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Masters course: Homological algebra

Homological algebra is the algebra that was invented in order to define and study the homology and cohomology of topological spaces, but it has applications all over mathematics.

My aim is to cover the properties of projective, injective and flat modules, complexes of modules and Ext and Tor groups, homological dimensions, homology and cohomology of groups, and more abstractly, abelian and triangulated categories.

Students are expected to already have some familiarity with rings and modules.

Some suggested books:

- C. A. Weibel, An introduction to homological algebra, CUP 1994.
- J. J. Rotman, An introduction to homological algebra, Springer 2009.
- M. S. Osborne, Basic homological algebra, Springer 2000.
- S. I. Gelfand and Yu. I. Manin, Methods of homological algebra, 2nd ed., Springer 2010.
- H. Krause, Homological theory of representations, CUP 2022.
- The Stacks Project, https://stacks.math.columbia.edu/

Contents

1 Abelian categories

The basic setting setting for homological algebra, for example used in the book by Henri Cartan and Samuel Eilenberg, 'Homological algebra', 1956, is complexes of additive groups, or more generally modules for a ring R.

Algebraic geometers also want to work with complexes of sheaves on an algebraic variety, and in his paper 'Sur quelques points d'algèbre homologique', Tohoku Math. J. 9 (1957), 119–221, Alexander Grothendieck showed that you can unify the two settings by working with abelian categories.

Although we won't work with sheaves, it is good to start with abelian categories: modern homological algebra uses triangulated categories and other concepts, and abelian categories are a necessary preparation.

We begin with the language of categories, although many students will have seen this already.

1.1 Categories and functors

Definition. A *category* $\mathcal{C}$ consists of

(i) a collection $\mathrm{ob}(\mathcal{C})$ of *objects*

(ii) For any $X, Y \in \mathrm{ob}(\mathcal{C})$, a set $\mathrm{Hom}(X, Y)$ (also denoted $\mathcal{C}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$) of *morphisms* $\theta : X \to Y$, and

(iii) For any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(\theta, \phi) \mapsto \theta\phi$.

satisfying

(a) Associativity: $(\theta\phi)\psi = \theta(\phi\psi)$ for $X \xrightarrow{\psi} Y \xrightarrow{\phi} Z \xrightarrow{\theta} W$, and

(b) For each object X there is an identity morphism $\mathrm{Id}_X \in \mathrm{Hom}(X, X)$, with $\mathrm{Id}_Y\theta = \theta = \theta\mathrm{Id}_X$ for all $\theta : X \to Y$.

An *isomorphism* is a morphism $\theta : X \to Y$ with an inverse, that is, if there is some $\phi : Y \to X$, $\theta\phi = \mathrm{Id}_Y$, $\phi\theta = \mathrm{Id}_X$. If so, then ϕ is uniquely determined, and denoted θ^{-1}.

Examples. (1) The categories of Sets, Groups, Rings, etc. The category R-Mod of (left) R-modules for a ring R. These are *concrete categories*: the objects are sets, possibly with extra structure, and the morphisms are maps of sets preserving the extra structure.

(2) If $\mathcal{C}$ is a category, the *opposite category* C^{op} is given by $\mathrm{ob}(\mathcal{C}^{op}) = \mathrm{ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$, with composition derived from that in $\mathcal{C}$.

(3) If $\mathcal{C}$ and $\mathcal{D}$ are categories, the product $\mathcal{C} \times \mathcal{D}$ is the category with $\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$ and $\mathrm{Hom}((X,U),(Y,V)) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{D}}(U,V)$.

(4) Given a group G or a ring R, there is a category with one object $*$, $\mathrm{Hom}(*,*) = G$ or R and composition given by multiplication.

(5) A partially ordered set $(S, \leq)$ gives a category with objects $s \in S$ and

$$\mathrm{Hom}(s,t) = \begin{cases} i_{st} & (s \leq t) \\ \emptyset & (s \not\leq t). \end{cases}$$

The composition must be given by $i_{tu} i_{st} = i_{su}$ for $s \leq t \leq u$, so $\mathrm{Id}_s = i_{ss}$.

(6) A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows and mappings $s, t : Q_1 \to Q_0$ giving the source and target of each arrow, so $s(a) \xrightarrow{a} t(a)$. It is like a category without a composition. The *path category* of a quiver has objects the vertices, and the morphisms $i \to j$ are the *paths* $a_n \dots a_2 a_1$ given by sequences of arrows

$$i = i_0 \xrightarrow{a_1} i_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} i_n = j.$$

There is also a trivial path Id_i for each vertex i. Composition is given by concatenation. For example the category given by the poset $(\mathbb{N}, \leq)$ is isomorphic to the path category of the quiver $0 \to 1 \to 2 \to \dots$.

Definition. Because of Russell's paradox, there is no set of all sets. One solution is to allow normal sets and 'big sets' called classes. There is a class of all sets.

- *Normal category:* $\mathrm{ob}(\mathcal{C})$ is a class, $\mathrm{Hom}(X,Y)$ are sets. For example the concrete categories above.
- *BIG category:* $\mathrm{ob}(\mathcal{C})$ is a class, $\mathrm{Hom}(X,Y)$ are classes. We only rarely need this.
- *Small category:* $\mathrm{ob}(\mathcal{C})$ is a set, $\mathrm{Hom}(X,Y)$ are sets. For example the category given by a partially ordered set.
- *Skeletally small category:* A normal category, such that there is a set S of objects such that every object is isomorphic to one in S.

Definition. A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ such that

- $\mathrm{ob}(\mathcal{D})$ is a subclass of $\mathrm{ob}(\mathcal{C})$.

- $\operatorname{Hom}_{\mathcal{D}}(X,Y) \subseteq \operatorname{Hom}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \operatorname{ob}(\mathcal{D})$.
- $\operatorname{Id}_X^{\mathcal{C}} \in \operatorname{Hom}_{\mathcal{D}}(X,X)$ for all $X \in \operatorname{ob}(\mathcal{D})$.
- Composition in $\mathcal{D}$ is the same as composition in $\mathcal{C}$.

It is a *full subcategory* if $\operatorname{Hom}_{\mathcal{D}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \operatorname{ob}(\mathcal{D})$. Thus a full subcategory of $\mathcal{C}$ is determined by a subclass $\operatorname{ob}(\mathcal{D})$ of $\operatorname{ob}(\mathcal{C})$.

Examples. (a) The category Ab of abelian groups is a full subcategory of the category of all groups.

(b) The category R-mod is a of finitely generated R-modules is a full subcategory of R-Mod. It is skeletally small, with $S = \{R^n/U : n \in \mathbb{N}, U \subseteq R^n\}$.

(c) The category whose objects are sets and with $\operatorname{Hom}(X,Y) =$ the injective functions $X \to Y$ is a subcategory of the category of sets.

Definition. A *monomorphism* in a category is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Z \to X$, if $\theta\alpha = \theta\beta$ then $\alpha = \beta$.

An *epimorphism* is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Y \to Z$, if $\alpha\theta = \beta\theta$ then $\alpha = \beta$.

Examples. (1) In the categories of sets or of R-modules, monomorphism = injective map, epimorphism = surjective map. For example we show epi = surjection for modules. Say $\theta : X \to Y$ is surjective and $\alpha\theta = \beta\theta$. Since θ is surjective, for all $y \in Y$ there is $x \in X$ with $\theta(x) = y$. Then $\alpha(y) = \alpha(\theta(x)) = \beta(\theta(x)) = \beta(y)$. Thus $\alpha = \beta$. Say $\theta : X \to Y$ is an epimorphism. The natural map $Y \to Y/\operatorname{Im}\theta$ and the zero map have the same composition with θ, so they are equal. Thus $\operatorname{Im}\theta = Y$.

(2) In the category of rings, the inclusion map $\theta : \mathbb{Z} \to \mathbb{Q}$ is not surjective, but it is an epimorphism.

