1 Introduction

Let $K$ be a field. Often it will be algebraically closed. Let $A$ be a $K$-algebra. Any $A$-module $M$ becomes a vector space. We are interested in the classification of finite-dimensional $A$-modules, especially in the case when $A$ is finite-dimensional.

References:
D. J. Benson, Representations and cohomology
Assem, Simson and Skowronski, Elements of the representation theory of associative algebras I
Auslander, Reiten and Smalo, Representation theory of artin algebras
R. Schiffler, Quiver representations

1.1 Examples of algebras

(1) The group algebra $A = KG$ for a group $G$. Then $A$-modules are the same as representations of the group $G$, so group homomorphisms $G \to \text{GL}(V)$.

(2) Let $V = K^n$ and let $q : V \to K$ be a quadratic form. The associated Clifford algebra $C_K(V, q)$ is $TV/I$, where $TV$ is the tensor algebra on $V$ over $K$ and $I$ is the ideal generated by $v^2 - q(v)1$ for all $v \in V$. For example if $V$ has basis $e_1, \ldots, e_n$ and $q(\sum_{i=1}^n \lambda_i e_i) = \sum_{i=1}^n a_i \lambda_i^2$ for all $\lambda_1, \ldots, \lambda_n \in K$, then $C(V, q)$ is generated by $e_1, \ldots, e_n$ subject to the relations $e_i^2 = a_i 1$.
and $e_i e_j = -e_j e_i$ for $i \neq j$. It has basis the products $e_{i_1} e_{i_2} \ldots e_{i_k}$ with $i_1 < i_2 < \cdots < i_k$.

In particular the exterior algebra is the case when $q = 0$.

For example $C_\mathbb{R}(\mathbb{R}^2, q)$ with $q(x_1, x_2) = -x_1^2 - x_2^2$ is the quaternions.

(3) The Temperley-Lieb algebra $TL_n(\delta)$ for $n \geq 1$ and $\delta \in K$ has basis the diagrams with two vertical rows of $n$ dots, connected by $n$ nonintersecting curves. Two are considered equal if the same vertices are connected. The product $ab$ is defined by

$$ab = \delta^r \cdot \text{(diagram obtained by joining } a \text{ and } b \text{ and removing any loops)},$$

where $r$ is the number of loops removed.

Picture for $n = 3$. Basis elements 1, $u_1$, $u_2$, $p$, $q$.

1 = \begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}, \quad u_1 = \begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}, \quad u_2 = \begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}, \quad p = \begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}, \quad q = \begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}

One can show that $TL_n(\delta)$ is generated by $u_1 \ldots, u_{n-1}$ subject to the relations $u_i^2 = \delta u_i$, $u_i u_{i+1} u_i = u_i$ and $u_i u_j = u_j u_i$ if $|i - j| > 1$.

The Temperley-Lieb algebra was invented to study Statistical Mechanics. It is now also important in Knot Theory. (Fields medal for Vaughan Jones in 1990.)

(4) $A = KQ$, the path algebra of a finite quiver $Q$. We are interested in classification of configurations of vector spaces and linear maps. It is finite-dimensional provided $Q$ has no oriented cycles.

If $Q$ is one vertex and a loop, a representation is given by a vector space and an endomorphism. If finite-dimensional, choosing a basis, an $n$-dimensional representation is given by an $n \times n$ matrix. Two representations are isomorphic if the matrices are similar. If the field $K$ is algebraically closed, any matrix is similar to one in Jordan normal form. Thus any representation is isomorphic to a direct sum of Jordan block representations.

If $Q$ is two vertices with an arrow between them, a representation is given by two vector spaces and a linear map $a : X \to Y$. A finite-dimensional representation is given by an $m \times n$ matrix $M$. Two representations given by $M, M'$ are isomorphic if there are invertible matrices $P, Q$ with $PM = M'Q$. Equivalently if $M, M'$ are related by row and column operations.
Can always transform to block matrix of 1s and 0s. Thus finite-dimensional representations are determined by the dimension vector and the rank.

(5) If $I$ is an ideal in $KQ$ then representations of $A = KQ/I$ correspond to configurations of vector spaces and linear maps satisfying the relations in $I$. For example if $Q$ is

$$
\begin{array}{cccccc}
1 & a_1 & 2 & a_2 & \ldots & a_{n-1} \\
\end{array}
$$

and $I = (a_2 a_1, a_3 a_2, \ldots, a_{n-1} a_{n-2})$ then $A$-modules correspond to complexes of vector spaces with $n$ terms.

The quiver with one vertex and loop $x$ with relation $x^n = 0$. This is the algebra $K[x]/(x^n)$. Any module is a direct sum of modules given by Jordan blocks with eigenvalue 0 and size up to $n$.

1.2 Radical and socle

Theorem/Definition. If $R$ is a ring and $x \in R$, the following are equivalent

(i) $xS = 0$ for any simple left module $S$.

(ii) $x \in I$ for every maximal left ideal $I$

(iii) $1 - ax$ has a left inverse for all $a \in R$.

(iv) $1 - ax$ is invertible for all $a \in R$.

(i')-(iv') The right-hand analogues of (i)-(iv).

The set of such elements is called the (Jacobson) radical $J(R)$ of $R$. It is an ideal in $R$.

Proof (i) implies (ii). If $I$ is a maximal left ideal in $R$, then $R/I$ is a simple left module, so $x(R/I) = 0$, so $x(I + 1) = I + 0$, so $x \in I$.

(ii) implies (iii). If there is no left inverse, then $R(1 - ax)$ is a proper left ideal in $R$, so contained in a maximal left ideal $I$ by Zorn’s Lemma. Now $x \in I$, and $1 - ax \in I$, so $1 \in I$, so $I = R$, a contradiction.

(iii) implies (iv) $1 - ax$ has a left inverse $u$, and $1 + uax$ has a left inverse $v$. Then $u(1 - ax) = 1$, so $u = 1 + uax$, so $vu = 1$. Thus $u$ has a left and right inverse, so it is invertible and these inverses are equal, and are themselves invertible. Thus $1 - ax$ is invertible.

(iv) implies (i). If $s \in S$ and $xs \neq 0$, then $Rxs = S$ since $S$ is simple, so $s = axs$ for some $a \in R$. Then $(1 - ax)s = 0$, but then $s = 0$ by (iv).
(iv) implies (iv'). If \( b \) is an inverse for \( 1 - ax \), then \( 1 + xba \) is an inverse for \( 1 - xa \). Namely \( (1 - ax)b = b(1 - ax) = 1 \), so \( axb = b - 1 = bax \), and then \( (1 + xba)(1 - xa) = 1 + xba - xa - xbaxa = 1 \), and \( (1 - xa)(1 + xba) = 1 - xaxba - xa + xba = 1 \).

Examples.
(a) For the ring \( \mathbb{Z} \), the simple modules are \( \mathbb{Z}/p\mathbb{Z} \). Then \( J(\mathbb{Z}) = \bigcap_p p\mathbb{Z} = 0 \).
(b) Similarly, for the ring \( K[x] \), the simple modules are \( K[x]/(f(x)) \), \( f(x) \) irreducible. Then \( J(K[x]) = 0 \).
(c) If \( K \) is a field and \( Q \) a finite quiver, then \( J(KQ) \) has as basis the paths from \( i \) to \( j \) such that there is no path from \( j \) to \( i \).

Lemma 1. If \( I \) is an ideal in which every element is nilpotent (a ‘nil ideal’), then \( I \subseteq J(R) \).

Proof. If \( x \in I \) and \( a \in R \) then \( ax \in I \), so \( (ax)^n = 0 \) for some \( n \). Then \( 1 - ax \) is invertible with inverse \( 1 + ax + (ax)^2 + \cdots + (ax)^{n-1} \). Thus \( x \in J(R) \).

If \( M \) is an \( R \)-module and \( I \) an ideal in \( R \), we write \( IM \) for the set of sums of products of elements in \( M \).

Nakayama’s Lemma. Suppose \( M \) is a finitely generated \( R \)-module.
(i) If \( J(R)M = M \), then \( M = 0 \).
(ii) If \( N \subseteq M \) is a submodule with \( N + J(R)M = M \), then \( N = M \).

Proof. (i) Suppose \( M \neq 0 \). Let \( m_1, \ldots, m_n \) be generators with \( n \) minimal. Since \( J(R)M = M \) we can write \( m_n = \sum_{i=1}^n r_i m_i \) with \( r_i \in J(R) \). This writes \( (1 - r_n)m_n \) in terms of the others. But \( 1 - r_n \) is invertible, so it writes \( m_n \) in terms of the others. Contradiction.

(ii) Apply (i) to \( M/N \).

Definition. The socle of a module \( M \) is the sum of its simple submodules,
\[
\text{soc} M = \sum_{S \subseteq M \text{ simple}} S
\]
The radical of a module \( M \) is the intersection of its maximal submodules.
\[
\text{rad} M = \bigcap_{N \subseteq M, M/N \text{ simple}} N
\]
Thus \( J(R) = \text{rad}(R) \). The quotient \( M/\text{rad} M \) is called the top of \( M \).
Lemma 2.
(i) soc $M$ is the unique largest semisimple submodule of $M$.
(ii) If $N \subseteq M$ and $M/N$ is semisimple, then rad $M \subseteq N$.
Note that in general $M/N$ is not semisimple, eg $M = \mathbb{Z}/2\mathbb{Z}$.

Proof. (i) Use that a module is semisimple if and only if it is a sum of simple submodules.

(ii) Suppose $M/N = \bigoplus S_j$. For each $j$ let $U_j$ be the kernel of the map $M \rightarrow M/N \rightarrow \bigoplus S_i \rightarrow S_j$. Then $M/U_j \cong S_j$, so $U_j$ is a maximal submodule. If $m \in \text{rad } M$, then $m \in U_j$ for all $j$, so $m$ is sent to zero in $M/N \cong \bigoplus S_i$, so $m \in N$.

1.3 Radical and socle (finite-dimensional case)

Let $K$ be a field and let $A$ be a $K$-algebra. Any $A$-module $M$ becomes a vector space. Often in this section $M$ or $A$ is finite-dimensional. The results in this section generalize to finite length or artinian modules and artinian rings.

Lemma 1. If $M$ is a finite-dimensional $A$-module, then $M/\text{rad } M$ is semisimple, so rad $M$ is the unique smallest submodule $N$ of $M$ with $M/N$ semisimple.

Proof. Since $M$ is finite-dimensional, it has DCC on submodules, so we can write $\text{rad } M$ as a finite intersection of maximal submodules $U_1 \cap \cdots \cap U_n$. Then $M/\text{rad } M$ embeds in $(M/U_1) \oplus \cdots \oplus (M/U_n)$, so it is semisimple.

Wedderburn’s Theorem. For a finite-dimensional algebra $A$ the following are equivalent
(i) $A$ is a semisimple module
(ii) Every $A$-module is semisimple
(iii) $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ with the $D_i$ division algebras.
(iv) $J(A) = 0$.

The Artin-Wedderburn Theorem generalizes this to arbitrary rings $R$. Part (iv) becomes that the ring is left artinian and $J(R) = 0$.

Proof. We already had the equivalence of (i) to (iii), and (i) if and only if (iv) follows from Lemma 1.
Example. If $G$ is a finite group and $K$ has characteristic 0 (or not dividing the order of $G$), then $KG$ is semisimple. (Maschke’s Theorem). Also Clifford algebras associated to non-degenerate quadratic forms are semisimple and the Temperley-Lieb algebras over $\mathbb{C}$ are semisimple for generic $\delta$.

The product of two ideals is $IJ = \text{sums of products } ij$ with $i \in I$ and $j \in J$. The powers of an ideal are defined inductively by $I^n = I^{n-1}I$. An ideal is \textit{nilpotent} if $I^n = 0$ for some $n$. If so, then every element of $I$ is nilpotent.

Proposition. Suppose that $A$ is a finite-dimensional algebra.
(i) $J(A)$ is a nilpotent ideal in $A$ and $A/J(A)$ is a semisimple algebra.
(ii) If $I$ is an ideal in which every element is nilpotent and $A/I$ is semisimple, then $I = J(A)$.

Proof. (i) Since $A$ is finite-dimensional, we have 
$$J(A) \supseteq J(A)^2 \supseteq \cdots \supseteq J(A)^n = J(A)^{n+1} = \ldots$$
for some $n$. Then $J(A)J(A)^n = J(A)^n$, so $J(A)^n = 0$ by Nakayama’s lemma. Now use Lemma 1.

(ii) Use §1.2 Lemmas 1 and 2.

Example. If $A$ is the algebra of matrices of shape
$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$
then $A = S \oplus I$ where $S$ and $I$ consist of matrices of shape
$$S = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$ 
It is easy to check that $I$ is an ideal in $A$, and it consists of nilpotent elements. Also one can check that $S$ is a subalgebra in $A$, and it is clearly isomorphic to $M_2(K) \times K$, so semisimple. Thus $A/I \cong S$ is semisimple. Thus $J(A) = I$.

Lemma 2. If $A$ is a finite-dimensional algebra and $M$ is an $A$-module, then
(i) $\text{rad } M = J(A)M$.
(ii) $\text{soc } M = \{m \in M : J(A)m = 0\}$.

Proof (i) If $N$ is a maximal submodule of $M$, then $M/N$ is simple, and so $J(A)(M/N) = 0$. Thus $J(A)M \subseteq N$, and hence $J(A)M \subseteq \text{rad } M$. On the
other hand $M/J(A)M$ is an $A/J(A)$-module, so semisimple as an $A/J(A)$-module, so also semisimple as an $A$-module, so rad $M \subseteq J(A)M$ by §1.2 Lemma 2.

(ii) Any simple submodule $S$ of $M$ satisfies $J(A)S = 0$, so $J(A)m = 0$ for all $m \in \text{soc} M$, so $\text{soc} M$ is contained in the RHS. Now the RHS is an $A/J(A)$-module, so semisimple as an $A/J(A)$-module, so also semisimple as an $A$-module, so rad $M \subseteq J(A)M$ by §1.2 Lemma 2.

Let $Q$ be a finite quiver and write $(KQ)_+$ for the ideal in $KQ$ spanned by the non-trivial paths. Clearly $(KQ)^n_+$ is the ideal spanned by paths of length $\geq n$, and $KQ/(KQ)_+ \cong K \times \cdots \times K$.

Definition. An ideal $I \subseteq KQ$ is admissible if
1. $I \subseteq (KQ)^2_+$
2. $(KQ)^n_+ \subseteq I$ for some $n$.

Lemma 3. If $I$ is an admissible ideal in $KQ$ then $A = KQ/I$ is finite-dimensional, $J(A) = (KQ)_+/I$, and $A/J(A) \cong K \times \cdots \times K$.

Proof. Clear.

Example. Let $Q$ be the quiver with one vertex and one loop $x$, so $KQ = K[x]$. Then the admissible ideals in $KQ$ are $(x^n)$ for $n \geq 2$, and the radical of $KQ/(x^n)$ is $(x)/(x^n)$.

1.4 Local rings

Lemma/Definition. A ring $R$ is called a local ring if it satisfies the following equivalent conditions.
(i) $R/J(R)$ is a division ring.
(ii) The non-invertible elements of $R$ form an ideal.
(iii) There is a unique maximal left ideal in $R$.

