Noncommutative Algebra 3: Geometric methods for representations of algebras

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William Crawley-Boevey

Introduction

This is part three of a masters sequence on Noncommutative Algebra, with emphasis on representation theory of finite-dimensional algebras and quivers. This part is devoted to links with geometry. The course begins with an introduction to the appropriate sort of geometry, which is algebraic geometry. Suitable literature is as follows.

- G. R. Kempf, 'Algebraic varieties', London Mathematical Society Lecture Note Series, 172. Cambridge University Press, Cambridge, 1993. (A short book, useful for this course. I introduce varieties as spaces with functions, as in this book.)
- U. Görtz and T. Wedhorn, 'Algebraic geometry I. Schemes with examples and exercises', Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. (I haven't used this book, but it was recommended by students who took this course before.)
- R. Hartshorne, 'Algebraic geometry', Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
- D. Mumford, 'Algebraic geometry. I. Complex projective varieties', Grundlehren der Mathematischen Wissenschaften, No. 221. Springer-Verlag, Berlin-New York, 1976.
- D. Mumford, 'The red book of varieties and schemes', Lecture Notes in Mathematics, 1358. Springer-Verlag, Berlin, 1988.

One uses geometry to study varieties of module and algebra structures, and also quiver grassmannians. Examples of applications are as follows. I won't cover much of this, but I will include a few of the ideas as examples when introducing algebraic geometry.

- K. Bongartz, A geometric version of the Morita equivalence, J. Algebra 139 (1991), no. 1, 159–171.
- K. Bongartz, On degenerations and extensions of finite-dimensional modules, Adv. Math. 121 (1996), no. 2, 245–287.
- P. Gabriel, Finite representation type is open, in 'Representations of Algebras', Springer Lecture Notes vol. 488, 1975
- A. Hubery, Irreducible components of quiver Grassmannians, Trans. Amer. Math. Soc. 369 (2017), no. 2, 13951458.
- C. Riedtmann, Degenerations for representations of quivers with relations, Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 2, 275–301.
- A. Schofield, General representations of quivers. Proc. London Math. Soc. (3) 65 (1992), no. 1, 46–64.
- G. Zwara, Degenerations for modules over representation-finite algebras, Proc. Amer. Math. Soc. 127 (1999), no. 5, 1313–1322.
- G. Zwara, Degenerations of finite-dimensional modules are given by extensions, Compositio Math. 121 (2000), no. 2, 205–218.

Symmetry is of course essential, and this appears as actions of algebraic groups on varieties. Quotient spaces in this setup are unfortunately rather complicated, leading to geometric invariant theory and moduli spaces. We discuss only a little of this. References

- A. Borel, 'Linear algebraic groups', Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.
- A. D. King, Moduli of representations of finite-dimensional algebras. Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 515–530.
- S. Mukai, An introduction to invariants and moduli. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury. Cambridge Studies in Advanced Mathematics, 81. Cambridge University Press, Cambridge, 2003.

- D. Mumford, J. Fogarty and F. Kirwan, 'Geometric invariant theory', Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34. Springer-Verlag, Berlin, 1994.

Kac's Theorem is one of the outstanding results about representations of quivers, and the proof uses geometry in an essential way. There were a number of conjectures that remained, but they have now been solved. References.

- V. G. Kac, Root systems, representations of quivers and invariant theory, in 'Invariant theory' (Montecatini, 1982), 74–108, Lecture Notes in Math., 996, Springer, Berlin, 1983.
- T. Hausel, Kac's conjecture from Nakajima quiver varieties, Invent. Math. 181 (2010), no. 1, 21–37.
- T. Hausel, E. Letellier and F. Rodriguez-Villegas, Positivity for Kac polynomials and DT-invariants of quivers, Ann. of Math. (2) 177 (2013), no. 3, 1147–1168.

A key example of this setup is Nakajima's 'Quiver varieties', which are constructed using ideas from Symplectic geometry. Alternatively they arise from preprojective algebras. They were used by Nakajima to construct representations of Kac-Moody Lie algebras. They also give a very nice construction of deformations and desingularizations of Kleinian singularities. References.

- H. Cassens and P. Slodowy, On Kleinian singularities and quivers, in 'Singularities' (Oberwolfach, 1996), 263–288, Progr. Math., 162, Birkhäuser, Basel, 1998
- V. Ginzburg, Lectures on Nakajima's quiver varieties, in 'Geometric methods in representation theory. I', 145–219, Smin. Congr., 24-I, Soc. Math. France, Paris, 2012.
- A. Kirillov, Jr., 'Quiver representations and quiver varieties', Graduate Studies in Mathematics, 174. American Mathematical Society, Providence, RI, 2016.
- H. Nakajima, Quiver varieties and Kac-Moody algebras, Duke Math. J. 91 (1998), no. 3, 515–560.

This is a rather long wish-list. Through lack of time and energy, we shall only touch on any of these topics.

Contents

1 Varieties

[Start of LECTURE 1 on 20 April 2020]

We fix an algebraically closed field K of arbitrary characteristic.

1.1 Spaces with functions

A) **Topological spaces.** Recall that a *topological space* is given by a set X together with a set of subsets of X, the *open sets* such that

- $\emptyset$ and X are open.

- Any union of open sets is open

- A finite intersection of open sets is open.

There are notions of closed subsets, continuous mappings, neighbourhoods, etc.

Any subset Y of a topological space X becomes a topological space with the *induced topology*, in which the open sets are the sets of the form $Y \cap U$ with U an open subset of X.

B) **Definition.** If U is a set, then the set of functions $U \to K$ becomes a commutative K-algebra under the pointwise operations

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x).$$

[See Kempf] A *space with functions* consists of a topological space X and an assignment for each open set $U \subseteq X$ of a K-subalgebra $\mathcal{O}(U)$ of the algebra of functions $U \to K$, satisfying:

(a) If U is a union of open sets, $U = \bigcup U_\alpha$, then $f \in \mathcal{O}(U)$ iff $f|_{U_\alpha} \in \mathcal{O}(U_\alpha)$ for all α.

(b) If $f \in \mathcal{O}(U)$ then $D(f) = \{u \in U \mid f(u) \neq 0\}$ is open in U and $1/f \in \mathcal{O}(D(f))$.

Elements of $\mathcal{O}(U)$ are called *regular functions*. We sometimes write it as $\mathcal{O}_X(U)$.

A *morphism* of spaces with functions is a continuous map $\theta : X \to Y$ with the property that for any open subset U of Y, and any $f \in \mathcal{O}(U)$, the composition

$$\theta^{-1}(U) \xrightarrow{\theta} U \xrightarrow{f} K$$

is in $\mathcal{O}(\theta^{-1}(U))$. In this way one gets a category of spaces with functions.

C) **Examples.**
(1) Let X be a topological space, and choose any topology on the field K (compatible with the field operations). Let $\mathcal{O}(U)$ be the set of continuous functions $U \to K$. Morphisms between such spaces with functions are continuous maps.
(2) X manifold, $\mathcal{O}(U)$ = infinitely differentiable functions $U \to \mathbb{R}$. Morphisms are infinitely differential maps between manifolds.
(3) X complex manifold, eg the complex plane, $\mathcal{O}(U)$ = analytic functions $U \to \mathbb{C}$.

D) **Subsets.** If X is a space with functions and Y is a subset of X, one defines $\mathcal{O}(Y)$ to be the set of functions $f : Y \to K$ such that each $y \in Y$ has an open neighbourhood U in X such that $f|_{Y\cap U} = g|_{Y\cap U}$ for some $g \in \mathcal{O}(U)$.

Any subset Y of a space with functions X has an *induced structure as a space with functions* by equipping Y with the subspace topology and open subsets of Y with the induced sets of functions.

Lemma. The inclusion $i : Y \to X$ is a morphism of spaces with functions, and if Z is a space with functions, then $\theta : Z \to Y$ is a morphism if and only if $i\theta : Z \to X$ is a morphism.

Proof. Exercise.

E) **Theorem.** If X and Y are spaces with functions, then the set $X \times Y$ can be given the structure of a space with functions, so that it becomes a product of X and Y in the category of spaces with functions.

Proof. See Kempf, Lemma 3.1.1. The topology is not the usual product topology. Instead a basis of open sets is given by the sets

$$\{(u, v) \in U \times V : f(u, v) \neq 0\}$$

where U is open in X, V is open in Y and $f(x, y) = \sum_{i=1}^{n} g_i(x)h_i(y)$ with $g_i \in \mathcal{O}(U)$ and $h_i \in \mathcal{O}(V)$

Lemma. The image of an open set under the projection $p : X \times Y \to X$ is open.

Proof. For $y \in Y$, the categorical product gives a morphism $i_y : X \to X \times Y$ with $i_y(x) = (x, y)$. Now if $U \subseteq X \times Y$, then $p(U) = \bigcup_{y\in Y} i_y^{-1}(U)$, which is open.

F) **Definition.** A space with functions X is *separated* if the diagonal

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in $X \times X$.

Here the product $X \times X$ is the corresponding space with functions, and the topology is not the product topology. On the other hand Δ_X is closed in $X \times X$ with the product topology if and only if X is Hausdorff.

Note that separatedness passes to subsets of a space with functions equipped with the induced structure, for if Y is a subset of X, then $\Delta_Y = (Y \times Y) \cap \Delta_X$ in $X \times X$.

1.2 Affine space

A) **Definition.** *Affine n-space* is $\mathbb{A}^n = K^n$ considered as a space with functions

- The topology is the *Zariski topology.* Closed sets are of the form

$$V(S) = \{(x_1, \ldots, x_n) \in K^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in S\}$$

where S is a subset of the polynomial ring $K[X_1, \ldots, X_n]$. Observe that $V(S) = V(I)$, where I is the ideal generated by S.

Equivalently, the open sets are unions of sets of the form

$$D(f) = \{(x_1, \ldots, x_n) \in K^n \mid f(x_1, \ldots, x_n) \neq 0\}$$

with $f \in K[X_1, \ldots, X_n]$.

This is a topology since $D(1) = K^n$, $D(0) = \emptyset$ and $D(f) \cap D(g) = D(fg)$, so

$$(\bigcup_\lambda D(f_\lambda)) \cap (\bigcup_\mu D(g_\mu)) = \bigcup_{\lambda,\mu} D(f_\lambda g_\mu).$$

For example, for $\mathbb{A}^1$, if $0 \neq f \in K[X]$ then $V(f)$ is a finite set. Thus the closed subsets of $\mathbb{A}^1$ are $\emptyset$, finite subsets, and $\mathbb{A}^1$. Thus the nonempty open sets in $\mathbb{A}^1$ are the cofinite subsets $\mathbb{A}^1 \setminus \{a_1, \ldots, a_k\}$. This is NOT Hausdorff.

- If U is an open subset of $\mathbb{A}^n$ then the set of regular functions $\mathcal{O}(U)$ consists of the functions $f : U \to K$ such that each point $u \in U$ has an open neighbourhood $W \subseteq U$ such that $f|_W = p/q$ with $p, q \in K[X_1, \ldots, X_n]$ and $q(x_1, \ldots, x_n) \neq 0$ for all $(x_1, \ldots, x_n) \in W$.

B) **Theorem.** This turns $\mathbb{A}^n$ into a space with functions. Moreover any open subset of $\mathbb{A}^n$ is a finite union $D(f_1) \cup \cdots \cup D(f_m)$.

Proof. The first claim is straightforward, since the regular functions are defined locally. If U is an open set, say $U = \mathbb{A}^n \setminus V(S)$, then

$$V(S) = V(I) = V((f_1, \dots, f_m)) = V(f_1) \cap \dots \cap V(f_m)$$

since any ideal I is finitely generated, so $U = D(f_1) \cup \dots \cup D(f_m)$.

[End of LECTURE 1 on 20 April 2020]

C) **Theorem.** If X is a space with functions, then a mapping

$$\theta : X \to \mathbb{A}^n, \quad \theta(x) = (\theta_1(x), \dots, \theta_n(x))$$

is a morphism of spaces with functions iff the θ_i are regular functions on X.

Proof. Since the ith projection $\pi_i : \mathbb{A}^n \to K$ is regular, if θ is a morphism then $\theta_i = \pi_i \theta$ is regular.

Suppose $\theta_1, \dots, \theta_n$ are regular. Let U be an open subset of $\mathbb{A}^n$ and $f = p/q \in \mathcal{O}(U)$ with $q(u) \neq 0$ for $u \in U$. We need to show that $f\theta$ is regular on $\theta^{-1}(U)$. Now by assumption $p\theta = p(\theta_1(x), \dots, \theta_n(x))$ and $q\theta$ are regular on U. Also $q\theta$ is non-vanishing on $\theta^{-1}(U)$. Thus $p\theta/q\theta$ is regular on $\theta^{-1}(U)$.

Corollary 1. $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$.

Corollary 2. $\mathbb{A}^n$ is separated.

Proof. The diagonal for $\mathbb{A}^n$ is

$$\Delta_{\mathbb{A}^n} = \{(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{A}^{2n} : x_1 = y_1, \dots, x_n = y_n\}$$

so it is closed.

D) **Coordinate-free description.** If V is an n-dimensional vector space, then by choosing a basis we can identify $V \cong \mathbb{A}^n$, and then V becomes a space with functions. Choosing a different basis gives the same space with functions.

Examples. $M_n(K)$, $\mathrm{End}_K(V)$.

E) **Representations of a quiver.** The space of representations of a quiver Q of dimension vector $\alpha \in \mathbb{N}^{Q_0}$ is $\mathrm{Rep}(Q, \alpha) = \prod_{a \in Q_1} \mathrm{Hom}_K(K^{\alpha_{t(a)}}, K^{\alpha_{h(a)}})$. This is an affine space.

To each element $x \in \mathrm{Rep}(Q, \alpha)$ there corresponds the representation K_x in which the vector space at vertex i is K^{α_i} and the linear map corresponding to an arrow $a : i \to j$ is given by the matrix x_a.

The group $\mathrm{GL}(\alpha) = \prod_{i \in Q_0} \mathrm{GL}_{\alpha_i}(K)$ acts by conjugation,

$$(g \cdot x)_a = g_{h(a)} x_a (g_{t(a)})^{-1}$$

and the orbits correspond to the isomorphism classes of represntations of Q of dimension vector α. We sometimes write O_X for the orbit corresponding to a representation X - don't confuse it with the ring of functions!

1.3 Affine varieties

A) **Definition.** An *affine variety* is a space with functions which is, or is isomorphic to, a closed subset of $\mathbb{A}^n$.

The *coordinate ring* of an affine variety X is $K[X] := \mathcal{O}(X)$.

B) **Definition.** The *radical* of an ideal I in a commutative ring A is

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n > 0\}$$

It is an ideal. The ideal I is *radical* if $I = \sqrt{I}$. Equivalently, if the factor ring A/I is *reduced*, that is, it has no nonzero nilpotent elements.

Since $K[X_1, \dots, X_n]$ is a UFD, if f is an irreducible polynomial in $K[X_1, \dots, X_n]$, then (f) is a prime ideal, so $K[X_1, \dots, X_n]/(f)$ is a domain, so (f) is a radical ideal.

C) **Theorem.** If I is an ideal in $K[X_1, \dots, X_n]$ and $X = V(I)$ is the corresponding closed subset of $\mathbb{A}^n$, then the natural map

$$K[X_1, \dots, X_n] \to K[X]$$

is surjective, and has kernel $\sqrt{I}$. In particular, if X is an affine variety, then $K[X]$ is a finitely generated K-algebra which is reduced.

Proof. For surjectivity, adapt Hartshorne, Proposition II.2.2. The statement about the kernel is Hilbert's Nullstellensatz.

D) **Theorem.** If X is an affine variety, and Z is a space with functions, then the map

$$\mathrm{Hom}_{\text{spaces with functions}}(Z, X) \to \mathrm{Hom}_{K\text{-algebras}}(K[X], \mathcal{O}(Z))$$

sending $\theta : Z \to X$ to the composition map $f \mapsto f\theta$, is a bijection.

Proof. Implicit in Kempf.

Corollary. There is an anti-equivalence between the categories of affine varieties and finitely generated reduced commutative K-algebras. The variety corresponding to a finitely generated reduced K-algebra A is denoted $\mathrm{Spec}\, A$.

Proof. It just remains to observe that all finitely generated reduced K-algebras arise.

E) **Determinantal varieties.** A matrix in $M_{n\times m}(K)$ has rank $\le r$ if and only if all minors of size $r+1$ vanish. These are polynomials in the entries of the matrix, so the matrices of rank $\le r$ form a closed subset of $M_{n\times m}(K)$, so an affine variety. The coordinate-free version is that if V and W are f.d. vector spaces then the space $\mathrm{Hom}(V,W)_{\le r}$ of linear maps of rank $\le r$ is closed in $\mathrm{Hom}(V,W)$, so an affine variety.

1.4 Module varieties

Let A be a finitely generated associative K-algebra, and $d \in \mathbb{N}$.

A) **Definition.** The set of A-module structures on K^d is $\mathrm{Mod}(A,d) = \mathrm{Hom}_{K\text{-algebra}}(A, M_d(K))$. Each element $\theta \in \mathrm{Mod}(A,d)$ turns K^d into an A-module via restriction, so the module is ${}_\theta K^d$.

Lemma. (i) $\mathrm{Mod}(A,d)$ has a natural structure as an affine variety.

(ii) Given any $a \in A$, the map $\mathrm{Mod}(A,d) \to M_d(K)$, sending $\theta : A \to M_d(K)$ to $\theta(a)$, is a morphism of varieties.

(iii) There is an action of $\mathrm{GL}_d(K)$ on $\mathrm{Mod}(A,d)$ by conjugation, so given by $(g\cdot\theta)(a) = g\theta(a)g^{-1}$. The orbits correspond to isomorphism classes of d-dimensional modules.

Proof. (i) We choose a presentation $A \cong K\langle x_1,\dots,x_k\rangle/I$. A homomorphism $\theta : A \to M_d(K)$ is determined by the matrices $A_i = \theta(x_i)$, so

$$\mathrm{Mod}(A,d) = \{(A_1,\dots,A_k) \in M_d(K)^k : p(A_1,\dots,A_k) = 0 \text{ for all } p \in I\}.$$

Here any $p \in K\langle x_1,\dots,x_k\rangle$ is a noncommutative polynomial in $x_1,\dots,x_k$, and then $p(A_1,\dots,A_k)$ is a $k\times k$ matrix of ordinary polynomials in the entries of the A_i. This is a closed subset of the affine space $M_d(K)^k$, so an affine variety.

It remains to check that the structure doesn't depend on the presentation of A. For this we use (ii), which is clear. Now if $\mathrm{Mod}(A,d)'$ is the same set but with the variety structure given by a different presentation, then (ii) shows that the identity maps $\mathrm{Mod}(A,d) \to \mathrm{Mod}(A,d)'$ and $\mathrm{Mod}(A,d)' \to \mathrm{Mod}(A,d)$ are morphisms of varieties, giving (i).

(iii) Clear.

B) **Examples.** (1) $\mathrm{Mod}(A,1)$ consists of the homomorphisms $\theta : A \to K$. Since K is commutative, θ kills any commutator $[a,b] = ab-ba$, so it descends to a homomorphism $A/I \to K$ where where I is the ideal generated by all commutators. Observe that A/I is commutative. Since K is reduced, any homomorphism $A/I \to K$ kills any nilpotent element of A/I, so descends to a homomorphism $(A/I)/\sqrt{0} \to K$. It follows that $\mathrm{Mod}(A,1)$ is the affine variety given by the commutative ring $(A/I)/\sqrt{0}$.

(2) The *nilpotent variety* consists of the $d \times d$ nilpotent matrices over K. In fact that dth power of such a matrix must be zero, so the nilpotent variety is

$$N_d = \{A \in M_d(K) : A^d = 0\} = \mathrm{Mod}(K[x]/(x^d), d)$$

(3) The *commuting variety* consists of the pairs of commuting matrices

$$C_d = \{(A,B) \in M_d(K)^2 : AB = BA\} = \mathrm{Mod}(K[x,y], d).$$

C) **Version with dimension vectors.** Suppose A is a finitely generated K-algebra and $e_1, \dots, e_n$ is a complete set of orthogonal idempotents in A (not necessarily primitive). Thus $e_i e_j = \delta_{ij} e_i$ and $\sum e_i = 1$.

It is equivalent that A can be presented as KQ/I where Q is a (finite) quiver with vertex set $\{1, \dots, n\}$, with the e_i corresponding to the trivial paths.

If M is any A-module, then $M = \bigoplus_{i=1}^n e_i M$. The *dimension vector* of M is the vector $\alpha \in \mathbb{N}^n$ with $\alpha_i = \dim e_i M$.

Given a dimension vector $\alpha \in \mathbb{N}^n$, we define $\mathrm{Mod}(A,\alpha)$ to be the subset of $\mathrm{Rep}(Q,\alpha)$ consisting of those representations of Q satisfying the relations I. It is a closed subset, so an affine variety. The structure does not depend on the presentation of A.

The group $\mathrm{GL}(\alpha)$ acts on $\mathrm{Mod}(A,\alpha)$ by conjugation, and the orbits correspond to isomorphism classes of A-modules of dimension vector α.

Example. Let Q be the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ and I the ideal generated by ba.

$$\mathrm{Mod}(KQ/I, (2,2,1)) = \{(a,b) \in M_{2\times 2}(K) \times M_{1\times 2}(K) : ba = 0\}$$

$$= \{(a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}) \in K^6 : b_{11}a_{1i} + b_{12}a_{2i} = 0, i = 1, 2\}.$$

[End of LECTURE 2 on 23 April 2020]

1.5 Abstract varieties

A) **Definition.** A *variety* is a space with functions X which is separated and with a finite open covering $X = U_1 \cup \cdots \cup U_n$ by affine varieties.

If X is a variety and $x \in X$, then the singleton set $\{x\}$ is closed in X. This is easy to see for affine space, it follows immediately for X an affine variety, and then for X an arbitrary variety.

Any variety is a *noetherian* topological space, that is it has the ascending chain condition on open subsets. The noetherian property of polynomial rings proves this for affine space, and then it follows for affine varieties and then for arbitrary varieties.

In particular, any variety is *quasi-compact*, meaning that any open covering has a finite subcovering. (Usually this is just called compactness, but in this context it is called quasi-compactness, apparently to make clear that the topological spaces needn't be Hausdorff.)

B) **Definition.** A subset S of a topological space X is *locally closed* if the following equivalent conditions hold:
(i) S is an open subset of a closed subset of X
(ii) S is open in its closure
(iii) S is the intersection of an open and a closed subset of X.

Proof of equivalence. Exercise.

Lemma. If $X \subseteq Y \subseteq Z$ and Y is locally closed in Z, then X is locally closed in Y iff it is locally closed in Z.

Proof. Exercise.

Definition. A *subvariety* Y of a variety X is a locally closed subset equipped with the induced structure as a space with functions. A *quasi-affine variety* is an open subvariety of an affine variety, or equivalently a subvariety of affine space.

C) **Proposition.** If $f \in K[X_1, \dots, X_n]$, then the open subvariety $D(f)$ of $\mathbb{A}^n$ is isomorphic to the affine variety

$$\{(x_1, \dots, x_n, t) \in \mathbb{A}^{n+1} : f(x_1, \dots, x_n) \cdot t = 1\},$$

so

$$\begin{aligned}\mathcal{O}(D(f)) &\cong K[X_1, \dots, X_n, T]/(f(X_1, \dots, X_n) \cdot T - 1) \\ &\cong K[X_1, \dots, X_n, 1/f(X_1, \dots, X_n)].\end{aligned}$$

Proof. The maps are the projection $(x_1, \dots, x_n, t) \mapsto (x_1, \dots, x_n)$ and the map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 1/f(x_1, \dots, x_n))$. Now $1/f \in \mathcal{O}(D(f))$, so both are morphisms.

Corollary. If X is an affine variety and $f \in K[X]$, then the open subset $D(f) = \{x \in X : f(x) \neq 0\}$ is an affine variety.

Proof. X is a closed subset in some $\mathbb{A}^n$ and $K[X] = K[X_1, \dots, X_n]/I$, so f lifts to an element $\hat{f} \in K[X_1, \dots, X_n]$ with $D_X(f) = X \cap D_{\mathbb{A}^n}(\hat{f})$. We know that there is an isomorphism between $D_{\mathbb{A}^n}(\hat{f})$ and an affine variety Y. Under this, $D_X(f)$ corresponds to a closed subset Z of Y, so Z is an affine variety.

Corollary. Any subvariety of a variety is a variety.

Proof. Suppose $Y \subseteq X$. We need to show that Y is a finite union of affine open subsets. Since X is a finite union of affine opens, we may reduce to the case when X is affine. We may also assume that Y is open in X. But then $Y = X \cap U$ with $U = D(f_1) \cup \dots \cup D(f_m)$, and then $Y = V_1 \cup \dots \cup V_m$ with $V_i = X \cap D(f_i)$ a closed subset of the affine variety $D(f_i)$, hence affine.

D) **Remark.** The example of $D(f)$ shows that some quasi-affine varieties are again affine. But this is not always true. For example $U = \mathbb{A}^2 \setminus \{0\} = D(X_1) \cup D(X_2)$ is quasi-affine but not affine.

To see this, we show first that $\mathcal{O}(U) = K[X_1, X_2]$. A function $f \in \mathcal{O}(U)$ is determined by its restrictions f_i to $D(X_i)$ $(i = 1, 2)$. Now $f_i \in \mathcal{O}(D(X_i)) = K[X_1, X_2, X_i^{-1}]$. Moreover the restrictions of f_1 and f_2 to $D(X_1) \cap D(X_2) = D(X_1 X_2)$ are equal, so f_1 and f_2 are equal as elements of $K[X_1, X_2, 1/X_1 X_2]$. But this is only possible if they are both in $K[X_1, X_2]$, and equal. Thus $f \in K[X_1, X_2]$.

Now the inclusion morphism $\theta : U \to \mathbb{A}^2$ induces a homomorphism $\mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(U)$ which is actually an isomorphism. Now the corollary in the last section says that the category of affine varieties is anti-equivalent to the category of finitely generated reduced K-algebras. If U were affine, then since the map on coordinate rings is an isomorphism, θ would have to be an isomorphism. But is isn't.

E) **Example.** If V and W are vector spaces, the set of injective linear maps $\mathrm{Inj}(V, W)$ is an open in $\mathrm{Hom}(V, W)$, since the complement is $\mathrm{Hom}_{\leq r}(V, W)$ where $r = \dim V - 1$. Thus $\mathrm{Inj}(V, W)$ is a quasi-affine variety.

F) **Theorem.** A product of varieties $X \times Y$ is a variety.

Proof. Recall that the product $X \times Y$ exists for any two spaces with functions. It is straightforward that if $U \subseteq X$ and $V \subseteq Y$ are open (resp. closed) subsets, then $U \times V$ is open (resp. closed) in $X \times Y$. Moreover with the induced structure as a space with functions it is a categorical product.

Since any variety is a finite union of affine open subsets, decomposing X and Y it suffices to prove that a product of affine varieties is affine. Now if X is closed in $\mathbb{A}^n$ and Y is closed in $\mathbb{A}^m$ then $X \times Y$ is closed in $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$, so affine.

Assuming that X and Y are separated, $\Delta_{X\times Y}$ is identified with $\Delta_X \times \Delta_Y$ which is closed in $(X \times X) \times (Y \times Y)$.

G) **Definition.** An *embedding* or *immersion* of varieties is a morphism $\theta : X \to Y$ whose image is locally closed, and such that $X \to \mathrm{Im}(\theta)$ is an isomorphism.

For example, for any variety there is a diagonal morphism $X \to X \times X$ and X is separated if and only if the diagonal morphism is a closed embedding. The point is that the natural map $\Delta_X \to X$ is always a morphism, since it factors as the inclusion morphism into $X \times X$ followed by either projection to X.

1.6 The variety of algebras

Let V be a vector space of dimension n, with basis $e_1, \dots, e_n$.

A) **Definition.** We write $\mathrm{Bil}(n)$ for the set of bilinear maps $V \times V \to V$. A map $\mu \in \mathrm{Bil}(n)$ is given by its structure constants $(c_{ij}^k) \in K^{n^3}$ with

$$\mu(e_i, e_j) = \sum_k c_{ij}^k e_k.$$

Equivalently $\mathrm{Bil}(n) \cong \mathrm{Hom}(V \otimes V, V)$, Thus it is affine space $\mathbb{A}^{n^3}$.

We write $\mathrm{Ass}(n)$ for the subset consisting of associative multiplications. This is a closed subset of $\mathrm{Bil}(n)$, hence an affine variety, since it is defined by the equations

$$\mu(\mu(e_i, e_j), e_k) = \mu(e_i, \mu(e_j, e_k)),$$

that is

$$\sum_p c_{ij}^p c_{pk}^s = \sum_q c_{iq}^s c_{jk}^q$$

for all s.

We write $\mathrm{Alg}(n)$ for the subset of associative unital multiplications, so algebra structures on V.

Theorem. $\mathrm{Alg}(n)$ is an affine open subset of $\mathrm{Ass}(n)$, hence an affine variety.

Proof. (i) We use that a vector space A with an associative multiplication has a 1 if and only if there is some $a \in A$ for which the maps $\ell_a, r_a : A \to A$ of left and right multiplication by a are invertible.

Namely, if $u = \ell_a^{-1}(a)$, then $au = a$. Thus $aub = ab$ for all b, so since ℓ_a is invertible, $ub = b$. Thus u is a left 1. Similarly there is a right 1, and they must be equal.

(ii) For the algebra V with multiplication μ, write ℓ_a^μ and r_a^μ for left and right multiplication by $a \in V$. Then $\mathrm{Alg}(n) = \bigcup_{a \in V} D(f_a)$ where $f_a(\mu) = \det(\ell_a^\mu)\det(r_a^\mu)$. Thus $\mathrm{Alg}(n)$ is open in $\mathrm{Ass}(n)$.

(iii) The map

$$\mathrm{Alg}(n) \to V, \quad \mu \mapsto \text{the 1 for } \mu$$

is a morphism of varieties, since on $D(f_a)$ it is given by $(\ell_a^\mu)^{-1}(a)$, whose components are rational functions, with $\det(\ell_a^\mu)$ in the denominator.

(iv) $\mathrm{Alg}(n)$ is affine. In fact

$$\mathrm{Alg}(n) \cong \{(\mu, u) \in \mathrm{Ass}(n) \times V \mid u \text{ is a 1 for } \mu\}.$$

The right hand side is a closed subset, hence it is affine. Certainly there is a bijection, and the maps both ways are morphisms.

The structure of $\mathrm{Alg}(n)$ is known for small n. See P. Gabriel, Finite representation type is open, 1974.

