} = (X, \bar{\pi}_{2j+1,j}, 0)$.

1.3. **A lifting property.** We denote by $\mathcal{M}_0$ (resp. $\mathcal{M}_1$) the full thick sub-category of $\mathcal{M}$ generated by the Chow motives of points (resp. the $\mathfrak{h}_1$'s of smooth projective curves over k). For a motive $P \in \mathcal{M}$, let $\bar{P}$ denote its image in $\bar{\mathcal{M}}$. (This notation is abusive since we previously denoted numerical motives with a bar and it is not known if all numerical motives admit a lift to rational equivalence). Let us also write $\bar{\mathcal{M}}_0$ (resp. $\bar{\mathcal{M}}_1$) for the image of $\mathcal{M}_0$ (resp. $\mathcal{M}_1$) in $\bar{\mathcal{M}}$. The functors $\mathcal{M}_0 \to \bar{\mathcal{M}}_0$ and $\mathcal{M}_1 \to \bar{\mathcal{M}}_1$ are equivalence of categories (see [20, Corollary 3.4]) and as such the categories $\mathcal{M}_0$ and $\mathcal{M}_1$ are abelian semi-simple.

Proposition 1.1. *Let M be an object in $\mathcal{M}_0$ (resp. in $\mathcal{M}_1$). Let N be any motive in $\mathcal{M}$. Then any morphism $f : M \to N$ induces a splitting $N = N_1 \oplus N_2$ with N_1 isomorphic to an object in $\mathcal{M}_0$ (resp. in $\mathcal{M}_1$) and $\bar{N}_1 \cong \mathrm{im}\, \bar{f}$.*

Proof. The morphism $M \to N$ induces a morphism $\bar{M} \to \bar{N}$ and it is known that any morphism in an abelian semi-simple category is a direct sum of a zero morphism and of an isomorphism (cf. [2, A.2.13]). Let us thus write

$$\bar{M} = \bar{M}_1 \oplus \bar{M}_2 \xrightarrow{\bar{f} \oplus 0} \bar{N}_1 \oplus \bar{N}_2 = \bar{N}$$

where $\bar{f}$ is an isomorphism $\bar{M}_1 \xrightarrow{\sim} \bar{N}_1$. The composition $(\bar{f}^{-1} \oplus 0) \circ (\bar{f} \oplus 0) \in \mathrm{End}(\bar{M})$ is therefore equal to the projector $\bar{M} \to \bar{M}_1 \to \bar{M}$. Let then $g : N \to M$ be any lift of $\bar{f}^{-1} \oplus 0 : \bar{N} \to \bar{M}$

and let M_1 be any lift of $\bar{M}_1$. Then $g \circ f \in \mathrm{End}(M)$. But it is a fact that $\mathrm{End}(M) = \mathrm{End}(\bar{M})$. Therefore, $g \circ f$ is a projector on M_1. We now claim that $f \circ g \circ f \circ g$ defines a projector in $\mathrm{End}(N)$ onto an object isomorphic to M_1. Indeed,

$$(f \circ g \circ f \circ g) \circ (f \circ g \circ f \circ g) = f \circ \big((g \circ f) \circ (g \circ f) \circ (g \circ f)\big) \circ g = f \circ (g \circ f) \circ g = f \circ g \circ f \circ g$$

and we have the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "g"] & M \arrow[r, "f"] \arrow[dr] & N \arrow[r, "g"] & M \arrow[r, "f"] \arrow[dr] & N \arrow[r, "g"] & M \arrow[r, "f"] \arrow[dr] & N \arrow[r, "g"] & M \arrow[r, "f"] & N \\
 & & M_1 \arrow[ur] \arrow[rr, "\mathrm{id}"] & & M_1 \arrow[ur] \arrow[rr, "\mathrm{id}"] & & M_1 \arrow[ur] & &
\end{tikzcd}$$

showing that indeed $f \circ g \circ f \circ g$ projects onto M_1 (since it has a retraction). $\square$

1.4. **The idempotents $\pi_{2j,j}$ and $\pi_{2j+1,j}$.** Proposition 1.1 shows that it is actually possible to choose the correspondences $\pi_{2j,j}$ and $\pi_{2j+1,j}$ above to be idempotents in $\mathrm{CH}_d(X \times X)$. In other words, Proposition 1.1 shows that it is possible to define direct summands $(X, \pi_{2j,j}, 0)$ and $(X, \pi_{2j+1,j}, 0)$ of the Chow motive $\mathfrak{h}(X)$ of X whose reduction modulo numerical equivalence are the direct summands $M_{2j,j}$ and $M_{2j+1,j}$ defined above.

We won't be giving the details here but it can be shown that, if Grothendieck's Lefschetz standard conjecture B (see below for a precise statement of this conjecture) is true for all smooth projective varieties, then the idempotents $\pi_{2j,j}$ and $\pi_{2j+1,j}$ constructed here coincide modulo homological equivalence with the idempotents $\Pi_{2j,j}$ and $\Pi_{2j+1,j}$ of §2.

1.5. **A remark about the Künneth projectors.** We would like to explain how it is possible to construct cycles whose numerical classes are the expected Künneth projectors, i.e. whose homological classes are expected to be the projectors $H_*(X) \to H_i(X) \to H_*(X)$. We proceed by induction on $d = \dim X$. If $X = \operatorname{Spec} k$, we define π_0^X to be the cycle $X \times X$ inside $X \times X$. Suppose we have constructed projectors modulo numerical equivalence $\pi_0^Y, \pi_1^Y, \ldots, \pi_{2(\dim Y)-2}^Y$ for all smooth projective varieties Y of dimension $\dim Y < d$. Then, for all $i \in \{0, \ldots, d-1\}$, we define the cycle $\pi_i^X \in \mathrm{CH}_d(X \times X)/\mathrm{num}$ to be the projector such that

$$(X, \pi_i^X, 0) = \bigcup_{f : Y \to X} \mathrm{im}\left(f_* : (Y, \pi_i^Y, 0) \to \bar{\mathfrak{h}}(X)\right),$$

where the sum runs through all smooth projective varieties Y of dimension i and all morphisms $f : Y \to X$. We then set $\pi_{2d-i}^X = {}^t\pi_i^X$ and $\pi_d^X = \mathrm{id}_X - \sum_{i \neq d} \pi_i^X$.

If Grothendieck's standard conjecture B is true, then it can be checked that those define the expected Künneth projectors.

2. The projectors $\Pi_{2i,i}$ and $\Pi_{2i+1,i}$

In this section, we fix an algebraically closed field k with an embedding $k \hookrightarrow \mathbb{C}$ and we prove Theorems 1, 2 and 3. We start with a lemma which shows that many varieties do satisfy the assumptions of these theorems.

Let X be a d-dimensional smooth projective variety over k. Let $\iota : H \to X$ be a smooth hyperplane section of X and let $\Gamma_\iota \in \mathrm{CH}_{d-1}(H \times X)$ be its graph and let ${}^t\Gamma_\iota$ be the transpose of Γ_ι. We define $L := \Gamma_\iota \circ {}^t\Gamma_\iota \in \mathrm{CH}_{d-1}(X \times X)$. The hard Lefschetz theorem asserts that the map $L^i : H_{d+i}(X) \to H_{d-i}(X)$ given by intersecting i times with H is an isomorphism for all $i \geq 0$. The variety X is said to satisfy property B if the inverse morphism is induced by an algebraic correspondence for all $i \geq 0$. It is one of Grothendieck's standard conjectures that all smooth projective varieties should satisfy B.

Lemma 2.1. *Let i be an integer in $\{d, \ldots, 2d\}$. The cup product pairing $N^{\lfloor (2d-i)/2 \rfloor} H_{2d-i}(X) \times N^{\lfloor i/2 \rfloor} H_i(X) \to \mathbb{Q}$ is non degenerate in either of the following cases:*

- *X satisfies property B.*
- *$N^{\lfloor i/2 \rfloor} H_i(X) = H_i(X)$.*

In particular the pairing $N^{d-1}H_{2d-1}(X) \times N^0 H_1(X) \to \mathbb{Q}$ is non degenerate for all X.

Proof. In the case X satisfies property B, the Hodge index theorem is a crucial ingredient and the lemma is a special case of [22, Prop. 1.4]. The other case is obvious. $\square$

Therefore the results in this section hold for curves, surfaces, abelian varieties, complete intersections, uniruled threefolds, rationally connected fourfolds and any smooth hypersurface section, product or finite quotient thereof. All those varieties satisfy property B. Lemma 3.1 shows that the results in this section also hold for varieties for which some cycle class map is surjective.

2.1. **Setup.** We are given a smooth projective variety X over k of dimension d. By definition $N^i H_{2i}(X)$ coincides with the image of the cycle class map $\mathrm{CH}_{2i}(X) \to H_{2i}(X)$. For each integer i, let $d_i = \dim_{\mathbb{Q}} N^i H_{2i}(X)$ and let P_i be the disjoint union of d_i copies of Spec k. Notice that $d_i > 0$ because $N^i H_{2i}(X)$ always contains the $(d-i)$-fold intersection of a hyperplane section. We then fix $\Gamma_{2i} \in \mathrm{CH}_i(P_i \times X)$ such that

$$(\Gamma_{2i})_* : H_0(P_i) \xrightarrow{\simeq} N^i H_{2i}(X)$$

is bijective. This amounts to fixing a basis of $N^i H_{2i}(X) = \mathrm{im}\,(cl_i : \mathrm{CH}_i(X) \to H_{2i}(X))$.

For each integer i, we also fix a smooth projective curve (not necessarily connected) C_i and a correspondence $\Gamma_{2i+1} \in \mathrm{CH}_{i+1}(C_i \times X)$ such that

$$(\Gamma_{2i+1})_* H_1(C_i) = N^i H_{2i+1}(X).$$

Let $C_{i,l}$ be the connected components of C_i and for all l let $z_{i,l}$ be a rational point on $C_{i,l}$. Up to composing Γ_{2i+1} with the correspondence $\Delta_{C_i} - \sum_l \left(\{z_{i,l}\} \times C_{i,l} + C_{i,l} \times \{z_{i,l}\}\right) \in \mathrm{CH}_1(C_i \times C_i)$, we can and we will assume moreover that

$$(\Gamma_{2i+1})_* H_0(C_i) = (\Gamma_{2i+1})_* H_2(C_i) = 0.$$

In order to establish the Lefschetz isomorphism of Theorems 1, 2 and 3 we will make use of the following easy lemma.