Proof. (i) implies (ii). The elements of $J(R)$ are not invertible, so it suffices to show that any $x \notin J(R)$ is invertible. Now $J(R) + x$ is an invertible element in $R/J(R)$, say with inverse $J(R) + a$. Then $1 - ax, 1 - xa \in J(R)$. But this implies $ax$ and $xa$ are invertible, hence so is $x$.

(ii) implies (iii). Clear.
(iii) implies (i). Assuming (iii), \( J(R) \) is the unique maximal left ideal, so \( \overline{R} = R/J(R) \) is a simple \( R \)-module, and so a simple \( \overline{R} \)-module. Then \( \overline{R} \cong \text{End}_{\overline{R}}(\overline{R}) \), which is a division ring by Schur’s Lemma.

Lemma. A finite-dimensional algebra is local if and only if every element is invertible or nilpotent.

Proof. If local, then every element is invertible or nilpotent, since elements of \( J(A) \) are nilpotent.

Conversely, suppose every element of \( A \) is nilpotent or invertible. The nilpotent elements form an ideal, for if \( ax \) is invertible, with \( x^n = 0 \), then \( 0 = [(ax)^{-1}a x^n = 1 \), and if there are \( x, y \) nilpotent with \( x + y \) invertible, then we may assume that \( x + y = 1 \). But then \( x = 1 - y \) is invertible.

1.5 Indecomposable modules

Recall that an \( A \)-module \( M \) is indecomposable if there is no direct sum decomposition \( M = X \oplus Y \) with \( X \) and \( Y \) non-zero submodules of \( M \). It is equivalent that \( \text{End}(M) \) contains no idempotents except 0,1.

Fitting’s Lemma. If \( M \) is a finite-dimensional \( A \)-module (or more generally of finite length) and \( \theta \in \text{End}_A(M) \), then there is a decomposition

\[
M = M_0 \oplus M_1
\]

such that \( \theta |_{M_0} \) is a nilpotent endomorphism of \( M_0 \) and \( \theta |_{M_1} \) is an invertible endomorphism of \( M_1 \). In particular, if \( M \) is indecomposable, then any endomorphism is invertible or nilpotent or invertible, so \( \text{End}_A(M) \) is local.

Proof. There are chains of submodules

\[
\text{Im}(\theta) \supseteq \text{Im}(\theta^2) \supseteq \text{Im}(\theta^3) \supseteq \ldots
\]

\[
\text{Ker}(\theta) \subseteq \text{Ker}(\theta^2) \subseteq \text{Ker}(\theta^3) \subseteq \ldots
\]

which must stabilize since \( M \) is finite dimensional. Thus there is some \( n \) with \( \text{Im}(\theta^n) = \text{Im}(\theta^{2n}) \) and \( \text{Ker}(\theta^n) = \text{Ker}(\theta^{2n}) \). We show that

\[
M = \text{Ker}(\theta^n) \oplus \text{Im}(\theta^n).
\]

If \( m \in \text{Ker}(\theta^n) \oplus \text{Im}(\theta^n) \) then \( m = \theta^n(m') \) and \( \theta^{2n}(m') = \theta^n(m) = 0 \), so \( m' \in \text{Ker}(\theta^{2n}) = \text{Ker}(\theta^n) \), so \( m = \theta^n(m') = 0 \). If \( m \in M \) then \( \theta^n(m) \in \text{Im}(\theta^n) \).
\[ \text{Im}(\theta^n) = \text{Im}(\theta^{2n}), \text{ so } \theta^n(m) = \theta^{2n}(m'') \text{ for some } m''. \] Then \( m = (m - \theta^n(m'')) + \theta^n(m'') \in \text{Ker}(\theta^n) + \text{Im}(\theta^n). \]

Now it is easy to see that the restriction of \( \theta \) to \( \text{Ker}(\theta^n) \) is nilpotent, and its restriction to \( \text{Im}(\theta^n) \) is invertible.

We now apply the idea of the Jacobson radical to the module category.

**Lemma/Definition.** If \( X \) and \( Y \) are \( A \)-modules, we define \( \text{rad}_A(X,Y) \) to be the set of all \( \theta \in \text{Hom}_A(X,Y) \) satisfying the following equivalent conditions.

(i) \( 1_X - \phi \theta \) is invertible for all \( \phi \in \text{Hom}_A(Y,X) \).

(ii) \( 1_Y - \theta \phi \) is invertible for all \( \phi \in \text{Hom}_A(Y,X) \).

Thus by definition \( \text{rad}(X,X) = J(\text{End}(X)) \).

Proof of (i) implies (ii). If \( u \) is an inverse for \( 1_X - \phi \theta \) then \( 1_Y + \theta u \phi \) is an inverse for \( 1_Y - \theta \phi \).

**Lemma 1.**

(a) \( \text{rad}_A(X,Y) \) is a subspace of \( \text{Hom}_A(X,Y) \).

(b) Given maps \( X \to Y \to Z \), if one is in the radical, so is the composition.

(c) \( \text{rad}_A(X \oplus X', Y) = \text{rad}_A(X,Y) \oplus \text{rad}_A(X',Y) \) and \( \text{rad}_A(X,Y \oplus Y') = \text{rad}_A(X,Y) \oplus \text{rad}_A(X,Y') \).

Proof. (a) For a sum \( \theta + \theta' \), let \( f \) be an inverse for \( 1 - \phi \theta \). Then \( 1 - \phi(\theta + \theta') = (1 - \phi \theta)(1 - f \phi \theta') \), a product of invertible maps.

(b) Clear.

(c) Straightforward.

**Definition.** A module map \( \theta : X \to Y \) is a split mono if there is a map \( \phi : Y \to X \) with \( \phi \theta = 1_X \). Equivalently if \( \theta \) is an isomorphism of \( X \) with a direct summand of \( Y \).

A module map \( \theta : X \to Y \) is a split epi if there is a map \( \psi : Y \to X \) with \( \theta \psi = 1_Y \). Equivalently if \( \theta \) identifies \( Y \) with a direct summand of \( X \).

**Lemma 2.**

(i) If \( X \) is indecomposable, then \( \text{rad}_A(X,Y) \) is the set of maps which are not split monos.

(ii) If \( Y \) is indecomposable, then \( \text{rad}_A(X,Y) \) is the set of maps which are not split epis.

(iii) If \( X \) and \( Y \) are indecomposable, then \( \text{rad}_A(X,Y) \) is the set of non-
isomorphisms.

Proof. (i) Suppose \( \theta \in \text{Hom}(X,Y) \). If \( \theta \) is a split mono there is \( \phi \in \text{Hom}(Y,X) \) with \( \phi \theta = 1_X \), so \( 1 - \phi \theta \) is not invertible. Conversely if there is some \( \phi \) with \( f = 1 - \phi \theta \) not invertible, then \( f \) is nilpotent, and so \( \phi \theta = 1 - f \) is invertible. Then \( (\phi \theta)^{-1} \phi \theta = 1_X \), so \( \theta \) is split mono.

(ii) is dual and (iii) follows.

Krull-Remak-Schmidt Theorem. Every finite-dimensional \( A \)-module is isomorphic to a direct sum of indecomposable modules,

\[ M \cong X_1 \oplus \cdots \oplus X_n. \]

Moreover if \( M \cong Y_1 \oplus \cdots \oplus Y_m \) is another decomposition into indecomposables, then \( m = n \) and the \( X_i \) and \( Y_j \) can be paired off so that corresponding modules are isomorphic.

Proof. Given any two modules \( X \) and \( Y \), we can define a vector space

\[ t(X,Y) = \text{Hom}_A(X,Y) / \text{rad}_A(X,Y). \]

If \( X \) is indecomposable, then \( D = \text{End}(X)/J(\text{End}(X)) \) is a division algebra, and we define

\[ \mu_X(Y) = \frac{\dim t(X,Y)}{\dim D}. \]

In fact \( t(X,Y) \) is an \( \text{End}(Y) \)-\( \text{End}(X) \)-bimodule, and in fact an \( \text{End}(Y)/J(\text{End}(Y)) \)-\( \text{End}(X)/J(\text{End}(X)) \)-bimodule. In particular it is a right \( D \)-module, and we are taking its dimension as a vector space over \( D \).

Now

\[ t(X,M) = t(X, X_1 \oplus \cdots \oplus X_n) \cong t(X, X_1) \oplus \cdots \oplus t(X, X_n) \]

so \( \mu_X(M) = \mu_X(X_1 \oplus \cdots \oplus X_n) = \mu_X(X_1) + \cdots + \mu_X(X_n) \). Also

\[ \mu_X(X_i) = \begin{cases} 1 & (X_i \cong X) \\ 0 & (X_i \not\cong X) \end{cases}. \]

Thus \( \mu_X(M) \) is the number of the \( X_i \) which are isomorphic to \( X \). Similarly, it is the number of \( Y_j \) which are isomorphic to \( X \). Thus these numbers are equal.

Notation. If \( M \) is a f.d. module we write \( \text{add} M \) for the subcategory of \( A \)-modules consisting of the direct summands of \( M^n \). Equivalently it is the
modules which are isomorphic to a direct sum of copies of the indecomposable direct summands of $M$.

For example add $A$ is exactly the f.d. projective left $A$-modules.

We say that a module is basic if in the decomposition into indecomposables, the summands are non-isomorphic.

Thus up to isomorphism, given $M$ there is a unique basic module $B$ with add $M = \text{add } B$.

### 1.6 Projectives and injectives for f.d. algebras

Let $A$ be a finite-dimensional algebra. Henceforth, unless explicitly stated otherwise, we only consider finite-dimensional modules.

The functor $D(-) = \text{Hom}_K(-, K)$ gives an antiequivalence between the categories of finite-dimensional left and right $A$-modules. It restricts to give an equivalence between projective modules on one side and injective modules on the other side.

The functor Hom$_A(-, A)$ gives an antiequivalence between the categories of finite-dimensional left and right projective $A$-modules.

**Definition.** The Nakayama functor is

$$\nu(-) = D \text{Hom}_A(-, A) : A - \text{mod} \to A - \text{mod}.$$ 

**Lemma 1.** (i) $\nu$ is naturally isomorphic to $DA \otimes_A -$.

(ii) $\nu$ has right adjoint

$$\nu^*(-) = \text{Hom}_A(D(-), A) \cong \text{Hom}_A(DA, -) : A - \text{mod} \to A - \text{mod}.$$ 

(iii) $\nu$ restricts to an equivalence from projective left modules to injective left modules, with inverse equivalence given by $\nu^{-}$.

(iv) $\text{Hom}(X, \nu P) \cong D \text{Hom}(P, X)$ for $X, P$ left $A$-modules, $P$ projective.

**Proof.**

(i) $D(DA \otimes_A X) \cong \text{Hom}_A(X, DDA) \cong \text{Hom}(X, A)$. Now apply $D$.

(ii), (iii) Clear.
(iv) The composition
\[ \text{Hom}(P, A) \otimes_A X \cong \text{Hom}(P, A) \otimes_A \text{Hom}(A, X) \to \text{Hom}(P, X) \]
is an isomorphism, since it is for \( P = A \). Thus
\[ D \text{Hom}(P, X) \cong \text{Hom}_K(\text{Hom}(P, A) \otimes_A X, K) \]
\[ \cong \text{Hom}(X, \text{Hom}_K(\text{Hom}(P, A), K)) = \text{Hom}(X, \nu P). \]

Definition. Let \( \theta : X \to Y \) be a map of \( A \)-modules. We say that \( \theta \) is \textit{left minimal} if for \( \alpha \in \text{End}(Y) \), if \( \alpha \theta = \theta \), then \( \alpha \) is invertible. We say that \( \theta \) is \textit{right minimal} if for \( \beta \in \text{End}(X) \), if \( \theta \beta = \theta \), then \( \beta \) is invertible.

Lemma 2. Given a map \( \theta : X \to Y \) of finite-dimensional \( A \)-modules
(i) There is a decomposition \( Y = Y_0 \oplus Y_1 \) such that \( \text{Im}(\theta) \subseteq Y_1 \) and \( X \to Y_1 \) is left minimal.
(ii) There is a decomposition \( X = X_0 \oplus X_1 \) such that \( \theta(X_0) = 0 \) and \( X_1 \to Y \) is right minimal.

Proof. (i) Of all decompositions \( Y = Y_0 \oplus Y_1 \) with \( \text{Im}(\theta) \subseteq Y_1 \) choose one with \( Y_1 \) of minimal dimension. Let \( \theta_1 \) be the map \( X \to Y_1 \). Let \( \alpha \in \text{End}(Y_1) \) with \( \alpha \theta_1 = \theta_1 \). By the Fitting decomposition, \( Y_1 = \text{Im}(\alpha^n) \oplus \text{Ker}(\alpha^n) \) for \( n \gg 0 \). Now \( \alpha^n \theta_1 = \theta_1 \), so \( \text{Im}(\theta_1) \subseteq \text{Im}(\alpha^n) \), and we have another decomposition \( Y = [Y_0 \oplus \text{Ker}(\alpha^n)] \oplus \text{Im}(\alpha^n) \). By minimality, \( \text{Ker}(\alpha^n) = 0 \), so \( \alpha \) is injective, and hence an isomorphism.

Proposition 1. Let \( M \) be a module and let \( \theta : M \to I \) be a map with \( I \) injective. The following are equivalent
(i) Any map from \( M \) to an injective factors through \( \theta \), and \( \theta \) is left minimal (i.e. \( \theta \) is an injective envelope).
(ii) \( \theta \) is a monomorphism and left minimal.
(iii) The induced map \( \theta|_{\text{soc} M} : \text{soc} M \to \text{soc} I \) is an isomorphism.

For every module \( M \) there exists a map \( \theta \) with these properties. It is unique up to a (non-unique) isomorphism.

Proof. (i) iff (ii). Clear, using that any module can be embedded in an injective module.

(ii) implies (iii). Identify \( M \) as a submodule of \( I \). We use that the inclusion \( M \to I \) is an essential extension. Clearly \( \theta|_{\text{soc} M} \) is injective. Suppose not onto. Then by semisimplicity \( \text{soc} I = \text{soc} M \oplus X \) for some \( X \neq 0 \). But then \( X \cap M = 0 \), for any simple submodule of it is in \( \text{soc} M \cap X \). Contradiction.
(iii) implies (ii). Any simple submodule of Ker \( \theta \) is in Ker(\( \theta \mid_{\text{soc} M} \)), so zero. Thus \( \theta \) is injective. If \( \alpha \) is an endomorphism of \( I \) with \( \alpha \theta = \theta \), then \( \alpha \mid_{\text{soc} I} \in \text{End}(\text{soc} I) \), and \( \alpha \mid_{\text{soc} I} \theta \mid_{\text{soc} M} = \theta \mid_{\text{soc} M} \). This forces \( \alpha \mid_{\text{soc} I} \) to be injective, so \( \alpha \) is injective, so an isomorphism by dimensions.

Using the duality \( D \) to turn statement about projectives into injectives, we get

Proposition 2. Let \( M \) be a module and let \( \theta : P \to M \) be a map with \( P \) projective. The following are equivalent

(i') Any map from a projective to \( M \) factors through \( \theta \), and \( \theta \) is right minimal (i.e. \( \theta \) is a projective cover).

(ii') \( \theta \) is surjective and right minimal.

(iii') The induced map \( \bar{\theta} : P/\text{rad} P \to M/\text{rad} M \) is an isomorphism.