[End of LECTURE 3 on 27 April 2020]

1.7 Projective space and projective varieties

A) **Definition.** *Projective n-space* $\mathbb{P}^n$ is the set of 1-dimensional subspaces of K^{n+1}; alternatively it is the set of equivalence classes for the relation $\sim$ on $K^{n+1} \setminus \{0\}$, where $x \sim y$ if and only if $y = \lambda x$ for some $0 \neq \lambda \in K$. We write $[x_0 : x_1 : \cdots : x_n]$ for the element of $\mathbb{P}^n$ corresponding to $(x_0, x_1, \ldots, x_n) \in K^{n+1} \setminus \{0\}$. Now $\mathbb{P}^n$ becomes a space with functions:

- $\mathbb{P}^n$ is equipped with its Zariski topology, in which the closed subsets are $V'(S) = \{[x_0 : \cdots : x_n] \mid F(x_0, \ldots, x_n) = 0 \text{ for all } F \in S\}$ where S is a set of homogeneous polynomials. Recall that a polynomial $F \in K[X_0, \ldots, X_n]$ is homogeneous of degree d provided all monomials in it have total degree d, or equivalently

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n)$$

for all $\lambda, x_i \in K$.

Equivalently the open sets are unions of sets of the form

$$D'(F) = \{[x_0 : \cdots : x_n] \mid F(x_0, \ldots, x_n) \neq 0\}$$

with F a homogeneous polynomial.

- If U is an open subset of $\mathbb{P}^n$, then $\mathcal{O}(U)$ consists of the functions $f : U \to K$ such that any point $u \in U$ has an open neighbourhood W in U such that $f|_W = P/Q$ with $P, Q \in K[X_0, \dots, X_n]$ homogeneous of the same degree and $Q(x_0, \dots, x_n) \neq 0$ for all $[x_0 : \dots : x_n] \in W$.

B) **Theorem.** (i) $\mathbb{P}^n$ is a space with functions.

(ii) For $0 \leq i \leq n$ the set $U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0\}$ is an open subset of $\mathbb{P}^n$ which is isomorphic to $\mathbb{A}^n$.

(iii) $\mathbb{P}^n = U_0 \cup \dots \cup U_n$ and $\mathbb{P}^n$ is separated. Thus $\mathbb{P}^n$ is a variety.

(iv) The map $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a morphism of varieties. A subset U of $\mathbb{P}^n$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{A}^{n+1} \setminus \{0\}$. If so, then a function $f : U \to K$ is in $\mathcal{O}(U)$ if and only if $f\pi \in \mathcal{O}(\pi^{-1}(U))$.

Proof. (i) Clear.

(ii) There are inverse maps between U_i and $\mathbb{A}^n$ sending $[x_0 : \dots : x_n]$ to $(x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i)$ and $(y_1, \dots, y_n)$ to $[y_1 : \dots : y_i : 1 : y_{i+1} : \dots : y_n]$. One needs to check that the regular functions correspond.

(iii) The union is clear.

For separatedness, given distinct points u, w, we need to find open neighbourhoods U and W and a function $f(x, y)$ on $U \times W$ of the form $\sum_i g_i(x)h_i(y)$ with g_i and h_i regular, such that $f(u, w) \neq 0$ but $f(x, x) = 0$ for all $x \in U \cap W$.

There must be indices i, j with $u_i w_j \neq u_j w_i$, and without loss of generality $u_i w_j \neq 0$. Take $U = \{[x_0 : \dots : x_n] : x_i \neq 0\}$, $W = \{[y_0 : \dots : y_n] : y_j \neq 0\}$ and

$$f(x, y) = \frac{x_j y_i - x_i y_j}{x_i y_j}.$$

(iv) It is clear that π is a morphism. We show that as subset U of $\mathbb{P}^n$ is open if and only if its inverse image $\pi^{-1}(U)$ is open in $X = \mathbb{A}^{n+1} \setminus \{0\}$. We leave the rest as an exercise. First observe that $\pi^{-1}(D'(F)) = X \cap D(F)$, so if U is open, so is $\pi^{-1}(U)$. Conversely suppose that $\pi^{-1}(U)$ is open, so

$$\pi^{-1}(U) = X \cap \bigcup_{f \in S} D(f)$$

for some subset $S \subseteq K[X_0, \dots, X_n]$. Suppose $x = [x_0 : \dots : x_n] \notin U$. Let $f \in S$. Then $(\lambda x_0, \dots, \lambda x_n) \notin \pi^{-1}(U)$ for all $0 \neq \lambda \in K$. Thus $f(\lambda x_0, \dots, \lambda x_n) = 0$ for all $\lambda \neq 0$. Writing f as a sum of homogeneous polynomials, say $f = \sum_d f_d$ with f_d homogeneous of degree d, we have $\sum_d f_d(x_0, \dots, x_n)\lambda^d = 0$ for all $\lambda \neq 0$. This forces $f_d(x_0, \dots, x_n) = 0$ for all d. It follows that

$$U = \bigcup_{f \in S} \bigcup_{d} D'(f_d),$$

so U is open.

C) **Coordinate-free description.** The set $\mathbb{P}(V)$ of 1-dimensional subspaces of V a vector space of dimension $n+1$ has a natural structure as a variety isomorphic to $\mathbb{P}^n$.

D) **Lemma.** $\mathbb{P}^n$ is a disjoint union $U_0 \cup V_0$ where
$U_0 = \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\}$ is an open subvariety isomorphic to $\mathbb{A}^n$.
$V_0 = \{[x_0 : \cdots : x_n] \mid x_0 = 0\}$ is a closed subvariety isomorphic to $\mathbb{P}^{n-1}$.

Repeating, we can write $\mathbb{P}^n$ as a disjoint union of copies of $\mathbb{A}^n$, $\mathbb{A}^{n-1}$, $\dots$, $\mathbb{A}^0 = \{pt\}$.

Example. $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ where $\lambda \in \mathbb{A}^1$ coresponds to $[1 : \lambda]$ and $\infty = [0 : 1]$. For $K = \mathbb{C}$ one identifies $\mathbb{P}^1$ with the Riemann sphere by stereographic projection.

The closed subsets are $\emptyset$, finite subsets, and $\mathbb{P}^1$. Thus the nonempty open sets are the cofinite subsets $\mathbb{P}^1 \setminus \{a_1, \dots, a_k\}$.

We show that $\mathcal{O}(\mathbb{P}^1) = K$. A regular function $f \in \mathcal{O}(\mathbb{P}^1)$ induces a regular functions on $U_0 \cong \mathbb{A}^1$ and on $U_1 \cong \mathbb{A}^1$. The coordinate ring of $\mathbb{A}^1$ is the polynomial ring $K[X]$. Thus there are polynomials $p, q \in K[X]$ with $f([x_0 : x_1]) = p(x_1/x_0)$ for $x_0 \neq 0$ and $f([x_0 : x_1]) = q(x_0/x_1)$ for $x_1 \neq 0$. Thus $p(t) = q(1/t)$ for $t \neq 0$. Thus both are constant polynomials.

E) **Definition.** A *projective variety* is (a variety isomorphic to) a closed subset in projective space. A *quasiprojective variety* is (a variety isomorphic to) a locally closed subset in projective space.

F) **Example.** A curve in $\mathbb{A}^2$, for example

$$\{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + x\},$$

can be homogenized to give a curve in $\mathbb{P}^2$

$$\{[w : x : y] \in \mathbb{P}^2 : y^2 w = x^3 + x w^2\}.$$

Recall that $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. On the affine space part $w \neq 0$, we recover the original curve. On the line at infinity $w = 0$ the equation is $x^3 = 0$, which has solution $x = 0$, giving rise to one point at infinity $[w : x : y] = [0 : 0 : 1]$.

For the curve $y^3 = x^3 + x$, the points at infinity are $[0 : 1 : \epsilon]$ where $\epsilon^3 = 1$.

G) **Theorem (Segre).** There is a closed embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$, given by

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \mapsto [x_0 y_0 : \cdots : x_i y_j : \cdots : x_n y_m].$$

Proof. See Kempf, Theorem 3.2.1.

Corollary. A product of (quasi-)projective varieties is (quasi-)projective.

[End of LECTURE 4 on 30 April 2020]

2 Algebraic groups and actions

2.1 Algebraic groups

A) **Definition.** An *algebraic group* is a group which is also a variety, such that multiplication $G \times G \to G$ and inversion $G \to G$ are morphisms of varieties. A morphism of algebraic groups is a map which is a group homomorphism and a morphism of varieties.

The *general linear group* is $\mathrm{GL}_n(K)$ is an affine variety since it is the affine open subset $D(\det)$ of $M_n(K)$. Inversion is a morphism thanks to the formula $g^{-1} = \operatorname{adj} g / \det g$, so it is an algebraic group.

When considering an action of an algebraic group on a variety X we shall suppose that the map $G \times X \to X$ is a morphism of varieties. For example this is true for the action of $\mathrm{GL}_n(K)$ by left multiplication or conjugation on $M_n(K)$. Similarly the action of $\mathrm{GL}_n(K)$ by base change on $\mathrm{Mod}(A, n)$ or $\mathrm{Alg}(n)$ is a morphism of varieties.

B) **Definition.** A *linear algebraic group* is an algebraic group which is isomorphic to a closed subgroup of $\mathrm{GL}_n(K)$. For example
- the special linear group $\mathrm{SL}_n(K) = \{g \in \mathrm{GL}_n(K) : \det g = 1\}$,
- the orthogonal group $O_n(K) = \{g \in \mathrm{GL}_n(K) : g^{-1} = g^T\}$,
- the multiplicative group $G_m = (K^*, \times) = GL_1(K)$,
- the additive group $G_a = (K, +)$, since it is isomorphic to $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) : a \in K\}$
- any finite group,
- any finite product of these, as $\mathrm{GL}_n(K) \times \mathrm{GL}_m(K)$ embeds in $\mathrm{GL}_{n+m}(K)$.

Thus $\mathrm{GL}(\alpha)$ is an linear algebraic group, and of course its action on $\mathrm{Rep}(Q, \alpha)$ is a morphism of varieties.

C) **Connectedness.** Recall that a topological space X is *connected* if it cannot be written as a disjoint union $X = Y \cup Z$ with Y and Z non-empty open sets (so Y and Z are both also closed sets).

The noetherian property implies that any variety is a finite disjoint union of its connected components, which are open and closed subvarieties.

If G is an algebraic group, one of its connected components contains the identity element, call it G^0. The images of the multiplication map $G^0 \times G^0 \to G$ and inversion map $G^0 \to G$ are connected and contain 1, so contained in G^0. Thus G^0 is an open and closed subgroup of G. Similarly, it is a normal subgroup. The other connected components of G are the cosets of G^0. Thus G^0 is a subgroup of finite index in G.

For example any matrix in $G = O_n(K)$ has determinant ± 1, and the matrices with determinant 1 give the subgroup $G^0 = SO_n(K)$.

D) **Remark.** Clearly any linear algebraic group is an affine variety, and conversely one can show that any affine algebraic group is linear, see for example Borel, Linear algebraic groups, Chapter I, Proposition 1.10.

There are algebraic groups which are not affine varieties. One can show that a connected algebraic group which is a projective variety must be commutative. It is called an 'abelian variety'. Most famous are elliptic curves, which are non-singular cubics in $\mathbb{P}^2$.

Henceforth, all algebraic groups we consider will be linear algebraic groups.

E) **Hopf algebras.** A *coalgebra* is a vector space C equipped with a comultiplication $\mu : C \to C \otimes C$ and a counit $\epsilon : C \to K$ satisfying the axioms
(i) (coassociativity) $(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$ as maps $C \to C \otimes C \otimes C$, and
(ii) (counitality) $(\epsilon \otimes 1)\mu = 1$ and $(1 \otimes \epsilon)\mu = 1$ as maps $C \to C$.

A *Hopf algebra* is a coalgebra which is also an algebra in a compatible way, which means that the maps μ and ϵ are algebra maps, and with in addition an *antipode* $S : C \to C$, a linear map with the property that if $\mu(c) = \sum c_i \otimes c'_i$ then $\sum S(c_i)c'_i = \epsilon(c)1 = \sum c_i S(c'_i)$.

Let G be a linear algebraic group. The multiplication map $G \times G \to G$ is a map of affine varieties, so corresponds to an algebra homomorphism $\mu : K[G] \to K[G] \otimes K[G]$, where the right hand side is identified with the coordinate ring of $G \times G$. Thus $\mu(f) = \sum_i f_i \otimes f'_i$ if and only if $f(gg') = \sum_i f_i(g) f'_i(g')$ for all $g, g' \in G$. The inclusion of the identity element into G, $\{1\} \to G$ is a map of affine varieties, so corresponds to an algebra map $\epsilon : K[G] \to K$, with $\epsilon(f) = f(1)$. These turn $K[G]$ into a coalgebra. It is a Hopf algebra with antipode $S : K[G] \to K[G]$ corresponding to the map $G \to G$, $g \mapsto g^{-1}$.

In this way, the category of linear algebraic groups is opposite to the category of Hopf algebras which are finitely generated reduced and commutative as algebras.

Examples. (i) $K[G_a] = K[X]$, $\mu(X) = X \otimes 1 + 1 \otimes X$, $\epsilon(X) = 0$.
(ii) $K[G_m] = K[T, T^{-1}]$, $\mu(T) = T \otimes T$, $\epsilon(T) = 1$.
(iii) $K[\mathrm{GL}_2(K)] = K[X_{11}, X_{12}, X_{21}, X_{22}, 1/d]$ where $d = X_{11}X_{22} - X_{12}X_{21}$, $\mu(X_{ij}) = \sum_{k=1}^{2} X_{ik} \otimes X_{kj}$, $\mu(1/d) = 1/d \otimes 1/d$, $\epsilon(X_{ij}) = \delta_{ij}$, $\epsilon(d) = 1$.

F) **Note.** For any affine variety X and $x \in X$, the inclusion $\{x\} \to X$ corresponds to the evaluation map $ev_x : K[X] \to K$, $ev_x(f) = f(x)$. Applying this to $X = G$, we see that $\epsilon = ev_1$. Also $ev_{g_1 g_2}(f) = f(g_1 g_2) = (ev_{g_1} \otimes ev_{g_2})\mu(f)$.

2.2 Representations of algebraic groups

Let G be a linear algebraic group.

A) **Definition.** A KG-module is *rational* provided that any finite-dimensional subspace is contained in a finite dimensional submodule U such that the corresponding representation $G \to \mathrm{GL}(U)$ is a morphism of algebraic groups.

Theorem. Any submodule or quotient of a rational KG-module is rational.

Proof. Suppose W is a submodule and U is a finite-dimensional submodule as in the definition. Then $U \cap W$ is a submodule of U, and the representation $G \to \mathrm{GL}(U)$ takes block triangular form

$$R(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & D(g) \end{pmatrix}$$

with $A(g) \in \mathrm{GL}(U \cap W)$ and $D(g) \in \mathrm{GL}(U/(W \cap U)) \cong \mathrm{GL}((U+W)/W)$. Now if R is a map of algebraic groups, so are A and D.

B) **Definition.** If C is a coalgebra, a C-*comodule* is a vector space V equipped with a *coaction*, a map $\rho : V \to V \otimes C$ such that $(1 \otimes \mu)\rho = (\rho \otimes 1)\rho$ as maps $V \to V \otimes C \otimes C$ and $(1 \otimes \epsilon)\rho = 1$ as maps $V \to V$. There is a natural category of comodules, subcomodules, etc.

Lemma. Any comodule is a union of finite-dimensional subcomodules.

Proof. A sum of subcomodules is again a subcomodule, so it suffices to show that each $v \in V$ is contained in a finite-dimensional subcomodule. Let (c_i) be a basis of C. Write

$$\rho(v) = \sum v_i \otimes c_i$$

with all but finitely many of the v_i zero. Write $\mu(c_i) = \sum_{j,k} a_{ijk}\, c_j \otimes c_k$. Then

$$\sum_i \rho(v_i) \otimes c_i = (\rho \otimes 1)\rho(v) = (1 \otimes \mu)\rho(v) = \sum_{i,j,k} a_{ijk}\, v_i \otimes c_j \otimes c_k.$$

Comparing coefficients of c_k we get $\rho(v_k) = \sum_{i,j} a_{ijk} v_i \otimes c_j$, so the subspace spanned by the v_i is a subcomodule.

C) **Theorem.** Any $K[G]$-comodule V becomes a KG-module via

$$g.v = (1 \otimes ev_g)\rho(v).$$

This defines an equivalence from the category of $K[G]$-comodules to the category of rational KG-modules.

[End of LECTURE 5 on 4 May 2020]

Sketch. We have

$$g_1.(g_2.v) = g_1.(1 \otimes ev_{g_2})\rho(v) = (1 \otimes ev_{g_1})\rho((1 \otimes ev_{g_2})\rho(v))$$

$$= (1 \otimes ev_{g_1})(\rho \otimes ev_{g_2})\rho(v)$$

$$= (1 \otimes ev_{g_1})(1 \otimes 1 \otimes ev_{g_2})(\rho \otimes 1)\rho(v)$$

$$= (1 \otimes ev_{g_1} \otimes ev_{g_2})(1 \otimes \mu)\rho(v)$$

$$= (1 \otimes ev_{g_1 g_2})\rho(v) = (g_1 g_2).v.$$

Similarly for $1.v$.

This clearly defines a faithful functor. Observe that if $x \in U \otimes K[G]$ and $(1 \otimes ev_g)(x) = 0$ for all $g \in G$, then $x = 0$. Namely, write $x = \sum u_i \otimes f_i$ with the u_i linearly independent. Then $\sum f_i(g)u_i = 0$ for all g, so $f_i(g) = 0$, so $f_i = 0$ for all i. It follows that the functor is full. Namely, if V and V' are comodules and $\theta : V \to V'$ satisfies $g.\theta(v) = \theta(g.v)$ for all g, then

$$(1 \otimes ev_g)\rho'(\theta(v)) = g.\theta(v) = \theta(g.v) = \theta((1 \otimes ev_g)\rho(v)) = (1 \otimes ev_g)(\theta \otimes 1)\rho(v).$$

Applying the observation above to the difference between the two sides of this equation, we deduce that $\rho'(\theta(v)) = (\theta \otimes 1)\rho(v))$, which is the condition for θ to be a homomorphism of comodules.

To show that any comodule V is sent to a rational KG-module, by the lemma, we may suppose that V is finite dimensional. Take a basis of $e_1, \dots e_n$ of V. Let $\rho(e_j) = \sum_i e_i \otimes f_{ij}$ for suitable $f_{ij} \in K[G]$. Then the matrix (f_{ij}) corresponds to a morphism of varieties $\theta : G \to M_n(K)$. Once checks easily that this is the representation given by V, so it actually goes into $\mathrm{GL}_n(K)$, and is a morphism of varieties.

Conversely if V is a rational KG-module, we want to show that it comes from some comodule structure on V, so we need to define $\rho : V \to V \otimes K[G]$. Given $v \in V$, choose a finite-dimensional submodule U containing V such that $G \to \mathrm{GL}(U)$ is a morphism. Take a basis $e_1, \dots, e_n$ of U. Then the map $G \to M_n(K)$ is a morphism, so given by a matrix of regular maps (f_{ij}). Then we define ρ on U by $\rho(e_j) = \sum_i e_i \otimes f_{ij}$.

D) **Example.** If G acts on an affine variety X then it acts as algebra automorphism on $K[X]$. This turns $K[X]$ into a rational KG-module, because the action $G \times X \to X$ gives a coaction $K[X] \to K[X] \otimes K[G]$.

E) **Theorem.** Any rational representation of the multiplicative group G_m is a direct sum of copies of the one-dimensional representations

$$\theta_n : G_m \to \mathrm{GL}_1(K), \quad \theta_n(\lambda) = \lambda^n \quad (n \in \mathbb{Z}).$$

In this way one gets an equivalence between the category of rational representations of G_m and the category of vector spaces V equipped with a direct sum decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$.

Proof. We have $K[G_m] = K[T, T^{-1}]$ with $\mu(T^n) = T^n \otimes T^n$ and $\epsilon(T^n) = 1$. Then θ_n corresponds to the comodule $K \to K \otimes K[G_m]$, $1 \mapsto 1 \otimes T^n$.

If V is a comodule, let $V_n = \{v \in V : \rho(v) = v \otimes T^n\}$. Clearly this is a subcomodule isomorphic to a direct sum of copies of θ_n. We show that $V = \bigoplus_{n\in Z} V_n$. The sum is clearly direct. Now if $v \in V$, write $\rho(v) = \sum_n v_n \otimes T^n$ for suitable $v_n \in V$. Then

$$(1 \otimes \mu)\rho(v) = \sum_n v_n \otimes T^n \otimes T^n$$

and it also equals

$$(\rho \otimes 1)\rho(v) = \sum_n \rho(v_n) \otimes T^n$$

so $\rho(v_n) = v_n \otimes T^n$. Thus $v_n \in V_n$ and $v = (1 \otimes \epsilon)\rho(v) = \sum v_n$.

Conversely given a graded vector space, we turn it into a KG_m-module by making G_m act on V_n as $\lambda.v = \lambda^n v$.

F) **Example.** If R is a K-algebra, a grading of R is the same thing as a rational action of G_m as algebra automorphisms of R.

Suppose G_m acts as automorphisms. Let R_n be the subspace on which $\lambda \in G_m$ acts as λ^n. Then for $x \in R_n$, $y \in R_r$ we have $\lambda.(xy) = (\lambda.x)(\lambda.y) = (\lambda^n x)(\lambda^r y) = \lambda^{n+r}(xy)$ so $xy \in R_{n+r}$.

Conversely if we have a grading, so $R_n R_r \subseteq R_{n+r}$ and we make G_m act via $\lambda.x = \lambda^n x$ for $x \in R_n$, then G_m acts as algebra automorphisms on R.

Special case. The actions of G_m on an affine variety X correspond to the gradings of $K[X]$.

2.3 Geometric quotients

A) **Definitions.** Suppose that a linear algebraic group G acts on a variety X.

Let X/G be the set of orbits and let $\pi : X \to X/G$ be the quotient map. We can turn X/G into a space with functions via

- A subset U of X/G is open iff $\pi^{-1}(U)$ is open in X. (Thus also U is closed iff $\pi^{-1}(U)$ is closed in X.)

- A function $f : U \to K$ is in $\mathcal{O}(U)$ iff $f\pi \in \mathcal{O}(\pi^{-1}(U))$.

This ensures that π is a morphism. If with this structure X/G is a variety, we call it a *geometric quotient*.

A morphism $X \to Y$ is a *categorical quotient* if it is constant on G-orbits, and any morphism $\phi : X \to Z$ which is constant on G-orbits factors uniquely as a composition

$$X \xrightarrow{\pi} Y \xrightarrow{\psi} Z.$$

B) **Example.** The group G_m acts on $X = \mathbb{A}^{n+1} \setminus \{0\}$ by rescaling. The quotient X/G is isomorphic to $\mathbb{P}^n$, so is a variety. This was part (iv) of Theorem 1.7B.

On the other hand the orbits of G_m acting on $\mathbb{A}^{n+1}$ are not all closed, so $\mathbb{A}^{n+1}/G_m$ is not a geometric quotient by the following. In fact since the closure of any orbit of G_m on $\mathbb{A}^{n+1}$ contains 0, any morphism $\mathbb{A}^{n+1} \to Z$ which is constant on orbits must be constant. It follows that the map $\mathbb{A}^{n+1} \to \{pt\}$ is a categorical quotient.

C) **Lemma.**
(i) If X/G is a geometric quotient, the orbits of G must be closed in X.
(ii) A geometric quotient X/G is a *categorical quotient*.
(iii) If Y is a variety and G acts on $G \times Y$ by $g(g', y) = (gg', y)$, then $(G \times Y)/G \cong Y$.

Proof. (i) Any orbit of G in X is the inverse image of a point in X/G, and any point in a variety is closed.

(ii) There is a unique map $X/G \to Z$. It is a morphism by definition.

(iii) The image of an open set under the projection map $p : G \times Y \to Y$ is open by the lemma in section 1.1E. Thus a set U is open in Y if and only if $p^{-1}(U)$ is open. Also a function f on an open set U of Y is regular if and only if fp is regular on $G \times U$. Namely, if it is regular on $G \times U$ then so is its composition with the map $U \to G \times U$, $x \mapsto (1, x)$.

D) **Remark.** If the orbits aren't closed, one needs a different approach. This is 'geometric invariant theory'. More later.

Even if the orbits of G are closed, there may not be a geometric quotient. See for example H. Derksen, Quotients of algebraic group actions, in: Automorphisms of affine spaces, 1995. Maybe you need to work with algebraic spaces rather than varieties. See for example J. Kollár, Quotient spaces modulo algebraic groups, Ann. of Math. 1997.

[End of LECTURE 6 on 7 May 2020]

E) **Quotients of groups.** One case that is understood, however, is quotients of groups. Let G be a linear algebraic group and let H be a closed subgroup. We consider the action of H on G by left multiplication (respectively by the formula $h \cdot g = gh^{-1}$). The quotient G/H is then the set of right (respectively left) cosets of H in G. It is known that:

- G/H is a quasi-projective variety, so a geometric quotient. See T. A. Springer, Linear Algebraic Groups, Second edition, 1998, Corollary 5.5.6.

- If H is a normal subgroup, G/H is an affine variety, so a linear algebraic group. Springer, Proposition 5.5.10.

- G/H is a projective variety if and only if H contains a Borel subgroup (a maximal closed connected soluble subgroup of G). Springer, Theorem 6.2.7. In this case H is called a *parabolic subgroup.*

F) **Definition.** Let X be a variety with an action of G, and let $\pi : X \to Y$ be a morphism which is constant on G-orbits. We say that π is a *Zariski-locally-trivial principal G-bundle* if locally it looks like a projection of a product of G with a variety, that is, each point in Y has an open neighbourhood U and an isomorphism

$$\phi : G \times U \to \pi^{-1}(U)$$

such that $\pi(\phi(g, u)) = u$ for $u \in U$ and such that ϕ commutes with the natural G-action, $\phi(g'g, u) = g'\phi(g, u)$ for $g, g' \in G$ and $u \in U$.

Clearly if π is a *Zariski-locally-trivial principal G-bundle*, then each fibre $\pi^{-1}(y)$ is isomorphic to G.

Remark. A basic reference for fibre bundles in algebraic geometry is J.-P. Serre, Espaces fibrés algébriques, Séminaire Claude Chevalley, 1958. In general a principal G-bundle need not be Zariski-locally-trivial, but only locally trivial for the 'étale topology'; but be warned, this is a 'Grothendieck topology', which is not a topology in the usual sense. However, Serre showed that $\mathrm{SL}_n(K)$ and $\mathrm{GL}_n(K)$ are 'special' groups, meaning that any principal bundle for these groups is automatically Zariski-locally-trivial.

G) **Lemma.** Let $\pi : X \to Y$ be a Zariski-locally-trivial principal G-bundle.
(i) π induces an isomorphism $X/G \cong Y$, so X/G is a geometric quotient.
(ii) π is *universally open*, that is, if Z is a variety, and U is an open subset of $X \times Z$, then its image in $Y \times Z$ is open.
(iii) π is *universally submersive*, that is, if Z is a variety, and V is a subset of $Y \times Z$, then V is open if and only if its inverse image in $X \times Z$ is open.

Proof. (i) Use part (iii) of Lemma C above. (ii) Use the lemma in section 1.1E. (ii) implies (iii) is trivial since π is onto.

H) **Remark.** The book Mumford, Fogarty and Kirwan, Geometric Invariant Theory, 3rd edition, 1994, claims in remark (4) on page 6 that any geometric quotient is universally open. But this does not seem to be true. In the first edition universally submersive was included as part of the definition of a geometric quotient. The definition was changed in the second edition, but the remark was not.

2.4 Grassmannians

A) **Definition.** If V is a vector space of dimension n, the Grassmannian $\mathrm{Gr}(V,d)$ is the set of subspaces of V of dimension d.

We write $\mathrm{Inj}(K^d,V)$ for the set of injective linear maps $K^d \to V$. It is open in $\mathrm{Hom}(K^d,V)$, so a quasi-affine variety. The group $\mathrm{GL}_d(K)$ act by $g\cdot\theta = \theta g^{-1}$. We have a natural bijection $\mathrm{Inj}(K^d,V)/\,\mathrm{GL}_d(K) \to \mathrm{Gr}(V,d)$ since two injective maps are in the same orbit if and only if they have the same image. This turns $\mathrm{Gr}(V,d)$ into a space with functions.

Fixing a basis $e_1,\dots,e_n$ of V, we identify $\mathrm{Inj}(K^d,V)$ with the set of $n\times d$ matrices of rank d. Let I be a subset of $\{1,\dots,n\}$ with d elements. If $A\in\mathrm{Inj}(K^d,V)$, we write A_I for the square matrix obtained by selecting the rows of A in I. Then $\det(A_I)$ is a minor of A. We write A'_I for the $(n-d)\times d$ matrix obtained by deleting the rows in I.

Let $N = \binom{n}{d}-1$. We write elements of $\mathbb{P}^N$ in the form $[x_I]$ with $x_I\in K$, not all zero, where I runs through the subsets of $\{1,\dots,n\}$ of size d. We consider the map $f:\mathrm{Inj}(K^d,V)\to\mathbb{P}^N$ sending A to $[\det(A_I)]$. The action of $g\in\mathrm{GL}_d(K)$ on $\mathrm{Inj}(K^d,V)$ sends A to Ag^{-1}, and $\det((Ag^{-1})_I) = \det(A_I)\det(g)^{-1}$, so f is constant on the orbits of $\mathrm{GL}_d(K)$. Thus it induces a map $\overline{f}:\mathrm{Gr}(V,d)\to\mathbb{P}^N$ called the *Plücker map*.

B) **Theorem.** (i) The Plücker map $\overline{f}:\mathrm{Gr}(V,d)\to\mathbb{P}^N$ is a closed embedding, so the Grassmannian $\mathrm{Gr}(V,d)$ is a projective variety.
(ii) The natural map $\pi:\mathrm{Inj}(K^d,V)\to\mathrm{Gr}(V,d)$ is a Zariski-locally-trivial principal bundle.

We use the following facts. (I should have defined embeddings for spaces with functions, not just for varieties.)

Lemma 1. Given a mapping $\theta: X\to Y$ between spaces with functions and an open covering $Y=\bigcup U_\lambda$, the map θ is a closed embedding if and only if its restrictions $\theta_\lambda : \theta^{-1}(U_\lambda) \to U_\lambda$ are closed embeddings.

Proof. Suppose the θ_λ are closed embeddings. Then $Y \setminus \operatorname{Im}\theta$ is the union of the sets $U_\lambda \setminus \operatorname{Im}\theta_\lambda$, so it is open in Y, hence $\operatorname{Im}\theta$ is closed.

Clearly θ is 1-1, so it defines a bijective morphism $X \to \operatorname{Im}\theta$. We need to show that the inverse map $g : \operatorname{Im}\theta \to X$ is a morphism. But $\operatorname{Im}\theta$ has an open covering by sets of the form $U_\lambda \cap \operatorname{Im}\theta$, and the restriction of g to each of these sets is a morphism, hence so is g.

Lemma 2. If $g : X \to Y$ is a morphism of spaces with functions and Y is separated, then the map $X \to X \times Y$, $x \mapsto (x, g(x))$ is a closed embedding.

