Noncommutative Algebra 2: Representations of finite-dimensional algebras

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William Crawley-Boevey

Contents

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1 Some basics

Let A be a K-algebra.

Usually we are interested in finite-dimensional A-modules. That is, we assume that K is a field. We write A-mod for the category of finite-dimensional A-modules. It is an abelian category. The functor $D(-) = \mathrm{Hom}_K(-, K)$ gives an antiequivalence between the categories A-mod and A^{op}-mod.

We are especially interested in the case that A is finite-dimensional. Often K will be an algebraically closed field. In this chapter we do some basics. Later we shall cover Auslander-Reiten theory, representations of quivers and other topics.

1.1 Fitting and Krull-Remak-Schmidt

Let A be a K-algebra. For the rest of this subsection we consider f.d. A-modules, with K a field. In this case, indecomposable modules have local endomorphism ring. We proved this before in another way. Here is the usual way to see it.

Fitting's Lemma. If M is a finite-dimensional A-module and $\theta \in \mathrm{End}_A(M)$, then there is a decomposition

$$M = M_0 \oplus M_1$$

such that $\theta|_{M_0}$ is a nilpotent endomorphism of M_0 and $\theta|_{M_1}$ is an invertible endomorphism of M_1. In particular, if M is indecomposable, then any endomorphism is invertible or nilpotent, so $\mathrm{End}_A(M)$ is a local ring.

Proof. There are chains of submodules

$$\mathrm{Im}(\theta) \supseteq \mathrm{Im}(\theta^2) \supseteq \mathrm{Im}(\theta^3) \supseteq \dots$$

$$\mathrm{Ker}(\theta) \subseteq \mathrm{Ker}(\theta^2) \subseteq \mathrm{Ker}(\theta^3) \subseteq \dots$$

which must stabilize since M is finite dimensional. Thus there is some n with $\mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$ and $\mathrm{Ker}(\theta^n) = \mathrm{Ker}(\theta^{2n})$. We show that

$$M = \mathrm{Ker}(\theta^n) \oplus \mathrm{Im}(\theta^n).$$

If $m \in \mathrm{Ker}(\theta^n) \oplus \mathrm{Im}(\theta^n)$ then $m = \theta^n(m')$ and $\theta^{2n}(m') = \theta^n(m) = 0$, so $m' \in \mathrm{Ker}(\theta^{2n}) = \mathrm{Ker}(\theta^n)$, so $m = \theta^n(m') = 0$. If $m \in M$ then $\theta^n(m) \in \mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$, so $\theta^n(m) = \theta^{2n}(m'')$ for some m''. Then $m = (m - \theta^n(m'')) + \theta^n(m'') \in \mathrm{Ker}(\theta^n) + \mathrm{Im}(\theta^n)$.

Now it is easy to see that the restriction of θ to $\mathrm{Ker}(\theta^n)$ is nilpotent, and its restriction to $\mathrm{Im}(\theta^n)$ is invertible.

Definition/Lemma. If X and Y are A-modules, we define $\mathrm{rad}_A(X, Y)$ to be the set of all $\theta \in \mathrm{Hom}_A(X, Y)$ satisfying the following equivalent conditions.
(i) $1_X - \phi\theta$ is invertible for all $\phi \in \mathrm{Hom}_A(Y, X)$.
(ii) $1_Y - \theta\phi$ is invertible for all $\phi \in \mathrm{Hom}_A(Y, X)$.
Thus by definition $\mathrm{rad}_A(X, X) = J(\mathrm{End}_A(X))$.

Proof of (i) implies (ii). If u is an inverse for $1_X - \phi\theta$ then $1_Y + \theta u\phi$ is an inverse for $1_Y - \theta\phi$.

Lemma 1.
(a) The radical forms an ideal in the module category, that is, $\mathrm{rad}_A(X,Y)$ is closed under addition, and given maps $X \to Y \to Z$, if one is in the radical, so is the composition.
(b) The radical commutes with finite direct sums, that is, $\mathrm{rad}_A(X \oplus X', Y) = \mathrm{rad}_A(X,Y) \oplus \mathrm{rad}_A(X',Y)$ and $\mathrm{rad}_A(X, Y \oplus Y') = \mathrm{rad}_A(X,Y) \oplus \mathrm{rad}_A(X,Y')$.

Proof. (a) For a sum $\theta + \theta'$, let f be an inverse for $1 - \phi\theta$. Then $1 - \phi(\theta + \theta') = (1 - \phi\theta)(1 - f\phi\theta')$, a product of invertible maps. The composition is straightforward.

(b) If you keep one variable fixed, it is a K-linear functor, so preserves direct sums.

Lemma 2. For f.d. modules we have
(i) If X is indecomposable, then $\mathrm{rad}_A(X,Y)$ is the set of maps which are not split monos. ($\theta : X \to Y$ is a *split mono* if there is a map $\phi : Y \to X$ with $\phi\theta = 1_X$, Equivalently if θ is an isomorphism of X with a direct summand of Y.)
(ii) If Y is indecomposable, then $\mathrm{rad}_A(X,Y)$ is the set of maps which are not split epis. ($\theta : X \to Y$ is a *split epi* if there is a map $\psi : Y \to X$ with $\theta\psi = 1_Y$. Equivalently if θ identifies Y with a direct summand of X.)
(iii) If X and Y are indecomposable, then $\mathrm{rad}_A(X,Y)$ is the set of non-isomorphisms.

Proof. We use Fitting's Lemma. (i) Suppose $\theta \in \mathrm{Hom}(X,Y)$. If θ is a split mono there is $\phi \in \mathrm{Hom}(Y,X)$ with $\phi\theta = 1_X$, so $1 - \phi\theta$ is not invertible. Conversely if there is some ϕ with $f = 1 - \phi\theta$ not invertible, then f is nilpotent, and so $\phi\theta = 1 - f$ is invertible. Then $(\phi\theta)^{-1}\phi\theta = 1_X$, so θ is split mono. (ii) is dual and (iii) follows.

Krull-Remak-Schmidt Theorem. Any f.d. module can be written as a direct sum of indecomposable modules,

$$M \cong X_1 \oplus \cdots \oplus X_n.$$

Moreover if $M \cong Y_1 \oplus \cdots \oplus Y_m$ is another decomposition into indecomposables, then $m = n$ and the X_i and Y_j can be paired off so that corresponding modules are isomorphic.

Proof. Given modules X and M, with X indecomposable, we can define a vector space

$$t(X,M) = \mathrm{Hom}_A(X,M)/\,\mathrm{rad}_A(X,M).$$

This is naturally a right $\mathrm{End}_A(X)$-module, and in fact a module for the division algebra $D = \mathrm{End}(X)/J(\mathrm{End}(X))$ is a division algebra. Thus it is a free right D-module of a certain rank. In fact the rank is

$$\mu_X(Y) = \frac{\dim t(X,Y)}{\dim D}.$$

Clearly if Y is indecomposable, then

$$\mu_X(Y) = \begin{cases} 1 & (Y \cong X) \\ 0 & (Y \ncong X). \end{cases}$$

Now $t(X, M) = t(X, X_1 \oplus \cdots \oplus X_n) \cong t(X, X_1) \oplus \cdots \oplus t(X, X_n)$, so

$$\mu_X(M) = \mu_X(X_1 \oplus \cdots \oplus X_n) = \mu_X(X_1) + \cdots + \mu_X(X_n)$$

Thus $\mu_X(M)$ is the number of the X_i which are isomorphic to X. Similarly, it is the number of Y_j which are isomorphic to X. Thus these numbers are equal.

Definition. Let $\theta : X \to Y$ be a map of A-modules.
We say that θ is *left minimal* if for $\alpha \in \mathrm{End}(Y)$, if $\alpha\theta = \theta$, then α is invertible.
We say that θ is *right minimal* if for $\beta \in \mathrm{End}(X)$, if $\theta\beta = \theta$, then β is invertible.

Lemma 3. Given a map $\theta : X \to Y$ of finite-dimensional A-modules
(i) There is a decomposition $Y = Y_0 \oplus Y_1$ such that $\mathrm{Im}(\theta) \subseteq Y_1$ and $X \to Y_1$ is left minimal.
(ii) There is a decomposition $X = X_0 \oplus X_1$ such that $\theta(X_0) = 0$ and $X_1 \to Y$ is right minimal.

Proof. (i) Of all decompositions $Y = Y_0 \oplus Y_1$ with $\mathrm{Im}(\theta) \subseteq Y_1$ choose one with Y_1 of minimal dimension. Let θ_1 be the map $X \to Y_1$. Let $\alpha \in \mathrm{End}(Y_1)$ with $\alpha\theta_1 = \theta_1$. By the Fitting decomposition, $Y_1 = \mathrm{Im}(\alpha^n) \oplus \mathrm{Ker}(\alpha^n)$ for $n \gg 0$. Now $\alpha^n\theta_1 = \theta_1$, so $\mathrm{Im}(\theta_1) \subseteq \mathrm{Im}(\alpha^n)$, and we have another decomposition $Y = [Y_0 \oplus \mathrm{Ker}(\alpha^n)] \oplus \mathrm{Im}(\alpha^n)$. By minimality, $\mathrm{Ker}(\alpha^n) = 0$, so α is injective, and hence an isomorphism.

Lemma 4. If $\theta_i : X_i \to Y_i$ are finitely many right (respectively left) minimal maps, then so is $\bigoplus_i X_i \to \bigoplus_i Y_i$.

Proof. If not, then by the lemma, there is a non-zero summand X' of $\bigoplus_i X_i$ on which the map is zero. We may assume that X' is indecomposable. Let

$f_i : X' \to X_i$ be the projections. Since $\theta(X') = 0$ we have $\theta_i f_i = 0$ for all i. Since X' is a summand there are $g_i : X_i \to X'$ with $1_{X'} = \sum_i g_i f_i$. Thus some $g_i f_i$ is invertible, so f_i is a split mono. But $\theta_i f_i = 0$, which is impossible for θ_i right minimal. Namely, let $h_i f_i = 1_{X'}$. Then $1_{X_i} - f_i h_i$ is not an automorphism, and $\theta_i(1_{X_i} - f_i h_i) = \theta_i$.

1.2 Socle, radical and top of a module

Definition. The *socle* of a module M is the sum of its simple submodules,

$$\operatorname{soc} M = \sum_{S \subseteq M \text{ simple}} S$$

The *radical* of a module M is the intersection of its maximal submodules.

$$\operatorname{rad} M = \bigcap_{N \subseteq M,\ M/N \text{ simple}} N$$

Thus the Jacobson radical is $J(A) = \operatorname{rad}({}_A A)$. The *top* of M is the quotient $\operatorname{top} M = M/\operatorname{rad} M$.

These constructions are functorial: a homomorphism $\theta : M \to N$ induces homomorphisms $\operatorname{soc} M \to \operatorname{soc} N$, $\operatorname{rad} M \to \operatorname{rad} N$ and $\operatorname{top} N \to \operatorname{top} N$.

Moreover the functors are K-linear, so $\operatorname{soc}(M \oplus N) \cong \operatorname{soc} M \oplus \operatorname{soc} N$, etc.

Recall that a submodule N is an *essential* submodule of M if for all $L \subseteq M$, $L \neq 0$ implies $L \cap N \neq 0$. It is a *superfluous* submodule if for all $L \subseteq M$, $L \neq M$ implies $L + N \neq M$.

Lemma 1. Let L be a submodule of M (f.d.)
(i) $\operatorname{soc} M$ is semisimple and L is semisimple iff $L \subseteq \operatorname{soc} M$,
(ii) L is essential in M iff $\operatorname{soc} M \subseteq L$,
(iii) The functor soc is right adjoint to the inclusion of the category of semisimple modules in A-mod,
(iv) $\operatorname{soc} M = 0$ iff $M = 0$. If M has simple socle then M is indecomposable,
(i') $\operatorname{top} M$ is semisimple and M/L is semisimple iff $\operatorname{rad} M \subseteq L$,
(ii') L is superfluous in M iff $L \subseteq \operatorname{rad} M$,
(iii') The functor top is left adjoint to the inclusion of the category of semisimple modules in A-mod,
(iv') $\operatorname{top} M = 0$ iff $M = 0$. If M has simple top then M is indecomposable.

Remark. For arbitrary modules over an arbitrary ring $\operatorname{top} M$ need not be semisimple, eg $M = {}_{\mathbb{Z}}\mathbb{Z}$. Also $\operatorname{soc} M$ need not be essential, but it is the intersection of the essential submodules. And $\operatorname{rad} M$ need not be superfluous, but it is the sum of the superfluous submodules. See Anderson and Fuller, Rings and categories of modules.

Proof. (i)-(iv) Clear. Use that any non-zero module has a simple submodule.

Now a submodule L of M gives a submodule $L^\perp$ of DM, and under the identification $M \cong DDM$, L is identified with $(L^\perp)^\perp$. Now $(L+N)^\perp = L^\perp \cap N^\perp$, and by duality $(L \cap N)^\perp = L^\perp + N^\perp$. Thus (i')-(iv') follow by duality.

Lemma 2. If A is a finite-dimensional algebra and M is an A-module, then
(i) $\operatorname{rad} M = J(A)M$.
(ii) $\operatorname{soc} M = \{m \in M : J(A)m = 0\}$.

Proof. Use that M is semisimple $\Leftrightarrow J(A)M = 0$.

Special case. If $A = KQ/I$ where Q is a quiver and I is an admissible ideal, then $J(A)$ is generated as a left or right ideal by the arrows. Thus

$$\operatorname{rad} M = \sum_{a \in Q_1} aM \qquad \operatorname{soc} M = \bigcap_{a \in Q_1} \{m \in M : am = 0\}.$$

Any A-module M corresponds to a representation of Q with vector spaces $M_i = e_i M$ and linear maps $M_a : M_i \to M_j$ for $a : i \to j$. We have

$$(\operatorname{rad} M)_i = \sum_{h(a)=i} \operatorname{Im} M_a \qquad (\operatorname{soc} M)_i = \bigcap_{t(a)=i} \operatorname{Ker} M_a.$$

1.3 Approximations, covers and envelopes

Definition. We shall call a subcategory $\mathcal{C}$ of A-mod a *module class* provided
(i) It is a full subcategory,
(ii) It is closed under isomorphisms, that is, if $X \cong Y$ and $X \in \mathcal{C} \Rightarrow Y \in \mathcal{C}$,
(iii) It is closed under sums and summands, that is, $X \oplus Y \in \mathcal{C}$ iff $X, Y \in \mathcal{C}$.

A module class is determined by the indecomposables it contains.

Examples.
(1) All modules, the zero module, the semisimple modules.

(2) If $\mathcal{M}$ is any collection of modules, $\operatorname{add}\mathcal{M}$ is the smallest module class containing $\mathcal{M}$, so it consists of all modules isomorphic to a direct summand of a finite direct sum of modules in $\mathcal{M}$.

Definition. Let $\mathcal{C}$ be a module class and X a module, not necessarily in $\mathcal{C}$.

A *left $\mathcal{C}$-approximation* (or preenvelope) of X is a morphism $\theta : X \to C$ with $C \in \mathcal{C}$, such that for any $\theta' : X \to C'$ with C' in $\mathcal{C}$, there is $f : C \to C'$ with $\theta' = f\theta$.

A *$\mathcal{C}$-envelope* of X is a left minimal left $\mathcal{C}$-approximation of X.

A *right $\mathcal{C}$-approximation* (or precover) of X is a morphism $\theta : C \to X$ with $C \in \mathcal{C}$, such that for any $\theta' : C' \to X$ with C' in $\mathcal{C}$, there is $f : C' \to C$ with $\theta' = \theta f$.

It *$\mathcal{C}$-cover* of X is a right minimal right $\mathcal{C}$-approximation.

Lemma.
(i) If X has a left $\mathcal{C}$-approximation, it has a $\mathcal{C}$-envelope. Moreover any two $\mathcal{C}$-envelopes of X are isomorphic. If $X_i \to C_i$ are envelopes, so is $\bigoplus_i X_i \to \bigoplus_i C_i$
(ii) If X has a right $\mathcal{C}$-approximation, it has a $\mathcal{C}$-cover. Moreover any two $\mathcal{C}$-covers of X are isomorphic. If $C_i \to X_i$ are covers, so is $\bigoplus_i C_i \to \bigoplus_i X_i$

Proof. Use lemmas about left and right minimal maps.

Examples.
(i) $M \to \operatorname{top} M$ is a semisimple-envelope and $\operatorname{soc} M \to M$ is a semisimple cover.

(ii) More generally, if the inclusion $i : \mathcal{C} \to A$-mod has a left adjoint L, then the natural map $M \to i(LM)$ is a $\mathcal{C}$-envelope, and if it has a right adjoint R, then $i(RM) \to M$ is a $\mathcal{C}$-cover.

(iii) For any modules M, X the map $X \to M^n$ given by a spanning set of the vector space $\operatorname{Hom}(X, M)$ is a left $\operatorname{add} M$-approximation of X and the map $M^m \to X$ given by a spanning set of $\operatorname{Hom}(M, X)$ is a right $\operatorname{add} M$-approximation of X.

1.4 Projectives and injectives for f.d. algebras

Now let A be a finite-dimensional algebra. By module we mean f.d. module.

The projective modules form the module class $\mathcal{P}_A = \operatorname{add} A$ and the injectives form the module class $\mathcal{I}_A = \operatorname{add} D(A_A)$. (Note that $D(A_A)$ really is an injective module, not just with respect to other finite-dimensional modules, since $\operatorname{Hom}_A(-, DA) \cong \operatorname{Hom}_K(A \otimes_A -, K) \cong D(-)$ is exact.)

Lemma 1. Any module M has an injective envelope $M \to E(M)$ and a projective cover $P(M) \to M$.

Proof. Already known, or use left and right $\operatorname{add} N$-approximations, where $N = A$ or DA.

Lemma 2. Let $\theta : M \to I$ be a map with I injective. The following are equivalent
(i) θ is an injective envelope (that is, an $\mathcal{I}_A$-envelope).
(ii) θ is 1-1 and $\operatorname{Im}\theta$ is essential in I.
(iii) the induced map $\operatorname{soc} M \to \operatorname{soc} I$ is an isomorphism.

Proof. (i) $\Rightarrow$ (ii) Recall that any module M embeds in some injective module (for if $(A^{op})^n \twoheadrightarrow DM$ then $M \hookrightarrow (DA)^n$). By the injective approximation property, θ must be 1-1. Suppose the X is a non-zero submodule of I with $X \cap \operatorname{Im}\theta = 0$. The projection onto $\operatorname{Im}\theta$ defines an endomorphism on $X \oplus \operatorname{Im}\theta$, and by the injective property it extends to an endomorphism ϕ of I. Now $\phi\theta = \theta$ but $\phi(X) = 0$, so ϕ is not an automorphism, contradicting minimality.

(ii) $\Rightarrow$ (i) By the injective property, θ is a left injective-approximation. Moreover if $\phi \in \operatorname{End}(I)$ and $\phi\theta = \theta$, then $\operatorname{Im}\theta \cap \operatorname{Ker}\phi = 0$, so $\operatorname{Ker}\phi = 0$, so by dimensions, ϕ is an automorphism. Thus θ is left minimal.

(ii) $\Rightarrow$ (iii). Clearly the map $\operatorname{soc} M \to \operatorname{soc} I$ is injective. Since $\operatorname{soc} M$ is essential in θ, $\theta(\operatorname{soc} M) = \operatorname{soc}\theta(M)$ is essential in $\operatorname{Im}\theta$. Now this is essential in M, and hence $\theta(\operatorname{soc} M)$ is essential in M. Thus it contains $\operatorname{soc} I$.

(iii) $\Rightarrow$ (ii). $\operatorname{Ker}\theta \cap \operatorname{soc} M = 0$ so since $\operatorname{soc} M$ is essential in M we must have $\operatorname{Ker}\theta = 0$, so θ is 1-1. If $\operatorname{Im}\theta \cap X = 0$, with X a non-zero submodule of I, then X has a simple submodule S and $\theta(\operatorname{soc} M) \cap S = 0$. Then $\theta(\operatorname{soc} M)$ is strictly contained in $\theta(\operatorname{soc} M) \oplus S \subseteq \operatorname{soc} I$.

Lemma 2'. Let $\theta : P \to M$ be a map with P projective. The following are

equivalent
(i) θ is a projective cover (that is, a $\mathcal{P}_A$-cover).
(ii) the induced map $\operatorname{top} P \to \operatorname{top} M$ is an isomorphism.
(iii) θ is onto and $\operatorname{Ker}\theta$ is superfluous in P.

Proof. Dual.

Lemma 3. Injectives are indecomposable iff they have simple socle. Projectives are indecomposable iff they have simple top. There are 1:1 correspondences between indecomposable projectives, simple modules, and indecomposable injectives, $S \mapsto P(S)$, $P \mapsto \operatorname{top} P$, $S \mapsto E(S)$, $I \mapsto \operatorname{soc} I$.

Proof. $E(S)$ has simple socle S, so it must be indecomposable.

If I is injective and has simple S as a submodule, then one gets a map $E(S) \to I$, and since S is essential in $E(S)$, it must be injective. Thus $E(S)$ is a summand of I. Thus if I is indecomposable injective it has simple socle, and if it has simple socle S then $I \cong E(S)$.

Notation. The algebra $A/J(A)$ is semisimple, so it has Wedderburn decomposition $M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$. There are simple modules $S[i] = D_i^{n_i}$ $(i = 1, \dots, r)$. They have projective covers $P[i]$ and injective envelopes $I[i]$. Moreover $D_i \cong \operatorname{End}_A(S[i])^{op}$, $n_i = \dim S_i / \dim D_i$.

Since $\operatorname{top}{}_A A \cong \bigoplus S[i]^{n_i}$, we have ${}_A A \cong \bigoplus P[i]^{n_i}$.

If $A/J(A) \cong K \times \cdots \times K$ (for example if A is basic and K is algebraically closed), then $A \cong KQ/I$ for some quiver Q and admissible ideal I. In this case the simple modules $S[i]$ correspond to the vertices, $P[i] = Ae_i$ and $I[i] = D(e_i A)$.

Example. The commutative square algebra with source 1 and sink 4 has $P[1] \cong I[4]$.

Definition. The *Nakayama functor* is

$$\nu(-) = D \operatorname{Hom}_A(-, A) : A\text{-mod} \to A\text{-mod}.$$

Lemma 4. (i) ν is naturally isomorphic to $DA \otimes_A -$.
(ii) ν has right adjoint

$$\nu^-(-) = \operatorname{Hom}_A(D(-), A) \cong \operatorname{Hom}_A(DA, -) : A\text{-mod} \to A\text{-mod}.$$

(Here $\operatorname{Hom}_A(D(-), A)$ is a space of homomorphisms between right A-modules.)
(iii) ν induces an equivalence $\mathcal{P}_A \to \mathcal{I}_A$ with inverse equivalence given by ν^-.

(iv) $\mathrm{Hom}(X, \nu P) \cong D\,\mathrm{Hom}(P, X)$ for X, P left A-modules, P projective.
(v) $\nu(P[i]) \cong I[i]$.

Proof. (i) $D(DA \otimes_A X) \cong \mathrm{Hom}_A(X, DDA) \cong \mathrm{Hom}(X, A)$. Now apply D.

(ii) Clear.

(iii) Recall that the contravariant functor $\mathrm{Hom}_A(-, A) : A\text{-mod} \to A^{op}\text{-mod}$ gives an antiequivalence $\mathcal{P}_A \to \mathcal{P}_{A^{op}}$. Now duality gives an antiequivalence $\mathcal{P}_{A^{op}} \to \mathcal{I}_A$.

(iv) The composition

$$\mathrm{Hom}(P, A) \otimes_A X \cong \mathrm{Hom}(P, A) \otimes_A \mathrm{Hom}(A, X) \to \mathrm{Hom}(P, X)$$

is an isomorphism, since it is for $P = A$. Thus

$$D\,\mathrm{Hom}(P, X) \cong \mathrm{Hom}_K(\mathrm{Hom}(P, A) \otimes_A X, K)$$

$$\cong \mathrm{Hom}(X, \mathrm{Hom}_K(\mathrm{Hom}(P, A), K)) = \mathrm{Hom}(X, \nu P).$$

(v) $\nu(P[i])$ is indecomposable injective and we have $\mathrm{Hom}(S[i], \nu(P[i])) \cong D\,\mathrm{Hom}(P[i], S[i]) \neq 0$.

1.5 Projective-injective modules

Example. The commutative square algebra with source 1 and sink 4 has $P[1] \cong I[4]$. But the other indecomposable projectives are not injective.

Modules which are both projective and injective can be useful.

Lemma. Let P be a projective-injective module, and for simplicity suppose it is a direct summand of A. Let $I = SA \subseteq A$ be the ideal generated by $S = \mathrm{soc}\, P$. If M is an indecomposable A-module, then either M is a direct summand of P or $IM = 0$.

Proof. If $IM \neq 0$, then $SM \neq 0$. Thus there is some $m \in M$ with $Sm \neq 0$. Identifying $M \cong \mathrm{Hom}_A(A, M)$, m corresponds to a homomorphism $\theta : A \to M$ with $\theta(S) \neq 0$. Write $P = \bigoplus_i P_i$ with the P_i indecomposable. Then some $\theta(\mathrm{soc}\, P_i) \neq 0$. Thus the restriction of θ to P_i is injective. Thus P_i embeds in M. But P_i is an injective module, so it is a direct summand of M.

In the example, any indecomposable module is either $P[1]$ or a module for the algebra give by a square with two zero relations.

Definitions. We define the following classes of algebras with the obvious implications. They are all left-right symmetric.

$$A \text{ symmetric} \Rightarrow A \text{ Frobenius} \Rightarrow A \text{ self-injective} \Rightarrow A \text{ QF-3}$$

(i) A is *symmetric* if ${}_AA_A \cong {}_ADA_A$. Equivalently if there is a bilinear form $(-,-) : A \times A \to K$ which is
- non-degenerate: $(a,b) = 0 \forall b \Rightarrow a = 0$, $(a,b) = 0 \forall a \Rightarrow b = 0$,
- associative: $(ab,c) = (a,bc)$, and
- symmetric: $(a,b) = (b,a)$.

The corresponding map $A \to DA$ is $a \mapsto (a,-)$. It follows that $I[i] = \nu(P[i]) \cong DA \otimes_A P[i] \cong A \otimes_A P[i] \cong P[i]$.

(ii) A is *Frobenius* if ${}_AA \cong {}_ADA$. Equivalently if there is a bilinear form which is non-degenerate and associative.

(iii) A is *self-injective* (or *quasi-Frobenius*) if ${}_AA$ is an injective module. Equivalently $\text{add}\,{}_AA = \text{add}\,{}_ADA$. Equivalently the modules $P[i]$ and $I[j]$ are the same, up to a permutation.

(iv) A is *QF-3* (in the sense of Thrall) if A has a faithful projective-injective module, or equivalently A embeds in a projective-injective module, or equivalently the injective envelope of any projective module is projective.

A module M is *faithful* if $am = 0$ for all $m \in M$ implies $a = 0$, that is, if the map $A \to \text{End}_K(M)$ is injective.

A f.d. A-module M is faithful if and only if there is an embedding $A \to M^n$ for some n. Namely, if $A \hookrightarrow M^n$, $a \in A$ and $am = 0$ for all $m \in M$, then $ax = 0$ for all $x \in M^n$, so $a1 = 0$ for $1 \in A$. Thus $a = 0$. Conversely, if M is faithful, choose a basis $m_1, \dots, m_n$ of M. This gives a map $A \to M^n$, $a \mapsto (am_1, \dots, am_n)$. If $a \mapsto 0$, then $am_i = 0$ for all i, so $am = 0$ for all $m \in M$.

Examples. (1) The group algebra KG of a finite group is symmetric with $(a,b) = \lambda_1$ where $ab = \sum_{g \in G} \lambda_g g$.

(2) If Q is the cyclic quiver with n vertices then $KQ/(KQ_+)^{k+1}$ is Frobenius, and it is symmetric iff $n|k$. The bilinear form (a,b) is the sum of coefficients of paths of length k in ab.

(3) The commutative square algebra with source 1 and sink 4 is QF-3 because any indecomposable projective has socle $S[4]$, so embeds in $I[4] \cong P[1]$.

(4) For a commutative algebra the concepts are the same ((ii)$\Rightarrow$(i) since $(a, b) = (1a, b) = (1, ab) = (1, ba) = (b, a)$, (iii)$\Rightarrow$(ii), since the algebra is basic, and (iv)$\Rightarrow$(iii) since , then $(1-e)Ae = 0$, so it Ae is a faithful projective-injective, then $e = 1$). Commutative Frobenius algebras appear in topological quantum field theory.

1.6 Uniserial modules and Nakayama algebras

Definition. A module M is *uniserial* if its submodules are totally ordered by inclusion, that is, if $N, N' \subseteq M$, then either $N \subseteq N'$ or $N' \subseteq N$.

A *composition series* of a module M is a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that each M_i/M_{i-1} is simple. Since M is f.d., any chain of submodules can be refined to give a composition series. Now M is uniserial iff it has a unique composition series.

Example. If S and T are simple modules and $0 \to S \to M \to T \to 0$ is non-split, then M is uniserial. (If L is a submodule with $L \neq 0, S, M$, then $L + S = M$, and $L \cap S = 0$, so the sequence splits.)

Lemma.
(i) Any uniserial module is indecomposable, with simple top and socle, and only finitely many submodules.
(ii) Any submodule or quotient of a uniserial module is uniserial.
(iii) M is a uniserial A-module iff $D(M)$ is a uniserial A^{op}-module.
(iv) M is uniserial iff the chain

$$M \supseteq \operatorname{rad} M \supseteq \operatorname{rad}^2 M \supseteq \cdots \supseteq \operatorname{rad}^{n-1} M \supseteq \operatorname{rad}^n M = 0$$

is a composition series for suitable n.

Proof. (i)-(iii) trivial. (iv) It suffices to show that if the chain is a composition series, then every submodule L of M is equal to $\operatorname{rad}^i M$, some i. Let i be maximal with $L \subseteq \operatorname{rad}^i M$ If $i = n$ then $L = 0$, otherwise $\operatorname{rad}^i M / \operatorname{rad}^{i+1} M$ is simple, so $\operatorname{rad}^{i+1} M$ is the unique maximal submodule of $\operatorname{rad}^i M$. Since L is not contained in $\operatorname{rad}^{i+1} M$, we must have $L = \operatorname{rad}^i M$.

Definition. A f.d. algebra A is a *Nakayama algebra* if the indecomposable projective left and right A-modules are uniserial. It is equivalent that the indecomposable projective modules and the indecomposable injective modules are all uniserial.

Proposition 1. If $A = KQ/I$ with Q connected and I admissible, then A is Nakayama iff Q is a linear or cyclic quiver.

Proof. If the quiver is linear or cyclic, then for each vertex i there is a unique maximal path $a_n \dots a_1$ with tail at i and not in I. Then $\operatorname{rad}^j P[i]$ is spanned by the paths $a_k \dots a_1$ with $k \geq j$. Thus the radical series is a composition series. Thus $P[i]$ is uniserial. Similarly for the indecomposable projective right modules.

If two arrows a, b have tail at i then the submodules Aa and Ab of $Ae_i = P[i]$ are incomparable, for if $Aa \subseteq Ab$, then there is $x \in KQ$ with $a - xb \in I \subseteq (KQ_+)^2$, which is impossible. Similarly for right modules if two arrows have tail at i.

Proposition 2.
(i) A Nakayama $\Rightarrow$ A/I Nakayama for any ideal I.
(ii) A Nakayama $\Rightarrow$ A QF-3.
(iii) $A/J(A)^2$ QF-3 $\Rightarrow$ A Nakayama.
Thus, for example, A is Nakayama $\Leftrightarrow$ A/I is QF-3 for all I.

Proof. (i) Write $A = \bigoplus P_i$ with P_i indecomposable projective. Then $A/I = \bigoplus P_i/IP_i$, a direct sum of uniserial modules, so the indecomposable projective left A/I-modules are uniserial. Similarly for right modules.

(ii) We show that if P is indecomposable projective, then so is $E(P)$. Since P has simple socle, so does $E(P)$. Thus it is indecomposable. Thus it is uniserial, so it has simple top. If $\theta : P' \to E(P)$ is its projective cover, then P' is indecomposable. This gives an exact sequence $0 \to \operatorname{Ker}\theta \to \theta^{-1}(P) \to P \to 0$. Now $\theta^{-1}(P)$ is uniserial, so indecomposable, but this sequence splits, so we must have $\operatorname{Ker}\theta = 0$.

(iii) We show the indecomposable projective left modules are uniserial. For right modules it is dual.