Lemma 2.2. *Let i be an integer in $\{d+1, \ldots, 2d\}$ and assume that the cup product pairing $N^{\lfloor (2d-i)/2 \rfloor} H_{2d-i}(X) \times N^{\lfloor i/2 \rfloor} H_i(X) \to \mathbb{Q}$ is non degenerate. Then $L^{i-d} : H_i(X) \to H_{2d-i}(X)$ maps isomorphically $N^{\lfloor i/2 \rfloor} H_i(X)$ to $N^{\lfloor (2d-i)/2 \rfloor} H_{2d-i}(X)$.*

Proof. The non degeneracy assumption says in particular that the two $\mathbb{Q}$-vector spaces $N^{\lfloor i/2 \rfloor} H_i(X)$ and $N^{\lfloor (2d-i)/2 \rfloor} H_{2d-i}(X)$ have same dimension. The Lefschetz isomorphism L restricts to an injective map $N^{\lfloor i/2 \rfloor} H_i(X) \to H_{2d-i}(X)$ and, by definition of N, maps $N^{\lfloor i/2 \rfloor} H_i(X)$ into $N^{\lfloor (2d-i)/2 \rfloor} H_{2d-i}(X)$. $\square$

Remark 2.3. In fact if the pairing $N^{\lfloor (2d-i)/2 \rfloor} H_{2d-i}(X) \times N^{\lfloor i/2 \rfloor} H_i(X) \to \mathbb{Q}$ is non degenerate, then more is true. Namely, as a consequence of the Lefschetz isomorphisms of Propositions 2.4 and 2.8, we have that the isomorphism $L^{i-d} : N^{\lfloor i/2 \rfloor} H_i(X) \to N^{\lfloor (2d-i)/2 \rfloor} H_{2d-i}(X)$ has its inverse induced by a correspondence.

If for an integer i such that $2i \in \{d, \ldots, 2d\}$, the cup product pairing $N^{d-i} H_{2d-2i}(X) \times N^i H_{2i}(X) \to \mathbb{Q}$ is non degenerate, Lemma 2.2 makes it possible to furthermore assume that $P_i = P_{d-i}$ and $\Gamma_{2d-2i} = L^{2i-d} \circ \Gamma_{2i}$.

Likewise if for an integer i such that $2i+1 \in \{d, \ldots, 2d\}$, the cup product pairing $N^{d-i-1} H_{2d-2i-1}(X) \times N^i H_{2i+1}(X) \to \mathbb{Q}$ is non degenerate, Lemma 2.2 makes it possible to furthermore assume that $C_i = C_{d-i-1}$ and $\Gamma_{2d-2i-1} = L^{2i+1-d} \circ \Gamma_{2i+1}$.

2.2. **Proof of Theorem 1.** In this paragraph we consider an integer i with $2i \geq d$ and a smooth projective variety X of dimension d for which the pairing $N^{d-i}H_{2d-2i}(X) \times N^i H_{2i}(X) \to \mathbb{Q}$ is non degenerate.

The correspondence $\Gamma_{2d-2i} := L^{2i-d} \circ \Gamma_{2i}$ induces by duality a bijective map

$$({}^t\Gamma_{2d-2i})_* : \big(N^{d-i}H_{2d-2i}(X)\big)^\vee \xrightarrow{\simeq} H_0(P_i)^\vee.$$

By the non degeneracy assumption $\big(N^{d-i}H_{2d-2i}(X)\big)^\vee$ identifies with $N^iH_{2i}(X)$ and $H_0(P_i)^\vee$ identifies with $H_0(P_i)$. Therefore the composition ${}^t\Gamma_{2d-2i} \circ \Gamma_{2i}$ induces a $\mathbb{Q}$-linear isomorphism $H_0(P_i) \xrightarrow{\simeq} H_0(P_i)$. It is then clear that there exists a correspondence $\gamma_{2i} \in \mathrm{CH}_0(P_i \times P_i)$ such that $\gamma_{2i} \circ {}^t\Gamma_{2d-2i} \circ \Gamma_{2i}$ acts as identity on $H_0(P_i)$. Because $L = {}^tL$ we also have

$$\gamma_{2i} = {}^t\gamma_{2i}.$$

We then set

$$\Pi_{2i,i} := \Gamma_{2i} \circ \gamma_{2i} \circ {}^t\Gamma_{2d-2i} \in \mathrm{CH}_d(X \times X).$$

Since $\gamma_{2i} \circ {}^t\Gamma_{2d-2i} \circ \Gamma_{2i} = \mathrm{id} \in \mathrm{CH}_0(P_i \times P_i)$, it is clear that $\Pi_{2i,i}$ is an idempotent and that it induces the projector $H_*(X) \to N^iH_{2i}(X) \to H_*(X)$ in homology. Also, it is clear that $\Pi_{2d-2i,d-i} := {}^t\Pi_{2i,i}$ (Notice that if $2i = d$ then $\Pi_{d,d/2} = {}^t\Pi_{d,d/2}$) defines an idempotent which induces the projector $H_*(X) \to N^{d-i}H_{2d-2i}(X) \to H_*(X)$ in homology. □

Proposition 2.4. *The correspondence $\Pi_{2d-2i,d-i} \circ L^{2i-d} \circ \Pi_{2i,i} : (X, \Pi_{2i,i}, 0) \to (X, \Pi_{2d-2i,d-i}, 2i-d)$ is an isomorphism of Chow motives.*

Proof. Using the identities $L = {}^tL$, $\gamma_{2i} = {}^t\gamma_{2i}$, $\Pi_{2d-2i,d-i} = {}^t\Pi_{2i,i}$ and the fact that $\Pi_{2i,i} = \Gamma_{2i} \circ \gamma_{2i} \circ {}^t\Gamma_{2i} \circ {}^tL^{2i-d}$ is an idempotent, one can easily check that $\Pi_{2i,i} \circ \Gamma_{2i} \circ \gamma_{2i} \circ {}^t\Gamma_{2i} \circ \Pi_{2d-2i,d-i}$ is the inverse of $\Pi_{2d-2i,d-i} \circ L^{2i-d} \circ \Pi_{2i,i}$. □

Proposition 2.5. *Let $\Pi_{2i} \in \mathrm{CH}_d(X \times X)$ be an idempotent which factors through a zero-dimensional variety P_i as $\Pi_{2i} = \Gamma \circ \alpha$ with $\Gamma \in \mathrm{CH}_i(P_i \times X)$ and $\alpha \in \mathrm{CH}_{d-i}(X \times P_i)$, and whose action on $H_*(X)$ is the orthogonal projection on $N^iH_{2i}(X)$. Then the Chow motive $(X, \Pi_{2i}, 0)$ is isomorphic to $(\mathbb{L}^{\otimes i})^{\oplus d_i}$.*

Proof. The cycle class map $\mathrm{CH}_0(P_i) \to H_0(P_i)$ is an isomorphism. Let $\pi := \alpha \circ \Gamma \in \mathrm{CH}_0(P_i \times P_i)$. By functoriality of the cycle class map, we see that π is an idempotent such that $(P_i, \pi, 0) = \mathbb{1}^{\oplus d_i}$. The correspondence Γ is an element of $\mathrm{CH}_i(P_i \times X) = \mathrm{Hom}_k\big(\mathfrak{h}(P_i)(i), \mathfrak{h}(X)\big)$ and it can easily be checked that the correspondence $\Pi_{2i} \circ \Gamma \circ \pi \in \mathrm{Hom}_k\big((P_i, \pi, i), (X, \Pi_{2i}, 0)\big)$ is an isomorphism with inverse $\pi \circ \alpha \circ \Pi_{2i}$. □

Proposition 2.6. *Let Q_{2i} be a correspondence in $\mathrm{CH}_d(X \times X)$ such that Q_{2i} acts as the identity on $N^iH_{2i}(X)$ and such that Q_{2i} is supported on $X \times Z$ with Z a sub-variety of X of dimension i. Then $\mathrm{CH}_i(X)_{\mathrm{hom}} = \ker\big(Q_{2i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\big)$. In particular*

$$\mathrm{CH}_i(X)_{\mathrm{hom}} = \ker\big(\Pi_{2i,i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\big).$$

Proof. By functoriality of the cycle class map, we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}_i(X) & \longrightarrow & \mathrm{CH}^0(Z) & \longrightarrow & \mathrm{CH}_i(X) \\ \downarrow{\scriptstyle cl_i} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle cl_i} \\ H_{2i}(X) & \longrightarrow & H^0(Z) & \longrightarrow & H_{2i}(X) \end{array}$$

The composition of the two arrows of the top row is the map induced by Q_{2i} and the composition of the two arrows of the bottom row is the identity on $\mathrm{im}\,(cl_i)$. The proposition follows easily. □

2.3. **Proof of Theorem 2.**

2.3.1. *The Albanese and the Picard varieties.* Let X be a smooth projective variety over a field k. The Albanese variety attached to X and denoted Alb_X is an abelian variety universal for maps $X \to A$ from X to abelian varieties A sending a fixed point $x_0 \in X$ to $0 \in A$. The Picard variety Pic^0_X of X is the abelian variety parametrizing numerically trivial line bundles on X (i.e. those with vanishing Chern class). These define respectively a covariant and a contravariant functor from the category of smooth projective varieties to the category of abelian varieties.

The abelian varieties Alb_X and Pic^0_X are dual and are isogenous in the following way. Let C be a curve which is a smooth linear section of X. Then the map

$$\Psi : \mathrm{Pic}^0_X \to \mathrm{Pic}^0_C \xrightarrow{\Theta} \mathrm{Alb}_C \to \mathrm{Alb}_X$$

is an isogeny, where Θ is the map induced by the theta-divisor on the curve C.

The following proposition is essential to the construction of the idempotents $\Pi_{2i+1,i}$.

Proposition 2.7 (*cf.* Th. 3.9 and Prop. 3.10 of [20])**.** *Let Y and Z be connected smooth projective varieties and let $\zeta \in \mathrm{CH}_0(Y)$ and $\eta \in \mathrm{CH}_0(Z)$ be 0-cycles of positive degree. Then there is an isomorphism*

$$\Omega : \mathrm{Hom}(\mathrm{Alb}_Y, \mathrm{Pic}^0_Z) \otimes \mathbb{Q} \xrightarrow{\simeq} \{c \in \mathrm{CH}^1(Y \times Z), \quad c(\zeta) = {}^t c(\eta) = 0\}.$$

Moreover, Ω is functorial in the following sense. Let $\phi : Y' \to Y$ and $\psi : Z' \to Z$ be morphisms of varieties and let ζ' and η' be positive 0-cycles on Y' and Z' with direct image ζ and η on Y and Z. If $\beta : \mathrm{Alb}_Y \to \mathrm{Pic}^0_Z$ is a homomorphism, then

$$\Omega(\mathrm{Pic}_\psi \circ \beta) = \psi^* \circ \Omega(\beta) \quad \text{and} \quad \Omega(\beta \circ \mathrm{Alb}_\phi) = \Omega(\beta) \circ \phi_*,$$

where Ω is taken with respect to the chosen 0-cycles.