For every module \( M \) there exists a map \( \theta \) with these properties. It is unique up to a (non-unique) isomorphism.

1.7 Structure of f.d. algebras

Let \( A \) be a finite-dimensional algebra over \( K \). We decompose \( AA \) into indecomposables, and collect isomorphic terms, so

\[
A \cong P[1]^r_1 \oplus \cdots \oplus P[n]^r_n
\]

with the \( P[i] \) non-isomorphic modules.

Properties.

(1) The modules \( P[1], \ldots, P[n] \) are a complete set of non-isomorphic indecomposable projective modules. They are called principal indecomposable modules (pims). The finite-dimensional projective \( A \)-modules are exactly the direct sums of copies of the \( P[i] \).

This is just the Krull-Remak-Schmidt Theorem,

(2) Let \( S[i] = P[i]/\text{rad} P[i] \). The modules \( S[1], \ldots, S[n] \) are a complete set of non-isomorphic simple \( A \)-modules.

Namely, \( S[i] \) is semisimple, but any endomorphism \( \theta \) of it lifts to \( P[i] \), so is nilpotent or invertible, so \( \theta \) is nilpotent or invertible, so \( S[i] \) is indecomposable, so simple. Inverse isomorphisms between \( S[i] \) and \( S[j] \) lift to maps \( P[i] \to P[j] \to P[i] \) whose composition can't be nilpotent, so must be invertible, so \( P[i] \cong P[j] \), so \( i = j \). Any simple module \( S \) has a non-zero map from
some $P[i]$, but then the map $P[i] \to S$ must give a non-zero map $S[i] \to S$, and this must be an isomorphism.

(3) Let $D_i = \text{End}_A(P[i])/J(\text{End}_A(P[i]))^{\text{op}} \cong \text{End}_A(S[i])^{\text{op}}$. It is a division algebra. If $K$ is algebraically closed, then $D_i = K$.

Use that $P[i]$ is indecomposable, or Schur’s Lemma. If $K$ is algebraically closed and $0 \neq x \in D$, then the map $D \to D$ of multiplication by $x$ must have an eigenvalue $\lambda \in K$. Then $(x - \lambda)y = 0$ for some $0 \neq y \in D$. Since $D$ is a division algebra, $x = \lambda \in K$.

(4) $A/J(A) \cong \text{End}_A(A/J(A))^{\text{op}} \cong \text{End}_A(\bigoplus_i (P[i]/\text{rad} P[i]))^{\text{op}}$.

Then $S[i] \cong \text{Hom}(A, S[i]) \cong \bigoplus_j \text{Hom}(P[j], S[i])^{\text{tr}} \cong \text{End}(S[i])^{\text{tr}}$.

(5) Define $I[i] = \nu P[i]$. It is an injective module with soc $I[i] \cong S[i]$. The modules $I[1], \ldots, I[n]$ are a complete set of non-isomorphic indecomposable injective modules, and any injective module is a direct sum of copies of them.

We check $\text{Hom}(S[j], I[i]) \cong D \text{Hom}(P[i], S[j])$ which is $D_i$ if $j = i$ and otherwise 0.

Examples. (1) For the algebra

$$A = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad J(A) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

we get $A = Ae^{11} \oplus Ae^{22} \oplus Ae^{33}$,

$$P[1] = Ae^{11} = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \cong Ae^{22}, \quad P[2] = Ae^{33} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

$$\text{rad} P[1] = J(A)P[1] = 0, \quad \text{rad} P[2] = J(A)P[2] = \begin{pmatrix} * \\ 0 \end{pmatrix} \cong P[1].$$


(2) If an algebra is given as $A = KQ/I$ with $I$ admissible and $Q_0 = \{1, \ldots, n\}$, then we get
\( P[i] = Ae_i \). As a representation of \( Q \), the vector space at vertex \( j \) is \( e_j(KQ/I)e_i \), so has as basis the paths from \( i \) to \( j \) modulo the relations.

\( \text{rad} P[i] = J(A)e_i \). The vector space at \( j \) has as basis the non-trivial paths from \( i \) to \( j \) modulo the relations.

\( S[i] = Ae_i / \text{rad} Ae_i \). As a representation of \( Q \) the vector space at \( i \) is \( K \), the other vector spaces are zero, and the linear maps corresponding to arrows are zero.

\( I[i] = D(e_i A) \). As a representation of \( Q \), the vector space at vertex \( j \) is \( D(e_i(KQ/I)e_j) \), is has as basis the dual basis vectors corresponding to paths from \( j \) to \( i \) modulo the relations.

(3) For the quiver 1 \( \xrightarrow{a} \) 2 \( \xrightarrow{b} \) 3 \( \xrightarrow{c} \) 4 with relations \( I = (ba) \) we have

\[
P[1] = K \to K \to 0 \to 0, P[2] = 0 \to K \to K \to K,
\]

\[
P[3] = 0 \to 0 \to K \to K, P[4] = 0 \to 0 \to 0 \to K.
\]

for example \( P[1] = Ae_1 \) has basis \( e_1, a \) (but not \( ba \) or \( cba \) as they are in the ideal \( I \)). Now \( e_1 \in e_1 Ae_1 \) and \( a \in e_2 Ae_1 \) so they belong to the vector spaces at vertices 1 and 2 for \( P[1] \).

\[
S[1] = K \to 0 \to 0 \to 0, S[2] = 0 \to K \to 0 \to 0,
\]

\[
S[3] = 0 \to 0 \to K \to 0, S[4] = 0 \to 0 \to 0 \to K.
\]

Also

\[
I[1] = K \to 0 \to 0 \to 0, I[2] = K \to K \to 0 \to 0,
\]

\[
I[3] = 0 \to K \to K \to 0, I[4] = 0 \to K \to K \to K.
\]

(4) The commutative square quiver \( ba = dc \), vertices 1(source),2,3,4(sink).

The representation \( P[1] \cong I[4] \) is 1-dimensional at each vertex.

(5) A cyclically oriented square

\[
\begin{array}{c}
1 \xrightarrow{a} 2 \\
\downarrow d \quad \downarrow b \\
4 \xleftarrow{c} 3
\end{array}
\]

with relations \( cba \) and \( dc \). For example \( P[1] \) has basis the paths starting 1, modulo the relations, so \( e_1, a, ba \). These basis elements belong to the vector
spaces at vertices 1, 2, 3. Moreover \( ae_1 = a \), so the arrow \( a \) sends the basis element corresponding to \( e_1 \) to the basis element corresponding to \( a \). Thus

\[
P[1] = \begin{array}{c}
\downarrow 1 \\
K \\
\uparrow 1 \\
0 \\
\end{array}
\]

Similarly

\[
P[2] = \begin{array}{c}
\downarrow 1 \\
K \\
\uparrow 1 \\
K \\
\end{array}
\quad P[3] = \begin{array}{c}
\downarrow 1 \\
K \\
\uparrow 1 \\
K \\
\end{array}
\quad P[4] = \begin{array}{c}
\downarrow 1 \\
K \\
\uparrow 1 \\
K \\
\end{array}
\]

For example in \( P[4] \) the arrow \( c \) sends the basis element \( bad \) in the vector space at vertex 3 to \( cbad = 0 \), and not to \( e_4 \), which is the basis element of the vector space at vertex 4.

Definition. We say that a finite-dimensional algebra \( A \) is basic if \( _AA \) is basic. It is equivalent that \( A/J(A) \cong D_1 \times \cdots \times D_n \) with the \( D_i \) division algebras.

Theorem. Any f.d. algebra is Morita equivalent to a basic one.

Proof. Let \( P = P[1] \oplus \cdots \oplus P[n] \) be the basic module with \( \text{add} \ P = \text{add} \ A \). The \( P \) is a finitely generated projective generator for \( A-mod \), so \( A \) is Morita equivalent to \( B = \text{End}(A(P))^{\text{op}} \). Now

\[
B/J(B) \cong \text{End}_A(P/\text{rad} \ P)^{\text{op}} \cong \text{End}_A(S[1] \oplus \cdots \oplus S[n])^{\text{op}} \cong D_1 \times \cdots \times D_n.
\]

Theorem (Gabriel’s less famous theorem about quivers). If \( A \) is a f.d. \( K \)-algebra, and \( A/\text{rad} \ A \cong K \times \cdots \times K \) (for example if \( A \) is basic and \( K \) is algebraically closed), then \( A \cong KQ/I \) for some quiver \( Q \) and admissible ideal \( I \).

Proof. We have a decomposition \( A = P[1] \oplus \cdots \oplus P[n] \) without multiplicities. Using the isomorphism \( A \cong \text{End}(A)^{\text{op}} \), the projections onto the \( P[i] \) give a complete set of inequivalent primitive orthogonal idempotents \( e_1, \ldots, e_n \). Let \( J = J(A) \). We have

\[
J = \bigoplus_{i,j} e_j Je_i.
\]

and

\[
J^2 = \bigoplus_{i,j} e_j J^2 e_i
\]

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\[ J/J^2 \cong \bigoplus_{i,j} (e_j Je_i)/(e_j J^2 e_i). \]

Let \( Q \) be the quiver with \( Q_0 = \{1, \ldots, n\} \) and with
\[
\dim(e_j Je_i)/(e_j J^2 e_i)
\]
arrows from \( i \) to \( j \), for all \( i, j \). Define an algebra homomorphism
\[
\theta : KQ \rightarrow A
\]
sending \( e_i \) to \( e_i \), and sending the arrows from \( i \) to \( j \) to elements in \( e_j Je_i \) inducing a basis of the quotient. Let \( U = \theta(KQ_+) \). We have \( U \subseteq J \) and \( U + J^2 = J \), using that \( J = \bigoplus_{i,j} e_j Je_i \).

Thus by Nakayama’s Lemma, \( U = J \). It follows that \( \theta \) is surjective. Let \( I = \text{Ker} \theta \). If \( m \) is sufficiently large that \( J^m = 0 \), then \( \theta(KQ_+^m) = U^m = 0 \), so \( KQ_+^m \subseteq I \). Suppose \( x \in I \). Write it as \( x = u + v + w \) where \( u \) is a linear combination of \( e_i \)'s, \( v \) is a linear combination of arrows, and \( w \) is in \( KQ_+^2 \). Since \( \theta(e_i) = e_i \) and \( \theta(v), \theta(w) \in J \), we must have \( u = 0 \). Now \( \theta(v) = -\theta(w) \in J^2 \), so that \( \theta(v) \) induces the zero element of \( J/J^2 \). Thus \( v = 0 \). Thus \( x = w \in KQ_+^2 \).

### 1.8 Homological algebra for finite-dimensional algebras

Let \( A \) be a finite-dimensional algebra.

Definition. A projective resolution
\[
\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \rightarrow 0
\]
is minimal if at each stage, the map \( \epsilon : P_0 \rightarrow M, d_1 : P_1 \rightarrow \text{Ker}(\epsilon), d_2 : P_2 \rightarrow \text{Ker}(d_1) \) and so on, is a projective cover. Dually for an injective resolution
\[
0 \rightarrow M \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d_1} I^2 \rightarrow \cdots,
\]
the maps \( \epsilon : M \rightarrow I^0, I^0/\text{Im}(\epsilon) \rightarrow I^1, I^1/\text{Im}(d^0) \rightarrow I^2 \) and so on must be injective envelopes.
The minimal projective and injective resolutions of $M$ are unique up to (non-unique) isomorphism.

Lemma 1. $\dim \text{Ext}^k(S[i], M)$ is equal to $\dim D_i$ times the multiplicity of $I[i]$ as a summand of $I^k$ in the minimal injective resolution of $M$.

$\dim \text{Ext}^k(M, S[j])$ is equal to $\dim D_j$ times the multiplicity of $P[j]$ as a summand of $P_k$ in the minimal projective resolution of $M$.

Proof. By minimality, any element of soc $I^i$ is in the image of the map $I^{i-1} \to I^i$, so is killed by the map $I^i \to I^{i+1}$. Thus in the complex $\text{Hom}(S[i], I^*)$, the differential is zero.

Example. For the oriented cycle of example (5) in the last section, the simple modules have minimal projective resolutions

$0 \to P[1] \to P[4] \to P[2] \to P[1] \to S[1] \to 0$,

$0 \to P[3] \to P[2] \to S[2] \to 0$,

$0 \to P[1] \to P[4] \to P[3] \to S[3] \to 0$,

$0 \to P[1] \to P[4] \to S[4] \to 0$.

For example the projective cover of $S[1]$ is $P[1]$, giving an exact sequence

$0 \to \Omega^1 S[1] \to P[1] \to S[1] \to 0$

which is

$0 \to 0 \to K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \to 0$.

and the projective cover of $\Omega^1 S[1]$ is $P[2]$, giving an exact sequence

$0 \to \Omega^2 S[1] \to P[2] \to \Omega^1 S[1] \to 0$

which is

$0 \to 0 \to K \to 0 \to K \xrightarrow{1} K \xrightarrow{1} K \to 0$.

so $\Omega^2 S[1] \cong S[3]$, etc.
Lemma 2. If $A = KQ/I$ with $I$ admissible, then the number of arrows from $i$ to $j$ is $\dim \text{Ext}^1(S[i], S[j])$.

Proof. Since $I$ is admissible, $I \subseteq (KQ)_+^2$. Now $P[i] = (KQ/I)e_i$, so $\text{rad} P[i] = ((KQ)_+^2/I)e_i$, and $\text{rad} \text{rad} P[i] = ((KQ)_+^2/I)e_i$. Thus the top of $\text{rad} P[i] \cong ((KQ)_+^2/(KQ)_+^2)e_i \cong \bigoplus_j S[j]^{n_{ij}}$

where $n_{ij}$ is the number of arrows from $i$ to $j$. Then in the minimal projective resolution of $S[i]$,

$$\cdots \rightarrow P_1 \rightarrow P[i] \rightarrow S[i] \rightarrow 0$$

$P_1$ is the projective cover of $\text{rad} P[i]$, so also of top of $\text{rad} P[i]$, so the multiplicity of $P[j]$ is $n_{ij}$. Thus $\dim \text{Ext}^1(S[i], S[j]) = n_{ij}$.

Lemma 3. The following are equivalent for a module $M$

(i) proj. dim $M \leq n$
(ii) $\text{Ext}^{n+1}(M, S) = 0$ for all simples $S$.
(iii) the minimal projective resolution of $M$ has $P_k = 0$ for $k > n$.

Similarly for the injective dimension.

Proof. (i) implies (ii) is clear.

(ii) implies (iii). By the lemma above, the minimal projective resolution of $M$ has $P_{n+1} = 0$.

(iii) implies (i). Trivial.

Proposition. The global dimension of a f.d. algebra is the supremum of the projective dimensions of its simple modules.

Proof. If every simple $S$ has a projective resolution of length $\leq n$, then every semisimple module has a projective resolution of length $\leq n$, so every semisimple module has projective dimension $\leq n$. Now every module $X$ has a filtration $X \supseteq J(A)X \supseteq \cdots \supseteq J(A)^N X = 0$ in which the quotients are semisimple, and the long exact sequence shows that an extension of modules of projective dimension $\leq n$ again has projective dimension $\leq n$.

Corollary. For a f.d. algebra, the left and right global dimensions are the same.

Proof. If the right global dimension is $\leq n$, then the simple right modules have injective resolutions of length $\leq n$. Dualizing, the simple left modules
have projective resolutions of length $\leq n$. Thus the left global dimension is $\leq n$.