First we show that A/J^2 in Nakayama. Let P be an indecomposable projective A/J^2-module. Thus $\operatorname{rad} P = JP$ is semisimple. We show it is zero or simple. We have $P \subseteq E(P)$, where $E(P)$ denotes the injective envelope as an A/J^2-module, and $E(P)$ is projective for A/J^2. If $P \subseteq JE(P)$, then $JP = 0$. Thus suppose $P \not\subseteq JE(P)$. We decompose $E(P)$ into indecomposables, $E(P) = \bigoplus P_i$. Then one of the maps $\operatorname{top} P \to \operatorname{top} P_i$ is an isomorphism, so $P \to P_i$ is an isomorphism, so P is injective, so $E(P) = P$. Then JP is semisimple, but P has simple socle, so JP is simple or zero.

Now we show by induction on n that A/J^n is Nakayama for $n > 2$. Let P be an indecomposable projective for A/J^n. Then P/J^2P is a projective for A/J^2, and it has simple top, so it is indecomposable, so JP/J^2P is zero or simple. Thus JP is a module for A/J^{n-1} which is zero or has simple top, so it is uniserial. Thus P is uniserial.

Theorem. Any indecomposable module for a Nakayama algebra is uniserial. Thus any indecomposable module is a quotient of an indecomposable projective, so there are only finitely many indecomposable modules - Nakayama algebras have finite representation type.

Proof. We prove this for Nakayama algebras A by induction on $\dim A$.

Now A has an indecomposable projective-injective module P. We can embed it as an ideal in A. Let $I = SA$, the ideal generated by $S = \operatorname{soc} P$. Then any indecomposable module for A is either isomorphic to P, so uniserial, or an indecomposable module for A/I, so uniserial by induction.

Recall that a f.d. representation of a quiver is *nilpotent* if there is some m such that any path of lenght $\geq m$ is zero in the representation. For a quiver without oriented cycles all representations are nilpotent. If I is an admissible ideal then any KQ/I-module corresponds to a nilpotent representation of Q.

Corollary. (i) Any f.d. indecomposable nilpotent representation M of a linear or cyclic quiver Q is isomorphic to $(KQ/KQ_+^m)e_i$ for some vertex i and some m.
(ii) Any f.d. indecomposable representation of a cyclic quiver is either nilpotent or isomorphic to one of the form

$$V \xrightarrow{1} V \xrightarrow{1} \dots \xrightarrow{1} V \xrightarrow{x} V \quad \text{(the two ends identified)}$$

where $V = K[x]/(f(x)^n)$ with $f(x)$ a monic irreducible polynomial $\neq x$ in $K[x]$. In particular if K is algebraically closed, $f(x) = x - \lambda$, then $V \cong K^n$ and x corresponds to the Jordan block $J_n(\lambda)$.

Proof. (i) M is a module for $KQ/(KQ_+)^k$ for some k, which is Nakayama.

(ii) Let Q be cyclic with N vertices. Let $\alpha \in KQ$ be the sum of all paths of length N. Then α is a central element of KQ, so it induces an element of

$\mathrm{End}_{KQ}(M)$. By Fitting's Lemma, this element must be nilpotent or invertible. If nilpotent, then M is nilpotent. If invertible, then all paths of length N in M must be invertible. Thus all arrows in M must be invertible. Thus M is of the indicated form for some for some $K[x]$-module V on which x acts invertibly. Now V must be indecomposable, so it has the stated form.

1.7 Homological algebra for f.d. algebras

Definition. A projective resolution

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

is *minimal* if the maps $\epsilon : P_0 \to M$, $d_1 : P_1 \to \mathrm{Ker}(\epsilon)$, $d_2 : P_2 \to \mathrm{Ker}(d_1)$ and so on, are projective covers. Dually for an injective resolution

$$0 \to M \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d_1} I^2 \to \dots,$$

the maps $\epsilon : M \to I^0$, $I^0/\mathrm{Im}(\epsilon) \to I^1$, $I^1/\mathrm{Im}(d^0) \to I^2$ and so on must be injective envelopes.

Minimal projective and injective resolutions of M exist and are unique up to isomorphism.

Lemma 1.
The multiplicity of $I[i]$ in I^k is $\mu_{I[i]}(I^k) = \dim \mathrm{Ext}^k(S[i], M)/\dim D_i$, and
The multiplicity of $P[i]$ in P_k is $\mu_{P[i]}(P_k) = \dim \mathrm{Ext}^k(M, S[i])/\dim D_i$.

Proof. By minimality, any element of $\mathrm{soc}\, I^i$ is in the image of the map $I^{i-1} \to I^i$, so is killed by the map $I^i \to I^{i+1}$. Thus in the complex $\mathrm{Hom}(S[i], I^*)$, the differential is zero. Thus

$$\dim \mathrm{Ext}^k(S[i], M) = \dim \mathrm{Hom}(S[i], I^k) = \dim \mathrm{End}(S[i]).\mu_{I[i]}(I^k).$$

Lemma 2. If $A = KQ/I$ with I admissible, then the number of arrows from i to j in Q is $\dim \mathrm{Ext}^1(S[i], S[j])$.

Proof. Since I is admissible, $I \subseteq (KQ)_+^2$. Now $P[i] = (KQ/I)e_i$, so $\mathrm{rad}\, P[i] = ((KQ)_+/I)e_i$, and $\mathrm{rad}\,\mathrm{rad}\, P[i] = ((KQ)_+^2/I)e_i$. Thus

$$\mathrm{top}(\mathrm{rad}\, P[i]) \cong ((KQ)_+/(KQ)_+^2))e_i \cong \bigoplus_j S[j]^{n_{ij}}$$

where n_{ij} is the number of arrows from i to j. Then in the minimal projective resolution of $S[i]$,

$$\cdots \to P_1 \to P[i] \to S[i] \to 0$$

P_1 is the projective cover of $\operatorname{rad} P[i]$, so $\operatorname{top} P_1 \cong \operatorname{top}(\operatorname{rad} P[i])$, so the multiplicity of $P[j]$ is n_{ij}. Thus $\dim \operatorname{Ext}^1(S[i], S[j]) = n_{ij}$.

Lemma 3. The following are equivalent for a module M
(i) $\operatorname{proj.dim} M \leq n$
(ii) $\operatorname{Ext}^{n+1}(M, S) = 0$ for all simples S.
(iii) the minimal projective resolution of M has $P_k = 0$ for $k > n$.
Similarly for the injective dimension.

Proof. (i) implies (ii) is clear.

(ii) implies (iii). By the lemma above, the minimal projective resolution of M has $P_{n+1} = 0$.

(iii) implies (i). Trivial.

Proposition. The global dimension of a f.d. algebra is the maximum of the projective dimensions of its simple modules.

Proof. If the maximum is n, we need to show that every module, even infinite-dimensional, has projective dimension at most n. Now every simple S has a projective resolution of length $\leq n$. Thus every semisimple module has a projective resolution of length $\leq n$, so every semisimple module has projective dimension $\leq n$. Now every module X has a filtration $X \supseteq J(A)X \supseteq \cdots \supseteq J(A)^N X = 0$ in which the quotients are semisimple, and the long exact sequence shows that an extension of modules of projective dimension $\leq n$ again has projective dimension $\leq n$.

Corollary. For a f.d. algebra, the left and right global dimensions are the same.

Proof. If the right global dimension is $\leq n$, then the simple right modules have injective resolutions of length $\leq n$. Dualizing, the simple left modules have projective resolutions of length $\leq n$. Thus the left global dimension is $\leq n$.

Example. If $A = KQ/I$ with I admissible, and Q has no oriented cycles, then A has finite global dimension. We show by induction that if i is a vertex and every path starting at i has length $\leq n$ then $\operatorname{proj.dim} S[i] \leq n$. If $n = 0$

then i is a sink and $S[i] = P[i]$ is projective. Otherwise there is an exact sequence

$$0 \to \operatorname{rad} P[i] \to P[i] \to S[i] \to 0.$$

Now $\operatorname{rad} P[i]$ is an iterated extension of simples $S[j]$ for which there is a non-trivial path $i \to j$. Thus by induction $\operatorname{proj.dim} S[j] < n$. Thus $\operatorname{proj.dim} \operatorname{rad} P[i] < n$. Thus $\operatorname{proj.dim} S[i] \leq n$.

Recall that a *hereditary algebra* is one with global dimension ≤ 1. Recall that path algebras KQ are hereditary.

Theorem. If A is a f.d. hereditary algebra and $A/J(A) \cong K \times \cdots \times K$ (for example if A is basic and K is algebraically closed), then A is isomorphic to a path algebra KQ.

Proof. The algebra can be written as $A = KQ/I$ with I admissible. We show $I = 0$. Since $I.KQ_+ \subseteq I$ we have an exact sequence of KQ-modules

$$0 \to I/(I.KQ_+) \to KQ_+/(I.KQ_+) \to KQ_+/I \to 0.$$

The middle module is annihilated by I, so this is a sequence of A-modules. The RH module is a submodule of $A = KQ/I$, so it is projective as an A-module. Thus the sequence splits. Letting

$$M = KQ_+/(I.KQ_+), \quad N = I/(I.KQ_+) \oplus KQ_+/I.$$

we deduce that $M \cong N$. Thus $M/(KQ_+)M \cong N/(KQ_+)N$, which gives

$$KQ_+/KQ_+^2 \cong (I/(KQ_+.I + I.KQ_+)) \oplus (KQ_+/KQ_+^2).$$

Thus by dimensions, $I = KQ_+.I + I.KQ_+$. Now if $I \neq 0$ there is a maximal k such that $I \subseteq (KQ)_+^k$. But then $I = KQ_+.I + I.KQ_+ \subseteq (KQ)_+^{k+1}$, a contradiction.

2 Auslander-Reiten Theory

Throughout, A is a f.d. K-algebra, and we consider f.d, modules.

2.1 The transpose

We consider the contravariant functor $M \mapsto M^\vee = \mathrm{Hom}_A(M, A) : A\text{-mod} \to A^{op}\text{-mod}$. It gives an antiequivalence $\mathcal{P}_A \to \mathcal{P}_{A^{op}}$. Given a left (or right) module M, we fix a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0.$$

That is, $g : P_0 \to M$ and $f : P_1 \to \mathrm{Ker}(g)$ are projective covers. The *transpose* $\mathrm{Tr}\, M$ is the cokernel of the map $f^\vee : P_0^\vee \to P_1^\vee$. If M is a left A-module, then $\mathrm{Tr}\, M$ is a left A^{op}-module. Thus there is an exact sequence

$$0 \to M^\vee \to P_0^\vee \to P_1^\vee \to \mathrm{Tr}\, M \to 0$$

Note that Tr doesn't define a functor on the module categories.

Lemma.
(i) Up to isomorphism, $\mathrm{Tr}\, M$ doesn't depend on the choice of minimal projective presentation of M.
(ii) If P is projective, then $\mathrm{Tr}\, P = 0$.
(iii) $\mathrm{Tr}(M \oplus N) \cong \mathrm{Tr}\, M \oplus \mathrm{Tr}\, N$.
(iv) If M has no nonzero projective summand, the same is true for $\mathrm{Tr}\, M$, and $P_0^\vee \to P_1^\vee \to \mathrm{Tr}\, M \to 0$ is a minimal projective presentation.
(v) If M has no nonzero projective summand then $\mathrm{Tr}\,\mathrm{Tr}\, M \cong M$.

Proof. Two different minimal projective presentations of M fit in a commutative diagram

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \\ P_1' & \xrightarrow{f'} & P_0' & \xrightarrow{g'} & M & \longrightarrow & 0 \end{array}$$

and the minimality ensures that the vertical maps are isomorphisms. Applying $(-)^\vee$, one sees that the two different constructions of $\mathrm{Tr}\, M$ are isomorphic.

(ii) is clear.

(iii) Straightforward since the direct sum of minimal projective presentations of M and N gives a minimal projective presentation of $M \oplus N$.

(iv) Suppose Q is a non-zero projective summand of $\operatorname{Tr} M$. Then there is a split epi $P_1^\vee \to Q$ whose composition with $f^\vee$ is zero. Thus there is a split mono $Q^\vee \to P_1$ whose composition with f is zero. Contradicts that $P_1 \to \operatorname{Ker}(g)$ is a projective cover.

Suppose $P_1^\vee \to \operatorname{Tr} M$ is not a projective cover. Then there is a non-zero summand Q of $P_1^\vee$ with image zero in $\operatorname{Tr} M$. This gives a map $Q \to \operatorname{Im}(f^\vee)$. Since Q is projective and $P_0^\vee \to \operatorname{Im}(f^\vee)$ is onto, we get a map $Q \to P_0^\vee$ whose composition with $f^\vee$ is the inclusion of Q in $P_1^\vee$. Thus f composed with the map $P_0 \to Q^\vee$ is the projection $P_1 \to Q^\vee$. Thus $\operatorname{Ker}(g) = \operatorname{Im}(f)$ is not contained in $\operatorname{rad} P_0$. Contradicts that $g : P_0 \to M$ is a projective cover.

Suppose that $P_0^\vee \to \operatorname{Im}(f^\vee)$ is not a projective cover. Then there is a non-zero summand Q of $P_0^\vee$ whose composition with $f^\vee$ is zero. Then there is a split epimorphism $P_0 \to Q^\vee$ whose composition with f is zero. This induces a split epimorphism $M \to Q^\vee$, contradicting the fact that M has no non-zero projective summand.

(v). $\operatorname{Tr}\operatorname{Tr} M$ is the cokernel of the map $P_1^{\vee\vee} \to P_0^{\vee\vee}$, that is, $P_1 \to P_0$.

Proposition. Tr induces a bijection between isomorphism classes of indecomposable non-projective left A-modules and indecomposable non-projective left A^{op}-modules.

Definition. Given modules M, N, we denote by $\operatorname{Hom}^{\mathrm{proj}}(M, N)$ the set of all maps $M \to N$ which can be factorized through a projective module $M \to P \to N$.

Clearly $\operatorname{Hom}^{\mathrm{proj}}(M, N)$ is a subspace of $\operatorname{Hom}(M, N)$, for example if θ factors through P and θ' factors throught P' then $\theta + \theta'$ factors through $P \oplus P'$. Moreover $\operatorname{Hom}^{\mathrm{proj}}$ is an ideal in the module category.

We define $\underline{\operatorname{Hom}}(M, N) = \operatorname{Hom}(M, N)/\operatorname{Hom}^{\mathrm{proj}}(M, N)$. These form the Hom spaces in a category, the *stable module category*, denoted A-$\underline{\mathrm{mod}}$.

Theorem. The transpose defines inverse anti-equivalences

$$A\text{-}\underline{\mathrm{mod}} \rightleftarrows A^{op}\text{-}\underline{\mathrm{mod}}.$$

Proof. First we show that Tr defines a contravariant functor from A-mod to A^{op}-$\underline{\text{mod}}$. Any map $\theta : M \to M'$ can be lifted to a map of projective presentations

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ {\scriptstyle\theta_1}\downarrow & & {\scriptstyle\theta_0}\downarrow & & {\scriptstyle\theta}\downarrow & & \\ P'_1 & \xrightarrow{f'} & P'_0 & \xrightarrow{g'} & M' & \longrightarrow & 0 \end{array}$$

Applying $()^\vee$ there is an induced map ϕ.

$$\begin{array}{ccccccc} P_0'^\vee & \xrightarrow{f'^\vee} & P_1'^\vee & \xrightarrow{p'} & \operatorname{Tr} M' & \longrightarrow & 0 \\ {\scriptstyle\theta_0^\vee}\downarrow & & {\scriptstyle\theta_1^\vee}\downarrow & & {\scriptstyle\phi}\downarrow & & \\ P_0^\vee & \xrightarrow{f^\vee} & P_1^\vee & \xrightarrow{p} & \operatorname{Tr} M & \longrightarrow & 0 \end{array}$$

The map ϕ depends on θ_0 and θ_1, which are not uniquely determined. We show that any choices lead to the same element of $\underline{\text{Hom}}(\operatorname{Tr} M', \operatorname{Tr} M)$. For this we may assume that $\theta = 0$, and need to show that ϕ factors through a projective.

Thus assume that θ is zero. Then $g'\theta_0 = 0$. Thus there is $h : P_0 \to P'_1$ with $\theta_0 = f'h$. This gives $h^\vee : P_1'^\vee \to P_0^\vee$ with $\theta_0^\vee = h^\vee f'^\vee$. Now we have a commutative diagram

$$\begin{array}{ccccccc} P_0'^\vee & \xrightarrow{f'^\vee} & P_1'^\vee & \xrightarrow{p'} & \operatorname{Tr} M' & \longrightarrow & 0 \\ {\scriptstyle\theta_0^\vee}\downarrow & & {\scriptstyle f^\vee h^\vee}\downarrow & & {\scriptstyle 0}\downarrow & & \\ P_0^\vee & \xrightarrow{f^\vee} & P_1^\vee & \xrightarrow{p} & \operatorname{Tr} M & \longrightarrow & 0. \end{array}$$

Taking the difference of the vertical maps, there is also a commutative diagram

$$\begin{array}{ccccccc} P_0'^\vee & \xrightarrow{f'^\vee} & P_1'^\vee & \xrightarrow{p'} & \operatorname{Tr} M' & \longrightarrow & 0 \\ {\scriptstyle 0}\downarrow & & {\scriptstyle \theta_1^\vee - f^\vee h^\vee}\downarrow & & {\scriptstyle\phi}\downarrow & & \\ P_0^\vee & \xrightarrow{f^\vee} & P_1^\vee & \xrightarrow{p} & \operatorname{Tr} M & \longrightarrow & 0. \end{array}$$

But then $(\theta_1^\vee - f^\vee h^\vee)f'^\vee = 0$. Thus there is a map $s : \operatorname{Tr} M' \to P_1^\vee$ with $\theta_1^\vee - f^\vee h^\vee = sp'$. It follows that $psp' = \phi p'$, so since p' is surjective, $\phi = ps$, so ϕ factors through a projective.

Thus a morphism $g : M \to M'$ gives a well-defined morphism $\operatorname{Tr} g = [\phi] \in \underline{\text{Hom}}(\operatorname{Tr} M', \operatorname{Tr} M)$. It is straightforward that this construction behaves well on compositions of morphsms, so that the transpose defines a contravariant functor A-mod to A^{op}-$\underline{\text{mod}}$.

Now clearly the transpose sends any projective module to 0, so it sends any morphism factoring through a projective to 0, so it descends to a contravariant functor A-$\underline{\text{mod}}$ to A^{op}-$\underline{\text{mod}}$. Now it is straightforward that it is an antiequivalence.

2.2 Auslander-Reiten formula

Definition. We define A-$\overline{\text{mod}}$ as the category with Hom spaces

$$\overline{\text{Hom}}(M, N) = \text{Hom}(M, N)/\operatorname{Hom}^{\text{inj}}(M, N)$$

where $\text{Hom}^{\text{inj}}(M, N)$ is the maps factoring through an injective module.

Lemma 1. $\underline{\text{Hom}}(M, N) \cong \overline{\text{Hom}}(DN, DM)$, so D gives an antiequivalence between $\underline{\text{mod}}$-A and A-$\overline{\text{mod}}$.

Proof. Straightforward.

Definition. The *Auslander-Reiten translate* is $\tau = D\,\text{Tr}$ and the inverse construction is $\tau^{-} = \text{Tr}\,D$.

By the results of the previous section we have inverse bijections

$$\text{non-projective indec mods/iso} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \text{non-injective indec mods/iso}$$

and inverse equivalences

$$A\text{-}\underline{\text{mod}} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} A\text{-}\overline{\text{mod}}.$$

Applying D to the exact sequence defining $\text{Tr}\,M$, we see that there is an exact sequence

$$0 \to \tau M \to \nu(P_1) \to \nu(P_0) \to \nu(M) \to 0.$$

Thus τ con be computed by taking a minimal projective presentation of M, applying the Nakayama functor (which turns each $P[i]$ into $I[i]$) and taking the kernel.

Example. For the commutative square with source 1 and sink 4, the simple $S[2]$ has minimal projective presentation

$$P[4] \to P[2] \to S[2] \to 0$$

so we get

$$0 \to \tau S[2] \to I[4] \to I[2]$$

so $\tau S[2] \cong P[3]$.

Lemma 2. If M is an A-module, then
(i) $\operatorname{proj.dim} M \leq 1 \Leftrightarrow \operatorname{Hom}(DA, \tau M) = 0 \Leftrightarrow$ there is no non-zero map from an injective module to τM.
(ii) $\operatorname{inj.dim} M \leq 1 \Leftrightarrow \operatorname{Hom}(\tau^{-} M, A) = 0 \Leftrightarrow$ there is no non-zero map from $\tau^{-} M$ to a projective module.

Proof. (i) Recall that $\nu^{-}(-) = \operatorname{Hom}(DA, -)$, and that $\nu^{-}(\nu(P)) \cong P$. Thus we get $0 \to \nu^{-}(\tau M) \to \nu^{-}(\nu(P_1)) \to \nu^{-}(\nu(P_0))$ exact, so $0 \to \nu^{-}(\tau M) \to P_1 \to P_0$. Thus $\operatorname{proj.dim} M \leq 1$ iff $P_1 \to P_0$ is injective iff $\nu^{-}(\tau M) = 0$ iff $\operatorname{Hom}(DA, \tau M) = 0$.

(ii) Dual.

Lemma 3. Given a right A-module M, a left A-module N, $m \in M$ and $n \in N$ let $f_{mn} : M^{\vee} \to N$ be the map defined by $f_{mn}(\alpha) = \alpha(m)n$. It is a left A-module map. There is a natural transformation

$$\theta_{MN} : D\operatorname{Hom}(M^{\vee}, N) \to \operatorname{Hom}(M, DN), \quad \theta_{MN}(\xi) = (m \mapsto (n \mapsto \xi(f_{mn}))).$$

Then θ_{MN} is an isomorphism for M projective. And in general the image of θ_{MN} is $\operatorname{Hom}^{\text{proj}}(M, DN)$.

Proof. The first part is clear. Clearly θ_{MN} is well-defined. Both $D\operatorname{Hom}(M^{\vee}, N)$ and $\operatorname{Hom}(M, DN)$ define functors which are contravariant in M and N, and it is straightforward that θ_{MN} is natural in M and N.

For M projective, the map is an isomorphism, since it is for $M = A$. Thus given a map $f : M \to P$ with P projective, we get a commutative diagram

$$\begin{array}{ccc} D\operatorname{Hom}(P^{\vee}, N) & = & \operatorname{Hom}(P, DN) \\ {\scriptstyle b}\downarrow & & {\scriptstyle a}\downarrow \\ D\operatorname{Hom}(M^{\vee}, N) & \xrightarrow{\theta_{MN}} & \operatorname{Hom}(M, DN) \end{array}$$

where the top horizontal map is the natural isomorphism θ_{PN} and the vertical maps are induced by f. Any map $M \to DN$ factoring through P is in the image of a, so in $\mathrm{Im}(\theta_{MN})$.

Varying P, we get $\mathrm{Hom}^{\mathrm{proj}}(M, DN) \subseteq \mathrm{Im}(\theta_{MN})$.

Now take a basis of $M^\vee$. This defines a map $M \to P$, where $P = A^n$. Then $P^\vee \to M^\vee$ is onto. Thus $\mathrm{Hom}(M^\vee, N) \to \mathrm{Hom}(P^\vee, N)$ is 1-1. Thus b is onto. Thus $\mathrm{Im}(\theta_{MN}) = \mathrm{Im}(a) \subseteq \mathrm{Hom}^{\mathrm{proj}}(M, DN)$.

Theorem. There are isomorphisms

$$\underline{\mathrm{Hom}}(\tau^- N, M) \cong D\,\mathrm{Ext}^1(M, N) \cong \overline{\mathrm{Hom}}(N, \tau M).$$

Proof. Given a minimal projective presentation $P_1 \to P_0 \to M \to 0$, write $\Omega^1 M$ for the image of $P_1 \to P_0$, so there is

$$0 \to \Omega^1 M \to P_0 \to M \to 0$$

and hence

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(\Omega^1 M, N) \to \mathrm{Ext}^1(M, N) \to 0.$$

Also we have

$$0 \to M^\vee \to P_0^\vee \to P_1^\vee \to \mathrm{Tr}\, M \to 0$$

so

$$0 \to (\mathrm{Tr}\, M)^\vee \to P_1 \to P_0$$

so

$$0 \to (\mathrm{Tr}\, M)^\vee \to P_1 \to \Omega^1 M \to 0.$$

and hence

$$0 \to \mathrm{Hom}(\Omega^1 M, N) \to \mathrm{Hom}(P_1, N) \to \mathrm{Hom}((\mathrm{Tr}\, M)^\vee, N).$$

Thus we have a commutative diagram with exact rows and columns,

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & D\,\mathrm{Ext}^1(M,N) & & \\
 & & & & & & \downarrow & & \\
 & & D\,\mathrm{Hom}((\mathrm{Tr}\,M)^\vee, N) & \longrightarrow & D\,\mathrm{Hom}(P_1, N) & \longrightarrow & D\,\mathrm{Hom}(\Omega^1 M, N) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathrm{Tr}\,M, DN) & \longrightarrow & \mathrm{Hom}(P_1^\vee, DN) & \longrightarrow & \mathrm{Hom}(P_0^\vee, DN) & & \\
 & & \downarrow & & & & & & \\
 & & \underline{\mathrm{Hom}}(\mathrm{Tr}\,M, DN) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

By the Snake Lemma we get an isomorphism $D\,\mathrm{Ext}^1(M,N) \to \underline{\mathrm{Hom}}(\mathrm{Tr}\,M, DN)$.

Now use Lemma 1 to rewrite this as $\overline{\mathrm{Hom}}(N, D\,\mathrm{Tr}\,M)$, or use that Tr gives inverse anti-equivalences between A-$\underline{\mathrm{mod}}$ and A^{op}-$\underline{\mathrm{mod}}$ to rewrite it as $\underline{\mathrm{Hom}}(M, \mathrm{Tr}\,DN)$.

Corollary. If A is hereditary, we get

$$\mathrm{Hom}(\tau^- N, M) \cong D\,\mathrm{Ext}^1(M,N) \cong \mathrm{Hom}(N, \tau M).$$

Proof. Use Lemma 2. We have $\mathrm{Hom}(\tau^- N, M) \cong \underline{\mathrm{Hom}}(\tau^- N, M)$ if $\mathrm{inj.dim}\, N \leq 1$, and $\mathrm{Hom}(N, \tau M) \cong \overline{\mathrm{Hom}}(N, \tau M)$ if $\mathrm{proj.dim}\, M \leq 1$.

2.3 Auslander-Reiten sequences

Definitions. Given X, a map $f : X \to Y$ is a *source map* for X if it is left minimal, not a split mono, and any map $X \to M$ which is not split mono factors through f.

Given Z, a map $g : Y \to Z$ is a *sink map* for Z if it is right minimal, not a split epi, and any map $M \to Z$ which is not split epi factors through g.

Remarks.
(i) If X has a source map, then X is indecomposable and the map is unique

up to isomorphism, that is, if $X \to Y$ and $X \to Y'$ are source maps, then there is an isomorphism $Y \to Y'$ giving a commutative triangle. Similarly for sink maps.

(ii) $I[i] \to I[i]/S[i]$ is a source map for $I[i]$, and $\mathrm{rad}\, P[i] \to P[i]$ is a sink map for $P[i]$.

Definition. By an *Auslander-Reiten sequence* or *almost split sequence* we mean an exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

where f is a source map for X and g is a sink map for Z.

If an AR sequence exists, it is determined up to isomorphism by either of the end terms.

Lemma. If M is a (f.d.) A-B-bimodule, and $\mathrm{soc}({}_AM)$ and $\mathrm{soc}(M_B)$ are simple, then they are equal.

Proof. Since the socle is functorial, if $\theta \in \mathrm{End}_A(M)$ then $\theta(\mathrm{soc}({}_AM)) \subseteq \mathrm{soc}({}_AM)$. Thus $\mathrm{soc}({}_AM)$ is a B-submodule of M. Since $\mathrm{soc}(M_B)$ is simple, it must be contained in any non-zero B-submodule of M, so $\mathrm{soc}(M_B) \subseteq \mathrm{soc}({}_AM)$. Dually we get the other incluson.

Theorem. Let Z be a non-projective indecomposable A-module, and let $X = \tau Z$ be the corresponding non-injective indecomposable module. (Or equivalently let X be non-injective indecomposable and let $Z = \tau^- X$.) Then there exists an Auslander-Reiten sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0.$$

Proof. $\mathrm{Ext}^1(Z, X)$ is an $\mathrm{End}(X)$-$\mathrm{End}(Z)$-bimodule.

As a right $\mathrm{End}(Z)$ module it is isomorphic to $D\underline{\mathrm{End}}(Z)$, so has simple socle S, corresponding to the fact that $\underline{\mathrm{End}}(Z)$ as a left $\mathrm{End}(Z)$-module is a quotient of $\mathrm{End}(Z)$, so has simple top, since Z is indecomposable.

As a left $\mathrm{End}(X)$ module it is isomorphic to $D\overline{\mathrm{End}}(X)$, so has simple socle T, corresponding to the fact that $\overline{\mathrm{End}}(X)$ as a right $\mathrm{End}(X)$-module is a quotient of $\mathrm{End}(X)$, so has simple top, since X is indecomposable.

By the lemma, $S = T$. Let

$$\xi : 0 \to X \to Y \to Z \to 0$$

be an exact sequence corresponding to a non-zero element of S.

(a) Since $\xi \neq 0$ the map f is not a split mono and g is not a split epi.

(b) Suppose $M \to Z$ not a split epi. The map $\mathrm{Hom}(Z, M) \to \mathrm{End}(Z)$ has image contained in the radical of $\mathrm{End}(Z)$.

Thus the map $\underline{\mathrm{Hom}}(Z, M) \to \underline{\mathrm{End}}(Z)$ has image contained in the radical of $\underline{\mathrm{End}}(Z)$.

Thus the map $D\underline{\mathrm{End}}(Z) \to D\underline{\mathrm{Hom}}(Z, M)$ kills the socle of $D\underline{\mathrm{End}}(Z)$ as a $\mathrm{End}(Z)$-module.

Thus the map $\mathrm{Ext}^1(Z, X) \to \mathrm{Ext}^1(M, X)$ kills ξ. Thus the pullback of ξ by $M \to Z$ splits.

Using a section of this pullback we get a map $M \to Y$ whose composition is the original map $M \to Z$.

(b') By duality, if $X \to M$ is not a split mono, it factors through f.

(c) If g is not right minimal, then there is non-invertible $\alpha \in \mathrm{End}(Y)$ with $g\alpha = g$. Then g induces non-invertible $\beta \in \mathrm{End}(X)$ with $\alpha f = f\beta$. Now $\beta^n = 0$ for some n, so $0 = f\beta^n = \alpha^n f$, so $\alpha^n = rg$ for some $r : Z \to Y$. But then $g = g\alpha^n = grg$, so since g is epi, $gr = 1_Z$, contradicting that g is not split epi. Thus g is right minimal.

(c') Similarly f is left minimal.

Corollary. Every indecomposable module has a source map and a sink map.
(i) If X is indecomposable non-injective then the map $X \to Y$ in the AR sequence starting at X is a source map, and if $X = I[i]$ then $I[i] \to I[i]/S[i]$ is a source map.
(ii) If Z is indecomposable non-projective then the map $Y \to Z$ in the AR sequence ending at Z is a sink map, and if $Z = P[i]$ then $\mathrm{rad}\, P[i] \to P[i]$ is a sink map.

2.4 Irreducible maps

Definition. A map $f : X \to Y$ is *irreducible* if it is in $\mathrm{rad}(X, Y)$ and for any factorization $f = \phi\theta$ with $\theta : X \to M$ and $\phi : M \to Y$, either θ is split mono or ϕ is split epi.

Remarks.
(i) Any irreducible map is mono or epi. Otherwise it factors through its image.

(ii) The kernel/cokernel of an irreducible map is indecomposable (exercise).

(iii) Any source or sink map is irreducible. If $X \to Y$ is a source map for X, then the irreducible maps $X \to Z$ are the compositions $X \to Y \to Z$ with $Y \to Z$ split epi. Any irreducible map factors this way. Conversely, it suffices to show that if

$$\begin{pmatrix} \phi \\ \phi' \end{pmatrix} : X \to Z \oplus Z'$$

is irreducible, so is ϕ. For this, observe that if ϕ factors through M then $\binom{\phi}{\phi'}$ factors through $M \oplus Z'$.

