Noncommutative Algebra 2: Representations of finite-dimensional algebras

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1 Some basics

Let A be a K-algebra.

Usually we are interested in finite-dimensional A-modules. That is, we assume that K is a field. We write A-mod for the category of finite-dimensional A-modules. It is an abelian category. The functor $D(-) = \mathrm{Hom}_K(-, K)$ gives an antiequivalence between the categories A-mod and A^{op}-mod.

We are especially interested in the case that A is finite-dimensional. Often K will be an algebraically closed field. In this chapter we do some basics. Later we shall cover Auslander-Reiten theory, representations of quivers and other topics.

1.1 Fitting and Krull-Remak-Schmidt

Let A be a K-algebra. For the rest of this subsection we consider f.d. A-modules, with K a field. In this case, indecomposable modules have local endomorphism ring. We proved this before in another way. Here is the usual way to see it.

Fitting's Lemma. If M is a finite-dimensional A-module and $\theta \in \mathrm{End}_A(M)$, then there is a decomposition

$$M = M_0 \oplus M_1$$

such that $\theta|_{M_0}$ is a nilpotent endomorphism of M_0 and $\theta|_{M_1}$ is an invertible endomorphism of M_1. In particular, if M is indecomposable, then any endomorphism is invertible or nilpotent, so $\mathrm{End}_A(M)$ is a local ring.

Proof. There are chains of submodules

$$\mathrm{Im}(\theta) \supseteq \mathrm{Im}(\theta^2) \supseteq \mathrm{Im}(\theta^3) \supseteq \dots$$

$$\mathrm{Ker}(\theta) \subseteq \mathrm{Ker}(\theta^2) \subseteq \mathrm{Ker}(\theta^3) \subseteq \dots$$

which must stabilize since M is finite dimensional. Thus there is some n with $\mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$ and $\mathrm{Ker}(\theta^n) = \mathrm{Ker}(\theta^{2n})$. We show that

$$M = \mathrm{Ker}(\theta^n) \oplus \mathrm{Im}(\theta^n).$$

If $m \in \mathrm{Ker}(\theta^n) \oplus \mathrm{Im}(\theta^n)$ then $m = \theta^n(m')$ and $\theta^{2n}(m') = \theta^n(m) = 0$, so $m' \in \mathrm{Ker}(\theta^{2n}) = \mathrm{Ker}(\theta^n)$, so $m = \theta^n(m') = 0$. If $m \in M$ then $\theta^n(m) \in \mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$, so $\theta^n(m) = \theta^{2n}(m'')$ for some m''. Then $m = (m - \theta^n(m'')) + \theta^n(m'') \in \mathrm{Ker}(\theta^n) + \mathrm{Im}(\theta^n)$.

Now it is easy to see that the restriction of θ to $\mathrm{Ker}(\theta^n)$ is nilpotent, and its restriction to $\mathrm{Im}(\theta^n)$ is invertible.

Definition/Lemma. If X and Y are A-modules, we define $\mathrm{rad}_A(X,Y)$ to be the set of all $\theta \in \mathrm{Hom}_A(X,Y)$ satisfying the following equivalent conditions.
(i) $1_X - \phi\theta$ is invertible for all $\phi \in \mathrm{Hom}_A(Y,X)$.
(ii) $1_Y - \theta\phi$ is invertible for all $\phi \in \mathrm{Hom}_A(Y,X)$.
Thus by definition $\mathrm{rad}_A(X,X) = J(\mathrm{End}_A(X))$.

Proof of (i) implies (ii). If u is an inverse for $1_X - \phi\theta$ then $1_Y + \theta u \phi$ is an inverse for $1_Y - \theta\phi$.

Lemma 1.
(a) The radical forms an ideal in the module category, that is, $\operatorname{rad}_A(X,Y)$ is closed under addition, and given maps $X \to Y \to Z$, if one is in the radical, so is the composition.
(b) The radical commutes with finite direct sums, that is, $\operatorname{rad}_A(X \oplus X', Y) = \operatorname{rad}_A(X,Y) \oplus \operatorname{rad}_A(X',Y)$ and $\operatorname{rad}_A(X, Y \oplus Y') = \operatorname{rad}_A(X,Y) \oplus \operatorname{rad}_A(X,Y')$.

Proof. (a) For a sum $\theta + \theta'$, let f be an inverse for $1 - \phi\theta$. Then $1 - \phi(\theta + \theta') = (1 - \phi\theta)(1 - f\phi\theta')$, a product of invertible maps. The composition is straightforward.

(b) If you keep one variable fixed, it is a K-linear functor, so preserves direct sums.