2.3.2. *Intermediate Jacobians.* Given a smooth projective complex variety X, the i^{th} intermediate Jacobian attached to X is the compact complex torus

$$J_i(X) = \frac{H_{2i+1}(X, \mathbb{C})}{F^i H_{2i+1}(X, \mathbb{C}) + H_{2i+1}(X, \mathbb{Z})}.$$

It comes with a map

$$AJ_i : \mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{hom}} \to J_i(X)$$

defined on the integral Chow group called the i^{th} Abel–Jacobi map which was thoroughly studied by Griffiths [9]. In the cases $i = 0$ and $i = \dim X - 1$, we recover the notions of Albanese variety and Picard variety respectively. These intermediate Jacobians are however fairly different since they are of transcendental nature. While the Albanese and the Picard variety are algebraic tori, this is not the case in general for intermediate Jacobians. Precisely, let J_i^{alg} denote the maximal sub-torus inside $J_i(X)$ whose tangent space is included in $H_{i+1,i}(X, \mathbb{C})$. It is then a fact that J_i^{alg} is an abelian variety and that

$$J_i^{\mathrm{alg}}(X) = \frac{N_H^i H_{2i+1}(X, \mathbb{C})}{N_H^i H_{2i+1}(X, \mathbb{C}) \cap \big(F^i H_{2i+1}(X, \mathbb{C}) + H_{2i+1}(X, \mathbb{Z})\big)}$$

where $N_H^i H_{2i+1}(X)$ is the maximal sub-Hodge structure of $H_{2i+1}(X)$ contained in $H_{i+1,i}(X, \mathbb{C}) \oplus H_{i,i+1}(X, \mathbb{C})$. In particular, the intermediate Jacobian is algebraic if and only if $H_{2i+1}(X, \mathbb{C})$ is concentrated in degrees $(i, i+1)$ and $(i+1, i)$. As a consequence of the horizontality of normal functions associated to algebraic cycles [9], the cycles in $\mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{hom}}$ that are algebraically trivial map into $J_i^{\mathrm{alg}}(X)$ under the Abel–Jacobi map. The map $\mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{alg}} \to J_i^{\mathrm{alg}}(X)$ is surjective if $N^i H_{2i+1}(X) \supseteq N_H^i H_{2i+1}(X)$ (the reverse inclusion always holds), in particular if $N^i H_{2i+1}(X) = H_{2i+1}(X)$. In any case, let us write $J_i^a(X)$ for the image of the map $AJ_i : \mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{alg}} \to J_i^{\mathrm{alg}}(X)$.

It is an abelian sub-variety of the abelian variety $J_i^{\mathrm{alg}}(X)$ which is defined the same way as $J_i^{\mathrm{alg}}(X)$ with N_H replaced with N. We sum this up in the commutative diagram

$$\begin{tikzcd} \mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{hom}} \arrow[rr, "AJ_i"] & & J_i(X) \\ \mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{alg}} \arrow[u, hook] \arrow[r, two heads, "AJ_i"] & J_i^a(X) \arrow[r, hook] & J_i^{\mathrm{alg}}(X). \arrow[u, hook] \end{tikzcd}$$

Finally if X is defined over an algebraically closed sub-field k of $\mathbb{C}$, the image of the composite map

$$\mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{alg}} \to \mathrm{CH}_i^{\mathbb{Z}}(X_{\mathbb{C}})_{\mathrm{alg}} \to J_i^{\mathrm{alg}}(X_{\mathbb{C}})$$

defines an abelian variety over k that we denote $J_i^a(X)$.

2.3.3. *The projectors* $\Pi_{2i+1,i}$ *and* $\Pi_{2d-2i-1,d-i-1}$. Given any abelian varieties A and B, $\mathrm{Hom}(A,B)$ denotes the group of homomorphisms from A to B. Recall that the category of abelian varieties up to isogeny is the category whose objects are the abelian varieties and whose morphisms are given by $\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}$ for any abelian varieties A and B. This category is abelian semi-simple, cf. [23].

In the rest of this paragraph we consider an integer i with $2i+1 \geq d$ and a smooth projective variety X of dimension d for which the pairing $N^{d-i-1}H_{2d-2i-1}(X) \times N^iH_{2i+1}(X) \to \mathbb{Q}$ is non degenerate. In particular the dual of $J^a_{d-i-1}(X)$ identifies with $J_i^a(X)$. Lemma 2.2 implies that the correspondence L^{2i+1-d} induces an isogeny $\Lambda : J_i^a(X) \to J^a_{d-i-1}(X)$ and because $L = {}^tL$ we have $\Lambda = \Lambda^\vee$, i.e. Λ is equal to its dual.

Taking up what was said in §2.1 we have a smooth projective curve C_i over k and correspondences $\Gamma_{2i+1} \in \mathrm{CH}_{i+1}(C_i \times X)$ and $\Gamma_{2d-2i-1} := L^{2i+1-d}\circ\Gamma_{2i+1} \in \mathrm{CH}_{d-i}(C_i\times X)$ such that both maps

$$(\Gamma_{2i+1})_* : H_1(C_i) \to N^iH_{2i+1}(X) \quad \text{and} \quad (\Gamma_{2d-2i-1})_* : H_1(C_i) \to N^{d-i-1}H_{2d-2i-1}(X)$$

are surjective and such that both maps act trivially on $H_0(C_i)$ and on $H_2(C_i)$. The correspondence Γ_{2i+1} induces by functoriality of the Abel–Jacobi map a surjective homomorphism

$$(\Gamma_{2i+1})_* : \mathrm{Alb}_{C_i} \twoheadrightarrow J_i^a(X)$$

as well as a homomorphism with finite kernel

$$({}^t\Gamma_{2i+1})_* \circ \Lambda : J_i^a(X) \hookrightarrow \mathrm{Pic}^0_{C_i}.$$

By semisimplicity of the category of abelian varieties up to isogeny, there exists $\alpha \in \mathrm{Hom}(J_i^a(X), \mathrm{Alb}_{C_i})\otimes\mathbb{Q}$ such that $(\Gamma_{2i+1})_*\circ\alpha = \mathrm{id}_{J_i^a(X)}$. Let us consider

$$\Phi := \alpha\circ\Lambda^{-1}\circ\alpha^\vee \in \mathrm{Hom}(\mathrm{Pic}^0_{C_i}, \mathrm{Alb}_{C_i})\otimes\mathbb{Q}$$

so that

$$(\star) \qquad (\Gamma_{2i+1})_*\circ\Phi\circ({}^t\Gamma_{2i+1})_*\circ\Lambda = \mathrm{id}_{J_i^a(X)}.$$

We would now like to use Proposition 2.7 in order to give an algebraic origin to Φ. Decomposing C_i into the disjoint union of its connected components $C_{i,l}$, Proposition 2.7 gives a functorial isomorphism between $\mathrm{Hom}(\mathrm{Alb}_{C_i}, \mathrm{Pic}^0_{C_i})\otimes\mathbb{Q}$ and $\{c \in \mathrm{CH}^1(C_i\times C_i) \ / \ c(z_{i,l}) = {}^tc(z_{i,l}) = 0\}$ for $z_{i,l}$ the rational point on $C_{i,l}$ considered in §2.1. Here, Φ belongs to $\mathrm{Hom}(\mathrm{Pic}^0_{C_i}, \mathrm{Alb}_{C_i})\otimes\mathbb{Q}$ which is not quite the Hom group in the statement of Proposition 2.7. To correct this, let Θ denote the theta-divisor of the curve C_i. Then under the isomorphism of Proposition 2.7, Φ corresponds to a correspondence $\gamma_{2i+1} := \Theta\circ\Gamma'\circ\Theta^{-1} \in \mathrm{CH}^1(C_i\times C_i)$ satisfying $(\gamma_{2i+1})_*z_{i,l} = ({}^t\gamma_{2i+1})_*z_{i,l} = 0$ for all l. Because $\Phi = \Phi^\vee$ we have

$$\gamma_{2i+1} = {}^t\gamma_{2i+1}.$$

We now set

$$\Pi_{2i+1,i} := \Gamma_{2i+1} \circ \gamma_{2i+1} \circ {}^t\Gamma_{2d-2i-1} = \Gamma_{2i+1} \circ \gamma_{2i+1} \circ {}^t\Gamma_{2i+1} \circ L^{2i+1-d} \in \mathrm{CH}_d(X \times X).$$

By $(\star)$, the action of $\Pi_{2i+1,i}$ on $J_i^a(X)$ is given by $\mathrm{id}_{J_i^a(X)}$. The fact that $\Pi_{2i+1,i}$ defines a projector goes as follows. It is enough to prove that

$$\gamma_{2i+1} \circ {}^t\Gamma_{2i+1} \circ \Lambda \circ \Gamma_{2i+1} \circ \gamma_{2i+1} = \gamma_{2i+1}.$$

Thanks to Proposition 2.7, it is actually enough to prove $\Phi \circ ({}^t\Gamma_{2i+1})_* \circ L_*^{2i+1-d} \circ (\Gamma_{2i+1})_* \circ \Phi = \Phi : \mathrm{Pic}^0_{C_i} \to \mathrm{Alb}_{C_i}$. This last statement follows directly from $(\star)$.

Now because $\Pi_{2i+1,i}$ acts as the identity on $J_i^a(X)$ and because Γ_{2i+1} and $\Gamma_{2d-2i-1}$ act trivially on homology classes of degree $\neq 1$, we see that the homology class of $\Pi_{2i+1,i}$ is the projector $H_*(X) \to N^iH_{2i+1}(X) \to H_*(X)$.

Finally we set $\Pi_{2d-2i-1,d-i-1} := {}^t\Pi_{2i+1,i}$, which is licit in the case $2d-2i-1 = d$ since in this case $\gamma_{2i+1} = {}^t\gamma_{2i+1}$ implies $\Pi_{2i+1,i} = {}^t\Pi_{2i+1,i}$. It is then straightforward that $\Pi_{2d-2i-1,d-i-1}$ defines and idempotent that induces the projector $H_*(X) \to N^{d-i-1}H_{2d-2i-1}(X) \to H_*(X)$. $\square$

Proposition 2.8. *The correspondence* $\Pi_{2d-2i-1,d-i-1} \circ L^{2i+1-d} \circ \Pi_{2i+1,i} : (X, \Pi_{2i+1,i}, 0) \to (X, \Pi_{2d-2i-1,d-i-1}, 2i+1-d)$ *is an isomorphism of Chow motives.*

Proof. Using the identities $L = {}^tL$, $\gamma_{2i+1} = {}^t\gamma_{2i+1}$, $\Pi_{2d-2i-1,d-i-1} = {}^t\Pi_{2i+1,i}$ and the fact that $\Pi_{2i+1,i} = \Gamma_{2i+1} \circ \gamma_{2i+1} \circ {}^t\Gamma_{2i+1} \circ {}^tL^{2i+1-d}$ is an idempotent, one can easily check that $\Pi_{2i+1,i} \circ \Gamma_{2i+1} \circ \gamma_{2i+1} \circ {}^t\Gamma_{2i+1} \circ \Pi_{2d-2i-1,d-i-1}$ is the inverse of $\Pi_{2d-2i-1,d-i-1} \circ L^{2i+1-d} \circ \Pi_{2i+1,i}$. $\square$