Proof. Its image is the inverse image of the diagonal Δ_Y under the map $X \times Y \to Y \times Y$, $(x, y) \mapsto (g(x), y)$. Since Y is separated, this is closed. Now the projection from $X \times Y \to X$ gives an inverse map from the image to X.

Sketch of the theorem. (i) Let $X = \operatorname{Inj}(K^d, V)$, let $Y = \operatorname{Gr}(V, d)$, so $f : X \to \mathbb{P}^N$ and $\overline{f} : Y \to \mathbb{P}^N$ is the Plücker map.

We use the notation above for $\mathbb{P}^N$. Recall that $\mathbb{P}^N$ has an affine open covering by the sets $U_J = \{[x_I] : x_J \neq 0\}$ for J a subset of $\{1, \dots, n\}$ with d elements.

Let $X_J = f^{-1}(U_J)$ and $Y_J = (\overline{f})^{-1}(U_J) = X_J / \operatorname{GL}_d(K)$. By Lemma 1 it suffices to show that $Y_J \to U_J$ is a closed embedding.

Now X_J consists of the matrices A such that A_J is invertible. Thus there is an isomorphism of varieties

$$\phi_J : \operatorname{GL}_d(K) \times M_{(n-d)\times d}(K) \to X_J, \quad \phi_J(g, B) = \hat{B} g^{-1},$$

where given a matrix $B \in M_{(n-d)\times d}(K)$, we write $\hat{B}$ for the matrix A with $A_J = I_d$ and $A'_J = B$. This ensures that $\phi_J(g'g, B) = g' \cdot \phi_J(g, B)$ (where we recall that the action of $\operatorname{GL}_d(K)$ on X is given by $g' \cdot A = A(g')^{-1}$). By Lemma 2.3C(iii), $Y_J = X_J / \operatorname{GL}_d(K) \cong M_{(n-d)\times d}(K)$, so it is an affine variety. We can identify U_J with $\mathbb{A}^N = \{(x_I)_I : I \neq J\}$, and the map $Y_J \to U_J$ with the map

$$M_{(n-d)\times d}(K) \to \mathbb{A}^N, \quad B \mapsto (\det \hat{B}_I)_I.$$

Now observe that if we take I to be equal to J, except that we omit the jth element, and instead insert an element $i \in \{1, \dots, n\} \setminus J$, then $\det(\hat{B}_I) = \pm b_{ij}$. Thus, up to sign, this map is of the form $Y_J \to Y_J \times W$ for some W, as in Lemma 2. Thus it is a closed embedding.

(ii) The intersection of the U_J with the image of $\overline{f}$ give an open cover of $Gr(V, d)$, and the isomorphisms ϕ_J shows that the map $\pi : \operatorname{Inj}(K^d, V) \to \operatorname{Gr}(V, d)$ is locally a projection, as required.

[End of LECTURE 7 on 11 May 2020]

C) **Variation.** We can instead consider kernels of surjective linear maps, and realise $\mathrm{Gr}(V,d)$ as the quotient of $\mathrm{Surj}(V,K^c)$ by $\mathrm{GL}_c(K)$ where $c+d=\dim V$.

Proposition. The map $\mathrm{Surj}(V,K^c)\to\mathrm{Gr}(V,d)$, $\phi\mapsto\operatorname{Ker}\phi$ is a Zariski-locally-trivial principal $\mathrm{GL}_c(K)$-bundle.

Sketch. We show only that it is a morphism. We check this locally. Identify $\mathrm{Surj}(V,K^c)$ with the set of matrices $C\in M_{c\times n}(K)$ of rank c.

Given a subset I of $\{1,\dots,n\}$ of size d, let C_I be the $c\times c$ matrix obtained by deleting the columns in I and C'_I the $c\times d$ matrix obtained by keeping only the columns in I.

Let W_I be the open subset of $\mathrm{Surj}(V,K^c)$ consisting of the matrices C with C_I invertible. As I varies, this gives an open cover of $\mathrm{Surj}(V,K^c)$. Thus it suffices to show that the restriction to W_I is a morphism.

Now we have a map of varieties

$$W_I\xrightarrow{f}\mathrm{Inj}(K^d,V)$$

where $f(C)$ is the $n\times d$ matrix A with $A_I=I_d$ and $A'_I=-(C_I)^{-1}(C'_I)$. Observe that we have an exact sequence

$$0\to K^d\xrightarrow{A}K^n\xrightarrow{C}K^c\to 0.$$

The composition is zero since it is $C_IA'_I+C'_IA_I=0$. Thus the composition of f and the map $\mathrm{Inj}(K^d,V)\to\mathrm{Gr}(V,d)$ is the required map $W_I\to\mathrm{Gr}(V,d)$, and it is a morphism of varieties.

D) **Lemma.** The subset S of a product $\mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')\times\mathrm{Hom}(V,V')$ consisting of the triples (U,U',θ) satisfying $\theta(U)\subseteq U'$, is closed. Thus, fixing θ, the subset of $\mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')$ consisting of the pairs (U,U') satisfying the same condition is also closed.

Proof. For this we realise $\mathrm{Gr}(V,d)=\mathrm{Inj}(K^d,V')/\mathrm{GL}(d)$ and $\mathrm{Gr}(V',d')=\mathrm{Surj}(V',K^c)/\mathrm{GL}_c(K)$, where $c=\dim V'-d'$. Then we have a closed subset

$$C=\{(f,g,\theta)\in\mathrm{Inj}(K^d,V)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V'):g\theta f=0\}$$

whose complement C' is sent under the map

$$\pi:\mathrm{Inj}(K^d,V)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V')\to\mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')\times\mathrm{Hom}(V,V')$$

to the complement S' of S. To show this is open, we factorize π as

$$\mathrm{Inj}(K^d,V)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V') \xrightarrow{\pi_1} \mathrm{Gr}(V,d)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V')$$

$$\xrightarrow{\pi_2} \mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')\times\mathrm{Hom}(V,V').$$

Since $\mathrm{Inj}(K^d,V)\to\mathrm{Gr}(V,d)$ is a Zariski-locally-trivial principal bundle it is universally open, so $\pi_1(C')$ is open, and then since $\mathrm{Surj}(V',K^c)\to\mathrm{Gr}(V',d')$ is a Zariski-locally-trivial principal bundle it too is universally open, so $S'=\pi(C')=\pi_2(\pi_1(C'))$ is open.

E) **Remark.** The group $\mathrm{GL}(V)$ acts transitively on $\mathrm{Inj}(K^d,V)$ and on $\mathrm{Gr}(V,d)$, and the map $\pi:\mathrm{Inj}(K^d,V)\to\mathrm{Gr}(V,d)$ is $\mathrm{GL}(V)$-equivariant.

Fix $\theta_0\in\mathrm{Inj}(K^d,V)$ say with image W. By similar arguments, it is easy to check the following. The map

$$\mathrm{GL}(V)\to\mathrm{Inj}(K^d,V),\quad g\mapsto g\theta_0$$

is a Zariski-locally-trivial principal S-bundle, where

$$S=\{g\in\mathrm{GL}(V):g\theta_0=\theta_0\},$$

the pointwise stabilizer of W, and the map

$$\mathrm{GL}(V)\to\mathrm{Gr}(V,d),\quad g\mapsto g(W)(=\mathrm{Im}\,g\theta_0)$$

is a Zariski-locally-trivial principal H-bundle, where

$$H=\{g\in\mathrm{GL}(V):g(W)=W\},$$

the setwise stabilizer of W. Thus by Lemma 2.3G(i), $\mathrm{Inj}(K^d,V)$ and $\mathrm{Gr}(V,d)$ can be realized as quotients of $\mathrm{GL}(V)$ as in section 2.3E.

F) **Flag varieties.** Using the lemma in D, the flag variety

$$\mathrm{Flag}(V,d_1,\dots,d_k)=\{0\subseteq U_1\subseteq\cdots\subseteq U_k\subseteq V:\dim U_i=d_i\}$$

for $0\le d_1\le\cdots\le d_k\le\dim V$, is realized as a closed subset of $\prod_i\mathrm{Gr}(V,d_i)$, hence a projective variety. It can alternatively be realized as $\mathrm{GL}(V)/H$ where H is the stabilizer of a given flag, a 'parabolic' subgroup of $\mathrm{GL}(V)$.

2.5 Quiver Grassmannians.

A) **Definition.** Let A be an algebra and $e_1,\dots,e_n$ a complete set of orthogonal idempotents. Let M be a finite dimensional A-module. Recall that its dimension vector is $\alpha \in \mathbb{N}^n$ defined by $\alpha_i = \dim e_i M$. Let β be another dimension vector and let $m = \sum_{i=1}^n \beta_i$. We define

$$\mathrm{Gr}_A(M, \beta) = \{U \in \mathrm{Gr}(M, m) : U \text{ is an } A\text{-submodule of } M \text{ of dim. vector } \beta\}.$$

This is called a *Quiver Grassmannian*. This name is used even if A is not a path algebra, because we can always reduce to this case. Firstly, $\mathrm{Gr}_A(M, \beta) = \mathrm{Gr}_{A'}(M, \beta)$, where $A' = A/\operatorname{Ann}_A(M)$, so one can reduce to the case when A is finite-dimensional. Secondly, writing $A' = KQ/I$ where Q is a quiver, so that the idempotents e_i correspond to the trivial paths in Q, then M can be considered as a KQ-module annihiliated by I, and $\mathrm{Gr}_{A'}(M, \beta) = \mathrm{Gr}_{KQ}(M, \beta)$.

Lemma. $\mathrm{Gr}_A(M, \beta)$ is a closed subset of $\mathrm{Gr}(M, m)$, so a projective variety.

Proof. Being a submodule is a closed condition. Namely, given $a \in A$ we need $\hat{a}(U) \subseteq U$, where $\hat{a} : M \to M$ is the homothety $\hat{a}(m) = am$. The set of such U is closed by Lemma D (using the diagonal embedding of $\mathrm{Gr}(M, m)$ in $\mathrm{Gr}(M, m) \times \mathrm{Gr}(M, m)$).

Amongst the submodules U of dimension m, the ones of dimension vector β are those with $\hat{e}_i$ having rank $\leq \beta_i$. This is also a closed condition thanks to section 1.3E.

Alternatively, a submodule U is determined by the subspaces $e_i U \subseteq e_i M$, and so $\mathrm{Gr}_A(M, \beta)$ could be defined as a closed subset of $\prod_{i=1}^n \mathrm{Gr}(e_i M, \beta_i)$.

B) **Remark.** It is a theorem of M. Reineke that every projective variety is isomorphic to a quiver Grassmannian for an indecomposable representation of a quiver. See M. Reineke, Every projective variety is a quiver Grassmannian, Algebr. Represent. Theory 2013. It turned out that the result could have been known earlier, see for example the discussion in C. M. Ringel, Quiver Grassmannians and Auslander varieties for wild algebras, J. Algebra 2014.

C) **Remark.** We can vary the module M at the same time. Given A as before and dimension vectors α and β, the set

$$\operatorname{Mod}\mathrm{Gr}(A, \alpha, \beta) = \{(x, U) \in \operatorname{Mod}(A, \alpha) \times \mathrm{Gr}(K^n, m) : U \in \mathrm{Gr}_A(K_x, \beta)\}$$

is a closed subset of the product, so a variety. Then there is a morphism $\pi : \operatorname{Mod}\mathrm{Gr}(A, \alpha, \beta) \to \operatorname{Mod}(A, \alpha)$ whose fibres are $\pi^{-1}(x) = \mathrm{Gr}_A(K_x, \beta)$.

[End of LECTURE 8 on 14 May 2020]

3 Methods of algebraic geometry

3.1 Irreducible varieties

A) **Definition.** A topological space X is *irreducible* if it is non-empty, and $X = Y \cup Z$ with Y and Z closed subsets implies $Y = X$ or $Z = X$.

It is equivalent that X is non-empty and any two non-empty open subsets intersect. It is also equivalently that X is non-empty and any non-empty open subset is dense.

It follows that a non-empty open subset of an irreducible space is irreducible.

Clearly any irreducible set is connected. The variety $\{(x, y) \in \mathbb{A}^2 : xy = 0\}$ is the union of the x- and y-axes, so not irreducible, but it is connected.

It is easy to see that a topological space which is irreducible and Hausdorff is a point, so this concept is only interesting for non-Hausdorff spaces.

B) **Theorem.**
(i) An affine variety X is irreducible iff $K[X]$ is a domain. In particular $\mathbb{A}^n$ is irreducible.
(ii) If $f \in K[X_1, \ldots, X_n]$ is an irreducible polynomial, then $V(f) \subseteq \mathbb{A}^n$ is irreducible.
(iii) $\mathbb{P}^n$ is irreducible.

Proof. (i) If $K[X]$ is not a domain, then there are $0 \neq f, g \in K[X]$ with $fg = 0$. For all $x \in X$ we have $(fg)(x) = 0$, i.e. $f(x)g(x) = 0$, so $f(x) = 0$ or $g(x) = 0$. Then $X = \{x \in X : f(x) = 0\} \cup \{x \in X : g(x) = 0\}$, a union of two proper closed subsets, so X is not irreducible.

Conversely, suppose X is not irreducible. Since X is affine, $X = V(I) \subseteq \mathbb{A}^n$, with I a radical ideal in $K[X_1, \ldots, X_n]$. Now X has non-empty open subsets with empty intersection. We may suppose these sets are of the form $V(I) \cap D(f)$ and $V(I) \cap D(g)$ with $f, g \in K[X_1, \ldots, X_n]$. Since they are non-empty, $f, g \notin I$. But their intersection is empty, so fg vanishes on X. Thus $fg \in \sqrt{I} = I$. Thus $K[X] = K[X_1, \ldots, X_n]/I$ is not a domain.

(ii) Since $K[X_1, \ldots, X_n]/(f)$ is a domain, see section 1.3B.

(iii) It suffices to show that any two basic open sets intersect. Using that $D'(G) \cap D'(H) = D'(GH)$, it suffices to show that if $0 \neq F \in K[X_0, \ldots, X_n]$ is homogeneous, then $F(x_0, \ldots, x_n) \neq 0$ for some $x_0, \ldots, x_n$, not all zero. One can show this by induction, expanding F as a polynomial in x_n, with coefficients in $K[x_0, \ldots, x_{n-1}]$.

C) **Lemma.** If X and Y are irreducible, so is $X \times Y$.

Proof. Say X, Y are irreducible and $X \times Y = \bigcup_i Z_i$, a finite union of closed subsets. If $y \in Y$ then $X = \bigcup_i i_y^{-1}(Z_i)$, where $i_y : X \to X \times Y$, $x \mapsto (x, y)$. Thus by irreducibility $i_y^{-1}(Z_i) = X$ for some i. Thus $Y = \bigcup_i Y_i$ where $Y_i = \{y \in Y \mid i_y^{-1}(Z_i) = X\}$. Now $Y \setminus Y_i = p_Y((X \times Y) \setminus Z_i)$, where $p_Y : X \times Y \to Y$ is the projection. This is open by the lemma in section 1.1E. Thus the Y_i are closed, so some $Y_i = Y$. Then $Z_i = X \times Y$.

D) **Theorem.** Any variety X can be written in a unique way as a finite union of *irreducible components*, maximal irreducible closed subsets. For an affine variety, the irreducible components correspond to the minimal prime ideals in $K[X]$.

Proof. See Kempf section 2.3.

Examples. (i) The variety $\{(x, y) \in \mathbb{A}^2 : xy = 0\}$ is the union of the two coordinate axes. These are each isomorphic to $\mathbb{A}^1$, so irreducible.

(ii) $W = \{(x, y) \in \mathbb{A}^2 : xy^2 = x^4\} = V(x(y^2 - x^3)) = V(x) \cup V(y^2 - x^3)$. Since x and $y^2 - x^3$ are irreducible polynomials, the varieties they define are irreducible. Thus W has two irreducible components.

(iii) $Z = \{(x, y, z) \in \mathbb{A}^3 : xy = xz = 0\} = \{(0, y, z) : y, z \in K\} \cup \{(x, 0, 0) : x \in K\}$, a union of a plane and a line, copies of $\mathbb{A}^2$ and $\mathbb{A}^1$, so Z has two irreducible components.

E) **Lemma.** A connected algebraic group is an irreducible variety.

Proof. Write the group as a union of irreducible components $G = G_1 \cup \dots \cup G_n$. Since G_1 is not a subset of the union of the other components, some element $g \in G_1$ does not lie in any other component. Now any two elements of an algebraic group look the same, since multiplication by any $h \in G$ defines an isomorphism $G \to G$. It follows that every element of G lies in only one irreducible component. Thus G is the disjoint union of its irreducible components. But then the components are open and closed, and since G is connected, there is only one component.

3.2 Function fields

A) **Definition.** The *function field* $K(X)$ of an irreducible variety X is the direct limit of the rings $\mathcal{O}(U)$ where U runs through all non-empty open subsets of X. That is,

$$K(X) = \left(\bigcup \mathcal{O}(U) \right) / \sim$$

where $\sim$ is the equivalence relation that identifies $f_1 \in \mathcal{O}(U_1)$ and $f_2 \in \mathcal{O}(U_2)$ if $f_1|_V = f_2|_V$ for some non-empty open subset $V \subseteq U_1 \cap U_2$. The elements of $K(X)$ are called *rational functions* on X.

B) **Elementary properties.** (i) $K(X)$ is a field, for if a non-zero rational function is represented by $f \in \mathcal{O}(U)$, then $1/f \in \mathcal{O}(D(f))$ represents its inverse.

(ii) We identify $f_1 \in \mathcal{O}_X(U_1)$ with $f_2 \in \mathcal{O}_X(U_2)$ if they agree on a non-empty open subset of $U_1 \cap U_2$. But then they actually agree on all of $U_1 \cap U_2$, for

$$\{x \in U_1 \cap U_2 \mid f_1(x) = f_2(x)\}$$

is closed and dense in $U_1 \cap U_2$ (because X is irreducible, hence so is $U_1 \cap U_2$, and then any set containing a non-empty open subset of this is dense).

(iii) Because of (ii) and the noetherian property, for any given rational function, there is a unique maximal open subset of X on which it is defined.

(iv) If U is a nonempty open subset of X, then restriction induces an isomorphism $K(X) \to K(U)$. One can identify $\mathcal{O}(U)$ with the subset of $K(X)$ of rational functions defined on U.

C) **Lemma.** If X is irreducible and affine, then $K(X)$ is the quotient field of its coordinate ring $K[X]$.

Proof. An element of the quotient field of $K[X]$ is of the form f/g with $f, g \in K[X]$ and $g \neq 0$. Thus quotient makes sense as a regular function on $D(g)$, so defines a rational function on X. Conversely, we need to show that any rational function on X can be represented in the form f/g for some $f, g \in K[X]$. Write $X = V(I) \subseteq \mathbb{A}^n$ with I a radical ideal in $K[X_1, \dots, X_n]$, so $K[X] = K[X_1, \dots, X_n]/I$. A rational function on X is represented by a regular function on some non-empty open subset U of X. We may suppose this open set is of the form $X \cap D(h)$ for some $h \in K[X_1, \dots, X_n]$. This is a closed subset of $D(h)$, which is an affine variety by Proposition 1.5C. Now it follows from Theorem 1.3C, that if you have a closed subset of an affine variety, then any regular function on the closed subset extends to a regular function on the full variety. Thus any regular function on $X \cap D(h)$ is the restriction of a regular function on $D(h)$, and by Proposition 1.5C again, it can be written in the form f/h^m for some $f \in K[X_1, \dots, X_n]$. Considering f and h as regular functions on X, we get the result.

D) **Definition.** Two irreducible varieties are said to be *birational* if they have non-empty open subsets which are isomorphic.

For example $\mathbb{A}^2$, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birational, but not isomorphic.

Proposition. Irreducible varieties are birational if and only if they have isomorphic function fields.

Proof. One implication is trivial. For the other, assume that $K(X) \cong K(Y)$. We may assume that X is affine.

Take generators of $K[X]$, consider as elements of $K(Y)$, and choose an affine open subset Y' of Y on which all the elements are defined. Then $K[X]$ embeds in $K[Y']$.

Similarly $K[Y']$ embeds in $K[X']$ for an affine open X' in X.

These give maps

$$X' \xrightarrow{f} Y' \xrightarrow{g} X$$

such that gf is the inclusion, so an open embedding. But then the map $X' \to g^{-1}(X')$ is an isomorphism.

[End of LECTURE 9 on 18 May 2020]

3.3 Dimension

The results below can be found in D. Mumford, The red book of varieties and schemes, 1988. But the definitions there are done differently.

A) **Definition.** The *dimension* of a variety is the supremum of the n such that there is a chain of distinct irreducible (so non-empty) closed subsets $X_0 \subset X_1 \subset \cdots \subset X_n$ in X. (The empty set has dimension $-\infty$.)

Thus, if X is an affine variety, $\dim X$ is the Krull dimension of $K[X]$, the maximal length of a chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$.

B) **Theorem.** If X is an irreducible affine variety, then $\dim X$ is the transcendence degree of the field extension $K(X)/K$.

The proof is commutative algebra, so we omit it. As a consequence we get the following.

C) **Properties.** (i) $\dim \mathbb{A}^n = n$.

Proof. $K(\mathbb{A}^n) = K(X_1, \ldots, X_n)$ has transcendence degree n over K.

(ii) If $X \subseteq Y$ is a locally closed subset, then $\dim X \leq \dim Y$, strict if Y is irreducible and X is a proper closed subset.

Proof. If X_i is a chain of irreducible closed subsets in X, then their closures $\overline{X_i}$ in Y are a chain of irreducible closed subsets of Y. Moreover if $\overline{X_i} = \overline{X_{i+1}}$

then X_i is locally closed in Y, so X_i is open in $\overline{X_i}$ and

$$X_{i+1} = X_i \cup (X_{i+1} \cap (\overline{X_i} \setminus X_i))$$

a union of two closed subsets of X_{i+1}, so by irreducibility $X_i = X_{i+1}$.

(iii) If $X = Y_1 \cup \dots \cup Y_n$, with the Y_i locally closed in X, then $\dim X = \max\{\dim Y_i\}$.

For now we prove this only in the special case when the Y_i are open in X. Take a chain $X_0 \subset X_1 \subset \dots \subset X_n$ in X. Then X_0 meets some Y_i. Consider the chain $Y_i \cap X_0 \subset Y_i \cap X_1 \subset \dots \subset Y_i \cap X_n$ in Y_i. Now $Y_i \cap X_j$ is nonempty and open in X_j, hence irreducible. The terms are distinct, for if $Y_i \cap X_j = Y_i \cap X_{j+1}$ then $X_{j+1} = X_j \cup (X_{j+1} \setminus Y_i)$ is a proper decomposition. Thus $\dim Y_i \geq n$.

(iv) Any variety has finite dimension.

Proof. It follows from (i), (ii) and (iii) in the special case.

(v) If X is irreducible then $\dim X$ is the transcendence degree of $K(X)/K$. Thus if U is nonempty open in X, and X is irreducible, then $\dim U = \dim X$.

Proof. X is a union of affine opens. These all have function field $K(X)$, so dimension given by the transcendence degree.

Proof of (iii) in general. Suppose F is an irreducible closed subset of X. Then F is the union of the sets $\overline{F \cap Y_i}$. By irreducibility, some $\overline{F \cap Y_i} = F$. Thus $F \cap Y_i$ is open in F. Thus $\dim F = \dim F \cap Y_i \leq \dim Y_i$. Thus $\dim X \leq \max\{\dim Y_i\}$. The reverse inequality is given by (ii).

D) **Definition.** A morphism $\theta : X \to Y$ of varieties, with X and Y irreducible, is *dominant* if its image is dense in Y.

Lemma. If $\theta : X \to Y$ is a morphism of varieties and X is irreducible, then $Z = \overline{\operatorname{Im}\theta}$ is irreducible, the resulting map $\theta' : X \to Z$ is dominant and it induces an injection $K(Z) \to K(X)$. Thus $\dim Z \leq \dim X$.

Proof. Straightforward.

E) **Main Lemma.** If $\pi : X \to Y$ is a dominant morphism of irreducible varieties then any irreducible component of a fibre $\pi^{-1}(y)$ has dimension at least $\dim X - \dim Y$. Moreover, there is a nonempty open subset $U \subseteq Y$ with $\dim \pi^{-1}(u) = \dim X - \dim Y$ for all $u \in U$.

This can be reduced to the case when X, Y are affine, and then it is commutative algebra.

F) **Two special cases.** (1) $\dim X \times Y = \dim X + \dim Y$. Reduce to the case of irreducible varieties, and then consider the projection $X \times Y \to Y$.

(2) (Hypersurfaces in $\mathbb{A}^n$). The irreducible closed subsets of $\mathbb{A}^n$ of dimension $n-1$ are the zero sets $V(f)$ of irreducible polynomials $f \in K[X_1, \dots, X_n]$.

Namely, if f is irreducible then $V(f)$ is irreducible, a proper closed subset of $\mathbb{A}^n$, so of dimension $< n$, but a fibre of $f : \mathbb{A}^n \to K$, so of dimension $\geq n-1$.

Conversely if $X \subseteq \mathbb{A}^n$ is an irreducible closed subset of dimension $n-1$ then $X = V(I)$ for some I. Then $X \subseteq V(g)$ for any non-zero $g \in I$. But then $X \subseteq V(f)$ for some irreducible factor f of g. Then one has equality by dimensions.

G) **Example.** Recall from section 1.4B that the commuting variety C_d is the set of pairs of $d \times d$ commuting matrices.

Theorem (Motzkin and Taussky, 1955). C_d is irreducible of dimension d^2+d.

Our proof follows R. M. Guralnick, A note on commuting pairs of matrices, 1992. A $d \times d$ matrix A is *regular* or *non-derogatory* if in it's Jordan normal form, each Jordan block has a different eigenvalue. Equivalently if its minimal polynomial is equal to its characteristic polynomial. Equivalently if it defines a cyclic $K[X]$-module. Equivalently if all eigenspaces are at most one-dimensional. Equivalently the only matrices which commute with A are polynomials in A. Equivalently if $I, A, A^2, \dots, A^{d-1}$ are linearly independent. Thus the set of regular matrices is an open subset U of $M_d(K)$.

Suppose B is any matrix and R is regular. Consider the map

$$f : \mathbb{A}^1 \to M_d(K), \quad f(\lambda) = R + \lambda B.$$

The image meets U. Thus $f^{-1}(M_d(K) \setminus U)$ is a proper closed subset of $\mathbb{A}^1$, so finite. Thus $R + \lambda B$ is regular for all but finitely many λ. Thus $B + \nu R$ is regular for all but finitely many $\nu \in K$.

Every matrix A commutes with a regular matrix R. Namely, Jordan blocks of the same size clearly commute (since their difference is a multiple of the identity matrix), so if A has Jordan normal form

$$A = P \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \dots \\ 0 & J_{n_2}(\lambda_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} P^{-1},$$

then it commutes with any matrix of the form

$$R = P \begin{pmatrix} J_{n_1}(\mu_1) & 0 & \dots \\ 0 & J_{n_2}(\mu_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} P^{-1}$$

and this is regular if the μ_i are distinct.

We show that the set $C'_d = C_d \cap (M_d \times U)$ is dense in C_d. If not, then there is a non-empty open subset W of C_d not meeting C'_d. Choose $(A, B) \in W$ and let R be regular commuting with A. Let $g : \mathbb{A}^1 \to C_d$, $g(\nu) = (A, B + \nu R)$. Then $g^{-1}(C'_d)$ and $g^{-1}(W)$ are non-empty open subsets of $\mathbb{A}^1$. Thus they must meet; if they both contain ν, then $(A, B + \nu R) \in C'_d \cap W$, which is impossible.

Let P be the vector space of polynomials $f(t)$ of degree $\leq d - 1$. Now the map $h : P \times U \to C_d$, $(f(t), B) \mapsto (f(B), B)$ has image C'_d. Thus $C_d = \overline{\operatorname{Im} h}$, and since $P \times U$ is irreducible, so is C_d by Lemma D. Also the map h is injective, so $\dim C_d = \dim U + \dim P = d^2 + d$ by the Main Lemma.

[End of LECTURE 10 on 25 May 2020]

3.4 Constructible sets

A subset of a variety is *constructible* if it is a finite union of locally closed subsets.

A) **Lemma.** (i) The class of constructible subsets is closed under finite unions and intersections, complements, and inverse images.
(ii) If V is a constructible subset of X and $\overline{V}$ is irreducible, then there is a nonempty open subset U of $\overline{V}$ with $U \subset V$.

Proof. (i) Exercise. (ii) Write V as a finite union of locally closed subsets V_i. Then $\overline{V} = \bigcup_i \overline{V_i}$. Thus some $\overline{V_i} = \overline{V}$. Then V_i is open in $\overline{V}$.

Example. The punctured x-axis $X' = \{(x, 0) : x \neq 0\}$ is locally closed in $\mathbb{A}^2$. Its complement is not locally closed, but it is constructible, being the union of the plane minus the x-axis, and the origin: $\mathbb{A}^2 \backslash X' = \{(x, y) : y \neq 0\} \cup \{(0, 0)\}$.

B) **Chevalley's Constructibility Theorem.** The image of a morphism of varieties $\theta : X \to Y$ is constructible. More generally, the image of any constructible set is constructible.

Sketch. We may assume X irreducible. We may assume $Y = \overline{\operatorname{Im}(\theta)}$. The main lemma says that $\operatorname{Im}(\theta)$ contains a dense open subset U of Y. Thus it suffices to prove that the image under θ of $X \setminus \theta^{-1}(U)$ is constructible. Now work by induction on the dimension.

C) **Example.** The set $\{x \in \operatorname{Mod}(A, \alpha) : K_x \text{ is indecomposable }\}$ is constructible in $\operatorname{Mod}(A, \alpha)$. Here K_x denotes the A-module of dimension vector α corresponding to x.

If $\alpha = \beta + \gamma$, then there is a direct sum map

$$f : \mathrm{Mod}(A, \beta) \times \mathrm{Mod}(A, \gamma) \to \mathrm{Mod}(A, \alpha)$$

sending (x, y) to the module structure $A \to M_d(K)$ which has x and y as diagonal blocks. It is a morphism of varieties. Thus the map

$$\mathrm{GL}(\alpha) \times \mathrm{Mod}(A, \beta) \times \mathrm{Mod}(A, \gamma) \to \mathrm{Mod}(A, \alpha), \quad (g, x, y) \mapsto g.f(x, y)$$

has as image all modules which can be written as a direct sum of modules of dimensions β and γ. This is constructible. Thus so is the union of these sets over all non-trivial decompositions $\alpha = \beta + \gamma$. Hence so is its complement, the set of indecomposables.

3.5 Upper semicontinuity

A) **Definition.** A function $f : X \to \mathbb{Z}$ is *upper semicontinuous* if $\{x \in X \mid f(x) < n\}$ is open for all $n \in \mathbb{Z}$. Equivalently $\{x \in X \mid f(x) \geq n\}$ is closed.

