Pure motives with representable Chow groups

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November 9, 2010

Abstract

Let k be an algebraically closed field. We show using Kahn's and Sujatha's theory of birational motives that a Chow motive over k whose Chow groups are all representable (in the sense of definition 2.1) belongs to the full and thick subcategory of motives generated by the twisted motives of curves.

Résumé

Motifs purs dont les groupes de Chow sont représentables. Soit k un corps algébriquement clos. Nous prouvons, en nous servant de la théorie des motifs birationnels développée par Kahn et Sujatha, qu'un motif de Chow défini sur k dont les groupes de Chow sont tous représentables (au sens de la définition 2.1) appartient à la sous-catégorie pleine et épaisse des motifs engendrée par les motifs de courbes tordus.

Version française abrégée

Dans cette note, nous présentons une preuve de l'énoncé suivant.

Théorème 0.1 (cf. theorem 3.4)**.** *Soit k un corps algébriquement clos et soit $\Omega \supset k$ un domaine universel, i.e. un corps algébriquement clos de degré de transcendance infini sur k. Soit M un motif de Chow rationnel sur k dont les groupes de Chow $CH_j(M_\Omega)_{\mathrm{alg}}$ sont représentables pour tout entier j. Alors M est isomorphe à une somme directe de motifs de Lefschetz et de $\mathfrak{h}_1$ de variétés abéliennes tordus.*

Ici, $CH_j(M_\Omega)_{\mathrm{alg}}$ désigne le groupe des cycles algébriques de dimension j algébriquement triviaux modulo l'équivalence rationnelle. La notion de représentabilité est définie en 2.1. Un tel résultat est connu pour les motifs de surfaces et a été récemment prouvé pour les motifs de variétés lisses projectives de dimension 3 par Gorchinskiy et Guletskii [3]. Kimura [7] a prouvé qu'étant donné un motif M, si ses groupes de Chow rationnels $CH_j(M_\Omega)$ sont des $\mathbf{Q}$-espaces vectoriels de dimension finie alors M est isomorphe à une somme directe de motifs de Lefschetz, offrant ainsi une généralisation d'un théorème dû à Jannsen [5, Th. 3.5.]. Notre résultat généralise les résultats cités ci-dessus et redonne un théorème dû à Esnault et Levine [2] qui montre que pour une variété complexe lisse et projective, si l'application classe de Deligne rationnelle en tous degrés est injective alors elle est bijective.

Notre méthode repose sur l'existence de projecteurs de Chow relevant le plus grand facteur direct du motif numérique $\bar{M}$ isomorphe à un objet dans la sous-catégorie pleine et épaisse des motifs numériques engendrée par les motifs de courbes tordus. Une telle construction est l'objet de [10, §1]. Nous nous servons de façon essentielle de la théorie des motifs purs birationnels développée par Kahn et Sujatha. Nous aimons penser à cette théorie comme à une manière synthétique de procéder à une décomposition généralisée de la diagonale telle qu'elle a été mise en œuvre par Jannsen, et Esnault et Levine entre autres.

1 Introduction

Given a field k, there are three categories we will be dealing with.

- $\mathcal{M}^{\text{eff}}(k, \mathbf{Q})$, the category of effective Chow motives with coefficients in $\mathbf{Q}$.
- $\mathcal{M}(k, \mathbf{Q})$, the category of Chow motives with coefficients in $\mathbf{Q}$, see [9].
- $\mathcal{M}^\circ(k, \mathbf{Q})$, the category of birational Chow motives with coefficients in $\mathbf{Q}$, as defined by Kahn and Sujatha in [6].

Roughly, $\mathcal{M}(k, \mathbf{Q})$ is obtained from $\mathcal{M}^{\text{eff}}(k, \mathbf{Q})$ by inverting the Lefschetz motive and $\mathcal{M}^\circ(k, \mathbf{Q})$ is obtained from $\mathcal{M}^{\text{eff}}(k, \mathbf{Q})$ by killing the Lefschetz motive (modulo taking the pseudo abelian envelope).