Proposition 2.9. *Let $\Pi_{2i+1} \in \mathrm{CH}_d(X \times X)$ be an idempotent which factors through a curve C_i as $\Pi_{2i+1} = \Gamma \circ \alpha$ with $\Gamma \in \mathrm{CH}_{i+1}(C_i \times X)$ and $\alpha \in \mathrm{CH}_{d-i}(X \times C_i)$, and whose action on $H_*(X)$ is the orthogonal projection on $N^iH_{2i+1}(X)$. Then the Chow motive $(X, \Pi_{2i+1,i}, 0)$ is isomorphic to $\mathfrak{h}_1(J_i^a(X))(i)$.*

Proof. The assumption on the homology class of Π_{2i+1} implies that Π_{2i+1} acts as the identity on $J_i^a(X)$ and acts as zero on $H_{2i}(X)$. Therefore, by functoriality of the cycle class map, $\alpha \circ \Gamma \in \mathrm{CH}^1(C_i \times C_i)$ acts as zero on some positive degree zero-cycle ζ on C_i. Now a consequence of Proposition 2.7 is that given two abelian varieties J and J' over k, there is a canonical identification

$$\mathrm{Hom}(J, J') \otimes \mathbb{Q} = \mathrm{Hom}_k\big(\mathfrak{h}_1(J), \mathfrak{h}_1(J')\big).$$

Because Π_{2i+1} acts as the identity on $J_i^a(X)$, $\alpha \circ \Gamma$ defines an idempotent $\pi \in \mathrm{End}(\mathfrak{h}_1(C_i))$ such that $(C_i, \pi, 0) \cong \mathfrak{h}_1(J_i^a(X))$.
The correspondence Γ seen as a morphism of motives belongs to $\mathrm{Hom}_k\big(\mathfrak{h}(C_i)(i), \mathfrak{h}(X)\big)$. Let us show that

$$\Pi_{2i+1,i} \circ \Gamma \circ \pi \in \mathrm{Hom}_k\big((C_i, \pi, i), (X, \Pi_{2i+1,i}, 0)\big)$$

is an isomorphism. In fact, let us show that its inverse is given by $\pi \circ \alpha \circ \Pi_{2i+1,i}$, i.e that

$$(\Pi \circ \Gamma \circ \pi) \circ (\pi \circ \alpha \circ \Pi) = \Pi \ \text{ and } \ (\pi \circ \alpha \circ \Pi) \circ (\Pi \circ \Gamma \circ \pi) = \pi \text{ as correspondences,}$$

where for convenience we have dropped the subscripts "$2i+1$". But then this is obvious because $\Pi = \Gamma \circ \alpha$ and $\pi = \alpha \circ \Gamma$ are idempotents. $\square$

Proposition 2.10. *Let Q_{2i+1} be a correspondence in $\mathrm{CH}_d(X \times X)$ such that Q_{2i+1} acts as the identity on $N^iH_{2i+1}(X)$ and such that Q_{2i+1} is supported on $X \times Z$ with Z a sub-variety of X of dimension $i+1$. Then* $\ker\big(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbb{Q}\big) = \ker\big(Q_{2i+1} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\big)$. *In particular*

$$\ker\big(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbb{Q}\big) = \ker\big(\Pi_{2i+1,1} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\big).$$

Proof. The assumptions on Q_{2i+1} imply that the action of Q_{2i+1} on $\mathrm{CH}_i(X)$ factors through $\mathrm{CH}^1(\widetilde{Z})$ for some desingularization $\widetilde{Z} \to Z$; they also imply that the induced action of Q_{2i+1} on $J_i^a(X)$ is the identity.

We have thus the commutative diagram

$$\begin{tikzcd} \mathrm{CH}_i(X)_{\mathrm{alg}} \arrow[r, "A"] \arrow[d, "AJ_i"] & \mathrm{CH}^1(\widetilde{Z})_{\mathrm{alg}} \arrow[r, "B"] \arrow[d, "\simeq"] & \mathrm{CH}_i(X)_{\mathrm{alg}} \arrow[d, "AJ_i"] \\ J_i^a(X) \otimes \mathbb{Q} \arrow[r] & \mathrm{Pic}^0_{\widetilde{Z}} \otimes \mathbb{Q} \arrow[r] & J_i^a(X) \otimes \mathbb{Q} \end{tikzcd}$$

where A and B are correspondences such that $B \circ A = Q_{2i+1}$. The inclusion $\ker AJ_i \subseteq \ker \Pi_{2i+1,i}$ follows from the commutativity of the diagram, which itself is a consequence of the functoriality of the Abel–Jacobi map with respect to the action of correspondences. The reverse inclusion $\ker \Pi_{2i+1,i} \subseteq \ker AJ_i$ follows from the fact that the composite of the two lower horizontal arrows is the identity on $\mathrm{im}\, AJ_i = J_i^a(X) \otimes \mathbb{Q}$. □

Remark 2.11. An interesting question is to decide whether or not the action of an idempotent on homology determines its action on Chow groups. For example, given idempotents $\pi_{2i,i}$ and $\pi_{2i+1,i} \in \mathrm{CH}_d(X \times X)$ such that $(\pi_{2i,i})_* H_*(X) = N^i H_{2i}(X)$ and $(\pi_{2i+1,i})_* H_*(X) = N^i H_{2i+1}(X)$, do we have

$$\mathrm{CH}_i(X)_{\mathrm{hom}} = \ker\big(\pi_{2i,i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\big)$$

$$\text{and } \ker\big(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i^a(X) \otimes \mathbb{Q}\big) = \ker\big(\pi_{2i+1,i} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\big)?$$

It is shown in [22] that this is the case if X is finite dimensional in the sense of Kimura.

2.4. **Proof of Theorem 3.** In this section we are given a smooth projective variety X of dimension d for which the pairings are all non degenerate. As such, by Theorems 1 and 2 we can define all the idempotents $\Pi_{2i,i}$ and $\Pi_{2i+1,i}$. However these are not all necessarily pairwise orthogonal. We start with the following linear algebra lemma which makes it possible to modify the idempotents so as to make them pairwise orthogonal.

Lemma 2.12. *Let V be a $\mathbb{Q}$-algebra and let k be a positive integer. Let $\pi_0, \ldots, \pi_n$ be idempotents in V such that $\pi_j \circ \pi_i = 0$ whenever $j - i < k$ and $j \neq i$. Then the endomorphisms*

$$p_i := (1 - \frac{1}{2}\pi_n) \circ \cdots \circ (1 - \frac{1}{2}\pi_{i+1}) \circ \pi_i \circ (1 - \frac{1}{2}\pi_{i-1}) \circ \cdots \circ (1 - \frac{1}{2}\pi_0)$$

define idempotents such that $p_j \circ p_i = 0$ whenever $j - i < k + 1$ and $j \neq i$.

Proof. Let j and i be such that $j - i < k + 1$ and look at

$$\Pi := \pi_j \circ (1 - \frac{1}{2}\pi_{j-1}) \circ \cdots \circ (1 - \frac{1}{2}\pi_0) \circ (1 - \frac{1}{2}\pi_n) \circ \cdots \circ (1 - \frac{1}{2}\pi_{i+1}) \circ \pi_i.$$

Suppose first $j < i$. Because we have $\pi_r \circ \pi_s = 0$ for all $r < s$, we immediately see that $\Pi = 0$. Suppose $j = i$, it is also easy to see that in this case $\Pi = \pi_i$. Finally, suppose that $i < j < i+k+1$. Because $\pi_r \circ \pi_s = 0$ for all $r < s + k$, we can see after expanding Π that $\Pi = \pi_j \circ \pi_i - \frac{1}{2}\pi_j \circ \pi_i \circ \pi_i - \frac{1}{2}\pi_j \circ \pi_j \circ \pi_i = 0$. □

In our case of concern, we get

Theorem 2.13. *Let X be a smooth projective variety of dimension d. Let $i < d$ be an integer and let $\pi_0, \ldots, \pi_i \in \mathrm{CH}_d(X \times X)$ be idempotents such that $(\pi_j)_* H_*(X) = H_j(X)$ for all $0 \leq j \leq i$. Let $\pi_{2d-j} := {}^t\pi_j$ for $0 \leq j \leq i$. If $\pi_r \circ \pi_s = 0$ for all $0 \leq r < s \leq 2d$, then the Gram–Schmidt process of Lemma 2.12 gives mutually orthogonal idempotents $\{p_j\}_{j \in \{0,\ldots,i,2d-i,\ldots 2d\}}$ such* *that* $(p_j)_*H_*(X) = H_j(X)$ *and* $p_{2d-j} := {}^tp_j$ *for all* $j \in \{0, \ldots, i, 2d-i, \ldots 2d\}$. *Moreover, we have isomorphisms of Chow motives* $(X, \pi_j) \cong (X, p_j)$ *for all* j.

Proof. In order to get mutually orthogonal idempotents, it is enough to apply Lemma 2.12 $2i+2$ times. In order to prove the theorem, it suffices to prove each statement after each application of the process of Lemma 2.12. Everything is then clear, except perhaps for the last statement. The isomorphism is simply given by the correspondence $p_j \circ \pi_j$ and its inverse is $\pi_j \circ p_j$. □

Proposition 2.14. *The projectors of Theorems 1 and 2 satisfy*

- $\Pi_{2i,i} \circ \Pi_{2j,j} = 0$ *for* $i \neq j$.
- $\Pi_{2i+1,i} \circ \Pi_{2j+1,j} = 0$ *for* $|i-j| > 1$.
- $\Pi_{2i+1,i} \circ \Pi_{2j,j} = 0$ *for* $|i-j| > 1$.
- $\Pi_{2i,i} \circ \Pi_{2j+1,j} = 0$ *for* $|i-j| > 1$.

Proof. The proposition follows from looking at the dimension of ${}^t\Gamma_{2d-i} \circ \Gamma_j$. □

Proposition 2.15. *The projectors of Theorems 1 and 2 satisfy*

- $\Pi_{2i-1,i-1} \circ \Pi_{2i+1,i} = 0$ *for all* i.
- $\Pi_{2i,i} \circ \Pi_{2i+1,i} = 0$ *and* $\Pi_{2i+1,i} \circ \Pi_{2i+2,i+1} = 0$ *for all* i.

Proof. For the first point we have ${}^t\Gamma_{2d-2i+1} \circ \Gamma_{2i+1} \circ \gamma_{2i+1} \in \mathrm{CH}_2(C_i \times C_{i-1})$ and thus there exist rational numbers $a_{l,l'}$ such that ${}^t\Gamma_{2d-2i+1} \circ \Gamma_{2i+1} \circ \gamma_{2i+1} = \sum_{l,l'} a_{l,l'}[C_{i,l} \times C_{i-1,l'}]$. This yields $({}^t\Gamma_{2d-2i+1} \circ \Gamma_{2i+1} \circ \gamma_{2i+1})_* z_{i,l} = \sum_{l'} a_{l,l'}[C_{i-1,l'}]$. By definition of γ_{2i+1} we also have $(\gamma_{2i+1})_* z_{i,l} = 0$ for all l. Hence $a_{l,l'} = 0$ for all l and all l'. Therefore ${}^t\Gamma_{2d-2i+1} \circ \Gamma_{2i+1} \circ \gamma_{2i+1} = 0$.