Clearly if $f : X \to \mathbb{Z}$ is upper semicontinuous and $\phi : Y \to X$ is a morphism, then the composition $f\phi : Y \to \mathbb{Z}$ is upper semicontinuous. Also, if $f, g : X \to \mathbb{Z}$ are upper semicontinuous, so is $f + g$, since

$$\{x : f(x) + g(x) < n\} = \bigcup_p \{x : f(x) < p\} \cap \{x : g(x) < n + 1 - p\}.$$

Examples. (i) The map $\mathrm{Hom}(V, W) \to \mathbb{Z}$, $\theta \mapsto \dim \mathrm{Ker}\, \theta$ is upper semicontinuous.

Proof. The set where it is $\geq t$ is the set of maps of rank $\leq \dim V - t$, which is closed by section 1.3E.

(ii) On the variety $\{(\theta, \phi) \in \mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) : \phi\theta = 0\}$, the map $(\theta, \phi) \mapsto \dim(\mathrm{Ker}\, \phi / \mathrm{Im}\, \theta)$ is upper semicontinuous.

Proof. It is equal to $\dim \mathrm{Ker}\, \theta + \dim \mathrm{Ker}\, \phi - \dim U$.

B) **Definition.** The *local dimension* at $x \in X$, denoted $\dim_x X$ is the infemum of the dimensions of neighbourhoods of x. Equivalently it is the maximal dimension of an irreducible component containing x.

C) **Chevalley's Upper Semicontinuity Theorem.** If $\theta : X \to Y$ is a morphism then the function $X \to \mathbb{Z}$, $x \mapsto \dim_x \theta^{-1}(\theta(x))$ is upper semicontinuous.

Sketch. We may assume X is irreducible. We may assume $Y = \overline{\mathrm{Im}(\theta)}$. By the Main Lemma, the minimal value of the function is $\dim X - \dim Y$, and it takes this value on an open subset $\theta^{-1}(U)$ of X. Thus need to know for the morphism $X \setminus \theta^{-1}(U) \to Y \setminus U$. Now use induction.

D) **Cones.** Let V be a f.d. vector space. By a *cone* in V we mean a subset which contains 0 and is closed under multiplication by any $\lambda \in K$.

Any subspace of V, or union of subspaces, is a cone in V. Any subspace or finite union will also be closed (for the Zariski topology).

If C is a closed cone in V, then every irreducible component of C contains 0, so $\dim_0 C = \dim C$. Namely, if D is an irreducible component of C, there is a scaling map $f : K \times D \to C$, $(\lambda, d) \mapsto \lambda d$, so $D \subseteq \overline{\mathrm{Im}\, f} \subseteq C$. Now $\overline{\mathrm{Im}\, f}$ is irreducible and contains D, so equal to D, and clearly $\overline{\mathrm{Im}\, f}$ contains 0.

Corollary. Let X be a variety, and suppose that for each $x \in X$ we are given a cone $V_x \subseteq V$. Suppose that the set $Z = \{(x, v) : v \in V_x\}$ is closed in $X \times V$. Then the function $X \to \mathbb{Z}$, $x \mapsto \dim V_x$ is upper semicontinuous.

Proof. Consider the projection $\theta : Z \to X$. This gives an upper semicontinuous function $Z \to \mathbb{Z}$, $(x, v) \mapsto \dim_{(x,v)} \theta^{-1}(\theta(x))$. Composing with the zero section $X \to Z$, $x \mapsto (x, 0)$, the map

$$x \mapsto \dim_{(x,0)} \theta^{-1}(\theta(x, 0)) = \dim_0 V_x = \dim V_x$$

is upper semicontinuous.

E) **Example.** The function $\mathrm{Mod}(A, \alpha) \to \mathbb{Z}$, $x \mapsto \dim \mathrm{End}_A(K_x)$ is upper semicontinuous.

Since $\mathrm{Mod}(A, \alpha)$ is a closed subset of $\mathrm{Rep}(Q, \alpha)$ for a suitable quiver, it suffices to prove it for the map $\mathrm{Rep}(Q, \alpha) \to \mathbb{Z}$, $x \mapsto \dim \mathrm{End}_{KQ}(K_x)$. Now

$$\mathrm{End}_{KQ}(K_x) = \{\theta \in \prod_{i \in Q_0} \mathrm{End}(K^{\alpha_i}) : \theta_j x_a = x_a \theta_i \forall a : i \to j \in Q_1\}$$

This is a subspace of the vector space $V = \prod_{i \in Q_0} \mathrm{End}(K^{\alpha_i})$, so in particular it is a cone. The corollary now applies with $X = \mathrm{Rep}(Q, \alpha)$ and

$$Z = \{(x, \theta) \in X \times V : \theta_j x_a = x_a \theta_i \forall a : i \to j \in Q_1\}.$$

A variation: for a fixed finite-dimensional module M, the maps $\mathrm{Mod}(A, \alpha) \to \mathbb{Z}$, $x \mapsto \dim \mathrm{Hom}_A(M, K_x)$ and $\dim \mathrm{Hom}_A(K_x, M)$ are upper semicontinuous.

Another variation: the map $\mathrm{Mod}(A, \alpha) \times \mathrm{Mod}(A, \beta) \to \mathbb{Z}$ given by $(x, y) \mapsto \dim \mathrm{Hom}_A(K_x, K_y)$ is upper semicontinuous.

3.6 Completeness

A) **Definition** A variety X is *complete* or *proper over K* if, for any variety Y, the projection $X \times Y \to Y$ is a closed map (images of closed sets are closed).

The affine line is not complete since under the projection $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, $(x, y) \mapsto y$, the image of the hyperbola $\{(x, y) : xy = 1\}$ is the set $\{y : y \neq 0\}$.

Easy properties.
(i) A closed subvariety of a complete variety is complete.
(ii) A product of complete varieties is complete.
(iii) If X is complete and $\theta : X \to Y$ is a morphism, then $\operatorname{Im}\theta$ is closed and complete. (The image is the projection of the graph, which is closed by separatedness.)
(iv) If X is complete and connected, then any regular function on X is constant. (The image is a closed connected subset of $\mathbb{A}^1$, but not all of $\mathbb{A}^1$.)
(v) A complete affine or quasi-projective variety is projective. (There is an embedding $X \to \mathbb{P}^n$.)

B) **Theorem.** Projective varieties are complete.

Proof. It suffices to prove the $\mathbb{P}^n$ is complete. Thus, letting C be a closed subset of $\mathbb{P}^n \times X$, we need to show that its image under the projection to X is closed.

Letting $V = K^{n+1}$ and $V_* = V \setminus \{0\}$, there is a morphism $p : V_* \to \mathbb{P}^n$ sending a nonzero vector $(x_0, \dots, x_n)$ to $[x_0 : \cdots : x_n]$.

For each $x \in X$, the set $V_x = \{0\} \cup \{v \in V_* : (p(v), x) \in C\}$ is a cone in V. Moreover $\{(x, v) : (x, p(v)) \in C\}$ is closed in $X \times V_*$, so $\{(x, v) : v \in V_x\}$ is closed in $X \times V$. Thus the function $x \mapsto \dim V_x$ is upper semicontinuous. Thus the set $\{x \in X \mid \dim V_x \geq 1\}$ is closed. This is the image of C under the projection to X.

C) **Example.** Since Grassmannians and quiver Grassmannians are projective varieties, they are complete. Now the subset

$$\{x \in \operatorname{Mod}(A, \alpha) : K_x \text{ has a submodule of dimension } \beta\}$$

of $\operatorname{Mod}(A, \alpha)$ is the image of the projection $\operatorname{Mod}\operatorname{Gr}(A, \alpha, \beta) \to \operatorname{Mod}(A, \alpha)$ so it is closed. Taking the union over all $\beta \neq 0, \alpha$, we get that the set

$$\operatorname{Simple}(A, \alpha) = \{x \in \operatorname{Mod}(A, \alpha) : K_x \text{ is a simple module}\}$$

is open in $\operatorname{Mod}(A, \alpha)$.

[End of LECTURE 11 on 28 May 2020]

3.7 Local rings

A) **Definition.** Suppose p is a point in a variety X. The local ring of X at p, denoted $\mathcal{O}_{X,p}$, is the direct limit of the rings $\mathcal{O}(U)$ where U runs through all open neighbourhoods of p in X. That is,

$$\mathcal{O}_{X,p} = \left(\bigcup \mathcal{O}(U)\right) / \sim$$

where $\sim$ is the equivalence relation that identifies $f_1 \in \mathcal{O}(U_1)$ and $f_2 \in \mathcal{O}(U_2)$ if $f_1|_V = f_2|_V$ for some open neighbourhood $V \subseteq U_1 \cap U_2$ of p.

B) **Lemma.** (i) Evaluation at p defines a homomorphism $\mathcal{O}_{X,p} \to K$, $f \mapsto f(p)$, and $\mathcal{O}_{X,p}$ is a commutative local K-algebra with maximal ideal $\mathfrak{m}_p = \{f \in \mathcal{O}_{X,p} : f(p) = 0\}$.

(ii) If $\theta : X \to Y$ is a morphism and $p \in X$, then composition with θ induces a homomorphism $\theta^* : \mathcal{O}_{Y,\theta(p)} \to \mathcal{O}_{X,p}$ whose composition with evaluation at p is evaluation at $\theta(p)$.

(iii) θ^* is a *local homomorphism* in the sense that $\theta^*(\mathfrak{m}_{\theta(p)}) \subseteq \mathfrak{m}_p$ (or equivalently $(\theta^*)^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{\theta(p)}$).

Proof. (i) To show it is local, it suffices to show that any element of $\mathcal{O}_{X,p}$ not in $\mathfrak{m}_p$ is invertible. The element is represented by some $f \in \mathcal{O}(U)$ with $f(p) \neq 0$. But then $1/f \in \mathcal{O}(D(f))$, and $D(f)$ is also a neighbourhood of p, so f is invertible in $\mathcal{O}_{X,p}$. (ii) and (iii) are straightforward.

C) **Lemma.** If X is affine and $p \in X$ then $\mathcal{O}_{X,p}$ is the localization of $K[X]$ with respect to the the maximal ideal $\mathfrak{m}$ of all functions vanishing at p. Thus $\mathcal{O}_{X,p}$ consists of equivalence classes of fractions f/g with $f, g \in K[X]$ and $g \notin \mathfrak{m}$.

Proof. Similar to Lemma 3.2C.

D) **Lemma.** If U is an open neighbourhood of p in X, then the induced map $\mathcal{O}_{X,p} \to \mathcal{O}_{U,p}$ is an isomorphism. If Z is a locally closed subset of X containing p, then the induced map $\mathcal{O}_{X,p} \to \mathcal{O}_{Z,p}$ is surjective.

Proof. For the first part, if W is an open neighbourhood of p in X, then so is $U \cap W$. For the second part, we may suppose that Z is closed in X, and then that X is affine. Then it follows from (iii) and the fact (mentioned in the proof of Lemma 3.2C) that the map $K[X] \to K[Z]$ is surjective.

E) **Remark.** The prime ideals in $\mathcal{O}_{X,p}$ correspond to the irreducible closed subsets of X which contain p. The minimal primes correspond to the irreducible components containing p. The Krull dimension of $\mathcal{O}_{X,p}$ is $\dim_p X$, the local dimension of X at p.

Proof. Let U be an affine open neighbourhood of p. There is a 1-1 correspondence

$$\begin{matrix}\text{irreducible closed subsets} \\ C \text{ of } X \text{ containing } p\end{matrix} \quad \underset{D\mapsto \text{closure of } D \text{ in } X}{\overset{C\mapsto C\cap U}{\rightleftarrows}} \quad \begin{matrix}\text{irreducible closed subsets} \\ D \text{ of } U \text{ containing } p.\end{matrix}$$

Thus can reduce to the affine case, when it is commutative algebra.

F) **Lemma.** If X is irreducible, then $\mathcal{O}_{X,p}$ can be identified with a subalgebra of $K(X)$, the set of all rational functions on X defined at p. Moreover, if $p, q \in X$ and $\mathcal{O}_{X,p} \subseteq \mathcal{O}_{X,q}$ as subalgebras of $K(X)$, then $p = q$.

Proof. First part is clear. For the last part, we first show that if U and V are affine open subsets of X with $\mathcal{O}(U) \subseteq \mathcal{O}(V)$ inside $K(X)$, then $V \subseteq U$. Since X is separated, the natural map $U \cap V \to U \times V$ is a closed embedding. Thus $U \cap V$ is affine and the map $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is onto. Thus $\mathcal{O}(U \cap V)$ is generated as a K-subalgebra of $K(X)$ by $\mathcal{O}(U) \cup \mathcal{O}(V)$. The hypothesis then says it is generated by $\mathcal{O}(V)$, so is equal to $\mathcal{O}(V)$. Thus the inclusion $U \cap V \to V$ of affine varieties induces an isomorphism on rings of functions. Thus $U \cap V = V$, and hence $V \subseteq U$.

Now suppose $p \neq q$, and let U be an affine open neighbourhood of p not containing q. Let $f_1, \dots, f_n$ be generators of $\mathcal{O}(U)$. Now $f_1, \dots, f_n$ are defined at p, so at q by the assumption that $\mathcal{O}_{X,p} \subseteq \mathcal{O}_{X,q}$. Choose an affine open neighbourhood V of q on which the f_i are defined. Then $f_i \in \mathcal{O}(V)$. Thus $\mathcal{O}(U) \subseteq \mathcal{O}(V)$, so $V \subseteq U$. Contradiction.

3.8 Tangent spaces

A) **Definitions.** Recall that if A is a K-algebra and M is an A-A-bimodule, then a *derivation* $d : A \to M$ is a linear map with $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$. The set of all derivations forms a vector space $\mathrm{Der}(A, M)$.

Observe that $d(1) = 0$ since $d(1) = d(1 \cdot 1) = 1d(1) + d(1)1 = 2d(1)$. Letting A be commutative, any A-module M becomes an A-A-bimodule using the same action on both sides. Then $d(a^2) = ad(a) + d(a)a = 2ad(a)$, and by induction $d(a^n) = na^{n-1}d(a)$.

Let X is a variety and $p \in X$. A *point derivation* at p is a derivation $\xi : \mathcal{O}_{X,p} \to K$, where K is considered as an $\mathcal{O}_{X,p}$-module using the homomorphism $\mathcal{O}_{X,p} \to K$ of evaluation at p. Thus $\xi(fg) = f(p)\xi(g) + \xi(f)g(p)$ for $f, g \in \mathcal{O}_{X,p}$. The *tangent space* at p is

$$T_pX := \mathrm{Der}(\mathcal{O}_{X,p}, K),$$

the set of all point derivations at p. If $\theta : X \to Y$ is a morphism, then $\theta^* : \mathcal{O}_{Y,\theta(p)} \to \mathcal{O}_{X,p}$ induces a linear map

$$d\theta_p : T_pX \to T_{\theta(p)}Y, \quad \xi \mapsto \xi \circ \theta^*.$$

B) **Lemma.** (i) $T_pX \cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$, where * denotes duality into the field K.
(ii) If $X \xrightarrow{\theta} Y \xrightarrow{\phi} Z$, then $d(\phi\theta)_p$ is the composition

$$T_pX \xrightarrow{d\theta_p} T_{\theta(p)}Y \xrightarrow{d\phi_{\theta(p)}} T_{\phi\theta(p)}Z.$$

Proof. (i) Use that $\mathcal{O}_{X,p} = \mathfrak{m}_p \oplus K1$. If ξ is a point derivation then $\xi(1) = 0$; if $f, g \in \mathfrak{m}_p$, then $\xi(fg) = f(p)\xi(b) + \xi(a)g(p) = 0$. Conversely any linear map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to K$ defines a point derivation. (ii) This is the chain rule for differentiation!

C) **Lemma.** If U is an open neighbourhood of p in X, then the induced map $T_{U,p} \to T_{X,p}$ is an isomorphism. If Z is a locally closed subset of X containing p, then the induced map $T_{Z,p} \to T_{X,p}$ is injective.

Proof. Use Lemma 3.7D.

D) **Lemma.** If X is a locally closed subset of $\mathbb{A}^n$ and $p \in X$, we can identify

$$\begin{aligned} T_pX &= \{(v_1, \dots, v_n) \in K^n \mid \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p) = 0 \ \forall f \in I\} \\ &= \{v \in K^n \mid f(p + \epsilon v) = O(\epsilon^2) \text{ for all } f \in I\}, \end{aligned}$$

where I is the radical ideal in $K[X_1, \dots, X_n]$ of polynomials vanishing on $\overline{X}$, and the notation $\phi(\epsilon) = O(\epsilon^2)$ means that $\phi(\epsilon) = \epsilon^2\psi(\epsilon)$ with ψ regular on some neighbourhood of 0 in $\mathbb{A}^1$.

Remark. If $\overline{X} = V(S)$, one can instead consider

$$\begin{aligned} T_p^S X &:= \{(v_1, \dots, v_n) \in K^n \mid \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p) = 0 \ \forall f \in S\} \\ &= \{v \in K^n \mid f(p + \epsilon v) = O(\epsilon^2) \text{ for all } f \in S\}. \end{aligned}$$

This is the tangent space at p of a 'nonreduced scheme' with coordinate ring $K[X_1, \dots, X_n]/(S)$. One has $T_pX \subseteq T_p^S X$ since $I = \sqrt{(S)} \supseteq (S)$.

Proof. We may assume that X is closed, so affine. The map $K[X] \to \mathcal{O}_{X,p}$ induces map

$$T_pX = \mathrm{Der}(\mathcal{O}_{X,p}, K) \to \mathrm{Der}(K[X], K)$$

where K is considered as a module for $K[X]$ by evaluation at p. This is an isomorphism, because any derivation $\xi : K[X] \to K$ extends uniquely to a derivation on $\xi : \mathcal{O}_{X,p} \to K$. Namely, any element of $\mathcal{O}_{X,p}$ is locally a rational function, so can be written as f/g with $f, g \in K[X]$ and $g(p) \neq 0$. Then

$$\xi(f) = \xi(\frac{f}{g} \cdot g) = \xi(\frac{f}{g})g(p) + \frac{f(p)}{g(p)}\xi(g),$$

so

$$\xi(f/g) = \frac{g(p)\xi(f) - \xi(g)f(p)}{g(p)^2},$$

which can be used to extend ξ, and shows uniqueness. Then

$$\begin{aligned} T_pX &= \mathrm{Der}(K[X], K) = \mathrm{Der}(K[X_1, \dots, X_n]/I, K) \\ &= \{\xi \in \mathrm{Der}(K[X_1, \dots, X_n], K) \mid \xi(f) = 0 \ \forall f \in I\}. \end{aligned}$$

Now $\mathrm{Der}(K[X_1, \dots, X_n], K) \cong K^n$, sending a ξ to $v = (v_1, \dots, v_n)$ where $v_i = \xi(X_i)$, or $v = (v_1, \dots, v_n)$ to ξ with $\xi(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p)$.

[End of LECTURE 12 on 4 June 2020]

E) **Lemma.** Let $\theta : X \to Y$ be a morphism with X locally closed in $\mathbb{A}^n$ and Y locally closed in $\mathbb{A}^m$. Suppose that θ is the restriction of a morphism $U \to \mathbb{A}^m$ with U an open subset of $\mathbb{A}^n$, also denoted by θ. Then for $v \in T_pX \subseteq K^n$ the element $d\theta_p(v) \in T_pY \subseteq K^m$ is given by

$$\theta(p + \epsilon v) = \theta(p) + \epsilon\, d\theta_p(v) + O(\epsilon^2).$$

Proof. Straightforward.

F) **Examples.** (1) If $X = V(f)$ with $f \in K[X_1, \dots, X_n]$ an irreducible polynomial, then $I = (f)$, so if $p \in V(f)$, then

$$T_pX = \{(v_1, \dots, v_n) \in K^n \mid \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p) = 0\}.$$

For example if $f = x^2(x + 1) - y^2$ then $V(f)$ is a nodal curve in K^2. Then $\partial f/\partial x = 2x + 3x^2$ and $\partial f/\partial y = -2y$, so

$$T_{(0,0)}V(f) = \{(v_1, v_2) \in K^2 \mid 0v_1 + 0v_2 = 0\} = K^2$$

and

$$T_{(3,6)}V(f) = \{(v_1, v_2) \in K^2 \mid 33v_1 - 12v_2 = 0\}.$$

(2) If V is a vector space then $T_pV \cong V$ for all $p \in V$. Thus $T_pM_n(K) \cong M_n(K)$. The determinant map $\det : M_n(K) \to K$ has

$$\det(1 + \epsilon v) = 1 + \epsilon \operatorname{tr}(v) + \cdots + \epsilon^n \det(v)$$

so $d(\det)_1 = \operatorname{tr}$.

(3) Since $\mathrm{GL}_n(K)$ is open in $M_n(K)$, we have $T_g \mathrm{GL}_n(K) \cong M_n(K)$ for all g. If $\theta : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K)$ is the inversion map, then

$$(g + \epsilon v)^{-1} = g^{-1} + \epsilon\, d\theta_g(v) + O(\epsilon^2).$$

Multiplying by $g + \epsilon v$ gives

$$\begin{aligned} 1 &= (g + \epsilon v)(g + \epsilon v)^{-1} \\ &= (g + \epsilon v)(g^{-1} + \epsilon\, d\theta_g(v)) + O(\epsilon^2) \\ &= 1 + \epsilon(vg^{-1} + g\, d\theta_g(v)) + O(\epsilon^2). \end{aligned}$$

so $d\theta_g(v) = -g^{-1}vg^{-1}$. In particular $d\theta_1(v) = -v$.

G) **Lie algebras.** The *Lie algebra* of an algebraic group G is $\mathfrak{g} = T_1G$. If $g \in G$, there is a map $c^g : G \to G$, $x \mapsto gxg^{-1}$, and hence $d(c^g)_1 : \mathfrak{g} \to \mathfrak{g}$. This defines an action of G on $\mathfrak{g}$, the *adjoint action*

$$Ad : G \to \mathrm{GL}(\mathfrak{g}), g \mapsto d(c^g)_1.$$

Taking the tangent space map gives a linear map

$$ad = d(Ad)_1 : \mathfrak{g} \to \mathrm{End}_K(\mathfrak{g}).$$

Defining $[u, v] = ad(u)(v)$ turns $\mathfrak{g}$ into a Lie algebra.

Example. For $G = \mathrm{GL}_n(K)$, $c^g(1 + \epsilon v) = g(1 + \epsilon v)g^{-1} = 1 + \epsilon gvg^{-1}$, so $d(c^g)_1(v) = gvg^{-1}$ for $v \in M_n(K)$. Then

$$\begin{aligned} Ad(1 + \epsilon u)(v) &= (1 + \epsilon u)v(1 + \epsilon u)^{-1} \\ &= (1 + \epsilon u)v(1 - \epsilon u) + O(\epsilon^2) \\ &= v + \epsilon(uv - vu) + O(\epsilon^2) \end{aligned}$$

Thus $[u, v] = ad(u)(v) = uv - vu$.

3.9 Smooth varieties

A) **Definition.** A variety X is *smooth* (or *nonsingular*, or *regular*) at $p \in X$ if $\dim T_pX = \dim_p X$, or equivalently if the local ring $\mathcal{O}_{X,p}$ is a 'regular' local ring, which means that $\dim \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim \mathcal{O}_{X,p}$. The variety X is *smooth* if it is smooth at all points.

Clearly $\mathbb{A}^n$ and $\mathbb{P}^n$ are smooth.

B) **Theorem.** For a variety X we have:
(i) The function $X \to \mathbb{Z}$, $p \mapsto \dim T_pX$ is upper semicontinuous;
(ii) If X is irreducible, then $\dim T_pX = \dim X$ for all p in a nonempty open subset of X;
(iii) The set of smooth points of X is a dense open subset of X;
(iv) $\dim T_pX \geq \dim_p X$ for all $p \in X$.

Proof. (i) We may suppose that X is affine, say closed in $\mathbb{A}^n$. Now apply the cones theorem. The relevant set is

$$\{(p,v) \in \mathbb{A}^n \times K^n \mid f(p) = 0, \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p) = 0 \ \forall f \in I\}$$

where I is the set of all polynomials vanising on X.

(ii) We use that any irreducible variety of dimension $n-1$ is birational to a hypersurface in $\mathbb{A}^n$ (this follows from Proposition 3.2D and some theory of fields, see Hartshorne, Algebraic Geometry, Proposition I.4.9). Thus we only need to prove the statement for a hypersurface. Say $X = V(f)$ for $f \in K[X_1,\dots,X_n]$ an irreducible polynomial. For $p \in X$ we have

$$T_pX = \{(v_1,\dots,v_n) \in K^n \mid \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p) = 0\}.$$

which has the right dimension if some $\partial f/\partial X_i(p) \neq 0$.

In characteristic 0, if all partial derivatives $\partial f/\partial X_i$ are identically zero then f is constant. In characteristic ℓ this is not true, for example $a + bX_1^\ell + cX_2^\ell X_3^{2\ell}$, but all exponents must be multiples of ℓ, and choosing an ℓ-th root of each coefficient, one gets

$$f = \left(\sqrt[\ell]{a} + \sqrt[\ell]{b}X_1 + \sqrt[\ell]{c}X_2X_3^2\right)^\ell$$

contradicting irreducibility of f.

Thus some partial derivative $\partial f/\partial X_i$ is not identically zero. If it vanishes on X, then it is in (f), which is impossible by degrees. Thus $X \cap D(\partial f/\partial X_i)$ is a dense open subset of X with the right property.

(iii) Reduce to the irreducible case, which is (ii).

(iv) Reduce to the irreducible case, when it follows from (i), (ii).

C) **Remarks.** (1) Algebraic groups are smooth, since any two points look the same.

(2) Any point in an intersection of irreducible components cannot be smooth (since regular local rings are domains, so have a unique minimal prime ideal). Thus a smooth variety must be the disconnected union of its irreducible components.

D) **Key observation.** Suppose that $X = V(S)$, a closed subspace of $\mathbb{A}^n$ defined by a subset $S = \{f_1, \dots, f_m\} \subseteq K[X_1, \dots, X_n]$ with m elements. For $p \in X$ we have an inclusion

$$T_p X \subseteq T_p^S X = \{v \in K^n \mid f(p + \epsilon v) = O(\epsilon^2) \ \forall f \in S\}$$

which is in general strict, since $T_p X$ is defined using all $f \in \sqrt{(S)}$. However, suppose that $\dim T_p^S X = n - m$. Since X is the fibre over 0 of the map $f : \mathbb{A}^n \to \mathbb{A}^m$ given by the f_i, the Main Lemma say that each irreducible component of X has dimension at least $\dim \mathbb{A}^n - \dim \overline{\operatorname{Im} f}$. Thus

$$\dim T_p X \geq \dim_p X \geq \dim \mathbb{A}^n - \dim \overline{\operatorname{Im} f} \geq n - m = \dim T_p^S X \geq \dim T_p X,$$

so $\dim_p X = \dim T_p X$. Thus X is smooth at p and $T_p X = T_p^S X$.

E) **Example.** The orthogonal group is $G = O_n(K) = \{A \in M_n(K) : A^T A = 1\}$. This is a subset of $M_n(K) \cong \mathbb{A}^{n^2}$ given by $n(n+1)/2$ equations (since $A^T A$ is always symmetric). Then

$$T_1^S G = \{v \in M_n(K) : (1+\epsilon v)^T (1+\epsilon v) = 1+O(\epsilon^2)\} = \{v \in M_n(K) : v^T + v = 0\}.$$

This is the set of skew symmetric matrices, so it has dimension $n(n-1)/2$. Since this is equal to $n^2 - n(n+1)/2$, it is the tangent space $T_1 G$, so the Lie algebra of G.

F) **Theorem.** Let $\theta : X \to Y$ be a morphism between smooth irreducible varieties. If $d\theta_p$ is surjective for some $p \in X$, then θ is dominant. The converse holds if K has characteristic 0.

This is an analogue of Sard's Lemma for algebraic geometry. For a proof see T. A. Springer, Linear algebraic groups, second edition, 1998, Theorem 4.3.6.

[End of LECTURE 13 on 8 June 2020]

4 Hochschild cohomology and $\operatorname{Alg}\operatorname{Mod}$

4.1 Hochschild cohomology

This should have gone in my first lecture course on noncommutative algebra.

A) **Lemma.** For any algebra A there is an exact sequence

$$\dots \xrightarrow{b_3} A \otimes A \otimes A \xrightarrow{b_2} A \otimes A \xrightarrow{b_1} A \to 0,$$

$$b_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n,$$

of A-A-bimodules (where tensor products are over the base field K). On Wikipedia it is called the *standard complex*.

Proof. Define a map (of right A-modules) $h_n : A^{\otimes n} \to A^{\otimes n+1}$ by $h_n(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n$. One easily checks that $b_1 h_1 = 1$ and

$$b_{n+1} h_{n+1} + h_n b_n = 1 \quad (n \geq 1).$$

Also $b_1 b_2 = 0$ and then by induction $b_n b_{n+1} = 0$ for all $n \geq 1$ since

$$b_{n+1} b_{n+2} h_{n+2} = b_{n+1}(1 - h_{n+1} b_{n+1}) = b_{n+1} - b_{n+1} h_{n+1} b_{n+1} = b_{n+1} - (1 - h_n b_n) b_{n+1} = 0.$$

Now $\operatorname{Im}(h_{n+2})$ generates $A^{\otimes n+2}$ as a left A-module, and the b_i are left A-module maps (in fact bimodule maps), so $b_{n+1} b_{n+2} = 0$. Finally if $x \in \operatorname{Ker}(b_n)$ then $x = (b_{n+1} h_{n+1} + h_n b_n)(x)$ implies $x \in \operatorname{Im}(b_{n+1})$, giving exactness.

B) **Remarks.** (i) Recall that an A-A-bimodule is the same thing as a left $A \otimes A^{op}$-module. Thus if V is a vector space, then $A \otimes V \otimes A$ is a free A-A-bimodule. Thus, defining $P_n = A^{\otimes n+2}$, the standard complex gives a projective resolution

$$\cdots \to P_1 \to P_0 \to A \to 0$$

of A as an A-A-bimodule.

(ii) Tensoring the standard complex over A with a left A-module X, and using that $A \otimes_A X \cong X$ gives a projective resolution of X which can be written as

$$\cdots \to P_1 \otimes_A X \to P_0 \otimes_A X \to X \to 0$$

or as

$$\cdots \to A \otimes A \otimes X \to A \otimes X \to X \to 0.$$

In MacLane, Homology, this is called the *un-normalized bar resolution of* X. It is exact because the terms in the standard complex are projective right A-modules, so if you break it into short exact sequences of right A-modules, all of them are split.