Objects in $\mathcal{M}(k, \mathbf{Q})$ are triples (X, p, n) and morphisms are given by

$$\mathrm{Hom}_k((X, p, n), (Y, q, m)) := q \circ CH_{\dim X + n - m}(X \times Y) \circ p$$

where CH_i denotes the Chow group of i-dimensional cycles tensored with $\mathbf{Q}$. To any smooth projective variety X we associate functorially the motive $\mathfrak{h}(X) := (X, \mathrm{id}_X, 0)$. The category $\mathcal{M}^{\text{eff}}(k, \mathbf{Q})$ is the full subcategory of $\mathcal{M}(k, \mathbf{Q})$ whose objects have the form $(X, p, 0)$.

Let $\mathcal{L}$ be the ideal of $\mathcal{M}^{\text{eff}}(k, \mathbf{Q})$ generated by those morphisms that factor through an object of the form $N \otimes \mathbb{L}$ with N an effective Chow motive and $\mathbb{L} = (\mathrm{Spec}\, k, \mathrm{id}, 1)$ the Lefschetz motive. Kahn and Sujatha [6] define the $\mathbf{Q}$-linear tensor category $\mathcal{M}^\circ(k, \mathbf{Q})$ of pure birational Chow motives over k to be the pseudo-abelianization of the quotient category $\mathcal{M}^{\text{eff}}(k, \mathbf{Q})/\mathcal{L}$. The functor $\mathcal{M}^{\text{eff}}(k, \mathbf{Q}) \to \mathcal{M}^\circ(k, \mathbf{Q})$ will be denoted by $M \mapsto M^\circ$ and to any smooth projective variety X we associate functorially the motive $\mathfrak{h}^\circ(X)$.

For each of these three categories, we will write Hom_k for the groups of morphisms. It will be clear in which category this takes place. Note that since the functor $\mathcal{M}^{\text{eff}}(k, \mathbf{Q}) \to \mathcal{M}(k, \mathbf{Q})$ is fully faithful, it doesn't matter in which of these two categories we consider $\mathrm{Hom}_k(M, N)$ for two effective motives M and N.

Given a field extension L/k, there are base change functors for each of these three categories. Given a motive (either effective, pure or birational) M over k, we will write M_L for its image in the corresponding category of motives (either effective, pure or birational) over L. Moreover, for two motives M and N over k, we write

$$\mathrm{Hom}_L(M, N) := \mathrm{Hom}_L(M_L, N_L).$$

An essential feature of Kahn and Sujatha's category of birational motives is the following (cf. [6, (2.5)])

Theorem 1.1 (Kahn-Sujatha)**.** *Let X and Y be smooth projective varieties over k. Denote by $k(X)$ the function field of X. Then*

$$\mathrm{Hom}_k(\mathfrak{h}^\circ(X), \mathfrak{h}^\circ(Y)) = CH_0(Y_{k(X)}) = \mathrm{Hom}_{k(X)}(\mathbb{1}^\circ, \mathfrak{h}^\circ(Y)).$$

We now fix a field Ω containing k which is a universal domain, i.e. an algebraically closed field of infinite transcendence degree over its prime subfield. Before we start, we need to compute some Hom groups in the category of birational motives. Let X and Y be smooth projective varieties. Then by theorem 1.1 $\mathrm{Hom}_k(\mathfrak{h}^\circ(Y), \mathfrak{h}^\circ(X)) = \varinjlim CH^{\dim X}(U \times X)$, where the limit runs through all nonempty open subsets U of Y. Now assume $M = (X, p)$ is an effective Chow motive and denote by M° its image in $\mathcal{M}^\circ(k, \mathbf{Q})$. Then, we have

$$\begin{aligned}\mathrm{Hom}_k(\mathfrak{h}^\circ(Y), M^\circ) &= p \circ \mathrm{Hom}_k(\mathfrak{h}^\circ(Y), \mathfrak{h}^\circ(X)) = p \circ \varinjlim CH^{\dim X}(U \times X) \\ &= \varinjlim\, (\mathrm{id}_U \otimes p)_* CH^{\dim X}(U \times X) = (\mathrm{id}_{k(Y)} \otimes p)_* CH_0(X_{k(Y)}) \\ &= p_{k(Y)} \circ CH_0(X_{k(Y)}) = \mathrm{Hom}_{k(Y)}(\mathbb{1}^\circ, M^\circ).\end{aligned}$$

Lemma 1.2. *Let $M \in \mathcal{M}^{\mathrm{eff}}(k, \mathbf{Q})$ and let M° denote its image in $\mathcal{M}^\circ(k, \mathbf{Q})$. Then the following statements are equivalent.*