For the second point, up to transposing it is enough to prove one of the two equalities. Let us prove the second one. We have ${}^t\Gamma_{2d-2i-1} \circ \Gamma_{2i+2} \in \mathrm{CH}_1(P_{i+1} \times C_i)$. But then, because ${}^t\gamma_{2i+1}$ acts trivially on $z_{i.l} \in \mathrm{CH}_0(C_{i,l})$ for all l, we see that γ_{2i+1} acts trivially on $\mathrm{CH}_1(C_i)$. Therefore $\gamma_{2i+1} \circ {}^t\Gamma_{2d-2i-1} \circ \Gamma_{2i+2} = 0$. □

Remark 2.16. We have shown through the two previous propositions that ${}^t\Gamma_{2d-j} \circ \Gamma_i \circ \gamma_i = 0$ for $j - i < 0$ and in particular that $\Pi_{j,\lfloor j/2 \rfloor} \circ \Pi_{i,\lfloor i/2 \rfloor} = 0$ for $j - i < 0$.

Remark 2.17. The missing orthogonal relations are $\Pi_{2i+1,i} \circ \Pi_{2i,i} = 0$, $\Pi_{2i+2,i+1} \circ \Pi_{2i+1,i} = 0$ or $\Pi_{2i+1,i} \circ \Pi_{2i-1,i-1} = 0$. There is no reason that these should hold true for the idempotents constructed in §§2.2. and 2.3.

Before we proceed to the proof of Theorem 3 we need a lemma.

Lemma 2.18. *If* $i \geq d$ *then* ${}^t\Pi_j \circ L^{i-d} \circ \Pi_{j'} = 0$ *for* $j, j' \geq i$ *except in the case* $i = j = j'$.

Proof. Up to transposing we only have to prove ${}^t\Pi_j \circ L^{i-d} \circ \Pi_{j'} = 0$ for $j' \geq j \geq i$ not all equal. In fact it is enough to prove $\gamma_j \circ {}^t\Gamma_j \circ L^{i-d} \circ \Gamma_{j'} \circ \gamma_{j'} = 0$ for $j' \geq j \geq i$ not all equal. The correspondence $\gamma_j \circ {}^t\Gamma_j \circ L^{i-d} \circ \Gamma_{j'} \circ \gamma_{j'}$ is a cycle of dimension $j + j' - 2i$ in the Chow group of $P_{\lfloor j' \rfloor} \times P_{\lfloor j \rfloor}$, $C_{\lfloor j' \rfloor} \times P_{\lfloor j \rfloor}$, $P_{\lfloor j' \rfloor} \times C_{\lfloor j \rfloor}$ or $C_{\lfloor j' \rfloor} \times C_{\lfloor j \rfloor}$ depending on the parity of j and j'. Notice that $j + j' - 2i \geq 1$, and that $j + j' - 2i = 1$ implies that $j' = i+1$ and $j = i$, and that $j + j' - 2i = 2$ implies that j and j' have same parity (in fact $j = j' = i+1$ or $j' = j+2 = i+2$). The proof that $\gamma_j \circ {}^t\Gamma_j \circ L^{i-d} \circ \Gamma_{j'} \circ \gamma_{j'} = 0$ in each of these cases is then similar to the cases treated in the proof of the previous proposition. □

Proof of Theorem 3. We proceed by induction on $k \geq 0$ to prove property
$\mathcal{P}_k$: There exist idempotents $\Pi_i \in \mathrm{CH}_d(X \times X)$ for $0 \leq i \leq 2d$ such that

- $\Pi_j \circ \Pi_i = 0$ if $j - i < k$ and $j \neq i$.
- Π_{2i} satisfies the properties listed in Theorem 1 for all i.
- Π_{2i+1} satisfies the properties listed in Theorem 2 for all i.
- The Π_i's satisfy the conclusion of Lemma 2.18.

Clearly if property $\mathcal{P}_{2d+1}$ holds, then the idempotents Π_i are mutually orthogonal. (Actually it is enough to settle $\mathcal{P}_3$ by Remark 2.17). If we set $\Pi_i := \Pi_{i,\lfloor i/2 \rfloor}$, we see thanks to Theorems 1 and 2, Remark 2.16 and Lemma 2.18 that property $\mathcal{P}_0$ holds. Let us suppose that property $\mathcal{P}_k$ holds and let us prove that $\mathcal{P}_{k+1}$ holds.

We set

$$P_i := (1-\frac{1}{2}\Pi_{2d}) \circ (1-\frac{1}{2}\Pi_{2d-1}) \circ \cdots \circ (1-\frac{1}{2}\Pi_{i+1}) \circ \Pi_i \circ (1-\frac{1}{2}\Pi_{i-1}) \circ \cdots \circ (1-\frac{1}{2}\Pi_0).$$

By Lemma 2.12, these define idempotents such that $P_j \circ P_i = 0$ if $j - i < k+1$ and $j \neq i$.. It remains to check that P_i enjoys the same properties as Π_i.

It is straightforward from the formula that we have ${}^tP_i = P_{2d-i}$. It is also straightforward that P_i induces the projector $H_*(X) \to N^{\lfloor i/2 \rfloor}H_i(X) \to H_*(X)$ in homology.

Let us now consider an integer $i \geq d$ and prove that the Lefschetz correspondence L^{i-d} induces an isomorphism of Chow motives $(X, P_i, 0) \to (X, P_{2d-i}, i-d)$. In fact, we are going to show that ${}^tP_i \circ L^{i-d} \circ P_i$ admits $P_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^tP_i$ as an inverse, i.e. that

$$(P_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^tP_i) \circ ({}^tP_i \circ L^{i-d} \circ P_i) = P_i$$

and

$$({}^tP_i \circ L^{i-d} \circ P_i) \circ (P_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^tP_i) = {}^tP_i.$$

Because $L = {}^tL$ and $\gamma_i = {}^t\gamma_i$, the second equality is the transpose of the first one. Therefore it is enough to establish the first equality. Thanks to Remark 2.16 we have $\Pi_j \circ \Gamma_i \circ \gamma_i = 0$ for all $j < i$ and by transposing $\gamma_i \circ {}^t\Gamma_i \circ {}^t\Pi_j = 0$ for all $j < i$. Expanding P_i, we therefore see that

$$P_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^tP_i = (1-\frac{1}{2}\Pi_{2d}) \circ \cdots \circ (1-\frac{1}{2}\Pi_{i+1}) \circ \Pi_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^t\Pi_i \circ (1-\frac{1}{2}{}^t\Pi_{i+1}) \circ \cdots \circ (1-\frac{1}{2}{}^t\Pi_{2d}).$$

On the other hand, Lemma 2.18 implies that

$${}^tP_i \circ L^{i-d} \circ P_i = (1-\frac{1}{2}{}^t\Pi_0) \circ \cdots \circ (1-\frac{1}{2}{}^t\Pi_{i+1}) \circ {}^t\Pi_i \circ L^{i-d} \circ \Pi_i \circ (1-\frac{1}{2}\Pi_{i-1}) \circ \cdots \circ (1-\frac{1}{2}\Pi_0).$$

Put altogether, this gives

$$(P_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^tP_i) \circ ({}^tP_i \circ L^{i-d} \circ P_i) =$$

$$(1-\frac{1}{2}\Pi_{2d}) \circ \cdots \circ (1-\frac{1}{2}\Pi_{i+1}) \circ \Pi_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^t\Pi_i \circ L^{i-d} \circ \Pi_i \circ (1-\frac{1}{2}\Pi_{i-1}) \circ \cdots \circ (1-\frac{1}{2}\Pi_0).$$

By Proposition 2.4 if i is even and by Proposition 2.8 if i is odd, we have $\Pi_i \circ \Gamma_i \circ \gamma_i \circ {}^t\Gamma_i \circ {}^t\Pi_i \circ L^{i-d} \circ \Pi_i = \Pi_i$. This finishes the proof of the Lefschetz isomorphism.

Let us now prove that the P_i's satisfy the conclusion of Lemma 2.18. A careful look at the proof of Lemma 2.18 shows that it is enough to show that P_j factors through $\Gamma_j \circ {}^t\gamma_j$ if Π_j does. This can be read immediately from the formula defining Π_j.

If the projectors Π_{2i} factor through a 0-dimensional variety and if the projectors Π_{2i+1} factor through a curve for all i, then it is clear from the formula that so will the projectors P_{2i} and P_{2i+1}. On the one hand, Proposition 2.6 gives $\mathrm{CH}_i(X)_{\mathrm{hom}} = \ker\big(P_{2i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\big)$ and Proposition 2.10 gives $\ker\big(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i^a(X) \otimes \mathbb{Q}\big) = \ker\big(P_{2i+1} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\big)$. On the other hand, Proposition 2.5 shows that $(X, P_{2i}, 0)$ is isomorphic to $(\mathbb{L}^{\otimes i})^{\oplus d_i}$ and Proposition 2.9 shows that $(X, P_{2i+1}, 0)$ is isomorphic to $\mathfrak{h}_1(J_i^a(X))(i)$. Alternately, the conclusion of Theorem 2.13 gives these isomorphisms of Chow motives. □

3. Representability of Chow groups and finite dimensional motives

Given a smooth projective complex variety X of dimension d, its i^{th} Deligne cohomology group $H_i^{\mathcal{D}}(X,\mathbb{Z}(p))$ is the $(2d-i)^{\text{th}}$ hypercohomology group of the complex $\mathbb{Z}_{\mathcal{D}}(d-p)$ given by $0 \to \mathbb{Z} \xrightarrow{\cdot(2i\pi)^{d-p}} \mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^{d-p-1} \to 0$. In other words,

$$H_i^{\mathcal{D}}(X,\mathbb{Z}(p)) = \mathbb{H}^{2d-i}(X,\mathbb{Z}_{\mathcal{D}}(d-p)).$$

Deligne cohomology comes with a cycle map $cl_i^{\mathcal{D}} : \mathrm{CH}_i^{\mathbb{Z}}(X) \to H_{2i}^{\mathcal{D}}(X,\mathbb{Z}(i))$ defined on the integral Chow group $\mathrm{CH}_i^{\mathbb{Z}}(X)$ which is functorial with respect to the action of correspondences and fits into an exact sequence

$$0 \to J_i(X) \to H_{2i}^{\mathcal{D}}(X,\mathbb{Z}(i)) \to \mathrm{Hdg}_{2i}(X) \to 0$$

where $\mathrm{Hdg}_{2i}(X)$ denotes the Hodge classes in $H_{2i}(X,\mathbb{Z})$ and $J_i(X)$ is Griffiths' i^{th} intermediate Jacobian. As proved in [5, Prop. 1], the following diagram with exact rows commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{hom}} \arrow[r] \arrow[d, "AJ_i"] & \mathrm{CH}_i^{\mathbb{Z}}(X) \arrow[r] \arrow[d, "cl_i^{\mathcal{D}}"] & \mathrm{CH}_i^{\mathbb{Z}}(X)/\mathrm{hom} \arrow[r] \arrow[d, "cl_i"] & 0 \\ 0 \arrow[r] & J_i(X) \arrow[r] & H_{2i}^{\mathcal{D}}(X) \arrow[r] & \mathrm{Hdg}_{2i}(X) \arrow[r] & 0. \end{tikzcd} \tag{1}$$

The homomorphism $cl_i : \mathrm{CH}_i^{\mathbb{Z}}(X)/\mathrm{hom} \to \mathrm{Hdg}_{2i}(X)$ is always injective by definition of homological equivalence. In particular the functoriality of the Deligne cycle class map implies the functoriality of the Abel–Jacobi map with respect to the action of correspondences.