Lemma 2. For f.d. modules we have
(i) If X is indecomposable, then $\operatorname{rad}_A(X,Y)$ is the set of maps which are not split monos. ($\theta : X \to Y$ is a *split mono* if there is a map $\phi : Y \to X$ with $\phi\theta = 1_X$, Equivalently if θ is an isomorphism of X with a direct summand of Y.)
(ii) If Y is indecomposable, then $\operatorname{rad}_A(X,Y)$ is the set of maps which are not split epis. ($\theta : X \to Y$ is a *split epi* if there is a map $\psi : Y \to X$ with $\theta\psi = 1_Y$. Equivalently if θ identifies Y with a direct summand of X.)
(iii) If X and Y are indecomposable, then $\operatorname{rad}_A(X,Y)$ is the set of non-isomorphisms.

Proof. We use Fitting's Lemma. (i) Suppose $\theta \in \operatorname{Hom}(X,Y)$. If θ is a split mono there is $\phi \in \operatorname{Hom}(Y,X)$ with $\phi\theta = 1_X$, so $1 - \phi\theta$ is not invertible. Conversely if there is some ϕ with $f = 1 - \phi\theta$ not invertible, then f is nilpotent, and so $\phi\theta = 1 - f$ is invertible. Then $(\phi\theta)^{-1}\phi\theta = 1_X$, so θ is split mono. (ii) is dual and (iii) follows.

Krull-Remak-Schmidt Theorem. Any f.d. module can be written as a direct sum of indecomposable modules,

$$M \cong X_1 \oplus \cdots \oplus X_n.$$

Moreover if $M \cong Y_1 \oplus \cdots \oplus Y_m$ is another decomposition into indecomposables, then $m = n$ and the X_i and Y_j can be paired off so that corresponding modules are isomorphic.

Proof. Given modules X and M, with X indecomposable, we can define a vector space

$$t(X,M) = \operatorname{Hom}_A(X,M)/\operatorname{rad}_A(X,M).$$

This is naturally a right $\mathrm{End}_A(X)$-module, and in fact a module for the division algebra $D = \mathrm{End}(X)/J(\mathrm{End}(X))$ is a division algebra. Thus it is a free right D-module of a certain rank. In fact the rank is

$$\mu_X(Y) = \frac{\dim t(X, Y)}{\dim D}.$$

Clearly if Y is indecomposable, then

$$\mu_X(Y) = \begin{cases} 1 & (Y \cong X) \\ 0 & (Y \not\cong X). \end{cases}$$

Now $t(X, M) = t(X, X_1 \oplus \cdots \oplus X_n) \cong t(X, X_1) \oplus \cdots \oplus t(X, X_n)$, so

$$\mu_X(M) = \mu_X(X_1 \oplus \cdots \oplus X_n) = \mu_X(X_1) + \cdots + \mu_X(X_n)$$

Thus $\mu_X(M)$ is the number of the X_i which are isomorphic to X. Similarly, it is the number of Y_j which are isomorphic to X. Thus these numbers are equal.

Definition. Let $\theta : X \to Y$ be a map of A-modules.
We say that θ is *left minimal* if for $\alpha \in \mathrm{End}(Y)$, if $\alpha\theta = \theta$, then α is invertible.
We say that θ is *right minimal* if for $\beta \in \mathrm{End}(X)$, if $\theta\beta = \theta$, then β is invertible.

Lemma 3. Given a map $\theta : X \to Y$ of finite-dimensional A-modules
(i) There is a decomposition $Y = Y_0 \oplus Y_1$ such that $\mathrm{Im}(\theta) \subseteq Y_1$ and $X \to Y_1$ is left minimal.
(ii) There is a decomposition $X = X_0 \oplus X_1$ such that $\theta(X_0) = 0$ and $X_1 \to Y$ is right minimal.

Proof. (i) Of all decompositions $Y = Y_0 \oplus Y_1$ with $\mathrm{Im}(\theta) \subseteq Y_1$ choose one with Y_1 of minimal dimension. Let θ_1 be the map $X \to Y_1$. Let $\alpha \in \mathrm{End}(Y_1)$ with $\alpha\theta_1 = \theta_1$. By the Fitting decomposition, $Y_1 = \mathrm{Im}(\alpha^n) \oplus \mathrm{Ker}(\alpha^n)$ for $n \gg 0$. Now $\alpha^n\theta_1 = \theta_1$, so $\mathrm{Im}(\theta_1) \subseteq \mathrm{Im}(\alpha^n)$, and we have another decomposition $Y = [Y_0 \oplus \mathrm{Ker}(\alpha^n)] \oplus \mathrm{Im}(\alpha^n)$. By minimality, $\mathrm{Ker}(\alpha^n) = 0$, so α is injective, and hence an isomorphism.

Lemma 4. If $\theta_i : X_i \to Y_i$ are finitely many right (respectively left) minimal maps, then so is $\bigoplus_i X_i \to \bigoplus_i Y_i$.