- $M^\circ = 0$.
- $\mathrm{Hom}_\Omega(\mathbb{1}^\circ, M^\circ) = 0$.
- *There exists an effective Chow motive $N \in \mathcal{M}^{\mathrm{eff}}(k, \mathbf{Q})$ such that M is isomorphic to $N(1)$.*

Proof. The first and the third statements are equivalent by [6]. Moreover, the first statement obviously implies the second one. It remains to prove that the second statement implies the first one. Suppose $M = (X, p)$. Then $\mathrm{Hom}_\Omega(\mathbb{1}^\circ, M^\circ) \supset \mathrm{Hom}_{k(X)}(\mathbb{1}^\circ, M^\circ) = \mathrm{Hom}_k(\mathfrak{h}^\circ(X), M^\circ) \supset \mathrm{Hom}(M^\circ, M^\circ) = \mathrm{End}(M^\circ)$. Consequently $\mathrm{End}(M^\circ) = 0$ and thus $M^\circ = 0$. □

Lemma 1.3 (see also [3], lemma 1)**.** *Let $M \in \mathcal{M}(k, \mathbf{Q})$. Then the following statements are equivalent.*

- $M = 0$.
- $\mathrm{Hom}_\Omega(\mathbb{1}(i), M) = 0$ *for all* $i \in \mathbf{Z}$.

Proof. Assume $M = (X, p, n)$ is non-zero, $\dim X = d$ and that M is isomorphic to some effective motive $N = (Y, q, 0)$. Then $\mathrm{End}(M) \simeq \mathrm{Hom}(M, N) \subseteq \mathrm{Hom}(\mathfrak{h}(X)(n), N) \simeq \mathrm{Hom}(\mathbb{1}(n+d), \mathfrak{h}(X) \otimes N) \subseteq CH_{n+d}(X \times Y)$ and hence $\mathrm{End}(M) \neq 0$ (i.e. $M \neq 0$) implies $n \geq -d$.

Thus there is an integer j which is the smallest integer such that $M(j)$ is effective. Then, by assumption, $\mathrm{Hom}_\Omega(\mathbb{1}, M(j)) = 0$. Therefore, $\mathrm{Hom}_\Omega(\mathbb{1}^\circ, M(j)^\circ) = 0$. By lemma 1.2, this implies there exists an effective motive N such that $M(j) \simeq N(1)$. Hence, $M(j-1) \simeq N$ is effective, contradicting the choice of j. □

2 Representability

Let $M = (X, p, j)$ be a motive in $\mathcal{M}(k, \mathbf{Q})$ and let $CH_i(M)_\sim := \mathrm{Hom}_k(\mathbb{1}, M(-i))_\sim = p_* CH_{i-j}(X)_\sim$ be the subgroup of $CH_i(M)$ made of those cycles ~ 0 for an adequate equivalence relation $\sim$. Such a definition is unambiguous in the following sense : it can be checked that if $p \in \mathrm{End}(\mathfrak{h}(X))$ is an idempotent then $(p_* CH_i(X))_\sim = p_*(CH_i(X)_\sim)$. In what follows we will be mainly interested in $\sim =$ alg where alg denotes algebraic equivalence.

Definition 2.1. *Let Ω be a universal domain over k. We say that the Chow group $CH_i(M)_{\mathrm{alg}}$ of algebraically trivial cycles of a motive $M = (X, p, j) \in \mathcal{M}(k, \mathbf{Q})$ is* representable *if there is a smooth projective curve C over Ω (not necessarily connected) and a correspondence $\Gamma \in \mathrm{Hom}_\Omega(\mathfrak{h}_1(C), M(-i))$ such that $\Gamma_* : \mathrm{Hom}_\Omega(\mathbb{1}, \mathfrak{h}_1(C)) \to \mathrm{Hom}_\Omega(\mathbb{1}, M(-i))_{\mathrm{alg}}$ is surjective. We say that the total Chow group $CH_*(M)_{\mathrm{alg}}$ of a motive $M \in \mathcal{M}(k, \mathbf{Q})$ is* representable *if $CH_i(M)_{\mathrm{alg}}$ is representable for all i.*

Notice that we do not require the curve C to be defined over k (which would have been more restrictive). The notion of representability chosen here seems to be the most appropriate in the language of motives. Proposition 2.1 below, which generalizes Jannsen's [5, 1.6] where idempotents are not being dealt with, shows that most notions of representability for zero-cycles are the same. First we need a lemma whose proof can be found in [8, (1.4)–(1.7)].