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is given by

- For each object $X \in \operatorname{ob}(\mathcal{C})$, an object $F(X) \in \operatorname{ob}(\mathcal{D})$, and
- For each morphism $\theta : X \to Y$ in $\mathcal{C}$, a morphism $F(\theta) : F(X) \to F(Y)$ in $\mathcal{D}$

such that $F(\operatorname{Id}_X) = \operatorname{Id}_{F(X)}$ for all $X \in \operatorname{ob}(\mathcal{C})$ and $F(\theta\phi) = F(\theta)F(\phi)$ for composable morphisms $X \xrightarrow{\phi} Y \xrightarrow{\theta} Z$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the same thing as a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$. Thus it is an assignment of

- For each object $X \in \operatorname{ob}(\mathcal{C})$, an object $F(X) \in \operatorname{ob}(\mathcal{D})$, and

- For each morphism $\theta : X \to Y$ in $\mathcal{C}$ a morphism $F(\theta) : F(Y) \to F(X)$ in $\mathcal{D}$,

such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ and $F(\theta\phi) = F(\phi)F(\theta)$ for composable morphisms $X \xrightarrow{\phi} Y \xrightarrow{\theta} Z$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- *faithful* if the map $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective for all $X, Y \in \mathrm{ob}(\mathcal{C})$,
- *full* if the map $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective for all $X, Y \in \mathrm{ob}(\mathcal{C})$,
- *dense* if every object in $\mathcal{D}$ is isomorphic to $F(X)$ for some object X in $\mathcal{C}$.
- An *isomorphism* if it has an inverse, or equivalently if it is full, faithful and a bijection on objects.
- An *equivalence* if it is full, faithful and dense.

Examples. (1) The inclusion functor of a subcategory, for example Ab to *Group*, is always faithful. It is full if and only if the subcategory is full.

(2) A composition of functors is a functor. (Thus there is a category of small categories.)

(3) There are many examples of *forgetful functors* for concrete categories, which forget some structure. For example *Group* $\to$ *Set*, or R-Mod $\to$ Ab. They are faithful.

(4) Given a ring homomorphism $\theta : R \to S$, restriction defines a faithful functor S-Mod $\to$ R-Mod. [It is full if and only if θ is a epimorphism in the category of rings, but that is another story.]

(5) If K is a field, then duality $V \rightsquigarrow V^* = \mathrm{Hom}_K(V, K)$ gives a contravariant functor K-Mod to K-Mod.

Definition. Let $\mathcal{C}$ be a category and let $\mathrm{Hom}(X,Y)$ denote the Hom sets for $\mathcal{C}$. Fix an object $X \in \mathrm{ob}(\mathcal{C})$. The *representable functor* $F = \mathrm{Hom}(X, -)$ is the functor $C \to Set$ sending an object Y to $F(Y) = \mathrm{Hom}(X,Y)$, and sending a morphism $\theta \in \mathrm{Hom}(Y,Z)$ to the mapping $F(\theta) : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$ defined by $F(\theta)(\phi) = \theta\phi$.

Dually, fixing Y, we get a contravariant functor $\mathrm{Hom}(-, Y)$ from $\mathcal{C}$ to *Set*.

Varying both X and Y, we get a functor $\mathrm{Hom}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to Set$.

1.2 Natural transformations and functor categories

Definition. Let F, G be functors $\mathcal{C} \to \mathcal{D}$. A *natural transformation* $\alpha : F \to G$ is given by morphisms $\alpha_X : F(X) \to G(X)$ for all $X \in \mathrm{ob}(\mathcal{C})$ such that $G(\theta)\alpha_X = \alpha_Y F(\theta)$ for every morphism $\theta : X \to Y$ in $\mathcal{C}$.

It is a *natural isomorphism* if all α_X are isomorphisms in $\mathcal{D}$.

Examples. (1) Clearly we have an identity natural transformation $\mathrm{Id}_F : F \to F$ and a composition $\beta\alpha$ of natural transformations $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ is a natural transformation.

(2) If K is a field and V is a K-vector space, there is a natural map $V \to V^{**}$, $v \mapsto (\theta \mapsto \theta(v))$. This is a natural transformation $\mathrm{Id} \to (-)^{**}$ of functors from K-Mod to K-Mod. If we used K-mod, it would be a natural isomorphism.

Lemma (Yoneda's Lemma)**.** *For a functor $F : \mathcal{C} \to Set$ and $X \in \mathrm{ob}(\mathcal{C})$ there is a 1-1 correspondence between natural transformations $\alpha : \mathrm{Hom}(X, -) \to F$ and elements $f \in F(X)$.*

Proof. A natural transformation α gives a morphism $\alpha_X : \mathrm{Hom}(X, X) \to F(X)$, and hence an element $f = \alpha_X(\mathrm{Id}_X) \in F(X)$. Conversely, given $f \in F(X)$ and $Y \in \mathrm{ob}(\mathcal{C})$ we get a morphism $\alpha_Y : \mathrm{Hom}(X, Y) \to F(Y)$, $\theta \mapsto F(\theta)(f)$. This defines a natural transformation α. These constructions are inverses. $\square$

Definition. The *functor category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has objects the functors $F : \mathcal{C} \to \mathcal{D}$. The morphisms are the natural transformations.

Remarks. (1) In general this is a BIG category. To get a normal category, we can take $\mathcal{C}$ small, or more generally skeletally small. We need to check that the collection of natural transformations $F \to G$ is a set. Every object in $\mathcal{C}$ is isomorphic to an object in a set S. A natural transformation $\alpha : F \to G$ is determined by the morphisms α_X for $X \in S$, for if $\theta : Y \to X$ is an isomorphism, then $\alpha_Y = G(\theta^{-1})\alpha_X F(\theta)$.

(2) The natural isomorphisms $F \to G$ are the isomorphisms in this category, e.g. if α is a natural isomorphism, it has inverse α^{-1} defined by $(\alpha^{-1})_X = (\alpha_X)^{-1}$.

(3) Any morphism $\theta : X \to Y$ in $\mathcal{C}$ defines a natural transformation of representable functors $\mathrm{Hom}(\theta, -) : \mathrm{Hom}(Y, -) \to \mathrm{Hom}(X, -)$, sending $f \in \mathrm{Hom}(Y, Z)$ to $f\theta \in \mathrm{Hom}(X, Z)$. Thus we get a functor $\mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, Set)$, sending $X \in \mathrm{ob}(\mathcal{C})$ to $\mathrm{Hom}(X, -)$ and sending $\theta : X \to Y$ in $\mathcal{C}$ to $\mathrm{Hom}(\theta, -)$. By Yoneda's Lemma this functor is full and faithful. Thus two representable functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(Y, -)$ are naturally isomorphic if and only if X and Y are isomorphic.

Definition. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say that (F, G) is an *adjoint pair*, or that F is *left adjoint* to G or G is *right adjoint* to F if there is a natural isomorphism $\alpha : \mathrm{Hom}(F(-), -) \to \mathrm{Hom}(-, G(-))$ of functors $\mathcal{C}^{op} \times \mathcal{D} \to Sets$.

Thus one needs bijections

$$\alpha_{X,Y} : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, G(Y))$$

for all $X \in \mathrm{ob}(\mathcal{C})$ and $Y \in \mathrm{ob}(\mathcal{D})$, such that

$$\begin{array}{ccc} \mathrm{Hom}(F(X'), Y) & \xrightarrow{\alpha_{X',Y}} & \mathrm{Hom}(X', G(Y)) \\ {\scriptstyle \cdot F(\theta)}\downarrow & & {\scriptstyle \cdot\theta}\downarrow \\ \mathrm{Hom}(F(X), Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}(X, G(Y)) \end{array}$$

commutes for all $\theta : X \to X'$, and

$$\begin{array}{ccc} \mathrm{Hom}(F(X), Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}(X, G(Y)) \\ {\scriptstyle \phi\cdot}\downarrow & & {\scriptstyle G(\phi)\cdot}\downarrow \\ \mathrm{Hom}(F(X), Y') & \xrightarrow{\alpha_{X,Y'}} & \mathrm{Hom}(X, G(Y')) \end{array}$$

commutes for all $\phi : Y \to Y'$.