C) **Definition.** Let M be an A-A-bimodule. Applying $\mathrm{Hom}_{A-A}(-, M)$ to the projective resolution P_* of A, and using that $\mathrm{Hom}_{A-A}(A \otimes A, M) \cong M$ and $\mathrm{Hom}_{A-A}(A \otimes V \otimes A, M) \cong \mathrm{Hom}_K(V, M)$, we get a complex

$$0 \to M \xrightarrow{\phi_0} \mathrm{Hom}_K(A, M) \xrightarrow{\phi_1} \mathrm{Hom}_K(A \otimes A, M) \to \dots$$

called the *Hochschild cohain complex*, where the maps are given by

$$\phi_0(m)(a) = am - ma, \quad \phi_1(\theta)(a \otimes b) = a\theta(b) - \theta(ab) + \theta(a)b$$

$$\begin{aligned}\phi_n(\theta)(a_0 \otimes \cdots \otimes a_n) = {} & a_0\theta(a_1 \otimes a_2 \otimes \cdots \otimes a_n) - \theta(a_0a_1 \otimes a_2 \otimes \cdots \otimes a_n) + \dots \\ & + (-1)^n\theta(a_0 \otimes \cdots \otimes a_{n-1}a_n) - (-1)^n\theta(a_0 \otimes \cdots \otimes a_{n-1})a_n.\end{aligned}$$

The cohomology of this complex is *Hochschild cohomology* $H^i(A, M)$. Since the standard complex is a projective resolution of A as an A-A-bimodule, we have $H^i(A, M) \cong \mathrm{Ext}^i_{A-A}(A, M)$.

Small cases:

(i) $H^0(A, M) = \{m \in M : am = ma\, \forall a \in A\}$. Thus for example $H^0(A, A) = Z(A)$.

(ii) $H^1(A, M) = \mathrm{Der}(A, M)/Inn(A, M)$, where $Inn(A, M)$ is the set of inner derivations, those of the form $a \mapsto am - ma$ for some $m \in M$,

(iii) $H^2(A, M) = \{\text{2-cocycles } \theta : A \otimes A \to M\}/\{\text{2-coboundaries}\}$, where θ is a 2-cocycle if $a\theta(b \otimes c) - \theta(ab \otimes c) + \theta(a \otimes bc) - \theta(a \otimes b)c = 0$ for all a, b, c, and it is a 2-coboundary if it is of the form $a \otimes b \mapsto a\psi(b) - \psi(ab) + \psi(a)b$ for some $\psi : A \to M$.

It classifies the algebra structures on $\Lambda := A \oplus M$ in which M becomes an ideal of square zero. Explicitly, the multiplication $*_\theta$ on Λ defined by

$$(a, m) *_\theta (b, n) = (ab, an + mb + \theta(a \otimes b))$$

is associative if and only if θ is a 2-cocycle, and two 2-cocycles θ, θ' differ by a 2-coboundary if and only if there is an isomorphism

$$F : (\Lambda, *_\theta) \to (\Lambda, *_{\theta'})$$

of the form $F(a, m) = (a, m + \psi(a))$.

D) **Lemma.** If X and Y are A-modules, and $\mathrm{Hom}_K(X,Y)$ is considered as an A-A-bimodule in the natural way, then

$$H^n(A, \mathrm{Hom}_K(X,Y)) \cong \mathrm{Ext}^n_A(X,Y).$$

Proof. If P_* is the projective resolution of A as an A-A-bimodule above, then $P_* \otimes_A X$ is a projective resolution of X by Remark B, and the terms in the Hochschild cohain complex are

$$\mathrm{Hom}_K(A^{\otimes n}, \mathrm{Hom}_K(X,Y)) \cong \mathrm{Hom}_K(A^{\otimes n} \otimes X, Y) \cong \mathrm{Hom}_A(P_n \otimes_A X, Y).$$

4.2 Application to $\mathrm{Mod}(A,d)$

A) **Tangent space.** For simplicity let A be a finite-dimensional algebra, so we can identify $\mathrm{Mod}(A,d) = \mathrm{Hom}_{K\text{-alg}}(A, M_d(K))$. Recall that the module corresponding to $\theta \in \mathrm{Mod}(A,d)$ is ${}_\theta K^d$.

Clearly $\mathrm{Mod}(A,d)$ is the subset of all θ in the affine space $\mathrm{Hom}_K(A, M_d(K))$ satisfying the conditions $f_0(\theta) = 0$ and $f_{ab}(\theta) = 0$ for all $a, b \in A$, where

$$f_0(\theta) = \theta(1) - 1, \quad f_{ab}(\theta) := \theta(ab) - \theta(a)\theta(b).$$

Identifying $\mathrm{Hom}_K(A, M_d(K))$ with affine space $\mathbb{A}^N$, the set of these functions corresponds to a subset S of the polynomial ring $K[X_1, \dots, X_N]$, and $V(S)$ corresponds to $\mathrm{Mod}(A,d)$.

Let $\theta \in \mathrm{Mod}(A,d)$. Recall that $T_\theta \mathrm{Mod}(A,d) \subseteq T^S_\theta \mathrm{Mod}(A,d)$. To compute $T^S_\theta \mathrm{Mod}(A,d)$, we examine $\theta + \epsilon\phi$ for $\phi \in \mathrm{Hom}_K(A, M_d(K))$. We have

$$f_0(\theta + \epsilon\phi) = (\theta + \epsilon\phi)(1) - 1 = \epsilon\phi(1)$$

and

$$\begin{aligned} f_{ab}(\theta + \epsilon\phi) &= (\theta + \epsilon\phi)(ab) - (\theta + \epsilon\phi)(a)(\theta + \epsilon\phi)(b) \\ &= \epsilon(\phi(ab) - \phi(a)\theta(b) - \theta(a)\phi(b)) + O(\epsilon^2). \end{aligned}$$

The terms which are linear in ϵ vanish precisely when ϕ is a derivation from A to $M_d(K)$, considered as an A-A-bimodule via $avb = \theta(a)v\theta(b)$ for $v \in M_d(K)$. Thus $T^S_\theta \mathrm{Mod}(A,d)$ can be identified with $\mathrm{Der}(A, M_d(K))$.

If $\theta \in \mathrm{Mod}(A,d)$, the action of $\mathrm{GL}_d(K)$ on $\mathrm{Mod}(A,d)$ defines a morphism $m : \mathrm{GL}_d(K) \to \mathrm{Mod}(A,d)$ by $m(g) = {}^g\theta$, where $({}^g\theta)(a) = g\theta(a)g^{-1}$ for

$a \in A$. Now

$$\begin{aligned}({}^{1+\epsilon v}\theta)(a) &= (1+\epsilon v)\theta(a)(1+\epsilon v)^{-1} \\ &= (1+\epsilon v)\theta(a)(1-\epsilon v) + O(\epsilon^2) \\ &= \theta(a) + \epsilon(v\theta(a) - \theta(a)v) + O(\epsilon^2)\end{aligned}$$

so $dm_1 : M_d(K) \to T_\theta \operatorname{Mod}(A,d)$ is given by $dm_1(v) = (a \mapsto v\theta(a) - \theta(a)v)$, so $\operatorname{Im}(dm_1)$ is the set of inner derivations from A to $M_d(K)$. Thus

$$\begin{aligned}\operatorname{Coker}(dm_1) = \frac{T_\theta \operatorname{Mod}(A,d)}{\operatorname{Im}(dm_1)} &\subseteq \frac{T_\theta^S \operatorname{Mod}(A,d)}{\operatorname{Im}(dm_1)} \cong \frac{\operatorname{Der}(A, M_d(K))}{Inn(A, M_d(K))} \\ &\cong H^1(A, M_d(K)) \cong \operatorname{Ext}^1_A({}_\theta K^d, {}_\theta K^d).\end{aligned}$$

Similar results hold for $\operatorname{Mod}(A,\alpha)$ for A a finitely generated algebra.

[End of LECTURE 14 on 15 June 2020]

B) **Smoothness.** One can show that if $\theta \in \operatorname{Mod}(A,d)$ and $\operatorname{Ext}^2({}_\theta K^d, {}_\theta K^d) = 0$, then $\operatorname{Mod}(A,d)$ is smooth at θ, and moreover the scheme-theoretic structure is reduced, so $T_\theta \operatorname{Mod}(A,d) = T_\theta^S \operatorname{Mod}(A,d)$ in this case.

We omit the proof. For details see section 6.4 of Crawley-Boevey and Sauter, On quiver Grassmannians and orbit closures for representation-finite algebras, 2016, or the work of Geiß cited there. The proof uses $H^2(A, \operatorname{End}_K(X))$.

Again, similar results hold for $\operatorname{Mod}(A,\alpha)$ for A a finitely generated algebra.

One special case is clear. If $A = KQ$, then A is hereditary, so Ext^2 vanishes. Now $\operatorname{Mod}(A,\alpha) = \operatorname{Rep}(Q,\alpha)$, which is an affine space, so smooth.

4.3 $\operatorname{Alg}\operatorname{Mod}$ and a theorem of Gabriel

A) **Definition.** Let $r, d \in \mathbb{N}$. Any element $a \in \operatorname{Alg}(r)$ turns the vector space K^r into an algebra, denoted K_a. We define $\operatorname{Alg}\operatorname{Mod}(r,d)$ to be the set of pairs $(a,x) \in \operatorname{Alg}(r) \times \operatorname{Hom}_K(K^r, M_d(K))$ such that x is a K-algebra map from K_a to $M_d(K)$. This is a closed subset, so an affine variety. If $\pi : \operatorname{Alg}\operatorname{Mod}(r,d) \to \operatorname{Alg}(r)$ is the projection and $a \in \operatorname{Alg}(r)$, then

$$\pi^{-1}(a) \cong \operatorname{Hom}_{K\text{-algebra}}(K_a, M_d(K)) \cong \operatorname{Mod}(K_a, d).$$

The group $\operatorname{GL}_d(K)$ acts on $\operatorname{Alg}\operatorname{Mod}(r,d)$ by conjugation on the second factor.

B) **Definition.** Let a group G act on a set X. We say that a subset Y of X is a *G-subset* if $gy \in Y$ for all $g \in G$ and $y \in Y$. Equivalently if it is a union of orbits.

C) **Lemma.** If X is a variety, then under the projection $X \times \mathrm{Inj}(K^d, V) \to X$, the image of a closed $\mathrm{GL}_d(K)$-subset is a closed subset. Similarly for the projection $X \times \mathrm{Surj}(V, K^d) \to X$.

Proof. We factor it as

$$X \times \mathrm{Inj}(K^d, V) \to X \times \mathrm{Gr}(V, d) \to X.$$

The map $\mathrm{Inj}(K^d, V) \to \mathrm{Gr}(V, d)$ is universally submersive, so open $\mathrm{GL}_d(K)$-subsets of $X \times \mathrm{Inj}(K^d, V)$ correspond to open subsets of $X \times \mathrm{Inj}(K^d, V)$. Thus closed $\mathrm{GL}_d(K)$-subsets of $X \times \mathrm{Inj}(K^d, V)$ correspond to closed subsets of $X \times \mathrm{Inj}(K^d, V)$. Now use that $\mathrm{Gr}(V, d)$ is complete.

D) **Theorem** (Gabriel). Under the projection $\pi : \mathrm{Alg}\,\mathrm{Mod}(r, d) \to \mathrm{Alg}(r)$, the image of a closed $\mathrm{GL}_d(K)$-subset is a closed subset.

This is a reformulation of Lemma 3.2 in P. Gabriel, Finite representation type is open. This reformulation is mentioned in C. Geiss, On degenerations of tame and wild algebras, 1995.

Proof. Let

$$W = \{(a, \theta) \in \mathrm{Alg}(r) \times \mathrm{Surj}(K^{rd}, K^d) : \mathrm{Ker}\,\theta \text{ is a } K_a\text{-submodule of } (K_a)^r\}.$$

This is a closed subset of the product. We have a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & \mathrm{Alg}(r) \times \mathrm{Surj}(K^{rd}, K^d) \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ \mathrm{Alg}\,\mathrm{Mod}(r, d) & \xrightarrow{\pi} & \mathrm{Alg}(r) \end{array}$$

where p is the projection and g sends (a, θ) to the pair consisting of a and the induced K_a-module structure on K^d. Now g is onto since any d-dimensional K_a-module is a quotient of a free module of rank d.

One can check using the affine open covering of $\mathrm{Surj}(K^{rd}, K^d)$ that g is a morphism of varieties.

Suppose $Z \subseteq \mathrm{Alg}\,\mathrm{Mod}(r, d)$ is a closed $\mathrm{GL}_d(K)$-subset. Then $g^{-1}(Z)$ is also. Thus it is a closed $\mathrm{GL}_d(K)$-subset of $\mathrm{Alg}(r) \times \mathrm{Surj}(K^{rd}, K^d)$. Thus $\pi(Z) = p(g^{-1}(Z))$ is closed by the lemma.

4.4 Application to global dimension

A) **Lemma.** For any i, the map

$$\mathrm{Alg}\,\mathrm{Mod}(r, d) \to \mathbb{Z}, \quad (a, x) \mapsto \dim \mathrm{Ext}^i_{K_a}(K_x, K_x)$$

is upper semicontinuous.

Proof. Use Lemma 4.1D with $A = K_a$ and $X = Y = K_x$ for $(a, x) \in \operatorname{Alg}\operatorname{Mod}(r, d)$. The terms in the Hochschild cochain complex are fixed vector spaces V^i, and the maps are given by morphisms $f_i : \operatorname{Alg}\operatorname{Mod}(r, d) \to \operatorname{Hom}_K(V^i, V^{i+1})$. Thus we get a morphism

$$\operatorname{Alg}\operatorname{Mod}(r, d) \to \{(\theta, \phi) \in \operatorname{Hom}(V^{i-1}, V^i) \times \operatorname{Hom}(V^i, V^{i+1}) : \phi\theta = 0\}.$$

Now use that the map $(\theta, \phi) \mapsto \dim(\operatorname{Ker}\phi / \operatorname{Im}\theta)$ is upper semicontinuous.

B) **Notation.** Given any module M, we define $\operatorname{gr} M$ to be the associated graded module for any composition series of M. Thus

$$\operatorname{gr} M = \bigoplus_S S^{n_S},$$

where S runs through the simple A-modules and n_S is the multiplicity of S as a composition factor of M. It is semisimple, other the same dimension as M.

C) **Theorem** (Schofield). The algebras of global dimension $\leq g$ form an open subset of $\operatorname{Alg}(r)$, as do the algebras of finite global dimension. There is an integer N_r, depending on r, such that any algebra of dimension r of finite global dimension has global dimension $\leq N_r$.

Proof. A has global dimension $\leq g$
$\Leftrightarrow \operatorname{Ext}^{g+1}_A(M, N) = 0$ for all M, N
$\Leftrightarrow \operatorname{Ext}^{g+1}_A(M, N) = 0$ for all simple M and N
$\Leftrightarrow \operatorname{Ext}^{g+1}_A(\operatorname{gr} A, \operatorname{gr} A) = 0$
$\Leftrightarrow \operatorname{Ext}^{g+1}_A(M, M) = 0$ for all M of dimension r.

Consider the pairs $(a, x) \in \operatorname{Alg}\operatorname{Mod}(r, r)$ such that $\operatorname{Ext}^{g+1}_{K_a}(K_x, K_x) \neq 0$. By upper semicontinuity this is a closed subset of $\operatorname{Alg}\operatorname{Mod}(r, r)$. It is also a $\operatorname{GL}_r(K)$-subset, so its image in $\operatorname{Alg}(r)$ is closed. This is the set of algebras of global dimension $> g$. Thus the algebras of global dimension $\leq g$ form an open subset D_g. Now since varieties are noetherian topological spaces, the chain of open sets

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$$

stabilizes, so the set of algebras of finite global dimension is $\bigcup_i D_i = D_{N_r}$ for some integer N_r.

[End of LECTURE 15 on 18 June 2020]

5 Orbits in $\mathrm{Mod}(A, \alpha)$

5.1 Orbits for algebraic group actions

Let G be a (linear) algebraic group. For simplicity we assume G is connected.

A) **Theorem.** Suppose that G acts on a variety X and $x \in X$. Then
(i) The orbit $Gx = \{gx : g \in G\}$ is a smooth locally closed subset of X.
(ii) Gx and $\overline{Gx}$ are irreducible varieties.
(iii) The stabilizer $\mathrm{Stab}_G(x) = \{g \in G : gx = x\}$ is a closed subgroup of G,
(iv) $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$.
(v) The closure $\overline{Gx}$ is the union of Gx with orbits of smaller dimension.
(vi) The closure $\overline{Gx}$ contains a closed orbit.

Proof. (i) The map $G \to X$, $g \mapsto gx$ is a morphism, so its image Gx is constructible. Since G is connected, it is an irreducible variety, so its closure $\overline{Gx}$ is irreducible. Thus Gx contains a nonempty open subset U of $\overline{Gx}$. Left multiplication by $g \in G$ induces an isomorphism $X \to X$, so gU is an open subset of $g\overline{Gx} = \overline{Gx}$. Thus $Gx = \bigcup_{g \in G} gU$ is an open subset of $\overline{Gx}$. Thus Gx is locally closed. Now it is smooth since, by the action of G, all points look the same.

(ii) Here we really use that G is connected, so an irreducible variety. Consider the map $G \to X$, $g \mapsto gx$. Lemma 3.3D implies that $\overline{Gx}$ is irreducible. Then Gx is a non-empty open subset of this, so irreducible.

(iii) Clear.

(iv) The morphism $G \to Gx$, $g \mapsto gx$ is surjective. Its fibres are cosets of $\mathrm{Stab}_G(x)$, so all are isomorphic as varieties to $\mathrm{Stab}_G(x)$, so they have the same dimension. Then the Main Lemma gives $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$.

(v) Clearly $\overline{Gx}$ is a G-subset, so a union of orbits. If Gy is one of them and $\dim Gy \not< \dim Gx$, then $\overline{Gy} = \overline{Gx}$, so Gy is open in $\overline{Gx}$, so $C = \overline{Gx} \setminus Gy$ is closed in X. If $Gy \neq Gx$ then C contains Gx, which is nonsense.

(vi) If $Gy \subseteq \overline{Gx}$ is of minimal dimension, it must be closed.

B) **Proposition.** The map $X \to \mathbb{Z}$, $x \mapsto \dim \mathrm{Stab}_G(x)$ is upper semicontinuous. Thus the set

$$X_{\leq s} = \{x \in X : \dim \mathrm{Stab}_G(x) \leq s\} = \{x \in X : \dim Gx \geq \dim G - s\}$$

is open and the set

$$X_s = \{x \in X : \dim \mathrm{Stab}_G(x) = s\} = \{x \in X : \dim Gx = \dim G - s\}$$

is locally closed.

Proof. Let $Z = \{(g,x) \in G \times X : gx = x\}$ and let $\pi : Z \to X$ be the projection. Now

$$\dim_{(1,x)} \pi^{-1}\pi(1,x) = \dim_1 \operatorname{Stab}_G(x) = \dim \operatorname{Stab}_G(x)$$

since $\operatorname{Stab}_G(x)$ is a group, so every point looks the same.

5.2 Orbits and degeneration in $\operatorname{Mod}(A,\alpha)$

A) **Setup.** We consider a f.g. algebra A with a complete set of orthogonal idempotents $e_1,\dots,e_n$ and $\alpha \in \mathbb{N}^n$. The group $\mathrm{GL}(\alpha) = \prod_i \mathrm{GL}_{\alpha_i}(K)$ of dimension $\sum \alpha_i^2$ acts on $\operatorname{Mod}(A,\alpha)$. Recall that the orbits of correspond to isomorphism classes of modules of dimension vector α. We write $\mathcal{O}_M$ for the orbit corresponding to a module M, so $\mathcal{O}_M = \{x \in \operatorname{Mod}(A,\alpha) : K_x \cong M\}$. Now

$$\operatorname{Aut}_A(M) = \mathrm{GL}_K(M) \cap \operatorname{End}_A(M) \subseteq \operatorname{End}_K(M)$$

and $\operatorname{End}_A(M)$ is a vector space, so an affine space, so an irreducible variety. Then $\operatorname{Aut}_A(M)$ is a non-empty open subvariety of $\operatorname{End}_A(M)$, so also an irreducible variety. It is also a closed subgroup of $\mathrm{GL}_K(M)$, so a connected algebraic group. If $x \in \operatorname{Mod}(A,\alpha)$, then $\operatorname{Stab}_{\mathrm{GL}(\alpha)}(x) = \operatorname{Aut}(K_x)$. Thus

$$\dim \mathrm{GL}(\alpha) - \dim \mathcal{O}_M = \dim \operatorname{Aut}_A(M) = \dim \operatorname{End}_A(M).$$

B) **Definition.** Let M and N be A-modules of the same dimension vector α. We say that M *degenerates to* N if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$.

This defines a partial order on the set of isomorphism classes of modules of dimension α: it is clearly reflexive and transitive, and it is antisymmetric since if M degenerates to N and $M \not\cong N$, then $\dim \mathcal{O}_N < \dim \mathcal{O}_M$.

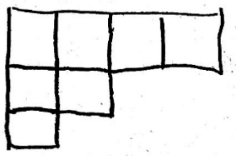
More generally, given any linear algebraic group G acting on a variety X, we say that $x \in X$ *degenerates to* $y \in X$ if $y \in \overline{Gx}$.

C) **Example.** Recall that the nilpotent variety is

$$N_d = \{A \in M_d(K) : A^d = 0\} = \operatorname{Mod}(K[T]/(T^d), d)$$

There are only finitely many orbits under the action of $\mathrm{GL}_d(K)$. They are $\mathcal{O}_{M(\lambda)}$ where $\lambda = (\lambda_1, \lambda_2, \dots,)$ is a partition of d, and $M(\lambda)$ is the $K[T]/(T^d)$-module with vector space K^d, with T acting as the matrix involving a Jordan block $J_i(0)$ of eigenvalue 0 and size i for each column of length i in the Young diagram of shape λ (so with rows of length λ_i).

For example for $d = 7$, the partition $\lambda = (4, 2, 1)$ has Young diagram



and $M(\lambda)$ is given by the matrix with diagonal blocks $J_3(0), J_2(0), J_1(0), J_1(0)$,

$$\left(\begin{array}{ccc|cc|c|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

The module $M(1^d) \cong K[T]/(T^d)$ given by a Jordan block of size d degenerates into any other module. Namely, given λ and $t \in K$, consider the module M_t given by the same matrix as $M(\lambda)$, but with zeros on the superdiagonal changed into ts, for example

$$\left(\begin{array}{ccc|cc|c|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & t \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

This defines a morphism $f : \mathbb{A}^1 \to N_d$, $t \mapsto M_t$.

Clearly $M_t \cong M_1 \cong M(1^d)$ for $t \neq 0$ and $M_0 = M(\lambda)$. Thus $f^{-1}(\overline{\mathcal{O}_{M(1^d)}})$ is a closed subset of $\mathbb{A}^1$ containing all $t \neq 0$. So it equals $\mathbb{A}^1$, so $M(\lambda) = f(0) \in \overline{\mathcal{O}_{M(1^d)}}$.

Thus $\overline{\mathcal{O}_{M(1^d)}}$ contains the orbit $\mathcal{O}_{M(\lambda)}$ for all λ, and hence $\overline{\mathcal{O}_{M(1^d)}} = N_d$. Thus N_d is irreducible of dimension $\dim \mathcal{O}_{M(1^d)} = d^2 - \dim \operatorname{End}(M(1^d)) = d^2 - d$.

[End of LECTURE 16 on 22 June 2020]

D) **Theorem.** Given A-modules M and N of the same dimension vector, we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), where:
(i) There are modules $M = M_0, M_1, \dots, M_n = N$ and exact sequences $0 \to L_i \to M_i \to L_i' \to 0$ with $M_{i+1} \cong L_i \oplus L_i'$.
(ii) M degenerates to N
(iii) $\dim \operatorname{Hom}(X, M) \leq \dim \operatorname{Hom}(X, N)$ for all X.

Proof. (ii) $\Rightarrow$ (iii). Use that $\dim \operatorname{Hom}_A(X, -)$ is upper semicontinuous.

(i) $\Rightarrow$ (ii). If M degenerates to N and N degenerates to L, then certainly M degenerates to L. Thus it suffices to prove that if $0 \to L \to M \to L' \to 0$ then M degenerates to $L \oplus L'$. For simplicity we do $\operatorname{Mod}(A, d)$. An element $x \in \operatorname{Mod}(A, d)$ is defined by matrices x_a where a runs through a set of generators of A. Taking a basis of L and extending it to a basis of M, each matrix x_a has the form

$$x_a = \begin{pmatrix} y_a & w_a \\ 0 & z_a \end{pmatrix}$$

with $K_y \cong L$ and $K_z \cong L'$.

For $t \in K$ define an element x^t by the formula

$$x_a^t = \begin{pmatrix} y_a & tw_a \\ 0 & z_a \end{pmatrix}.$$

It is not immediately obvious that these matrices satisfy the relations of the algebra A, and hence define an element of $\operatorname{Mod}(A, d)$. But they clearly do for $t = 0$, and for $t \neq 0$, x^t is the conjugation of x by the diagonal matrix $\left(\begin{smallmatrix} tI & 0 \\ 0 & I \end{smallmatrix}\right) \in \operatorname{GL}_d(K)$, so again they must satisfy the relations.

Now $x^t \in \mathcal{O}_M$ for $t \neq 0$, so $x^0 \in \overline{\mathcal{O}_M}$, and $K_{x^0} \cong L \oplus L'$.

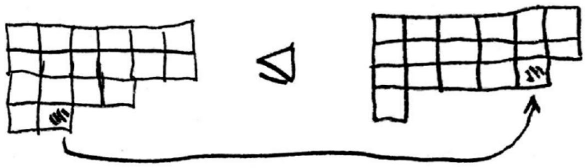
E) **Remark.** Two difficult theorems by G. Zwara say:
- M degenerates to $N \Leftrightarrow \exists$ an exact sequence $0 \to Z \to Z \oplus M \to N \to 0$ for some module Z.
- If A has finite representation type, then (ii) and (iii) in the theorem are equivalent.

F) **Special case.** For the nilpotent variety, so the algebra $K[T]/(T^d)$, or more generally the algebra $K[T]$, conditions (i),(ii),(iii) are all equivalent (Gerstenhaber-Hesselink). Moreover if $M = M(\lambda)$ and $N = M(\mu)$ then condition (iii) becomes that $\lambda \trianglelefteq \mu$ in the *dominance ordering* of partitions.

Firstly, $\dim \operatorname{Hom}(K[T]/(T^i), M(\lambda)) = \lambda_1 + \cdots + \lambda_i$, so condition (iii) says that $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i, and this is the dominance ordering.

Now the dominance order is generated by the following move: $\lambda \trianglelefteq \mu$ if μ is obtained from λ by moving a corner block from a column of length j

to a column further to the right to make it of length $i < j$, for example $(6,6,4,2) \trianglelefteq (6,6,5,1)$ since



(See for example I. G. Macdonald, Symmetric functions and Hall polynomials, I, (1.16).) We want to show in this case that there is an exact sequence

$$0 \to L \to M(\lambda) \to L' \to 0$$

with $M(\mu) \cong L \oplus L'$. Now $M(\lambda) = K[T]/(T^j) \oplus K[T]/(T^{i-1}) \oplus C$ and $M(\mu) = K[T]/(T^{j-1}) \oplus K[T]/(T^i) \oplus C$, so the exact sequence

$$0 \to K[T]/(T^i) \xrightarrow{\begin{pmatrix} -1 \\ T^{j-i} \end{pmatrix}} K[T]/(T^{i-1}) \oplus K[T]/(T^j) \xrightarrow{\begin{pmatrix} T^{j-i} & 1 \end{pmatrix}} K[T]/(T^{j-1}) \to 0$$

will do.

G) **Lemma.** If C is a finite-dimensional algebra, then the variety $N(C)$ of nilpotent elements in C is irreducible of dimension $\dim C - s$, where s is the sum of the dimensions of the simple C-modules.

Proof. Since K is algebraically closed, we can write $C = S \oplus J(C)$ where S is semisimple, so $S \cong M_{d_1}(K) \oplus \cdots \oplus M_{d_r}(K)$. Then $N(C) \cong N_{d_1} \times \ldots N_{d_r} \times J(C)$, so it is irreducible of dimension

$$\dim N(C) = \sum_i (d_i^2 - d_i) + \dim J(C) = \dim C - \sum_i d_i.$$

H) **Proposition.** If A is a finitely generated algebra, α a dimension vector, and $r \in \mathbb{N}$ then the set

$\mathrm{Ind}(A,\alpha)_r = \{x \in \mathrm{Mod}(A,\alpha) : K_x \text{ is indecomposable and } \dim \mathrm{End}_A(K_x) = r\}$

is a closed subset of

$$\mathrm{Mod}(A,\alpha)_{\leq r} = \{x \in \mathrm{Mod}(A,\alpha) : \dim \mathrm{End}_A(K_x) \leq r\},$$

which is an open subset of $\mathrm{Mod}(A,\alpha)$.

Proof. By the upper semicontinuity theorem for cones, the function

$$\mathrm{Mod}(A,\alpha) \to \mathbb{Z}, \quad x \mapsto \dim N(\mathrm{End}_A(K_x))$$

is upper semicontinuous. Now by the lemma $\mathrm{Ind}(A,\alpha)_r$ is equal to

$\{x \in \mathrm{Mod}(A,\alpha) : \dim \mathrm{End}_A(K_x) \leq r\} \cap \{x \in \mathrm{Mod}(A,\alpha) : \dim N(\mathrm{End}_A(K_x)) \geq r-1\}.$

5.3 Open orbits in $\mathrm{Mod}(A,\alpha)$

A) **Theorem.** $\mathrm{Ext}^1(M,M) = 0 \Rightarrow \mathcal{O}_M$ is open in $\mathrm{Mod}(A,\alpha)$. The converse holds if, as a scheme, $\mathrm{Mod}(A,\alpha)$ is reduced at any (hence every) point of $\mathcal{O}_M$.

Proof. Let $x \in \mathcal{O}_M$ and consider the map $GL(\alpha) \to \mathrm{Mod}(A,\alpha)$, $g \mapsto gx$. The map on tangent spaces

$$T_1 \,\mathrm{GL}(\alpha) \to T_x \,\mathrm{Mod}(A,\alpha)$$

has kernel $\mathrm{End}_A(M)$ and if $\mathrm{Ext}^1(M,M) = 0$, then by the discussion in section 4.2A, this map is surjective. Thus

$$\begin{aligned}\dim \mathcal{O}_M &= \dim_x \mathcal{O}_M \leq \dim_x \mathrm{Mod}(A,\alpha) \leq \dim T_x \,\mathrm{Mod}(A,\alpha)\\ &= \dim T_1 \,\mathrm{GL}(\alpha) - \dim \mathrm{End}_A(M) = \dim \mathcal{O}_M.\end{aligned}$$

Thus $\mathrm{Mod}(A,\alpha)$ is smooth at x, so contained in a single irreducible component C of $\mathrm{Mod}(A,\alpha)$. Then also $\mathcal{O}_M \subseteq C$ and $\dim C = \dim \overline{\mathcal{O}_M}$, so $C = \overline{\mathcal{O}_M}$. Any other irreducible component D of $\mathrm{Mod}(A,\alpha)$ may meet C, but it will not meet $\mathcal{O}_M$. It follows that $\mathcal{O}_M$ is open in $\mathrm{Mod}(A,\alpha)$, since

$$\mathcal{O}_M \subseteq_{\text{open}} C \setminus \bigcup_D D = \mathrm{Mod}(A,\alpha) \setminus \bigcup_D D \subseteq_{\text{open}} \mathrm{Mod}(A,\alpha).$$

Conversely, if $\mathcal{O}_M$ is open in $\mathrm{Mod}(A,\alpha)$, then $T_x\mathcal{O}_M = T_x \,\mathrm{Mod}(A,\alpha)$. It follows that the map $T_1 \,\mathrm{GL}(\alpha) \to T_x \,\mathrm{Mod}(A,\alpha)$ is onto. If also $\mathrm{Mod}(A,\alpha)$ is reduced at x, then in section 4.2D the tangent space is equal to the scheme-theoretic tangent space, and it follows that $\mathrm{Ext}^1(M,M) = 0$.