Theorem. If $A$ is a f.d. hereditary algebra and $A/J(A) \cong K \times \cdots \times K$ (for example if $A$ is basic and $K$ is algebraically closed), then $A$ is isomorphic to a path algebra $KQ$.

Proof. The algebra can be given as $A = KQ/I$ with $I$ admissible. Consider the exact sequence of $KQ$-modules

$$0 \to I/(I.KQ_+) \to KQ_+/I.KQ_+ \to KQ_+/I \to 0.$$ 

The middle module is annihilated by $I$, so this is a sequence of $A$-modules. The RH module is a submodule of $A = KQ/I$, so it is projective as an $A$-module. Thus the sequence splits. Letting

$$M = KQ_+/I.KQ_+, \quad N = I/(I.KQ_+) \oplus KQ_+/I.$$ 

we deduce that $M \cong N$. Thus $M/(KQ_+)M \cong N/(KQ_+)N$, which gives

$$KQ_+/KQ_+^2 \cong (I/(I.KQ_+I + I.KQ_+)) \oplus (KQ_+/KQ_+^2).$$

Thus by dimensions, $I = KQ_+.I + I.KQ_+$. Now if $I \neq 0$ there is a maximal $k$ such that $I \subseteq (KQ)_+^k$. But then $I = KQ_+.I + I.KQ_+ \subseteq (KQ)_+^{k+1}$, a contradiction.

1.9 Some homological properties and conjectures for f.d. algebras

Definitions. (i) An algebra is self-injective if $_AA$ is an injective module. Equivalently the modules $P[i]$ and $I[j]$ are the same, up to a permutation. This is left-right symmetric.

(ii) A Frobenius algebra is an algebra $A$ with a bilinear form

$$(\cdot, \cdot) : A \times A \to K$$

which is non-degenerate and associative, that is, $(ab,c) = (a,bc)$ for all $a,b,c \in A$. The form defines an isomorphism $_AA \to _{DA}A$, $a \mapsto (\cdot,a)$, so $A$ is injective. Conversely, any such isomorphism gives a suitable bilinear form.
(iii) \( A \) is a symmetric algebra if also \((a,b) = (b,a)\). In this case the form defines an isomorphism \( A \overline{A} \rightarrow \overline{A} A \overline{A} \), and conversely. For a symmetric algebra we have \( I[i] = \nu(P[i]) = \overline{A} \otimes \overline{A} P[i] \cong A \otimes A P[i] \cong P[i] \).

Examples. (1) The group algebra \( A = KG \) of a finite group is symmetric with \((a,b) = \lambda_i\) where \(ab = \sum_{g \in G} \lambda_g g\).

(2) The path algebra of a cyclic quiver with relation that all paths of length \( k \) equal to 0 is Frobenius. For example the oriented cycle with vertices 1,2 and arrows \( a : 1 \rightarrow 2 \) and \( b : 2 \rightarrow 1 \) with relations \( ab, ba \). The indecomposable projectives and injectives all have dimension 2.

(3) For a commutative algebra the three concepts are the same (for (i) and (ii), since the algebra is basic). Commutative Frobenius algebras appear in topological quantum field theory.

Lemma 1. If \( \text{inj.dim}_{A} A = n \), any \( A \)-module has \( \text{proj.dim} M \leq n \) or \( \infty \).

For example, every non-projective module for a self-injective algebra has infinite projective dimension.

Proof. Say \( \text{proj.dim} M = i < \infty \). There is some \( N \) with \( \text{Ext}^i(M,N) \neq 0 \). Choose \( 0 \rightarrow L \rightarrow P \rightarrow N \rightarrow 0 \) with \( P \) projective. The long exact sequence for \( \text{Hom}(M,-) \) gives

\[ \cdots \rightarrow \text{Ext}^i(M,P) \rightarrow \text{Ext}^i(M,N) \rightarrow \text{Ext}^{i+1}(M,L) \rightarrow \cdots \]

Now \( \text{Ext}^{i+1}(M,L) = 0 \), so \( \text{Ext}^i(M,P) \neq 0 \), so \( \text{Ext}^i(M,A) \neq 0 \), so \( i \leq n \).

Definition. An algebra \( A \) is (Iwanaga) Gorenstein if \( \text{inj.dim}_{A} A < \infty \) and \( \text{inj.dim}_{A} A < \infty \).

Conjecture (see Auslander and Reiten, Applications of contravariantly finite subcategories, Adv. Math 1991). If one is finite, so is the other.

Theorem. If \( \text{inj.dim}_{A} A = r \) and \( \text{inj.dim}_{A} A = s \) are both finite, they are equal.

Proof. \( \text{proj.dim}_{A} DA = \text{inj.dim}_{A} A = s \), so \( s \leq r \) by Lemma 1. Dually \( s \geq r \).

Also true for noetherian rings (Zaks, Injective dimension of semi-primary rings, J. Alg. 1969).

Definition. A module \( M \) is faithful if \( am = 0 \) for all \( m \in M \) implies \( a = 0 \),
that is, if the map $A \to \text{End}_K(M)$ is injective.

A f.d. $A$-module $M$ is faithful if and only if there is an embedding $A \to M^n$ for some $n$. Namely, if $A \hookrightarrow M^n$, $a \in A$ and $am = 0$ for all $m \in M$, then $ax = 0$ for all $x \in M^n$, so $a1 = 0$ for $1 \in A$. Thus $a = 0$. Conversely, if $M$ is faithful, choose a basis $m_1, \ldots, m_n$ of $M$. This gives a map $A \to M^n$, $a \mapsto (am_1, \ldots, am_n)$. If $a \mapsto 0$, then $am_i = 0$ for all $i$, so $am = 0$ for all $m \in M$.

Definition. An algebra is QF-3 (in the sense of Thrall) if it has a faithful projective-injective module.

Examples. Any self-injective algebra is QF-3.

The commutative square algebra is QF-3, because the socle of any indecomposable projective is the simple projective module $S$, so the algebra embeds in a direct sum of copies of the injective envelope $I$ of $S$, but this is also projective.

Remark. If $A$ is QF-3 then the faithful projective-injective module $M$ is unique (up to multiplicities). It must be a direct sum of indecomposable projective-injective modules, and every indecomposable projective-injective module $I$ must occur, because $I \hookrightarrow A \hookrightarrow M^n$. Since $I$ is injective, it is a direct summand of $M^n$, hence by Krull-Remak-Schmidt, $I$ is a summand of $M$.

Definition. Let $0 \to A \to I^0 \to I^1 \to \ldots$ be the minimal injective resolution of a f.d. algebra $A$. One says that $A$ has dominant dimension $\geq n$ if $I^0, \ldots, I^{n-1}$ are all projective.

Special case. dom. dim $A \geq 1$ iff $A$ is QF-3. For example if it is QF-3, with faithful projective-injective module $M$, then there is an embedding $A \to M^n$, and then the injective envelope of $A$ is a direct summand of $M^n$, so it is projective.

Nakayama conjecture (1958). If all $I^n$ are projective, i.e. dom. dim $A = \infty$, then $A$ is self-injective.

Generalized Nakayama conjecture (Auslander and Reiten 1975). For any f.d. algebra $A$, every indecomposable injective occur as a summand of some $I^n$.

It clearly implies the Nakayama conjecture, for if the $I^n$ are projective, and each indecomposable injective occurs as a summand of some $I^n$, then the
indecomposable injectives are projective.

Example. For the commutative square, vertices 1(source), 2, 3, 4(sink). There are injective resolutions

\[ 0 \rightarrow P[1] \rightarrow I[4] \rightarrow 0, \]
\[ 0 \rightarrow P[2] \rightarrow I[4] \rightarrow I[3] \rightarrow 0, \]
\[ 0 \rightarrow P[3] \rightarrow I[4] \rightarrow I[2] \rightarrow 0, \]

so


so all indecomposable injectives occur.

Finitistic Dimension Conjecture (see H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 1960) For any f.d. algebra \( A \),

\[ \text{fin. dim } A = \sup \{ \text{proj. dim } M \mid \text{proj. dim } M < \infty \} \]

is finite.

Note that \( \text{fin. dim } A \) is not necessarily the same as the maximum of the projective dimensions of the simple modules of finite projective dimension.

Lemma 2. If \( \text{inj. dim } A_A = n < \infty \) then \( \text{fin. dim } A < \infty \) implies \( A \) Gorenstein implies \( \text{fin. dim } A = n \).

Proof. We have \( \text{proj. dim } ADA = n < \infty \). Thus any injective module has projective dimension \( < \infty \). Take a minimal injective resolution of \( 0 \rightarrow A \rightarrow I^0 \rightarrow \ldots \). We show by induction on \( i \) that \( \text{proj. dim } \Omega^{-i}A < \infty \). There is an exact sequence

\[ 0 \rightarrow \Omega^{-(i-1)}A \rightarrow I^{i-1} \rightarrow \Omega^{-i}A \rightarrow 0. \]

Applying \( \text{Hom}_A(-, X) \) for a module \( X \) gives a long exact sequence

\[ \cdots \rightarrow \text{Ext}^m(\Omega^{-(i-1)}A, X) \rightarrow \text{Ext}^{m+1}(\Omega^{-i}A, X) \rightarrow \text{Ext}^{m+1}(I^{i-1}, X) \rightarrow \ldots \]

For \( m \) sufficiently large the outside terms are zero, hence so is the middle.

Suppose \( \Omega^{-i}A \neq 0 \). Let \( f : \Omega^{-i}A \rightarrow I^i \) be the inclusion. Then \( f \) belongs to the middle term in the complex

\[ \text{Hom}(\Omega^{-i}A, I^{i-1}) \rightarrow \text{Hom}(\Omega^{-i}A, I^i) \rightarrow \text{Hom}(\Omega^{-i}A, I^{i+1}) \]
and it is sent to zero in the third term. Now \( f \) is not in the image of the map from the first term, for otherwise the map \( I^{i-1} \to \Omega^{-i}A \) is a split epimorphism, so the inclusion \( \Omega^{-(i-1)}A \to I^i \) is a split monomorphism. But the resolution was minimal, so the inclusion is an isomorphism, and \( \Omega^{-i}A = 0 \). Thus \( \Ext^i(\Omega^{-i}A) \neq 0 \). Thus \( \text{proj. dim } \Omega^{-i}A \geq i \). Thus \( \text{fin. dim } A = \infty \).

For the second implication, the theorem and Lemma 1 gives \( \text{fin. dim } A \leq n \), and consideration of \( DA \) gives equality.

**Lemma 3.** The finitistic dimension conjecture implies the Nakayama conjecture.

**Proof.** Suppose all \( I^n \) are projective. Then the syzygies \( \Omega^{-n}A \) have finite projective dimension. In particular if the finitistic dimension is \( f \), then \( M = \Omega^{-(f+1)}A \) has projective dimension \( \leq f \), so \( \Omega^fM \cong \Omega^{-1}A \) is projective, so the sequence \( 0 \to A \to I^0 \to \Omega^{-1}A \to 0 \) splits, so \( A \) is a summand of \( I^0 \), so injective.

### 1.10 Generator correspondence

I discuss what I call ‘Generator correspondence’. I learnt it from Sauter and Pressland (work in progress). They attribute it to Kato and Tachikawa. It generalizes Morita-Tachikawa correspondence, see Ringel. It is essentially equivalent to Auslander’s Wedderburn correspondence.

H. Tachikawa, On splitting of module categories, Math. Z. 1969
C. M. Ringel, Artin algebras of dominant dimension at least 2, manuscript 2007, available from his Bielefeld homepage.

We consider pairs \( (A, AM) \) consisting of an algebra and a module (both f.d.), up to an equivalence which identifies \( (A, AM) \) with \( (A', AM') \) whenever there is an equivalence of categories \( A - \text{mod} \to A' - \text{mod} \) under which \( \text{add } M \) and \( \text{add } M' \) correspond.

Given a pair \( (A, M) \) we construct a new pair \( (\End_A(M)^{op}, DM) \). We call it the **dual pair**.

**Lemma.** If \( (A, AM) \) and \( (A', AM') \) are equivalent, then so are \( (\End_A(M)^{op}, DM) \) and \( (\End_{A'}(M')^{op}, D(M')) \).
The proof is an exercise in Morita equivalence. I couldn’t find an elegant argument, so I omit it.

Applying duality twice you recover the same pair (up to equivalence) provided that the map \( A \to \text{End} \text{End}_A(M)^{\text{op}}(DM) \) is an isomorphism.

Recall that an \( A \)-module \( M \) is a generator if for every module \( X \) there is a surjection \( M^{(l)} \to X \). It is equivalent that \( A \in \text{add} M \). Dually \( M \) is a cogenerator if this is always an injection \( X \to M^l \). For \( A, M \) finite-dimensional it is equivalent that \( DA \in \text{add}(M) \).

Given an injective \( B \)-module \( U \), we say that a \( B \)-module \( Z \) has \( U \)-dom \( \dim Z \geq n \) provided there is an exact sequence

\[
0 \to Z \to U^0 \to U^1 \to \cdots \to U^{n-1}
\]

with \( U^i \in \text{add}(U) \).

Theorem (Generator correspondence). The construction of dual pairs gives a 1-1 correspondence between

(a) equivalence classes of pairs \((A, AM)\) where \( M \) is a generator, and

(b) equivalence classes of pairs \((B, BU)\) where \( U \) is injective and \( U \)-dom \( \dim B \geq 2 \).

Moreover, in this case there are inverse equivalences

\[
A \mod \xrightarrow{\text{Hom}_A(M, -)} M \otimes_B - \text{Category of } B\text{-modules with } U \text{-dom} \dim \geq 2.
\]

Proof. (1). Suppose \((A, M)\) satisfies (a) and \( B = \text{End}_A(M)^{\text{op}} \) and \( U = BDM \) are the dual pair. We need to show that \( U \) is injective, \( U \)-dom \( \dim B \geq 2 \) and the natural map \( A \to \text{End}_B(U)^{\text{op}} \) is an isomorphism.

Now \( A \in \text{add}(M) \), so \( MB \cong \text{Hom}_A(A, M) \in \text{add(Hom}_A(M, M)) = \text{add}(B) \), so \( DM \in \text{add}(DB) \) is injective.

Now \( M \) has an injective resolution starting \( 0 \to M \to I^0 \to I^1 \). Apply \( \text{Hom}_A(M, -) \) to get an exact sequence of left \( B \)-modules \( 0 \to B \to \text{Hom}_A(M, I^0) \to \text{Hom}_A(M, I^1) \). Now \( I^1 \in \text{add}(DA) \), so \( \text{Hom}_A(M, I^1) \in \text{add(\text{Hom}_A(M, DA))} = \text{add}(DM) \).
For $X, Y$ $A$-modules there is a map
\[ \text{Hom}_A(X, Y) \rightarrow \text{Hom}_B(\text{Hom}_A(Y, M), \text{Hom}_A(X, M)) \]
This is a functorial in $X$ and $Y$ and an isomorphism for $Y = M$, so also for $Y \in \text{add}(M)$, so for $Y = A$. Taking also $X = A$ this gives an isomorphism $A \rightarrow \text{End}_B(M) \cong \text{End}_B(U)^{\text{op}}$.

(2) Suppose $(B, U)$ satisfies (b) and $A = \text{End}_B(U)^{\text{op}}$ and $M = _A DU$ are the dual pair. We need to show that $M$ is a generator and the natural map $B \rightarrow \text{End}_A(M)^{\text{op}}$ is an isomorphism.