Examples. (1) Let R be a ring. We have a forgetful functor $Forget_R : R\text{-Mod} \to Sets$. Given a set X, let $Free_R(X)$ be the free left R-module with basis X. Thus

$$Free_R(X) = \{\sum_{x \in X} r_x x : r_x \in R \text{ for } x \in X, \text{ all but finitely many zero}\}.$$

Any mapping $\phi : X \to Y$ gives a module homomorphism $Free_R(X) \to Free_R(Y)$. This gives a functor $Free_R : Sets \to R\text{-Mod}$. For M a left R-module, we have a bijection

$$\alpha_{X,M} : \mathrm{Hom}_R(Free_R(X), M) \to \mathrm{Hom}_{Sets}(X, Forget_R(M))$$

This is natural in both X and M, so it turns $(Free_R, Forget_R)$ into an adjoint pair of functors.

(2) By defining things with morphisms in the natural way, we get adjoint functors $(Path, Forget)$ where $Forget$ is the functor from small categories to quivers which forgets the composition and $Path$ sends a quiver to its path category.

Theorem. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if there is functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong \mathrm{Id}_{\mathcal{D}}$ and $GF \cong \mathrm{Id}_{\mathcal{C}}$. In this case the pairs (F, G) and (G, F) are adjoint pairs.*

The first part is proved in §1.3 of my Algebra II notes. Now for example if $X \in \mathrm{ob}(\mathcal{C})$ and $Y \in \mathrm{ob}(\mathcal{D})$ then since G is full and faithful, we get $\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(GF(X), G(Y))$, and since $GF \cong \mathrm{Id}_{\mathcal{C}}$ this is in bijection with $\mathrm{Hom}_{\mathcal{D}}(X, G(Y))$.

1.3 Limits and colimits

Definition. Let $\mathcal{C}$ be a category. Let $\mathcal{I}$ be a small category. An *$\mathcal{I}$-diagram* in $\mathcal{C}$ is a functor $M : \mathcal{I} \to \mathcal{C}$. For $i \in \mathrm{ob}(\mathcal{I})$, we write M_i instead of $M(i)$ and for a morphism $a : i \to j$ in $\mathcal{I}$, we write M_a for the morphism $M_i \to M_j$.

Given an object X in $\mathcal{C}$, the *constant functor* $c_X : \mathcal{I} \to \mathcal{C}$ sends every object of $\mathcal{I}$ to X and every morphism to Id_X. A morphism $\theta : X \to Y$ induces a natural transformstion $c_\theta : c_X \to c_Y$. Thus we get a functor $c : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$.

Given an $\mathcal{I}$-diagram M, a *limit* for M is an object

$$L = \lim_{i \in \mathcal{I}} M_i \in \mathrm{ob}(\mathcal{C})$$

together with a natural transformation $\alpha : c_L \to M$ such that any natural transformation $\beta : c_X \to M$ factors as αc_θ for a unique $\theta : X \to L$.

In other words, a limit is an object L equipped with morphisms $\alpha_i : L \to M_i$ for each $i \in \mathrm{ob}(\mathcal{I})$ such that $\alpha_j = M_a \alpha_i$ for any $a : i \to j$ and such that if $X \in \mathrm{ob}(\mathcal{C})$ and $\beta_i : X \to M_i$ satisfy $\beta_j = M_a \beta_i$ for any $a : i \to j$, then there is a unique $\theta : X \to L$ such that $\beta_i = \alpha_i \theta$ for all i.

If M has a limit, it is unique up to a unique isomorphism, so we can talk about *the limit*.

Remarks. (1) The limit $L = \lim_{i \in \mathcal{I}} M_i$ is an object giving a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X, L) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(c_X, M)$$

which is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(-, L) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(c(-), M)$, so to say that the limit exists is to say that the contravariant functor $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(c(-), M)$ is representable.

(2) Suppose $\phi : M \to N$ is a natural transformation between $\mathcal{I}$-diagrams, and suppose that $\lim_{i \in \mathcal{I}} M$ and $\lim_{i \in \mathcal{I}} N_i$ both exist. Then for each i we get a morphism

$$\lim_{i \in \mathcal{I}} M_i \xrightarrow{\alpha_i^M} M_i \xrightarrow{\phi_i} N_i$$

and these morphisms are compatible with the morphisms N_a. Thus we get a unique morphism

$$\lim_{i\in\mathcal{I}} \phi_i : \lim_{i\in\mathcal{I}} M_i \to \lim_{i\in\mathcal{I}} N_i$$

such that for any i the diagram

$$\begin{array}{ccc} \lim_{i\in\mathcal{I}} M_i & \xrightarrow{\alpha_i^M} & M_i \\ {\scriptstyle \lim_{i\in\mathcal{I}} \phi_i}\downarrow & & {\scriptstyle \phi_i}\downarrow \\ \lim_{i\in\mathcal{I}} N_i & \xrightarrow{\alpha_i^N} & N_i \end{array}$$

commutes.

Examples. (a) Let I be a set. A *product* of a family of objects $M_i \in \mathrm{ob}(\mathcal{C})$ $(i \in I)$ is an object $P = \prod_{i\in I} M_i \in \mathrm{ob}(\mathcal{C})$ equipped with morphisms $p_i : P \to M_i$ such that for any object X and morphisms $q_i : X \to M_i$ there is a unique morphism $\theta : X \to P$ with $q_i = p\theta$, that is, the map

$$\mathrm{Hom}(X, P) \to \prod_i \mathrm{Hom}(X, M_i), \quad \theta \mapsto (p_i\theta)$$

is a bijection. Here we take the category $\mathcal{I}$ with object set I and only identity morphisms.

(b) A *terminal object* in a category $\mathcal{C}$ is an object T such that for every object X there is a unique morphism $X \to T$. This is the same thing as a product of objects indexed by the empty set or a limit over an empty category.

(c) An *equalizer* of a pair of morphisms $f, g : U \to W$ consists of an object E and a morphism $p : E \to U$ with $fp = gp$ and with the universal property, that for all $q : X \to U$ with $fq = gq$ there is a unique $\theta : X \to E$ with $q = p\theta$. Here $\mathcal{I}$ is the category

$$\circ \rightrightarrows \circ$$

with two objects and two non-identity morphisms.

(d) A *pullback* of a diagram

$$\begin{array}{ccc} & & U \\ & & {\scriptstyle f}\downarrow \\ V & \xrightarrow{g} & W \end{array}$$

of objects and morphisms in $\mathcal{C}$ consists of an object X and morphisms p, q giving a commutative square

$$\begin{array}{ccc} X & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & {\scriptstyle f}\downarrow \\ V & \xrightarrow{g} & W \end{array}$$

and which is universal for such commutative squares, that is for any X', $p' : X' \to U$, $q' : X' \to V$ with $fp' = gq'$ there is a unique $\theta : X' \to X$ with $p' = p\theta$ and $q' = q\theta$.