Dually, the if $Y \to Z$ is a sink map for Z, the irreducible maps $X \to Z$ are the compositions $X \to Y \to Z$ with $X \to Y$ split mono.

(iv) This is not the classical definition of an irreducible map. For example if Y is indecomposable, then $0 \to Y$ is irreducible, but it would not be with the classical definition. Thus (iii) would be wrong.

Recall that we have defined $\mathrm{rad}(X,Y) \subseteq \mathrm{Hom}(X,Y)$. If X is indecomposable it is the set of maps which are not split monos. If Y is indecomposable it is the set of maps which are not split epis. If X and Y are indecomposable it is the set of non-isomorphisms.

Definitions. We define $\mathrm{rad}^2(X,Y)$ to be the set of all homomorphisms $X \to Y$ which can be written as a composition

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

with $f \in \mathrm{rad}(X,M)$ and $g \in \mathrm{rad}(M,Y)$. This is a subspace of $\mathrm{rad}(X,Y)$.

If X,Y are indecomposable, we define $\mathrm{irr}(X,Y) = \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$. It is naturally a D_Y-D_X-bimodule, where D_X is the division algebra $\mathrm{End}(X)/J(\mathrm{End}(X))$.

For X,Y indecomposable, $f : X \to Y$ is irreducible iff $f \in \mathrm{rad}(X,Y)$ and $f \notin \mathrm{rad}^2(X,Y)$. Thus there is an irreducible map $X \to Y$ iff $\mathrm{irr}(X,Y) \neq 0$.

Recall from the section on the Krull-Remak-Schmidt Theorem that the multiplicity of an indecomposable M as a direct summand of a module Y is

$$\mu_M(Y) = \frac{\dim t(M,Y)}{\dim D_M}, \qquad t(M,Y) = \frac{\mathrm{Hom}(M,Y)}{\mathrm{rad}(M,Y)}.$$

Theorem. Let M be indecomposable.
(i) If $f : X \to Y$ is a source map, then $\dim \operatorname{irr}(X, M) = \mu_M(Y). \dim D_M$,
(ii) If $g : Y \to Z$ is a sink map, then $\dim \operatorname{irr}(M, Z) = \mu_M(Y). \dim D_M$.
In particular, for given X and Z, there are only finitely many indecomposable M with $\operatorname{irr}(X, M)$ or $\operatorname{irr}(M, Z)$ non-zero.

Proof. (ii) Since g is a sink map, either $\operatorname{Ker} g$ is zero, or $\operatorname{Ker} g \to Y$ is a source map. Either way it is in $\operatorname{rad}(\operatorname{Ker} g, Y)$. Since g is a radical homomorphism, composition with g induces left exact sequences

$$0 \to \operatorname{Hom}(M, \operatorname{Ker} g) \to \operatorname{Hom}(M, Y) \to \operatorname{rad}(M, Z)$$

and

$$0 \to \operatorname{Hom}(M, \operatorname{Ker} g) \to \operatorname{rad}(M, Y) \to \operatorname{rad}^2(M, Z)$$

and since g is a sink map, these are exact on the right. For example any map $\theta \in \operatorname{rad}^2(M, Z)$ can be written as a sum $\theta = \sum_i \psi_i \phi_i$ with $\phi_i \in \operatorname{rad}(M, X_i)$ and $\psi_i \in \operatorname{rad}(X_i, Z)$ and X_i indecomposable. But then ψ_i is not split epi, so it factorizes as $g\chi_i$ for some $\chi_i \in \operatorname{Hom}(X_i, Y)$, and then $\theta = g(\sum_i \chi_i \phi_i)$, and $\sum_i \chi_i \phi_i \in \operatorname{rad}(M, Y)$. Thus

$$\dim \operatorname{irr}(M, Z) = \dim[\operatorname{Hom}(M, Y)/\operatorname{rad}(M, Y)] = \dim D_M.\mu_M(Y).$$

Corollary. If X is indecomposable, then

$$\left(\sum_M \frac{\dim \operatorname{irr}(X, M)}{\dim D_M} \dim M \right) - \dim X = \begin{cases} -\dim S[i] & (X \cong I[i]) \\ \dim \tau^- X & (X \text{ not injective}) \end{cases}$$

where the sum is over all indecomposable modules up to isomorphism. Moreover if $A = KQ/I$ then the same applies for dimension vectors.

2.5 Auslander-Reiten quiver

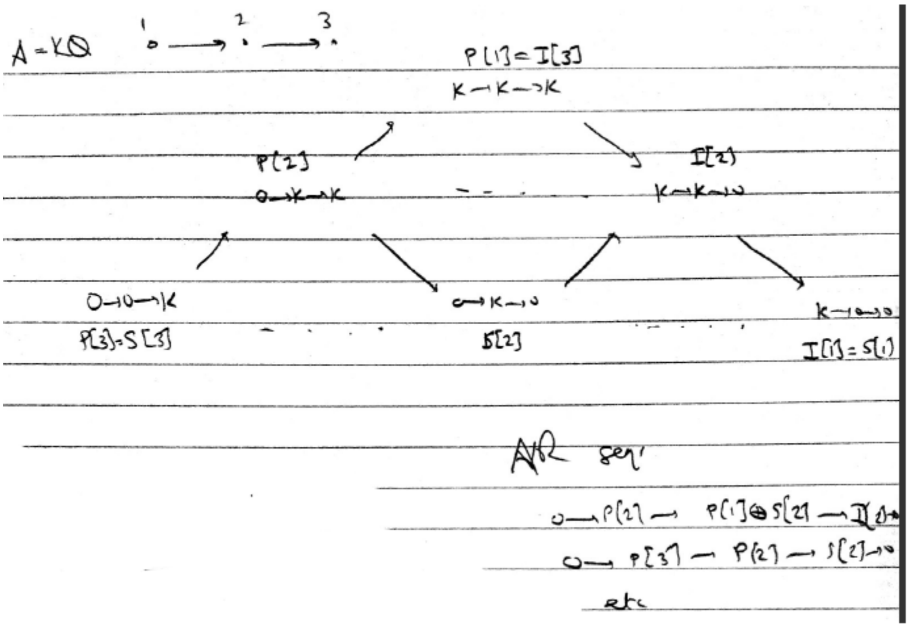
We now assume that K is algebraically closed. Otherwise we would have to deal with valued quivers. Thus for M indecomposable, $D_M = K$, so $\dim D_M = 1$.

Definition. Given a f.d. algebra A, the *Auslander-Reiten quiver* of A has vertices corresponding to the isomorphism classes of indecomposable A-modules, and the number of arrows $M \to N$ is $\dim \operatorname{irr}(M, N)$.

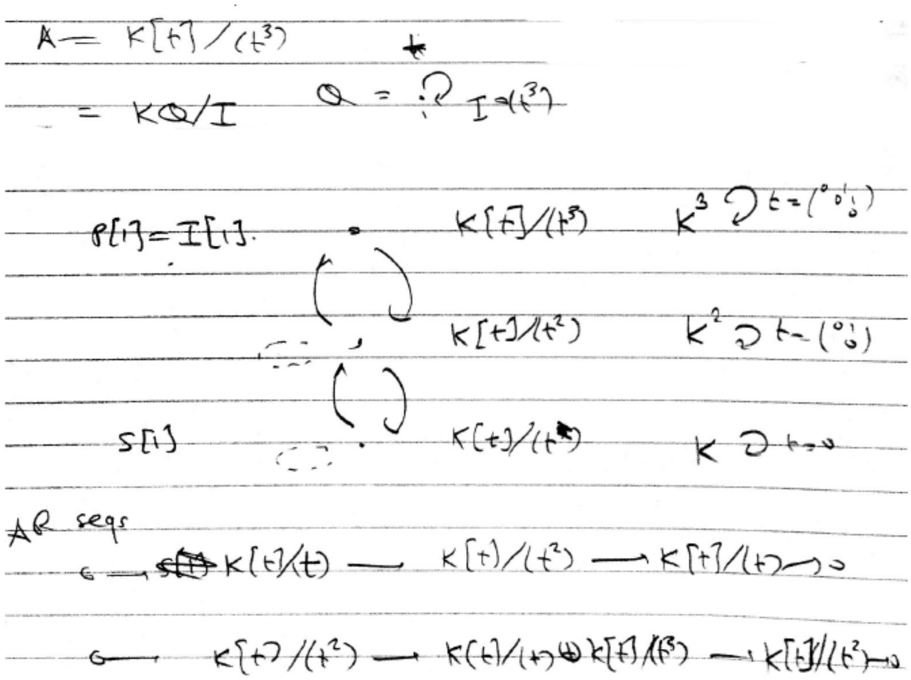
It is often useful to indicate the AR translate τ by a dotted line joining Z and $\tau Z = D\,\mathrm{Tr}\,Z$.

In general the AR quiver is not connected. It is finite iff the algebra has only finitely many indecomposable modules, that is, it has finite representation type.

Examples. For a Nakayama algebra, the irreducible maps between indecomposables are the monos $X \to Y$ with simple cokernel and the epis $X \to Y$ with simple kernel.



Linear quiver with three vertices.



$K[t]/(t^3)$.

Harada-Sai Lemma. A composition of $2^n - 1$ non-isomorphisms between indecomposables of dimension $\leq n$ must be zero.

Proof. We show for $m \leq n$ that a composition of $2^m - 1$ non-isomorphisms between indecomposables of dimension $\leq n$ has rank $\leq n - m$.

If $m = 1$ this is clear. If $m > 1$, a composition of $2^m - 1$ non-isomorphisms can be written as a composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

where f and h are compositions of $2^{m-1} - 1$ non-isomorphisms. By induction $\operatorname{rank} f, \operatorname{rank} h \leq n - m + 1$. If either has strictly smaller rank, we're done. Thus suppose that $\operatorname{rank} f = \operatorname{rank} h = \operatorname{rank} hgf = n - m + 1$.

This implies that $\operatorname{Ker} f = \operatorname{Ker} hgf$ and $\operatorname{Im} hgf = \operatorname{Im} h$. It follows that $Y = \operatorname{Ker} hg \oplus \operatorname{Im} f$ and $Z = \operatorname{Ker} h \oplus \operatorname{Im} gf$. For example if $y \in Y$ then $hg(y) = hgf(x)$, so $y = f(x) + (y - f(x)) \in \operatorname{Im} f + \operatorname{Ker} hg$, and if $y \in \operatorname{Im} f \cap \operatorname{Ker} hg$ then $y = f(x)$ and $hgf(x) = 0$, so $x \in \operatorname{Ker} hgf = \operatorname{Ker} f$, so $y = 0$.

By indecomposability f is onto and h is 1-1. Thus $\dim Y = \dim Z = n - m + 1$ and g is an isomorphism. Contradiction.

Theorem (Auslander). Assume that the algebra A is given by a connected quiver and an admissible ideal. If C is a connected component of the AR quiver, and there is a bound on the dimension of the indecomposable modules in C, then C is finite and is the whole of the AR quiver of A.

Proof. Suppose M, N are indecomposable modules with $\mathrm{Hom}(M, N) \neq 0$. For $i \geq 0$ we consider a chain of maps

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \dots \xrightarrow{f_i} M_i \xrightarrow{g_i} N$$

with the M_j indecomposable, f_j irreducible and $g_i f_i \dots f_1 \neq 0$. Such a chain exists if $i = 0$. If g_i is not an isomorphism, then it is not a split mono, so it factors through the source map $M_i \to E$. Then we get a chain of size $i + 1$ by taking M_{i+1} to be one of the summands of E.

Suppose all indecomposables in C have dimension $\leq n$.

If M is in C, then by Harada-Sai any such chain must have length $i < 2^n - 1$. Thus the construction must terminate, with g_i an isomorphism, for some $i < 2^n - 1$. Thus there is a chain of irreducible maps from M to N of length $< 2^n - 1$. Dually if N is in C.

Now choose some M in C. There is a projective $P[i]$ with $\mathrm{Hom}(P[i], M) \neq 0$, so $P[i] \in C$. If $i \to j$ is an arrow in Q there is a non-zero map $P[j] \to P[i]$, so if one is in C so is the other. Thus all projectives are in C. Thus C is the whole AR quiver.

Now for any indecomposable there is a chain of irreducible maps of length $< 2^n - 1$ from a projective $P[i]$. Thus C is finite.

Corollary (First Brauer-Thrall Conjecture). If there is a bound on the dimensions of indecomposable A-modules, then A has only finitely many indecomposable modules. (Finite representation type.)

Definition. An indecomposable module Z is *directing* if there is no cycle of non-zero non-isomorphisms between indecomposable modules that includes Z, so $Z \to Z_1 \to \cdots \to Z_k \to Z$ with $k \geq 0$.

Proposition. Let Z be an indecomposable module. Suppose there is a bound on the length of paths in the AR quiver ending at Z. Then Z is directing.

Proof. By induction on the bound. If zero, then Z is simple projective. But then there is no non-zero non-isomorphism from an indecomposable module to Z. Otherwise, decompose the sink map $Y_1 \oplus \cdots \oplus Y_m \to Z$. It there is a cycle, say $Z \to Z_1 \to \cdots \to Z_k \to Z$, then the map $Z_k \to Z$ factors through the sink map, so for some i there are non-zero map $Z_k \to Y_i \to Z$. Now the map $Z_k \to Y_i$ is either an isomorphism, or not. Either way we see that Y_i is in a cycle. Impossible by induction.

Definition. A module M is *sincere* if, considered as a representation of the quiver Q, the vector space at each vertex i is non-zero. Equivalently $\mathrm{Hom}(P[i], M) \neq 0$ for all i. Equivalently $\mathrm{Hom}(M, I[i]) \neq 0$ for all i.

Lemma. If M is sincere and directing, then $\operatorname{proj.dim} M \leq 1$, $\operatorname{inj.dim} M \leq 1$, $\mathrm{End}(M) = K$ and $\mathrm{Ext}^1(M, M) = 0$.

Proof. If $\operatorname{proj.dim} M \geq 2$ then there is a non-zero map $I[i] \to \tau M$ for some i. But then one gets a cycle $M \to I[i] \to \tau M \to E_i \to M$.

Similarly for injective dimension.

$\mathrm{End}(M) = K$ trivially.

$\mathrm{Ext}^1(M, M) \cong \overline{\mathrm{Hom}}(M, \tau M) = 0$, for if $M \to \tau M$ is non-zero, either it is an isomorphism, or not, and either way one gets a cycle.

Proposition. If M is directing and M' is indecomposable, of the same dimension vector, then $M \cong M'$.

Proof. Passing to $A/(\sum e_i)$ for all vertices where M is zero, we may suppose that M (and M') are sincere. Let $0 \to P_1 \to P_0 \to M \to 0$ be a projective presentation. Then for any module X we have

$$\dim \mathrm{Hom}(M, X) - \dim \mathrm{Ext}^1(M, X) = \dim \mathrm{Hom}(P_0, X) - \dim \mathrm{Hom}(P_1, X).$$

This only depends on the dimension vector of X, so

$$\dim \mathrm{Hom}(M, M') - \dim \mathrm{Ext}^1(M, M') = \dim \mathrm{Hom}(M, M) - \dim \mathrm{Ext}^1(M, M) = 1$$

so $\mathrm{Hom}(M, M') \neq 0$. Similarly $\mathrm{Hom}(M', M) \neq 0$. Contradicts the directing property.

2.6 Knitting construction

We work with an algebra $A = KQ/I$. Still K is algebraically closed.

Preparation. Compute the modules $\operatorname{rad} P[i]$ and decompose into indecomposable summands. We need to the dimension vectors of the summands.

Iterative construction. We suppose we have drawn a finite subquiver of the AR quiver with no oriented cycles and the property that if an indecomposable module X is in the subquiver, then so are all arrows ending at X, and suppose we know the dimension vectors of the modules we have drawn. All modules we have drawn are directing, so uniquely determined by their dimension vectors.

We start with the empty subquiver.

If we have drawn all the summands of $\operatorname{rad} P[i]$, but haven't yet drawn $P[i]$, we can now draw $P[i]$ and fill in the arrows ending at $P[i]$ with their multiplicities. In particular we can start by drawing the simple projective modules.

Suppose we have drawn an indecomposable module X. If we have drawn all projectives $P[i]$ such that X is a summand of $\operatorname{rad} P[i]$, and if we have drawn $\tau^- U$ for all non-injective undecomposables U with an arrow $U \to X$, then we can be sure that we have drawn all arrows starting at X.

If we have drawn all arrows starting at X then

$$-\underline{\dim} X + \sum_M \dim \operatorname{irr}(X, M) \, \underline{\dim} M = \begin{cases} -\underline{\dim} S[i] & (X \cong I[i]) \\ \underline{\dim} \tau^- X & (X \text{ not injective}) \end{cases}$$

so we know whether or not X is injective by the sign of the left hand side. If it is not injective, and we haven't yet drawn $\tau^- X$, we can now do so, and draw $\dim \operatorname{irr}(M, \tau^- X) = \dim \operatorname{irr}(X, M)$ arrows from M to X, for all M.

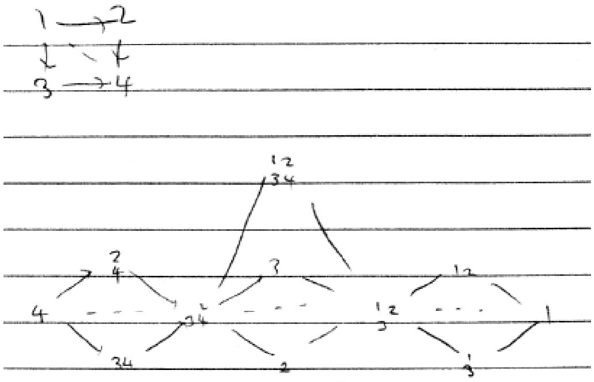
Repeat.

Several possibilities. (a) Get stuck, because either there is no simple projective, or we have written down some summands of $\operatorname{rad} P[i]$, for some projective P, but can't write down all summands, so can't write down $P[i]$.

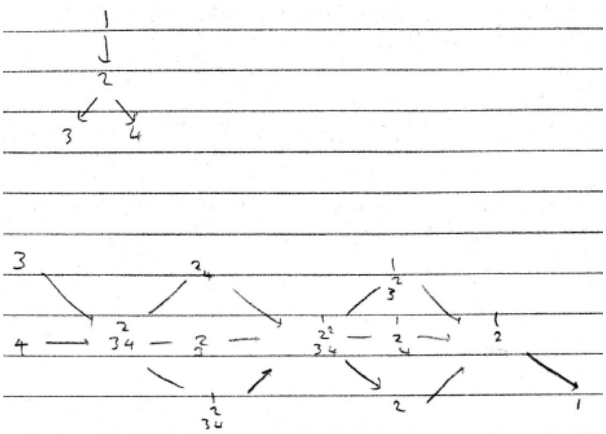
(b) Terminate after a finite number of steps. By Auslander's Theorem we have the whole AR quiver.

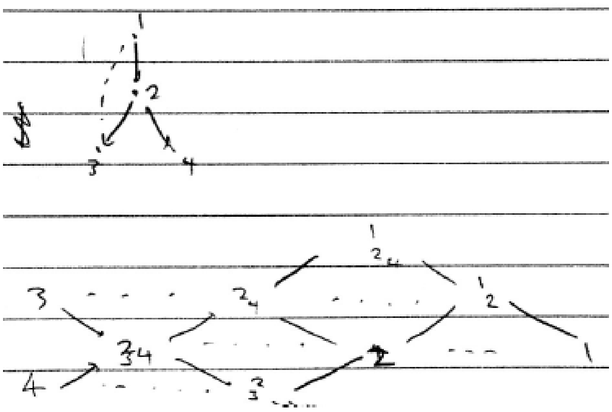
(c) Go on forever. In this case we have constructed one or more connected components of the AR quiver, called 'preprojective' components.

Examples.

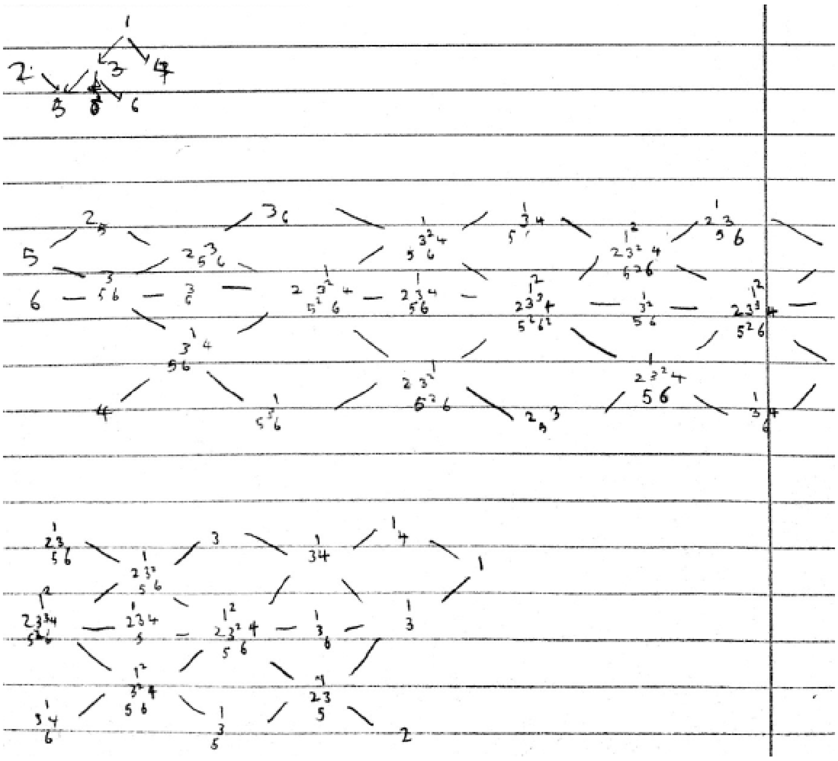


Quiver $\circ \leftarrow \circ \rightarrow \circ$.





E_6:

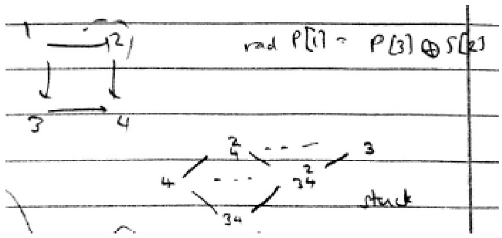


4-subspace,

Kronecker quiver,

Kronecker quiver with another vertex 3, such that the radical of $P[i]$ never appears.

Maybe one gets stuck. For example if some projective has decomposable radical:



Dually construct preinjective components starting with simple injective.

2.7 Covering theory via graded modules

The knitting procedure fails for many algebras. But a tool called 'covering theory', due to Gabriel (1981), can often be used to make it work. By Gordon and Green (1982) it is essentially equivalent to study graded modules.

Recall that A is a $\mathbb{Z}$-graded algebra if

$$A = \bigoplus_{n \in \mathbb{Z}} A_n, \qquad A_n.A_m \subseteq A_{n+m}.$$

It follows that $1 \in A_0$. We assume that A is f.d.. Thus only finitely many A_n are nonzero.

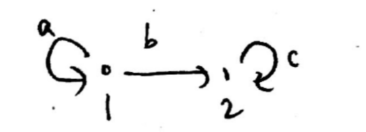
Recall that a $\mathbb{Z}$-graded A-module is an A-module

$$M = \bigoplus_{n \in \mathbb{Z}} M_n, \qquad A_n.M_m \subseteq M_{n+m}.$$

We only consider f.d. graded modules, and write A-grmod for the category of f.d. $\mathbb{Z}$-graded A-modules, with

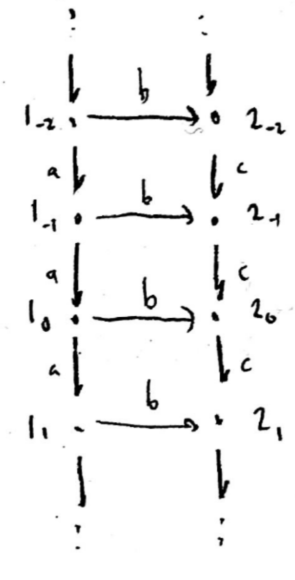
$$\mathrm{Hom}_{A\text{-grmod}}(M,N) = \{\theta \in \mathrm{Hom}_A(M,N) \mid \theta(M_n) \subseteq N_n \text{ for all } n \in \mathbb{Z}\}.$$

Given an algebra A such as

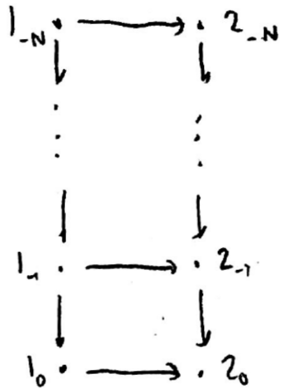


$$a^2 = 0\ ,\ c^2 = 0\ ,\ ba = cb$$

we can grade by setting $\deg a = \deg c = 1$ and $\deg b = 0$, and letting an arbitrary path have degree equal to the sum of the degrees of the arrows it involves (since with this choice the relations are homogeneous). Last semester we have seen that graded modules for an algebra correspond to modules for a suitable catalgebra $\hat{A}$ (an algebra which doesn't necessarily have a 1, but does have enough indempotents). In this example, the catalgebra is given by the quiver



with the corresponding relations. Given a graded A-module M, the vector space at vertex i_n is $e_i M_n$. So M_n is the direct sum of the vector spaces at vertices with subscript n. In particular, if we are only interested in graded modules living in degrees $-N \leq n \leq 0$, then we deal with the truncated catalgebra



This is now a f.d. algebra. Perhaps we can use knitting with it. We show how it can be used to understand modules for the original algebra A.

Theorem 1. A is local iff A_0 is local.

Proof. Suppose A is local. If I is a proper left ideal in A_0 then $AI \subseteq J(A)$ since it is a left ideal in A, and if $a \in A$ and $i \in I$, then i is not invertible in A_0, so not in A, so ai is not invertible, so it is in $J(A)$. Thus $I \subseteq J(A) \cap A_0$. Thus $J(A) \cap A_0$ is the unique maximal left ideal in A_0.

Now suppose that A_0 is local.

Let $I = \sum_{n \neq 0} A_n A_{-n} \subseteq A_0$.

If $a \in A_n$ and $b \in A_{-n}$ with $n \neq 0$, then a is nilpotent, so not invertible, so ab is not invertible in A, so it is not invertible in A_0, so $ab \in J(A_0)$. Thus $I \subseteq J(A_0)$.

Thus I is nilpotent. Say $I^N = 0$.

Let L be the ideal in A generated by all A_n ($n \neq 0$). Clearly $L = I \oplus \bigoplus_{n \neq 0} A_n$.

It suffices to show that L is nilpotent, for then $L \subseteq J(A)$, so that $A/J(A)$ is a quotient of $A/L \cong A_0/I$, which is local.

Suppose that A lives in d different degrees.

It suffices to show that any product $\ell_1 \ell_2 \ldots \ell_{dN}$ of homogeneous elements of L is zero.

Suppose not. Let d_i be the degree of $\ell_1 \ell_2 \ldots \ell_i$.

We have $dN+1$ numbers $d_0, d_1, \dots, d_{dN}$ taking at most d different values, so some value must occur at least $N+1$ times. Say

$$d_{i_1} = d_{i_2} = \cdots = d_{i_{N+1}}$$

with $i_1 < i_2 < \cdots < i_{N+1}$. Then we can write the product as

$$\ell_1 \dots \ell_{i_1}(\ell_{i_1+1} \dots \ell_{i_2})(\ell_{i_2+1} \dots \ell_{i_3}) \dots (\ell_{i_N+1} \dots \ell_{i_{N+1}})\ell_{i_{N+1}+1} \dots \ell_{dN}$$

But each of the bracketed terms has degree 0, so is in I, so their product is zero.

Definition. Given a graded module M and $i \in \mathbb{Z}$ we write $M(i)$ for the module with shifted grading $M(i)_n = M_{i+n}$. There is a forgetful functor F from A-grmod to A-mod which forgets the grading.

Lemma 1. If M, N are graded A-modules, then $\mathrm{Hom}_A(FM, FN)$ can be graded,

$$\mathrm{Hom}_A(FM, FN) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{A\text{-grmod}}(M, N(n)).$$

In this way

$$\mathrm{End}_A(FM) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{A\text{-grmod}}(M, M(n))$$

becomes a graded algebra.

Proof. Given a homomorphism $\theta : FM \to FN$, we get linear maps $\theta_n : M \to N$ defined by

$$\theta_n(m) = \sum_{i \in \mathbb{Z}} \theta(m_i)_{i+n}$$

where a subscript k applied to an element of a graded module picks out the degree k component of the element.

Now if $a \in A$ is homogeneous of degree d, then $(am)_i = a.m_{i-d}$, so

$$\theta_n(am) = \sum_i \theta((am)_i)_{i+n} = \sum_i \theta(a.m_{i-d})_{i+n} = \sum_i (a\theta(m_{i-d}))_{i+n}$$

$$= \sum_i a.\theta(m_{i-d})_{i+n-d} = \sum_j a.\theta(m_j)_{j+n} = a\theta_n(m).$$

Thus $\theta_n \in \mathrm{Hom}_{A\text{-grmod}}(M, N(n))$. Clearly θ is the sum of the θ_n. The rest is clear.

Corollary. (i) A graded module M is indecomposable iff the ungraded module FM is indecomposable.

(ii) If M and N are indecomposable graded modules with $FM \cong FN$ then M is isomorphic to $N(i)$ for some i.

Proof. (i) By Theorem 1, $\mathrm{End}_A(FM)$ is local iff its degree zero part is local. This is $\mathrm{End}_A(FM)_0 = \mathrm{End}_{A\text{-grmod}}(M)$. Now the ungraded module FM is indecomposable iff its endomorphism algebra $\mathrm{End}_A(FM)$ is local. The graded module M is indecomposable iff its endomorphism algebra $\mathrm{End}_{A\text{-grmod}}(M)$ has no non-trivial idempotents, and since it is f.d., it is equivalent that it is local.

(ii) Suppose $\theta : FM \to FN$ is an isomorphism. Then $\theta^{-1}\theta = 1_{FM}$, so $(\theta^{-1}\theta)_0 = 1_M$, so $\sum_i (\theta^{-1})_{-i}\theta_i = 1_M$. Since $\mathrm{End}(M)$ is local, some $(\theta^{-1})_i\theta_i$ is invertible, so $\theta_i : M \to N(i)$ is a split mono of graded modules, and hence also an isomorphism.

Setup. Let $A = KQ/I$ and grade it as above, assuming the relations are homogeneous. We suppose that A lives in non-negative degrees, so since it is f.d., it lives in degrees $[0, d]$ for some d.

Recall that graded A-modules correspond to modules for a catalgebra $\hat{A}$. Given $n \leq m$, graded modules living in degrees $[n, m] = \{i \in \mathbb{Z} : n \leq i \leq m]$ correspond to modules for a truncation of the catalgebra which is an actual algebra. It is

$$\tilde{A} = \begin{pmatrix} A_0 & 0 & \cdots & 0 & 0 \\ A_1 & A_0 & \cdots & 0 & 0 \\ A_2 & A_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{m-n-1} & A_{m-n-2} & \cdots & A_0 & 0 \\ A_{m-n} & A_{m-n-1} & \cdots & A_1 & A_0 \end{pmatrix}$$

We write F also for the functor from $\tilde{A}$-mod to A-mod.

Theorem 2. Suppose A lives in degrees $[0, d]$ with $d \geq 0$ and $\tilde{A}$ is the truncation corresponding to degrees $[n, m]$. If $\xi : 0 \to X \to Y \to Z \to 0$ is an AR sequence of $\tilde{A}$-modules, and Z lives in degrees $[n + d, m - 2d]$, then $F(\xi)$ is an AR sequence of A-modules.

Sketch of proof. The trivial idempotents $e_i \in A$ are homogeneous of degree 0, so the module $P_A[i] = Ae_i$ is graded, and lives in degrees $[0, d]$. Thus the module $P_A[i](j)$ lives in degrees $[j, j+d]$. Thus if $j \in [n, m-d]$ then $P_A[i](j)$ corresponds to an $\tilde{A}$-module. In fact it corresponds to $P_{\tilde{A}}[i_j]$. Thus $F(P_{\tilde{A}}[i_j]) \cong P_A[i]$.

Similarly, if $j \in [n+d, m]$ then $F(I_{\tilde{A}}[i_j] \cong I_A[i]$.

Take a minimal projective presentation

$$P_1 \to P_0 \to Z \to 0.$$

Now P_0 only involves projective covers of simples in degrees $[n+d, m-2d]$, so P_0 lives in degrees $[n+d, m-d]$. Then P_1 only involves projective covers of simples in degrees $[n+d, m-d]$. It follows that $F(P_i)$ are projective A-modules, and that

$$F(P_1) \to F(P_0) \to F(Z) \to 0$$

is a minimal projective presentation of $F(Z)$.