We first show the inverse equivalences. Clearly $M$ is an $A$-$B$-bimodule and it is projective as a right $B$-module.

As a right $A$-module
\[ D(M \otimes_B U) = \text{Hom}_K(M \otimes_B U, K) \cong \text{Hom}_B(U, \text{Hom}_K(M, K)) \cong \text{Hom}_B(U, U) = A \]
so $M \otimes U \cong DA$.

For any $B$-module $X$ there is a natural transformation $\phi_X : X \rightarrow \text{Hom}_A(M, M \otimes_B X)$, $\phi_X(x)(m) = m \otimes x$. This map is an isomorphism for $X = U$ since
\[ \text{Hom}_A(M, M \otimes_B U) \cong \text{Hom}_A(M, DA) \cong DM \cong U. \]
Thus $\phi_X$ is an isomorphism for $X \in \text{add}(U)$. Given a $B$-module $B X$ with $U - \text{dom. dim} X \geq 2$, we have an exact sequence
\[ 0 \rightarrow X \rightarrow U^0 \rightarrow U^1 \]
with $U^i \in \text{add}(U)$. Since $M_B$ is projective, this gives an exact sequence
\[ 0 \rightarrow M \otimes_B X \rightarrow M \otimes_B U^0 \rightarrow M \otimes_B U^1. \]
This is a sequence of $A$-modules, and applying $\text{Hom}_A(M, -)$ it gives a commutative diagram with exact rows
\[
\begin{array}{cccc}
0 & \rightarrow & X & \rightarrow & U^0 & \rightarrow & U^1 \\
\downarrow \phi_X & & \downarrow \phi_{U^0} & & \downarrow \phi_{U^1} \\
0 & \rightarrow & \text{Hom}_A(M, M \otimes_B X) & \rightarrow & \text{Hom}_A(M, M \otimes_B U^0) & \rightarrow & \text{Hom}_A(M, M \otimes_B U^1). \\
\end{array}
\]
Since $\phi_{U^0}$ and $\phi_{U^1}$ are isomorphisms, so is $\phi_X$. Thus starting from $X$, the composition of the two functors recovers $X$. 

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Given any $A$-module $Y$, there is a natural transformation $\psi_Y : M \otimes_B \text{Hom}_A(M,Y) \to Y$, $\psi_Y(m \otimes \theta) = \theta(m)$. This map is an isomorphism for $Y = DA$ since $M \otimes_B U \cong DA$. Thus $\psi_Y$ is an isomorphism for $Y$ an injective $A$-module. For general $Y$, choose an injective resolution
\[
0 \to Y \to I^0 \to I^1.
\]
Applying $\text{Hom}_A(M, -)$ gives an exact sequence
\[
0 \to \text{Hom}_A(M,Y) \to \text{Hom}_A(M,I^0) \to \text{Hom}_A(M,I^1)
\]
and $\text{Hom}(A,I^i) \in \text{add}(\text{Hom}_A(M,DA)) = \text{add}(DM) = \text{add}(U)$, so $U - \text{dom.dim} \text{Hom}_A(M,Y) \geq 2$. Also one gets a commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \to & M \otimes_B \text{Hom}_A(M,Y) & \to & M \otimes_B \text{Hom}_A(M,I^0) & \to & M \otimes_B \text{Hom}_A(M,I^1) \\
& & \psi_Y \downarrow & & \psi_{I^0} \downarrow & & \psi_{I^1} \downarrow & & \\
0 & \to & Y & \to & I^0 & \to & I^1
\end{array}
\]
Since $\psi_{I^0}$ and $\psi_{I^1}$ are isomorphisms, so is $\psi_Y$. Thus starting from $Y$, the composition of the two functors recovers $Y$.

Finally, since $U - \text{dom.dim} B \geq 2$ we can take $Y = B$, and we get
\[
B \cong \text{End}(B)^{op} \cong \text{End}_A(M \otimes_B B)^{op} \cong \text{End}_A(M)^{op}.
\]
Also $U \in \text{add}(BDB)$, so
\[
A = \text{Hom}_B(U,U) \in \text{add}(\text{Hom}_B(U,DB)) = \text{add}(DU) = \text{add}(M)
\]
so $M$ is a generator.

Special cases. Suppose $(A,M)$ corresponds to $(B,U)$ under the generator correspondence.

(i) (Morita equivalence) $M$ is also projective iff $U$ is also a cogenerator

If $M$ is projective then $B = \text{Hom}_A(M,M) \in \text{add}(\text{Hom}_A(A,M)) = \text{add}(MB)$.

If $U$ is a cogenerator then $B_B \in \text{add}(DU) = \text{add}(M)$, so $M = \text{Hom}_B(B,M) \in \text{add}(\text{Hom}_B(M,M)) = \text{add}(A)$.

(ii) (Morita-Tachikawa correspondence) $M$ is also a cogenerator iff $U$ is also projective.
If $DA \in \text{add}(M)$, then $U \cong \text{Hom}_A(M, DA) \in \text{add}(\text{Hom}_A(M, M)) = \text{add}(B)$.

If $U \in \text{add}(B)$ then $A_A = \text{Hom}_B(U, U) \in \text{add}(\text{Hom}_B(B, U)) = \text{add}(U) = \text{add}(DM)$, so $DA \in \text{add}(M)$.

Note that in this case $B$ is QF-3, and $U$ is uniquely determined up to equivalence as the direct sum of all indecomposable projective-injective modules (with nonzero multiplicities). Thus in this case we get a 1-1 correspondence:

\[
\text{Pairs } (A, M) \text{ with } M \text{ gen-cogen} \underset{\text{equiv}}{\leftrightarrow} \text{Algebras } B \text{ with } \text{dom. dim } B \geq 2 \underset{\text{Morita equiv}}{\leftrightarrow} \text{add}(U).
\]

(iii) (Müller, The classification of algebras by dominant dimension, Canad. J. Math 1968) $\text{Ext}_A^i(M, M) = 0$ for $1 \leq i < n$ iff $U = \text{dom. dim } B \geq n + 1$.

If the condition on Exts holds, then an injective resolution $0 \to M \to I^0 \to \ldots$ gives a complex

\[
0 \to \text{Hom}_A(M, M) \to \text{Hom}_A(M, I^0) \to \cdots \to \text{Hom}_A(M, I^n)
\]

which is in fact exact. Now as a left $B$-module we have $\text{Hom}_A(M, I^i) \in \text{add}(\text{Hom}_A(M, DA)) = \text{add}(U)$, so this sequence shows that $B = \text{End}_A(M)$ has $U = \text{dom. dim } \geq n + 1$.

Conversely suppose $0 \to B \to U^0 \to \cdots \to U^n$ is an exact sequence of $B$-modules with $U^i \in \text{add}(U)$. Since $M_B$ is projective we get an exact sequence

\[
0 \to M \to M \otimes_B U^0 \to \cdots \to M \otimes_B U^n.
\]

Now $M \otimes_B U^i \in \text{add}(M \otimes_B U) = \text{add}(DA)$, so this is the start of an injective resolution of $M$. Thus the cohomology of the complex

\[
\text{Hom}_A(M, M \otimes_B U^0) \to \cdots \to \text{Hom}_A(M, M \otimes_B U^n)
\]

at the term with $U^i$ is $\text{Ext}_A^i(M, M)$. But, as before, we have $U^i \cong \text{Hom}_A(M, M \otimes_B U^i)$, so this complex is exact. Thus $\text{Ext}_A^i(M, M) = 0$ for $1 \leq i < n$.

Proposition 1. The following are equivalent.

(i) The Nakayama conjecture (if $\text{dom. dim } B = \infty$ then $B$ is self-injective).

(ii) If $A_M$ is a generator-cogenerator and $\text{Ext}_A^i(M, M) = 0$ for all $i > 0$ then $M$ is projective.

Proof (i) implies (ii). Say $M$ satisfies the hypotheses. There is a corresponding $(B, U)$ with $U$ faithful projective-injective and $\text{dom. dim } B = \infty$. 

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Thus $B$ is self-injective, so $\text{add}(U) = \text{add}(B)$, so $U$ is a cogenerator, so $B$ is projective.

(ii) implies (i). Say $\text{dom.dim } B = \infty$. Thus $B$ is QF-3 and let $U$ be the faithful projective-injective module. Then there is corresponding $(A, M)$ with $M$ a generator-cogenerator and Exts vanish. Thus $M$ is projective, so $U$ is a cogenerator, so involves all indecomposable injectives, so they are all projective.

Proposition 2. The following are equivalent.

(i) The Generalized Nakayama Conjecture (every indecomposable injective occurs as a summand of some $I$ in the minimal injective resolution of $B$).

(ii) If $_AM$ is a generator and $\text{Ext}^i_A(M, M) = 0$ for all $i > 0$ then $M$ is projective.

Proof. (i) implies (ii). Suppose $M$ satisfies the conditions, then there is corresponding $(B, U)$ and $U - \text{dom.dim } B = \infty$. Thus by (i), $U$ must be a cogenerator, so $M$ is projective.

(ii) implies (i). Let $U$ be the sum of all indecomposable injectives occurring in the $I$. Then there is corresponding $(A, M)$ and Exts vanish for $M$. Thus $M$ is projective, so $U$ is a cogenerator, so all indecomposable injectives occur as a summand of $U$.

Boundedness Conjecture (Happel, Selforthogonal modules, 1995). If $M$ is an $A$-module with $\text{Ext}^n_A(M, M) = 0$ for all $n > 0$ then $\#M \leq \#A$, where $\#M$ denotes the number of non-isomorphic indecomposable summands of $M$.

This implies the GNC.

Special cases continued.

(iv) (Auslander, 1974) $\text{add}(M) = A - \text{mod}$ iff $\text{gl.dim } B \leq 2 \leq \text{dom.dim } B$ and $U$ is the faithful projective-injective.

For the condition on the left to be possible $A$ must have finite representation type, that is, only finitely many indecomposables. Then $M$ is uniquely determined up to multiplicities. Thus one gets a 1-1 correspondence

\[
\begin{array}{c}
\text{Algebras } A \text{ of finite representation type} \\
\text{Morita equiv}
\end{array} \leftrightarrow \begin{array}{c}
\text{Algebras } B \text{ with } \text{gl.dim } B \leq 2 \leq \text{dom.dim } B \\
\text{Morita equiv}
\end{array}
\]

The algebra $B$ is called the Auslander algebra of $A$.  
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Suppose \( \text{add}(M) = A \text{- mod.} \) Given a \( B \)-module \( Z \), choose a projective presentation
\[
P_1 \xrightarrow{f} P_0 \rightarrow Z \rightarrow 0.
\]
Since \( M_B \) is projective, tensoring with \( M \) gives an exact sequence
\[
0 \rightarrow M \otimes_B \text{Ker}(f) \rightarrow M \otimes_B P_1 \rightarrow M \times_B P_0 \rightarrow M \otimes_B Z \rightarrow 0.
\]
Applying \( \text{Hom}_A(M, -) \) we get a commutative diagram with exact rows
\[
\begin{array}{cccc}
0 & \rightarrow & \text{Ker}(f) & \rightarrow & P_1 & \rightarrow & P_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}(M, M \otimes \text{Ker}(f)) & \rightarrow & \text{Hom}(M, M \otimes P_1) & \rightarrow & \text{Hom}(M, M \otimes P_0)
\end{array}
\]
The two vertical maps on the right are isomorphisms, hence so is the first. Now \( M \otimes \text{Ker}(f) \) is an \( A \)-module, so in \( \text{add}(M) \), so \( \text{Ker}(f) \cong \text{Hom}_A(M, M \otimes \text{Ker}(f)) \in \text{add}(\text{Hom}_A(M, M)) = \text{add}(B) \), so it is projective. Thus \( \text{proj. dim } Z \leq 2 \). Thus \( \text{gl. dim } B \leq 2 \).

Conversely suppose \( \text{gl. dim } B \leq 2 \leq \text{dom. dim } B \). If \( Y \) is an \( A \)-module, it has an injective coresolution starting
\[
0 \rightarrow Y \rightarrow I^0 \rightarrow I^1.
\]
Applying \( \text{Hom}_A(M, -) \) we get an exact sequence of \( B \)-modules
\[
0 \rightarrow \text{Hom}_A(M, Y) \rightarrow \text{Hom}_A(M, I^0) \xrightarrow{g} \text{Hom}_A(M, I^1) \rightarrow \text{Coker}(g) \rightarrow 0
\]
and \( \text{Hom}_A(M, I^i) \in \text{add}(\text{Hom}_A(M, DA)) = \text{add}(DA) = \text{add}(U) \). Now \( BU \) is projective and \( \text{proj. dim } \text{Coker}(g) \leq 2 \), so \( \text{Hom}_A(M, Y) \) is a projective \( B \)-module. Then \( Y \cong M \otimes_B \text{Hom}_A(M, Y) \in \text{add}(M) \).

(v) (Iyama, 2007) \( M \) is a ‘\( n \)-cluster tilting module’ iff \( \text{gl. dim } B \leq n + 1 \leq \text{dom. dim } B \).

Details omitted.

### 1.11 No loops conjecture

No Loops Conjecture (Proved by Igusa 1990, based on Lenzing 1969). If \( A = KQ/I \), \( I \) admissible, and \( A \) has finite global dimension then \( Q \) has no loops (that is, \( \text{Ext}^1(S[i], S[i]) = 0 \) for all \( i \)).
Proof. We use the trace function of Hattori and Stallings. I only sketch the proof of its properties.

(1) For any matrix $\theta \in M_n(A)$ we consider its trace $\text{tr}(\theta) \in A/[A,A]$, where $[A,A]$ is the subspace of $A$ spanned by the commutators $ab-ba$. This ensures that $\text{tr}(\theta \phi) = \text{tr}(\phi \theta)$. This works also for matrices of size $m \times n$ and $n \times m$.

(2) If $P$ is a f.g. projective $A$-module it is a direct summand of a f.g. free module $F = A^n$. Let $p : F \to P$ and $i : P \to F$ be the projection and inclusion. One defines $\text{tr}(\theta)$ for $\theta \in \text{End}(P)$ to be $\text{tr}(i\theta p)$. This is well defined, for if $A^n = F' \xrightarrow{p'} P \xleftarrow{i'} F = A^m$ then $\text{tr}(i\theta p) = \text{tr}((ip')(i'\theta p)) = \text{tr}((i'\theta p)(ip')) = \text{tr}(i'\theta p')$.

(3) Any module $M$ has a finite projective resolution $P_* \to M$, and an endomorphism $\theta$ of $M$ lifts to a map between the projective resolutions

$$
\begin{array}{cccccccc}
0 & \to & P_n & \to & \ldots & \to & P_1 & \to & P_0 & \to & M & \to & 0 \\
& \theta_n & & & \theta_1 & & \theta_0 & & \theta & & \\
0 & \to & P_n & \to & \ldots & \to & P_1 & \to & P_0 & \to & M & \to & 0.
\end{array}
$$

Define $\text{tr}(\theta) = \sum_i (-1)^i \text{tr}(\theta_i)$. One can show that does not depend on the projective resolution or the lift of $\theta$.