Theorem. *A category $\mathcal{C}$ is* (finitely) complete, *meaning that for all (finite) small categories $\mathcal{I}$ and $\mathcal{I}$-diagrams M the limit exists in $\mathcal{C}$ if and only if $\mathcal{C}$ has products indexed by any (finite) set and equalizers.*

Proof. We will need the explicit construction of limits. Suppose M is an $\mathcal{I}$-diagram in $\mathcal{C}$. Consider the products and associated morphisms

$$\prod_{i \in \mathrm{ob}(\mathcal{I})} M_i \xrightarrow{p_i} M_i, \quad \prod_a M_{t(a)} \xrightarrow{p_a} M_{t(a)}$$

where the second product is indexed by the morphisms a in $\mathcal{I}$ and $s(a), t(a)$ are the source and target of a. By the universal property of the second product, there are unique morphisms

$$\prod_{i \in \mathrm{ob}(\mathcal{I})} M_i \; \overset{\phi}{\underset{\psi}{\rightrightarrows}} \; \prod_a M_{t(a)}$$

with $p_{t(a)} = p_a\phi$ and $M_a p_{s(a)} = p_a\psi$. Then the equalizer E of this diagram, equipped with the morphisms

$$E \xrightarrow{p} \prod_{i \in \mathrm{ob}(\mathcal{I})} M_i \xrightarrow{p_i} M_i$$

is $\lim_{i \in \mathcal{I}} M_i$. This is straightforward. $\square$

Examples. The categories *Set* and R-Mod are complete. The product is the usual one. The terminal object is a one-point set or the zero module. The equalizer of $f, g : U \to W$ is the inclusion

$$\{u \in U : f(u) = g(u)\} \to U.$$

For R-modules this is the same as $\mathrm{Ker}(f - g)$. The pullback is $\{(u, v) \in U \times V : f(u) = g(v)\}$, etc.

Lemma. *In an equalizer, p is mono. A pullback of a mono is a mono, that is, in a pullback diagram, if f is mono, so is the parallel morphism q.*

Proof. For the equalizer, suppose $\alpha, \beta : X \to E$ and $p\alpha = p\beta = p'$. Since $fp' = gp'$, there is a unique $\theta : X \to E$ with $p' = p\theta$. But both $\theta = \alpha$ and $\theta = \beta$ satisfy this, so $\alpha = \beta$.

For the pullback. Suppose $\alpha, \beta : X' \to X$ with $q\alpha = q\beta$. Then $gq\alpha = gq\beta$, so $fp\alpha = fp\beta$. Since f is mono, $p\alpha = p\beta$. Thus by the uniqueness part of the universal property for a pullback, $\alpha = \beta$. $\square$

Now we do the dual notion.

Definition. A *colimit* of a diagram $M : \mathcal{I} \to \mathcal{C}$ is the same thing as a limit of M considered as a functor $\mathcal{I}^{op} \to \mathcal{C}^{op}$. Thus it is an object

$$C = \operatorname*{colim}_{i \in \mathcal{I}} M_i \in \mathrm{ob}(\mathcal{C})$$

equipped with a natural transformation $\alpha : M \to c_C$ such that any natural transformation $\beta : M \to c_X$ factors as $c_\theta \alpha$ for a unique $\theta : C \to X$.

In other words, a colimit is an object C equipped with morphisms $\alpha_i : M_i \to C$ for each $i \in \mathrm{ob}(\mathcal{I})$ such that $\alpha_j M_a = \alpha_i$ for any $a : i \to j$ and such that if $X \in \mathrm{ob}(\mathcal{C})$ and $\beta_i : M_i \to X$ satisfy $\beta_j M_a = \beta_i$ for any $a : i \to j$, then there is a unique $\theta : C \to X$ such that $\beta_i = \theta \alpha_i$ for all i.

Examples. (a) A *coproduct* of a family of objects M_i $(i \in I)$ is an object $C = \coprod_{i \in I} M_i$ equipped with morphisms $i_i : M_i \to C$ such that for any object X and morphisms $j_i : M_i \to X$ there is a unique morphism $\theta : C \to X$ with $j_i = \theta i_i$. That is, the map

$$\mathrm{Hom}(C, X) \to \prod_i \mathrm{Hom}(M_i, X), \quad \theta \mapsto (\theta i_i)$$

is a bijection.

(b) An *initial* object is an object X with a unique morphism to any other object. It is a coproduct over the empty set or colimit over the empty category.

(c) A *coequalizer* of a pair of morphisms $f, g : U \to W$ consists of an object X and a morphism $p : W \to X$ with $pf = pg$ and the universal property.

(d) A *pushout* of a pair of morphisms $f : W \to U$ and $g : W \to V$, consists of an object X and morphisms $p : U \to X$ and $q : V \to X$ giving a commutative square $pf = qg$, and which is univeral for such commutative squares, that is for any X', $p' : U \to X'$, $q' : V \to X'$ with $p'f = q'g$ there is a unique $\theta : X \to X'$ with $p' = \theta p$ and $q' = \theta q$.

Definition. A category $\mathcal{C}$ is *(finitely) cocomplete* if all (finite) colimits exist. It is equivalent that $\mathcal{C}$ has all (finite) coproducts and coequalizers.

Examples. (i) The categories *Set* and R-Mod are cocomplete.

For *Sets* the coproduct is the disjoint union

$$\dot{\bigcup} X_i.$$

The initial object is the empty set. The coequalizer of morphisms $f, g : U \to W$ is $W \to W/\sim$ where $\sim$ is the smallest equivalence relation with $f(u) \sim g(u)$ for all $u \in U$. The pushout of morphisms $f : W \to U$ and $g : W \to V$ is $U \cup V/\sim$ where $\sim$ is the equivalence relation generated by $f(w) \sim g(w)$ for $w \in W$.

For R-Mod coproducts are direct sums

$$\bigoplus_{i \in I} X_i = \{(x_i) \in \prod_{i \in I} X_i : \text{all but finitely many } x_i = 0$$

The initial object is the the zero module 0. The coequalizer of morphisms $f, g : U \to W$ in R-Mod is the map $W \to W/\operatorname{Im}(f - g)$. The pushout of morphisms $f : W \to U$ and $g : W \to V$ is $(U \oplus V)/\operatorname{Im}\theta$, where $\theta : W \to U \oplus V$ is $\theta(w) = (f(w), -g(w))$.

Lemma. *A pushout of an epi is an epi, that is, in a pushout diagram, if f is epi, so is the parallel morphism q.*

Proposition. *If (L, R) is a pair of adjoint functors, $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$, then L preserves colimits and R preserves limits, if they exist.*

Proof. Suppose M is an $\mathcal{I}$-diagram in $\mathcal{D}$ and suppose that $\lim_{i \in \mathcal{I}} M_i$ exists in $\mathcal{D}$. This gives a bijection

$$\operatorname{Hom}_{\mathcal{D}}(X, \lim_{i \in \mathcal{I}} M_i) \cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{D})}(c_X, M)$$

which is natural in X. Now for $Y \in \operatorname{ob}(\mathcal{C})$ we get

$$\operatorname{Hom}_{\mathcal{C}}(Y, R(\lim_{i \in \mathcal{I}} M_i)) \cong \operatorname{Hom}_{\mathcal{D}}(L(Y), \lim_{i \in \mathcal{I}} M_i) \cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{D})}(c_{L(Y)}, M).$$

Now $c_{L(Y)} \cong Lc_Y$ and it is easy to see that

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{D})}(Lc_Y, M) \cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{I},\mathcal{C})}(c_Y, RM)$$

This is natural in Y, so it shows that $\lim_{i \in \mathcal{I}}(RM)_i$ exists and is isomorphic to $R(\lim_{i \in \mathcal{I}} M_i)$. Now the statement for L is dual, using that (R, L) is an adjoint pair of functors between $\mathcal{D}^{op}$ and $\mathcal{C}^{op}$. $\square$

1.4 Additive categories

Definition. Let K be a commutative ring. A *K-category* is a category $\mathcal{C}$ with the extra structure that the sets $\operatorname{Hom}(X, Y)$ are K-modules for all $X, Y \in \operatorname{ob}(\mathcal{C})$ and the multiplication maps

$$\operatorname{Hom}(Y, Z) \times \operatorname{Hom}(X, Y) \to \operatorname{Hom}(X, Z), \quad (\theta, \phi) \mapsto \theta\phi$$

are K-bilinear. In particular, for any objects $X, Y \in \mathrm{ob}(\mathcal{C})$, there is a zero morphism $0 \in \mathrm{Hom}(X, Y)$.