Now $\tau_{\tilde{A}} Z$ is computed using the exact sequence

$$0 \to \tau_{\tilde{A}} Z \to \nu_{\tilde{A}}(P_1) \to \nu_{\tilde{A}}(P_0).$$

Since the modules $\nu_{\tilde{A}}(P_i)$ only involve injective envelopes of simples in degrees $[n+d, m-d]$, $F(\nu_{\tilde{A}}(P_i))$ is injective, and isomorphic to $\nu_A(F(P_i))$. Thus

$$0 \to F(\tau_{\tilde{A}} Z) \to F(\nu_{\tilde{A}}(P_1)) \to F(\nu_{\tilde{A}}(P_0)),$$

is identified with the sequence

$$0 \to \tau_A F(Z) \to \nu_A(F(P_1)) \to \nu_A(F(P_0)).$$

Thus $\tau_A F(Z) \cong F(\tau_{\tilde{A}} Z) \cong F(X)$.

Now there is a homomorphism $\mathrm{End}_{\tilde{A}}(Z) \to \mathrm{End}_A(F(Z))$ whose image is the degree 0 part. It induces an isomorphism on tops.

This induces a map $\underline{\mathrm{End}}_{\tilde{A}}(Z) \to \underline{\mathrm{End}}_A(F(Z))$ giving an isomorphism on tops.

This gives $D\underline{\mathrm{End}}_A(F(Z)) \to D\underline{\mathrm{End}}_{\tilde{A}}(Z)$ giving an isomorphism on socles.

This gives a map $\mathrm{Ext}^1_A(F(Z), F(X)) \to \mathrm{Ext}^1_{\tilde{A}}(Z, X)$ giving an isomorphism on socles.

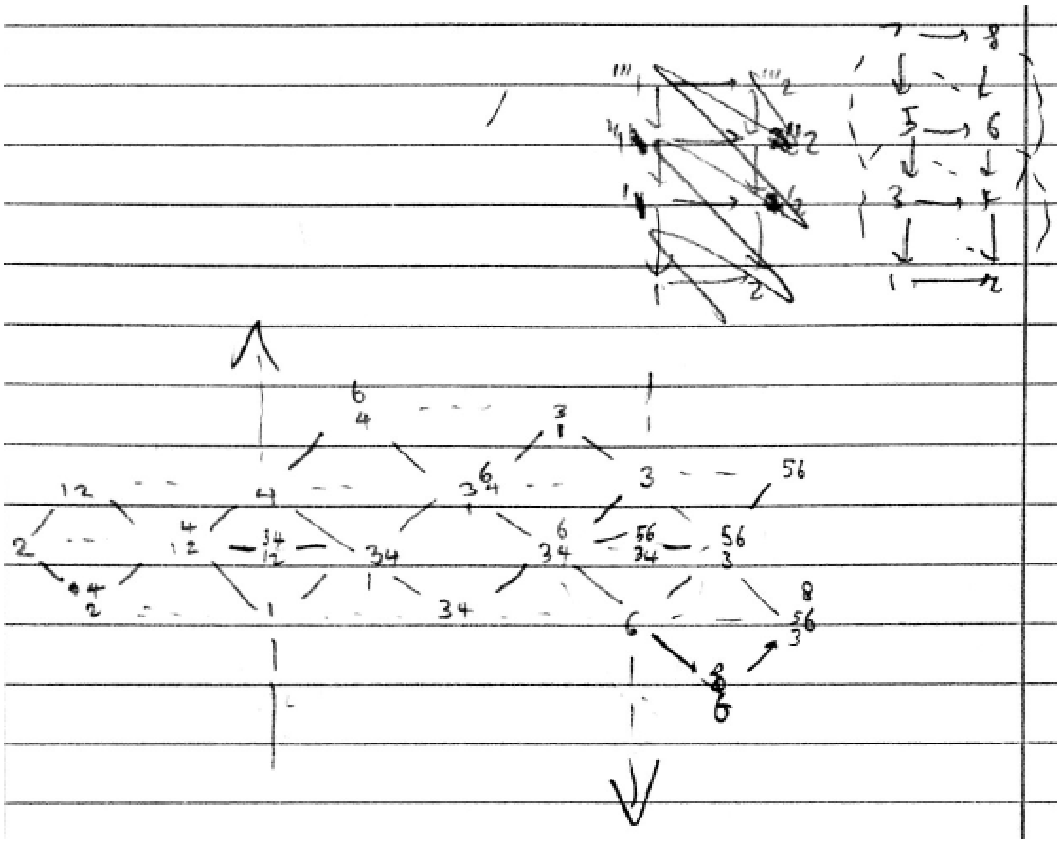
Now AR sequences are defined by elements of the socle, so the forgetful functor sends an AR sequence to an AR sequence.

Construction. Take a range of degrees $[-N, 0]$ with $N \gg 0$, which we consider to be finite, but arbitrarily large.

Now knit. If, eventually the knitted modules live in degrees $\leq -2d$, then the subsequent AR sequences are sent by the forgetful functor to AR sequences of A-modules.

If also the knitted modules are eventually all shifts of finitely many A-modules, then they give a finite connected component of the AR quiver of A. By Auslander's Theorem it is the whole AR quiver.

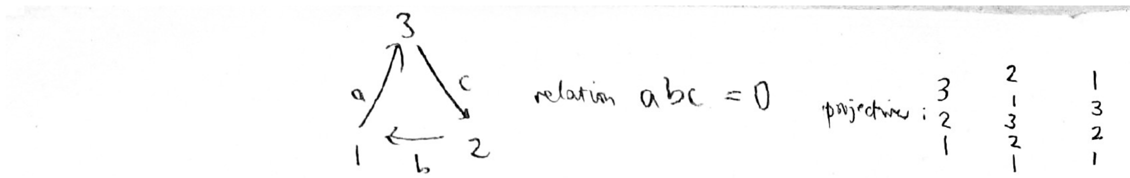
Examples.



Observe that the modules along the two vertices arrows correspond, with the modules on the right hand arrow being the shifts of the modules on the left hand arrow one place up the ladder. Moreover the modules to the right of each arrow also correspond. Thus you can be sure that all further knitting will follow the same pattern.

Now take the part of the AR quiver between the two vertices arrows. You can be sure that the forgetful functor sends it to a finite connected component of the AR quiver of A. Thus it is the whole of the AR quiver of A. You need to identify the two vertical arrows, giving a Möbius band.

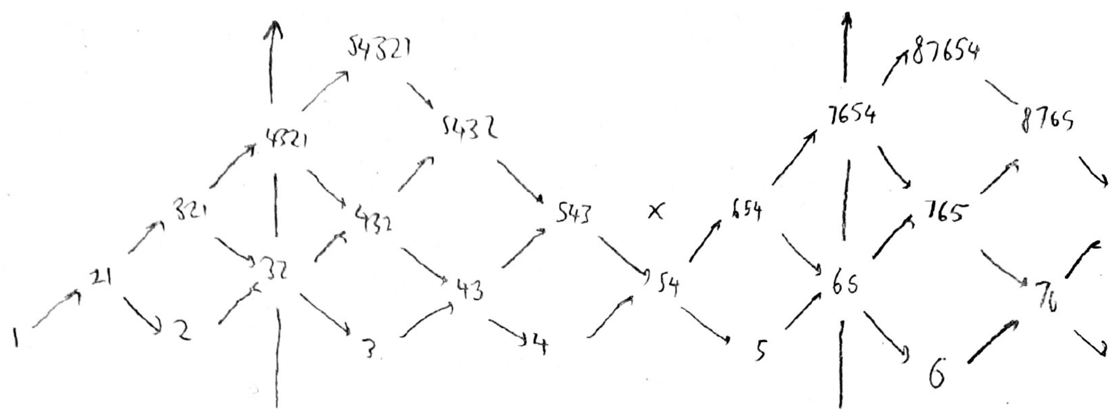
Another example: a Nakayama algebra (so we can compute its AR quiver anyway).



We grade it with $\deg a = 1$ and the other arrows of degree 0. Algebra $\tilde{A}$ is of the following form, where for simplicity we label the vertices $1_0, 2_0, 3_0, 1_{-1}, 2_{-1}, 3_{-1,-2}, \ldots$ as $1, 2, 3, 4, 5, 6, 7, \ldots$.

$$\to 9 \xrightarrow{c} 8 \xrightarrow{b} 7 \xrightarrow{a} 6 \xrightarrow{c} 5 \xrightarrow{b} 4 \xrightarrow{a} 3 \xrightarrow{c} 2 \xrightarrow{b} 1$$

Knitting gives the following.



Again we observe that the pattern repeats, so the AR quiver of A is the part between the two vertical arrows, with the arrows identified. The cross means that at that place it is not an AR sequence (since 654 is projective and 543 is injective).

Another example: Q with one vertex and loops p, q with relations $p^2 =$

$qpq, q^2 = pqp, p^3 = q^3 = 0$. There is no non-trivial grading, so can't get started.

In the case when this process works, every module is gradeable. In general that is not true.

For example the quiver with arrows from 1 to 2 and 3, and from 2 to 3. Grade it with the arrow from 1 to 3 of degree 1 and the others of degree 0. Then the module which is K at each vertex, identity for each arrow is not gradeable.

Theorem 3. If the field K has characteristic zero, and A is graded, then any A-module M with $\mathrm{Ext}^1(M, M) = 0$ is gradeable.

Proof. The result is possibly folklore. This proof comes from Keller, Murfet and van den Bergh, On two examples by Iyama and Yoshino.

Let $d : A \to A$ be the map defined by $d(a) = \deg(a)a$ for a homogeneous. It is a derivation since $d(ab) = \deg(ab)ab = (\deg(a)+\deg(b))ab = ad(b)+d(a)b$. It is called the *Euler derivation.*

Let $E = M \oplus M$ as a vector space, with A-module action given by $a(m, m') = (am, d(a)m + am')$. This is an A-module structure and there is an exact sequence

$$0 \to M \xrightarrow{\binom{0}{1}} E \xrightarrow{(1 0)} M \to 0$$

By assumption this is split, so there is a map $M \to E$ of the form $m \mapsto (m, \nabla(m))$. Moreover the map $\nabla : M \to M$ satisfies

$$\nabla(am) = d(a)m + a\nabla(m)$$

so it is a *connection on M with respect to d.* Since M is f.d.,

$$M = \bigoplus_{\lambda \in K} M_\lambda$$

where M_λ is the λ-generalised eigenspace for ∇. Now for any $\lambda \in K$ and a homogeneous we have

$$(\nabla - \lambda - \deg(a))^N(am) = a(\nabla - \lambda)^N(m)$$

for all $N \geq 1$, so $a(M_\lambda) \subseteq M_{\lambda+\deg(a)}$. Thus if we let T be a set of coset representatives for $\mathbb{Z}$ as a subgroup of K under addition, and set

$$M_n = \bigoplus_{\lambda \in T+n} M_\lambda$$

then $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded module.

2.8 An example of a self-injective algebra of finite representation type

Proposition. If P is an indecomposable projective-injective A-module which is not simple, then there is an AR-sequence

$$0 \to \operatorname{rad} P \xrightarrow{f} P \oplus \operatorname{rad} P/\operatorname{soc} P \xrightarrow{g} P/\operatorname{soc} P \to 0$$

where $f(x) = (x, \overline{x})$ and $g(x, y) = \overline{x} - y$.

Proof. Clearly $\operatorname{soc} P \subseteq \operatorname{rad} P$. Now $\operatorname{rad} P$ has simple socle, so is indecomposable and $P/\operatorname{soc} P$ has simple top, so is indecomposable. The sequence is clearly exact.

We show that any map $\theta : \operatorname{rad} P \to M$ which is not split mono factors through f. We may assume that M is indecomposable. If θ is not mono, the map factors through $\operatorname{rad} P/\operatorname{soc} P$, so through f. If θ is mono, then since P is injective, the inclusion $\operatorname{rad} P \to P$ lifts to a map $\phi : M \to P$. If ϕ is not epi, then it maps to $\operatorname{rad} P$ and is a section for θ, contradicting that θ is not a split mono. Thus ϕ is epi, so since P is projective, it is a split epi, so an isomorphism. Then using ϕ^{-1} we see that θ factors through f.

By symmetry any map $P/\operatorname{soc} P \to M$ factors through g.

Now g is right minimal by the argument of (c) in the proof of the theorem in section 2.3. Dually f is left minimal. Thus f is a source map and g a sink map. Thus the sequence is an AR sequence.

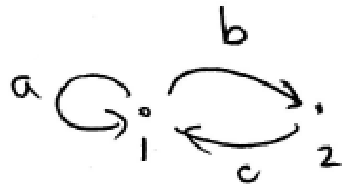
Lemma. If P is a projective-injective summand of A, $S = \operatorname{soc} P$ and $I = SA$, then if

$$0 \to X \to Y \to Z \to 0$$

is any AR sequence which is not of the form above for some summand of P, then X, Y, Z are killed by I, so this is also an AR sequence of A/I-modules.

Proof. Let P' be an indecomposable summand of P. It can't occur as X or Z since it is projective-injective. If it occurs are Y, then there is an irreducible map $X \to P'$. Thus X is a summand of $\operatorname{rad} P'$. Thus $X \cong \operatorname{rad} P'$. Thus the sequence is as in the last proposition.

Example. Consider the algebra with quiver



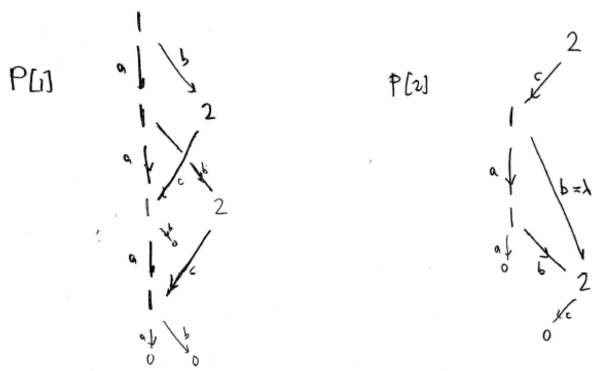
and relations $bacb = 0$, $bc = \lambda bac$ and $a^2 = cb$ for $\lambda \in K$. It is a special case of a *penny-farthing*.

Then $ba^2 = bcb = \lambda bacb = 0$, and hence also $a^4 = cbcb = 0$. Also $a^2c = cbc = \lambda cbac = \lambda a^3c$. Then $a^4 = 0 \Rightarrow a^3c = 0 \Rightarrow a^2c = 0$. Thus also $cbc = 0$.

If K has characteristic not 2, one can change generators to get $\lambda = 0$. If K has characteristic 2 this is not possible.

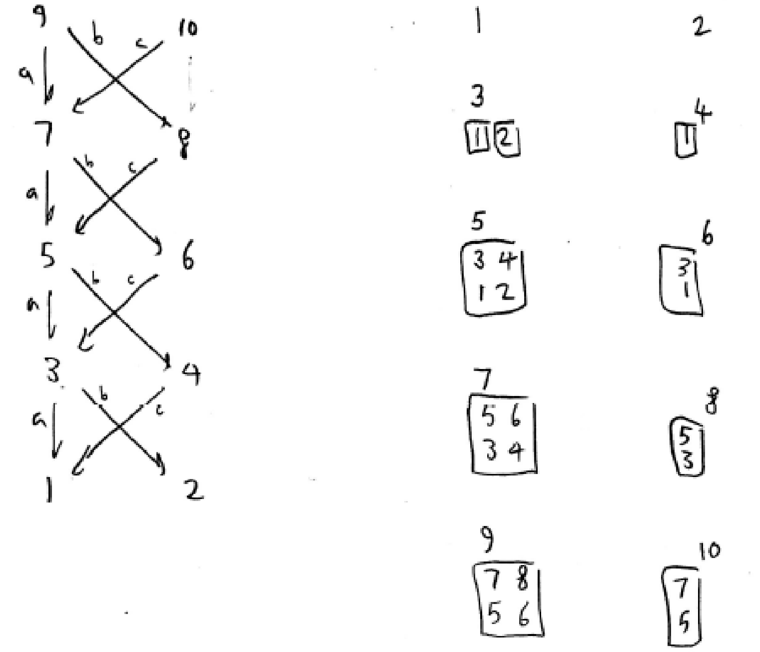
If $\lambda \neq 0$ there is no suitable grading.

The projectives are

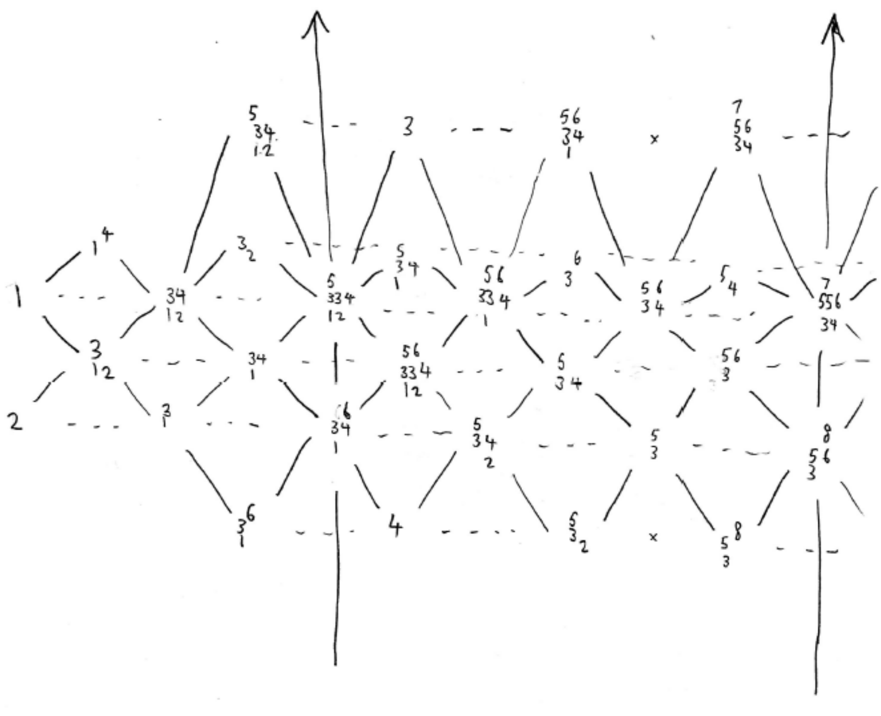


They have dimensions 6 and 4. Thus the algebra has dimension 10. Observe that the projectives have simples socles, and both simples occur. Thus the algebra embeds in the direct sum of the two injectives, which also has dimension 10. Thus the algebra is self-injective.

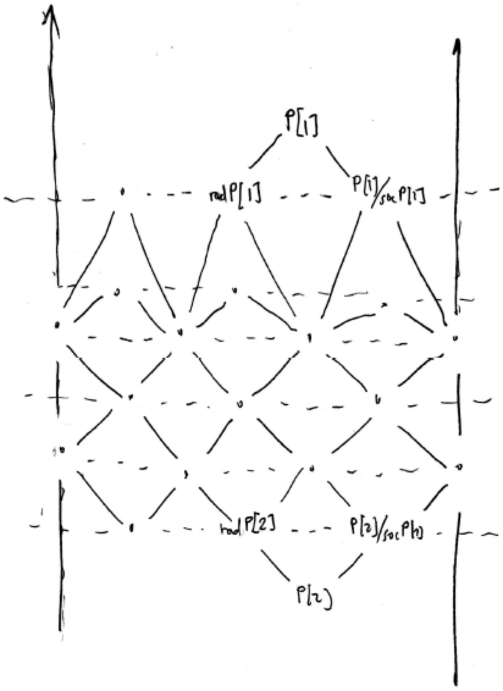
We pass to A/I where $I = \operatorname{soc} A$ (already an ideal), so add the relations $a^3 = cba = acb = 0$ and $bac = 0$, so $bc = 0$. According to the lemma of section 1.5, we only lose the two projective-injeective modules. The new algebra has a grading with all arrows of degree 1, so its covering and indecomposable projectives as follows (where we show the indecomposable summands of their radicals).



Knitting gives



Then we insert the original projective-injectives to get the AR quiver of A.



3 Homological topics

3.1 Faithfully balanced modules

See the following papers, and they papers they refer to.
T. Kato, Rings of U-dominant dimension ≥ 1, Tohoku Math. J. 1969.
B. Ma and J. Sauter, On faithfully balanced modules, F-cotilting and F-Auslander algebras, arXiv:1901.07855
Müller, The classification of algebras by dominant dimension, Canad. J. Math 1968.
C. M. Ringel, Artin algebras of dominant dimension at least 2, manuscript 2007, available from his Bielefeld homepage.

Definition. Let M be an A-module, and let $B = \mathrm{End}_A(M)$. Then M can be considered as a B-module, and there is a natural map $A \to \mathrm{End}_B(M)$. Clearly M is faithful iff this map is injective.

We say that M is a *balanced* A-module or that M has the *double centralizer property* if this map is onto, and that M is *faithfully balanced* (f.b.) if this map is an isomorphism.

Clearly M is a f.b. A-module iff DM is a f.b. A^{op}-module.

Definition. Given a module M, $\mathrm{gen}(M)$ denotes the module class consisting of quotients of direct sums of copies of M and $\mathrm{cogen}(M)$ the module class of submodules of a direct sum of copies of M.

We say M is a *generator* if $\mathrm{gen}(M) = A\text{-mod}$, or equivalently $A \in \mathrm{add}(M)$. We say M is a *cogenerator* if $\mathrm{cogen}(M) = A\text{-mod}$, or equivalently $DA \in \mathrm{add}(M)$.

Proposition/Definition 1. If M is an A-module and $n \geq 0$ then $\mathrm{gen}_n(M)$ is the module class consisting of the X satisfying the following equivalent conditions
(a) There is an exact sequence $M_n \to M_{n-1} \to \cdots \to M_0 \to X \to 0$ with $M_i \in \mathrm{add}\, M$ such that the sequence

$$\mathrm{Hom}_A(M, M_{n-1}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

is exact (note that this is automatic if M is projective).
(b) The natural map $\mathrm{Hom}_A(M, X) \otimes_B M \to X$ is surjective ($n = 0$) or an isomorphism ($n > 0$) and $\mathrm{Tor}_i^B(\mathrm{Hom}_A(M, X), M) = 0$ for $0 < i < n$.

Clearly $\mathrm{add}(M) \subseteq \dots \mathrm{gen}_2(M) \subseteq \mathrm{gen}_1(M) \subseteq \mathrm{gen}_0(M) = \mathrm{gen}(M)$.

Proof. Suppose (a). Then there is an exact sequence $M_n \to M_{n-1} \to \cdots \to M_0 \to X \to 0$ with

$$\mathrm{Hom}_A(M, M_{n-1}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

exact.

We first show that by changing M_n, if necessary, we may assume that the sequence

$$\mathrm{Hom}_A(M, M_n) \to \mathrm{Hom}_A(M, M_{n-1}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

is exact.

We are given an exact sequence

$$M_n \xrightarrow{\theta} M_{n-1} \xrightarrow{\phi} M_{n-2}$$

where we define $M_{-1} = X$ and $M_{-2} = 0$ if necessary. Thus θ factorizes as a composition

$$M_n \xrightarrow{\psi} \ker\phi \xrightarrow{i} M_{n-1}$$

where ψ is onto and i is the inclusion. Let $\psi' : M_n' \to \ker\phi$ be an $\mathrm{add}(M)$-approximation of $\ker\phi$. It must be onto since ψ factors through ψ'. Replacing θ by the composition $\theta' = i\psi'$, the sequence $M_n' \to M_{n-1} \to M_{n-2}$ is exact. By the approximation property, we have a surjection

$$\mathrm{Hom}_A(M, M_n') \twoheadrightarrow \mathrm{Hom}_A(M, \ker\psi) \cong \ker\left(\mathrm{Hom}_A(M, M_{n-1}) \to \mathrm{Hom}_A(M, M_{n-2})\right).$$

so the sequence $\mathrm{Hom}_A(M, M_n') \to \mathrm{Hom}_A(M, M_{n-1}) \to \mathrm{Hom}_A(M, M_{n-2})$ is exact.

Since $M_i \in \mathrm{add}(M)$, we have $\mathrm{Hom}_A(M, M^i) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}(B_B)$, so the exact sequence

$$\mathrm{Hom}_A(M, M_n) \to \mathrm{Hom}_A(M, M_{n-1}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

is part of a projective resolution of $\mathrm{Hom}_A(M, X)$ as a right B-module. We can use it to compute $\mathrm{Tor}_i^B(\mathrm{Hom}_A(M, X), M)$ for $i < n$ as the homology of the complex

$$\mathrm{Hom}(M, M_n) \otimes_B M \to \cdots \to \mathrm{Hom}(M, M_0) \otimes_B M.$$

Now for $M' \in \operatorname{add} M$, the natural map $\operatorname{Hom}(M, M') \otimes_B M \to M'$ is an isomorphism, so this complex is identified with

$$M_n \to \cdots \to M_0$$

This is exact at M_i for $0 < i < n$, and the homology at M_0 is isomorphic to X. Hence (b) holds

Conversely if (b) holds, take the start of a projective resolution of $\operatorname{Hom}_A(M, X)$ as a right B-module, say

$$P_n \to \cdots \to P_0 \to \operatorname{Hom}_A(M, X) \to 0$$

Applying $- \otimes_B M$ gives a complex, which by the hypotheses is exact:

$$M_n \to \cdots \to M_0 \to X \to 0,$$

where $M_i = P_i \otimes_B M \in \operatorname{add} M$. Applying $\operatorname{Hom}_A(M, -)$ to this, gives a complex

$$\operatorname{Hom}_A(M, M_n) \to \cdots \to \operatorname{Hom}_A(M, M_0) \to \operatorname{Hom}_A(M, X) \to 0.$$

Identifying $\operatorname{Hom}_A(M, M_i) = \operatorname{Hom}_A(M, P_i \otimes_B M) \cong P_i$, we see that this is the projective resolution we started with, so it is exact. Thus (a) holds.

Proposition/Definition 2. If M is an A-module and $n \geq 0$ then $\operatorname{cogen}^n(M)$ is the module class consisting of the X satisfying the following equivalent conditions
(a') There is an exact sequence $0 \to X \to M^0 \to \cdots \to M^n$ with $M^i \in \operatorname{add} M$ such that the sequence

$$\operatorname{Hom}(M^{n-1}, M) \to \cdots \to \operatorname{Hom}(M^0, M) \to \operatorname{Hom}(X, M) \to 0$$

is exact (this is automatic if M is injective).
(b') The natural map $X \to \operatorname{Hom}_B(\operatorname{Hom}_A(X, M), M)$ is a monomorphism ($n = 0$) or an isomorphism ($n > 0$) and $\operatorname{Ext}^i_B(\operatorname{Hom}_A(X, M), M) = 0$ for $0 < i < n$.

Clearly $\operatorname{add}(M) \subseteq \ldots \operatorname{cogen}^2(M) \subseteq \operatorname{cogen}^1(M) \subseteq \operatorname{cogen}^0(M) = \operatorname{cogen}(M)$.

The proof is dual.

Corollary 1.
(1) If M is an A-module, then M is f.b. iff $A \in \operatorname{cogen}^1(M)$ iff $DA \in \operatorname{gen}_1(M)$.
(2) Any generator or cogenerator is f.b.

Proof (1) Apply Prop/Def 2 with $X = A$ and $n = 1$. Now M is f.b. iff DM is f.b. iff $A^{op} \in \mathrm{cogen}^1({}_{A^{op}}DM)$ iff $DA \in \mathrm{gen}_1({}_AM)$.

(2) If M is a generator, then $A \in \mathrm{add}(M) \subseteq \mathrm{cogen}^1(M)$. If M is a cogenerator, then DM is a generator, so f.b., hence so is M.

Definition. By an *f.b. pair* we mean a pair (A, M) consisting of an algebra and a f.b. A-module.

Given an f.b. pair, we construct a new f.b. pair (B, M), its *endomorphism correspondent*, where $B = \mathrm{End}_A(M)$ and M is considered in the natural way as a B-module.

Repeating the construction twice, one recovers the original pair.

We say that f.b. pairs (A, M) and (A', M') are equivalent if there is an equivalence A-mod $\to A'$-mod sending $\mathrm{add}(M)$ to $\mathrm{add}(M')$. One can show that equivalent pairs have equivalent endomorphism correspondents.

Corollary 2. If (A, M) and (B, M) are f.b. pairs which are endomorphism correspondents, then $\mathrm{Hom}_A(-, M)$ and $\mathrm{Hom}_B(-, M)$ give inverse antiequivalences between $\mathrm{cogen}^1({}_AM)$ and $\mathrm{cogen}^1({}_BM)$.

Proof. In view of Prop/Def 2 (b') and the symmetrical role of A and B, it suffices to show that if $X \in \mathrm{cogen}^1({}_AM)$, then $\mathrm{Hom}_A(X, M) \in \mathrm{cogen}^1({}_BM)$. Take a free presentation of ${}_AX$, say $A^m \to A^n \to X \to 0$. Applying $\mathrm{Hom}_A(-, M)$ gives an exact sequence

$$0 \to \mathrm{Hom}_A(X, M) \to M^n \to M^m.$$

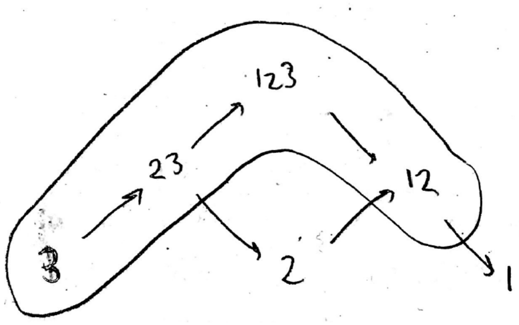
Applying $\mathrm{Hom}_B(-, M)$ to this gives

$$A^m \to A^n \to \mathrm{Hom}_B(\mathrm{Hom}_A(X, M), M) \to 0$$

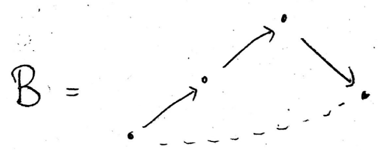
which is isomorphic to the original exact sequence, so exact. Thus $\mathrm{Hom}_A(X, M) \in \mathrm{cogen}^1({}_BM)$.

Remark. Composing with duality, we get that $\mathrm{Hom}_A(M, -)$ and $M \otimes_{B^{op}} -$ give inverse equivalences between $\mathrm{gen}_1({}_AM)$ and $\mathrm{cogen}^1({}_{B^{op}}DM)$.

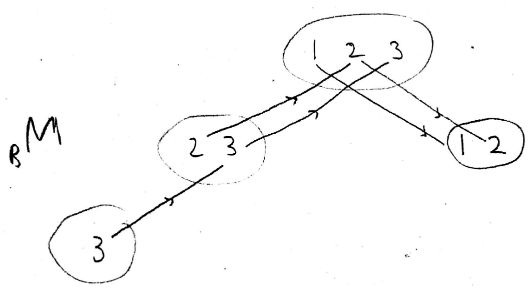
Example. Let A be the path algebra of the linear quiver $Q = 1 \to 2 \to 3$. We display its AR quiver below. Let ${}_AM$ be the direct sum of the circled indecomposables.



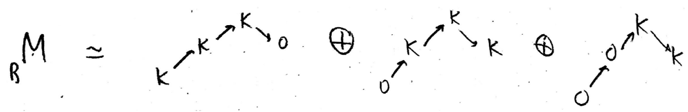
The endomorphism algebra of ${}_AM$ is



Considering M as a B-module, means to consider it as a representation of this quiver. The vector space at each vertex is the corresponding indecomposable A-module. In this example, the indecomposable A-modules are at most one-dimensional at each vertex of Q. In the following diagram we write i for the natural basis element at vertex i of Q. The arrows in the quiver for B correspond to homomorphisms of the indecomposable A-modules, and act on the basis elements as indicated below.



Thus



Observe that ${}_AM$ has all of the projective A-modules as summands, but not all injectives, so ${}_AM$ is a generator but not a cogenerator. On the other hand all of the summands of ${}_BM$ are projective, but one summand is not injective.

Proposition. Let (A, M) and (B, M) be f.b. pairs which are endomorphism correspondents.
(a) ${}_AM$ is a generator iff ${}_BM$ is projective.
(b) ${}_AM$ is a cogenerator iff ${}_BM$ is injective.
(c) $A \in \mathrm{cogen}^n({}_AM)$ iff $\mathrm{Ext}^i_B(M, M) = 0$ for $0 < i < n$.