B) **Remark.** This is much easier when $A = KQ$, so $\mathrm{Mod}(A,\alpha) = \mathrm{Rep}(Q,\alpha)$, a vector space. Namely, by section 4.1 in part 2 of the masters sequence, there is a quadratic form q with $q(\alpha) = \dim \mathrm{End}(M) - \dim \mathrm{Ext}^1(M,M)$. Moreover

$$q(\alpha) = \sum_{i=1}^{n} \alpha_i^2 - \sum_{a:i\to j} \alpha_i\alpha_j = \dim \mathrm{GL}(\alpha) - \dim \mathrm{Rep}(Q,\alpha).$$

Thus

$$\dim \mathcal{O}_M = \dim \mathrm{GL}(\alpha) - \dim \mathrm{End}(M) = \dim \mathrm{Rep}(Q,\alpha) - \dim \mathrm{Ext}^1(M,M).$$

from which the theorem follows.

5.4 Closed orbits in $\operatorname{Mod}(A, \alpha)$

A) **Lemma.** Given an A-module M and a simple module S, the multiplicity of S in M is given by

$$[M : S] = \frac{1}{\dim S} \min_{a \in \operatorname{Ann}(S)} \{\text{Order of zero at } t = 0 \text{ of } \chi_{\hat{a}_M}(t)\}$$

where $\hat{a}_M$ is the homothety $M \to M$, $m \mapsto am$ and $\chi_\theta(t) = \det(t1 - \theta)$ is the characteristic polynomial of $\theta \in \operatorname{End}_K(M)$.

Proof. Given an exact sequence $0 \to X \to Y \to Z \to 0$ of A-modules, the endomorphism $\hat{a}_Y$ has uppertriangular block form, so

$$\chi_{\hat{a}_Y}(t) = \chi_{\hat{a}_X}(t)\chi_{\hat{a}_Z}(t) = \chi_{\hat{a}_{X \oplus Z}}(t).$$

Thus we may assume that M is semisimple.

Next we may assume that $M \oplus S$ is faithful. Thus A is semisimple, Now if $M \cong S^k \oplus N$ with $[N : S] = 0$, then the smallest order we could hope to get is if a acts on S as 0 and invertibly on N. This is possible, for writing A as a product of matrix algebras we can take a to correspond to 0 in the block for S and 1 in the other blocks. With this order, the formula holds.

B) **Definition.** Given a module M of dimension d and $a \in A$, we define $c_i^a(M) \in K$ by

$$\chi_{\hat{a}_M}(t) = t^d + c_1^a(M)t^{d-1} + \cdots + c_d^a(M)$$

Thus $c_1^a(M) = -\operatorname{tr}(\hat{a}_M)$ and $c_d^a(M) = (-1)^d \det(\hat{a}_M)$. Then c_i^a defines a regular map $\operatorname{Mod}(A, \alpha) \to K$. Moreover it is constant on the orbits of $\operatorname{GL}(\alpha)$.

By the lemma, these functions determine the multiplicities of the simples in M. (In fact if K has characteristic zero, one only needs to know the trace c_1. This is character theory of groups.)

Recall that $\operatorname{gr} M$ is the semisimple module with the same composition multiplicities as M.

C) **Theorem.** $\overline{\mathcal{O}_M}$ contains a unique orbit of semisimple modules, namely $\mathcal{O}_{\operatorname{gr} M}$. It follows that $\mathcal{O}_M$ is closed if and only if M is semisimple.

Proof. By Theorem 5.2D, $\overline{\mathcal{O}_M}$ contains $\mathcal{O}_{\operatorname{gr} M}$. If $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$ then by continuity $c_i^a(N) = c_i^a(M)$, so M and N have the same composition multiplicities.

[End of LECTURE 17 on 25 June 2020]

6 Number of parameters and applications

6.1 Number of parameters

A) **Notation.** Let G be a connected algebraic group acting on a variety X. We define

$$X_{(d)} = \{x \in X : \dim Gx = d\}.$$

Since $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$, this is $X_{\dim G - d}$ in the notation of section 5.1B, so it is a locally closed G-subset of X. Similarly we define

$$X_{(\leq d)} = \{x \in X : \dim Gx \leq d\}.$$

This is the complement of $X_{\leq \dim G - d - 1}$, so a closed G-subset of X.

B) **Lemma.** If $Y \subseteq X$ is a constructible subset of X, then it can be written as a disjoint union

$$Y = Z_1 \cup \cdots \cup Z_n$$

with the Z_i being irreducible locally closed subsets of X. If Y is G-subset of X, then we may take the Z_i to be G-subsets.

Sketch. For the first part, by definition we can write Y as a not necessarily disjoint union $Y = Z_1 \cup \cdots \cup Z_n$. Replacing each Z_i by its irreducible components we may suppose the Z_i are irreducible. Then if this union is of the form $Y = Z \cup W$ where Z is irreducible of maximal dimension and W is the union of the other terms, then Y is the disjoint union of $Z \setminus \overline{W}$ and $W' = (Z \cap \overline{W}) \cup W$, and if the first term is non-empty, then $(Z \cap \overline{W})$ is a proper closed subset of Z, so has strictly smaller dimension than Z, so W' can be understood by induction.

For the last part, use that G is irreducible, so if $Z \subseteq Y$ is locally closed in X and irreducible, then $GZ = \bigcup_{g \in G} gZ$ is constructible, contained in Y and its closure $\overline{GZ}$ is irreducible, so there is an open subset U of $\overline{GZ}$ with $U \subseteq GZ$. But then GU is open in $\overline{GZ}$ and $GU \subseteq GZ$.

C) **Definition.** If Y is a constructible subset of a variety X, and it is written as a disjoint union of irreducible locally closed subsets Z_i, we define the dimension and number of top-dimensional irreducible components of Y by

$$\dim Y = \max\{\dim Z_i : 1 \leq i \leq n\},$$
$$\operatorname{top} Y = |\{1 \leq i \leq n : \dim Z_i = \dim Y\}|.$$

for a decomposition of Y as in the lemma (here we can take $G = 1$). This does not depend on the decomposition of Y.

Now suppose that G acts on X and assume that Y is a G-subset. We define the number of parameters and number of top-dimensional families by

$$\dim_G Y = \max\{\dim(Y \cap X_{(d)}) - d : d \geq 0\},$$

$$\operatorname{top}_G Y = \sum\{\operatorname{top}(Y \cap X_{(d)}) : d \geq 0, \dim(Y \cap X_{(d)}) - d = \dim_G Y\}.$$

D) **Easy properties.**

(i) If Y_1, Y_2 are G-subsets then $\dim_G(Y_1 \cup Y_2) = \max\{\dim_G Y_1, \dim_G Y_2\}$.

(ii) $\dim_G Y = 0$ if and only if Y contains only finitely many orbits, and if so, $\operatorname{top}_G Y$ is the number of orbits.

(iii) If Y contains a constructible subset Z meeting every orbit, then $\dim_G Y \leq \dim Z$.

(iv) If $f : Z \to X$ is a morphism and the inverse image of each orbit has dimension $\leq d$, then $\dim_G X \geq \dim Z - d$.

(v) $\dim_G Y = \max\{\dim(Y \cap X_{(\leq d)}) - d : d \geq 0\}$.

E) **Lemma.** Suppose that $\pi : X \to Y$ is constant on orbits, and suppose that the image of any closed G-subset of X is a closed subset of Y. Then the function $Y \to \mathbb{Z}$, $y \mapsto \dim_G(\pi^{-1}(y))$ is upper semicontinuous.

Proof. We prove it first for the function dim. By Chevalley's upper semicontinuity theorem, for any r the set

$$C_x = \{x \in X : \dim_x \pi^{-1}(\pi(x)) \geq r\}$$

is closed in X. It is also a G-subset, so by hypothesis $\pi(C_x)$ is closed. Now if $y \in Y$ then $\dim \pi^{-1}(y) = \max\{\dim_x \pi^{-1}(y) : x \in \pi^{-1}(y)\}$. Thus

$$\{y \in Y : \dim \pi^{-1}(y) \geq r\} = \pi(C_r),$$

so it is closed in Y. Thus the map $y \mapsto \dim \pi^{-1}(y)$ is upper semicontinuous.

Now $X_{(\leq d)} = \{x \in X : \dim Gx \leq d\}$ is closed in X, and π_d, which is the restriction of π to this set, sends closed G-subsets to closed subsets, so

$$\{y \in Y : \dim \pi_d^{-1}(y) \geq r\}$$

is closed in Y. Now

$$\{y \in Y : \dim_G \pi^{-1}(y) \geq r\} = \bigcup_d \{y \in Y : \dim \pi_d^{-1}(y) \geq d + r\}$$

which is closed.

6.2 Tame and wild

A) **Morphisms between module varieties.** Let A and B be finitely generated K-algebras and $d \in \mathbb{N}$.

Observe that there is a 1-1 correspondence between K-algebra homomorphisms $\theta : A \to M_d(B)$ up to conjugacy by an element of $\mathrm{GL}_d(B)$ and A-B-bimodules M which are free of rank d over B.

Now let $s \in \mathbb{N}$. Then a homomorphism θ induces a morphism of varieties

$$f : \mathrm{Mod}(B, s) \to \mathrm{Mod}(A, ds)$$

sending a K-algebra map $B \to M_s(K)$ to the composition $A \to M_d(B) \to M_d(M_s(K)) \cong M_{ds}(K)$. In terms of the corresponding A-B-bimodule M we have $M \otimes_B K_x \cong K_{f(x)}$ for all x.

B) **Remark.** Taking B to be a commutative and reduced, and $s = 1$, we have $\mathrm{Mod}(B, 1) = \operatorname{Spec} B$, so we can write this as

$$f : \operatorname{Spec} B \to \mathrm{Mod}(A, d).$$

Conversely any morphism of varieties of this form with B f.g. commutative and reduced comes from a homomorphism $A \to M_d(B)$.

Namely, first suppose that A is a free algebra $F = K\langle X_1, \dots, X_m\rangle$. Then a morphism $\operatorname{Spec} B \to \mathrm{Mod}(F, d) \cong M_d(K)^m$ corresponds to an algebra homomorphism $K[M_d(K)^m] \to B$. Since $K[M_d(K)^m]$ is a polynomial ring in indeterminates X_{ij}^k for $k = 1, \dots, m$ and $1 \le i, j \le d$, it is given by elements $b_{ij}^k \in B$, so a homomorphism $\theta : F \to M_d(B)$.

Now in general $A = F/I$ for some ideal I. A morphism $\operatorname{Spec} B \to \mathrm{Mod}(A, d)$ gives composition $\operatorname{Spec} B \to \mathrm{Mod}(A, d) \to \mathrm{Mod}(F, d)$, so a homomorphism $\theta : F \to M_d(B)$. Now for each homomorphism $B \to K$, the composition $F \to M_d(B) \to M_d(K)$ kills I. Since B is f.g. commutative and reduced, the intersections of the kernels of the homomorphisms $B \to K$ is zero. It follows that θ kills I, so induces a homomorphism $A \to M_n(B)$.

C) **Definition.** An algebra A is *tame* if for any d there are A-$K[T]$-bimodules $M_1, \dots, M_N$, finitely generated and free over $K[T]$, such that all but finitely many indecomposable A-modules of dimension $\le d$ are isomorphic to

$$M_i \otimes_{K[T]} K[T]/(T - \lambda)$$

for some i and some $\lambda \in K$.

Remarks. (i) Equivalently there are a finite number of morphisms $\mathbb{A}^1 \to \mathrm{Mod}(A, d)$ such that the union of the images meets all but finitely many orbits of indecopmposable modules.

(ii) In the definition of tame can delete the "but finitely many" by including additional maps $\mathbb{A}^1 \to \mathrm{Mod}(A, d)$ which are constant. In terms of bimodules it means including bimodules of the form $M = X \otimes_K K[T]$ where X is a given left A-module.

(iii) Any algebra of finite representation type is clearly tame by this definition. Sometimes the name tame representation type' is only used for algebras of infinite representation type.

[End of LECTURE 18 on 29 June 2020]

D) **Definition.** Let us say that a functor $F : B\text{-Mod} \to A\text{-Mod}$ is a *representation embedding* if

(i) X indecomposable $\Rightarrow F(X)$ indecomposable.

(ii) $F(X) \cong F(Y) \Rightarrow X \cong Y$.

(iii) F is naturally isomorphic to a tensor product functor $M \otimes_B -$ for an A-B-bimodule which is finitely generated projective over B, or equivalently it is an exact K-linear functor which preserves products and direct sums.

An algebra A is *wild* if there is a representation embedding $K\langle X, Y\rangle\text{-Mod} \to A\text{-Mod}$.

Remark. Traditionally one uses the categories of f.d. modules. The change to the categories of all modules is possible following Crawley-Boevey, Tame algebras and generic modules, 1991.

E) **Lemma.** (i) If I is an ideal in A then the natural functor $A/I\text{-Mod} \to A\text{-Mod}$ is a representation embedding.

(ii) For any n there is a representation embedding $K\langle X_1, \dots, X_n\rangle\text{-Mod} \to K\langle X, Y\rangle\text{-Mod}$.

Thus if A is wild there is a representation embedding $B\text{-Mod} \to A\text{-Mod}$ for any finitely generated algebra B.

Proof. (i) is trivial. For (ii) Let $B = K\langle X_1, \dots, X_n\rangle$. Consider the A-B-bimodule M corresponding to the homomorphism $\theta : A \to M_{n+2}(B)$ sending

X and Y to the matrices C and D,

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ X_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & X_2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & X_n & 1 & 0 \end{pmatrix}$$

These matrices are in S. Brenner, Decomposition properties of some small diagrams of modules, 1974. Thus $M \cong B^{n+2}$ as a right B-module, with the action of A given by the homomorphism. Suppose Z, Z' are B-modules and $f : M \otimes_B Z \to M \otimes_B Z'$. Then f is given by an $(n+2) \times (n+2)$ matrix of linear maps $Z \to Z'$, say $F = (f_{ij})$ such that $CF = FC$ and $DF = FD$. The condition $CF = FC$ gives

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & \dots \\ f_{21} & f_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & \dots \\ f_{21} & f_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

so $f_{i+1,j} = f_{i,j-1}$ for $1 \le i, j \le n+2$, where the terms are zero if i or j are out of range. This forces F to be constant on diagonals, and zero below the main diagonal,

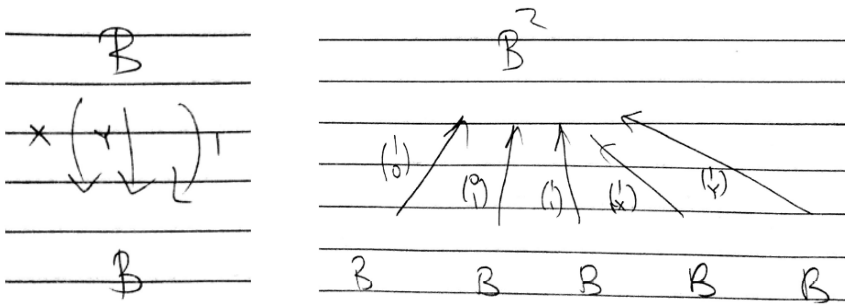
$$F = \begin{pmatrix} f_1 & f_2 & f_3 & \dots & f_{n+1} & f_{n+2} \\ 0 & f_1 & f_2 & \dots & f_n & f_{n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_1 & f_2 \\ 0 & 0 & 0 & \dots & 0 & f_1 \end{pmatrix}.$$

Now the condition $DF = FD$ gives $f_i = 0$ for $i > 1$ and $X_i f_1 = f_1 X_i$ for all i. Thus f_1 is a B-module map $Z \to Z'$.

If f is is an isomorphism, then so is f_1. Also, taking $Z = Z'$, if f is an idempotent endomorphism, then so is f_1. Thus Z is indecomposable, $f_1 = 0$ or 1, so $f = 0$ or 1, so $M \otimes_B Z$ is indecomposable.

F) **Examples.** (a) Path algebras of Dynkin and extended Dynkin quivers are tame. Other important classes of tame algebras are the tubular algebras and string algebras.

(b) Path algebras of other quivers are wild. For example, letting $B = K\langle X, Y\rangle$, for the path algebra A of the three arrow Kronecker quiver or five subspace quiver, consider the A-B-bimodule which is the direct sum of the indicated powers of B, with the natural action of B, and with the A-action given by the indicated matrices, acting as left multiplication.



(c) The algebra $A = K[x, y, z]/(x, y, z)^2$ is wild. (This argument is taken from Ringel, The representation type of local algebras, 1975) Given a $K\langle X, Y\rangle$-module V, we send it to the A-module V^2 with

$$x = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This is a tensor product functor. The image is contained in the subcategory C of A-modules M which are free over $K[z]/(z^2)$, or equivalently with $z^{-1}0_M = zM$. There is a functor from C to $K\langle X, Y\rangle$-modules, sending M to zM with X and Y given by the relations xz^{-1} and yz^{-1}. The composition

$$K\langle X, Y\rangle\text{-Mod} \xrightarrow{F} C \xrightarrow{G} K\langle X, Y\rangle\text{-Mod}$$

is isomorphic to the identity functor. Now if $M \in C$ and $G(M) = 0$ then $M = 0$. It follows that F is a representation embedding.

(d) The algebra $K[x, y]$ is wild (Gelfand and Ponomarev), in fact even the algebra $K[x, y]/(x^2, xy^2, y^3)$ is wild (Drozd).

G) **Drozd's Theorem.** Any finite dimensional algebra is tame or wild, and not both.

The proof of the first part is difficult, and outside the scope of these lectures. The second part follows from the following.

H) **Proposition.**
(i) If A is tame then $\dim_{\mathrm{GL}_d(K)} \mathrm{Mod}(A, d) \leq d$ for all d.
(ii) If A is wild then there is $r > 0$ with $\dim_{\mathrm{GL}_d(K)} \mathrm{Mod}(A, rd) \geq d^2$ for all d.

Proof. If A is wild, say given by a bimodule M, then since any f.g. projective $K\langle X,Y\rangle$-module is free, M is free of rank r over $K\langle X,Y\rangle$. Thus we have a map

$$\mathrm{Mod}(K\langle X,Y\rangle,d)\to \mathrm{Mod}(A,rd).$$

The inverse image of any orbit is an orbit, so

$$\dim_{\mathrm{GL}_{rd}(K)}\mathrm{Mod}(A,rd)\geq \dim_{\mathrm{GL}_d(K)}\mathrm{Mod}(K\langle X,Y\rangle,d).$$

Now $\dim \mathrm{Mod}(K\langle X,Y\rangle,d)=2d^2$, and every orbit in $\mathrm{Mod}(K\langle X,Y\rangle,d)$ has dimension $\leq d^2$. Thus there is some $s\leq d^2$ such that the set $\mathrm{Mod}(K\langle X,Y\rangle,d)_{(s)}$ consisting of the orbits of dimension s has dimension $2d^2$. Then

$$\dim_{\mathrm{GL}_d(K)}\mathrm{Mod}(K\langle X,Y\rangle,d)\geq 2d^2-s\geq d^2.$$

If A is tame, we can suppose that any d-dimensional module is isomorphic to a direct sum of

$$M_{i_1}\otimes K[T]/(T-\lambda_1)\oplus\cdots\oplus M_{i_m}\otimes K[T]/(T-\lambda_m)$$

where the sum of the ranks of the M_{i_j} is d. In particular $m\leq d$. This defines a map

$$\mathbb{A}^m\to \mathrm{Mod}(A,d).$$

The union of the images of these maps, over all possible choices is a constructible subset of $\mathrm{Mod}(A,d)$ of dimension $\leq d$ which meets every orbit, giving the claim.

[End of LECTURE 19 on 2 July 2020]

6.3 Degenerations of wild algebras are not tame

A) **Theorem (Geiß).** A degeneration of a wild algebra is not tame.

Thus, by Drozd's Tame and Wild Theorem, if an algebra degenerates to a tame algebra, it is tame.

Proof. By Proposition 6.2H, $\{x\in \mathrm{Alg}(r): K_x \text{ is wild}\}=\bigcup_d M_d$ where

$$M_d=\{x\in \mathrm{Alg}(r):\dim_{\mathrm{GL}_d(K)}\mathrm{Mod}(K_x,d)>d\}.$$

Recall Gabriel's Theorem 4.3D, that the image of a closed $\mathrm{GL}_d(K)$-subset under the map $\pi:\mathrm{Alg}\,\mathrm{Mod}(r,d)\to\mathrm{Alg}(r)$ is closed. Thus by Lemma 6.1E we have upper semicontinuity of the number of parameters of $\mathrm{GL}_d(K)$ acting on the fibres of π. Thus M_d is closed.

Suppose $x, y \in \mathrm{Alg}(r)$ and $y \in \overline{\mathrm{GL}_r(K)x}$. If K_x wild, then $x \in M_d$ for some d. Clearly M_d is a $\mathrm{GL}_r(K)$-subset, so the $\mathrm{GL}_r(K)$-orbit of x is contained in M_d, and hence so is the orbit closure. Thus $y \in M_d$, so K_y cannot be tame.

B) **Example.** The algebra

$$A = K\langle a, b\rangle/(a^2 - bab, b^2 - aba, (ab)^2, (ba)^2)$$

degenerates to

$$B = K\langle a, b\rangle/(a^2, b^2, (ab)^2, (ba)^2)$$

and B is known to be tame, hence so is A. The degeneration is given as follows. For $t \in K$ let $x^t \in \mathrm{Alg}(7)$ have basis $1, a, b, ab, ba, aba, bab$ with multiplication as indicated, and with $a^2 = tbab$, $b^2 = taba$, $(ab)^2 = 0$, $(ba)^2 = 0$. Then for $t \neq 0$ this is isomorphic to A, and for $t = 0$ it is B.

[At the moment, I know of no classification of the indecomposable modules for this algebra A.]

C) **Remark.** In the same way, a degeneration of an algebra with infinitely many isomorphism classes of modules of some dimension has the same property. Gabriel used this, together with the second Brauer-Thrall conjecture to prove that the set of algebras of finite representation type is open in $\mathrm{Alg}(r)$. (The second Brauer-Thrall conjecture says that if A is an algebra of infinite representation type, then there is some d (in fact infinitely many d) for which A has infinitely many non-isomorphic d-dimensional indecomposable modules. It has been proved by R. Bautista, On algebras of strongly unbounded representation type, Comment. Math. Helv. 1985, based on the fundamental paper R. Bautista, P. Gabriel, A.V. Roiter, L. Salmerón, Representation-finite algebras and multiplicative bases, Invent. Math. 1985.)

7 Kac's Theorem

7.1 Statement of Kac's Theorem

A) **Setup.** Let Q be a finite quiver. Recall that the *Ringel form* is the bilinear form on $\mathbb{Z}^{Q_0}$ given by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

The associated quadratic form is $q(\alpha) = \langle \alpha, \alpha \rangle$.

The associated symmetric bilinear form is $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

B) **Roots.** The *simple roots* are the coordinate vectors $\epsilon[i] \in \mathbb{Z}^{Q_0}$ with i a loopfree vertex. Thus $q(\epsilon[i]) = 1$. We display elements of $\mathbb{Z}^{Q_0}$ on the quiver, so for the quiver of type D_4

$$\begin{tikzcd} & 1 \arrow[d] & \\ & 2 \arrow[dl] \arrow[dr] & \\ 3 & & 4 \end{tikzcd}$$

the simple roots are

$$\epsilon[1] = \begin{smallmatrix} & 1 & \\ & 0 & \\ 0 & & 0 \end{smallmatrix}, \quad \epsilon[2] = \begin{smallmatrix} & 0 & \\ & 1 & \\ 0 & & 0 \end{smallmatrix}, \quad \epsilon[3] = \begin{smallmatrix} & 0 & \\ & 0 & \\ 1 & & 0 \end{smallmatrix}, \quad \epsilon[4] = \begin{smallmatrix} & 0 & \\ & 0 & \\ 0 & & 1 \end{smallmatrix}.$$

For each simple root there is a *reflection* $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ defined by

$$s_i(\alpha) = \alpha - (\alpha, \epsilon[i])\epsilon[i].$$

Thus $s_i(\alpha)$ is obtained from α by replacing the component α_i with $\sum_j \alpha_j - \alpha_i$ where j runs through the vertices adjacent to i. For example, there is a sequence of reflections

$$\epsilon[1] = \begin{smallmatrix} & 1 & \\ & 0 & \\ 0 & & 0 \end{smallmatrix} \xrightarrow{s_2} \begin{smallmatrix} & 1 & \\ & 1 & \\ 0 & & 0 \end{smallmatrix} \xrightarrow{s_3} \begin{smallmatrix} & 1 & \\ & 1 & \\ 1 & & 0 \end{smallmatrix} \xrightarrow{s_4} \begin{smallmatrix} & 1 & \\ & 1 & \\ 1 & & 1 \end{smallmatrix} \xrightarrow{s_2} \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 1 \end{smallmatrix}$$

Clearly $s_i^2 = 1$ and $s_i(\epsilon[i]) = -\epsilon[i]$. Reflections preserve the quadratic form, $q(s_i(\alpha)) = q(\alpha)$.

The *Weyl group* W is the subgroup of $\mathrm{Aut}(\mathbb{Z}^{Q_0})$ generated by the s_i.

A *real root* is a vector in $\mathbb{Z}^{Q_0}$ in the W-orbit of a simple root.

The *fundamental set* F is the set of non-zero $\alpha \in \mathbb{N}^{Q_0}$ whose support $\mathrm{Supp}(\alpha)$ is connected and with $(\alpha, \epsilon[i]) \leq 0$ for all vertices i.

An *imaginary root* is a vector in the W-orbit of $\pm\alpha$ for some $\alpha \in F$.

The set of *roots* in $\mathbb{Z}^{Q_0}$ is the union of the sets of real and imaginary roots. Note that thse sets do not depend on the orientation of Q.

Clearly α is a root iff $-\alpha$ is root. A root is *positive* if all components are ≥ 0 and *negative* if all are ≤ 0. In fact one can show that every root is positive or negative (either using Lie theory, or as a consequence of Kac's Theorem). Also any root has connected support (also as a consequence of Kac's Theorem).

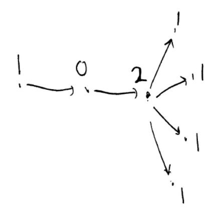
If α is a real root then $q(\alpha) = 1$ and if α is an imaginary root then $q(\alpha) \leq 0$ since $2q(\alpha) = (\alpha, \alpha) = \sum_{i \in Q_0} \alpha_i(\alpha, \epsilon[i]) \leq 0$ for $\alpha \in F$.

Examples. If Q is a Dynkin diagram then the fundamental set is empty, so all roots are real. One can show that α is a root if and only if $q(\alpha) = 1$. For example D_4 has 12 positive roots

$$\begin{smallmatrix} & 1 & \\ & 0 & \\ 0 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} & 0 & \\ & 1 & \\ 0 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} & 0 & \\ & 0 & \\ 1 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} & 0 & \\ & 0 & \\ 0 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & 1 & \\ & 1 & \\ 0 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} & 0 & \\ & 1 & \\ 1 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} & 0 & \\ & 1 & \\ 0 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & 1 & \\ & 1 & \\ 1 & & 0 \end{smallmatrix} \quad \begin{smallmatrix} & 1 & \\ & 1 & \\ 0 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & 0 & \\ & 1 & \\ 1 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & 1 & \\ & 1 & \\ 1 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 1 \end{smallmatrix}$$

If Q is an extended Dynkin diagram with radical vector δ, then the fundamental set is $F = \{m\delta : m \geq 1\}$. One can show that the roots are the α with $q(\alpha) \leq 1$; the imaginary roots are the $m\delta$ with $0 \neq m \in \mathbb{Z}$. For the Kronecker quiver with vertices 1 and 2, the roots are the vectors $(x, y) \in \mathbb{Z}^2$ with $|x - y| \leq 1$ and x, y not both zero.

For other quivers it is more complicated. For example, for the indicated quiver and dimension vector α,



one has that $q(\alpha) = 1$ but α is not a root, e.g. because it doesn't have connected support.

C) **Kac's Theorem.** The quiver Q has an indecomposable representation of dimension vector α (over an algebraically closed field K) if and only if α is a positive root.

If α is a positive real root there is a unique indecomposable representation, and if α is a positive imaginary root there are infinitely many non-isomorphic indecomposable representations.

More precisely, if α is a positive root and $\mathrm{Ind}(Q,\alpha)$ denotes the constructible subset of $\mathrm{Rep}(Q,\alpha)$ consisting of the indecomposable representations, then

$$\dim_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q,\alpha) = 1 - q(\alpha), \qquad \mathrm{top}_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q,\alpha) = 1.$$

Outline of proof. If α is a simple root, the assertion is clear.

The first step is to prove these formulas for α in the fundamental set. We do this in the next subsection using the methods we already developed.

It then suffices to prove that the numbers are unchanged by reflections. This is much harder, and uses many new ideas. This is done in the rest of the chapter.

For example, suppose there is an indecomposable representation whose dimension vector α is not a root. Choose α minimal with this property. Then no reflection $s_i(\alpha)$ can be smaller that α. If follows that $(\alpha, \epsilon[i]) \leq 0$ for all loopfree vertices i. But this inequality is automatic if there is a loop at i. Also $\mathrm{Supp}(\alpha)$ is connected since there is an indecomposable representation of dimension α. Thus α is in the fundamental set, so it is a root, a contradiction.

[End of LECTURE 20 on 6 July 2020]

7.2 Proof for the fundamental set

A) **Definition.** Let F' be the set of non-zero $\alpha \in \mathbb{N}^{Q_0}$ such that

$$q(\alpha) < q(\beta^{(1)}) + \cdots + q(\beta^{(r)})$$

whenever $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $r \geq 2$ and $\beta^{(1)}, \dots, \beta^{(r)}$ are nonzero elements of $\mathbb{N}^{Q_0}$.

B) **Lemma.** If $\alpha \in F$ then either $\alpha \in F'$ or $\mathrm{Supp}(\alpha)$ is extended Dynkin and $q(\alpha) = 0$.