Recall that a $\mathbb{Z}$-module is the same thing as an additive group. A $\mathbb{Z}$-category is also called a *preadditive* category, so any K-category is preadditive.

Examples. The category Ab of abelian groups is preadditive. So is R-Mod for a ring R. If R is a K-algebra, then R-Mod is a K-category.

Definition. If $\mathcal{C}$ and $\mathcal{D}$ are K-categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *K-linear* if the mapping

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of K-modules for all $X, Y \in \mathrm{ob}(\mathcal{C})$. A $\mathbb{Z}$-linear functor is also called an *additive* functor.

If $\mathcal{C}$ and $\mathcal{D}$ are K-categories, we denote by $\mathrm{Fun}_K(\mathcal{C}, \mathcal{D})$ the category whose objects are the K-linear functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations. It is naturally a K-category: if $\alpha, \alpha' : F \to G$ are natural transformations and $\lambda, \lambda' \in K$, we define $(\lambda\alpha + \lambda'\alpha')_X = \lambda\alpha_X + \lambda'\alpha'_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$.

Example. Let R be a ring, and consider it as a category with one object. It is preadditive, and

$$\mathrm{Fun}_{\mathbb{Z}}(R, \mathrm{Ab}) \cong R\text{-Mod}.$$

Remark. If $\mathcal{C}$ is a preadditive category and $X \in \mathrm{ob}(\mathcal{C})$, then the representable functor $\mathrm{Hom}(X, -)$ gives an additive functor $\mathcal{C} \to \mathrm{Ab}$, so an object in $\mathrm{Fun}_{\mathbb{Z}}(\mathcal{C}, \mathrm{Ab})$. An appropriate version of Yoneda's Lemma gives that if $F : \mathcal{C} \to \mathrm{Ab}$ is an additive functor, then there is a 1-1 correspondence between natural transformations $\mathrm{Hom}(X, -) \to F$ and elements $f \in F(X)$.

Definition. The *kernel* of a morphism $f : U \to W$ in a preadditive category is the equalizer of f and 0. Thus it is an object X and a morphism $p : X \to U$ with $fp = 0$, such that for any morphism $p' : X' \to U$ with $fp' = 0$ there is a unique morphism $\theta : X' \to X$ with $p' = p\theta$. Conversely the equalizer of f, g = kernel of $f - g$.

The cokernel of a morphism $f : U \to W$ in a preadditive category is the coequalizer of f and 0. Thus it is an object X and a morphism $p : W \to X$ with $pf = 0$, such that for any morphism $p' : W \to X'$ with $p'f = 0$ there is a unique morphism $\theta : X \to X'$ with $p' = \theta p$.

For example the cokernel of a morphism $f : U \to W$ in R-Mod is $W \to W/\operatorname{Im} f$.

Theorem. *For objects X, $X_1, \ldots, X_n$ $(n \geq 0)$ in a preadditive category the following are equivalent* *(i) X is the product of $X_1, \dots, X_n$ for some morphisms $p_i : X \to X_i$*

(ii) X is the coproduct of $X_1, \dots, X_n$ for some morphisms $i_i : X_i \to X$,

(iii) X is a biproduct *of $X_1, \dots, X_n$, meaning that there are morphisms $p_i : X \to X_i$ and $i_i : X_i \to X$ with $p_i i_i = \mathrm{Id}_{X_i}$, $p_i i_j = 0$ for $i \neq j$ and $\sum_{i=1}^n i_i p_i = \mathrm{Id}_X$.*

In this case we write $X = \bigoplus_{i=1}^n X_i$ and call it a direct sum.

Proof. (i)⇒(iii) For any object X' we have a bijection

$$\mathrm{Hom}(X', X) \to \prod_{i=1}^n \mathrm{Hom}(X', X_i), \quad \phi \mapsto (p_i \phi).$$

In particular, taking $X' = X_j$, there is a morphism $i_j : X_j \to X$ such that

$$p_i i_j = \begin{cases} \mathrm{Id}_{X_i} & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Now if $\phi = \sum_{i=1}^n i_i p_i$ then $p_j \phi = \sum_{i=1}^n p_j i_i p_i = p_j$, so $\phi = \mathrm{Id}_X$ by the uniqueness part of the definition of a product.

(iii)⇒(i) For any X' one has inverse bijections

$$\mathrm{Hom}(X', X) \underset{\phi \mapsto (p_i \phi)}{\overset{(\alpha_i) \mapsto \sum i_i \alpha_i}{\rightleftarrows}} \prod_{i=1}^n \mathrm{Hom}(X', X_i)$$

so the p_i turn X into a product.

(ii)⇔(iii) Dual. □

Remark. The case $n = 0$ gives the following. In a preadditive category, an object X is terminal if and only if it is initial if and only if $\mathrm{Id}_X = 0$. This is called a *zero* object, and denoted 0.

Definition. A category is *additive* if it is preadditive, it has a zero object and every pair of objects has a direct sum (equivalently it has all finite direct sums).

Examples. (1) Ab, R-Mod, R-mod.

(2) If $\mathcal{C}$ is a preadditive category and $\mathcal{D}$ is additive, then $\mathrm{Fun}_{\mathbb{Z}}(\mathcal{C}, \mathcal{D})$ is additive. The direct sum of functors $F_1, \dots, F_n$ is the functor F with

$$F(X) = F_1(X) \oplus \cdots \oplus F_n(X)$$

for $X \in \mathrm{ob}(\mathcal{C})$.

Corollary. *If F is an additive functor between additive categories, then F preserves finite direct sums, so $F(0) = 0$ and $F(X \oplus Y) \cong F(X) \oplus F(Y)$.*

Proof. If X is a biproduct of $X_1, \dots, X_n$, then clearly $F(X)$ is a biproduct of $F(X_1), \dots, F(X_n)$. □

1.5 Abelian categories

Definition. A category is *abelian* if

(i) it is additive,

(ii) every morphism has a kernel and a cokernel,

(iii) every epi is a cokernel and every mono is a kernel.

Remarks. (1) The opposite of an abelian category is abelian. This saves work in proofs.

(2) An abelian category has all finite limits and colimits.

(3) Every mono is the kernel of its cokernel and every epi is the cokernel of its kernel. For example, suppose $f : X \to Y$ is mono, say a kernel of $g : Y \to W$, and suppose f has cokernel $c : Y \to Z$. Then $g = kc$ for some $k : Z \to W$. Now if $s : U \to Y$ is a morphism with $cs = 0$, then $gs = kcs = 0$, so s factors through f. It follows that f is a kernel of c.

Lemma. *In an abelian category a pullback of an epi is an epi and a pushout of a mono is a mono.*

Proof. Say

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow{d} & W \end{array}$$

is a pullback with d epi. We want to show that a is epi. We have morphisms

$$X \xrightarrow{\binom{a}{-b}} Y \oplus Z \xrightarrow{(c\ d)} W$$

where $(c\ d)$ comes from considering $Y \oplus Z$ as the coproduct of Y and Z and $\binom{a}{-b}$ comes from considering $Y \oplus Z$ as the product of Y and Z. Since the square is a pullback, $\binom{a}{-b}$ is the kernel of $(c\ d)$. Since d is an epi, so is $(c\ d)$. Thus by the remark above, $(c\ d)$ is the cokernel of $(a\ -b)$. Thus the square is a pushout. Suppose $f : Y \to U$ is a morphism with $fa = 0$. Since $fa = 0 = 0b$, by the pushout property there is a unique morphism $h : W \to U$ with $hc = f$ and $hd = 0$. Since d is epi, $h = 0$. Thus $f = 0$. $\square$

Lemma. *Every morphism $f : X \to Y$ in an abelian category factors as a product $f = gh$ where h is an epi and g is a mono, and this decomposition is unique up to isomorphism, in fact h is a cokernel of the kernel of f and g is a kernel of the cokernel of f.*

Proof. Let $k : U \to X$ be a kernel of f and let $h : X \to Z$ be a cokernel of k. Let $h : X \to Z$ be a cokernel of the kernel $k : U \to X$ of f. Then f factors as gh for some $g : Z \to Y$. We show that g is mono, so suppose that $s : W \to Z$ is a morphism with $gs = 0$. Take the pullback

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle q}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ W & \xrightarrow{s} & Z \end{array}$$

By the previous result, q is an epi. Now $gsq = 0$, so $ghp = 0$, so $fp = 0$, so $p = kr$ for some $r : P \to U$. Then $sq = hp = hkr = 0$, so $s = 0$.