Lemma 3.1. *Let i be an integer such that $d \leq 2i \leq 2d$.*

- *If the map $cl : \mathrm{CH}_i(X) \to H_{2i}(X)$ is surjective then $H_{2i}(X) = N^i H_{2i}(X)$ and $H_{2d-2i}(X) = N^{d-i}H_{2d-2i}(X)$.*
- *If the map $cl^{\mathcal{D}} : \mathrm{CH}_i(X) \to H_{2i}^{\mathcal{D}}(X)$ is surjective then $H_{2i+1}(X) = N^i H_{2i+1}(X)$ and $H_{2d-2i-1}(X) = N^{d-i-1}H_{2d-2i-1}(X)$.*

Proof. If the map $cl : \mathrm{CH}_i(X) \to H_{2i}(X)$ is surjective then by definition $H_{2i}(X) = N^i H_{2i}(X)$. Because the Lefschetz isomorphism $L^{2i-d} : H_{2i}(X) \to H_{2d-2i}(X)$ is induced by a correspondence we also see that $H_{2d-2i}(X) = N^{d-i}H_{2d-2i}(X)$.

Now suppose that the map $cl^{\mathcal{D}} : \mathrm{CH}_i(X) \to H_{2i}^{\mathcal{D}}(X)$ is surjective. A simple diagram chase in diagram 1 shows that the Abel–Jacobi map $AJ_i : \mathrm{CH}_i(X)_{\mathrm{hom}} \to J_i(X) \otimes \mathbb{Q}$ is then surjective. The Griffiths group $\mathrm{Griff}_i(X)$ being countable, this is possible only if $J_i(X) \otimes \mathbb{Q} = J_i^{\mathrm{alg}}(X) \otimes \mathbb{Q}$. Therefore we have $J_i(X) \otimes \mathbb{Q} = J_i^a(X) \otimes \mathbb{Q}$ and hence $H_{2i+1}(X) = N^i H_{2i+1}(X)$. Again because the Lefschetz isomorphism $L^{2i-d+1} : H_{2i+1}(X) \to H_{2d-2i-1}(X)$ is induced by a correspondence we also see that $H_{2d-2i-1}(X) = N^{d-i-1}H_{2d-2i-1}(X)$. □

3.1. **From finite dimensionality to representability : proof of $2 \Rightarrow 1$ in Theorem 4.** First we need a standard lemma.

Lemma 3.2. *Let N be a finite dimensional Chow motive. If its homology groups $H_*(N)$ vanish then $N = 0$.*

Proof. The homology class of $\mathrm{id}_N \in \mathrm{End}_k(N)$ is then 0. Kimura [14, Prop. 7.2] proved that if a Chow motive N is finite dimensional then the ideal of correspondences in $\mathrm{End}_k(N)$ which are homologically trivial is a nilpotent ideal. Hence id_N is nilpotent i.e. $\mathrm{id}_N = 0$. □

Proof of 2 $\Rightarrow$ 1. Lemma 3.1 shows that $H_i(X) = N^{\lfloor i/2 \rfloor}H_i(X)$ for all i. Therefore by Lemma 2.1 the pairings $N^{\lfloor i/2 \rfloor}H_i(X) \times N^{\lfloor (2d-i)/2 \rfloor}H_{2d-i}(X) \to \mathbb{Q}$ are all non degenerate. Theorems 1, 2 and 3 then show that $A := \mathbb{1} \oplus \mathfrak{h}_1(\mathrm{Alb}_X) \oplus \mathbb{L}^{\oplus b_2} \oplus \mathfrak{h}_1(J_1^a(X))(1) \oplus (\mathbb{L}^{\otimes 2})^{\oplus b_4} \oplus \cdots \oplus \mathfrak{h}_1(J_{d-1}^a(X))(d-1) \oplus \mathbb{L}^{\otimes d}$ is a direct summand of the Chow motive $\mathfrak{h}(X)$ and that $H_*(A) = H_*(X)$. Let us write $\mathfrak{h}(X) = A \oplus N$. The property of being finite dimensional is stable by direct summand. Therefore N is a finite dimensional motive. Moreover $H_*(N) = 0$. Lemma 3.2 shows that $N = 0$. □

3.2. **Representability vs. injectivity of the Abel–Jacobi maps : proof of** 3 $\Leftrightarrow$ 4 **in Theorem 4.** The results in this section are seemingly well-known. Given a smooth projective complex variety X, we prove that the following statements are equivalent :

(1) $\mathrm{CH}_i(X)_{\mathrm{alg}}$ is representable for all i.
(2) The total Abel–Jacobi map $\bigoplus_i \mathrm{CH}_i(X)_{\mathrm{hom}} \to \bigoplus_i J_i(X) \otimes \mathbb{Q}$ is injective.
(3) The total Deligne cycle class map $cl_{\mathcal{D}} : \bigoplus_i \mathrm{CH}_i(X)_{\mathrm{hom}} \to \bigoplus_i H_{2i}^{\mathcal{D}}(X, \mathbb{Q}(i))$ is injective.
(4) The total Deligne cycle class map is bijective and $H_i(X) = N^{\lfloor i/2 \rfloor}H_i(X)$ for all i.

The equivalence (2 $\Leftrightarrow$ 3) follows immediately from diagram 1. The implication (4 $\Rightarrow$ 3) is obvious and the implication (3 $\Rightarrow$ 4) is due to Esnault and Levine [6] (Theorem 3.3 below together with Lemma 3.1). The main argument is a generalized decomposition of the diagonal as performed by Laterveer [16] and Paranjape [19] among others after Bloch's and Srinivas' original paper [4]. Proposition 3.4 proves the standard implication (2 $\Rightarrow$ 1). We couldn't find any reference for the implication (1 $\Rightarrow$ 2) so we include a proof of it, see Corollary 3.6. The proof goes through a generalized decomposition of the diagonal as done in [6, Theorem 1.2] with some minor changes (Theorem 3.5).

Theorem 3.3 (Esnault–Levine)**.** *Let s be an integer. Assume that the rational Deligne cycle class maps $cl_i^{\mathcal{D}} : \mathrm{CH}_i(X) \to H_{2i}^{\mathcal{D}}(X)$ are injective for all $i \leq s$. Then these are all surjective. Moreover the rational cycle class maps $cl_i : \mathrm{CH}_i(X) \to H_{2i}(X)$ are also surjective for all $i \leq s$.*

Proof. The fact that the rational Deligne cycle class maps are surjective for all $i \leq s$ is contained in Theorem 2.5 of [6] (the maps $cl_i^{\mathcal{D}}$ are denoted $cl_{0,0}^{2d-i}$ in [6]). The claim about the rational cycle class maps being surjective is Corollary 2.6 (which states that $N^iH_{2i}(X) = Hdg_{2i}(X) \otimes \mathbb{Q}$) together with Theorem 3.2 (which states in particular that $H_{2i}(X) = Hdg_{2i}(X) \otimes \mathbb{Q}$) of [6]. □

Proposition 3.4. *Given i, if the Abel–Jacobi map $\mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbb{Q}$ is injective, then $\mathrm{CH}_i(X)_{\mathrm{alg}}$ is representable.*

Proof. Let $J_i^a(X)$ be the image of the Abel–Jacobi map $AJ_i : \mathrm{CH}_i^{\mathbb{Z}}(X)_{\mathrm{alg}} \to J_i(X)$. By definition of algebraic equivalence, $\mathrm{CH}_i(X)_{\mathrm{alg}} := \sum \mathrm{im}\left(\Gamma_* : \mathrm{CH}_0(C)_{\mathrm{hom}} \to \mathrm{CH}_i(X)\right)$ where the sum runs through all smooth projective curves C and all correspondences $\Gamma \in \mathrm{CH}_{i+1}(C \times X)$. Therefore, by functoriality of the Abel–Jacobi map, we have $J_i^a(X) = \sum \mathrm{im}\left(\Gamma_* : J(C) \to J_i^a(X)\right)$. By finiteness properties of abelian varieties there exist a curve and a correspondence $\Gamma \in \mathrm{CH}_{i+1}(C \times X)$ such that $J_i^a(X) = \Gamma_* J(C)$. Therefore for this particular curve $\Gamma_*\mathrm{CH}_0(C)_{\mathrm{hom}} = \mathrm{CH}_i(X)_{\mathrm{alg}}$. □

Theorem 3.5. *Let s be an integer with $0 \leq s \leq d$ and let X be a d-dimensional smooth projective complex variety. Assume $\mathrm{CH}_i(X)_{\mathrm{alg}}$ is representable for all $i \leq s$. Then there is a decomposition*

$$\Delta_X = \gamma_0 + \gamma_1 + \cdots + \gamma_s + \gamma^{s+1} \in \mathrm{CH}_d(X \times X)$$

such that γ_i is supported on $D^i \times \Gamma_{i+1}$ for some sub-schemes D^i and Γ_{i+1} of X satisfying $\dim D^i = d - i$ and $\dim \Gamma_{i+1} = i + 1$ and γ^{s+1} is supported on $D^{s+1} \times X$ for some sub-scheme D^{s+1} of X satisfying $\dim D^{s+1} = d - s - 1$.

Proof. The proof is the same as the proof of [6, Lemma 1.1] once one remarks that the map $ch : \mathrm{CH}_0(\tilde{D}) \to \mathrm{CH}^n(X)$ on page 207 has image contained in $\mathrm{CH}^n(X)_{\mathrm{alg}}$ and therefore factors through the Albanese map $\mathrm{CH}_0(\tilde{D}) \to \mathrm{Alb}_{\tilde{D}}$, because $\mathrm{CH}^n(X)_{\mathrm{alg}}$ is representable and has thus the structure of an abelian variety. $\square$

Corollary 3.6. *Assume* $\mathrm{CH}_i(X)_{\mathrm{alg}}$ *is representable for all* $i \leq s$. *Then the Abel–Jacobi maps* $AJ_i : \mathrm{CH}_i(X)_{\mathrm{hom}} \to J_i(X) \otimes \mathbb{Q}$ *are injective for all* $i \leq s$.