Proof. We may assume $Q = \mathrm{Supp}(\alpha)$, and so Q is connected. If the condition fails, then $\sum(\alpha - \beta^{(i)}, \beta^{(i)}) = (\alpha,\alpha) - \sum(\beta^{(i)}, \beta^{(i)}) \geq 0$, so there is $0 \leq \beta \leq \alpha$, with $\beta \neq 0, \alpha$ and with $(\alpha - \beta, \beta) \geq 0$. Now

$$\sum_i (\alpha, \epsilon[i]) \beta_i (\alpha_i - \beta_i)/\alpha_i + \frac{1}{2} \sum_{i \neq j} (\epsilon[i], \epsilon[j]) \alpha_i \alpha_j \left(\frac{\beta_i}{\alpha_i} - \frac{\beta_j}{\alpha_j}\right)^2 \leq 0$$

since all terms in the sums are ≤ 0. Clearly the second sum can equally well be done over all pairs i, j, so this expands to

$$\sum_i (\alpha, \epsilon[i])\beta_i(\alpha_i - \beta_i)/\alpha_i + \frac{1}{2}\sum_{i,j}(\epsilon[i], \epsilon[j])\left(\frac{\beta_i^2\alpha_j}{\alpha_i} + \frac{\beta_j^2\alpha_i}{\alpha_j} - 2\beta_i\beta_j\right)$$

By symmetry, this becomes

$$\sum_i (\alpha, \epsilon[i])\beta_i(\alpha_i - \beta_i)/\alpha_i + \sum_{i,j}(\epsilon[i], \epsilon[j])\frac{\beta_j^2\alpha_i}{\alpha_j} - \sum_{i,j}(\epsilon[i], \epsilon[j])\beta_i\beta_j$$

$$= \sum_i (\alpha, \epsilon[i])\beta_i(\alpha_i - \beta_i)/\alpha_i + \sum_j (\alpha, \epsilon[j])\frac{\beta_j^2}{\alpha_j} - (\beta, \beta)$$

$$= \sum_i (\alpha, \epsilon[i])\beta_i - (\beta, \beta) = (\alpha, \beta) - (\beta, \beta) = (\alpha - \beta, \beta) \geq 0.$$

Thus all terms in the original sums are zero, so $\frac{\beta_j}{\alpha_j} = \frac{\beta_j}{\alpha_j}$ whenever $(\epsilon[i], \epsilon[j]) < 0$, i.e. if an arrow connects i with j. Thus α is a multiple of β. Now considering the terms in the first sum one sees that $(\alpha, \epsilon[i]) = 0$ for all i. This implies that Q is extended Dynkin.

C) **Lemma.** If $\alpha \in F'$, then the indecomposable representations form a dense subset $\mathrm{Ind}(KQ, \alpha)$ of $\mathrm{Rep}(Q, \alpha)$.

Proof. If $\alpha = \beta + \gamma$ $(\beta, \gamma \neq 0)$ then there is a map

$$\theta : \mathrm{GL}(\alpha) \times \mathrm{Rep}(Q, \beta) \times \mathrm{Rep}(Q, \gamma) \to \mathrm{Rep}(Q, \alpha), \quad (g, x, y) \mapsto g(x \oplus y).$$

This map is constant on the orbits of a free action of $H = \mathrm{GL}(\beta) \times \mathrm{GL}(\gamma)$, considered as a subgroup of $\mathrm{GL}(\alpha)$, given by

$$(h_1, h_2) \cdot (g, x_1, x_2) = (g(h_1, h_2)^{-1}, h_1x_1, h_2x_2),$$

so $\dim \overline{\mathrm{Im}(\theta)} \leq \dim \mathrm{LHS} - \dim H$. Now since $q(\alpha) = \dim \mathrm{GL}(\alpha) - \dim \mathrm{Rep}(Q, \alpha)$ one deduces that

$$\dim \mathrm{Rep}(Q, \alpha) - \dim \overline{\mathrm{Im}(\theta)} \geq q(\beta) + q(\gamma) - q(\alpha) > 0,$$

so $\overline{\mathrm{Im}(\theta)}$ is a proper subset of $\mathrm{Rep}(Q, \alpha)$. Thus $\mathrm{Im}(\theta)$ is a constructible subset of dimension strictly less than $\dim \mathrm{Rep}(Q, \alpha)$. Now as β and γ run over all possibilities, the union of these sets also has dimension strictly less than $\dim \mathrm{Rep}(Q, \alpha)$. Thus the complement $\mathrm{Ind}(KQ, \alpha)$ is dense.

D) **Notation.** Let $\mathrm{End}(\alpha) = \bigoplus_{i \in Q_0} M_{\alpha_i}(K)$.

Suppose that $\underline{\lambda} = (\lambda[i])$ is a collection of partitions, one for each vertex, where $\lambda[i]$ is a partition of α_i. We say that $\theta \in \mathrm{End}(\alpha)$ is of type $\underline{\lambda}$ if the maps $\theta_i \in M_{\alpha_i}(K)$ are nilpotent of type $\lambda[i]$ (so that $\lambda[i]_r$ is the number of Jordan blocks of size $\geq r$).

The zero element of $\mathrm{End}(\alpha)$ is of type $\underline{z}$, with $z[i]$ the partition $(\alpha_i, 0, \ldots)$.

We write $N_{\underline{\lambda}}$ for the set of $\theta \in \mathrm{End}(\alpha)$ of type $\underline{\lambda}$. It is a locally closed subset of $\mathrm{End}(\alpha)$.

If $\theta \in \mathrm{End}(\alpha)$ we define $\mathrm{Rep}_\theta = \{x \in \mathrm{Rep}(Q,\alpha) : \theta \in \mathrm{End}_{KQ}(K_x)\}$.

E) **Lemma.** (1) If $\theta \in N_{\underline{\lambda}}$ then $\dim \mathrm{Rep}_\theta = \sum_{a:i\to j} \sum_r \lambda[i]_r \lambda[j]_r$

(2) $\dim N_{\underline{\lambda}} = \dim \mathrm{GL}(\alpha) - \sum_{i \in Q_0} \sum_r \lambda[i]_r \lambda[i]_r$.

Proof. It is easy to check that if $f \in \mathrm{End}(V)$ and $g \in \mathrm{End}(W)$ are nilpotent endomorphisms of type μ and ν, then $\dim\{h : V \to W \mid gh = hf\} = \sum_r \mu_r \nu_r$. Part (1) follows immediately. For (2) note that $N_{\underline{\lambda}}$ is an orbit for the conjugation action of $\mathrm{GL}(\alpha)$ on $\mathrm{End}(\alpha)$, so if $\theta \in N_\lambda$ then

$$\dim N_{\underline{\lambda}} = \dim \mathrm{GL}(\alpha) - \dim\{g \in \mathrm{GL}(\alpha) \mid g\theta = \theta g\}$$

$$= \dim \mathrm{GL}(\alpha) - \dim\{g \in \mathrm{End}(\alpha) \mid g\theta = \theta g\}$$
$$= \dim \mathrm{GL}(\alpha) - \sum_i \sum_r \lambda[i]_r \lambda[i]_r.$$

F) **Notation.** Let $g = \dim \mathrm{GL}(\alpha) = \sum_{i \in Q_0} \alpha_i^2$. If $x \in \mathrm{Rep}(Q,\alpha)$, then its orbit has dimension $g - \dim \mathrm{End}_{KQ}(K_x)$.

Let $I = \mathrm{Ind}(KQ,\alpha) = \bigcup_{s<g} I_{(s)}$. Recall that $I_{(s)}$ is locally closed in $\mathrm{Rep}(Q,\alpha)$. Thus $I_{(g-1)}$ is the set of $x \in \mathrm{Rep}(\alpha)$ such that K_x is a brick (has 1-dimensional endomorphism algebra).

G) **Lemma.** If $\alpha \in F'$ and $s < g-1$ then $\dim_{\mathrm{GL}(\alpha)} I_{(s)} < 1 - q(\alpha)$.

Proof. Let N be the set of non-zero nilpotent $\theta \in \mathrm{End}(\alpha)$, so also the union $\bigcup_{\underline{\lambda} \neq \underline{z}} N_{\underline{\lambda}}$.

$RN = \{(x,\theta) \in \mathrm{Rep}(Q,\alpha) \times N \mid \theta \in \mathrm{End}_{KQ}(K_x)\} = \bigcup_{\underline{\lambda} \neq \underline{z}} RN_{\underline{\lambda}}$.

$I_{(s)}N = \{(x,\theta) \in I_{(s)} \times N \mid \theta \in \mathrm{End}_{KQ}(K_x)\} \subseteq RN$.

We show that $\dim RN < g - q(\alpha)$. It suffices to prove that $\dim RN_{\underline{\lambda}} < g - q(\alpha)$ for all $\underline{\lambda} \neq \underline{z}$. Let $\pi : RN_{\underline{\lambda}} \to N_{\underline{\lambda}}$ be the projection. Now $\pi^{-1}(\theta) = \mathrm{Rep}_\theta$ is of constant dimension, so

$$\dim RN_{\underline{\lambda}} \leq \dim N_{\underline{\lambda}} + \dim \mathrm{Rep}_\theta = g - \sum_r q(\lambda_r) < g - q(\alpha),$$

since $\alpha = \sum_r \lambda_r$, and at least two λ_r are non-zero since $\underline{\lambda} \neq \underline{z}$. Here λ_r denotes the dimension vector whose components are the $\lambda[i]_r$.

Now suppose that $s < g-1$. If $x \in I_{(s)}$ then K_x is indecomposable and not a brick, so has a non-zero nilpotent endomorphism. Thus the projection $I_{(s)}N \xrightarrow{\pi} I_{(s)}$ is onto. Now

$$\dim \pi^{-1}(x) = \dim \mathrm{End}_{KQ}(K_x) \cap N = \dim \mathrm{Rad}\,\mathrm{End}_{KQ}(K_x) = g - s - 1.$$

Thus $\dim I_{(s)} = \dim I_{(s)}N - (g-s-1) \leq \dim RN - (g-s-1) < s+1-q(\alpha)$.

H) **Lemma.** For $\alpha \in F'$ the set $I_{(g-1)}$ of bricks is a non-empty open subset of $\mathrm{Rep}(Q,\alpha)$.

Proof. It is the same as the set $\mathrm{Rep}(Q,\alpha)_{(\geq g-1)}$, so it is open. Now I is dense and constructible in $\mathrm{Rep}(Q,\alpha)$, so

$$\dim I = \dim \mathrm{Rep}(Q,\alpha) = \sum_{a \in Q_1} \alpha_{h(a)}\alpha_{t(a)} = g - q(\alpha).$$

On the other hand, if $s < g-1$ we have

$$\dim I_{(s)} = \dim_G I_{(s)} + s \leq 1 - q(\alpha) + s < g - q(\alpha)$$

so $I_{(g-1)}$ must be non-empty.

I) **Theorem.** If $\alpha \in F$ then we have $\dim_{\mathrm{GL}(\alpha)} \mathrm{Ind}(KQ,\alpha) = 1 - q(\alpha)$ and $\mathrm{top}_{\mathrm{GL}(\alpha)} \mathrm{Ind}(KQ,\alpha) = 1$.

Proof. If $\alpha \in F'$ it follows from above, since bricks dominate. Otherwise we may assume that Q is extended Dynkin and use the classification.

THE REST OF THIS CHAPTER IS NON-EXAMINABLE.

7.3 Counting representations over $\mathbb{F}_q$

Besides Kac's original papers, especially V. Kac, Root systems, representations of quivers and invariant theory. Invariant theory (Montecatini, 1982), 1983, in this subsection we cover material from J. Hua, Counting representations of quivers over finite fields, 2000. [I use the conjugate partition to Hua, so some formulas look different.] I also used notes of A. Hubery.

A) **Definition.** Let Q be a quiver. For notational simplicity we assume that $Q_0 = \{1, 2, \dots, n\}$, so dimension vectors are elements of $\mathbb{N}^n$. We consider the representations of Q over a finite field $K = \mathbb{F}_q$ with q elements.

Let $r(\alpha, q)$ be the number of isomorphism classes of representations of dimension vector α. Let $i(\alpha, q)$ be the number of isomorphism classes of indecomposable representations of dimension vector α. We consider the generating function

$$\sum_{\alpha\in\mathbb{N}^n} r(\alpha, q)X^\alpha \in \mathbb{Z}[[X_1, \dots, X_n]]$$

where $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$.

Examples. For the quiver consisting of a vertex 1 and no arrows, there is a unique representation of each dimension, so this is

$$1 + X_1 + X_1^2 + X_1^3 + \cdots = 1/(1 - X_1)$$

For the quiver $1 \to 2$ a dimension vector is a pair (a, b) and the number of representations is $r((a, b), q) = 1 + \min(a, b)$. So the generating function is

$$\sum_{a,b\geq 0} (1 + \min(a, b))X_1^a X_2^b.$$

This is

$$\sum_{m\geq 0}(1 + m)X_1^m X_2^m + \sum_{m\geq 0,k>0} (1 + m)X_1^{m+k} X_2^m + \sum_{m\geq 0,k>0} (1 + m)X_1^m X_2^{m+k}$$

This works out as

$$\frac{1}{(1 - X_1X_2)^2} + \sum_{k>0} \frac{X_1^k}{(1 - X_1X_2)^2} + \sum_{k>0} \frac{X_2^k}{(1 - X_1X_2)^2}$$
$$= \frac{1}{(1 - X_1X_2)^2}\left(1 + \frac{X_1}{1 - X_1} + \frac{X_2}{1 - X_2}\right)$$
$$= \frac{1}{(1 - X_1)(1 - X_2)(1 - X_1X_2)}.$$

B) **Proposition.** We have

$$\sum_{\alpha\in\mathbb{N}^n} r(\alpha, q)X^\alpha = \prod_{\beta\in\mathbb{N}^n} (1 - X^\beta)^{-i(\beta,q)}.$$

Proof. This follows from Krull-Remak-Schmidt, since if M_i $(i \in I)$ are a complete set of non-isomorphic indecomposable representations, we can write both sides as

$$\prod_{i\in I}(1 + X^{\underline{\dim} M_i} + X^{2\underline{\dim} M_i} + \dots).$$

C) **Notation.** Recall that $K = \mathbb{F}_q$. Let

$$X = \mathrm{Rep}(Q, \alpha) = \prod_{a \in Q_1} M_{\alpha_{h(a)} \times \alpha_{t(a)}}(K)$$

and

$$G = \mathrm{GL}(\alpha) = \prod_{i \in Q_0} \mathrm{GL}_{\alpha_i}(K).$$

Thus $r(\alpha, q) = |X/G|$, the number of orbits of G on X.

Recall that Burnside's Lemma says that if a group G acts on a finite set X, then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where X^g is the fixed points of g on X. Thus

$$|X/G| = \sum_{g \in G/\sim} \frac{|X^g|}{|C_G(g)|}$$

where the sum is over a set of representatives of the conjugacy classes $G/\sim$, and $C_G(g)$ is the centraliser of g in G.

D) **Lemma.** The conjugacy classes in G are in 1-1 correspondence with collections (M_i) of $K[T, T^{-1}]$-modules, with M_i of dimension α_i, up to isomorphism. For g in the corresponding conjugacy class, one has

$$X^g \cong \bigoplus_{a \in Q_1} \mathrm{Hom}_{K[T,T^{-1}]}(M_{t(a)}, M_{h(a)})$$

and

$$C_G(g) \cong \prod_{i \in Q_0} \mathrm{Aut}_{K[T,T^{-1}]}(M_i).$$

Proof. An element of $\mathrm{GL}_d(K)$ turns K^d into a $K[T, T^{-1}]$-module, and conjugate elements correspond to isomorphic modules. The rest follows.

E) **Notation.** Recall that the finite-dimensional indecomposable $K[T, T^{-1}]$-modules are the modules $K[T, T^{-1}]/(f^r)$ where $r \geq 1$ and f runs through the set Φ' of monic irreducible polynomials in $K[T]$, excluding the polynomial T. We write C_f for the full subcategory consisting of the direct sums of copies modules of the form $K[T, T^{-1}]/(f^r)$ with $r \geq 1$. Given a partition λ we define

$$M_f(\lambda) = \bigoplus_{i \geq 1} \left(K[T, T^{-1}]/(f^r)\right)^{\oplus \lambda_i - \lambda_{i+1}}$$

so the number of copies of $K[T,T^{-1}]/(f^r)$ is the number of columns of length r in the Young diagram for λ. These modules parameterize the isomorphism classes in C_f.

F) **Lemma.**
(i) $\dim M_f(\lambda) = d\ |\lambda|$ where $|\lambda| = \lambda_1 + \lambda_2 + \dots$ is the weight of λ and d is the degree of f.
(ii) We have

$$\dim \operatorname{Hom}(M_f(\lambda), M_g(\mu)) = \begin{cases} 0 & (f \neq g) \\ d\langle \lambda, \mu \rangle & (f = g) \end{cases}$$

where by definition $\langle \lambda, \mu \rangle = \sum_i \lambda_i \mu_i$.
(iii) $|\operatorname{Aut}(M_f(\lambda))| = q^{d\langle \lambda,\lambda\rangle} b_\lambda(q^{-d})$, where $b_\lambda(T) = \prod_{i\geq 1}\prod_{j=1}^{\lambda_i - \lambda_{i+1}}(1-T^j)$.

Proof. (iii) For all $i > 0$, the module $M_f(\lambda)$ has $\lambda_i - \lambda_{i+1}$ copies of the indecomposable module $K[T,T^{-1}]/(f^i)$ of length i. Thus

$$\operatorname{End}(M_f(\lambda))/\operatorname{Rad}\operatorname{End}(M_f(\lambda)) \cong \prod_i M_{\lambda_i - \lambda_{i+1}}(\mathbb{F}_{q^d}).$$

Thus

$$\dim \operatorname{Rad}\operatorname{End}(M_f(\lambda)) = d\left(\langle \lambda, \lambda\rangle - \sum_i (\lambda_i - \lambda_{i+1})^2\right).$$

Then

$$|\operatorname{Aut}(M_f(\lambda))| = |\operatorname{Rad}\operatorname{End}(M_f(\lambda)|. \prod_i |\operatorname{GL}_{\lambda_i-\lambda_{i+1}}(\mathbb{F}_{q^d})|$$

and $|\operatorname{GL}_n(\mathbb{F}_q)| = (q^n-1)(q^n-q)(q^n-q^2)\dots(q^n-q^{n-1})$.

G) **Theorem (Kac-Stanley, Hua).** We have

$$\sum_{\alpha\in\mathbb{N}^n} r(\alpha,q)X^\alpha = \prod_{d=1}^{\infty} P(X_1^d,\dots,X_n^d,q^d)^{\phi'_d(q)}$$

where $\phi'_d(q)$ is the number of polynomials in Φ' of degree d, so the number of monic irreducible polynomials in $K[T]$ of degree d, excluding T, and

$$P(X_1,\dots,X_n,q) = \sum_{\underline{\lambda}} \frac{\prod_{a\in Q_1} q^{\langle \lambda[t(a)],\lambda[h(a)]\rangle}}{\prod_{i\in Q_0} q^{\langle \lambda[i],\lambda[i]\rangle} b_\lambda(q^{-1})} X_1^{|\lambda[1]|}\dots X_n^{|\lambda[n]|}$$

where the sum is over collections of partitions $\underline{\lambda} = (\lambda[1],\dots,\lambda[n])$.

Proof. Burnside's Lemma and Lemma 7.3D give

$$r(\alpha,q)=\sum_{(M_i)}\frac{\prod_{a\in Q_1}|\operatorname{Hom}_{K[T,T^{-1}]}(M_{t(a)},M_{h(a)})|}{\prod_{i\in Q_0}|\operatorname{Aut}_{K[T,T^{-1}]}(M_i)|}$$

where the sum is over collections (M_i) of dimension α up to isomorphism. Thus the generating function is

$$\sum_{\alpha\in\mathbb{N}^n} r(\alpha,q)X^\alpha=\sum_{(M_i)}\frac{\prod_{a\in Q_1}|\operatorname{Hom}_{K[T,T^{-1}]}(M_{t(a)},M_{h(a)})|}{\prod_{i\in Q_0}|\operatorname{Aut}_{K[T,T^{-1}]}(M_i)|}X_1^{\dim M_1}\dots X_n^{\dim M_n}$$

where the sum is over all collections (M_i) of $K[T,T^{-1}]$-modules, up to isomorphism.

Since every $K[T,T^{-1}]$-module can be written uniquely as a direct sum of modules in C_f $(f\in\Phi')$ and there are no non-zero maps between the different C_f we obtain

$$\sum_\alpha r(\alpha,q)X^\alpha=\prod_{f\in\Phi}P_f$$

where

$$P_f=\sum_{(M_i)\in C_f}\frac{\prod_{a\in Q_1}|\operatorname{Hom}_{K[T,T^{-1}]}(M_{t(a)},M_{h(a)})|}{\prod_{i\in Q_0}|\operatorname{Aut}_{K[T,T^{-1}]}(M_i)|}X_1^{\dim M_1}\dots X_n^{\dim M_n}$$

where the sum is over all collections (M_i) in C_f, up to isomorphism. Now by Lemma 7.3F, if $f\in\Phi$ is of degree d, then $P_f=P(X_1^d,\dots,X_n^d,q^d)$.

H) **Notation.** The power series $P(X_1,\dots,X_n,q)\in\mathbb{Q}(q)[[X_1,\dots,X_n]]$ has constant term 1, so there are $h(\alpha,q)\in\mathbb{Q}(q)$ with

$$\log P(X_1,\dots,X_n,q)=\sum_{\alpha\in\mathbb{N}^n}\frac{h(\alpha,q)}{\overline{\alpha}}X^\alpha$$

where $\overline{\alpha}$ is the highest common factor of the coefficients of α.

I) **Corollary.** Letting $e(\alpha,q)=\sum_{d|\overline{\alpha}}d\phi'_d(q)h(\alpha/d,q^d)$, we have

$$\log(\sum_{\alpha\in\mathbb{N}^n}r(\alpha,q)X^\alpha)=\sum_{\alpha\in\mathbb{N}^n}\frac{e(\alpha,q)}{\overline{\alpha}}X^\alpha$$

and

$$e(\alpha,q)=\sum_{d|\overline{\alpha}}\frac{\overline{\alpha}}{d}i(\alpha/d,q),\quad i(\alpha,q)=\frac{1}{\overline{\alpha}}\sum_{d|\overline{\alpha}}\mu(d)e(\alpha/d,q).$$

Proof. Observe that

$$\log P(X_1^d, \dots, X_n^d, q^d) = \sum_{\alpha \in \mathbb{N}^n} \frac{h(\alpha, q^d)}{\overline{\alpha}} X^{d\alpha}$$

so the theorem gives the first part. Then by the proposition

$$\sum_{\alpha \in \mathbb{N}^n} \frac{e(\alpha, q)}{\overline{\alpha}} X^\alpha = \log(\sum_{\alpha \in \mathbb{N}^n} r(\alpha, q) X^\alpha) = \sum_{\beta \in \mathbb{N}^n} i(\beta, q) \log \frac{1}{1 - X^\beta}$$

$$= \sum_{\beta \in \mathbb{N}^n} \sum_{d=1}^{\infty} \frac{i(\beta, q)}{d} X^{d\beta}.$$

Comparing coefficients of X^α gives one equality. The other follows by Möbius inversion.

J) **Lemma.** $\phi'_n(q) \in \mathbb{Q}[q]$.

Proof. Any monic irreducible polynomial in $\mathbb{F}_q[T]$ of degree d corresponds to d elements which lie in $\mathbb{F}_{q^d}$ but not in any intermediate field between $\mathbb{F}_q$ and $\mathbb{F}_{q^d}$. Thus if the are $\phi_d(q)$ such polynomials, then

$$q^n = \sum_{d|n} d\phi_d(q).$$

By induction on d, or Möbius inversion, which gives

$$\phi_d(q) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d},$$

one deduces that $\phi_d(q) \in \mathbb{Q}[q]$. Then also $\phi'_d(q) \in \mathbb{Q}[q]$ since

$$\phi'_d(q) = \begin{cases} q - 1 & (d = 1) \\ \phi_d(q) & (d > 1) \end{cases}$$

(or $\phi'_d(q) = \frac{1}{n} \sum_{d|n} \mu(d)(q^{n/d} - 1)$).

K) **Corollary.** $i(\alpha, q)$ and $r(\alpha, q) \in \mathbb{Q}[q]$, and are independent of the orientation of Q.

Proof. Corollary 7.3I shows that $i(\alpha, q) \in \mathbb{Q}(q)$.

It takes integer values for q any prime power, so it must be a polynomial. (Note that you cannot deduce that it is in $\mathbb{Z}[q]$, for example $\frac{1}{2}q(q + 1)$.)

It is independent of the orientation since $P(X_1, \dots, X_n, q)$ only involves an arrow a through the bracket $\langle \lambda[t(a)], \lambda[h(a)] \rangle$, and this bracket is symmetric.

By Corollary 7.3I, we then have $e(\alpha, q) \in \mathbb{Q}[q]$ and then $r(\alpha, q) \in \mathbb{Q}[q]$ since

$$\sum_{\alpha} r(\alpha, q) X^{\alpha} = \exp\left(\sum_{\alpha \in \mathbb{N}^n} \frac{e(\alpha, q)}{\overline{\alpha}} X^{\alpha} \right)$$

7.4 Field extensions and absolutely indecomposable representations

A) **Setup.** Let L/K be a field extension. We consider the relationship between representations of Q over K and over L.

More generally we consider a K-algebra A and $A^L = A \otimes L$. (Unadorned tensor products are over K.) Since L is commutative, A^L-modules can be thought of as A-L-bimodules (with K acting centrally).

Any finite-dimensional A-module M gives a finite-dimensional A^L-module $M^L = M \otimes L$.

B) **Lemma.** We have $\mathrm{Hom}_{A^L}(M^L, (M')^L) \cong \mathrm{Hom}_A(M, M') \otimes L$. Moreover $\mathrm{top}\,\mathrm{End}_{A^L}(M^L) \cong \mathrm{top}\big((\mathrm{top}\,\mathrm{End}_A(M))^L\big)$.

Proof. We use that M is finite dimensional. There is a natural map

$$\mathrm{Hom}_A(M, M') \otimes L \to \mathrm{Hom}_{A^L}(M^L, (M')^L)$$

which is easily seen to be injective. We need to show it is onto. Say $\theta \in \mathrm{Hom}_{A^L}(M^L, (M')^L)$. Choose a basis ξ_i of L over K. Define θ_i by $\theta(m \otimes 1) = \sum_i \theta_i(m) \otimes \xi_i$. Clearly $\theta_i \in \mathrm{Hom}_A(M, M')$ and since M is finite-dimensional, only finitely many θ_i are non-zero. Then θ is the image of the element $\sum_i \theta_i \otimes \xi_i$.

For the last part we just observe that $(\mathrm{Rad}\,\mathrm{End}_A(M)) \otimes L$ is a nilpotent ideal in $\mathrm{End}_A(M) \otimes L \cong \mathrm{End}_{A^L}(M^L)$.

C) **Lemma.** Assume L/K is finite of degree n. Any finite-dimensional A^L-module N gives a finite-dimensional A-module N_K by restriction. If M is an A-module then $(M^L)_K \cong M^n$. If M, M' are A-modules and $M^L \cong (M')^L$, then $M \cong M'$.

Proof. Clear. For the last part use Krull-Remak-Schmidt, since $M^n \cong (M')^n$.

D) **Lemma.** Assume L/K is a finite separable extension. Then $\mathrm{top}\,\mathrm{End}(M^L) \cong (\mathrm{top}\,\mathrm{End}(M))^L$. If N is an A^L-module, then N is a direct summand of $(N_K)^L$.

Any indecomposable A^L-module N arises as a direct summand summand of an induced module M^L with M indecomposable. The module M is unique up to isomorphism.

Proof. The first part holds since, for a separable field extension, inducing up a semisimple K-algebra gives a semisimple L-algebra.

Since $L \otimes L$ is a semisimple algebra, the multiplication map $L \otimes L \to L$ is a split epimorphism of L-L-bimodules, so L is a direct summand of $L \otimes L$. It follows that if N is an A^L-module, then N is a direct summand of $(N_K)^L$.

If N arises as a summand of M^L and $(M')^L$ with M, M' indecomposable, then N_K is a summand of M^n and $(M')^n$. By Krull-Remak-Schmidt this implies $M \cong M'$.

E) **Lemma.** Assume L/K is Galois of degree n with group G. The map

$$L \otimes L \to \bigoplus_{g \in G} L, \quad a \otimes b \mapsto (ag(b))_g$$

is an isomorphism as K-algebras, and gives an isomorphism of L-L-bimodules $L \otimes L \cong \bigoplus_{g \in G} L_g$, where the L-action on the right is given by restriction via g.

Example. $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

Proof. I am indebted to Andrew Hubery for his help in many places in these notes, and especially with this lemma. The map is a map of K-algebras, and also a bimodule map for the indicated action. Thus we need it to be a bijection.

By the theorem of the primitive element we can write $L \cong K[x]/(f(x))$ with $f(x)$ irreducible over K. Let x correspond to an element $\alpha \in L$. Since G acts faithfully on L and α generates L over K, the elements $g(\alpha)$ are distinct, and in $L[x]$ we can factorize $f(x) = \prod_{g \in G}(x - g(\alpha))$.

Now we can identify $L \otimes L \cong L \otimes K[x]/(f(x)) \cong L[x]/(f(x))$, and the map sends elements of L (identified with $L \otimes 1$) to themselves, and x (identified with $1 \otimes \alpha$) to $(g(\alpha))_g$, so it sends any polynomial $p(x) \in L[x]$ to $(p(g(\alpha)))_g$. Thus if $p(x)$ is sent to zero, then $p(g(\alpha)) = 0$ for all $g \in G$. Thus $p(x)$ is divisible by $f(x)$. Thus $p(x) = 0$ in $L \otimes L$. Thus the map is injective, hence by dimensions a bijection.

F) **Theorem.** Suppose L/K is Galois with group G.

Then induction and restriction give a 1-1 correspondence between isomorphism classes of

- indecomposable A-modules M, and

- G-orbits of indecomposable A^L-modules.

Explicitly if M is an indecomposable A-module then the indecomposable summands of M^L form an orbit under G, perhaps occuring with multiplicity, and if N is an indecomposable A^L-module, then $N_K \cong M^r$ for some indecomposable A-module M and some r, and the modules in the orbit of N give the same module M.

Example. For the field extension $\mathbb{C}/\mathbb{R}$:

A	A^L	indec A-mods	G-orbits of indec A^L-mods
$A = \mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\{\mathbb{C}\}$
$A = \mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}$	$\{\mathbb{C}_1, \mathbb{C}_2\}$
$A = \mathbb{H}$	$M_2(\mathbb{C})$	$\mathbb{H}$	$\{\mathbb{C}^2\}$

Proof. The key formula is that if N is an A^L-module, then

$$(N_K)^L \cong N \otimes_L (L \otimes_K L) \cong N \otimes_L (\bigoplus_{g \in G} L_g) \cong \bigoplus_{g \in G} N_g.$$

where N_g is the A^L-module obtain from N with the L-action given by restriction via g.