For uniqueness suppose that f factors as $X \xrightarrow{h} Z \xrightarrow{g} Y$ with h an epi and g a mono. Since g is mono, the a kernel of f is also a kernel of h, so h is a cokernel of this. Similarly for g. □

Lemma. *A morphism in an abelian category is an isomorphism if and only if it is mono and epi.*

Proof. If $f : X \to Y$ is mono, then its kernel is $0 \to X$, and the cokernel of this is $X \to X$. □

Examples. (1) Ab is abelian and R-Mod is abelian. If R is a left noetherian ring, the category R-mod of finitely generated left modules is abelian. (The noetherian hypothesis ensures that the kernel of a morphism between f.g. modules is f.g.)

(2) If $\mathcal{C}$ is a preadditive category then $\mathrm{Fun}_{\mathbb{Z}}(\mathcal{C}, \mathrm{Ab})$ is abelian. Kernels and cokernels are computed objectwise: if $\alpha : F \to G$ is a natural transformation, then

$$(\operatorname{Ker}\alpha)(X) = \operatorname{Ker}(F(X) \to G(X)), \quad (\operatorname{Coker}\alpha)(X) = \operatorname{Coker}(F(X) \to G(X)).$$

Remark. A *subobject* of an object X in an abelian category is an equivalence class of monos to X, where $\alpha : U \to X$ is equivalent to $\alpha' : U' \to X \Leftrightarrow \alpha = \alpha'\theta$ for some isomorphism $\phi : U \to U'$. [There is possibly a set-theoretic problem here, which we ignore.]

Given a subobject $U \to X$ we denote its cokernel by $X \to X/U$.

Given a morphism $\theta : X \to Y$, the kernel of θ gives a subobject $\operatorname{Ker}\theta$ of X. The image $\operatorname{Im}\theta$ is the subobject of Y given by the morphism g in a factorization $\theta = gh$ with h epi and g mono.

We get analogues of the isomorphism theorems - details omitted.

1.6 Exact sequences

We work in an abelian category.

Definition. A sequence of objects and morphisms

$$\cdots \to L \xrightarrow{f} M \xrightarrow{g} N \to \cdots$$

is said to be *exact* at M if $\operatorname{Im} f = \operatorname{Ker} g$. The sequence is *exact* if it is exact at every place where morphisms come in and out. A *short exact sequence* is an exact sequence of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

Remarks. (1) Write f and g as compositions ba and dc with

$$L \overset{a}{\twoheadrightarrow} \operatorname{Im} f \overset{b}{\hookrightarrow} M \overset{c}{\twoheadrightarrow} \operatorname{Im} g \overset{d}{\hookrightarrow} N$$

Then we have: exact at M
$\Leftrightarrow$ b is a kernel for g (this is the definition)
$\Leftrightarrow$ b is a kernel for c (d is mono, so g and c have the same kernel)
$\Leftrightarrow$ c is a cokernel for b (since any epi is a cokernel for it kernel and any mono is a kernel for its cokernel)
$\Leftrightarrow$ c is a cokernel for f (since a is epi)

(2) $0 \to M \xrightarrow{g} N$ is exact at M if and only if g is a mono and $L \xrightarrow{f} M \to 0$ is exact at M if and only if f is an epi.

(3) A sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ is exact if and only if f is a kernel for g. A sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact if and only if g is a cokernel for f.

(4) $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence if and only if f is a kernel for g and g is a cocernel for f.

(5) Any subobject $U \to M$ gives a short exact sequence $0 \to U \to M \to M/U \to 0$.

(6) Any morphism $f : M \to N$ gives an exact sequence

$$0 \to \operatorname{Ker} f \to M \xrightarrow{f} N \to \operatorname{Coker} f \to 0$$

with $\operatorname{Coker} f = N/\operatorname{Im} f$ and short exact sequences

$$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0 \quad \text{and} \quad 0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0.$$

(7) If L and N are objects, their direct sum has morphisms

$$\begin{tikzcd} L \arrow[r, shift left, "i_L"] & L \oplus N \arrow[l, shift left, "p_L"] \arrow[r, shift right, "p_N"'] & N \arrow[l, shift right, "i_N"'] \end{tikzcd}$$

and the sequence

$$0 \to L \xrightarrow{i_L} L \oplus N \xrightarrow{p_N} N \to 0,$$

is exact. For example, if $\theta : L \oplus N \to X$ is a morphism with $\theta i_L = 0$, then

$$\theta = \theta \operatorname{Id}_{L\oplus N} = \theta(i_L p_L + i_N p_N) = \theta i_N p_N$$

so θ factors through p_N.

Lemma. *For a short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

in an abelian category, the following conditions are equivalent, in which case the sequence is said to be split.

(i) f *is a* split monomorphism, *meaning that it has a* retraction, *a morphism* $r : M \to L$ *with* $rf = \operatorname{Id}_L$.

(ii) g *is a* split epimorphism, *meaning that it has a* section, *a morphism* $s : N \to M$ *with* $gs = \operatorname{Id}_N$.

(iii) There are morphisms

$$\begin{tikzcd} L \arrow[r, shift right, "f"'] & M \arrow[l, shift right, "r"'] \arrow[r, shift left, "g"] & N \arrow[l, shift left, "s"] \end{tikzcd}$$

turning M *into a biproduct of* L *and* N.

(iv) There is an isomorphism $\theta : M \to L \oplus N$ *giving a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "\theta"'] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "i_L"] & L \oplus N \arrow[r, "p_N"] & N \arrow[r] & 0. \end{tikzcd}$$

Proof. (i)⇒(iii). We have $(\operatorname{Id}_M - fr)f = f - frf = f - f = 0$. Thus since g is a cokernel for f we have $\operatorname{Id}_M - fr = sg$ for some $s : N \to M$. Now $gsg = g(\operatorname{Id}_M - fr) = g = \operatorname{Id}_N g$, so $gs = \operatorname{Id}_N$ since g is epi. Also $rsg = r(\operatorname{Id}_M - fr) = r - r = 0$, so $rs = 0$ since g is epi.

(ii)⇒(iii) is dual.

(iii)⇒(iv) is clear, since M is identified with $L \oplus N$.

(iv)⇒(i) and (ii) taking $r = p_L\theta$ and $s = \theta^{-1} i_N$. □

Lemma (Snake Lemma)**.** *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
(0 & \longrightarrow &)L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N'(& \longrightarrow & 0)
\end{array}$$

there is a morphism $c : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ *giving an exact sequence*

$$(0 \to) \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{c} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma (\to 0).$$

Lemma (Five Lemma)**.** *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

If $\alpha, \beta, \delta, \epsilon$ *are isomorphisms, so is* γ.