(4) One can show that given a commutative diagram with exact rows

$$
\begin{array}{cccccccc}
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
& \theta' & & \theta & & \theta'' & & \\
0 & \to & M' & \to & M & \to & M'' & \to & 0
\end{array}
$$

one has $\text{tr}(\theta) = \text{tr}(\theta') + \text{tr}(\theta'')$.

(5) It follows that any nilpotent endomorphism has trace 0, since

$$
\begin{array}{cccccccc}
0 & \to & \text{Im } \theta & \to & M & \to & M/\text{Im } \theta & \to & 0 \\
& \theta_{|\text{Im } \theta} & & \theta & & 0 & & \\
0 & \to & \text{Im } \theta & \to & M & \to & M/\text{Im } \theta & \to & 0
\end{array}
$$

so $\text{tr}(\theta) = \text{tr}(\theta_{|\text{Im } \theta}) = \text{tr}(\theta_{|\text{Im } (\theta^2)}) = \cdots = 0$. 

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(6) Thus any element of $J(A)$ as a map $A \to A$ has trace 0, so $J(A) \subseteq [A, A]$. Thus $(KQ)_+ \subseteq I + [KQ, KQ]$.

(7) Any loop of $Q$ gives an element of $(KQ)_+$. But it is easy to see that

$$I + [KQ, KQ] \subseteq \text{span of arrows which are not loops} + (KQ)_+^2,$$

for example if $p, q$ are paths then $[p, q] \in (KQ)_+^2$ unless they are trivial paths or one is trivial and the other is an arrow. Thus there are no loops.

Strong no loops conjecture (proved by Igusa, Liu, Paquette 2011). If $S$ is a 1-dimensional simple module for a f.d. algebra and $S$ has finite injective or projective dimension, then $\text{Ext}^1(S, S) = 0$.

Extension Conjecture (Liu, Morin). If $S$ is simple module for a f.d. algebra and $\text{Ext}^1(S, S) \neq 0$ then $\text{Ext}^n(S, S) \neq 0$ for infinitely many $n$. 

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2 Auslander-Reiten Theory

From now on we work only with f.d. algebras, over an alg. closed field, and all modules are f.d.

2.1 The transpose

We write $M^\vee$ for $\text{Hom}_A(M, A)$. This defines a functor from modules on one side to modules on the other side. It gives an antiequivalence between the categories of finitely generated projective left and right $A$-modules.

Given a left (or right) module $M$, we fix a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \rightarrow 0.$$

That is, $g : P_0 \rightarrow M$ and $f : P_1 \rightarrow \text{Ker}(g)$ are projective covers. The transpose $\text{Tr} M$ is the cokernel of the map $f^\vee : P_0^\vee \rightarrow P_1^\vee$. It is a module on the other side. Thus there is an exact sequence

$$0 \rightarrow M^\vee \rightarrow P_0^\vee \rightarrow P_1^\vee \rightarrow \text{Tr} M \rightarrow 0$$

Note that $\text{Tr}$ doesn’t define a functor on the module categories.

Theorem.
(i) Up to isomorphism, $\text{Tr} M$ doesn’t depend on the choice of minimal projective presentation of $M$.
(ii) If $P$ is projective, then $\text{Tr} P = 0$.
(iii) $\text{Tr}(M \oplus N) \cong \text{Tr} M \oplus \text{Tr} N$.
(iv) If $M$ has no nonzero projective summand, the same is true for $\text{Tr} M$, and $P_0^\vee \rightarrow P_1^\vee \rightarrow \text{Tr} M \rightarrow 0$ is a minimal projective presentation.
(v) If $M$ has no nonzero projective summand then $\text{Tr} \text{Tr} M \cong M$.

Proof. Two different minimal projective presentations of $M$ fit in a commutative diagram

$$
\begin{array}{c}
P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \rightarrow 0 \\
\downarrow \quad \downarrow \quad \| \\
\end{array}
\begin{array}{c}
P'_1 \xrightarrow{f'} P'_0 \xrightarrow{g'} M \rightarrow 0 \\
\end{array}
$$

and the minimality ensures that the vertical maps are isomorphisms. Applying $(-)^\vee$, one sees that the two different constructions of $\text{tr} M$ are isomorphic.
(ii) is clear.

(iii) Straightforward since the direct sum of minimal projective presentations of $M$ and $N$ gives a minimal projective presentation of $M \oplus N$.

(iv) Suppose $Q$ is a non-zero projective summand of $\text{Tr} M$. Then there is a split epi $P^1_0 \to Q$ whose composition with $f^\vee$ is zero. Thus there is a split mono $Q^\vee \to P_1$ whose composition with $f$ is zero. Contradicts that $P_1 \to \text{Ker}(g)$ is a projective cover.

Suppose $P^1_0 \to \text{Tr} M$ is not a projective cover. Then there is a non-zero summand $Q$ of $P^1_0$ with image zero in $\text{Tr} M$. This gives a map $Q \to \text{Im}(f^\vee)$. Since $Q$ is projective and $P^0_0 \to \text{Im}(f^\vee)$ is onto, we get a map $Q \to P^0_0$ whose composition with $f^\vee$ is the inclusion of $Q$ in $P^1_0$. Thus $f$ composed with the map $P_0 \to Q^\vee$ is the projection $P_1 \to Q^\vee$. Thus $\text{Ker}(g) = \text{Im}(f)$ is not contained in $\text{rad} P_1$. Contradicts that $g : P_0 \to M$ is a projective cover.

Suppose that $P^0_0 \to \text{Im}(f^\vee)$ is not a projective cover. Then there is a non-zero summand $Q$ of $P^0_0$ whose composition with $f^\vee$ is zero. Then there is a split epimorphism $P_0 \to Q^\vee$ whose composition with $f$ is zero. This induces a split epimorphism $M \to Q^\vee$, contradicting the fact that $M$ has no non-zero projective summand.

(v) $\text{Tr Tr} M$ is the cokernel of the map $P^\vee_1 \to P^\vee_0$, that is, $P_1 \to P_0$.

Corollary. $\text{Tr}$ induces a bijection between isomorphism classes of indecomposable non-projective left and right $A$-modules.

Definition. Given modules $M, N$, we denote by $\text{Hom}^{\text{proj}}(M, N)$ the set of all maps $M \to N$ which can be factorized through a projective module $M \to P \to N$.

Clearly $\text{Hom}^{\text{proj}}(M, N)$ is a subspace of $\text{Hom}(M, N)$, for example if $\theta$ factors through $P$ and $\theta'$ factors throught $P'$ then $\theta + \theta'$ factors through $P \oplus P'$. Moreover $\text{Hom}^{\text{proj}}$ is an ideal in the module category.

We define $\text{Hom}(M, N) = \text{Hom}(M, N)/\text{Hom}^{\text{proj}}(M, N)$. These form the Hom spaces in a category, the stable module category, denoted $A\text{-mod}$.

Theorem. The transpose defines inverse anti-equivalences

$$A\text{-mod} \cong \text{mod-}A.$$
Proof. First we show that $\text{Tr}$ defines a contravariant functor from $A$-mod to mod-$A$. Any map $\theta : M \to M'$ can be lifted to a map of projective presentations

$$
P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

$$
P'_1 \xrightarrow{f'} P'_0 \xrightarrow{g'} M' \longrightarrow 0$$

Applying $(\cdot)^\vee$ there is an induced map $\phi$.

$$
P'^\vee_0 \xrightarrow{f'^\vee} P'^\vee_1 \xrightarrow{p'} \text{Tr} M' \longrightarrow 0$$

$$\theta'^\vee_0 \downarrow \quad \theta'^\vee_1 \downarrow \quad \phi \downarrow$$

$$
P'^\vee_0 \xrightarrow{f'^\vee} P'^\vee_1 \xrightarrow{p'} \text{Tr} M \longrightarrow 0$$

The map $\phi$ depends on $\theta_0$ and $\theta_1$, which are not uniquely determined. We show that any choices lead to the same element of $\text{Hom}(\text{Tr} M', \text{Tr} M)$. For this we may assume that $\theta = 0$, and need to show that $\phi$ factors through a projective.

Thus assume that $\theta$ is zero. Then $g'\theta_0 = 0$. Thus there is $h : P_0 \to P'_1$ with $\theta_0 = f'h$. This gives $h^\vee : P'^\vee_1 \to P'^\vee_0$ with $\theta'^\vee_0 = h^\vee f'^\vee$. Now we have a commutative diagram

$$
P'^\vee_0 \xrightarrow{f'^\vee} P'^\vee_1 \xrightarrow{p'} \text{Tr} M' \longrightarrow 0$$

$$\theta'^\vee_0 \downarrow \quad f'^\vee h^\vee \downarrow \quad 0 \downarrow$$

$$
P'^\vee_0 \xrightarrow{f'^\vee} P'^\vee_1 \xrightarrow{p'} \text{Tr} M \longrightarrow 0.$$ 

Taking the difference of the vertical maps, there is also a commutative diagram

$$
P'^\vee_0 \xrightarrow{f'^\vee} P'^\vee_1 \xrightarrow{p'} \text{Tr} M' \longrightarrow 0$$

$$0 \downarrow \quad \theta'^\vee_1 - f'^\vee h^\vee \downarrow \quad \phi \downarrow$$

$$
P'^\vee_0 \xrightarrow{f'^\vee} P'^\vee_1 \xrightarrow{p'} \text{Tr} M \longrightarrow 0.$$ 

But then $(\theta'^\vee_1 - f'^\vee h^\vee) f'^\vee = 0$. Thus there is a map $s : \text{Tr} M' \to P'^\vee_1$ with $\theta'^\vee_1 - f'^\vee h^\vee = sp'$. It follows that $psp' = \phi p'$, so since $p'$ is surjective, $\phi = ps$, so $\phi$ factors through a projective.

Thus a morphism $g : M \to M'$ gives a well-defined morphism $\text{Tr} g = [\phi] \in \text{Hom}(\text{Tr} M', \text{Tr} M)$. It is straightforward that this construction behaves well
on compositions of morphisms, so that the transpose defines a contravariant functor \( \text{A-mod} \) to \( \text{mod-}A \).

Now clearly the transpose sends any projective module to 0, so it sends any morphism factoring through a projective to 0, so it descends to a contravariant functor \( \text{A-mod} \) to \( \text{mod-}A \). Now it is straightforward that it is an antiequivalence.

### 2.2 Auslander-Reiten formula

**Definition.** We define \( \text{A-mod} \) as the category with Hom spaces

\[
\overline{\text{Hom}}(M, N) = \text{Hom}(M, N)/\text{Hom}^{\text{inj}}(M, N)
\]

where \( \text{Hom}^{\text{inj}}(M, N) \) is the maps factoring through an injective module.

**Lemma 1.** \( \text{Hom}(M, N) \cong \overline{\text{Hom}}(DN, DM) \), so \( D \) gives an antiequivalence between \( \text{mod-}A \) and \( \text{A-mod} \).

**Proof.** Straightforward.

**Definition.** The *Auslander-Reiten translate* is \( \tau = D \text{Tr} \) and the inverse construction is \( \tau^{-} = \text{Tr} D \).

By the results of the previous section we have inverse bijections

\[
\text{non-projective indec mods/iso} \xrightarrow{\tau} \text{non-injective indec mods/iso}
\]

and inverse equivalences

\[
\text{A-mod} \xrightarrow{\tau} \overline{\text{A-mod}}.
\]

Applying \( D \) to the exact sequence defining \( \text{Tr} M \), we see that there is an exact sequence

\[
0 \to \tau M \to \nu(P_{1}) \to \nu(P_{0}) \to \nu(M) \to 0.
\]

Thus \( \tau \) can be computed by taking a minimal projective presentation of \( M \), applying the Nakayama functor (which turns each \( P[i] \) into \( I[i] \)) and taking the kernel.
Example. For the commutative square with source 1 and sink 4, the simple $S[2]$ has minimal projective presentation


so we get

$$0 \to \tau S[2] \to I[4] \to I[2]$$


Lemma 2. If $M$ is an $A$-module, then

(i) $\text{proj. dim } M \leq 1 \iff \text{Hom}(DA, \tau M) = 0 \iff$ there is no non-zero map from an injective module to $\tau M$.

(ii) $\text{inj. dim } M \leq 1 \iff \text{Hom}(\tau^{-} M, A) = 0 \iff$ there is no non-zero map from $\tau^{-} M$ to a projective module.

Proof. (i) Recall that $\nu^{-}(-) = \text{Hom}(DA, -)$, and that $\nu^{-}(\nu(P)) \cong P$. Thus we get $0 \to \nu^{-}(\tau M) \to \nu^{-}(\nu(P_1)) \to \nu^{-}(\nu(P_0))$ exact, so $0 \to \nu^{-}(\tau M) \to P_1 \to P_0$. Thus proj. dim $M \leq 1$ iff $P_1 \to P_0$ is injective iff $\nu^{-}(\tau M) = 0$ iff $\text{Hom}(DA, \tau M) = 0$.

(ii) Dual.

Lemma 3. Given a right $A$-module $M$, a left $A$-module $N$, $m \in M$ and $n \in N$ let $f_{mn} : M^\vee \to N$ be the map defined by $f_{mn}(\alpha) = \alpha(m)n$. It is a left $A$-module map. There is a natural transformation

$$\theta_{MN} : D \text{Hom}(M^\vee, N) \to \text{Hom}(M, DN), \quad \theta_{MN}(\xi) = (m \mapsto (n \mapsto \xi(f_{mn}))).$$

Then $\theta_{MN}$ is an isomorphism for $M$ projective. And in general the image of $\theta_{MN}$ is $\text{Hom}^{\text{proj}}(M, DN)$.

Proof. The first part is clear. Clearly $\theta_{MN}$ is well-defined. Both $D \text{Hom}(M^\vee, N)$ and $\text{Hom}(M, DN)$ define functors which are contravariant in $M$ and $N$, and it is straightforward that $\theta_{MN}$ is natural in $M$ and $N$.

For $M$ projective, the map is an isomorphism, since it is for $M = A$. Thus given a map $f : M \to P$ with $P$ projective, we get a commutative diagram

$$\begin{array}{ccc}
D \text{Hom}(P^\vee, N) & \longrightarrow & \text{Hom}(P, DN) \\
\downarrow b & & \downarrow a \\
D \text{Hom}(M^\vee, N) & \xrightarrow{\theta_{MN}} & \text{Hom}(M, DN)
\end{array}$$

Any map $M \to DN$ factoring through $P$ is in the image of $a$, so in $\text{Im}(\theta)$. 37
Varying $P$, we get $\text{Hom}^{\text{proj}}(M, DN) \subseteq \text{Im}(\theta)$.

Now take a basis of $M^\vee$. This defines a map $M \to P$, where $P = A^n$. Then $P^\vee \to M^\vee$ is onto. Thus $\text{Hom}(M^\vee, N) \to \text{Hom}(P^\vee, N)$ is 1-1. Thus $b$ is onto. Thus $\text{Im}(\theta) = \text{Im}(a) \subseteq \text{Hom}^{\text{proj}}(M, DN)$.