Lemma 2.2. *Let X and Y be smooth projective varieties over an algebraically closed field F. Then there exists an albanese map $\mathrm{alb}_X : CH_0(X)_{\mathrm{alg}} \to \mathrm{Alb}_X(F)$. Moreover, if $\alpha \in \mathrm{Hom}_F(\mathfrak{h}(X), \mathfrak{h}(Y))$ then α in-duces a homomorphism $\bar{\alpha} : \mathrm{Alb}_X(F) \to \mathrm{Alb}_Y(F)$ satisfying $\bar{\alpha} \circ \mathrm{alb}_X = \mathrm{alb}_Y \circ \alpha_* : CH_0(X)_{\mathrm{alg}} \to \mathrm{Alb}_Y(F)$.*

Proposition 2.1. *Let $M = (X, p) \in \mathcal{M}^{\mathrm{eff}}(k, \mathbf{Q})$. The following statements are equivalent.*

i) $CH_0(M)_{\mathrm{alg}}$ is representable.

i′) There is a smooth projective curve C over k and a correspondence $\Gamma \in \mathrm{Hom}_k(\mathfrak{h}_1(C), M)$ such that $(\Gamma_\Omega)_* : CH_0(C_\Omega)_{\mathrm{alg}} \to (p_\Omega)_* CH_0(X_\Omega)_{\mathrm{alg}}$ is surjective.

i″) $(p_\Omega \circ \iota_\Omega)_* CH_0(B_\Omega)_{\mathrm{alg}} = (p_\Omega)_* CH_0(X_\Omega)_{\mathrm{alg}}$ where $\iota : B \hookrightarrow X$ is a smooth linear section of X of dimension 1.

ii) There exists a closed subvariety $Y \subset X_\Omega$ of dimension 1 such that for all $\gamma \in CH_0(X_\Omega)$, $(p_\Omega)_*\gamma$ has vanishing restriction in $CH_0(X_\Omega - Y)$.

ii′) There exists a closed subvariety $Y \subset X$ of dimension 1 such that for all $\gamma \in CH_0(X_\Omega)$, $(p_\Omega)_*\gamma$ has vanishing restriction in $CH_0((X - Y)_\Omega)$.

iii) There is a decomposition $p_\Omega = p_1 + p_2$ with $p_1, p_2 \in CH_d(X_\Omega \times X_\Omega)$ such that p_1 is supported on $X_\Omega \times Y$ and p_2 is supported on $D \times X_\Omega$ for some curve $Y \subset X_\Omega$ and some divisor $D \subset X_\Omega$.

iii′) There is a decomposition $p = p_1 + p_2$ with $p_1, p_2 \in CH_d(X \times X)$ such that p_1 is supported on $X \times Y$ and p_2 is supported on $D \times X$ for some curve $Y \subset X$ and some divisor $D \subset X$.

iv) The albanese map $\mathrm{alb} : CH_0(X_\Omega)_{\mathrm{alg}} \to \mathrm{Alb}_{X_\Omega}(\Omega)$ is injective when restricted to $(p_\Omega)_* CH_0(X_\Omega)_{\mathrm{alg}}$.

Proof. Clearly $i'' \Rightarrow i' \Rightarrow i$, $ii' \Rightarrow ii$ and $iii' \Rightarrow iii$.

$i' \Rightarrow ii'$. Let C and Γ be as in (i'). Clearly, $(\Gamma_\Omega)_* CH_0(C_\Omega)$ is supported on Y_Ω for Y the projection on X of a representative of Γ on $C \times X$. Therefore, by localization, this group vanishes in $CH_0((X - Y)_\Omega)$.