Proof. (a) If ${}_AM$ is a generator, then $A \in \mathrm{add}({}_AM)$, so ${}_BM \cong \mathrm{Hom}_A(A, M) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}(B)$, so ${}_BM$ is projective.

Conversely if ${}_BM$ is projective, then ${}_BM \in \mathrm{add}(B)$, so $A \cong \mathrm{Hom}_B(M, M) \in \mathrm{add}(\mathrm{Hom}_B(B, M)) = \mathrm{add}({}_AM)$.

(b) Apply (a) to DM.

(c) Prop/Def 2 with $X = A$.

Definition. Given an algebra B, we take the minimal injective resolution

$$0 \to B \to I^0 \to I^1 \to \dots$$

of the module ${}_BB$. We say that B has *dominant dimension* $\geq n$ if $I^0, \dots, I^{n-1}$ are projective.

Recall that a QF-3 algebra is an algebra with a faithful projective-injective module. Thus A is QF-3 iff $\mathrm{dom.\,dim}\, A \geq 1$. The faithful projective-injective module P is unique, up to multiplicities, since it is the direct sum of all indecomposable projective-injective modules. Moreover $\mathrm{add}(P) = \mathcal{P}_A \cap \mathcal{I}_A$, the module class of all projective-injective modules.

Theorem (Morita-Tachikawa correspondence). There is a 1:1 correspondence between equivalence classes of pairs (A, M) where ${}_AM$ is a generator-cogenerator and Morita equivalence classes of algebras B with $\mathrm{dom.\,dim}\, B \geq 2$.

It is given by endomorphism correspondence between (A, M) and (B, M), where ${}_BM$ is the faithful projective-injective module, which is unique up to multiplicities.

In particular $B = \mathrm{End}_A(M)$.

Proof. By the proposition, ${}_AM$ is a generator-cogenerator iff ${}_BM$ is projective-injective. Now ${}_BM$ is also f.b., so $B \in \mathrm{cogen}^1({}_BM)$, so $\mathrm{dom.\,dim}\, B \geq 2$.

Conversely if $\mathrm{dom.\,dim}\, B \geq 2$, and M is its faithful projective-injective module, then there is an exact sequence $0 \to B \to M^0 \to M^1$ with $M^i \in \operatorname{add}(M)$. Since M is injective, $\operatorname{Hom}(-, M)$ is exact on this. Thus $B \in \operatorname{cogen}^1({}_BM)$, so ${}_BM$ is f.b.. Let (A, M) be the endomorphism correspondent. Then ${}_AM$ is a generator-cogenerator.

Definition Let $n \geq 1$. A module ${}_AM$ is an *n-cluster tilting object* if
(i) $\operatorname{Ext}^i_A(M, M) = 0$ for $0 < i < n$
(ii) $\operatorname{Ext}^i_A(U, M) = 0$ for $0 < i < n$ implies $U \in \operatorname{add} M$
(iii) $\operatorname{Ext}^i_A(M, U) = 0$ for $0 < i < n$ implies $U \in \operatorname{add} M$

Clearly (ii) implies $A \in \operatorname{add} M$ and (iii) implies $DA \in \operatorname{add} M$, so any n-cto is a generator-cogenerator.

Example. For the algebra with quiver

$$0 \to 1 \to 2 \to \cdots \to n$$

with all paths of length 2 zero, the module $S[0]$ has projective resolution

$$0 \to P[n] \to P[n-1] \to \cdots \to P[1] \to P[0] \to S[0] \to 0$$

so $\dim \operatorname{Ext}^i(S[0], S[j]) = \delta_{ij}$. It follows that

$$M = S[0] \oplus P[0] \oplus \cdots \oplus P[n-1] \oplus P[n] \cong I[0] \oplus I[1] \oplus \cdots \oplus I[n] \oplus S[n]$$

is an n-cto. It's endomorphism algebra B is the path algebra of the quiver

$$n \to \cdots \to 1 \to 0 \to *$$

with all paths of length 2 zero. It has global dimension $n+1$. The projectives $P[n], \ldots, P[0]$ are injective, and $P[*]$ has injective resolution

$$0 \to P[*] \to I[*] \to I[0] \to \cdots \to I[n-1] \to I[n] \to 0.$$

Now $I[*] \cong P[0]$, $I[0] \cong P[1]$, $\ldots$, $I[n-1] \cong P[n]$ and $I[n] \cong S[n]$ is not projective, so $\operatorname{dom.dim} B = n + 1$.

Theorem (Iyama, 2007). There is a 1:1 correspondence between equivalence classes of pairs (A, M) where ${}_AM$ is an n-cto and Morita equivalence classes of algebras B with $\operatorname{gl.dim} B \leq n + 1 \leq \operatorname{dom.dim} B$.

As before, it is given by endomorphism correspondence between $(A, {}_AM)$ and $(B, {}_BM)$, so $B = \operatorname{End}_A(M)$.

Proof. We are in the setting of Morita-Tachikawa correspondence.

Now $\operatorname{Ext}^i_A(M,M)=0$ for $1<i<n$ corresponds to $B\in\operatorname{cogen}^n({}_BM)$, and since ${}_BM$ is the faithful projective-injective, this corresponds to $\operatorname{dom.dim} B\geq n+1$.

Suppose $\operatorname{gl.dim} B\leq n+1$.

We show that if $\operatorname{Ext}^i_A(U,M)=0$ for $0<i<n$ then $U\in\operatorname{add}M$. Take the start of a projective resolution of U, say

$$P_n\to\cdots\to P_0\to U\to 0.$$

Applying $\operatorname{Hom}_A(-,M)$ gives a complex

$$0\to\operatorname{Hom}_A(U,M)\to\operatorname{Hom}_A(P_0,M)\to\cdots\to\operatorname{Hom}_A(P_n,M)$$

which is exact because the Exts vanish. Since ${}_BM$ is injective, applying $\operatorname{Hom}_B(-,M)$ gives an exact sequence

$$\operatorname{Hom}_B(\operatorname{Hom}_A(P_n,M),M)\to\cdots\to\operatorname{Hom}_B(\operatorname{Hom}_A(P_0,M),M)\to\operatorname{Hom}_B(\operatorname{Hom}_A(U,M),M)\to 0.$$

Now the maps $P_i\to\operatorname{Hom}_B(\operatorname{Hom}_A(P_i,M),M)$ are isomorphisms since $P_i\in\operatorname{add}M$. Thus the map $U\to\operatorname{Hom}_B(\operatorname{Hom}_A(U,M),M)$ is an iso (so $U\in\operatorname{cogen}^1({}_AM)$). Also $\operatorname{Hom}_A(P_i,M)\in\operatorname{add}(\operatorname{Hom}_A(A,M))=\operatorname{add}({}_BM)$. Thus, since $\operatorname{gl.dim}B\leq n+1$, the B-module $\operatorname{Hom}_A(U,M)$ must be projective, so it is in $\operatorname{add}({}_BB)$, and then $U\cong\operatorname{Hom}_B(\operatorname{Hom}_A(U,M),M)\in\operatorname{add}(\operatorname{Hom}_B(B,M))=\operatorname{add}({}_AM)$.

Next we show that if $\operatorname{Ext}^i_A(M,U)=0$ for $0<i<n$ then $U\in\operatorname{add}M$. Take the start of an injective resolution of U, say

$$0\to U\to I^0\to\cdots\to I^n.$$

Applying $\operatorname{Hom}_A(M,-)$ gives a complex

$$0\to\operatorname{Hom}_A(M,U)\to\operatorname{Hom}_A(M,I^0)\to\cdots\to\operatorname{Hom}_A(M,I^n)$$

which is exact because the Exts vanish. Since ${}_BM$ is projective, applying $-\otimes_B M$ gives an exact sequence

$$0\to\operatorname{Hom}_A(M,U)\otimes_BM\to\operatorname{Hom}_A(M,I^0)\otimes_BM\to\cdots\to\operatorname{Hom}_A(M,I^n)\otimes_BM.$$

Now the maps $I^i\to\operatorname{Hom}_A(M,I^i)\otimes_B M$ are isomorphisms since $I^i\in\operatorname{add}M$. Thus the map $U\to\operatorname{Hom}_A(M,U)\otimes_B M$ is an iso. Also $\operatorname{Hom}_A(M,I^i)\in\operatorname{add}(\operatorname{Hom}_A(M,M))=\operatorname{add}(B_B)$. Thus, since $\operatorname{gl.dim}B\leq n+1$, the right

B-module $\mathrm{Hom}_A(M,U)$ must be projective, so it is in $\mathrm{add}(B_B)$, and then $U \cong \mathrm{Hom}_A(M,U) \otimes_B M \in \mathrm{add}(B \otimes_B M) = \mathrm{add}({}_A M)$.

Now suppose that M is an n-cto. Given a B-module Z, choose a projective presentation

$$P_1 \xrightarrow{f} P_0 \to Z \to 0.$$

Applying $\mathrm{Hom}_B(-,M)$ gives an exact sequence

$$0 \to \mathrm{Hom}_B(Z,M) \to \mathrm{Hom}_B(P_0,M) \xrightarrow{g} \mathrm{Hom}_B(P_1,M) \to \mathrm{Coker}(g) \to 0.$$

Let $C^0 = \mathrm{Coker}(g)$. Applying $\mathrm{Hom}_A(-,M)$ we get a commutative diagram with bottom row exact

$$\begin{array}{ccccccc} & & & & P_1 & \xrightarrow{f} & P_0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_A(C^0,M) & \longrightarrow & \mathrm{Hom}_A(\mathrm{Hom}_B(P_1,M),M) & \longrightarrow & \mathrm{Hom}_A(\mathrm{Hom}_B(P_0,M),M) \end{array}$$

The two vertical maps are isomorphisms, so $\mathrm{Ker}(f) \cong \mathrm{Hom}_A(C^0,M)$.

Now since M is a cogenerator, by repeatedly taking left M-approximations we can get an exact sequence

$$0 \to C^0 \to M^0 \to \cdots \to M^{n-2}$$

such that the sequence

$$\mathrm{Hom}_A(M^{n-2},M) \to \cdots \to \mathrm{Hom}_A(M^0,M) \to \mathrm{Hom}_A(C^0,M) \to 0$$

is exact. Let C^i be the cosyzygies for this sequence, so

$$0 \to C^i \to M^i \to C^{i+1} \to 0.$$

Then

$$\mathrm{Hom}(M^i,M) \twoheadrightarrow \mathrm{Hom}(C^i,M) \to \mathrm{Ext}^1(C^{i+1},M) \to \mathrm{Ext}^1(M^i,M) = 0 \to \dots,$$

so by dimension shifting

$$\mathrm{Ext}^{n-1}(C^{n-1},M) \cong \mathrm{Ext}^{n-2}(C^{n-1},M) \cong \ldots \cong \mathrm{Ext}^1(C^1,M) = 0$$

and similarly $\mathrm{Ext}^i(C^{n-1},M) = 0$ for $0 < i < n$. Thus $C^{n-1} \in \mathrm{add}\, M$. Thus Z has projective resolution

$$0 \to \mathrm{Hom}_A(C^{n-1},M) \to \mathrm{Hom}_A(M^{n-2},M) \to \cdots \to \mathrm{Hom}_A(M^0,M) \to P_1 \to P_0 \to Z \to 0.$$

Thus $\operatorname{proj.dim} Z \leq n+1$. Thus $\operatorname{gl.dim} B \leq n+1$.

Observe that M is a 1-cto iff $\operatorname{add}(M) = A$-mod. This is only possible if A has finite representation type, and then ${}_AM$ is unique up to multiplicities. Thus we recover.

Theorem (Auslander, 1974). There is a 1-1 correspondence between algebras A of finite representation type up to Morita equivalence and algebras B with $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$ up to Morita equivalence.

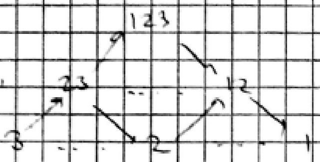
The correspondence sends A to $B = \operatorname{End}_A(M)$ where ${}_AM$ is the direct sum of all the indecomposable A-modules, and it sends B to $A = \operatorname{End}_B(M)$ where ${}_BM$ is the faithful projective-injective B-module.

The algebra B is called the *Auslander algebra* of A.

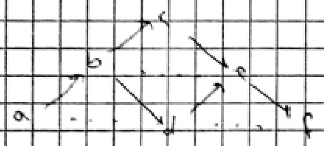
Example. We can check $\operatorname{gl.dim} B = 2 = \operatorname{dom.dim} B$ for the Auslander algebra of the linear quiver with three vertices.

$A = K(1 \longrightarrow 2 \longrightarrow 3)$

${}_A M = \bigoplus$ of all modules



$B = \mathrm{End}_A(M) = K\Big($



$\Big)$

$P[a] = I[c]$

$P[b] = I[e]$

$P[c] = I[f]$

$P[d] = {}_d{}^e$

$P[e] = {}_e{}_f$

$P[f] = f$

$0 \to P[d] \to I[e] \to I[c] \to I[a] \to 0$

$0 \to P[e] \to I[f] \to I[c] \to I[b] \to 0$

$0 \to P[f] \to I[f] \to I[e] \to I[d] \to 0$

proj-inj

$0 \to P[d] \to P[b] \to P[a] \to S[a] \to 0$

$0 \to P[e] \to P[c] \oplus P[d] \to P[b] \to S[b] \to 0$

$0 \to P[e] \to P[c] \to S[c] \to 0$

$0 \to P[f] \to P[e] \to P[d] \to S[d] \to 0$

$0 \to P[f] \to P[e] \to S[e] \to 0$

$0 \to P[f] \to S[f] \to 0$

3.2 Homological conjectures for f.d. algebras

Lemma. If $\operatorname{inj.dim}{}_A A = n$, any A-module has $\operatorname{proj.dim} M \leq n$ or ∞.

For example, every non-projective module for a self-injective algebra has infinite projective dimension.

Proof. Say $\operatorname{proj.dim} M = i < \infty$. There is some N with $\operatorname{Ext}^i(M,N) \neq 0$. Choose $0 \to L \to P \to N \to 0$ with P projective. The long exact sequence for $\operatorname{Hom}(M,-)$ gives

$$\dots \to \operatorname{Ext}^i(M,P) \to \operatorname{Ext}^i(M,N) \to \operatorname{Ext}^{i+1}(M,L) \to \dots$$

Now $\operatorname{Ext}^{i+1}(M,L) = 0$, so $\operatorname{Ext}^i(M,P) \neq 0$, so $\operatorname{Ext}^i(M,A) \neq 0$, so $i \leq n$.

Definition. An algebra A is *(Iwanaga) Gorenstein* if $\operatorname{inj.dim}{}_A A < \infty$ and $\operatorname{inj.dim} A_A < \infty$.

Gorenstein Symmetry Conjecture (see Auslander and Reiten, Applications of contravariantly finite subcategories, Adv. Math 1991). If one is finite, so is the other.

Proposition 1. If $\operatorname{inj.dim}{}_A A = r$ and $\operatorname{inj.dim} A_A = s$ are both finite, they are equal.

Proof. $\operatorname{proj.dim}{}_A DA = \operatorname{inj.dim} A_A = s$, so $s \leq r$ by the lemma. Dually $s \geq r$.

Also true for noetherian rings (Zaks, Injective dimension of semi-primary rings, J. Alg. 1969).

Recall that if $0 \to A \to I^0 \to I^1 \to \dots$ is the minimal injective resolution of a f.d. algebra A, one says that A has dominant dimension $\geq n$ if $I^0, \dots, I^{n-1}$ are all projective.

Nakayama conjecture (1958). If all I^n are projective, i.e. $\operatorname{dom.dim} A = \infty$, then A is self-injective.

Generalized Nakayama conjecture (Auslander and Reiten 1975). For any f.d. algebra A, every indecomposable injective occur as a summand of some I^n.

It clearly implies the Nakayama conjecture, for if the I^n are projective, and each indecomposable injective occurs as a summand of some I^n, then the indecomposable injectives are projective.

Example. For the commutative square, vertices 1(source),2,3,4(sink). There are injective resolutions

$$\begin{aligned}
0 \to & P[1] \to I[4] \to 0, \\
0 \to & P[2] \to I[4] \to I[3] \to 0, \\
0 \to & P[3] \to I[4] \to I[2] \to 0, \\
0 \to & P[4] \to I[4] \to I[2] \oplus I[3] \to I[1] \to 0,
\end{aligned}$$

so

$$0 \to A \to I[4]^4 \to I[2]^2 \oplus I[3]^2 \to I[1] \to 0,$$

so all indecomposable injectives occur.

Proposition 2. The following are equivalent.
(i) The Nakayama conjecture (if $\operatorname{dom.dim} B = \infty$ then B is self-injective).
(ii) If ${}_AM$ is a generator-cogenerator and $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$ then M is projective.

Proof (i) implies (ii). Say ${}_AM$ satisfies the hypotheses. Let (B, M) be the endomorphism correspondent. Then ${}_BM$ is projective-injective and $B \in \operatorname{cogen}^n(M)$ for all n. Thus for all n there is an exact sequence

$$0 \to B \to I^0 \to \cdots \to I^n$$

with the I^i projective-injective. Thus $\operatorname{dom.dim} B = \infty$. Thus B is self-injective, so $\operatorname{add}(M) = \operatorname{add}(B)$, so ${}_BM$ is a generator, so ${}_AM$ is projective.

(ii) implies (i). Say $\operatorname{dom.dim} B = \infty$. Thus B is QF-3 and let ${}_BM$ be the faithful projective-injective module. Let ${}_AM$ be the endomorphism correspondent. It is a generator-cogenerator. Then $B \in \operatorname{cogen}^1(M)$ for all n, so $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$. Thus by (ii), ${}_AM$ is projective, so ${}_BM$ is a generator. Thus $B \in \operatorname{add}(M)$ is injective.

Proposition 3. The following are equivalent.
(i) The Generalized Nakayama Conjecture (every indecomposable injective occurs as a summand of some I^i in the minimal injective resolution of B).
(ii) If ${}_AM$ is a cogenerator and $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$ then M is injective.

Proof. (i) implies (ii). Suppose ${}_AM$ satisfies the conditions. Then there is corresponding ${}_BM$ which is injective, and $B \in \operatorname{cogen}^n(M)$ for all n. Thus by (i) every indecomposable injective is a summand of ${}_BM$. Thus ${}_BM$ is a cogenerator. Thus ${}_AM$ is injective.

(ii) implies (i). Let ${}_AM$ be the sum of all indecomposable injectives occuring in the I^i. Then $B \in \mathrm{cogen}^n(M)$ for all n. Let ${}_AM$ be the endomorphism correspondent. Then ${}_AM$ is a cogenerator and $\mathrm{Ext}^i_A(M,M) = 0$ for all $i > 0$. Thus by (ii) ${}_AM$ is injective. Thus ${}_BM$ is a cogenerator. Thus all indecomposable injectives occur as a summand of ${}_AM$.

Boundedness Conjecture (Happel, Selforthogonal modules, 1995). If M is an A-module with $\mathrm{Ext}^i_A(M,M) = 0$ for all $i > 0$ then $\#M \leq \#A$, where $\#M$ denotes the number of non-isomorphic indecomposable summands of M.

This implies the GNC.

Finitistic Dimension Conjecture (see H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 1960) For any f.d. algebra A,

$$\mathrm{fin.\,dim}\, A = \sup\{\mathrm{proj.\,dim}\, M \mid \mathrm{proj.\,dim}\, M < \infty\}$$

is finite.

Note that $\mathrm{fin.\,dim}\, A$ is not necessarily the same as the maximum of the projective dimensions of the simple modules of finite projective dimension.

Example. If A is Gorenstein, with $\mathrm{inj.\,dim}\, {}_AA = n = \mathrm{inj.\,dim}\, A_A$, then $\mathrm{fin.\,dim}\, A = n$. For the lemma implies that any A-module M has $\mathrm{proj.\,dim}\, M \leq n$ or ∞, and $\mathrm{proj.\,dim}\, D(A_A) = n$.

Proposition 4. The finitistic dimension conjecture implies the Gorenstein symmetry conjecture.

Proof. Assuming $\mathrm{inj.\,dim}\, A_A = n < \infty$, we want to prove that $\mathrm{inj.\,dim}\, {}_AA < \infty$. We have $\mathrm{proj.\,dim}\, {}_ADA = n < \infty$. Thus any injective module has projective dimension $< \infty$. Take a minimal injective resolution $0 \to {}_AA \to I^0 \to \dots$. We show by induction on i that $\mathrm{proj.\,dim}\, \Omega^{-i}A < \infty$. There is an exact sequence

$$0 \to \Omega^{-(i-1)}A \to I^{i-1} \to \Omega^{-i}A \to 0.$$

Applying $\mathrm{Hom}_A(-, X)$ for a module X gives a long exact sequence

$$\dots \to \mathrm{Ext}^m(\Omega^{-(i-1)}A, X) \to \mathrm{Ext}^{m+1}(\Omega^{-i}A, X) \to \mathrm{Ext}^{m+1}(I^{i-1}, X) \to \dots$$

For m sufficiently large the outside terms are zero, hence so is the middle.

If some $\Omega^{-i}A = 0$, or is injective, then $\mathrm{inj.\,dim}\, {}_AA < \infty$, as desired, so suppose otherwise. Let $f : \Omega^{-i}A \to I^i$ be the inclusion. Then f belongs to the middle term in the complex

$$\mathrm{Hom}(\Omega^{-i}A, I^{i-1}) \to \mathrm{Hom}(\Omega^{-i}A, I^{i}) \to \mathrm{Hom}(\Omega^{-i}A, I^{i+1})$$

and it is sent to zero in the third term. Now f is not in the image of the map from the first term, for otherwise the map $I^{i-1} \to \Omega^{-i}A$ is a split epimorphism, so the inclusion $\Omega^{-(i-1)}A \to I^i$ is a split monomorphism. But $\Omega^{-(i-1)}A$ is not injective, a contradiction. Thus $\mathrm{Ext}^i(\Omega^{-i}AA) \neq 0$. Thus $\mathrm{proj.\,dim}\,\Omega^{-i}A \geq i$. This contradicts that $\mathrm{fin.\,dim}\,A < \infty$.

Proposition 5. The finitistic dimension conjecture implies the generalized Nakayama conjecture.

Proof. Assume the FDC. We show that if ${}_AM$ is a module and $\mathrm{Ext}^n(M, A) = 0$ for all $n \geq 0$ then $M = 0$ (the *strong Nakayama conjecture*). Taking $M = S[i]$, by Lemma 1 of section 1.7, this gives the GNC.

Take a minimal projective resolution $\to P_1 \to P_0 \to M \to 0$. By assumption the sequence

$$0 \to \mathrm{Hom}_A(P_0, A) \xrightarrow{f_0} \mathrm{Hom}_A(P_1, A) \xrightarrow{f_1} \mathrm{Hom}(P_2, A) \to \dots$$

is exact. Let $\mathrm{fin.\,dim}\,A^{op} = n < \infty$. Then $\mathrm{Coker}(f_n)$ has projective resolution

$$0 \to \mathrm{Hom}_A(P_0, A) \xrightarrow{f_0} \mathrm{Hom}_A(P_1, A) \to \dots \to \mathrm{Hom}_A(P_{n+1}, A) \to \mathrm{Coker}(f_n) \to 0$$

so it has finite projective dimension, so projective dimension $\leq n$, so by dimension shifting $\mathrm{Im}\, f_1$ is projective, so f_0 must be a split mono. But $\mathrm{Hom}_A(-, A)$ is an antiequivalence from P_A to $\mathcal{P}_{A^{op}}$. Thus the map $P_1 \to P_0$ must be a split epi, so $M = 0$.

3.3 No loops conjecture

In this section we do not assume that K is algebraically closed, but we do assume that $A = KQ/I$ with I admissible.

No Loops Conjecture (Proved by Igusa 1990, based on Lenzing 1969). If $\mathrm{gl.\,dim}\,A < \infty$ then Q has no loops (that is, $\mathrm{Ext}^1(S[i], S[i]) = 0$ for all i).

Proof. We use the trace function of Hattori and Stallings. I only sketch the proof of its properties.

(1) For any matrix $\theta \in M_n(A)$ we consider its trace $\mathrm{tr}(\theta) \in A/[A,A]$, where $[A,A]$ is the subspace of A spanned by the commutators $ab - ba$. This ensures that $\mathrm{tr}(\theta\phi) = \mathrm{tr}(\phi\theta)$. This equality holds also for $\theta \in M_{m\times n}(A)$ and $\phi \in M_{n\times m}(A)$.

(2) If P is a f.g. projective A-module it is a direct summand of a f.g. free module $F = A^n$. Let $p : F \to P$ and $i : P \to F$ be the projection and inclusion. One defines $\mathrm{tr}(\theta)$ for $\theta \in \mathrm{End}(P)$ to be $\mathrm{tr}(i\theta p)$. This is well defined, for if

$$A^n = F \underset{i}{\overset{p}{\rightleftarrows}} P \underset{p'}{\overset{i'}{\rightleftarrows}} F' = A^m$$

with $pi = 1_P = p'i'$, then $\mathrm{tr}(i\theta p) = \mathrm{tr}((ip')(i'\theta p)) = \mathrm{tr}((i'\theta p)(ip')) = \mathrm{tr}(i'\theta p')$.

(3) Any module M has a finite projective resolution $P_* \to M$, and an endomorphism θ of M lifts to a map between the projective resolutions

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_n} & & & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Define $\mathrm{tr}(\theta) = \sum_i (-1)^i \,\mathrm{tr}(\theta_i)$. One can show that does not depend on the projective resolution or the lift of θ.

(4) One can show that given a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta''} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

one has $\mathrm{tr}(\theta) = \mathrm{tr}(\theta') + \mathrm{tr}(\theta'')$.

(5) It follows that any nilpotent endomorphism has trace 0, since

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\,\theta & \longrightarrow & M & \longrightarrow & M/\mathrm{Im}\,\theta & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta|_{\mathrm{Im}\,\theta}} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathrm{Im}\,\theta & \longrightarrow & M & \longrightarrow & M/\mathrm{Im}\,\theta & \longrightarrow & 0 \end{array}$$

so $\mathrm{tr}(\theta) = \mathrm{tr}(\theta|_{\mathrm{Im}\,\theta}) = \mathrm{tr}(\theta|_{\mathrm{Im}(\theta^2)}) = \dots = 0$.

(6) Thus any element of $J(A)$ as a map $A \to A$ has trace 0, so $J(A) \subseteq [A,A]$. Thus $(KQ)_+ \subseteq I + [KQ, KQ]$.

(7) Any loop of Q gives an element of $(KQ)_+$. But it is easy to see that

$$I + [KQ, KQ] \subseteq \text{span of arrows which are not loops} + (KQ)_+^2,$$

for example if p, q are paths then $[p, q] \in (KQ)_+^2$ unless they are trivial paths or one is trivial and the other is an arrow. Thus there are no loops.

Strong no loops conjecture (proved by Igusa, Liu, Paquette 2011). If S is a 1-dimensional simple module for a f.d. algebra and S has finite injective or projective dimension, then $\mathrm{Ext}^1(S, S) = 0$.

Extension Conjecture (stated by Liu, Morin). If S is simple module for a f.d. algebra and $\mathrm{Ext}^1(S, S) \neq 0$ then $\mathrm{Ext}^n(S, S) \neq 0$ for infinitely many n.

Addendum to section 1.3

Definition. A module class $\mathcal{C}$ in A-mod is:

(i) *covariantly finite* if every A-module X has a left $\mathcal{C}$-approximation. That is, a map $X \to C$ with $C \in \mathcal{C}$ such that for all $C' \in \mathcal{C}$ the map $\mathrm{Hom}(C, C') \to \mathrm{Hom}(X, C')$ is onto.

(ii) *contravariantly finite* if every A-module X has a right $\mathcal{C}$-approximation. That is, a map $C \to X$ with $C \in \mathcal{C}$ such that for all $C' \in \mathcal{C}$ the map $\mathrm{Hom}(C', C) \to \mathrm{Hom}(C', X)$ is onto.

(iii) *functorially finite* if it is covariantly finite and contravariantly finite.

Examples.
(1) If M is a module, then $\mathrm{add}\, M$ is functorially finite. In particular, if A has finite representation type, every module class is of this form (for M the direct sum of the indecomposable modules it contains), so every module class is functorially finite.

Proof. The example at the end of section 1.3 applies.

(2) If the inclusion $\mathcal{C} \to A$-mod has a right (respectively left) adjoint then $\mathcal{C}$ is contravariantly (respectively covariantly) finite.

Proof. Say R is right adjoint to the inclusion, then $\mathrm{Hom}(X, RY) \cong \mathrm{Hom}(X, Y)$ for all $X \in \mathcal{C}$. For all Y there is a counit $RY \to Y$, and any morphism $X \to Y$ with $X \in \mathcal{C}$ factors through it.

Lemma. For any module M, $\operatorname{gen} M$ is covariantly finite (and dually $\operatorname{cogen} M$ is contravariantly finite).

Proof. Given X, take projective cover $P \to X$. Take a left $\operatorname{add} M$-approximation $P \to M'$. Take the pushout

$$\begin{array}{ccc} P & \longrightarrow & M' \\ \downarrow & & \downarrow \\ X & \longrightarrow & G \end{array}$$

Since $P \to X$ is onto, so is $M' \to G$, so $G \in \operatorname{gen} M$. If $f : X \to G'$ with $G' \in \operatorname{gen} M$, then there is a map from M^n onto G'. Since P is projective, the composition $P \to X \to G'$ lifts to a map $P \to M^n$. Since the map $P \to M'$ is an approximation, the map $P \to M^n$ factors as $P \to M' \to M^n$. Now the two maps $X \to G'$ and $M' \to G'$ agree on P, so there is an induced map of the pushout $G \to G'$. Thus the map $X \to G'$ factors as $X \to G \to G'$. Thus the map $X \to G$ is a left $\operatorname{gen} M$-approximation.

3.4 Torsion theories and τ-rigid modules

This is motivated by the paper of Adachi, Iyama and Reiten, τ-tilting theory, 2014, but I am not able to cover any of the content of that paper. Instead many of the results come from the two papers by Auslander and Smaløin 1981.

A *torsion theory* in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$, the *torsion* and *torsion-free* classes, such that
(i) $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0$.
(ii) Any object X has a subobject $t_{\mathcal{T}}X \in \mathcal{T}$ with $X/t_{\mathcal{T}}X \in \mathcal{F}$ (so it fits in an exact sequence $0 \to t_{\mathcal{T}}X \to X \to X/t_{\mathcal{T}}X \to 0$ with first term in $\mathcal{T}$ and last term in $\mathcal{F}$).

Examples.
(1) The torsion and torsion-free modules give a torsion theory in the category of $\mathbb{Z}$-modules.

(2) For the path algebra of the quiver $1 \to 2$, there is a torsion theory $(\operatorname{add} S[2], \operatorname{add} S[1])$ in A-mod.

(3) Let A be an algebra whose AR quiver is obtained by knitting, so A is of finite representation type and all of its indecomposables are directing.

Partition the indecomposables into two sets T, F with $\mathrm{Hom}(T, F) = 0$. Then $(\mathrm{add}\, T, \mathrm{add}\, F)$ is a torsion theory in A-mod.

Notation. For an a set $\mathcal{C}$ of modules in A-mod or more generally of objects in an abelian category

$$\mathcal{C}^{\perp i,j,\dots} = \{X : \mathrm{Ext}^n(M, X) = 0 \text{ for all } M \in \mathcal{C} \text{ and } n = i, j, \dots\},$$

$$^{\perp i,j,\dots}\mathcal{C} = \{X : \mathrm{Ext}^n(X, M) = 0 \text{ for all } M \in \mathcal{C} \text{ and } n = i, j, \dots\}.$$

Recall that $\mathrm{Ext}^0 = \mathrm{Hom}$.

Properties. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory.

(i) $\mathcal{T} = {}^{\perp 0}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp 0}$ so either of the classes determines the other.

(ii) $\mathcal{T}$ is closed under quotients and extensions; $\mathcal{F}$ is closed under subobjects and extensions.

(iii) The assignment sending X to $t_{\mathcal{T}}X$ defines a functor $\mathcal{A} \to \mathcal{T}$ which is a right adjoint to the inclusion $\mathcal{T}$ in $\mathcal{A}$. The assigment sending X to $X/t_{\mathcal{T}}X$ defines a functor $\mathcal{A} \to \mathcal{F}$ which is a left adjoint to the inclusion $\mathcal{F}$ in $\mathcal{A}$.