Theorem. There are isomorphisms

$$\text{Hom}(\tau^- N, M) \cong D \text{Ext}^1(M, N) \cong \overline{\text{Hom}}(N, \tau M).$$

Proof. Given a minimal projective presentation $P_1 \to P_0 \to M \to 0$, write $\Omega^1 M$ for the image of $P_1 \to P_0$, so there is

$$0 \to \Omega^1 M \to P_0 \to M \to 0$$

and hence

$$0 \to \text{Hom}(M, N) \to \text{Hom}(P_0, N) \to \text{Hom}(\Omega^1 M, N) \to \text{Ext}^1(M, N) \to 0.$$

Also we have

$$0 \to M^\vee \to P_0^\vee \to P_1^\vee \to \text{Tr} M \to 0$$

so

$$0 \to (\text{Tr} M)^\vee \to P_1 \to P_0$$

so

$$0 \to (\text{Tr} M)^\vee \to P_1 \to \Omega^1 M \to 0.$$

and hence

$$0 \to \text{Hom}(\Omega^1 M, N) \to \text{Hom}(P_1, N) \to \text{Hom}((\text{Tr} M)^\vee, N).$$
Thus we have a commutative diagram with exact rows and columns,

\[
\begin{array}{ccccccc}
0 & \rightarrow & D \text{Ext}^1(M, N) & \rightarrow & D \text{Hom}((\text{Tr} M)^{\vee}, N) & \rightarrow & D \text{Hom}(P_1, N) & \rightarrow & D \text{Hom}(\Omega^1 M, N) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Hom}(\text{Tr} M, DN) & \rightarrow & \text{Hom}(P_{\text{iv}}^{\vee}, DN) & \rightarrow & \text{Hom}(P_{\text{iv}}^{\vee}, DN) & \rightarrow & \text{Hom}(\text{Tr} M, DN) & \rightarrow & 0 \\
\end{array}
\]

The argument of the snake lemma then gives an isomorphism \( D \text{Ext}^1(M, N) \rightarrow \text{Hom}(\text{Tr} M, DN) \).

Now use that Lemma 1 to rewrite this as \( \text{Hom}(N, D \text{Tr} M) \), or use that \( \text{Tr} \) gives inverse anti-equivalences between \( A\text{-mod} \) and \( \text{mod}-A \) to rewrite it as \( \text{Hom}(M, \text{Tr} DN) \).

Corollary. If \( A \) is hereditary, we get
\[
\text{Hom}(\tau^{-}N, M) \cong D \text{Ext}^1(M, N) \cong \text{Hom}(N, \tau M).
\]

Proof. Use Lemma 2. We have \( \text{Hom}(\tau^{-}N, M) \cong \text{Hom}(\tau^{-}N, M) \) if inj. dim \( N \leq 1 \), and \( \text{Hom}(N, \tau M) \cong \text{Hom}(N, \tau M) \) if proj. dim \( M \leq 1 \).

### 2.3 Auslander-Reiten sequences

Definition. By an Auslander-Reiten sequence or almost split sequence we mean an exact sequence
\[
0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0
\]
satisfying
(i) It is not split.
(ii) $X$ and $Z$ are indecomposable
(iii) Every map $M \to Z$, which is not a split epi, factors through $Y \to Z$.
(iv) Every map $X \to N$, which is not a split mono, factors through $X \to Y$.

Proposition. An Auslander-Reiten sequence, if it exists, is determined up to isomorphism by either of the end terms. That is, if

$$0 \to X' \to Y' \to Z \to 0$$

is another Auslander-Reiten sequence ending at $Z$, there is a commutative diagram in which the vertical maps are isomorphisms

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0 \\
\end{array}
$$

and dually for another Auslander-Reiten sequence starting with $X$.

Proof. By assumption there is a map $Y \to Y'$. Dually there is a map $Y' \to Y$. These induce maps $X \to X'$ and $X' \to X$. If the composition $X \to X$ isn’t an isomorphism, then it is nilpotent, so some power is zero. But then the sequence is split. But then taking $\theta$ to be the corresponding power of the map $Y \to Y$ we get

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \theta & \longrightarrow & Z & \longrightarrow & 0 \\
0 & \downarrow & \vphantom{\theta} & \downarrow & \theta & \vphantom{\downarrow} & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \theta & \longrightarrow & Z & \longrightarrow & 0 \\
\end{array}
$$

and this can only happen if the sequence is split, for $\theta$ factors as $hg$ for some $h : Z \to Y$, so $ghg = g\theta = g1$, so since $g$ is onto, $gh = 1$.

Theorem. Let $Z$ be an non-projective indecomposable $A$-module and let $X = \tau Z$ be the corresponding non-injective indecomposable module. (Or equivalently let $X$ be non-injective indecomposable and let $Z = \tau^{-1}X$.) Then there exists an Auslander-Reiten sequence

$$0 \to X \to Y \to Z \to 0.$$

Proof. $\text{Ext}^1(Z, X)$ is an $\text{End}(X)$-$\text{End}(Z)$-bimodule.

First we consider it as a right $\text{End}(Z)$-module. Since $Z$ is indecomposable, $\text{End}(Z)$ is a local ring. Since $Z$ is not projective $\text{End}^{\text{proj}}(Z)$ is contained in
the maximal ideal of \( \text{End}(Z) \). Thus \( \text{End}(Z) \) has simple top as a left \( \text{End}(Z) \)-module. Thus \( D\text{End}(Z) \) has simple socle as a right \( \text{End}(Z) \)-module. Thus by the AR formula \( \text{Ext}^1(Z,X) \) has simple socle \( S \) as a right \( \text{End}(Z) \)-module. Since \( K \) is algebraically closed, \( \dim S = 1 \).

Now we consider \( \text{Ext}^1(Z,X) \) as a left \( \text{End}(X) \)-module. By the same argument it has simple socle \( T \) as a left \( \text{End}(X) \)-module.

But if \( U \) is an \( A \)-\( B \)-bimodule, then any endomorphism of \( U_B \) sends \( \text{soc}(U_B) \) into itself. Thus \( \text{soc}(U_B) \) is an \( A \)-submodule of \( U \). Thus \( S \) is 1-dimensional \( \text{End}(X) \)-submodule of \( \text{Ext}^1(Z,X) \), so \( S = T \).

Let

\[
\xi : 0 \to X \to Y \to Z \to 0
\]

be an exact sequence corresponding to a non-zero element of \( S \).

(i) Since \( \xi \neq 0 \). (ii) Trivial.

(iii) Suppose \( M \to Z \) not a split epi.

The map \( \text{Hom}(Z,M) \to \text{End}(Z) \) has image contained in the radical of \( \text{End}(Z) \).

Thus the map \( \text{Hom}(Z,M) \to \text{End}(Z) \) has image contained in the radical of \( \text{End}(Z) \).

Thus the map \( D\text{End}(Z) \to D\text{Hom}(Z,M) \) kills the socle of \( D\text{End}(Z) \) as a \( \text{End}(Z) \)-module.

Thus the map \( \text{Ext}^1(Z,X) \to \text{Ext}^1(M,X) \) kills \( \xi \). Thus the pullback of \( \xi \) by \( M \to Z \) splits.

(iv) By duality.

### 2.4 Irreducible maps

Recall that we have defined \( \text{rad}(M,N) \subseteq \text{Hom}(M,N) \). If \( M \) is indecomposable it is the set of maps which are not split monos. If \( N \) is indecomposable it is the set of maps which are not split epis. If \( M \) and \( N \) are indecomposable it is the set of non-isomorphisms.

Definition. Given modules \( M, N \) we define \( \text{rad}^2(M,N) \) to be the set of all
homomorphisms $M \to N$ which can be written as a composition

$$M \xrightarrow{f} X \xrightarrow{g} N$$

with $f \in \text{rad}(M, X)$ and $g \in \text{rad}(X, N)$. This is a subspace of $\text{rad}(M, N)$.

Suppose that $M$ and $N$ are indecomposable. We say that a map $M \to N$ is irreducible if it is in $\text{rad}(M, N)$, but not in $\text{rad}^2(M, N)$. It is equivalent that it is not an isomorphism, and whenever it factorizes as $gf$, either $f$ is a split mono, or $g$ is a split epi.

We define the multiplicity of irreducible maps from $M$ to $N$ to be

$$\operatorname{irr}(M, N) = \dim [\text{rad}(M, N) / \text{rad}^2(M, N)].$$

Recall from the section on the Krull-Remak-Schmidt Theorem that the multiplicity of $M$ as a direct summand of a module $U$ is

$$\mu_M(U) = \dim \frac{\text{Hom}(M, U)}{\text{rad}(M, U)}.$$

Recall that the indecomposable projective $P[i]$ has submodule $\text{rad} P[i]$ with $P[i]/\text{rad} P[i] \cong S[i]$, and that the indecomposable injective $I[i]$ has socle $S[i]$.

**Theorem.** Let $M$ be indecomposable and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an AR sequence.

(i) $\operatorname{irr}(M, P[i]) = \mu_M(\text{rad} P[i])$, the number of copies of $M$ in the decomposition of $\text{rad} P[i]$.

(ii) $\operatorname{irr}(M, Z) = \mu_M(Y)$, the number of copies of $M$ in the decomposition of the middle term of the AR sequence ending at $Z$.

(i') $\operatorname{irr}(I[i], M) = \mu_M(I[i]/S[i])$, the number of copies of $M$ in the decomposition of $I[i]/S[i]$.

(ii') $\operatorname{irr}(X, M) = \mu_M(Y)$, the number of copies of $M$ in the decomposition of the middle term of the AR sequence starting at $X$.

**Proof.** (i) If $f : M \to P[i]$ is not an isomorphism, then it can't be onto, so it maps into $\text{rad} P[i]$. Thus composition with the inclusion $\text{rad} P[i] \to P[i]$ induces an isomorphism $\text{Hom}(M, \text{rad} P[i]) \to \text{rad}(M, P[i])$. This restricts to an isomorphism $\text{rad}(M, \text{rad} P[i]) \to \text{rad}^2(M, P[i])$. Thus

$$\operatorname{irr}(M, P[i]) = \dim [\text{Hom}(M, \text{rad} P[i])/ \text{rad}(M, \text{rad} P[i])] = \mu_M(\text{rad} P[i]).$$
(ii) Using that \( f \) and \( g \) are radical homomorphisms, one gets left exact sequences

\[
0 \to \text{Hom}(M, X) \to \text{Hom}(M, Y) \to \text{rad}(M, Z)
\]

and

\[
0 \to \text{Hom}(M, X) \to \text{rad}(M, Y) \to \text{rad}^2(M, Z).
\]

Now both of these sequences are actually right exact by the AR property. For example any map \( \theta \in \text{rad}^2(M, Z) \) factorizes as \( \theta = \psi\phi \) with \( \phi \in \text{rad}(M, U) \) and \( \psi \in \text{rad}(U, Z) \). But then \( \phi \) factors as \( g\chi \) for some \( \chi \in \text{Hom}(U, Y) \), and then \( \theta = g(\chi\phi) \), and \( \chi\phi \in \text{rad}(M, Y) \). Thus

\[
\text{irr}(M, Z) = \dim[\text{Hom}(M, Y)/\text{rad}(M, Y)] = \mu_M(Y).
\]

Corollary If \( X \) is indecomposable, then

\[
-\dim X + \sum_M \text{irr}(X, M) \dim M = \begin{cases} 
-\dim S[i] & (M \cong I[i]) \\
\dim \tau^{-}X & (M \text{ not injective})
\end{cases}
\]

where the sum is over all indecomposable modules up to isomorphism. Moreover if \( A = KQ/I \) then the same applies for dimension vectors.

### 2.5 Auslander-Reiten quiver

Definition. Given a f.d. algebra \( A \), the **Auslander-Reiten quiver** of \( A \) has vertices corresponding to the isomorphism classes of indecomposable \( A \)-modules, and the number of arrows \( M \to N \) is \( \text{irr}(M, N) \).

It is often useful to indicate the AR translate \( \tau \) by a dotted line joining \( Z \) and \( \tau Z = D\text{Tr}Z \).

In general the AR quiver is not connected. It is finite iff the algebra has only finitely many indecomposable modules, that is, it has finite representation type.

Examples.
Harada-Sai Lemma. A composition of \(2^n - 1\) non-isomorphisms between indecomposables of dimension \(\leq n\) must be zero.
Proof. We show for $m \leq n$ that a composition of $2^m - 1$ non-isomorphisms between indecomposables of dimension $\leq n$ has rank $\leq n - m$.

If $m = 1$ this is clear. If $m > 1$, a composition of $2^m - 1$ non-isomorphisms can be written as a composition

$$ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W $$

where $f$ and $h$ are compositions of $2^{m-1} - 1$ non-isomorphisms. By induction rank $f, \text{rank } h \leq n - m + 1$. If either has strictly smaller rank, we’re done.

Thus suppose that rank $f = \text{rank } h = \text{rank } hgf = n - m + 1$.

This implies that $\text{Ker } f = \text{Ker } hgf$ and $\text{Im } hgf = \text{Im } h$. It follows that $Y = \text{Ker } hg \oplus \text{Im } f$ and $Z = \text{Ker } h \oplus \text{Im } gf$. For example if $y \in Y$ then $hg(y) = hgf(x)$, so $y = f(x) + (y - f(x)) \in \text{Im } f + \text{Ker } hg$, and if $y \in \text{Im } f \cap \text{Ker } hg$ then $y = f(x)$ and $hgf(x) = 0$, so $x \in \text{Ker } hgf = \text{Ker } f$, so $y = 0$.

By indecomposability $f$ is onto and $h$ is 1-1, but then $g$ is an isomorphism. Contradiction.

Theorem (Auslander). If $C$ is a connected component of the AR quiver containing modules of bounded dimension, then there are no nonzero maps between indecomposables in $C$ and indecomposables not in $C$.

If in addition $A = KQ/I$ with $I$ admissible and $Q$ connected, then $C$ is the whole of the AR quiver of $A$.

Proof. By duality, suppose that $\theta : M \rightarrow Z$ is a nonzero map with $M$ not in $C$ and $Z$ in $C$.

If $Z$ is projective, then $\theta$ maps into rad $Z$, and so there is an indecomposable summand $Z'$ of $\text{rad } Z$ and maps $M \rightarrow Z' \rightarrow Z$ with nonzero composition.

If $Z$ is non-projective, with AR sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ then $\theta$ factors through $Y$, and there is an indecomposable summand $Z'$ of $Y$ and maps $M \rightarrow Z' \rightarrow Z$ with nonzero composition.

Now repeat with $Z'$ to get maps $M \rightarrow Z'' \rightarrow Z'$ with non-zero composition.

Get contradiction.

Now suppose $A = KQ/I$ with $Q$ connected. Let $X \in C$. 45
Choose any vertex \( i \) with \( e_i X \neq 0 \). Then there is a nonzero map \( P[i] \to X \), so \( P[i] \in C \).

Now every \( P[j] \in C \), for if there is an arrow between \( i \) and \( j \) then there is a nonzero map between \( P[i] \) and \( P[j] \).

Finally, if \( Y \) is any indecomposable there is a map from some \( P[j] \) to \( Y \), so \( Y \in C \).

### 2.6 Knitting construction

We work with an algebra \( A = KQ/I \).

**Preparation.** Compute the modules \( \text{rad} P[i] \) and decompose into indecomposable summands.

**Iterative construction.** We suppose we have drawn a subquiver of the AR quiver with the property that if an indecomposable module \( X \) is in the subquiver, then so are all arrows ending at \( X \), and suppose we know the dimension vectors of the modules we have drawn.

We start with the empty subquiver.

If we have drawn all the summands of \( \text{rad} P[i] \), but haven’t yet drawn \( P[i] \), we can now draw \( P[i] \) and fill in the arrows ending at \( P[i] \) with their multiplicities. In particular we can start by drawing the simple projective modules.