$ii' \Rightarrow iii'$. We use Bloch's and Srinivas' technique [1]. Let $\tilde{p}$ be the image of p under the natural map $CH_d(X \times X) \to CH_0(k(X) \times_k X)$. Fix an embedding $k(X) \subset \Omega$ that extends that of k. The natural map $CH_0(Y_k) \to CH_0(Y_L)$ is known to be injective for any smooth variety Y over k and any field extension L/k. This combined to the fact that $\tilde{p} = p_*\eta_X$ for η_X the generic point of X seen as a rational point over $k(X)$ implies that, for Y as in ii', $\tilde{p}$ has vanishing restriction in $CH_0(k(X) \times_k (X - Y))$. By the localization exact sequence $\tilde{p}$ is supported on $k(X) \times_k Y$. Let p_1 be an element of $CH_d(X \times_k Y)$ mapping to $\tilde{p}$. Then $p - p_1$ has vanishing restriction in $CH_0(k(X) \times_k X)$ and thus, again by the localization exact sequence, is supported on $D \times X$ for some divisor D on X. Set $p_2 := p - p_1$.

$iii' \Rightarrow i'$. Thanks to Chow's lemma on 0-cycles, $(p_2)_\Omega$ acts trivially on $CH_0(X_\Omega)$, hence $((p_1)_\Omega)_* CH_0(X_\Omega) = (p_\Omega)_* CH_0(X_\Omega)$. If C is the normalization of Y and α is the pullback of p_1 in $CH_d(X \times C)$, we can write $p_1 = \beta \circ \alpha$ with $\beta \in \mathrm{Hom}_k(\mathfrak{h}(C), \mathfrak{h}(X))$. Then we have $((p \circ \beta)_\Omega)_* CH_0(C_\Omega)_{\mathrm{alg}} \supseteq ((p \circ \beta)_\Omega) \circ (\alpha_\Omega)_* CH_0(X_\Omega)_{\mathrm{alg}} = (p_\Omega \circ p_\Omega)_* CH_0(X_\Omega)_{\mathrm{alg}} = (p_\Omega)_* CH_0(X_\Omega)_{\mathrm{alg}}$. We also clearly have $((p \circ \beta)_\Omega)_* CH_0(C_\Omega)_{\mathrm{alg}} \subseteq (p_\Omega)_* CH_0(X_\Omega)_{\mathrm{alg}}$. Therefore ($i'$) follows by taking the curve C and the correspondence $\Gamma = p \circ \beta$.

By working over Ω instead of k, the exact same arguments prove $i \Rightarrow ii \Rightarrow iii \Rightarrow i$.

$iii \Rightarrow iv$. As for the implication (iii') $\Rightarrow$ (i'), p_2 acts trivially on $CH_0(X_\Omega)$ and p_1 factors as $\alpha \circ \beta$ with $\alpha \in \mathrm{Hom}_\Omega(\mathfrak{h}(X), \mathfrak{h}(C))$ and $\beta \in \mathrm{Hom}_\Omega(\mathfrak{h}(C), \mathfrak{h}(X))$ for some smooth projective curve C over Ω. We have to prove that if $x \in CH_0(X_\Omega)_{\mathrm{alg}}$ satisfies $\mathrm{alb}(p_*x) = 0$ then $p_*x = 0$. Obviously $\mathrm{alb}(p_*x) = 0$ implies $\bar{\alpha} \circ \mathrm{alb}(p_*x) = 0$. By lemma 2.2 $\mathrm{alb}_C((\alpha \circ p)_*x) = 0$ and because

alb_C is an isomorphism we get $(\alpha \circ p)_* x = 0$. Thus $(\beta \circ \alpha \circ p)_* x = 0$, that is $p_* \circ p_* x = 0$ i.e. $p_* x = 0$.

$iv \Rightarrow i''$. Fix $x \in CH_0(X_\Omega)_{\mathrm{alg}}$. We want to show that there exists $z \in CH_0(B_\Omega)_{\mathrm{alg}}$ such that $(p_\Omega \circ \iota_\Omega)_* z = (p_\Omega)_* x$. It is a fact [9, 4.3] that the induced map $\mathrm{Alb}_{B_\Omega}(\Omega) \to \mathrm{Alb}_{X_\Omega}(\Omega)$ is surjective. Thus, using also the bijectivity of alb_{B_Ω}, there exists $z \in CH_0(B_\Omega)_{\mathrm{alg}}$ such that $\mathrm{alb}((p_\Omega)_* x) = \bar{\iota}_\Omega \circ \mathrm{alb}(z)$. Now, thanks to lemma 2.2, $\mathrm{alb}((p_\Omega \circ \iota_\Omega)_* z) = \bar{p}_\Omega(\bar{\iota}_\Omega \circ \mathrm{alb}(z)) = \bar{p}_\Omega(\mathrm{alb} \circ (p_\Omega)_* x) = \mathrm{alb}((p_\Omega)_* x)$. By assumption on the map alb we get $(p_\Omega \circ \iota_\Omega)_* z = (p_\Omega)_* x$. $\square$