Proof. By assumption made on $\mathrm{CH}_*(X)_{\mathrm{alg}}$, the diagonal Δ_X admits a decomposition as in Theorem 3.5. For all $i \leq s$, let $\widetilde{\Gamma}_{i+1}$ be a desingularization of Γ_{i+1}. The action of the correspondence γ_i on $\mathrm{CH}_j(X)_{\mathrm{hom}}$ then factors through $\mathrm{CH}_j(\widetilde{\Gamma}_{i+1})_{\mathrm{hom}}$. For dimension reasons γ_i acts possibly non trivially only on $\mathrm{CH}_i(X)$ and $\mathrm{CH}_{i+1}(X)$. Also for dimension reasons, the correspondence γ^{s+1} acts trivially on $\mathrm{CH}_i(X)$ for $i \leq s$. Therefore the cycle γ_i acts non trivially on $\mathrm{CH}_j(X)_{\mathrm{hom}}$ only if $i = j$. Thus the action of γ_i on $\mathrm{CH}_i(X)_{\mathrm{hom}}$ is identity. Finally, by functoriality of the algebraic Abel–Jacobi map, we have the following commutative diagram for all $i \leq s$

$$\begin{tikzcd}
\mathrm{CH}_i(X)_{\mathrm{hom}} \arrow[r] \arrow[d, "AJ_i"] & \mathrm{CH}_i(\widetilde{\Gamma}_{i+1})_{\mathrm{hom}} \arrow[r] \arrow[d, "\simeq"] & \mathrm{CH}_i(X)_{\mathrm{hom}} \arrow[d, "AJ_i"] \\
J_i(X) \arrow[r] & J_i(\widetilde{\Gamma}_{i+1}) \arrow[r] & J_i(X).
\end{tikzcd}$$

The composition of the two maps on each row is induced by γ_i and is equal to identity up to torsion. A diagram chase then shows that $AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i^a(X) \otimes \mathbb{Q}$ is injective. $\square$

Remark 3.7. Given i, I cannot prove that if $\mathrm{CH}_i(X)_{\mathrm{alg}}$ is representable then the Abel–Jacobi map $AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbb{Q}$ restricted to algebraically trivial cycles is injective.

Remark 3.8. Bloch and Srinivas proved [4, Theorem 1(i)] that if $\mathrm{CH}_0(X)_{\mathrm{alg}}$ is representable then so is $\mathrm{CH}^2(X)_{\mathrm{alg}}$. A generalized decomposition of the diagonal shows that if $\mathrm{CH}_0(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_s(X)_{\mathrm{alg}}$ are representable then $\mathrm{CH}^2(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}^{2+s}(X)_{\mathrm{alg}}$ are also representable. Therefore, if d is the dimension of X, it is enough to know that $\mathrm{CH}_0(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{\lfloor d/2 \rfloor - 1}(X)_{\mathrm{alg}}$ are representable in order to deduce that $\mathrm{CH}_*(X)_{\mathrm{alg}}$ is representable.

3.3. **From representability to finite dimensionality : proof of $4 \Rightarrow 1$ in Theorem 4.** In order to prove the implication $4 \Rightarrow 1$ of Theorem 4, we again use our projectors $\Pi_{2i,i}$ and $\Pi_{2i+1,i}$, together with the following lemma which appears in [8, Lemma 1].

Lemma 3.9. *Let N be a Chow motive over a field k and let Ω be a universal domain over k, i.e. an algebraically closed field of infinite transcendence degree over k. If* $\mathrm{CH}_*(N_\Omega) = 0$, *then* $N = 0$. $\square$

Proof of $4 \Rightarrow 1$. If $\mathrm{CH}_*(X_{\mathbb{C}})_{\mathrm{alg}}$ is representable then Corollary 3.6 shows that the Deligne cycle class maps $cl_i^{\mathcal{D}}$ are all injective. By Esnault and Levine's Theorem 3.3, the Deligne cycle class maps $cl_i^{\mathcal{D}}$ and the cycle class maps cl_i are surjective for all i. Now Lemma 3.1 shows that $H_{2i}(X) = N^i H_{2i}(X)$ and $H_{2i+1}(X) = N^i H_{2i+1}(X)$ for all i. Thanks to Lemma 2.1 we can therefore apply Theorems 1, 2 and 3 to cut out the motive $\mathbb{1} \oplus \mathfrak{h}_1(\mathrm{Alb}_X) \oplus \mathbb{L}^{\oplus b_2} \oplus \mathfrak{h}_1(J_1^a(X))(1) \oplus (\mathbb{L}^{\otimes 2})^{\oplus b_4} \oplus \cdots \oplus \mathfrak{h}_1(J_{d-1}^a(X))(d-1) \oplus \mathbb{L}^{\otimes d}$ from $\mathfrak{h}(X)$. These two motives have same rational Chow groups when the base field is extended to $\mathbb{C}$, Lemma 3.9 implies they are equal. $\square$

As a corollary, we obtain a result proved independently by Kimura [15] (Kimura's result works more generally for any pure Chow motive over $\mathbb{C}$).

Proposition 3.10. *Let X be a d-dimensional smooth projective variety over k. If $\mathrm{CH}_*(X_{\mathbb{C}})$ is a finite dimensional $\mathbb{Q}$-vector space, then*

$$\mathfrak{h}(X) \cong \bigoplus_{i=0}^{d} (\mathbb{L}^{\otimes i})^{\oplus b_{2i}}.$$

Moreover, the cycle class maps $cl_i : \mathrm{CH}_i(X) \to H_{2i}(X)$ are all isomorphisms

Proof. Indeed, if $\mathrm{CH}_*(X_{\mathbb{C}})$ is a finite dimensional $\mathbb{Q}$-vector space then it is representable. Apply Theorem 4 to see that $\mathfrak{h}(X)$ is a direct sum of Lefschetz motives and twisted $\mathfrak{h}_1$'s of abelian varieties. Now for a complex abelian variety J, $\mathrm{CH}_0(\mathfrak{h}_1(J)) = J \otimes \mathbb{Q}$ which is an infinite dimensional $\mathbb{Q}$-vector space if $J \neq 0$. Therefore $\mathfrak{h}(X)$ is a direct sum of Lefschetz motives only. □

4. Chow–Künneth decompositions

A smooth projective variety X of dimension d is said to have a Chow–Künneth decomposition (CK decomposition for short) if there exist mutually orthogonal idempotents $\Pi_0, \Pi_1, \ldots \Pi_{2d} \in \mathrm{CH}_d(X \times X)$ adding to the identity Δ_X such that $(\Pi_i)_* H_*(X) = H_i(X)$ for all i. In this section, we wish to give explicit examples of varieties having a Chow–Künneth decomposition. For this purpose we use the projectors of Theorems 1 and 2. Along the way we are able to establish Grothendieck's standard conjectures and Kimura's finite dimensionality conjecture in some new cases. In [22], we prove Murre's conjectures for all the varieties considered in §§4.2 and 4.3 (and some others as well).

4.1. **The basic theorem.** Here, X denotes a smooth projective variety of dimension d. All the varieties for which we will be able to show that they admit a CK decomposition will actually also be endowed with a Chow–Lefschetz decomposition in the following sense.

Definition 4.1. The variety X is said to have a *Chow–Lefschetz decomposition* if it admits a CK decomposition $\{\Pi_i\}_{0 \leq i \leq 2d}$ such that, for all $i > d$ the morphism of Chow motives $(X, \Pi_i, 0) \to (X, \Pi_{2d-i}, i-d)$ given by intersecting $i-d$ times with a hyperplane section is an isomorphism.

It is immediate to see that if X has a Chow–Lefschetz decomposition then it satisfies the Lefschetz standard conjecture. Since in characteristic zero Grothendieck's standard conjectures for X reduce to the standard Lefschetz conjecture for X [1, 5.4.2.2], we get that X satisfies all of Grothendieck's standard conjectures.

The key result of this section is the following.

Theorem 4.2. *If $H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$ for all $i > d$, then X has a Chow–Lefschetz decomposition $\{P_i\}_{0 \leq i \leq 2d}$ where the idempotents P_i for $i \neq d$ satisfy all the properties listed in Theorems 1 and 2.*

Proof. If $H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$ for $i > d$, then by intersecting with a linear section of dimension $i-d$ we find that $H_{2d-i}(X) = N^{\lfloor \frac{2d-i}{2} \rfloor} H_{2d-i}(X)$ for $i > d$. Thus by Poincaré duality the pairings $N^{\lfloor \frac{i}{2} \rfloor} H_i(X) \times N^{\lfloor \frac{2d-i}{2} \rfloor} H_{2d-i}(X) \to \mathbb{Q}$ are non degenerate for all $i > d$. The motivic Lefschetz isomorphisms of Theorems 1 and 2 then imply that X satisfies the Lefschetz standard conjecture. Therefore, by Lemma 2.1, the pairing $N^{\lfloor d/2 \rfloor} H_d(X) \times N^{\lfloor d/2 \rfloor} H_d(X) \to \mathbb{Q}$ is also non degenerate. By Theorem 3, we get mutually orthogonal idempotents $\{\Pi_i\}_{0 \leq i \leq 2d}$ that satisfy the motivic Lefschetz isomorphisms. Let us set $P_i := \Pi_i$ for $i \neq d$ and $P_d := \Delta_X - \sum_{i \neq d} \Pi_i$. Then $\{P_i\}_{0 \leq i \leq 2d}$ is the required Chow–Lefschetz decomposition for X. □

4.2. **Some examples of varieties having a Chow–Lefschetz decomposition.** An immediate consequence to Theorem 4.2 is the following.

Corollary 4.3. *Let Y be a 3-fold with $H^2(Y, O_Y) = 0$, e.g. a Calabi–Yau 3-fold that is a 3-fold with trivial canonical bundle and vanishing first Betti number. Then Y has a Chow–Lefschetz decomposition.*

Proof. By the Lefschetz $(1,1)$-theorem, $H^2(Y, O_Y) = 0$ implies $H_4(Y) = N^2 H_4(Y)$. Thus $H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$ for all $i > 3$. □

In Theorems 4.5 and 4.8 below, in addition to proving that some X has a Chow–Lefschetz decomposition, we give some information on the support of the middle CK projector of X. Such information will be used in [22] to further prove Murre's conjectures in the cases covered by the theorems. For this purpose we need the following lemma stated in [21, Lemma 1.2] and which is due to Kahn and Sujatha [12].