Proof. Easy. For example for (i), if $\mathrm{Hom}(\mathcal{T}, X) = 0$, then applying $\mathrm{Hom}(t_{\mathcal{T}}X, -)$ to the torsion exact sequence gives an exact sequence

$$0 \to \mathrm{Hom}(t_{\mathcal{T}}X, t_{\mathcal{T}}X) \to \mathrm{Hom}(t_{\mathcal{T}}X, X) \to \mathrm{Hom}(t_{\mathcal{T}}X, X/t_{\mathcal{T}}X) = 0$$

so $\mathrm{Hom}(t_{\mathcal{T}}X, t_{\mathcal{T}}X) \cong \mathrm{Hom}(t_{\mathcal{T}}X, X) = 0$, so $t_{\mathcal{T}}X = 0$ so $X \cong X/t_{\mathcal{T}}X \in \mathcal{F}$.

For (ii), for $\mathcal{T}$ given an exact sequence $0 \to X \to Y \to Z \to 0$, apply $\mathrm{Hom}(-, C)$ for $C \in \mathcal{C}$ to get an exact sequence

$$0 \to \mathrm{Hom}(Z, C) \to \mathrm{Hom}(Y, C) \to \mathrm{Hom}(X, C).$$

Now if $X, Z \in \mathcal{T}$, then $\mathrm{Hom}(X, C) = \mathrm{Hom}(Z, C) = 0$, so $\mathrm{Hom}(Y, C) = 0$, so $Y \in \mathcal{T}$. Also, if $Y \in \mathcal{T}$, then $\mathrm{Hom}(Y, C) = 0$, so $\mathrm{Hom}(Z, C) = 0$, so $Z \in \mathcal{T}$.

For (iii) observe that any map $\theta : X \to Y$ induces a map $t_{\mathcal{T}}X \to t_{\mathcal{T}}Y$ since the composition $t_{\mathcal{T}}X \to X \to Y \to Y/t_{\mathcal{T}}Y$ must be zero.

Proposition 1. For a module class $\mathcal{T}$ in A-mod the following are equivalent.
(i) $\mathcal{T}$ is a torsion class for some torsion theory in A-mod.
(ii) $\mathcal{T} = {}^{\perp 0}(\mathcal{T}^{\perp 0})$.
(iii) $\mathcal{T} = {}^{\perp 0}\mathcal{C}$ for some module class $\mathcal{C}$.
(iv) $\mathcal{T}$ is closed under quotients and extensions.

Proof. (i) implies (ii) implies (iii) implies (iv). Straightforward.

(iv) implies (i). Define $\mathcal{F} = \mathcal{T}^{\perp 0}$. Given any module X, let T be a submodule of X in $\mathcal{T}$ of maximal dimension. Then $\mathrm{Hom}(\mathcal{T}, X/T) = 0$, for if T'/T is the image of such a map, then T'/T is in $\mathcal{T}$, hence so is T', contradicting maximality. Thus $X/T \in \mathcal{F}$.

Lemma (Auslander-Smalø). For modules M, N, tfae:
(i) $\mathrm{Hom}(N, \tau M) = 0$.
(ii) $\mathrm{Ext}^1(M, \mathrm{gen}\, N) = 0$ (that is, $\mathrm{Ext}^1(M, G) = 0$ for all $G \in \mathrm{gen}\, N$).

Proof. (i)⇒(ii). If $\mathrm{Hom}(N, \tau M) = 0$, then $\mathrm{Hom}(G, \tau M) = 0$ for all $G \in \mathrm{gen}\, N$), so $\overline{\mathrm{Hom}}(G, \tau N) = 0$, so $\mathrm{Ext}^1(N, G) = 0$ by the Auslander-Reiten formula.

(ii)⇒(i). Say $f : N \to \tau M$ is a non-zero map. Factorize it as a surjection $g : N \to G$ followed by a mono $h : G \to \tau M$.

Suppose that h factors through an injective. Then it factors through the injective envelope $E(G)$ of G. Since τM has no injective summand, the induced map $E(G) \to \tau M$ cannot be injective, so its kernel is non-zero. Since G is essential in $E(G)$, the kernel meets G. Thus $G \to \tau M$ has non-zero kernel. Contradiction.

Thus $\overline{\mathrm{Hom}}(G, \tau M) \neq 0$, so $\mathrm{Ext}^1(M, G) \neq 0$.

Definition. Given a module class $\mathcal{C}$ in A-mod and $M \in \mathcal{C}$, we say that
(i) M is *Ext-projective* in $\mathcal{C}$ if $\mathrm{Ext}^1(M, \mathcal{C}) = 0$.
(ii) M is *Ext-injective* in $\mathcal{C}$ if $\mathrm{Ext}^1(\mathcal{C}, M) = 0$.

Lemma 1. If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in A-mod, then
(i) $X \in \mathcal{T}$ is Ext-projective for $\mathcal{T}$ iff $\tau X \in \mathcal{F}$.
(ii) $X \in \mathcal{F}$ is Ext-injective for $\mathcal{F}$ iff $\tau^- X \in \mathcal{T}$.
(iii) There are bijections

$$\text{Non-proj indec Ext-projs in } \mathcal{T} \text{ up to iso} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \text{Non-inj indec Ext-injs in } \mathcal{F} \text{ up to iso}$$

Proof. (i) Say $X \in \mathcal{T}$. Then $\tau X \in \mathcal{F} \Leftrightarrow \mathrm{Hom}(T, \tau X) = 0$ for all $T \in \mathcal{T} \Leftrightarrow \mathrm{Ext}^1(X, \mathrm{gen}\, T) = 0$ for all $T \in \mathcal{T} \Leftrightarrow X$ is Ext-projective. (ii) is dual. (iii) follows.

Lemma 2. If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in A-mod, then

(i) The Ext-injectives for $\mathcal{T}$ are the modules $t_{\mathcal{T}}I$ with I injective. The indecomposable Ext-injectives are the modules $t_{\mathcal{T}}I[i]$ with $I[i] \notin \mathcal{F}$.
(ii) The Ext-projectives for $\mathcal{F}$ are the modules $P/t_{\mathcal{T}}P$ with P projective. The indecomposable Ext-projectives are the modules $P[i]/t_{\mathcal{T}}P[i]$ with $P[i] \notin \mathcal{T}$.

Proof. (i) $t_{\mathcal{T}}I$ is in $\mathcal{T}$, and it is Ext-injective since if $T \in \mathcal{T}$ and $0 \to t_{\mathcal{T}}I \to E \to T \to 0$ is an exact sequence, then the pushout along $t_{\mathcal{T}}I \to I$ splits, giving a map $E \to I$. But $E \in \mathcal{T}$, so it gives a map $E \to t_{\mathcal{T}}I$, which is a retraction for the given sequence. Conversely suppose X is Ext-injective in $\mathcal{T}$ and $X \to I$ is its injective envelope. Then we have an injection $X \to t_{\mathcal{T}}I$, so X is a direct summand of $t_{\mathcal{T}}I$, and we have equality since X is essential in I. (ii) is dual.

Definition. A module M is *τ-rigid* if $\operatorname{Hom}(M, \tau M) = 0$.

Proposition 2. Tfae
(i) M is τ-rigid.
(ii) $\operatorname{Ext}^1(M, \operatorname{gen} M) = 0$.
(iii) $\operatorname{gen} M$ is a torsion class and M is Ext-projective in $\operatorname{gen} M$.
(iv) M is Ext-projective in some torsion class.

Proof. (i)$\Leftrightarrow$(ii). The lemma.

(ii)$\Rightarrow$(iii). Suppose M is τ-rigid. To show that $\operatorname{gen} M$ is a torsion class, it suffices to show that if $0 \to X \to Y \to Z \to 0$ is exact and $X, Z \in \operatorname{gen} M$, then so is Y. Choose a surjection $M^n \to Z$. By (ii) The pullback sequence splits, so the middle term of it is in $\operatorname{gen} M$, and hence so is Y. Now $\operatorname{Ext}^1(M, \operatorname{gen} M) = 0$, so M is Ext-projective.

(iii)$\Rightarrow$(iv). Trivial.

(iv)$\Rightarrow$(ii). If M is Ext-projective in $\mathcal{T}$, then $\operatorname{Ext}^1(M, \operatorname{gen} M) = 0$ since $\operatorname{gen} M \subseteq \mathcal{T}$.

Example. Let A be the path algebra of $1 \to 2 \to 3$. Let $M = 2 \oplus 123$. It is τ-rigid. Then $\mathcal{T} = \operatorname{gen} M$ contains $123, 12, 2, 1$. The torsion-free class is $\mathcal{F} = \mathcal{T}^{\perp 0} = M^{\perp 0}$. It contains 3 and 23.

The Ext-projectives in $\mathcal{T}$ are $2, 12, 123$. The Ext-injectives in $\mathcal{T}$ are $1, 12, 123$. The Ext-projectives in $\mathcal{F}$ are $3, 23$. The Ext-injectives in $\mathcal{F}$ are $3, 23$.

Remark. Any torsion class in A-mod is contravariantly finite, since the inclusion has a right adjoint. If M is τ-rigid, then $\operatorname{gen} M$ is a functorially finite torsion class. The next thereom shows that any functorially finite torsion class $\mathcal{T}$ arises this way.

Theorem (Auslander-Smalø, Almost split sequences in subcategories, 1981, dual of Theorem 4.1(c).) Let $\mathcal{T}$ be a torsion class which is functorially finite, let $f : A \to M$ be a minimal left $\mathcal{T}$-approximation of A, and let $M' = M \oplus \operatorname{Coker}(f)$. Then
(i) $\mathcal{T} = \operatorname{gen} M = \operatorname{gen} M'$.
(ii) If $\theta : T \twoheadrightarrow M$ with $T \in \mathcal{T}$, then θ is a split epi. (M is a splitting projective in $\mathcal{T}$.)
(iii) M and M' are Ext-projective in $\mathcal{T}$, so they are τ-rigid.
(iv) Any module $T \in \mathcal{T}$ is a quotient of a module in $\operatorname{add} M'$ by a submodule in $\mathcal{T}$.
(v) Any Ext-projective in $\mathcal{T}$ is in $\operatorname{add} M'$.

Proof. (i) If $T \in \mathcal{T}$ then there is a map $A^n \twoheadrightarrow T$, and each component factors through M, so $M^n \twoheadrightarrow T$.

(ii) Since A is projective, the map $f : A \to M$ lifts to a map $A \to T$. By the approximation property, this factors as $A \to M \to T$. Now the composition $M \to T \to M$ must be an isomorphism by minimality.

(iii) If $0 \to T \to E \to M \to 0$ is an exact sequence with $T \in \mathcal{T}$, then $E \in \mathcal{T}$, so the sequence splits by (ii). Thus M is Ext-projective.

Now $0 \to \operatorname{Im} f \xrightarrow{i} M \xrightarrow{c} \operatorname{Coker} f \to 0$ gives

$$\operatorname{Hom}(M,T) \to \operatorname{Hom}(\operatorname{Im} f,T) \to \operatorname{Ext}^1(\operatorname{Coker} f,T) \to \operatorname{Ext}^1(M,T) = 0.$$

The composition $\operatorname{Hom}(M,T) \to \operatorname{Hom}(\operatorname{Im} f,T) \to \operatorname{Hom}(A,T)$ is surjective, and $\operatorname{Hom}(\operatorname{Im} f,T) \to \operatorname{Hom}(A,T)$ is injective, so $\operatorname{Hom}(M,T) \to \operatorname{Hom}(\operatorname{Im} f,T)$ is surjective. Thus $\operatorname{Ext}^1(\operatorname{Coker} f,T) = 0$.

(iv) (My thanks to Andrew Hubery for this argument). Take a right $\operatorname{add}(M')$-approximation $\phi : W \to T$. Since $T \in \operatorname{gen} M$, this map is onto, so it gives an exact sequence

$$0 \to U \xrightarrow{\theta} W \xrightarrow{\phi} T \to 0.$$

Given $u \in U$ there is a map $r : A \to U$, $a \mapsto au$. Since $A \to M$ is a $\mathcal{T}$-approximation and $W \in \mathcal{T}$, there is a map p, and hence a map q giving a commutative diagram

$$\begin{array}{ccccccccc}
 & & A & \xrightarrow{f} & M & \xrightarrow{c} & \operatorname{Coker} f & \longrightarrow & 0 \\
 & & {\scriptstyle r}\big\downarrow & & {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow & & \\
0 & \longrightarrow & U & \xrightarrow{\theta} & W & \xrightarrow{\phi} & T & \longrightarrow & 0.
\end{array}$$

Since ϕ is an approximation, $q = \phi h$ for some $h : \operatorname{Coker} f \to W$. Then $\phi(p - hc) = 0$. Thus $p - hc = \theta\ell$ for some $\ell : M \to U$. Then $\theta(r - \ell f) = 0$, so since θ is mono, $r = \ell i$. Thus $u \in \operatorname{Im}(\ell)$. Repeating for a basis of U, we get a map $M^n \twoheadrightarrow U$, so $U \in \mathcal{T}$.

(v) Follows.

3.5 Tilting modules

Definitions.
M is a *partial tilting module* if $\operatorname{proj.dim} M \leq 1$ and $\operatorname{Ext}^1(M, M) = 0$.

A partial tilting module M is a *tilting module* if there is an exact sequence $0 \to A \to M^0 \to M^1 \to 0$ with $M^i \in \operatorname{add} M$. (Later we will see that it is equivalent that $\#M = \#A$.)

M is a *partial cotilting module* if $\operatorname{inj.dim} M \leq 1$ and $\operatorname{Ext}^1(M, M) = 0$.

A partial cotilting module is a *cotilting module* if there is an exact sequence $0 \to M_1 \to M_0 \to DA \to 0$ with $M_i \in \operatorname{add} M$. (Again, it is equivalent that $\#M = \#A$.)

Clearly M is a (partial) tilting A-module iff DM is a (partial) cotilting A^{op}-module.

Remark. We deal only with *classical* tilting theory. There is a version allowing higher projective dimension.

Lemma. If M is a partial tilting module then M is τ-rigid. Conversely if M is τ-rigid, then $\operatorname{Ext}^1(M, M) = 0$, and if M is in addition faithful, then $\operatorname{proj.dim} M \leq 1$ so it is a partial tilting module.

Proof. Use the AR formula $D\operatorname{Ext}^1(M, N) \cong \overline{\operatorname{Hom}}(N, \tau M)$.

If $\operatorname{proj.dim} M \leq 1$ then $\operatorname{Hom}(DA, \tau M) = 0$, so the AR formula takes the form $D\operatorname{Ext}^1(M,N) \cong \operatorname{Hom}(N,\tau M)$.

Suppose M is τ-rigid. If M is faithful, then so is DM, so $A^{op} \hookrightarrow DM^n$, for some n, so $M^n \twoheadrightarrow DA$. Applying $\operatorname{Hom}(-,\tau M)$ we get $\operatorname{Hom}(DA,\tau M) \hookrightarrow \operatorname{Hom}(M^n,\tau M) = 0$. Thus $\operatorname{Hom}(DA,\tau M) = 0$, so $\operatorname{proj.dim} M \leq 1$.

Theorem of Bongartz. Let M be a partial tilting module. Take a basis of $\xi_1,\dots,\xi_n$ of $\operatorname{Ext}^1(M,A)$, consider the tuple $(\xi_1,\dots,\xi_n)$ as an element of $\operatorname{Ext}^1(M^n,A)$, and let

$$0 \to A \to E \to M^n \to 0.$$

be the corresponding *universal* extension. Then $T = E \oplus M$ is a tilting module.

Thus every partial tilting module is a direct summand of a tilting module, and by duality every partial cotilting module is a direct summand of a cotilting module.

Proof. The long exact sequence for $\operatorname{Hom}(M,-)$ gives

$$\operatorname{Hom}(M,M^n) \xrightarrow{\xi} \operatorname{Ext}^1(M,A) \to \operatorname{Ext}^1(M,E) \to \operatorname{Ext}^1(M,M^n),$$

the map ξ is onto, and $\operatorname{Ext}^1(M,M^n) = 0$, so $\operatorname{Ext}^1(M,E) = 0$. From the long exact sequence for $\operatorname{Hom}(-,M)$ one gets $\operatorname{Ext}^1(E,M) = 0$, from the long exact sequence for $\operatorname{Hom}(-,E)$ one gets $\operatorname{Ext}^1(E,E) = 0$. Also A and M^n have projective dimension ≤ 1, hence so does E.

A partial tilting module M is τ-rigid, so gives a torsion theory $(\operatorname{gen} M, M^{\perp 0})$. Moreover $\operatorname{gen}_1 M \subseteq \operatorname{gen} M \subseteq M^{\perp 1}$.

Proposition 1. For a partial tilting module M, tfae:
(i) M is a tilting module.
(ii) $M^{\perp 0,1} = 0$.
(iii) $\operatorname{gen} M = M^{\perp 1}$.
(iv) $\operatorname{gen}_1 M = M^{\perp 1}$.
(v) X is Ext-projective in $M^{\perp 1} \Leftrightarrow X \in \operatorname{add} M$.

Proof of equivalence. (i) $\Rightarrow$ (ii). If $X \in M^{\perp 0,1}$, apply $\operatorname{Hom}(-,X)$ to the exact sequence $0 \to A \to M^0 \to M^1 \to 0$, to deduce that $\operatorname{Hom}(A,X) = 0$.

(ii) $\Rightarrow$ (iii). Suppose $X \in M^{\perp 1}$. Take a basis of $\operatorname{Hom}(M,X)$ and use it to form the universal map $f : M^n \to X$. Then $\operatorname{Im} f \in \operatorname{gen} M$. Consider the exact sequence $0 \to \operatorname{Im} f \to X \to X/\operatorname{Im} f \to 0$. Apply $\operatorname{Hom}(M,-)$ giving an exact sequence

$$0 \to \mathrm{Hom}(M, \mathrm{Im}\, f) \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, X/\mathrm{Im}\, f) \to \mathrm{Ext}^1(M, \mathrm{Im}\, f).$$

By construction the map $\mathrm{Hom}(M, M^n) \to \mathrm{Hom}(M, X)$ is onto, hence so is the map $\mathrm{Hom}(M, \mathrm{Im}\, f) \to \mathrm{Hom}(M, X)$. Also $\mathrm{Ext}^1(M, \mathrm{Im}\, f) = 0$ since M is τ-rigid. Thus $\mathrm{Hom}(M, X/\mathrm{Im}\, f) = 0$. Also $\mathrm{Ext}^1(M, X/\mathrm{Im}\, f) = 0$. Thus $X/\mathrm{Im}\, f \in M^{\perp 0,1}$. Thus $X/\mathrm{Im}\, f = 0$, so f is onto, so $X \in \mathrm{gen}\, M$.

(iii) $\Rightarrow$ (iv). Suppose $X \in M^{\perp 1}$. Then it is in $\mathrm{gen}\, M$. Let L be the kernel of the universal map $M^n \to X$. Then applying $\mathrm{Hom}(M, -)$ we see that $L \in M^{\perp 1}$, so $L \in \mathrm{gen}\, M$. Say $M'' \twoheadrightarrow L$. Now the sequence $M'' \to M^n \to X \to 0$ shows that $X \in \mathrm{gen}_1 M$.

(iv) $\Rightarrow$ (v). Clearly M and so any $X \in \mathrm{add}(M)$ is in $M^{\perp 1}$ and Ext-projective. Conversely if X is in $M^{\perp 1}$ and Ext-projective, then by (iv) there is an exact sequence $M'' \xrightarrow{f} M' \to X \to 0$. This gives an exact sequence $0 \to \mathrm{Im}\, f \to M' \to X \to 0$ with $\mathrm{Im}\, f \in \mathrm{gen}\, M \subseteq M^{\perp 1}$. By assumption this sequence splits, so $X \in \mathrm{add}\, M$.

(v) $\Rightarrow$ (i). It suffices to show that E in Bongartz's sequence is in $\mathrm{add}\, M$, and for this it suffices to show it is Ext-projective in $M^{\perp 1}$. We know it is in $M^{\perp 1}$. If $Y \in M^{\perp 1}$, apply $\mathrm{Hom}(-, Y)$ to the Bongartz sequence to get $\mathrm{Ext}^1(M^n, Y) \to \mathrm{Ext}^1(E, Y) \to \mathrm{Ext}^1(A, Y)$, so $\mathrm{Ext}^1(E, Y) = 0$.

Dually, a partial cotilting module M gives a torsion theory $({}^{\perp 0}M, \mathrm{cogen}\, M)$. Moreover $\mathrm{cogen}^1 M \subseteq \mathrm{cogen}\, M \subseteq {}^{\perp 1}M$.

Proposition 2. For a partial cotilting module M, tfae:
(i') M is a cotilting module.
(ii') ${}^{\perp 0,1}M = 0$.
(iii') $\mathrm{cogen}\, M = {}^{\perp 1}M$.
(iv') $\mathrm{cogen}^1 M = {}^{\perp 1}M$.
(v') X is Ext-injective in ${}^{\perp 1}M \Leftrightarrow X \in \mathrm{add}\, M$.

Proof. Dual of Proposition 1.

Proposition 3. If ${}_AM$ is a (co)tilting module, then it is f.b. and if $B = \mathrm{End}_A(M)$ then ${}_BM$ is also a (co)tilting module.

Proof. If ${}_AM$ is tilting, then $\mathrm{gen}_1 M = M^{\perp 1}$, which contains DA, so ${}_AM$ is f.b.

(i) Applying $\mathrm{Hom}_A(-, M)$ to the exact sequence $0 \to A \to M^0 \to M^1 \to 0$

gives

$$0 \to \mathrm{Hom}_A(M^1, M) \to \mathrm{Hom}_A(M^0, M) \to M \to 0$$

and $\mathrm{Hom}_A(M^i, M) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}({}_B B)$, so $\mathrm{proj.\,dim}\,{}_B M \leq 1$.

(ii) The tilting sequence $0 \to A \to M^0 \to M^1 \to 0$ stays exact on applying $\mathrm{Hom}(-, M)$. Thus $A \in \mathrm{cogen}^2({}_B M)$. Thus $\mathrm{Ext}^1_B(M, M) = 0$ by the proposition in section 3.1.

(iii) Applying $\mathrm{Hom}_A(-, M)$ to a projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of M gives an exact sequence

$$0 \to B \to M^0 \to M^1 \to 0$$

where $M^i = \mathrm{Hom}_A(P_i, M) \in \mathrm{add}({}_B M)$. Thus ${}_B M$ is a tilting module.

Dually for cotilting.

3.6 The Brenner-Butler Theorem

Let ${}_A M$ be a cotilting module and $B = \mathrm{End}_A(M)$, so ${}_B M$ is also cotilting.

In A-mod we have a torsion theory $(\mathcal{T}_A, \mathcal{F}_A) = ({}^{\perp 0}{}_A M, \mathrm{cogen}\,{}_A M)$. Since ${}_A M$ is cotilting we have

$$\begin{aligned} \mathcal{F}_A &= \mathrm{cogen}\,M = \mathrm{cogen}^1\,M = {}^{\perp 1}M \\ &= \{X \in A\text{-mod} : \mathrm{Ext}^1(X, M) = 0\} \\ &= \{X \in A\text{-mod} : \underline{\mathrm{Hom}}(\tau^- M, X) = 0\} \\ &= \{X \in A\text{-mod} : \mathrm{Hom}(\tau^- M, X) = 0\} \end{aligned}$$

(where the last step is because $\mathrm{inj.\,dim}\,M \leq 1$, so there is no non-zero map from $\tau^- M$ to a projective module).

In B-mod we have a torsion theory $(\mathcal{T}_B, \mathcal{F}_B) = ({}^{\perp 0}{}_B M, \mathrm{cogen}\,{}_B M)$. Since ${}_B M$ is cotilting we have the equivalent alternative descriptions of $\mathcal{F}_B$.

Brenner-Butler Theorem. There are antiequivalences

$$\mathcal{F}_A \underset{\mathrm{Hom}_B(-,M)}{\overset{\mathrm{Hom}_A(-,M)}{\rightleftarrows}} \mathcal{F}_B \quad \text{and} \quad \mathcal{T}_A \underset{\mathrm{Ext}^1_B(-,M)}{\overset{\mathrm{Ext}^1_A(-,M)}{\rightleftarrows}} \mathcal{T}_B.$$

Proof. Since M is cotilting, $\mathcal{F} = \operatorname{cogen}^1 M$, so the first antiequivalence is Corollary 2 in section 3.1.

Given a module ${}_AX$ with $\operatorname{Hom}_A(X, M) = 0$ we show that $\operatorname{Hom}_B(\operatorname{Ext}^1_A(X, M), M) = 0$ and construct a natural isomorphism

$$X \to \operatorname{Ext}^1_B(\operatorname{Ext}^1(X, M), M).$$

Indeed, take a projective cover of X to get a sequence $0 \to L \to P \to X \to 0$. It gives an exact sequence of B-modules

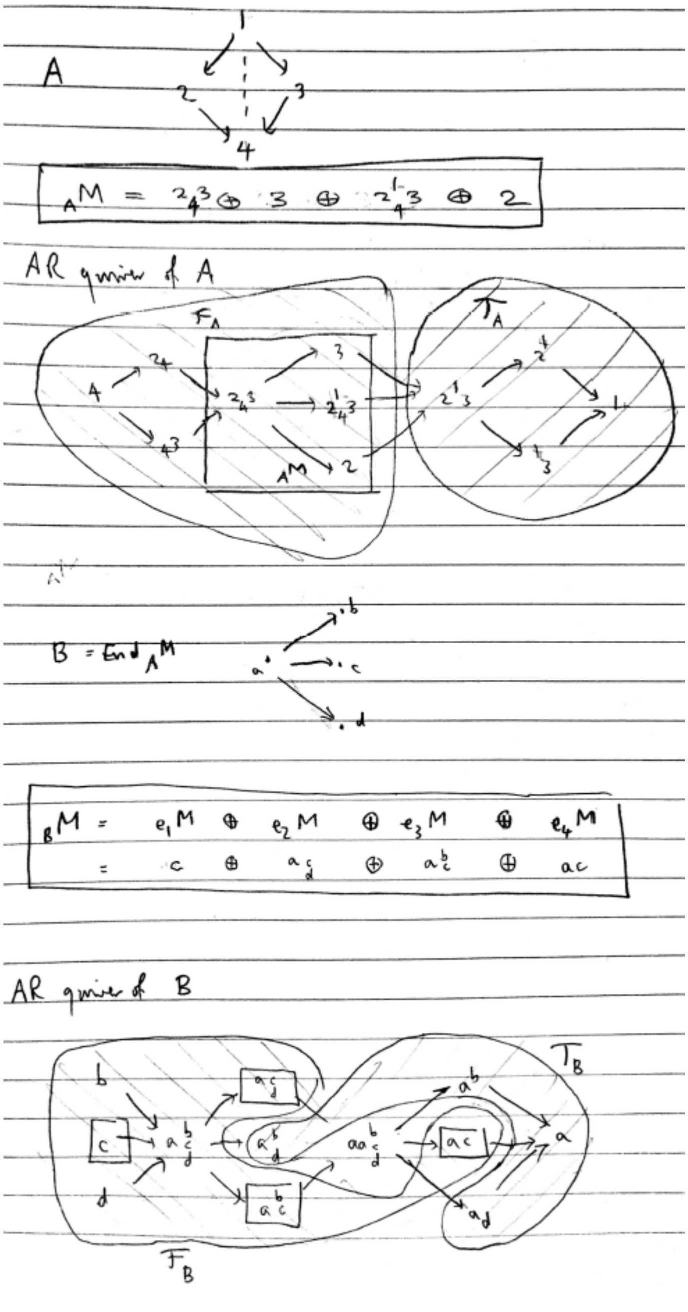
$$0 \to \operatorname{Hom}_A(P, M) \to \operatorname{Hom}_A(L, M) \to \operatorname{Ext}^1_A(X, M) \to 0$$

Now $P, L \in \operatorname{cogen} M = \operatorname{cogen}^1 M$, so the natural maps $P \to \operatorname{Hom}_B(\operatorname{Hom}_A(P, M), M)$ and $L \to \operatorname{Hom}_B(\operatorname{Hom}_A(L, M), M)$ are isomorphisms and $\operatorname{Hom}_A(L, M) \in \operatorname{cogen}^1({}_BM) = {}^{\perp 1}({}_BM)$ so $\operatorname{Ext}^1_B(\operatorname{Hom}(L, M), M) = 0$. Thus we get a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & X & \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & (^1(X, M), M) & \longrightarrow & ((L, M), M) & \longrightarrow & ((P, M), M) & \longrightarrow & {}^1(^1(X, M), M) & \longrightarrow 0 \end{array}$$

(where we omit the words Hom and Ext) with exact rows and in which the vertical maps are isomorphisms. Thus $\operatorname{Hom}_B(\operatorname{Ext}^1_A(X, M), M) = 0$ and there is an induced isomorphism $X \to \operatorname{Ext}^1_B(\operatorname{Ext}^1_A(X, M), M)$. One also needs to show that this is a natural isomorphism, but we omit the proof of this.

Example.



Remark. The usual form of the Brenner-Butler Theorem is as follows. Suppose ${}_AT$ is a tilting A-module and let $\Gamma = \mathrm{End}_A(T)^{op}$. Then DT is a cotilting A^{op}-module, so a cotilting Γ-module. Composing the functors with duality we have equivalences

$\mathrm{Hom}(T,-)$ from $T^{\perp 1} = \mathrm{gen}\, T \subseteq A$-mod to ${}^{\perp 1}DT = \mathrm{cogen}\, DT = \{Y : \mathrm{Tor}_1^\Gamma(T,Y) = 0\} \subseteq \Gamma$-mod. The inverse equivalence is given by $T \otimes_\Gamma -$.

$\mathrm{Ext}^1_A(T,-)$ from $T^{\perp 0} \subseteq A$-mod to ${}^{\perp 0}DT = \{Y : T \otimes_\Gamma Y = 0\} \subseteq \Gamma$-mod. The inverse equivalence is given by $\mathrm{Tor}_1^\Gamma(T,-)$.

Definition. The *Grothendieck group* $K_0(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the additive group generated by symbols $[X]$ for each object X in $\mathcal{A}$, modulo the relations $[Y] = [X] + [Z]$ for any exact sequence $0 \to X \to Y \to Z \to 0$.

Lemma. $K_0(A\text{-mod})$ is the free $\mathbb{Z}$-module on the symbols $[S]$, with S running through the simple modules up to isomorphism.

Proof. Use that every object has a composition series and the Jordan-Hölder theorem.

Corollary 1. If ${}_AM$ is a cotilting module and $B = \mathrm{End}_A(M)$, then there is an isomorphism

$$\theta : K_0(A\text{-mod}) \to K_0(B\text{-mod}), \quad [X] \mapsto [\mathrm{Hom}_A(X,M)] - [\mathrm{Ext}^1_A(X,M)].$$

(Thus the canonical basis of $K_0(B\text{-mod})$ gives a new basis of $K_0(A\text{-mod})$, hence the name "tilting".)

Proof. If we apply $\mathrm{Hom}_A(-,M)$ to a short exact sequence of A-modules, say $0 \to X \to Y \to Z \to 0$ we get a long exact sequence of B-modules

$$0 \to \mathrm{Hom}(Z,M) \to \mathrm{Hom}(Y,M) \to \mathrm{Hom}(X,M) \to$$

$$\mathrm{Ext}^1(Z,M) \to \mathrm{Ext}^1(Y,M) \to \mathrm{Ext}^1(X,M) \to 0.$$

Now the relations for $K_0(B\text{-mod})$ imply that

$$\theta([Y]) = [\mathrm{Hom}_A(Y,M)] - [\mathrm{Ext}^1_A(Y,M)] = [\mathrm{Hom}_A(X,M)] - [\mathrm{Ext}^1_A(X,M)]$$

$$+[\mathrm{Hom}_A(Z,M)] - [\mathrm{Ext}^1_A(Z,M)] = \theta([X]) + \theta([Z])$$

so that θ is well-defined.

Swapping the roles of A and B there is a map ϕ in the reverse direction.

If $X \in \mathrm{cogen}\, M$ or $X \in {}^{\perp 0}M$, then $\phi(\theta([X])) = [X]$. Because any X belongs to a short exact sequence whose ends are torsion and torsion-free, it follows that $\phi\theta = 1$. Similarly $\theta\phi = 1$.

We write $\#M$ for the number of isomorphism classes of indecomposable summands of M. Thus $\#A$ is the number of isomorphism classes of indecomposable projective A-modules, so the number of isomorphism classes of simple A-modules.

Corollary 2. Any partial (co)tilting module M has $\#M \leq \#A$, with equality if and only if M is (co)tilting.