3 Main theorem

We denote by $\mathcal{M}_0(k, \mathbf{Q})$ (resp. $\mathcal{M}_1(k, \mathbf{Q})$) the full thick subcategory of $\mathcal{M}(k, \mathbf{Q})$ generated by the $\mathfrak{h}_0$'s (resp. $\mathfrak{h}_1$'s) of smooth projective varieties over k. Equivalently, $\mathcal{M}_0(k, \mathbf{Q})$ is generated by the motives of points and $\mathcal{M}_1(k, \mathbf{Q})$ is generated by the $\mathfrak{h}_1$'s of curves, see [9]. We write $\mathcal{M}_{\mathrm{num}}(k, \mathbf{Q})$ for the category of motives for numerical equivalence with rational coefficients. Jannsen [4] famously proved that this category is abelian semi-simple. For a motive $P \in \mathcal{M}(k, \mathbf{Q})$, let $\bar{P}$ denote its image in $\mathcal{M}_{\mathrm{num}}(k, \mathbf{Q})$.

Proposition 3.1 (see [10] for a proof)**.** *Let M be an object in $\mathcal{M}_0(k, \mathbf{Q})$ (resp. in $\mathcal{M}_1(k, \mathbf{Q})$). Let N be any motive in $\mathcal{M}(k, \mathbf{Q})$. Then any morphism $f : M \to N$ induces a splitting $N = N_1 \oplus N_2$ with N_1 isomorphic to an object in $\mathcal{M}_0(k, \mathbf{Q})$ (resp. in $\mathcal{M}_1(k, \mathbf{Q})$) and $\bar{N}_1 \simeq \mathrm{Im}\,\bar{f}$.*

From now on, k is an algebraically closed field and Ω denotes a universal domain over k. Recall that if M and N are two Chow motives over k, then $\mathrm{Hom}_k(\bar{M}, \bar{N}) = \mathrm{Hom}_\Omega(\bar{M}, \bar{N})$.

Lemma 3.2. *Let $M \in \mathcal{M}^{\mathrm{eff}}(k, \mathbf{Q})$ and let n be the dimension of the finite dimensional vector space $\mathrm{Hom}_\Omega(\bar{\mathbb{1}}, \bar{M})$. Then $\mathbb{1}^{\oplus n}$ is a direct summand of M.*

Proof. Pick a basis $(\bar{e}_i)_{1 \leq i \leq n}$ of the group $\mathrm{Hom}_k(\bar{\mathbb{1}}, \bar{M}) = \mathrm{Hom}_\Omega(\bar{\mathbb{1}}, \bar{M})$ of 0-cycles modulo numerical equivalence on M. Lift it to a family $(e_i)_{1 \leq i \leq n}$ of the Chow group $\mathrm{Hom}_k(\mathbb{1}, M)$ and consider the morphism $\oplus e_i : \mathbb{1}^{\oplus n} \longrightarrow M$. By proposition 3.1, M has then a direct summand N isomorphic to an object in $\mathcal{M}_0$ whose reduction modulo numerical equivalence is $\bar{\mathbb{1}}^{\oplus n}$. Therefore $N \simeq \mathbb{1}^{\oplus n}$. $\square$

Lemma 3.3. *Let $M = (X, p) \in \mathcal{M}^{\mathrm{eff}}(k, \mathbf{Q})$ be such that $\mathrm{Hom}_k(\mathbb{1}, M)_{\mathrm{alg}}$ is representable. Assume moreover that $\bar{M}$ has no direct factor of the form $\bar{\mathfrak{h}}_1(J)$ for an abelian variety J. Then $\mathrm{Hom}_\Omega(\mathbb{1}, M)_{\mathrm{alg}} = 0$.*