Suppose we have drawn an indecomposable module \( X \). If we have drawn all projectives \( P[i] \) such that \( X \) is a summand of \( \text{rad} P[i] \), and if we have drawn \( \tau^- U \) for all non-injective indecomposables \( U \) with an arrow \( U \to X \), then we can be sure that we have drawn all arrows starting at \( U \).

If we have drawn all arrows starting at \( X \) then

\[
- \dim X + \sum_M \text{irr}(X, M) \dim M = \begin{cases} 
- \dim S[i] & (X \cong I[i]) \\
\dim \tau^- X & (X \text{ not injective}) 
\end{cases}
\]

so we know whether or not \( X \) is injective by the sign of the left hand side. If it is not injective, and we haven’t yet drawn \( \tau^- X \), we can now do so, and draw \( \text{irr}(M, \tau^- X) = \text{irr}(X, M) \) arrows from \( M \) to \( X \), for all \( M \).

Repeat.
Several possibilities. (a) Get stuck, because either there is no simple projective, or we have written down some summands of rad $P[i]$, for some projective $P$, but can’t write down all summands, so can’t write down $P[i]$.

(b) Terminate after a finite number of steps. By Auslander’s Theorem we have the whole AR quiver.

(c) Go on forever. In this case we have constructed one or more connected components of the AR quiver, called ‘preprojective’ components.

Examples.

Commutative square

A3

A3 with another orientation

D4
D4 with zero relation

E6

4-subspace,

Kronecker quiver,

Example with decomposable radical, etc.
Maybe one gets stuck.

Dually construct preinjective components starting with simple injective.

2.7 Graded modules

The knitting procedure fails for many algebras. But a tool called 'covering theory' can often be used to make it work. By Gordon and Green (1982) it is essentially equivalent to study graded modules.

Recall that $A$ is a $\mathbb{Z}$-graded algebra if

$$A = \bigoplus_{n \in \mathbb{Z}} A_n, \quad A_n A_m \subseteq A_{n+m}.$$ 

It follows that $1 \in A_0$. We assume that $A$ is f.d.. Thus only finitely many $A_n$ are nonzero.

Theorem 1. $A$ is local iff $A_0$ is local.

Proof. Suppose $A$ is local. If $I$ is a proper left ideal in $A_0$ then $AI \subseteq J(A)$ since it is a left ideal in $A$, and if $a \in A$ and $i \in I$, then $i$ is not invertible in $A_0$, so not in $A$, so $ai$ is not invertible, so it is in $J(A)$. Thus $I \subseteq J(A) \cap A_0$. Thus $J(A) \cap A_0$ is the unique maximal left ideal in $A_0$.

Now suppose that $A_0$ is local.
Let \( I = \sum_{n \neq 0} A_n A_{-n} \subseteq A_0 \).

If \( a \in A_n \) and \( b \in A_{-n} \) with \( n \neq 0 \), then \( a \) is nilpotent, so not invertible, so \( ab \) is not invertible in \( A \), so it is not invertible in \( A_0 \), so \( ab \in J(A_0) \). Thus \( I \subseteq J(A_0) \).

Thus \( I \) is nilpotent. Say \( I^N = 0 \).

Let \( L \) be the ideal in \( A \) generated by all \( A_n \) (\( n \neq 0 \)). Clearly \( L = I \oplus \bigoplus_{n \neq 0} A_n \).

It suffices to show that \( L \) is nilpotent, for then \( L \subseteq J(A) \), so that \( A/J(A) \) is a quotient of \( A/L \cong A_0/I \), which is local.

Suppose that \( A \) lives in \( d \) different degrees.

It suffices to show that any product \( \ell_1 \ell_2 \cdots \ell_{dN} \) of homogeneous elements of \( L \) is zero.

Suppose not. Let \( d_i \) be the degree of \( \ell_1 \ell_2 \cdots \ell_i \).

We have \( dN + 1 \) numbers \( d_0, d_1, \ldots, d_{dN} \) taking at most \( d \) different values, so some value must occur at least \( N + 1 \) times. Say

\[
d_{i_1} = d_{i_2} = \cdots = d_{i_{N+1}}
\]

with \( i_1 < i_2 < \cdots < i_{N+1} \). Then we can write the product as

\[
\ell_1 \cdots \ell_{i_1}(\ell_{i_1+1} \cdots \ell_{i_2})(\ell_{i_2+1} \cdots \ell_{i_3}) \cdots (\ell_{i_N+1} \cdots \ell_{i_{N+1}})\ell_{i_{N+1}+1} \cdots \ell_{dN}
\]

But each of the bracketed terms has degree 0, so is in \( I \), so their product is zero.

Definition. Recall that a \( \mathbb{Z} \)-graded \( A \)-module is an \( A \)-module

\[
M = \bigoplus_{n \in \mathbb{Z}} M_n, \quad A_n \cdot M_m \subseteq M_{n+m}.
\]

We only consider f.d. graded modules, and write \( A\text{-grmod} \) for the category of f.d. \( \mathbb{Z} \)-graded \( A \)-modules, with

\[
\text{Hom}_{A\text{-grmod}}(M, N) = \{ \theta \in \text{Hom}_A(M, N) \mid \theta(M_n) \subseteq N_n \text{ for all } n \in \mathbb{Z} \}.
\]

Given a graded module \( M \) and \( i \in \mathbb{Z} \) we write \( M(i) \) for the module with shifted grading \( M(i)_n = M_{i+n} \). There is a functor \( F \) from \( A\text{-grmod} \) to \( A\text{-mod} \) which forgets the grading.
Lemma 1. If $M, N$ are graded $A$-modules, then $\text{Hom}_A(FM, FN)$ can be graded,

$$\text{Hom}_A(FM, FN) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{A-grmod}}(M, N(n)).$$

In this way

$$\text{End}_A(FM) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{A-grmod}}(M, M(n))$$

becomes a graded algebra.

Proof. Given a homomorphism $\theta : FM \to FN$, we get linear maps $\theta_n : M \to N$ defined by

$$\theta_n(m) = \sum_{i \in \mathbb{Z}} \theta(m_i)_{i+n}$$

where a subscript $k$ applied to an element of a graded module picks out the degree $k$ component of the element.

Now if $a \in A$ is homogeneous of degree $d$, then $(am)_i = a.m_{i-d}$, so

$$\theta_n(am) = \sum_i \theta((am)_i)_{i+n} = \sum_i \theta(a.m_{i-d})_{i+n} = \sum_i (a\theta(m_{i-d}))_{i+n}$$

$$= \sum_i a.\theta(m_{i-d})_{i+n-d} = \sum_j a.\theta(m_j)_{j+n} = a\theta_n(m).$$

Thus $\theta_n \in \text{Hom}_{\text{A-grmod}}(M, N(n))$. Clearly $\theta$ is the sum of the $\theta_n$. The rest is clear.

Corollary. (i) A graded module $M$ is indecomposable iff the ungraded module $FM$ is indecomposable.

(ii) If $M$ and $N$ are indecomposable graded modules with $FM \cong FN$ then $M$ is isomorphic to $N(i)$ for some $i$.

Proof. (i) By Theorem 1, $\text{End}_A(FM)$ is local iff its degree zero part is local. This is $\text{End}_A(FM)_0 = \text{End}_{\text{A-grmod}}(M)$. Now the ungraded module $FM$ is indecomposable iff its endomorphism algebra $\text{End}_A(FM)$ is local. The graded module $M$ is indecomposable iff its endomorphism algebra $\text{End}_{\text{A-grmod}}(M)$ has no non-trivial idempotents, and since it is f.d., it is equivalent that it is local.

(ii) Suppose $\theta : FM \to FN$ is an isomorphism. Then $\theta^{-1} = 1_{FM}$, so $(\theta^{-1})_0 = 1_M$, so $\sum_i (\theta^{-1})_{-i} \theta_i = 1_M$. Since $\text{End}(M)$ is local, some $(\theta^{-1})_{-i} \theta_i$ is
invertible, so $\theta_i : M \to N(i)$ is a split mono of graded modules, and hence also an isomorphism.

Setup. Let $A = KQ/I$ be a f.d. algebra. Suppose that $A$ is graded in such a way that the trivial paths have degree 0 and the arrows are homogeneous. (Recall that this defines a grading of $KQ$, and one just needs to check that the generators of $I$ are homogeneous.)

Graded $A$-modules correspond to representations of an infinite quiver with relations. The vertex set is $Q_0 \times \mathbb{Z}$. Given a module $X$, the vector space at vertex $(i,n)$ is $e_i X_n$.

We discussed this before - it is modules for an algebra with enough idempotents.

Now truncate this: given an integer range $[n,m] = \{n, n+1, \ldots, m\}$, the graded modules living in degrees $[n,m]$ correspond to representations of a finite quiver with relations, with vertex set $Q_0 \times [n,m]$. Let $\tilde{A}$ be the corresponding algebra.

We write $F$ also for the functor from $\tilde{A}$-mod to $A$-mod.

We suppose that $A$ lives in non-negative degrees, so since it is f.d., it lives in degrees $[0,d]$.

Example. Vertices 1,2. Loops $p, r$ at 1,2. Arrow $q : 1 \to 2$ drawn going down. Relations $p^2 = r^2 = 0$, $qp = rq$. $\deg p = \deg r = 1$, $\deg q = 0$. The algebra lives in degrees $[0,1]$. etc.

Theorem 2. If $\xi : 0 \to X \to Y \to Z \to 0$ is an AR sequence of $\tilde{A}$-modules, and $Z$ lives in degrees $[n+d,m-2d]$, then $F(\xi)$ is an AR sequence of $A$-modules.

Sketch. The trivial idempotents $e_i \in A$ are homogeneous of degree 0, so the module $P_A[i] = Ae_i$ is graded, and lives in degrees $[0,d]$.

Thus the module $P_A[i](-j)$ lives in degrees $[j,j+d]$. Thus if $j \in [m,n-d]$ then $P_A[i](-j)$ corresponds to an $\tilde{A}$-module. In fact it corresponds to $P_{\tilde{A}}[(i,j)]$. Thus $F(P_{\tilde{A}}[(i,j)]) \cong P_A[i]$.

Similarly, if $j \in [m+d,n]$ then $F(I_{\tilde{A}}[(i,j)]) \cong I_A[i]$. 

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Take a minimal projective presentation

\[ P_1 \to P_0 \to Z \to 0. \]

Now \( P_0 \) only involves projective covers of simples in degrees \([n + d, m - 2d]\), so \( P_0 \) lives in degrees \([n + d, m - d]\). Then \( P_1 \) only involves projective covers of simples in degrees \([n + d, m - d]\). It follows that \( F(P_i) \) are projective \( A \)-modules, and that

\[ F(P_1) \to F(P_0) \to F(Z) \to 0 \]

is a minimal projective presentation of \( F(Z) \).

Now \( \tau_{\tilde{A}}Z \) is computed using the exact sequence

\[ 0 \to \tau_{\tilde{A}}Z \to \nu_{A}(P_1) \to \nu_{A}(P_0). \]

Since the modules \( \nu_{A}(P_i) \) only involve injective envelopes of simples in degrees \([n + d, m - d]\), \( F(\nu_{A}(E_i)) \) is injective, and isomorphic to \( \nu_{A}(F(E_i)) \). Thus

\[ 0 \to F(\tau_{\tilde{A}}Z) \to F(\nu_{A}(P_1)) \to F(\nu_{A}(P_0)), \]

is identified with the sequence

\[ 0 \to \tau_{A}F(Z) \to \nu_{A}(F(P_1)) \to \nu_{A}(F(P_0)). \]

Thus \( \tau_{A}F(Z) \cong F(\tau_{\tilde{A}}Z) \cong F(X) \).

Now \( \text{End}_{\tilde{A}}(Z) \to \text{End}_{A}(F(Z)) \) as the degree 0 part.

This induces a map \( \text{End}_{\tilde{A}}(Z) \to \text{End}_{A}(F(Z)). \)

This gives \( D\text{End}_{A}(F(Z)) \to D\text{End}_{\tilde{A}}(Z). \)

Thus \( \text{Ext}^1_{\tilde{A}}(F(Z), F(X)) \to \text{Ext}^1_{A}(Z, X). \)

The AR sequences are defined by 1-dimensional subspaces, and one needs to check that these correspond.

Construction. Take a range of degrees \([n, m]\) with \( m = 0 \) and \( n \ll 0 \) and knit.

If, eventually the knitted modules live in degrees \( \leq -2d \), then the subsequent AR sequences forget to AR sequences of \( A \)-modules.
If also the knitted modules are eventually all shifts of finitely many, then this gives a finite connected component of the AR quiver. By Auslander’s Theorem it is the whole AR quiver.

Examples.

You only keep the piece between the vertical arrows, and identify them, to get a Möbius band. But in the lecture I drew the algebra $\tilde{A}$ with the different graded pieces going horizontally, not vertically.
Here identify again. But I also drew this differently.

In the case when this process works, every module is gradeable. In general that is not true.

For example the quiver with arrows from 1 to 2 and 3, and from 2 to 3. Grade it with the arrow from 1 to 3 of degree 1 and the others of degree 0. Then the module which is $K$ at each vertex, identity for each arrow is not gradeable.

Another example, $Q$ with one vertex and loops $p,q$ with relations $p^2 = qpq, q^2 = pqp, p^3 = q^3 = 0$. There is no non-trivial grading, so can't get started.

Theorem 3. If the field $K$ has characteristic zero, and $A$ is graded, then any $A$-module $M$ with $\text{Ext}^1(M,M) = 0$ is gradeable.

Proof. The result is probably folklore, but this proof comes from Keller, Murfet and van den Bergh, On two examples by Iyama and Yoshino.

Let $d : A \to A$ be the map defined by $d(a) = \deg(a)a$ for $a$ homogeneous. It is a derivation since $d(ab) = \deg(ab)ab = (\deg(a) + \deg(b))ab = ad(b) + d(a)b$.

It is called the Euler derivation.

Let $E = M \oplus M$ as a vector space, with $A$-module action given by $a(m,m') = $
\((am, d(a)m + am')\). This is an \(A\)-module structure and there is an exact sequence
\[
0 \to M \xrightarrow{(a)} E \xrightarrow{(10)} M \to 0
\]
By assumption this is split, so there is a map \(M \to E\) of the form \(m \mapsto (m, \nabla(m))\). Moreover the map \(\nabla : M \to M\) satisfies
\[
\nabla(am) = d(a)m + a\nabla(m)
\]
so it is a \textit{connection on} \(M\) \textit{with respect to} \(d\). Since \(M\) is f.d.,
\[
M = \bigoplus_{\lambda \in K} M_{\lambda}
\]
where \(M_{\lambda}\) is the \(\lambda\)-generalised eigenspace for \(\nabla\). Now for any \(\lambda \in K\) and \(a\) homogeneous we have
\[
(\nabla - \lambda - \deg(a))^N(am) = a(\nabla - \lambda)^N(m)
\]
for all \(N \geq 1\), so \(a(M_\lambda) \subseteq M_{\lambda+\deg(a)}\). Thus if we let \(T\) be a set of coset representatives for \(Z\) as a subgroup of \(K\) under addition, and set
\[
M_n = \bigoplus_{\lambda \in T+n} M_{\lambda}
\]
then \(M = \bigoplus_{n \in \mathbb{Z}} M_n\) is a graded module.