Proof. Any tilting module has $\#A$ summands, and by Bongartz, any partial tilting module is a summand to a tilting module.

Definition. An algebra A is a *tilted algebra* if it has a tilting module ${}_AM$ such that $B = \mathrm{End}_A(M)$ is hereditary. In this case one can show that every indecomposable A-module is in $\mathcal{F}_A$ or in $\mathcal{T}_A$.

3.7 Wide module classes

In this section A is an arbitrary algebra, not even necessarily f.d., but we only consider f.d. modules.

Definition. A module class $\mathcal{C}$ of A-mod is *wide* if it is closed under kernels, cokernels and extensions.

Lemma.
(i) If $\mathcal{C}$ is wide, then it is also closed under images, and it is an abelian category in its own right, and the inclusion functor is exact.
(ii) If A is hereditary, then full subcategory is wide if and only if it is closed under direct summands and has the 2 out of three property for short exact sequences. That is, if $0 \to X \to Y \to Z \to 0$ is exact, and if two of X, Y, Z are in $\mathcal{C}$, then so is the third.

I learnt (ii) from Andrew Hubery. Full subcategories closed under direct summands and satisfying the 2 out of three property are sometimes called *thick subcategories*.

Proof. (i) If $\theta : X \to Y$ then $X \to \mathrm{Im}\,\theta$ is the cokernel of $\mathrm{Ker}\,\theta \to X$.

(ii) If $\mathcal{C}$ is wide, it has the 2 out of 3 property. Conversely, if $\theta : X \to Y$, one gets $\xi : 0 \to \mathrm{Ker}\,\theta \to X \to \mathrm{Im}\,\theta \to 0$ and $\zeta : 0 \to \mathrm{Im}\,\theta \to Y \to \mathrm{Coker}\,\theta$. Applying $\mathrm{Hom}(\mathrm{Coker}\,\theta, -)$ to ξ gives

$$\to \mathrm{Ext}^1(\mathrm{Coker}\,\theta, X) \to \mathrm{Ext}^1(\mathrm{Coker}\,\theta, \mathrm{Im}\,\theta) \to \mathrm{Ext}^2(\mathrm{Coker}\,\theta, \mathrm{Ker}\,\theta) = 0$$

so ζ comes from some η. Thus ζ is the pushout of eta, so there is a commutative diagram

$$\begin{array}{ccccccccc} \eta : 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & \operatorname{Coker}\theta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ \zeta : 0 & \longrightarrow & \operatorname{Im}\theta & \longrightarrow & Y & \longrightarrow & \operatorname{Coker}\theta & \longrightarrow & 0 \end{array}$$

By diagram chasing it follows that there is an exact sequence $0 \to X \to E \oplus \operatorname{Im}\theta \to Y \to 0$. Since $\mathcal{C}$ is closed under extensions and summands, it follows that $\operatorname{Im}\theta \in \mathcal{C}$. Then also $\operatorname{Ker}\theta$ and $\operatorname{Coker}\theta \in \mathcal{C}$.

Remark. Wide subcategories appear naturally. The recent interest is because of a paper of C. Ingalls and H. Thomas, Noncrossing partitions and representations of quivers, 2009, and subsequent developments, but this is beyond the scope of the course.

Definition. We say that an A-module X is $\mathcal{C}$*-simple* if it is in $\mathcal{C}$, and is simple as an object of $\mathcal{C}$. Thus there is no exact sequence $0 \to U \to X \to V \to 0$ with U, V non-zero and in $\mathcal{C}$.

Clearly any $\mathcal{C}$-simple is a *brick*, that is, its endomorphism algebra is a division algebra (so K, if it is algebraically closed).

If $\mathcal{B}$ is a collection of A-modules, we write $\mathcal{F}(\mathcal{B})$ for the full subcategory of A-mod consisting of the modules X with filtrations by submodules

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that each X_i/X_{i-1} is isomorphic to a module in B. Clearly $\mathcal{F}(\mathcal{B})$ is closed under extensions.

Proposition 1. (Ringel, Representations of K-species and bimodules, 1976) Suppose that $\mathcal{B}$ is a set of orthogonal bricks, with orthogonality meaning that $\operatorname{Hom}(X, Y) = 0$ if $X, Y \in \mathcal{B}$ and $X \not\cong Y$. Then $\mathcal{F}(\mathcal{B})$ is a wide subcategory of A-mod, and the $\mathcal{F}(\mathcal{B})$-simples are the modules isomorphic to modules in $\mathcal{B}$.

Clearly any wide subcategory $\mathcal{C}$ arises as $\mathcal{F}(\mathcal{B})$ where one takes $\mathcal{B}$ to be the set of $\mathcal{C}$-simple objects.

Special case. If $\mathcal{B} = \{X\}$ with X is a brick without self-extensions, then $\mathcal{F}(\mathcal{B}) = \operatorname{add} X$. This is wide, and equivalent to K-mod, and X is $\mathcal{C}$-simple.

Proof. Let $f : X \to Y$ be a morphism where X and Y have filtrations of lengths n and m.

We prove by induction on n that $\operatorname{Ker} f$ has a filtration. The result for $\operatorname{Coker} f$ is dual.

We can assume $f(X_1) \neq 0$, for otherwise $X_1 \subseteq \operatorname{Ker} f$ and $\operatorname{Ker} f/X_1$ is the kernel of the induced map $f' : X/X_1 \to Y$, and by induction this kernel is in $\mathcal{F}(\mathcal{B})$, hence $\operatorname{Ker} f \in \mathcal{F}(\mathcal{B})$.

We may also assume $f(X_1) = Y_1$, for there is some i with $f(X_1) \subseteq Y_i$ but $f(X_1) \not\subseteq Y_{i-1}$. Then

$$X_1 \to Y_i \to Y_i/Y_{i-1}$$

is a non-zero map between modules isomorphic to modules in $\mathcal{B}$, so an isomorphism. Thus $f|_{X_1}$ is injective, so $f(X_1) \cong X_1 \cong Y_i/Y_{i-1}$, and $Y_i = Y_{i-1} \oplus f(X_1)$. Now if the filtration of Y is $0 \subset Y_1 \subset \cdots \subset Y_m = Y$, then Y has another filtration

$$0 \subset f(X_1) \subset f(X_1) \oplus Y_1 \subset \cdots \subset f(X_1) \oplus Y_{i-1} = Y_i \subset Y_{i+1} \subset \cdots \subset Y_m = Y.$$

whose successive quotients are $f(X_1), Y_1, Y_2/Y_1, \ldots, Y_{i-1}/Y_{i-2}, Y_{i+1}/Y_i, \ldots, Y_m/Y_{m-1}$ which are all isomorphic to modules in $\mathcal{B}$, as required.

Now let $\overline{f} : X/X_1 \to Y/Y_1$ be the induced map. The map $X \to X/X_1$, $x \mapsto X_1 + x$ induces a map $g : \operatorname{Ker} f \to \operatorname{Ker} \overline{f}$. If $\overline{f}(X_1 + x) = 0$, then $f(x) \in Y_1$ so $f(x) = f(x_1)$ for some $x_1 \in X_1$, so $x' = x - x_1 \in \operatorname{Ker} f$ and $X_1 + x = X_1 + x'$, so g is onto. Also if $x \in \operatorname{Ker} f \cap X_1$ then $x = 0$ since $f|_{X_1}$ is injective, so g is mono.

Thus $\operatorname{Ker} f \cong \operatorname{Ker} \overline{f}$, and by induction this is in $\mathcal{F}(\mathcal{B})$.

Proposition 2. If $\mathcal{X}$ is a set of modules of projective dimension ≤ 1, then

$$\mathcal{X}^{\perp 0,1} = \{M \in A\text{-mod} : \operatorname{Hom}(X, M) = \operatorname{Ext}^1(X, M) = 0 \text{ for all } X \in \mathcal{X}\}.$$

is a wide subcategory.

Proof. Say $\theta : M \to N$ is in $\mathcal{X}^{\perp 0,1}$ and $X \in \mathcal{X}$. We get

$$0 \to \operatorname{Hom}(X, \operatorname{Ker} \theta) \to \operatorname{Hom}(X, M) \to \operatorname{Hom}(X, \operatorname{Im} \theta)$$
$$\to \operatorname{Ext}^1(X, \operatorname{Ker} \theta) \to \operatorname{Ext}^1(X, M) \to \operatorname{Ext}^1(X, \operatorname{Im} \theta) \to 0.$$

Also one has

$$0 \to \operatorname{Hom}(X, \operatorname{Im} \theta) \to \operatorname{Hom}(X, N) \to \operatorname{Hom}(X, \operatorname{Coker} \theta)$$

$$\to \operatorname{Ext}^1(X, \operatorname{Im}\theta) \to \operatorname{Ext}^1(X, N) \to \operatorname{Ext}^1(X, \operatorname{Coker}\theta) \to 0.$$

Thus $\operatorname{Hom}(X, \operatorname{Ker}\theta) = 0$ and $\operatorname{Ext}^1(X, \operatorname{Ker}\theta) \cong \operatorname{Hom}(X, \operatorname{Im}\theta) = 0$, and similarly for $\operatorname{Coker}\theta$. Closure under extensions is easy.

Remark. This is the *perpendicular category* of W. Geigle and H. Lenzing, Perpendicular categories with applications to representations and sheaves, 1991. In fact assuming that the modules in $\mathcal{X}$ are cokernels of monomorphisms between finitely generated projective modules, there is an epimorphism of rings, $A \to A_{\mathcal{X}}$, called the *universal localization*, which has the effect of inverting the maps between projectives. It follows that restriction induces an equivalence $A_{\mathcal{X}}$-mod $\to \mathcal{X}^{\perp 0,1}$. For more about universal localization see chapter 4 of the book A. H. Schofield, Representations of rings over skew fields, 1985.

Definitions. By a *stability function* for A we mean a group homomorphism $\theta : K_0(A\text{-mod}) \to \mathbb{R}$. We write $\theta(X)$ for $\theta([X])$. Thus $\theta(0) = 0$, $\theta(X) = \theta(Y)$ if $X \cong Y$, and $\theta(Y) = \theta(X) + \theta(Z)$ for a short exact sequence $0 \to X \to Y \to Z \to 0$. It is equivalent to fix $\theta(S) \in \mathbb{R}$ for each simple module S.

An A-module X is said to be *θ-semistable* if $\theta(X) = 0$ and $\theta(Y) \leq 0$ for all $Y \subseteq X$. It is *θ-stable* if $\theta(X) = 0$ and $\theta(Y) < 0$ for all non-zero proper submodules Y of X. Observe that X is θ-semistable if and only if $\theta(X) = 0$ and $\theta(Z) \geq 0$ for any quotient Z of X.

Remark. The notion comes from "geometric invariant theory". See the paper A. D. King, Moduli of representations of finite-dimensional algebras, 1994.

Proposition 3. The θ-semistable modules form a wide subcategory $\mathcal{C}$. The $\mathcal{C}$-simples are the θ-stables.

Proof. Let $f : X \to Y$ is a map between θ-semistable modules. We have $\theta(\operatorname{Im} f) \leq 0$ since it is a submodule of Y, and $\theta(\operatorname{Im} f) \geq 0$ since it as a quotient of X. Thus $\theta(\operatorname{Im} f) = 0$. Thus by additivity $\theta(\operatorname{Ker} f) = \theta(\operatorname{Coker} f) = 0$. Now any submodule U of $\operatorname{Ker} f$ is a submodule of X, so $\theta(U) \leq 0$. Thus $\operatorname{Ker} f$ is θ-semistable. Similarly any quotient V of $\operatorname{Coker} f$ is a quotient of Y so $\theta(V) \geq 0$. Thus $\operatorname{Coker} f$ is θ-semistable.

Suppose $X \subseteq Y$. If X and Y/X are θ-semistable, we need to show that Y is θ-semistable. By additivity we have $\theta(Y) = 0$. Now if $U \subseteq Y$ then $U \cap X \subseteq X$ and $U/(U \cap X) \cong (U + X)/X \subseteq Y/Z$, so both have $\theta \leq 0$, hence $\theta(U) \leq 0$.

4 Representations of quivers

Let Q be a quiver and let $A = KQ$. For simplicity throughout K is algebraically closed. We consider f.d. A-modules.

4.1 Bilinear and quadratic forms

We consider $\mathbb{Z}^{Q_0}$ as column vectors, with rows indexed by Q_0. Let $\epsilon[i]$ be the coordinate vector associated to a vertex $i \in Q_0$. Thus $\epsilon[i]_j = \delta_{ij}$.

The dimension vector of a module X is $\underline{\dim}\, X \in \mathbb{Z}^{Q_0}$.

Definition. The *Ringel form* is the bilinear form $\langle -, - \rangle$ on $\mathbb{Z}^{Q_0}$ defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}$$

The corresponding quadratic form $q(\alpha) = \langle \alpha, \alpha \rangle$ is called the *Tits form*. There is a corresponding symmetric bilinear form

$$(\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

Note that q and $(-, -)$ don't depend on the orientation of Q.

The *radical* of q is $\operatorname{rad} q = \{\alpha \in \mathbb{Z}^{Q_0} : (\alpha, \beta) = 0 \text{ for all } \beta \in \mathbb{Z}^{Q_0}\}$.

Theorem (Standard resolution) If X is a KQ-module (not necessarily f.d.) then it has projective resolution

$$0 \to \bigoplus_{a \in Q_1} KQe_{h(a)} \otimes_K e_{t(a)} X \to \bigoplus_{i \in Q_0} KQe_i \otimes_K e_i X \to X \to 0.$$

Proof. We had this before.

This shows KQ is left hereditary and

Corollary. If X and Y are (f.d.) KQ-modules, then

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \dim \operatorname{Hom}(X, Y) - \dim \operatorname{Ext}^1(X, Y).$$

Proof. Apply $\mathrm{Hom}(-,Y)$ to the projective resolution to get an exact sequence

$$0 \to \mathrm{Hom}(X,Y) \to \bigoplus_{i\in Q_0} \mathrm{Hom}(KQe_i \otimes_K e_iX, Y) \to$$

$$\to \bigoplus_{a\in Q_1} \mathrm{Hom}(KQe_{h(a)} \otimes_K e_{t(a)}X, Y) \to \mathrm{Ext}^1(X,Y) \to 0.$$

Now $\mathrm{Hom}(KQe_j \otimes_K e_iX, Y) \cong \mathrm{Hom}_K(e_iX, \mathrm{Hom}(KQe_j, Y)) \cong \mathrm{Hom}_K(e_iX, e_jY)$ so it has dimension $(\underline{\dim}\, X)_i(\underline{\dim}\, Y)_j$.

4.2 Cartan and Coxeter matrices

Suppose that Q has no oriented cycles, so $A = KQ$ is f.d. and so are the projective modules $P[i] = KQe_i$.

Definition. The *Cartan matrix* C has rows and column indexed by Q_0, and is defined by

$$C_{ij} = \dim \mathrm{Hom}(P[i], P[j]) = \dim e_iKQe_j$$
$$= \text{number of paths from } j \text{ to } i.$$

Thus the jth column is $C\epsilon[j] = \underline{\dim}\, P[j]$, and the jth row is $C^T\epsilon[j] = \underline{\dim}\, I[j]$. Namely, $(C\epsilon[j])_i = C_{ij} = \dim e_iKQe_j = \dim e_iP[j]$ and $(C^T\epsilon[j])_i = C^T_{ij} = C_{ji} = \dim D(e_jKQe_i) = \dim e_iI[j]$.

Lemma 1. For any α we have $\langle \underline{\dim}\, P[j], \alpha\rangle = \alpha_j = \langle \alpha, \underline{\dim}\, I[j]\rangle$. It follows that C is invertible, with $(C^{-1})_{ij} = \langle \epsilon[j], \epsilon[i]\rangle$.

Proof. When $\alpha = \underline{\dim}\, X$, we have

$$\langle \underline{\dim}\, P[j], \alpha\rangle = \dim \mathrm{Hom}(P[j], X) - \dim \mathrm{Ext}^1(P[j], X) = \dim e_jX$$

$$\langle \alpha, \underline{\dim}\, I[j]\rangle = \dim \mathrm{Hom}(X, I[j]) - \dim \mathrm{Ext}^1(X, I[j]) =$$
$$= \dim \mathrm{Hom}(P[j], X) = \dim e_jX$$

It follows for all α by additivity.

Now using that $\underline{\dim}\, P[j] = \sum_i C_{ij}\epsilon[i]$, the equality $\langle \underline{\dim}\, P[j], \epsilon[k]\rangle = \delta_{jk}$ gives that $\sum_i C_{ij}\langle \epsilon[i], \epsilon[k]\rangle = \delta_{jk}$.

Definition. The *Coxeter matrix* is $\Phi = -C^TC^{-1}$. That is, it is the matrix with $\Phi\, \underline{\dim}\, P[i] = -\underline{\dim}\, I[i]$ for all i. Thus $\Phi\, \underline{\dim}\, P = -\underline{\dim}\, \nu(P)$ for any projective module P.

Lemma 2. If X has no projective summand, then $\underline{\dim}\,\tau X = \Phi\,\underline{\dim}\,X$.

Proof. If $0 \to P_1 \to P_0 \to X \to 0$ is the minimal projective resolution, then $P_1 \to P_0 \to X \to 0$ is a minimal projective presentation, so one gets a sequence

$$0 \to \tau X \to \nu(P_1) \to \nu(P_0) \to \nu(X) \to 0$$

Since X has no projective summand, $\mathrm{Hom}(X, A) = 0$, so $\nu(X) = 0$. Thus

$$\underline{\dim}\,\tau X = \underline{\dim}\,\nu(P_1) - \underline{\dim}\,\nu(P_0)$$

$$= \Phi(\underline{\dim}\,P_0 - \underline{\dim}\,P_1) = \Phi\,\underline{\dim}\,X.$$

Recall that we have $\mathrm{Hom}(\tau^- X, Y) \cong D\,\mathrm{Ext}^1(Y, X) \cong \mathrm{Hom}(X, \tau Y)$.

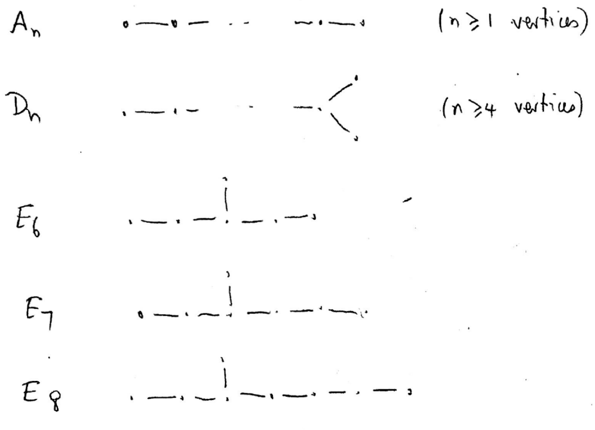
Lemma 3. We have $\langle\alpha, \beta\rangle = -\langle\beta, \Phi\alpha\rangle = \langle\Phi\alpha, \Phi\beta\rangle$. Moreover $\Phi\alpha = \alpha$ if and only if $\alpha \in \mathrm{rad}\, q$.

Proof. $\langle\underline{\dim}\,P[i], \beta\rangle = \beta_i = \langle\beta, \underline{\dim}\,I[i]\rangle = -\langle\beta, \Phi\,\underline{\dim}\,P[i]\rangle$, and now use that the $\underline{\dim}\,P[i]$ span $\mathbb{Z}^{Q_0}$.

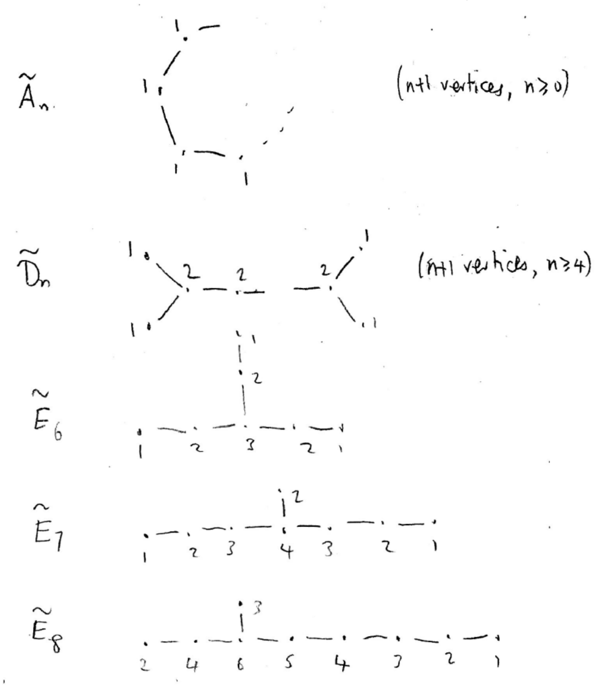
$\Phi\alpha = \alpha$ iff $\langle\beta, \alpha - \Phi\alpha\rangle = 0$ for all β. But this is $\langle\beta, \alpha\rangle + \langle\alpha, \beta\rangle$.

4.3 Classification of quivers

A quiver is *Dynkin* if it is obtained by orienting one of the following graphs (each with n vertices):



A quiver is *extended Dynkin* if it is obtained by orienting one of the following (each with $n+1$ vertices). In each case we define $\delta \in \mathbb{N}^{Q_0}$.



Properties. (1) Any extended Dynkin quiver has at least one vertex i with $\delta_i = 1$. Such a vertex is called an *extending vertex*. Deleting an extending vertex one obtains the corresponding Dynkin quiver.

(2) δ is in the radical of q. For this, we need to check that $(\delta, \epsilon[i]) = 0$ for all i. That is, $2\delta_i = \Sigma_{j-i}\delta_j$.

Lemma 1. Every connected quiver is either Dynkin, or has an extended Dynkin subquiver.

Proof. This is a case-by-case analysis. If there is a loop, it contains $\tilde{A}_0$. If there is a cycle it contains $\tilde{A}_n$. If there is a vertex of valency 4 it contains $\tilde{D}_4$. If there are two vertices of valency 3 it contains $\tilde{D}_n$. Thus (unless it is A_n) it is a star with three arms. If all arms have length > 1 then contains $\tilde{E}_6$. If two arms have length 1 then it is Dynkin. Thus suppose one arm has length 1. If both remaining arms have length > 2 then it contains $\tilde{E}_7$. Thus suppose one has length 2. If the other length is 2,3,4 then it is Dynkin, if > 4 it contains $\tilde{E}_8$.

Theorem. (i) If Q is Dynkin, q is positive definite, that is $q(\alpha) > 0$ for all $0 \neq \alpha \in \mathbb{Z}^{Q_0}$.
(ii) If Q is extended Dynkin quivers, q is positive semidefinite, that is $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^{Q_0}$. Moreover $\alpha \in \operatorname{rad} q \Leftrightarrow q(\alpha) = 0 \Leftrightarrow \alpha \in \mathbb{Z}\delta$.
(iii) If Q is connected and not Dynkin or extended Dynkin, then there is $\alpha \in \mathbb{N}^{Q_0}$ with $(\alpha, \epsilon[i]) \leq 0$ for all i and $q(\alpha) < 0$.

Proof. (ii) For $i \neq j$ we have $(\epsilon[i], \epsilon[j]) \leq 0$. Thus

$$0 \leq -\frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \delta_j \left(\frac{\alpha_i}{\delta_i} - \frac{\alpha_j}{\delta_j} \right)^2$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \frac{\alpha_j^2}{\delta_j} - \frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_j \frac{\alpha_i^2}{\delta_i}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \frac{\alpha_j^2}{\delta_j}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \sum_{j} \left(\sum_{i \neq j} (\epsilon[i], \epsilon[j]) \delta_i \right) \frac{\alpha_j^2}{\delta_j}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j - \sum_{j} \left((\delta, \epsilon[j]) - (\epsilon[j], \epsilon[j]) \delta_j \right) \frac{\alpha_j^2}{\delta_j}$$

$$= \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j + \sum_{j} (\epsilon[j], \epsilon[j]) \alpha_j^2$$

$$= \sum_{i,j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j = (\alpha, \alpha) = 2q(\alpha).$$

Thus q is positive semidefinite.

If $q(\alpha) = 0$ then α_i / δ_i is independent of i, so α is a multiple of δ. Since some $\delta_i = 1$, $\alpha \in \mathbb{Z}\delta$.

Trivially $\alpha \in \mathbb{Z}\delta \Rightarrow \alpha \in \operatorname{rad} q \Rightarrow q(\alpha) = 0$.

(i) Follows by embedding in the corresponding extended Dynkin diagram.

(iii) Take an extended Dynkin subquiver Q' with radical vector δ. If all vertices of Q are in Q', take $\alpha = \delta$. If i is a vertex not in Q' but connected to Q' by an arrow, take $\alpha = 2\delta + \epsilon[i]$.

Definition. We suppose that Q is Dynkin or extended Dynkin. The *roots* are the elements of

$$\Delta = \{\alpha \in \mathbb{Z}^{Q_0} \mid \alpha \neq 0, q(\alpha) \leq 1\}.$$

(One can define roots for arbitrary Q, but the definition is more complicated.)

A root α is *real* if $q(\alpha) = 1$, otherwise it is *imaginary*. In the Dynkin case all roots are real. In the extended Dynkin case the imaginary roots are $r\delta$ with $r \neq 0$.

Lemma 2. Any root α is positive or negative (that is, α or $-\alpha \in \mathbb{N}^{Q_0}$).

Proof. Write $\alpha = \alpha^+ - \alpha^-$ with $\alpha^+, \alpha^- \in \mathbb{N}^{Q_0}$ having disjoint support, then $(\alpha^+, \alpha^-) \leq 0$. But then

$$1 \geq q(\alpha) = q(\alpha^+) + q(\alpha^-) - (\alpha^+, \alpha^-) \geq q(\alpha^+) + q(\alpha^-)$$

so one of α^+, α^- is an imaginary root, hence a multiple of δ. Impossible if disjoint support.

Lemma 3. If Q is Dynkin, then Δ is finite.

Proof. Embed in an extended Dynkin quiver with radical vector δ and extending vertex i. Roots α for Q correspond to roots with $\alpha_i = 0$. Now

$$q(\alpha \pm \delta) = q(\alpha) \pm (\alpha, \delta) + q(\delta) = q(\alpha) = 1$$

so $\beta = \alpha \pm \delta$ is a root, and hence positive or negative. Now $\beta_i = \pm 1$. Thus $-\delta_j \leq \alpha_j \leq \delta_j$ for all j.

(Alternatively, Δ is a discrete subset of the closed bounded (hence compact) subset $\{\alpha \in \mathbb{R}^{Q_0} : q(\alpha) \leq 1\}$ of $\mathbb{R}^{Q_0}$.)

Lemma 4. If Q is Dynkin then $\Phi^N = 1$ for some $N > 0$.

Proof. $q(\Phi\alpha) = q(\alpha)$, so Φ induces a map from the set of roots Δ to itself. Since Φ is invertible this map is injective, and since Δ is finite, this map is a permutation. Thus it has finite order, say $\Phi^N(\alpha) = \alpha$ for all $\alpha \in \Delta$. Since $\epsilon[i] \in \Delta$, it follows that $\Phi^N(\alpha) = \alpha$ for all α.

4.4 Gabriel's Theorem

Gabriel's Theorem.
(i) If Q is a connected quiver, then KQ has finite representation type if and

only if Q is Dynkin.
(ii) If Q is Dynkin, then the assigment $X \rightsquigarrow \underline{\dim} X$ gives a 1-1 correspondence between indecomposable modules and positive roots.

Recall that there is $N > 0$ with $\Phi^N = 1$.

Lemma 1. If Q is Dynkin, then every indecomposable module X is of the form $\tau^{-m} P[i]$ for some projective module $P[i]$ and some $0 \leq m < N$.

Proof. If not, then $X, \tau X, \ldots, \tau^{N-1} X$ are all indecomposable and non-projective, so their direct sum has dimension vector $\beta = \alpha + \Phi\alpha + \cdots + \Phi^{N-1}\alpha$, where $\alpha = \underline{\dim} X$. But then $\beta = \Phi\beta$, so $\beta \in \operatorname{rad} q$ so $q(\beta) = 0$, so $\beta = 0$, which is impossible.

Definition. If Q is extended Dynkin and X is a KQ-module, we define

$$\operatorname{defect}(X) = \langle \delta, \underline{\dim} X \rangle = -\langle \underline{\dim} X, \delta \rangle.$$

Observe that this only depends on the dimension vector of X, so it is additive on short exact sequences.

Lemma 2. Suppose Q is extended Dynkin without oriented cycles. Then $\operatorname{defect}(\tau^{-m} P[i]) = -\delta_i < 0$ and $\operatorname{defect}(\tau^m I[i]) > 0$ for all i, m.

Proof. Clearly $\operatorname{defect}(P[i]) = -\delta_i < 0$ and $\operatorname{defect}(I[i]) > 0$. Also, if X is indecomposable and non-projective, then $\operatorname{defect}(X) = \operatorname{defect}(\tau X)$. By induction on m, none of the modules $\tau^{-m} P[i]$ can be injective, so they all have defect $-\delta_i$. Similarly none of the modules $\tau^m I[i]$ can be projective, so they all have defect δ_i.

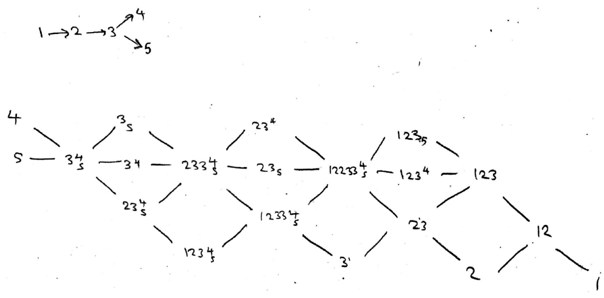
Proof of Gabriel's Theorem (i). If Q is Dynkin, then Lemma 1 shows KQ has finite representation type. Suppose Q is non-Dynkin. Want to show KQ infinite representation type. We may assume Q has no oriented cycle, else true. We may assume Q is extended Dynkin. Now Lemma 2 ensures infinitely many non-isomorphic indecomposables.

Remark. Suppose Q is connected, without oriented cycles. Now $\operatorname{rad} P[i] = \bigoplus_{t(a)=i} P[h(a)]$ and $I[i]/\operatorname{soc} I[i] = \bigoplus_{h(a)=i} I[t(a)]$, so each arrow $a : i \to j$ gives irreducible maps $P[j] \to P[i]$ and $I[j] \to I[i]$.

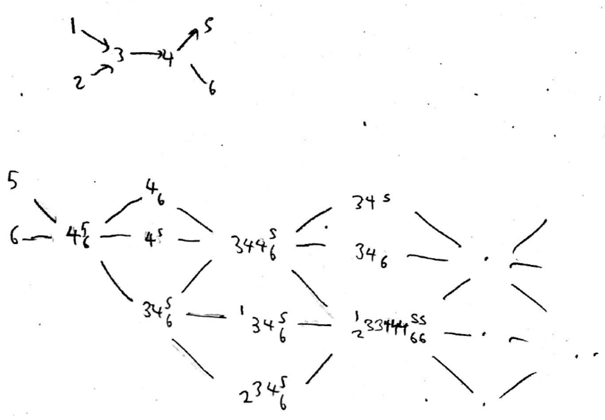
Starting from the projectives $P[i]$ one can knit without obstructions to form the preprojective component.

Starting from the injectives $I[i]$ one can knit without obstructions to form the preinjective component.

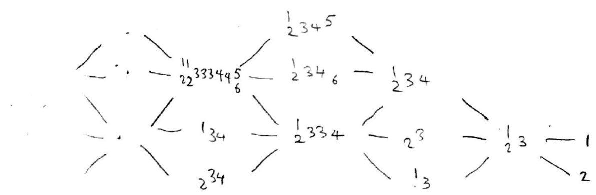
In the Dynkin case these are the same



In the non-Dynkin case they are disjoint



and



Lemma 3. If $\xi : 0 \to X \to Y \to Z \to 0$ is a non-split exact sequence of f.d. modules for any algebra, then

$$\dim \operatorname{End}(Y) < \dim \operatorname{End}(X \oplus Z).$$

Proof. Applying $\operatorname{Hom}(-, Y)$ to the short exact sequence gives a long exact sequence

$$0 \to \operatorname{Hom}(Z, Y) \to \operatorname{Hom}(Y, Y) \to \operatorname{Hom}(X, Y) \to \dots$$

so that

$$\dim \operatorname{Hom}(Y,Y) \le \dim \operatorname{Hom}(Z,Y) + \dim \operatorname{Hom}(X,Y).$$

Similarly, applying $\operatorname{Hom}(X,-)$ gives

$$\dim \operatorname{Hom}(X,Y) \le \dim \operatorname{Hom}(X,X) + \dim \operatorname{Hom}(X,Z).$$

Now applying $\operatorname{Hom}(Z,-)$ gives the long exact sequence

$$0 \to \operatorname{Hom}(Z,X) \to \operatorname{Hom}(Z,Y) \to \operatorname{End}(Z) \xrightarrow{f} \operatorname{Ext}^1(Y,X)$$

and the connecting map f is nonzero since it sends 1_Z to the element in $\operatorname{Ext}^1(Y,X)$ represented by ξ, so

$$\dim \operatorname{Hom}(Z,Y) < \dim \operatorname{Hom}(Z,X) + \dim \operatorname{Hom}(Z,Z).$$

Combining these three inequalities we get the result.