Proof. For the category R-Mod, these are most easily proved by diagram chasing. For proofs in general, see §1 of B. Iversen, Cohomology of sheaves, Springer 1986. Alternatively, in the exercises starting on page 118 of Gelfand and Manin, Methods of Homological Algebra, Springer 2002, the results are proved by a generalized type of diagram chasing. □

Lemma. *Given a short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

The pullback of g *along a morphism* $\theta : N' \to N$ *fits in a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\theta} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0
\end{array}$$

and the pushout of f *along a morphism* $\phi : L \to L''$ *fits in a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} & & \| & & \\
0 & \longrightarrow & L'' & \xrightarrow{f''} & M'' & \xrightarrow{g''} & N & \longrightarrow & 0.
\end{array}$$

Proof. Given θ there is a pullback given by g' and θ' and we have already seen that g' is epi. By the pullback property there is f' such that $\theta' f' = f$ and $g' f' = 0$. Now f' is clearly mono. It is a kernel for g', for if $h : X \to M'$ and $g'h = 0$ then $g\theta' h = \theta g' h = 0$, so $\theta' h = fk$ for some $k : X \to L$. Thus $\theta' f' k = \theta' h$. Now $f' k = h$ by the uniqueness property of the pullback. □

1.7 Exact functors

Definition. If F is an additive functor between abelian categories, we say that F is *exact* (respectively *left exact*, respectively *right exact*) if given any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$

is exact (respectively $0 \to F(X) \to F(Y) \to F(Z)$ is exact, respectively $F(X) \to F(Y) \to F(Z) \to 0$ is exact).

Similarly, if F is a contravariant functor, we want the sequence

$$0 \to F(Z) \to F(Y) \to F(X) \to 0$$

to be exact (respectively $0 \to F(Z) \to F(Y) \to F(X)$ exact, respectively $F(Z) \to F(Y) \to F(X) \to 0$ exact).

Remarks. (i) Any additive functor between abelian categories sends split exact sequences to split exact sequences.

(ii) An exact functor sends any exact sequence (not just a short exact sequence) to an exact sequence.

(iii) A left exact functor sends an exact sequence $0 \to X \to Y \to Z$ to an exact sequence $0 \to F(X) \to F(Y) \to F(Z)$. Similarly for right exact.

Lemma. *For an abelian category,* $\mathrm{Hom}(-,-)$ *gives a left exact functor in each variable. That is, if M is an object and $0 \to X \to Y \to Z \to 0$ is exact, then so are*

$$0 \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y) \to \mathrm{Hom}(M, Z)$$

and

$$0 \to \mathrm{Hom}(Z, M) \to \mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M).$$

Proof. The first sequence is exact at $\mathrm{Hom}(M, Y)$ since $X \to Y$ is a kernel for $Y \to Z$, and it is exact at $\mathrm{Hom}(M, X)$ since $X \to Y$ is a mono. □

Lemma. *If (L, R) are a pair of adjoint functors between abelian categories, $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$, then L is right exact and R is left exact.*

Proof. R is a right adjoint, so preserves limits, so preserves kernels, so it is left exact. Dually L is a left adjoint, so preserves colimits, so preserves cokernels, so it is right exact. More explicitly, suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact. For any object U in $\mathcal{D}$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Z, R(U)) \to \mathrm{Hom}_{\mathcal{C}}(Y, R(U)) \to \mathrm{Hom}_{\mathcal{C}}(X, R(U))$$

is exact. Hence so is

$$0 \to \mathrm{Hom}_{\mathcal{C}}(L(Z), U) \to \mathrm{Hom}_{\mathcal{C}}(L(Y), U) \to \mathrm{Hom}_{\mathcal{C}}(L(X), U).$$

Thus $L(g)$ is a cokernel of $L(f)$, so $L(X) \to L(Y) \to L(Z) \to 0$ is exact. Thus L is right exact. □

1.8 Filtered colimits

Remark. A poset $(I, \leq)$ is *directed* if it is non-empty and for all $x, y \in I$ there exists $z \in I$ with $x \leq z$ and $y \leq z$. For example the poset $\mathbb{N}$ is directed.

An *inverse limit* is a limit over the opposite of a directed poset. For example the ring of p-adic integers is

$$\hat{\mathbb{Z}}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/\mathbb{Z}p^n \quad \text{where} \quad \cdots \to \mathbb{Z}/\mathbb{Z}p^3 \to \mathbb{Z}/\mathbb{Z}p^2 \to \mathbb{Z}/\mathbb{Z}p \to \mathbb{Z}/\mathbb{Z}1.$$

On the other hand, a *direct limit* is a colimit over a directed poset. For example

$$\underset{n \in \mathbb{N}}{\underrightarrow{\mathrm{colim}}}\, \mathbb{Z}/\mathbb{Z}p^n \quad \text{where} \quad \mathbb{Z}/\mathbb{Z}1 \xrightarrow{p} \mathbb{Z}/\mathbb{Z}p \xrightarrow{p} \mathbb{Z}/\mathbb{Z}p^2 \xrightarrow{p} \mathbb{Z}/\mathbb{Z}p^3 \to \dots$$

is the union of the groups, the Prüfer group $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$.

More generally we shall consider colimits over small filtered categories. In fact any filtered colimit can be turned into a direct limit, see Proposition 8.1.6 in Exposé I of SGA 4 or H. Andréka and I. Németi, Direct limits and filtered colimits are strongly equivalent in all categories, Banach Center Publications 1982.

Definition. A category $\mathcal{I}$ is *filtered* if

- it is non-empty
- for any objects i, j are is an object k and morphisms $i \to k$ and $j \to k$, and

- for any morphisms $a, b : i \to j$ there is a morphism $c : j \to k$ with $ca = cb$.

Lemma. *Let $\mathcal{I}$ be a small filtered category and M an $\mathcal{I}$-diagram in R-*Mod*. On the disjoint union*

$$\dot{\bigcup_{i \in \mathrm{ob}(\mathcal{I})}} M_i$$

consider the equivalence relation $\sim$ generated by the condition that $M_a(m) \sim m$ whenever $a : i \to j$ is a morphism in $\mathcal{I}$ and $m \in M_i$. Then

(i) $\mathrm{colim}_{i \in \mathcal{I}} M \cong C := (\dot\bigcup_{i \in \mathrm{ob}(\mathcal{I})} M_i)/\sim$*, equipped with the mappings $\alpha_i : M_i \to C$, $m \mapsto [m]$.*

(ii) $m \in M_i \sim m' \in M_j \Leftrightarrow$ there exist $i \xrightarrow{a} k \xleftarrow{b} j$ in $\mathcal{I}$ with $M_a(m) = M_b(m')$. In particular, if $m \in M_i$, then $[m] = 0$ if and only if there is a morphism $a : i \to k$ in $\mathcal{I}$ such that $M_a(m) = 0$.

The same thing works for filtered colimits in the category of sets.

Proof. (ii) Consider the relation R defined by this condition. It is clearly reflexive and symmetric. It suffices to show that it is transitive. Suppose mRm' and $m'Rm''$ with $m \in M_i$, $m' \in M_j$, $m'' \in M_k$. By filteredness there are

$$i \xrightarrow{a} p \xleftarrow{b} j \xrightarrow{c} q \xleftarrow{d} k$$

with $M_a(m) = M_b(m')$ and $M_c(m') = M_d(m'')$. By filteredness there are morphisms $p \xrightarrow{a'} r \xleftarrow{d'} q$. And then $a'b$ and $d'c$ are morphisms $j \to r$, so there is a morphism $f : r \to s$ with $fa'b = fd'c$. Then $M_{fa'a}(m) = M_{fa'b}(m') = M_{fd'c}(m') = M_{fd'd}(m'')$, so mRm''.

(i) We turn C into an R-module as follows:
- If $m \in M_i$ and $r \in R$, then $r[m] := [rm]$.
- If $m \in M_i$ and $m' \in M_j$ then

$$[m] + [n] := [M_a(m) + M_b(m')]$$

for morphisms $i \xrightarrow{a} k \xleftarrow{b} j$ in $\mathcal{I}$.