Proof. Thanks to proposition 2.1 and its proof (specifically the statement $(i) \Rightarrow (iii')$ plus an extra argument included in the proof of $(iii') \Rightarrow (i')$) the representability assumption on $\mathrm{Hom}_k(\mathbb{1}, M)_{\mathrm{alg}}$ yields a decomposition $p = p_1 + p_2 \in CH_d(X \times X)$ such that p_1 factors through a smooth projective curve C over k and p_2 is supported on $D \times X$ for some proper subscheme D of X. In particular $(p_2)_\Omega$ acts trivially on 0-cycles on X_Ω. Let's write $p_1 = \beta \circ \alpha$ with $\alpha \in \mathrm{Hom}_k(\mathfrak{h}(X), \mathfrak{h}(C))$ and $\beta \in \mathrm{Hom}_k(\mathfrak{h}(C), \mathfrak{h}(X))$. The correspondence $(p_1)_\Omega$ acts as the identity on $\mathrm{Hom}_\Omega(\mathbb{1}, M)$. If π_1 denotes the projector on $\mathfrak{h}_1(C)$ with respect to the choice of a 0-cycle of degree 1 on C (see e.g. [9]) and if $q_1 := p \circ \beta \circ \pi_1 \circ \alpha$ then $(q_1)_\Omega$ acts as the identity on $\mathrm{Hom}_\Omega(\mathbb{1}, M)_{\mathrm{alg}}$. Therefore $(q_1 \circ q_1)_\Omega$ also acts as the identity on $\mathrm{Hom}_\Omega(\mathbb{1}, M)_{\mathrm{alg}}$. By assumption on M, the map $p \circ \beta \circ \pi_1$ must be numerically trivial. Hence the map $\pi_1 \circ \alpha \circ p \circ \beta \circ \pi_1 \in \mathrm{End}(\mathfrak{h}_1(C))$ is also numerically trivial. Because $\mathrm{End}(\mathfrak{h}_1(C)) = \mathrm{End}(\bar{\mathfrak{h}}_1(C))$ we get that $\pi_1 \circ \alpha \circ p \circ \beta \circ \pi_1 = 0$ and therefore that $q_1 \circ q_1 = p \circ \beta \circ \pi_1 \circ \alpha \circ p \circ \beta \circ \pi_1 \circ \alpha = 0$. This proves that $\mathrm{Hom}_\Omega(\mathbb{1}, M)_{\mathrm{alg}} = 0$. $\square$

Theorem 3.4. *Let $M \in \mathcal{M}(k,\mathbf{Q})$. Then $CH_*(M)_{\mathrm{alg}}$ is representable if and only if M is isomorphic to a sum of Lefschetz motives and twisted $\mathfrak{h}_1$'s of abelian varieties.*

Proof. Assume $M = (X,p,n)$ with X a smooth projective variety over k. Up to tensoring with $\mathbb{1}(-n)$ we can assume that M is effective. The integers r for which $CH_r(M_\Omega) := \mathrm{Hom}_\Omega(\mathbb{1}(r), M)$ is possibly non-zero are non-negative. We proceed by induction on $\mu(M) := \max\{r \ : \ \mathrm{Hom}_\Omega(\mathbb{1}(r), M) \neq 0\} \in \{-\infty\} \cup \mathbf{Z}_{\geq 0}$.

In the case $\mu(M) = -\infty$, that is by definition in the case when $CH_r(M_\Omega) = 0$ for all integers r, we conclude directly by lemma 1.3 that $M = 0$. Let then M be an effective motive with $\mu(M) > -\infty$. Let n be the dimension of the $\mathbf{Q}$-vector space $\mathrm{Hom}_\Omega(\bar{\mathbb{1}}, \bar{M})$. By lemma 3.2, there exists a motive M' over k such that $M = \mathbb{1}^n \oplus M'$ and $\mathrm{Hom}_\Omega(\bar{\mathbb{1}}, \bar{M}') = 0$.