Proof of Gabriel's Theorem (ii). Every indecomposable is preprojective, so uniquely determined by its dimension vector. Also every indecomposable is a brick with no self-extensions, so its dimension vector is a root since

$$q(\underline{\dim}\, X) = \dim \operatorname{End}(X) - \dim \operatorname{Ext}^1(X,X) = 1.$$

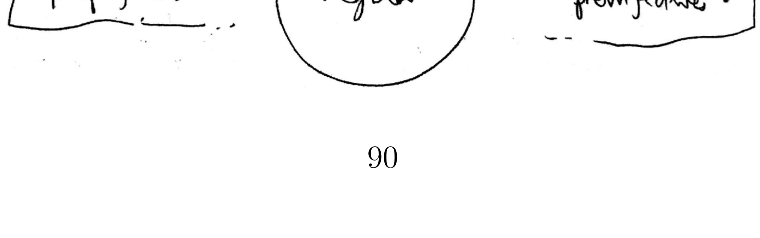
Now suppose α is a positive root. There are modules of dimension vector α, so let X be one with $\dim \operatorname{End}(X)$ minimal. If it decomposes, say $X = U \oplus V$, then $\operatorname{Ext}^1(U,V) = \operatorname{Ext}^1(V,U) = 0$ by Lemma 3. Thus

$$\begin{aligned} 1 = q(\alpha) &= \dim \operatorname{End}(U \oplus V) - \dim \operatorname{Ext}^1(U \oplus V, U \oplus V) \\ &= q(\underline{\dim}\, U) + q(\underline{\dim}\, V) + \dim \operatorname{Hom}(U,V) + \dim \operatorname{Hom}(V,U) \\ &\ge 1 + 1 + 0 + 0 = 2, \end{aligned}$$

a contradiction.

4.5 Preprojective, regular and preinjective modules

Proposition. Let Q be a connected non-Dynkin quiver without oriented cycles.
(a) The module category divides into the three classes



the indecomposable preprojectives are the modules $\tau^{-m}P[i]$,
the indecomposable preinjectives are the modules $\tau^{m}I[i]$,
the indecomposable regular modules are all the rest.

Thus a module X is preprojective $\Leftrightarrow \tau^N X = 0$ for $N \gg 0$,
a module X is preinjective $\Leftrightarrow \tau^{-N} X = 0$ for $N \gg 0$,
a module X is regular $\Leftrightarrow X \cong \tau^{-N}\tau^N X = 0$ for all $N \in \mathbb{Z}$.

(b) The indecomposable preprojectives and preinjectives are directing modules, so they are bricks without self-extensions, and they are uniquely determined by their dimension vectors.

(c) There are no non-zero maps from right to left in the diagram.

Proof. (a), (b) Follow from previous discussion.

(c) If X, Y are indecomposable, Y is preprojective and X is not, then $X \cong \tau^{-m}\tau^m X$ for $i \geq 0$. Thus

$$\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(\tau^{-m}\tau^m X, Y) \cong \operatorname{Hom}(\tau^m X, \tau^m Y) = 0$$

for $m \gg 0$.

From now on we suppose that Q is extended Dynkin without oriented cycles.

Lemma 1. There is $N > 0$ such that $\Phi^N \underline{\dim} X = \underline{\dim} X$ for regular X.

Proof. $\mathbb{Z}\delta$ is an additive subgroup of $\mathbb{Z}^{Q_0}$. Since $\delta \in \operatorname{rad} q$,

$$\Delta \cup \{0\} = \{\alpha \in \mathbb{Z}^{Q_0} : q(\alpha) \leq 1\}.$$

is a union of cosets of $\mathbb{Z}\delta$.

Let e be an extending vertex. If $\alpha \in \Delta \cup \{0\}$, then the coset of α contains $\beta = \alpha - \alpha_e \delta$, a vector with $\beta_e = 0$, which is either the zero vector, or a root for the corresponding Dynkin quiver.

Thus the set of these cosets $(\Delta \cup \{0\})/\mathbb{Z}\delta$ is finite,

Recall that $\Phi\alpha = \alpha$ if and only if α is radical, and that $q(\Phi\alpha) = q(\alpha)$. Thus Φ induces a permutation of the finite set $(\Delta \cup 0)/\mathbb{Z}\delta$.

Thus there is some $N > 0$ with Φ^N the identity on $(\Delta \cup 0)/\mathbb{Z}\delta$. Since $\epsilon[i] \in \Delta$ it follows that Φ^N is the identity on $\mathbb{Z}^{Q_0}/\mathbb{Z}\delta$.

Let $\Phi^N \underline{\dim} X - \underline{\dim} X = r\delta$. An induction shows that $\Phi^{mN} \underline{\dim} X = \underline{\dim} X + mr\delta$ for all $m \in \mathbb{Z}$. If $r < 0$ this is not positive for $m \gg 0$, so X must be preprojective. If $r > 0$ this is not positive for $m \ll 0$, so X is preinjective. Thus $r = 0$.

Lemma 2. An indecomposable module is preprojective, regular, or preinjective according to whether its defect is $< 0, 0$ or > 0.

Proof. We have seen that the preprojectives have defect < 0 and the preinjectives have defect > 0. Thus we must show that if X is regular, then $\text{defect}(X) = 0$.

Say $\underline{\dim} X = \alpha$. Then $\Phi^N\alpha = \alpha$. Let $\beta = \alpha + \Phi\alpha + \dots \Phi^{N-1}\alpha$. Clearly $\Phi\beta = \beta$, so $\beta = r\delta$. Now

$$0 = \langle \beta, \delta \rangle = \sum_{i=0}^{N-1} \langle \Phi^i \alpha, \delta \rangle = N \langle \alpha, \delta \rangle,$$

so $\langle \alpha, \delta \rangle = 0$.

Lemma 3. Suppose α is a positive real root. If $\langle \delta, \alpha \rangle \neq 0$ or $\alpha \leq \delta$, then there is an indecomposable of dimension α.

If $\langle \delta, \alpha \rangle \neq 0$ then X is preprojective or preinjective, so a directing module, so a brick without self-extensions, and the unique indecomposable of this dimension vector.

Proof. Pick a module X of dimension α with $\dim \text{End}(X)$ minimal. We show that if X decomposes, then $\langle \delta, \alpha \rangle = 0$ and $\delta \leq \alpha$, contrary to the assumptions.

Say $X = U \oplus V$. By minimality, $\text{Ext}^1(U, V) = \text{Ext}^1(V, U) = 0$. Then

$$1 = q(\alpha) = q(\underline{\dim} U) + q(\underline{\dim} V) + \dim \text{Hom}(U, V) + \dim \text{Hom}(V, U).$$

Thus, without loss of generality, $q(\underline{\dim} U) = 0$, so $\underline{\dim} U \in \mathbb{Z}\delta$, so $\delta \leq \alpha$. Now $q(\underline{\dim} V) = q(\alpha) = 1$, so the Hom spaces must be zero. Thus $\langle \underline{\dim} V, \underline{\dim} U \rangle = 0$, so $\langle \underline{\dim} V, \delta \rangle = 0$. Thus $\langle \alpha, \delta \rangle = 0$.

4.6 Uniserial structure

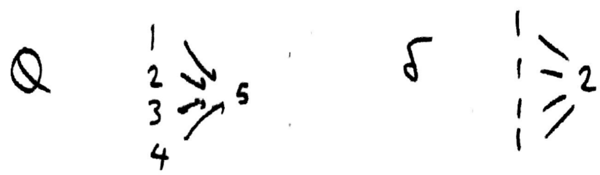
We continue with $A = KQ$ with Q extended Dynkin without oriented cycles. Let $\mathcal{C}$ be the module class of regular modules.

Lemma 1. $\mathcal{C}$ is exactly the defect-semistable modules, so is a wide module class. Moreover τ and τ^{-} define inverse equivalences from $\mathcal{C}$ to itself.

Proof. If $X \in \mathcal{C}$ is indecomposable and U is a submodule, then any indecomposable summand of U has a non-zero map $U \to X$, so U is not preinjective, so defect$(U) \leq 0$. Now τ is fully faithful and dense as a functor $\mathcal{C} \to \mathcal{C}$.

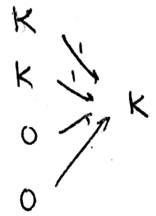
Definition. We have defined the notion of $\mathcal{C}$-simple objects for a wide module class $\mathcal{C}$, hence "regular-simple modules". We say that a regular module X is *regular-uniserial* if if is uniserial in the abelian category of regular modules. We can define the regular-top, regular-socle, a regular-composition series and the regular-length of a regular module.

Example. Consider the 'four subspace quiver' of type $\tilde{D}_4$



If $\underline{\dim}\, X = \alpha$ then defect$(X) = \langle \delta, \alpha \rangle = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\alpha_5$.

(a) The module S_{12} given by



is regular-simple. There are 6 modules like this, denoted S_{ij} with $0 \leq i < j \leq 4$ where the vertices i and j are copies of K. The minimal projective resolution of S_{12} is

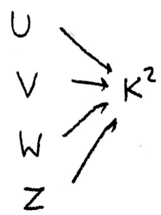
$$0 \to P[5] \to P[1] \oplus P[2] \to 0$$

so τS_{12} is given by the exact sequence

$$0 \to \tau S_{12} \to I[5] \to I[1] \oplus I[2]$$

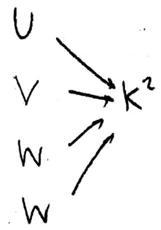
so $\tau S_{12} \cong S_{34}$.

(b) If U, V, W, Z are distinct 1-dimensional subspaces of K^2, then the module

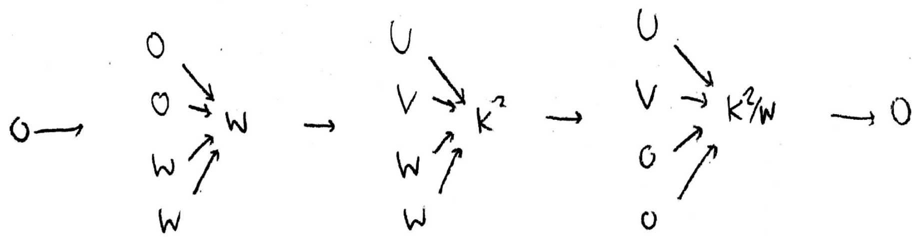


is indecomposable and of defect 0, so regular, and it is regular-simple since any proper non-trivial regular submodule N of dimension α must have $\alpha_5 = 1$, so two of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ must be 1 and two must be 0. But then N_5 must contain two of U, V, W, Z, so $N_5 = K^2$, a contradiction.

(c) Using the subspace W twice gives a module M via



(It is indecomposable since U, V, W already give an indecomposable representation of the Dynkin quiver D_4.) The module M is indecomposable and regular. It fits into an exact sequence



that is, $0 \to S_{34} \to M \to S_{12} \to 0$. This sequence shows the only proper non-trivial regular submodule of M, so it is regular-uniserial. Since $S_{34} \cong \tau S_{12}$ and $\mathrm{Ext}^1(S_{12}, S_{34}) \cong D\,\mathrm{End}(S_{12}) \cong K$, the exact sequence must be the AR sequence ending at S_{12}.

Lemma 2. If S is a regular-simple, then
(i) $\alpha = \underline{\dim}\, S$ is a root.
(ii) $\tau^j S$ is regular-simple for all $j \in \mathbb{Z}$.

(iii) $\tau S \cong S$ iff $\underline{\dim}\, S$ is an imaginary root.
(iv) $\tau^N S \cong S$

Proof. (i) S is a brick by Schur's Lemma for the abelian category $\mathcal{C}$, so $q(\alpha) = \dim \operatorname{End}(S) - \dim \operatorname{Ext}^1(S,S) \leq 1$.

(ii) Since τ and τ^- are equivalences on the abelian category $\mathcal{C}$, they send regular-simples to regular-simples.

(iii) If $\tau S \cong S$ then $\Phi\alpha = \alpha$, so α is radical, so $\alpha = r\delta$ is an imaginary root.

If α is an imaginary root, then $\operatorname{Ext}^1(S,S) \neq 0$, so $\operatorname{Hom}(S, \tau S) \neq 0$. Since S and τS are regular simples, Schur's Lemma for the abelian category $\mathcal{C}$ implies that $S \cong \tau S$.

(iv) If α is a real root, then $\langle \alpha, \Phi^N \alpha \rangle = \langle \alpha, \alpha \rangle = 1$, so $\operatorname{Hom}(S, \tau^N S) \neq 0$, so $S \cong \tau^N S$ by Schur's Lemma. If α is an imaginary root, it follows from (iii).

Lemma 3. If X is regular-uniserial, S is regular-simple and

$$\xi : 0 \to S \to E \xrightarrow{f} X \to 0$$

is non-split, then E is regular-uniserial.

Proof. Let T be the regular-socle of X. Now E is regular since $\mathcal{C}$ is closed under extensions. It suffices to prove that if $U \subseteq E$ is a regular submodule and U is not contained in S, then $S \subseteq U$. Now we have $f(U) \neq 0$, so $T \subseteq f(U)$. Then $f^{-1}(T) = S + U \cap f^{-1}(T)$ by the modular law.

Since $\tau^- S$ is regular-simple, the inclusion $T \hookrightarrow X$ gives an isomorphism $\operatorname{Hom}(\tau^- S, T) \to \operatorname{Hom}(\tau^- S, X)$. Thus it gives an isomorphism

$$\operatorname{Ext}^1(X,S) \cong D \operatorname{Hom}(\tau^- S, X) \cong D \operatorname{Hom}(\tau^- S, T) \cong \operatorname{Ext}^1(T,S),$$

so the pullback sequence

$$\begin{array}{ccccccccc} \zeta : 0 & \longrightarrow & S & \longrightarrow & f^{-1}(T) & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ \xi : 0 & \longrightarrow & S & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

is non-split. Thus the sum $f^{-1}(T) = S + U \cap f^{-1}(T)$, cannot be a direct sum, so $S \cap U \cap f^{-1}(T) \neq 0$. Thus $S \subseteq U$.

Lemma 4. For each regular-simple T and $r \geq 1$ there is a unique regular-uniserial module $T\{r\}$ with regular-top T and regular-length r. Its regular-composition factors are (from the top) $T, \tau T, \dots, \tau^{r-1}T$.

Proof. We work by induction on r. Suppose $X = T\{r\}$ exists. Let S be regular-simple. Now

$$\operatorname{Ext}^1(X,S) \cong D\operatorname{Hom}(\tau^- S, X) \cong D\operatorname{Hom}(\tau^- S, \tau^{r-1}T) \cong \begin{cases} K & (S \cong \tau^r T) \\ 0 & (\text{otherwise}) \end{cases}$$

so there is a non-split sequence $\xi : 0 \to S \to E \to X \to 0$ if and only if $S \cong \tau^{r-1}T$. Moreover in this case, since the space of extensions is 1-dimensional, any non-zero $\xi \in \operatorname{Ext}^1(X,S)$ gives rise to the same module E. It is uniserial by the previous lemma, so it is $T\{r+1\}$. Uniqueness is straightforward.

Theorem. Every indecomposable regular module X is regular-uniserial (so of the form $T\{r\}$ for some T and r).

Proof. Induction on the dimension of X. If X is regular-simple there is nothing to prove, so suppose otherwise. Let $S \subseteq X$ be a regular-simple submodule of X. Write the quotient as a direct sum of indecomposables

$$X/S = \bigoplus_{i=1}^{r} Y_i$$

By induction the Y_i are regular-uniserial. Now

$$\operatorname{Ext}^1(X/S,S) \cong \bigoplus_{i=1}^{r} \operatorname{Ext}^1(Y_i,S)$$

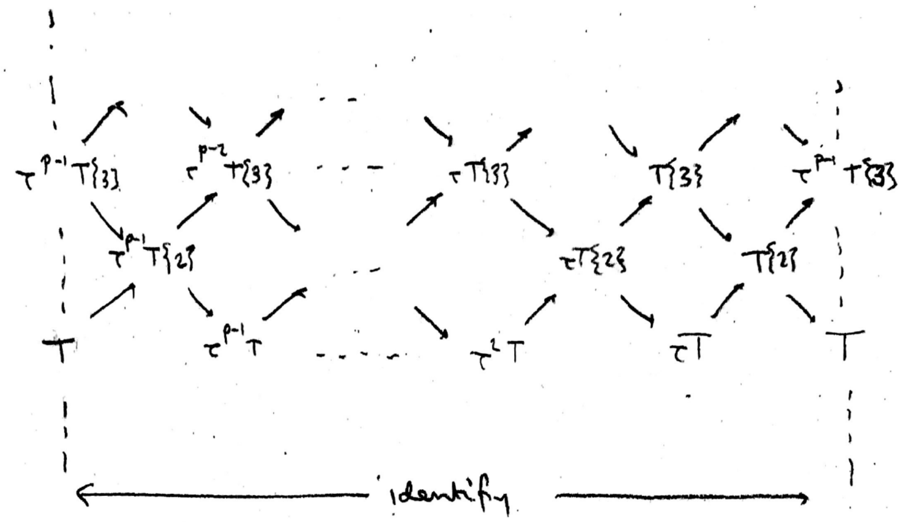
with the sequence $0 \to S \to X \to X/S \to 0$ corresponding to (ξ_i). Since X is indecomposable, all $\xi_i \neq 0$. Now

$$\operatorname{Ext}^1(Y_i,S) \cong \begin{cases} K & (\text{if regular-socle of } Y_i \text{ is } \tau^- S) \\ 0 & (\text{otherwise}) \end{cases}$$

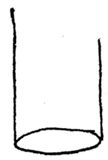
so all Y_i have regular-socle $t^- S$.

If $r = 1$ then X is regular uniserial, so suppose $r \geq 2$ for contradiction. We may assume that $\dim Y_1 \leq \dim Y_2$, and then by the classification in Lemma 4 of regular uniserials, there is an embedding $g : Y_1 \hookrightarrow Y_2$. This induces an isomorphism $\mathrm{Ext}^1(Y_2, S) \to \mathrm{Ext}^1(Y_1, S)$ so we can use g to adjust the decomposition of X/S to make one component ξ_i zero, a contradiction. Explicitly, we write $X/S = Y_1' \oplus Y_2 \oplus \cdots \oplus Y_r$ with $Y_1' = \{y_1 + \lambda g(y_1) : y_1 \in Y_1\}$ for some $\lambda \in K$.

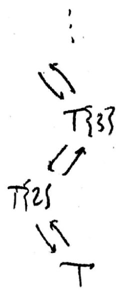
Corollary. A regular simple T of τ-period p is contained in a component of the AR quiver of the following shape



The dotted lines must be identified, to give a tube



In particular, for period 1, the component looks as follows



Every indecomposable regular module is contained in such a tube.

Proof. Details omitted.

4.7 Regular-simples and roots

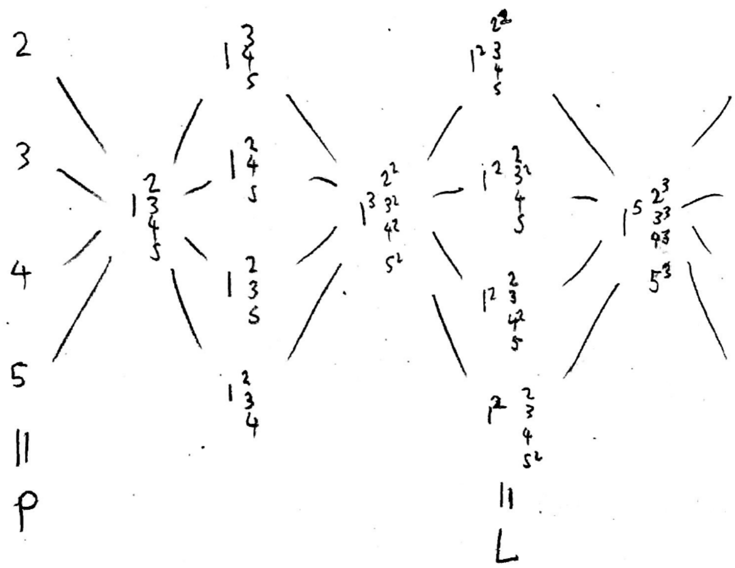
We continue with $A = KQ$ with Q extended Dynkin without oriented cycles. We show that the tubes are indexed by the projective line and the dimension vectors of indecomposable modules are exactly the positive roots.

Let e be an extending vertex, $P = P[e]$, $p = \underline{\dim}\, P$. Clearly $\langle p,p\rangle = 1 = \langle p,\delta\rangle$. Thus $\delta+p$ is a positive real root and $\langle \delta+p,\delta\rangle = 1$, so there is a unique indecomposable L of dimension $\delta + p$. Now P and L are preprojective, are bricks, and have no self-extensions.

Now $\mathrm{Hom}(L,P) = 0$, for if $\theta : L \to P$ then $\mathrm{Im}\,\theta$ is projective, so isomorphic to a summand of L, a contradiction. Also $\mathrm{Ext}^1(L,P) = 0$ since $\langle \underline{\dim}\, L, \underline{\dim}\, P\rangle = \langle p+\delta, p\rangle = \langle p,p\rangle - \langle p,\delta\rangle = 0$. Moreover $\dim \mathrm{Hom}(P,L) = 2$ since $\langle p, p+\delta\rangle = 2$.

Example. For the quiver

with extending vertex $e = 5$, the preprojective component starts as follows



Lemma 1. If $0 \neq \theta \in \mathrm{Hom}(P, L)$ then θ is mono, $\mathrm{Coker}\,\theta$ is a regular indecomposable of dimension δ, and $[\text{reg-top}(\mathrm{Coker}\,\theta)]_e \neq 0$.

Proof. Suppose θ is not mono. Now $\mathrm{Ker}\,\theta$ and $\mathrm{Im}\,\theta$ are preprojective (since they embed in P and L), and so they have defect ≤ -1. Now the sequence

$$0 \to \mathrm{Ker}\,\theta \to P \to \mathrm{Im}\,\theta \to 0$$

is exact, so $-1 = \mathrm{defect}(P) = \mathrm{defect}(\mathrm{Ker}\,\theta) + \mathrm{defect}(\mathrm{Im}\,\theta) \leq -2$, a contradiction.

Let $X = \mathrm{Coker}\,\theta$, and consider $\xi : 0 \to P \xrightarrow{\theta} L \to X \to 0$. Apply $\mathrm{Hom}(-, P)$ to get $\mathrm{Ext}^1(X, P) = K$. Apply $\mathrm{Hom}(-, L)$ to get $\mathrm{Hom}(X, L) = 0$. Apply $\mathrm{Hom}(X, -)$ to get X a brick. If X has regular top T, then $\dim T_e = \dim \mathrm{Hom}(P, T) = \langle p, \underline{\dim}\, T\rangle = \langle p + \delta, \underline{\dim}\, T\rangle = \dim \mathrm{Hom}(L, T) \neq 0$.

Lemma 2. If X is regular, $X_e \neq 0$ then $\mathrm{Hom}(\mathrm{Coker}\,\theta, X) \neq 0$ for some $0 \neq \theta \in \mathrm{Hom}(P, L)$.

Proof. $\mathrm{Ext}^1(L, X) = 0$, so

$$\dim \mathrm{Hom}(L, X) = \langle p + \delta, \underline{\dim}\, X\rangle = \langle p, \underline{\dim}\, X\rangle = \dim \mathrm{Hom}(P, X) \neq 0.$$

Let α, β be a basis of $\mathrm{Hom}(P, L)$. These give maps $a, b : \mathrm{Hom}(L, X) \to \mathrm{Hom}(P, X)$.

If a is an iso, let λ be an eigenvalue of $a^{-1}b$ and set $\theta = \beta - \lambda\alpha$. If a is not an iso, let $\theta = \alpha$. Either way, there is $0 \neq \phi \in \mathrm{Hom}(L, X)$ with $\phi \circ \theta = 0$. Thus there is an induced non-zero map $\overline{\phi} : \mathrm{Coker}\,\theta \to X$.

Theorem 1.
(a) If S is regular simple of τ-period p, then

$$\underline{\dim}\, S + \underline{\dim}\, \tau S + \cdots + \underline{\dim}\, \tau^{p-1} S = \delta.$$

(b) Regular simples S, T of the same dimension $\alpha \neq \delta$ must be isomorphic.
(c) All but finitely many regular simples have dimension δ, so all but finitely many tubes have period 1.

Proof. (a) We show that $\alpha := \underline{\dim}\, S \leq \delta$. If $\alpha_e \neq 0$, this holds since there is a non-zero map $\mathrm{Coker}\,\theta \to S$ which must be onto. If $\alpha_e = 0$, then $\delta - \alpha$ is a root, and $(\delta - \alpha)_e = 1$, so $\delta - \alpha$ is a positive root, so again $\alpha \leq \delta$.

If $\alpha = \delta$, then $S \cong \tau S$, so $p = 1$ and we are done. Thus we may suppose α is a real root. Now $\delta - \alpha$ is a positive real root, so by §4.5 Lemma 3 there is an indecomposable module R of dimension $\delta - \alpha$; regular since $\langle \delta, \delta - \alpha\rangle = 0$.

As $\langle\alpha,\delta-\alpha\rangle = -1$, $0 \neq \mathrm{Ext}^1(S,R) \cong D\,\mathrm{Hom}(R,\tau S)$, so reg-top $R \cong \tau S$. As $\langle\delta-\alpha,\alpha\rangle = -1$, $0 \neq \mathrm{Ext}^1(R,S) \cong D\,\mathrm{Hom}(\tau^- S,R)$, so reg-soc $R \cong \tau^- S$. It follows that R must at least involve $\tau S, \tau^2 S, \ldots, \tau^{p-1}S$, so

$$\underline{\dim}\, S + \underline{\dim}\, \tau S + \cdots + \underline{\dim}\, \tau^{p-1} S \leq \underline{\dim}\, S + \underline{\dim}\, R = \delta.$$

Also the sum is invariant under Φ, so it is a multiple of δ.

(b) $\mathrm{Hom}(S,T) \neq 0$ since $1 = \langle\alpha,\alpha\rangle = \dim \mathrm{Hom}(S,T) - \dim \mathrm{Ext}^1(S,T)$, so $S \cong T$.

(c) There are only finitely many positive roots $< \delta$.

Theorem 2. The set of tubes is in bijection with the projective line. Explicitly (i) each tube contains a unique module in the set

$$\Omega = \{\text{isoclasses of indecs } X \text{ with } \underline{\dim}\, X = \delta \text{ and reg-top}(X)_e \neq 0\}$$

(ii) There is a bijection, $\mathbb{P}\,\mathrm{Hom}(P,L) \to \Omega$, $K\theta \mapsto \mathrm{Coker}\,\theta$, where $\mathbb{P}V$ denotes the set of 1-dimensional subspaces of V.

Proof. (i) By Theorem 1(a), exactly one regular simple T in the tube has $T_e \neq 0$. Clearly Ω contains the module with regular top T and regular length p, the period of T, and no other modules from the tube.

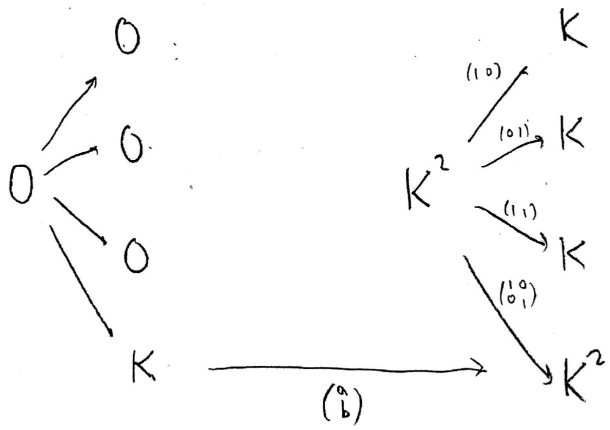
(ii) By Lemma 1, $\mathrm{Coker}\,\theta \in \Omega$. Conversely if $X \in \Omega$ then by Lemma 2, for some θ there is a non-zero map $\mathrm{Coker}\,\theta \to X$. This must be surjective, since any proper regular quotient of $\mathrm{Coker}\,\theta$ is non-zero at e, but any proper regular submodule of X is zero at e. Thus the map is an isomorphism.

If $0 \neq \theta, \theta' \in \mathrm{Hom}(P,L)$ and $\mathrm{Coker}\,\theta \cong \mathrm{Coker}\,\theta'$, then since $\mathrm{Ext}^1(L,P) = 0$ one gets a commutative diagram

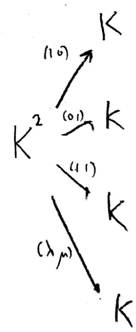
$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\theta'} & L & \longrightarrow & \mathrm{Coker}\,\theta' & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \| & & \\
0 & \longrightarrow & P & \xrightarrow{\theta} & L & \longrightarrow & \mathrm{Coker}\,\theta & \longrightarrow & 0
\end{array}$$

Now f and g are non-zero multiples of the identity, so $K\theta = K\theta'$.

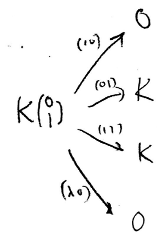
In the example before, the module L can be given by the indicated matrices, and then a map $P \to L$ is given by a linear map $K \to K^2$ as indicated



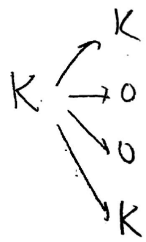
with $a, b \in K$ not both zero. The cokernel is isomorphic to



where $\lambda, \mu \in K$ are not both zero and satisfy $a\lambda + b\mu = 0$. This module is regular simple unless $(\lambda\ \mu)$ is a multiple of one of the other maps, say $(1\ 0)$, when the module is not regular simple since it has regular submodule



with quotient isomorphic to



Theorem 3.
(1) If X is indecomposable then $\underline{\dim} X$ is a root.
(2) If α is a positive imaginary root there are infinitely many many indecomposable modules X with $\underline{\dim} X = \alpha$.
(3) If α is a positive real root there is a unique indecomposable module X with $\underline{\dim} X = \alpha$.

Proof. (Omitted in the lecture.) (1) If X is a brick, this is clear. If X is not a brick, it is regular. Let X have period p and regular length $rp + q$ with $1 < q \leq p$. The regular submodule Y of X with regular length q is a brick, and so $\underline{\dim} X = \underline{\dim} Y + r\delta$ is a root.

(2) We have $\alpha = r\delta$. If T is a tube of period p, then the indecomposables in T of regular length rp have dimension $r\delta$, and there are infinitely many tubes.

(3) We have proved this already in case $\langle \alpha, \delta \rangle \neq 0$, so suppose $\langle \alpha, \delta \rangle = 0$. We can write $\alpha = r\delta + \beta$ for some real root β with $0 \leq \beta < \delta$. We know that there is a regular indecomposable Y of dimension β. Suppose it corresponds to a module of regular-length q in a tube of period p. Then $q < p$. Let X be the regular uniserial containing Y and with regular length $rp+q$. Clearly $\underline{\dim} X = r\delta + \underline{\dim} Y = \alpha$.

Finally suppose that there are two regular-uniserials X, Y that have the same dimension vector α, a real root. Then $\langle \underline{\dim} X, \underline{\dim} Y \rangle = q(\alpha) = 1$, so $\mathrm{Hom}(X, Y) \neq 0$. Thus X and Y are in the same tube, say of period p. Since α is not a multiple of δ, the regular-length of X is not a multiple of p. Thus if S is the regular socle of X, we have $\mathrm{Hom}(S, X) \neq 0$ and $\mathrm{Ext}^1(S, X) = 0$. Thus $\langle \underline{\dim} S, \alpha \rangle > 0$. Thus $\mathrm{Hom}(S, Y) \neq 0$. Thus Y has regular socle S. Thus since $\underline{\dim} X = \underline{\dim} Y$ we have $X \cong Y$.