Lemma 4.4. *Let $M = (X, p, 0)$ be a Chow motive over k and let Ω be a universal domain over k. If $\mathrm{CH}_0(N_\Omega) = 0$, then there exists a Chow motive $N = (Y, q, 0)$ such that $M = (Y, q, 1)$.* □

Theorem 4.5. *Let X be a smooth projective variety of even dimension $d = 2n$. If $\mathrm{CH}_0(X)_{\mathrm{alg}}, \mathrm{CH}_1(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{n-2}(X)_{\mathrm{alg}}$ are representable, then X has a Chow–Lefschetz decomposition $\{\Pi_i\}$. Moreover, the idempotents Π_i are as in Theorems 1 and 2 for $i \neq d$ and the idempotent Π_d has a representative supported on $X \times Z$ with Z a sub-variety of X of dimension $n+1$.*

Proof. By a generalized decomposition of the diagonal (as performed for instance by Laterveer [16, 2.1]), the assumption on the Chow groups of X implies that $H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$ for all $i > d$. We can therefore apply Theorem 4.2 to get a CK decomposition $\{\Pi_i\}_{0 \leq i \leq 2d}$ for X where the idempotents Π_i for $i \neq d$ satisfy all the properties listed in Theorems 1 and 2. By Corollary 3.6 if $\mathrm{CH}_0(X)_{\mathrm{alg}}, \mathrm{CH}_1(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{n-2}(X)_{\mathrm{alg}}$ are representable, then the Abel–Jacobi maps $AJ_i : \mathrm{CH}_i(X)_{\mathrm{hom}} \to J_i(X) \otimes \mathbb{Q}$ are injective for all $i \leq n-2$. Esnault and Levine's Theorem 3.3 then implies that the Abel–Jacobi maps are bijective. Thanks to the properties of the CK projectors, we thus get $\mathrm{CH}_i(X) = (\Pi_{2i} + \Pi_{2i+1})_* \mathrm{CH}_i(X)$ for all $i \leq n-2$. As such, the idempotent Π_d acts trivially on $\mathrm{CH}_i(X)$ for all $i \leq n-2$. By applying $n-1$ times Lemma 4.4, we get that $(X, \Pi_d, 0)$ is isomorphic to some Chow motive $(Y, q, n-1)$. This means that there exists a correspondence $f \in \mathrm{Hom}((X, \Pi_d, 0), (Y, q, n-1))$ such that $\Pi_d = \Pi_d \circ f^{-1} \circ q \circ f \circ \Pi_d$. In particular Π_d factors through Y and a straightforward analysis of the dimensions shows that Π_d has a representative supported on $X \times Z$ with Z a sub-variety of X of dimension $n+1$. □

Corollary 4.6. *Every fourfold X with $\mathrm{CH}_0(X)_{\mathrm{alg}}$ representable has a Chow–Lefschetz decomposition. In particular, if X is a smooth projective fourfold which is either rationally connected or admits a curve C as a base for its maximal rationally connected fibration (i.e. if there exists a rational map $f : X \dashrightarrow C$ with rationally connected general fiber), then X has a Chow–Lefschetz decomposition.* □

Remark 4.7. Arapura [3] proved the Lefschetz standard conjecture for unirational fourfolds. He does so by proving that a unirational fourfold is *motivated* by surfaces. More generally, Arapura proves that any variety which is motivated by a surface (this means that the cohomology of X is generated by the cohomology of product of surfaces via correspondences) satisfies the standard Lefschetz conjecture. Corollary 4.6 is more precise for unirational fourfolds because we obtain the Lefschetz isomorphism modulo rational equivalence (rather than just modulo homological equivalence). Moreover, Corollary 4.6 includes the case of rationally connected fourfolds as well as the case of fourfolds admitting a curve as a base for their maximal rationally connected fibration.

Let us also mention that in what follows Arapura's technique doesn't seem to apply to prove the Lefschetz standard conjecture because the middle cohomology of the varieties in question is not necessarily generated by the cohomology of products of surfaces.

Theorem 4.8. *Let X be a smooth projective variety of odd dimension $d = 2n+1$ with $H^{n+1}(X, \Omega_X^{n-1}) = 0$. If $\mathrm{CH}_0(X)_{\mathrm{alg}}, \mathrm{CH}_1(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{n-2}(X)_{\mathrm{alg}}$ are representable, then X has a Chow–Lefschetz decomposition. Moreover, the idempotents Π_i are as in Theorems 1 and 2 for $i \neq d$ and the idempotent Π_d has a representative supported on $X \times Z$ with Z a sub-variety of X of dimension $n+2$.*

Proof. As for the proof of Theorem 4.5, a generalized decomposition of the diagonal argument shows that the assumption on the Chow groups of X implies that $H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$ for $i > d+1$ and that $H_{d+1}(X) = N^{\frac{d-1}{2}} H_{d+1}(X)$. This last equality means that there is a smooth projective variety S of dimension $n+2$ and a map $f : S \to X$ such that $f_* H^2(S) = H_{d+1}(X)$. Because $H^{n+1}(X, \Omega_X^{n-1}) = 0$, we see that $H_{d+1}(X)$ is made of Hodge classes. By the Lefschetz $(1,1)$-theorem applied to S, we see that $H_{d+1}(X)$ is spanned by algebraic cycles, i.e. that $H_{d+1}(X) = N^{\frac{d+1}{2}} H_{d+1}(X)$. We can thus apply Theorem 4.2 to get a Chow–Lefschetz decomposition $\{\Pi_i\}_{0 \leq i \leq 2d}$. The proof of the fact that Π_d has a representative supported on $X \times Z$ for Z a sub-variety of X of dimension $n+2$ goes along the same lines as the proof of Theorem 4.5. □

Corollary 4.9. *Let X be a smooth projective fivefold. If $\mathrm{CH}_0(X)_{\mathrm{alg}}$ is representable and if $H^3(X, \Omega_X^1) = 0$, then X has a Chow–Lefschetz decomposition. In particular, if X is a smooth projective rationally connected fivefold with $H^3(X, \Omega_X^1) = 0$, then X has a Chow–Lefschetz decomposition.* □

4.3. **Hypersurfaces of very small degree are Kimura finite dimensional.** Otwinowska [18] proved that if X is a smooth hyperplane section of a hypersurface in $\mathbf{P}^{n+1}$ covered by l-planes then $\mathrm{CH}_i(X)_{\mathrm{hom}} = 0$ for $i \leq l-1$ (see also Esnault, Levine and Viehweg [7]). Therefore when $l = \lfloor n/2 \rfloor$ the Chow groups $\mathrm{CH}_i(X)_{\mathrm{alg}}$ are all representable by Remark 3.8. As a direct application of Theorem 4 we get

Theorem 4.10. *Let $l = \lfloor n/2 \rfloor$ and let X be a smooth hyperplane section of a hypersurface in $\mathbf{P}^{n+1}$ covered by l-planes. Then,*

- *if $n-1$ is even,* $\mathfrak{h}(X) = \mathbb{1} \oplus \mathbb{L} \oplus \mathbb{L}^{\otimes 2} \oplus \cdots \oplus \mathbb{L}^{\otimes n-1}$.
- *if $n-1$ is odd,* $\mathfrak{h}(X) = \mathbb{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes l} \oplus \mathfrak{h}_1(J_l^{\mathrm{alg}})(l) \oplus \mathbb{L}^{\otimes l+1} \oplus \cdots \oplus \mathbb{L}^{\otimes n-1}$.

Moreover, in any case, $\mathfrak{h}(X)$ is finite dimensional in the sense of Kimura.

Remark 4.11. Otwinowska also mentions that if $k(n-l) - \binom{d+l}{d} + 1 \geq 0$, any smooth projective hypersurface of degree d in $\mathbf{P}^n_{\mathbb{C}}$ is covered by linear projective varieties of dimension l.

Examples 4.12. Here are some varieties for which Theorem 4 and the results of [7] make it possible to prove that they have finite dimensional Chow motive :

- Cubic 5-folds.
- A 5-fold which is the smooth intersection of a cubic and a quadric.
- A 7-fold which is the smooth intersection of two quadrics.

Further examples of varieties with finite dimensional Chow motive can be constructed as follows. Let X be a variety as in the theorem above. Consider smooth projective varieties obtained from X by successively blowing up smooth curves. Then, by the blowing-up formula for Chow motives, such varieties have finite dimensional Chow motive. Moreover any variety Y which is dominated by a product of such varieties has finite dimensional Chow motive.

References

[1] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2004.

[2] Y. André and B. Kahn. (with an appendix by P. O'Sullivan) Nilpotence, radicaux et structures monoïdales. *Rend. Sem. Mat. Univ. Padova*, 108:107–291, 2002.

[3] D. Arapura. Motivation for Hodge cycles. *Adv. Math.*, 207(2):762–781, 2006.

[4] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. *Amer. J. Math.*, 105(5):1235–1253, 1983.

[5] F. El Zein and S. Zucker. Extendability of normal functions associated to algebraic cycles. In *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982)*, volume 106 of *Ann. of Math. Stud.*, pages 269–288. Princeton Univ. Press, Princeton, NJ, 1984.

[6] H. Esnault and M. Levine. Surjectivity of cycle maps. *Astérisque*, (218):203–226, 1993. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992).

[7] H. Esnault, M. Levine, and E. Viehweg. Chow groups of projective varieties of very small degree. *Duke Math. J.*, 87(1):29–58, 1997.

[8] S. Gorchinskiy and V. Guletskii. Motives and representability of algebraic cycles on threefolds over a field. *J. Algebraic Geom. 21(2):347–373, 2012* .

[9] P. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math. (2) 90 (1969), 460-495; ibid. (2)*, 90:496–541, 1969.

[10] U. Jannsen. Motives, numerical equivalence, and semi-simplicity. *Invent. Math.*, 107(3):447–452, 1992.

[11] U. Jannsen. Motivic sheaves and filtrations on Chow groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 245–302. Amer. Math. Soc., Providence, RI, 1994.

[12] B. Kahn and R. Sujatha. Birational motives, I pure birational motives. Preprint, February 27, 2009.

[13] B. Kahn, J. Murre, and C. Pedrini. On the transcendental part of the motive of a surface. In *Algebraic cycles and motives. Vol. 2*, volume 344 of *London Math. Soc. Lecture Note Ser.*, pages 143–202. Cambridge Univ. Press, Cambridge, 2007.

[14] S.-I. Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.

[15] S.-I. Kimura. Surjectivity of the cycle map for Chow motives. In *Motives and algebraic cycles*, volume 56 of *Fields Inst. Commun.*, pages 157–165. Amer. Math. Soc., Providence, RI, 2009.

[16] R. Laterveer. Algebraic varieties with small Chow groups. *J. Math. Kyoto Univ.*, 38(4):673–694, 1998.

[17] J. Murre. On the motive of an algebraic surface. *J. Reine Angew. Math.*, 409:190–204, 1990.

[18] A. Otwinowska. Remarques sur les groupes de Chow des hypersurfaces de petit degré. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(1):51–56, 1999.

[19] K. Paranjape. Cohomological and cycle-theoretic connectivity. *Ann. of Math. (2)*, 139(3):641–660, 1994.

[20] A. J. Scholl. Classical motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 163–187. Amer. Math. Soc., Providence, RI, 1994.

[21] C. Vial. Pure motives with representable Chow groups. *C. R. Math. Acad. Sci. Paris*, 348(21-22):1191–1195, 2010.

[22] C. Vial. Niveau and coniveau filtrations on cohomology groups and Chow groups. *Proc. Lond. Math. Soc. (3)*, 106(2):410–444, 2013.

[23] A. Weil. *Variétés abéliennes et courbes algébriques*. Actualités Sci. Ind., no. 1064 = Publ. Inst. Math. Univ. Strasbourg 8 (1946). Hermann & Cie., Paris, 1948.

DPMMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK
E-mail address: C.Vial@dpmms.cam.ac.uk