Let C be a curve over k and $\Gamma \in \mathrm{Hom}_k(\mathfrak{h}_1(C), M)$ be such that $\bar{\Gamma} \in \mathrm{Hom}_k(\bar{\mathfrak{h}}_1(C), \bar{M}')$ has maximal image inside $\bar{M}'$ among all curves C' and all morphisms in $\mathrm{Hom}_k(\bar{\mathfrak{h}}_1(C'), \bar{M}')$. By proposition 3.1, Γ induces a splitting $M' = \mathfrak{h}_1(J) \oplus N$ for some abelian variety J and some effective motive N satisfying $\mathrm{Hom}_k(\bar{\mathfrak{h}}_1(C'), \bar{N}) = 0$ for all curves C'. Since N is a direct summand of M, the group $\mathrm{Hom}_k(\mathbb{1}, N)_{\mathrm{alg}}$ is representable. By lemma 3.3, $\mathrm{Hom}_\Omega(\mathbb{1}, N)_{\mathrm{alg}} = 0$. Moreover, because N is a direct summand of M', we have $\mathrm{Hom}_\Omega(\bar{\mathbb{1}}, \bar{N}) = 0$. Algebraic equivalence and numerical equivalence agree on 0-cycles. Therefore we have a decomposition $M = \mathbb{1}^{\oplus n} \oplus \mathfrak{h}_1(J) \oplus N$ with $\mathrm{Hom}_\Omega(\mathbb{1}, N) = 0$. Hence $\mathrm{Hom}_\Omega(\mathbb{1}^\circ, N^\circ) = 0$. Therefore, by lemma 1.2, there exists an effective Chow motive $N' \in \mathcal{M}^{\mathrm{eff}}(k,\mathbf{Q})$ such that $M = \mathbb{1}^{\oplus n} \oplus \mathfrak{h}_1(J) \oplus N'(1)$. The Chow group of the motive N'_Ω is a subgroup of the Chow group of M_Ω, it is therefore representable. Clearly $\mu(N') \leq \mu(M) - 1$ which concludes the proof by induction. $\square$

Corollary 3.5 (Kimura [7])**.** *Let $M \in \mathcal{M}(k,\mathbf{Q})$. Then $CH_*(M_\Omega)$ is a finite dimensional $\mathbf{Q}$-vector space if and only if M is isomorphic to a sum of Lefschetz motives.*

Let X be a smooth projective variety of dimension d over an algebraically closed subfield k of $\mathbf{C}$. The Abel-Jacobi map $AJ_i : CH_i(X_{\mathbf{C}})_{\mathrm{hom}} \to J_i(X_{\mathbf{C}})$ defined by Griffiths (here Chow groups are not tensored with $\mathbf{Q}$) restricts to $CH_i(X_{\mathbf{C}})_{\mathrm{alg}}$ and the image of the composite map $CH_i(X)_{\mathrm{alg}} \to CH_i(X_{\mathbf{C}})_{\mathrm{alg}} \to J_i(X_{\mathbf{C}})$ defines an abelian variety over k that we denote $J_i^a(X)$.

Corollary 3.6. *Assume that the total Chow group of X is representable. Then,*

$$\mathfrak{h}(X) = \mathbb{1} \oplus \mathfrak{h}_1(\mathrm{Alb}_X) \oplus \mathbb{L}^{\oplus b_2} \oplus \mathfrak{h}_1(J_1^a(X))(1) \oplus (\mathbb{L}^{\otimes 2})^{\oplus b_4} \oplus \ldots \oplus \mathfrak{h}_1(J_{d-1}^a(X))(d-1) \oplus \mathbb{L}^{\otimes d}$$

where b_i denotes the i^{th} Betti number of X. Moreover, algebraic equivalence agrees with numerical equivalence on X and the generalized Hodge conjecture holds for X.

Corollary 3.7 (Esnault-Levine [2])**.** *Let X be a complex smooth projective variety. Suppose that the total rational Deligne cycle class map $cl_{\mathcal{D}} : \oplus_i CH^i(X) \to \oplus_i H^{2i}_{\mathcal{D}}(X, \mathbf{Q}(i))$ is injective. Then it is surjective.*

Acknowledgements

Thanks to Magdalene College, Cambridge and to the EPSRC for financial support under the grant EP/H028870/1. The idea of using birational motives grew out of a discussion with Bruno Kahn at the workshop "Finiteness results for motives and motivic cohomology" in Regensburg in February 2009. I am grateful to Bruno Kahn for explaining to me lemma 1.2 and related ideas, as well as to Uwe Jannsen for organizing the workshop and giving me the opportunity to present my work. I also wish to thank the referee for his very useful and numerous comments, and also for providing a reference to lemma 2.2.

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