Induction. If N is one of the summands of M^L, then N_K is a summand of $(M^L)_K \cong M^n$, so $N_K \cong M^r$, some r. Then $(M^L)^r \cong (N_K)^L \cong \bigoplus_{g \in G} N_g$.

Restriction. If $N_K = \bigoplus_i M_i$, then $\bigoplus_i M_i^L \cong (N_K)^L \cong \bigoplus_{g \in G} N_g$. Thus

$$M_i^L \cong \bigoplus_{g \in S_i} N_g$$

where the S_i are a partition of G. Then

$$M_i^n \cong (M_i^L)_K \cong N_K^{|S_i|}.$$

Thus N_K is isomorphic to a direct sum of copies of M_i, so all the summands M_i are isomorphic, say to M, and the sets S_i all have the same size s with $s|n$. Then $M^{n/s} \cong N_K$.

G) **Definition.** We say that an A-module M is *absolutely indecomposable* if M^L is an indecomposable A^L-module for any field extension L/K (equivalently for the algebraic closure $\overline{K}/K$).

Clearly any absolutely indecomposable module is indecomposable, but not every indecomposable module is absolutely indecomposable.

If $\operatorname{top}\operatorname{End}(M) \cong K$ then M is absolutely indecomposable. If the base field K is finite, then the converse holds. Namely, if M is absolutely indecomposable, then $\operatorname{top}\operatorname{End}(M)$ is a division algebra. But it is also finite dimensional over the finite field K, so by Wedderburn's Theorem it must be a field L. Now the extension L/K is necessarily Galois. Then as an algebra $L \otimes L \cong L \times \cdots \times L$ ($\dim L$ copies), showing that M^L has $\dim L$ indecomposable summands, so $L = K$.

H) **Corollary.** Suppose that A is an algebra over $K = \mathbb{F}_q$. Consider the field extension L/K where $L = \mathbb{F}_{q^n}$, and let $s|n$. Then induction and restriction give a 1-1 correspondence between isomorphism classes of

- indecomposable A-modules M with $\operatorname{top}\operatorname{End}(M) \cong \mathbb{F}_{q^s}$, and

- G-orbits of size s of absolutely indecomposable A^L-modules.

Explicitly M^L is the direct sum of one copy of each of the modules in the orbit, and if N is in the orbit then $N_K \cong M^{n/s}$.

Proof. If $\dim \operatorname{top}\operatorname{End}(M) \cong \mathbb{F}_{q^s}$ then $\operatorname{top}\operatorname{End}(M) \otimes_K \mathbb{F}_{q^s} \cong (\mathbb{F}_{q^s})^s$, so $\operatorname{top}\operatorname{End}(M) \otimes_K L \cong L^s$, so M^L splits as a direct sum of s non-isomorphic indecomposables with $\operatorname{top}\operatorname{End}(N) \cong L$.

Conversely if N comes from an orbit of size s of absolutely indecomposables, then $N_K \cong M^r$ for some indecomposable A-module M and some r. Now $(M^L)^r \cong (N_K)^L \cong \bigoplus_{g\in G} N_g$. Suppose $\operatorname{top}\operatorname{End}(M) = D$. Since there are no finite division algebras, D is a field. Thus $\operatorname{top}\operatorname{End}(M^L) = D^L$ is commutative. Thus M^L consists of one copy of each indecomposable in the orbit of N, so $r = n/s$. Then also $D^L \cong L^s$. Thus $\dim D = s$, so $D \cong \mathbb{F}_{q^s}$.

We return to representations of quivers.

I) **Definition.** We write $a(\alpha, q)$ for the number of absolutely indecomposable representations of Q of dimension α over $\mathbb{F}_q$.

J) **Corollary ([Hua, Corollary 4.2]).** We have

$$\sum_{d|\alpha} \frac{1}{d} i(\alpha/d, q) = \sum_{d|\alpha} \frac{1}{d} a(\alpha/d, q^d).$$

Proof. If M is an indecomposable representation of Q of dimension α, then for each vertex i, the vector space at i becomes a module for $\operatorname{End}(M)$. It follows that if $\operatorname{top}\operatorname{End}(M) = \mathbb{F}_{q^s}$, then $s|\overline{\alpha}$. Thus apply Corollary 7.4H with $n = \overline{\alpha}$.

Namely take n to be the hcf of components of α. An indecomposable of dimension α/r over $\mathbb{F}_q$ with $\operatorname{top}\operatorname{End}(M) \cong \mathbb{F}_{q^s}$ contributes $1/r$ to the LHS for $d = r$. For any n divisible by s it corresponds to an orbit of size s of absolutely indecomposable reps over $\mathbb{F}_{q^n}$ of dimension α/rs. This contributes $1/r$ to the term $d = rs$ on the RHS.

K) **Corollary.** We have

$$i(\alpha, q) = \sum_{d|\alpha} \frac{1}{d} \sum_{r|d} \mu(\frac{d}{r}) a(\frac{\alpha}{d}, q^r), \quad a(\alpha, q) = \sum_{d|\alpha} \frac{1}{d} \sum_{r|d} \mu(r) i(\frac{\alpha}{d}, q^r)$$

where μ is the Möbius function.

Proof. The formula

$$\sum_{d|\alpha} \frac{1}{d} i(\alpha/d, q) = \sum_{d|\alpha} \frac{1}{d} a(\alpha/d, q^d)$$

can be written for $\alpha = n\beta$ with β coprime as follows (multiplying it by n)

$$h(n) = \sum_{d|n} \frac{n}{d} i(n\beta/d, q) = \sum_{d|n} \frac{n}{d} a(n\beta/d, q^d)$$

The first of these can be written as

$$\sum_{e|n} ei(e\beta, q).$$

Then by Möbius inversion

$$ni(n\beta, q) = \sum_{d'|n} \mu(\frac{n}{d'}) h(d')$$

$$= \sum_{d'|n} \mu(\frac{n}{d'}) \sum_{r|d'} \frac{d'}{r} a(d'\beta/r, q^r).$$

Now rewrite this as a sum over $r|d|n$ where $d/r = n/d'$, and it becomes

$$\sum_{d|n} \sum_{r|d} \mu(\frac{d}{r}) \frac{n}{d} a(n\beta/d, q^r).$$

Giving the first formula. The second formula follows by another Möbius inversion.

7.5 Passing between finite and algebraically closed fields

A) **Definition.** Let X be a variety over an algebraically closed field K, and let k be a subfield of K. There is the notion of X being *defined over* k. For example if X is a (quasi) affine or projective variety in $\mathbb{A}^n$ or $\mathbb{P}^n$ this means that X can be defined using polynomials with coefficients in k. (A more formal definition might be to require that X is isomorphic to $(Y^K)_{\text{red}}$ for some reduced algebraic k-scheme Y.)

Given an intermediate field $k \subseteq L \subseteq K$, we write $X(L)$ for the L-valued points of X. In the (quasi) affine or projective case, this is the points in L^n or points of the form $[x_0 : \cdots : x_n]$ with $x_i \in L$, not all zero, satisfying the polynomial conditions.

If K is an algebraically closed field of characteristic $p > 0$, and $q = p^s$, then it contains a copy of $\mathbb{F}_q = \{x \in K : x^q = x\}$, and any field $\mathbb{F}_{q^r}$ is an intermediate field $\mathbb{F}_q \subseteq \mathbb{F}_{q^r} \subseteq K$.

B) **Definition.** The *Zeta function* of a variety X defined over $\mathbb{F}_q$ is

$$Z(X;t) = \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})|.t^r/r\right) \in \mathbb{Q}[[t]].$$

Examples.
$Z(\mathbb{A}^n;t) = \exp(\sum q^{rn}t^r/r) = \exp\log 1/(1-q^n t) = 1/(1-q^n t)$.
$Z(\mathbb{P}^1;t) = \exp(\sum(q^r+1)t^r/r) = 1/(1-qt)(1-t)$.

C) **Weil Conjectures (Weil, 1949).**
If X is a smooth projective variety of dimension n, then:
Rationality: $Z(X;t)$ is a rational function of t.
Functional equation: $Z(X;1/q^n t) = \pm q^{nE/2}t^E Z(X;t)$ for suitable E.
Analogue of Riemann hypothesis:

$$Z(X;t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)}$$

where $P_0(t) = 1-t$, $P_{2n}(t) = 1-q^n t$ and the other $P_i(t) \in \mathbb{Z}[t]$ and have roots which are algebraic integers with absolute value $q^{i/2}$.

D) **Theorem (Dwork, 1960).** Rationality holds for any X defined over $\mathbb{F}_q$ (not necessarily smooth or projective).

Later work of Grothendieck and Deligne gives the rest of the Weil conjectures, and much more, but here we only need rationality.

E) **Proposition.** If X is a variety defined over $\mathbb{F}_q$, and $|X(\mathbb{F}_{q^r})| = P(q^r)$ for some $P(t) \in \mathbb{Q}(t)$ then $P(t) \in \mathbb{Z}[t]$.

Proof. As argued before, in the proof of Corollary 7.3K, since $P(q^r) \in \mathbb{Z}$ for all r, we must have $P(t) \in \mathbb{Q}[t]$. Say $P(t) = \sum_i a_i t^i$. Then

$$Z(X;t) = \exp(\sum_r \sum_i a_i q^{ri} t^r / r) = \prod_i \frac{1}{(1-q^i t)^{a_i}}$$

But by Dwork, this must be a rational function, so $a_i \in \mathbb{Z}$.

Returning to quiver representations.

F) **Theorem.** $a(\alpha, q) \in \mathbb{Z}[q]$.

Proof. Let K be an algebraically closed field. Recall that representations of Q over K of dimension vector α correspond to elements of the affine variety $R = \mathrm{Rep}(Q, \alpha)$, and the isomorphism classes correspond to orbits under the natural action of the algebraic group $G = \mathrm{GL}(\alpha)$.

In fact these varieties can be defined over the prime subfield k of K, and then for any intermediate field $k \subseteq L \subseteq K$ we have

$$R(L) = \mathrm{Rep}(Q, \alpha)(L) = \prod_{a \in Q_1} \mathrm{Hom}_L(L^{\alpha_{t(a)}}, L^{\alpha_{h(a)}})$$

and

$$G(L) = \mathrm{GL}(\alpha)(L) = \prod_{i \in Q_0} \mathrm{GL}_{\alpha_i}(L),$$

so the orbits of $G(L)$ on $R(L)$ correspond to the isomorphism classes of representations of Q over L of dimension α.

Now R has a constructible subset $I = \mathrm{Ind}(Q, \alpha)$ of indecomposable representations. Moreover $I(L) = I \cap \mathrm{Rep}(Q, \alpha)(L)$ corresponds to the absolutely indecomposable representations of Q over L.

We would like to apply the proposition to I/G, but this is not a variety. Kac quotes a theorem of Rosenlicht. We would like to avoid this complication.

We have $I = \bigcup_s I_{(s)}$, where $I_{(s)}$ is the union of the orbits of dimension s, and set

$$I_{(s)}G = \{(x, g) \in I_{(s)} \times G : gx = x\}.$$

This is a locally closed subset of $R_{(s)} \times G$. The fibre over x of the projection $I_{(s)}G \to I_{(s)}$ is $\mathrm{Stab}_G(x)$. Now the elements of the orbit Gx are in 1:1 correspondence with cosets of $\mathrm{Stab}_x(G)$ in G. Thus, corresponding to each orbit in $I_{(s)}$ there are elements of $I_{(s)}G$ in bijection with G.

Let X be the disconnected union of the sets $I_{(s)}$ as s varies. This is defined over the prime subfield k of K so we can consider the L-valued points for any $k \subseteq L \subseteq K$. In particular, in case K is algebraically closed of characteritic p, we can take $L = \mathbb{F}_q$ and obtain

$$|X(\mathbb{F}_q)| = |G(\mathbb{F}_q)|.a(\alpha, q).$$

Now $|G(\mathbb{F}_q)| \in \mathbb{Z}[q]$ and it is monic (for example $|\operatorname{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$). By Corollaries 7.3K and 7.4K, we have $a(\alpha, q) \in \mathbb{Q}[q]$. Thus $|X(\mathbb{F}_q)| \in \mathbb{Q}[q]$. Thus by the Proposition it is in $\mathbb{Z}[q]$. But then $a(\alpha, q) \in \mathbb{Z}[q]$ by Gauss's Lemma.

G) **Theorem (Lang-Weil, 1954).** There is a constant $A(n, k, d)$ depending only on n, k, d, such that if X is an irreducible closed subvariety of projective space $\mathbb{P}^n$ of degree k and dimension d, defined over $\mathbb{F}_q$, then

$$||X(\mathbb{F}_q)| - q^d| \leq (k-1)(k-2)q^{d-\frac{1}{2}} + A(n, k, d)q^{d-1}.$$

The degree of a projective variety is defined using the Hilbert series of its coordinate ring. They remark that for curves, this is equivalent to the Riemann Hypothesis for function fields.

H) **Corollary.** If X is a variety which is defined over a finite field, then

$$|X(\mathbb{F}_q)| \sim tq^d$$

where $d = \dim X$ and $t = \operatorname{top} X$, where this notation means that for all $\epsilon > 0$ there is some finite field $\mathbb{F}_{q_0}$ over which X is defined, such that

$$1 - \epsilon < \frac{|X(\mathbb{F}_q)|}{tq^d} < 1 + \epsilon$$

for all $\mathbb{F}_q$ containing $\mathbb{F}_{q_0}$.

Sketch. One proves this by induction on the dimension.

It is true for irreducible projective varieties. It follows for all projective varieties. Note that the irreducible components of X are defined over a (possibly larger) finite field.

Any irreducible affine variety X can be embedded in projective space, and then we know the result for it's closure $\overline{X}$ and for the complement $\overline{X} \setminus X$.

Now any irreducible variety is the union of an affine open and a variety of smaller dimension. Then get it for all varieties.

I) **Theorem.** For any algebraically closed field K, $\dim_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q,\alpha)$ is the degree of $a(\alpha,q)$ and $\mathrm{top}_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q,\alpha)$ is its leading coefficient.

Sketch. For K of characteristic p this follows from Lang-Weil applied to the disconnected union X of the $I_{(s)}G$, as in the proof of Theorem 7.5F.

For K of characteristic 0, one needs to argue that X comes from a scheme over $\mathbb{Z}$, and that the behavour over an algebraically closed field of characteristic 0 is the same as the behavoiur over algebraically closed fields of large positive characteristic.

J) **Reflection Functors (Bernstein, Gelfand and Ponomarev).** Let i be a sink in Q and let Q' be the quiver obtained by reversing all arrows incident at i. There is a reflection functor which gives a bijection between isomorphism classes

Indecomposables of Q except $S_i \leftrightarrow$ Indecomposables of Q' except S_i

Moreover the functor acts on dimension vectors as s_i.

K) **Theorem.** $i(\alpha,q)$, $a(\alpha,q)$, $r(\alpha,q)$ are invariant under reflections.

Proof. $i(\alpha,q)$ is independent of the orientation of Q by Corollary 7.3K, so in order to apply the reflection s_i we may first change the orientation to turn i into a sink. Then we can apply the the reflection functor of Bernstein, Gelfand and Ponomarev, and then change the orientation back again. The numbers $a(\alpha,q)$ and $r(\alpha,q)$ are determined by $i(\alpha,q)$ (for all α,q) by Corollary 7.4K and Proposition 7.3B.

L) **Corollary.** The numbers $\dim_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q,\alpha)$ and $\mathrm{top}_{\mathrm{GL}(\alpha)} \mathrm{Ind}(Q,\alpha)$ are invariant under reflections.

Proof. Combine the last two theorems.

This was the result needed to complete the proof of Kac's Theorem.

[End of LECTURE 21 on 9 July 2020]

8 Moduli spaces

8.1 Reductive groups

A) **Notation.** Let G be a linear algebraic group. If V acts on a set X we write X^G for the fixed points. If V is a KG-module then V^G is a submodule. If G acts on an algebra R then R^G is a subalgebra.

B) **Definitions.** An *algebraic torus* is an algebraic group isomorphic to a finite product of copies of the multiplicative group G_m.

An algebraic group G is *reductive* if its radical (its unique maximal connected normal solvable subgroup) is an algebraic torus (see Borel, Linear algebraic groups, §11.21).

Examples. Classical groups like $\mathrm{GL}_n(K)$, $SL_n(K)$, $SO_n(K)$ are reductive. Products of reductive groups are reductive.

G is *linearly reductive* if any rational KG-module is semisimple. Thus every short exact sequence of rational KG-modules is split exact, so the functor $V \to V^G$ from rational KG-modules to vector spaces is exact. (Conversely, one can show that if this functor is exact, then G is linearly reductive)

By Theorem 2.2E, the multiplicative group G_m is linearly reductive, and more generally algebraic tori are linearly reductive. Over a field K of characteristic zero, reductive groups are linearly reductive (Weyl), but this fails if K has positive characteristic.

If V is a finite-dimensional rational KG-module, then it can be considered as an affine variety. Choosing coordinates, its algebra of regular functions $K[V]$ is isomorphic to a polynomial algebra, and this is graded by the degree of the polynomial. In coordinate free terms,

$$K[V] = \bigoplus_{d=0}^{\infty} K[V]_d$$

where the homogeneous component $K[V]_d$ is the set of regular functions with $f(\lambda v) = \lambda^d f(v)$ for $\lambda \in K$ and $v \in V$.

G is *geometrically reductive* if for any finite-dimensional rational KG-module V and non-zero $w \in V^G$ there is some non-constant G-invariant homogeneous $f \in K[V]$ with $f(w) \neq 0$. Thus f is a morphism of varieties $V \to K$ with $f(\lambda v) = \lambda^d f(v)$ and $f(gv) = f(v)$ for all $v \in V$, $\lambda \in K$ and $g \in G$, for some $d > 0$ and $f(w) \neq 0$.

Linearly reductive implies geometrically reductive. Namely, if V is a rational

KG-module, then V is semisimple. Thus every submodule has a complement, so $V = V^G \oplus W$ for some submodule W. It follows that the map $\mathrm{Hom}_{KG}(V, K) \to \mathrm{Hom}_{KG}(V^G, K) = \mathrm{Hom}_K(V^G, K)$ is onto. Now there is a linear map $V^G \to K$ which doesn't kill w. Hence there is a KG-module homomorphism $V \to K$ which doesn't kill v. This gives a G-invariant homogeneous regular function of degree 1.

C) **Theorem (Haboush, Nagata, Popov).** Given G, the following are equivalent.
- G is reductive
- G is geometrically reductive
- R^G is finitely generated for all finitely generated commutative K-algebras R with rational G-action.

The proof is beyond the scope of these lectures.

D) **Reynolds operator.** If G is linearly reductive, then, as mentioned, if V is a rational KG-module, then $V = V^G \oplus W$ for some W. In fact W is uniquely determined - it is the sum of all non-trivial simple submodules of V. The *Reynolds operator* is the unique KG-module map $E : V \to V$ which is the identity on V^G and zero on W. Thus $E^2 = E$ and $E(v) = v$ iff $v \in V^G$.

For characteristic $p > 0$ there is the following replacement: See M. Nagata, Invariants of a group in an affine ring, 1964, Lemma 5.1.B and 5.2.B. See also P. E. Newstead, Introduction to moduli problems and orbit spaces, Tata notes, 1978 Lemmas 3.4.1 and 3.4.2.

E) **Nagata Lemmas.** Suppose G is geometrically reductive, acting on a commutative K-algebra R, as a rational KG-module.

(1) If I is an ideal in R which is a KG-submodule, then $I + r \in (R/I)^G$ implies $r^d \in I + R^G$ for some positive integer d.

(2) If I is a finitely generated ideal in R^G and $r \in RI \cap R^G$, then $r^d \in I$ for some positive integer d.

Remark. If G is linearly reductive, then both hold with $d = 1$. For example in (1), since the map $R \to R/I$ is surjective, linear reductivity gives that $R^G \to (R/I)^G$ is surjective.

Proof. (1) We may suppose $r \notin I$. Choose a f.d. KG-submodule Y of R containing r, and let $X = Kr + (Y \cap I)$. Since $(I + r) \in (R/I)^G$ it follows that X is a KG-submodule of R. Now $X/(Y \cap I)$ is a one-dimensional trivial KG-module so there is a KG-module map $\lambda : X \to K$ with $\lambda(r) = 1$ and $\lambda(Y \cap I) = 0$. Apply the geometric reductivity hypothesis to $\lambda \in (DX)^G$. Let $y_1, \ldots, y_m$ be a basis of $Y \cap I$. Then $r, y_1, \ldots, y_m$ is a basis for X.

Thus we can identify the set $K[DX]$ of regular functions $DX \to K$ with $K[r, y_1, \ldots, y_m]$: given a polynomial f and $\xi \in DX$, the evaluation $f(\xi) \in K$ is given by applying ξ to each indeterminate. In particular $f(\lambda)$ is the sum of the coefficients of the powers of r. Now we have a G-invariant homogeneous f of degree d whose evaluation at λ is non-zero (so wlog 1). Thus $f = r^d +$ terms of lower degree in r. Now there is a natural map $p : K[r, y_1, \ldots, y_m] \to R$ and it is G-equivariant. It sends each indeterminate to the corresponding element of R, and a polynomial to the corresponding linear combination of products. Then $p(f) \in R^G$ and $p(y_i) \in I$, giving the result.

(2) We show by induction on s, that if $r_1, \ldots, r_s \in R^G$ and

$$r \in \left(\sum_{i=1}^{s} Rr_i\right) \cap R^G,$$

then $r^d \in \sum_{i=1}^{s} R^G r_i$ for some positive integer d.

For $s = 1$, we have $r = r' r_1$ for some $r' \in R$ and $({}^g r' - r')r_1 = 0$ for all $g \in G$. Consider the ideal $J = \{h \in R : hr_1 = 0\}$ in R. Then $J + r' \in (R/J)^G$, so by (1) applied to the ideal J, we obtain $r'' \in R^G$ and $d > 0$ with $(r'' - (r')^d)r_1 = 0$. Hence $r^d = (r')^d r_1^d = r'' r_1^d \in R^G r_1$.

Now suppose $s > 1$, let $J = Rr_1$ and $\overline{R} = R/J$. Now

$$\overline{r} \in \left(\sum_{i=2}^{s} \overline{R}\,\overline{r_i}\right) \cap \overline{R}^G$$

so by induction there is $d > 0$ with

$$(\overline{r})^d \in \sum_{i=2}^{s} \overline{R}^G \overline{r_i}.$$

Thus we can write $r^d = \sum_{i=1}^{s} h_i r_i$ with $h_i \in R$ and $\overline{h_2}, \ldots, \overline{h_s} \in \overline{R}^G$. Now $J + h_s = \overline{h_s} \in \overline{R}^G$, so by (1), there is $d' > 0$ and $h_s' \in R^G$ such that $h_s^d \in J + h_s'$. Now

$$r^{dd'} = \left(\sum_{i=1}^{s} h_i r_i\right)^{d'} \in h_s^{d'} r_s^{d'} + L = h_s' r_s^{d'} + L$$

where $L = \sum_{i=1}^{s-1} Rr_i$. It follows that $r^{dd'} - h_s' r_s^{d'} \in L \cap R^G$. Again by induction, there is a positive integer d'' with

$$(r^{dd'} - h_s' r_s^{d'})^{d''} \in \sum_{i=1}^{s-1} R^G r_i.$$

Thus $r^{dd'd''} \in \sum_{i=1}^{s} R^G r_i$ as required.

8.2 Good quotients and affine quotients

A) **Discussion.** Let a linear algebraic group G act on a variety X. We don't try to turn X/G into a variety. Instead we use the set of closed orbits, which we denote $X /\!/ G$. Recall that each orbit closure $\overline{Gx}$ contains a closed orbit.

Good example. If A is a finitely generated algebra and α is dimension vector, then $\mathrm{GL}(\alpha)$ acts on an affine variety $\mathrm{Mod}(A,\alpha)$. The closed orbits are those of semisimple modules. Each orbit closure contains a unique closed orbit. The quotient $\mathrm{Mod}(A,\alpha) /\!/ \mathrm{GL}(\alpha)$ classifies the semisimple modules of dimension vector α.

Bad example. Let

$$G = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & \mu \end{pmatrix} : \lambda \in K, \mu \in K^* \right\} \subseteq \mathrm{GL}_2(K)$$

acting on K^2. The orbits are $K \times K^*$ and $\{(x,0)\}$. The closure of the first orbit contains all the others.

B) **Definition.** An action of G on a variety X is said to have a *good quotient* if the following properties hold.

(1) For any $x \in X$, the orbit closure $\overline{Gx}$ contains a unique closed orbit.

Assuming this, we get a mapping $\phi : X \to X /\!/ G$, and we can turn $X /\!/ G$ into a space with functions:

Topology: $U \subseteq X /\!/ G$ is open iff $\phi^{-1}(U)$ is open in X.
Functions: $\mathcal{O}_{X /\!/ G}(U) = \mathcal{O}_X(\phi^{-1}(U))^G$.

Thus $\phi : X \to X /\!/ G$ is a morphism of spaces with functions.

(2) The space with functions $X /\!/ G$ is a variety.

(3) If W is a closed G-subset of X then $\phi(W)$ is closed in $X /\!/ G$. Equivalently $\{x \in X : \overline{Gx} \cap W \neq \emptyset\}$ is closed in X.

(4) We may also demand (Newstead, Geometric invariant theory, 2009, but not all others) that ϕ is an affine morphism, that is, $\phi^{-1}(U)$ is affine for any affine open subset U of $X /\!/ G$, or equivalently for the sets U in an affine open covering of $X /\!/ G$.

C) **Proposition.** If the action of G on X has a good quotient, then:
(i) Disjoint closed G-subsets of X have disjoint images under ϕ.
(ii) ϕ is a categorical quotient of X by G.
(iii) If G acts on X with closed orbits, then $Y = X/G$ is a geometric quotient of X by G.

Proof. (i) If a closed orbit Gu is in the image of closed G-subsets Z and Z', then there must be z, z' with $\overline{Gz}$ and $\overline{Gz'}$ both containing Gu. But then $u \in Z \cap Z'$.

(ii) Let $\psi : X \to Z$ be a morphism which is constant on G-orbits. If $\phi(x) = z$, then $\overline{Gx} \subseteq \psi^{-1}(z)$. It follows that $\psi = \chi\phi$ where $\chi : X /\!/ G \to Z$ sends a closed orbit Gu to $\psi(u)$. Now χ is a morphism by the definition of $X /\!/ G$ as a space with functions.

(iii) Clear.

D) **Lemma.** If a reductive group G acts on an affine variety X, and if W_1, W_2 are disjoint closed G-subsets of X, then there is a function $f \in K[X]^G$ with $f(W_1) = 0$ and $f(W_2) = 1$.

Proof. First we find a function in $K[X]$. If the ideals defining W_i are I_i then since the W_i are disjoint, $I_1 + I_2 = K[X]$. Thus we can write $1 = f_1 + f_2$ with $f_i \in I_i$. Thus f_1 is zero on W_1 and 1 on W_2.

Let V be the KG-submodule of $K[X]$ generated by f_1. It has basis $h_i = {}^{g_i}f_1$ for some elements $g_1, \dots, g_n \in G$. Consider the map $\alpha : X \to DV$, $x \mapsto (f \mapsto f(x))$. Then $\alpha(W_1) = 0$ and $\alpha(W_2) = \xi$ where ξ is the element with $\xi(h_i) = 1$ for all i.

Identify $K[DV] = K[h_1, \dots, h_n]$. Since G is geometrically reductive, there is some $p \in K[h_1, \dots, h_n]^G$, with $p(\xi) \neq 0$ and homogeneous of degree > 0, so with $p(0) = 0$. Rescaling, we may assume that $p(\xi) = 1$. Then the composition $f = p\alpha$ has the required properties.

E) **Theorem.** A reductive group G acting on an affine variety X has a good quotient, and $X /\!/ G$ is the affine variety with coordinate ring $K[X]^G$.

Proof. By Haboush and Nagata, the algebra $K[X]^G$ is finitely generated. It also has no nilpotent elements, so defines an affine variety Y, and the inclusion gives a morphism $\psi : X \to Y$.

First, ψ is constant on orbits, for if $\psi(gx) \neq \psi(x)$ then since Y is affine there is $f \in K[Y]$ with $f(\psi(gx)) \neq f(\psi(x))$. But this contradicts that $f \in K[X]^G$.

Next we show that ψ is onto. Let $y \in Y$ and let the maximal ideal in $K[Y] = K[X]^G$ corresponding to y be generated by $f_1, \dots, f_s$. Now Nagata's Lemma 8.1E(2) implies that

$$\sum_i f_i K[X] \neq K[X].$$

Hence some maximal ideal $\mathfrak{m}$ of $K[X]$ contains this ideal. Letting x be the corresponding point of X, since $\mathfrak{m} \cap K[Y]$ contains all the f_i, it follows that

$\psi(x) = y$.

Now if W_1, W_2 are disjoint closed G-subsets of X, then by the lemma there is $f \in K[X]^G$ with $f(W_1) = 0$ and $f(W_2) = 1$. Considering f as a map $Y \to K$, and composing with ψ, we see that $\psi(W_1)$ and $\psi(W_2)$ are disjoint.

It follows that every orbit closure contains a unique closed orbit, and the induced map ϕ, as a map of sets, coincides with ψ.

If W is a closed G-subset of X, then $\psi(W)$ is closed, for if $y \in \overline{\psi(W)} \setminus \psi(W)$, then $W_1 = W$ and $W_2 = \psi^{-1}(y)$ are disjoint closed G-subsets, but there is no function $f \in K[X]^G$ with $f(W_1) = 0$ and $f(W_2) = 1$.

It follows that the topology on Y coincides with that on $X /\!/ G$.

To identify Y with $X /\!/ G$ as a space with functions, we need to show that $\mathcal{O}_{X /\!/ G}(U) \cong \mathcal{O}_Y(U)$ for any open set U in Y. It suffices to do this for $U = D(f)$ with $f \in \mathcal{O}_Y(Y) = K[X]^G$. Then the LHS is $K[X][f^{-1}]^G$ and the RHS is $K[X]^G[f^{-1}]$, and these are isomorphic.

Remark. Thus the group $\mathrm{GL}(\alpha)$ acting on $\mathrm{Mod}(A, \alpha)$ has a good quotient $\mathrm{Mod}(A, \alpha) /\!/ \mathrm{GL}(\alpha)$, and as mentioned before, the points of this correspond to the isomorphism classes of semisimple A-modules of dimension vector α.

If A is finite-dimensional then this is is not so interesting, as it is a finite set, or even just a point. For this situation there are more interesting moduli spaces constructed using geometric invariant theory, but unfortunately we do not have time to discuss them. For more details see the original paper: A. D. King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford Ser. (2) 45 (1994), 515–530.

I also haven't had time to talk about Nakajima's quiver varieties, which are moduli spaces of representations of the preprojective algebra of a quiver, and their application to Kleinian singularities via McKay correspondence. For further reading, I suggest the book: A. Kirillov Jr., Quiver representations and quiver varieties, 2016.