Using filteredness one can show that this is well-defined. For example if $c : i \to i'$ we have $[m] = [M_c(m)]$, and we want

$$[M_a(m) + M_b(m')] = [M_{a'}(M_c(m)) + M_{b'}(m')]$$

where $i' \xrightarrow{a'} k' \xleftarrow{b'} j$. By filteredness there is are $k \xrightarrow{d} s \xleftarrow{d'} k'$ and then $f : s \to t$ such that $fda = fd'a'c$ and $fdb = fd'b'$. Then

$$[M_{a'}(M_c(m)) + M_{b'}(m')] = [M_{fd'}(M_{a'c}(m) + M_{b'}(m'))] = [M_{fd'a'c}(m) + M_{fd'b'}(m')]$$

$$= [M_{fda}(m) + M_{fdb}(m')] = [M_{fd}(M_a(m) + M_b(m'))] = [M_a(m) + M_b(m')].$$

Clearly this turns C into an R-module and the α_i are homomorphisms. We show it is a colimit for M. Clearly, if $a : i \to j$ then $\alpha_j M_a = \alpha_i$. Given a module X and homomorphisms $\beta_i : M \to X$ satisfying $\beta_j M_a = \beta_i$ for all $a : i \to j$, the β_i give a mapping

$$\dot{\bigcup}_{i \in \mathrm{ob}(\mathcal{I})} M_i \to X$$

and it is constant on equivalence classes, so it defines a homomorphism $\theta : C \to X$ satisfying $\theta\alpha_i = \beta_i$ for all i. Clearly θ is uniquely determined. Thus we have the universal property. $\square$

Theorem. *The category R-*Mod *has exact filtered colimits. That is, suppose $\mathcal{I}$ is a small filtered category. Let L, M, N be $\mathcal{I}$-diagrams in R-*Mod *and let $\alpha : L \to M$ and $\beta : M \to N$ be natural transformations. If for all i the sequences of R-modules*

$$0 \to L_i \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} N_i \to 0$$

are exact, then so is the induced sequence

$$0 \to \operatorname*{colim}_{i\in\mathcal{I}} L \to \operatorname*{colim}_{i\in\mathcal{I}} M \to \operatorname*{colim}_{i\in\mathcal{I}} N \to 0.$$

Proof. Follows directly from the lemma. Take an element $x \in \operatorname{colim}_{i\in\mathcal{I}} M$ sent to zero in $\operatorname{colim}_{i\in\mathcal{I}} N$. Now x is represented by an element $m \in M_i$. But $\beta_i(m)$ represents the zero element, so there is some $a : i \to j$ such that $N_a(\beta_i(m)) = 0$. Thus $\beta_i(M_a(m)) = 0$. Thus $M_a(m) = \alpha_j(\ell)$ for some $\ell \in L_j$. But then x is the image of the element in $\operatorname{colim}_{i\in\mathcal{I}} L$ represented by ℓ. $\square$

Definition. A *Grothendieck category* is an abelian category with the following additional properties:

- It is cocomplete. (Since it is abelian, it is equivalent that it has arbitrary coproducts, which is (AB3) in Grothendieck's terminology.)
- It has a *generator* that is, an object G such that for any object X there is an epimorphism from a coproduct of copies of G to X.
- It has exact filtered colimits (or equivalently, in Grothendieck's terminology, (AB5)).

Examples. Module categories are Grothendieck categories. As are functor categories with values in a Grothendieck category, such as Ab. Also categories of graded modules. Also the category of quasicoherent sheaves on a noetherian scheme.

Remarks. (1) Given exact sequences $0 \to X_i \to Y_i \to Z_i \to 0$ $(i \in I)$, the natural sequence

$$0 \to \coprod_i X_i \to \coprod_i Y_i \to \coprod_i Z_i \to 0$$

is in general only right exact, and the sequence

$$0 \to \prod_i X_i \to \prod_i Y_i \to \prod_i Z_i \to 0$$

is in general only left exact.

(2) Finite products and coproducts are the same, so we have exactness.

(3) In the category of R-modules, arbitrary products and coproducts are exact.

(4) In a cocomplete abelian category with exact filtered colimits, the sequence of coproducts is exact, since

$$\coprod_{i \in I} X_i \cong \operatorname*{colim}_{\text{finite } F \subseteq I} \coprod_{i \in F} X_i$$

where the colimit is over the directed poset of finite subsets F of I.

Definition. An R-module M is *finitely presented* (f.p.) if it is a quotient of a finitely generated free module by a finitely generated submodule. Equivalently if there is an exact sequence $R^m \to R^n \to M \to 0$.

Any quotient of a f.p. module by a f.g. submodule is f.p. If R is left noetherian, any f.g. left R-module is f.p.

Theorem. *Every R-module is a filtered colimit of f.p. modules. More generally, if M is a module and $\mathcal{C}$ is a full subcategory of the category of f.p. R-modules such that every map from a f.p. module to M factors through a module in $\mathcal{C}$, then M is a filtered colimit of modules in $\mathcal{C}$.*

Proof. We may assume that $\mathcal{C}$ is small. Let $\mathcal{I}$ be the category with:

- Objects are pairs (X, f) with $X \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathrm{Hom}(X, M)$.
- Morphisms $(X, f) \to (X', f')$ are morphisms $\theta : X \to X'$ with $f'\theta = f$.

This category is usually denoted $\mathcal{C}/M$. It is filtered:
- It is nonempty since the zero map $0 \to M$ must factor.
- Given objects (X, f) and (X', f'), the morphism

$$(f\ f') : X \oplus X' \to M$$

factors through an object in $\mathcal{C}$, say as

$$X \oplus X' \xrightarrow{(g\ g')} X'' \xrightarrow{f''} M.$$

Then we have morphisms $g : (X, f) \to (X'', f'')$ and $g' : (X', f') \to (X'', f'')$.
- Given morphisms $\alpha, \beta : (X, f) \to (X', f')$, we have $f'(\alpha - \beta) = 0$, so taking the cokernel

$$X \xrightarrow{\alpha-\beta} X' \xrightarrow{\gamma} \mathrm{Coker}(\alpha - \beta) \to 0,$$

we get $f' = h\gamma$ for some $h : \mathrm{Coker}(\alpha - \beta) \to M$. But then h factors through an object X'' in $\mathcal{C}$

$$\mathrm{Coker}(\alpha - \beta) \xrightarrow{\phi} X'' \xrightarrow{f''} M.$$

Then $\phi\gamma : (X', f) \to (X'', f'')$ and $\phi\gamma\alpha = \phi\gamma\beta$.

Let $F : \mathcal{I} \to R$-Mod be the $\mathcal{I}$-diagram sending an object (X, f) to X and a morphism θ to θ. Let

$$L = \operatorname*{colim}_{(X,f)\in\mathcal{I}} F(X, f) = \operatorname*{colim}_{(X,f)\in\mathcal{I}} X.$$

It is equipped with morphisms $\alpha_{(X,f)} : X \to L$ for each (X, f). For each object (X, f) in $\mathcal{I}$, we have the morphism $f : X \to M$. Thus by the universal property, there is a unique morphism $\beta : L \to M$ such that $\beta\alpha_{(X,f)} = f$ for each (X, f). We want to show β is an isomorphism.

For an element x in a module X we write $\hat{x}$ for the map $R \to X$, $r \mapsto rx$. For any $m \in M$, the map $\hat{m} : R \to M$ factors through an object X in $\mathcal{C}$, say as

$$R \xrightarrow{\hat{x}} X \xrightarrow{f} M.$$

Then (X, f) is an object in $\mathcal{I}$ and

$$m = f(x) = \beta(\alpha_{(X,f)}(x)) \in \mathrm{Im}(\beta)$$

so β is surjective. Suppose $\ell \in L$ and $\beta(\ell) = 0$. Then ℓ is represented by an element $x \in X$ for an object $(X, f) \in \mathrm{ob}(\mathcal{I})$. Since it is sent to 0 in M, we have $f(x) = 0$. Taking the cokernel

$$R \xrightarrow{\hat{x}} X \xrightarrow{\phi} \mathrm{Coker}(\hat{x}) \to 0$$

we have $f = h\phi$ for some $h : \mathrm{Coker}(\hat{x}) \to M$. Then h factors as

$$\mathrm{Coker}(\hat{x}) \xrightarrow{g} X' \xrightarrow{f'} M$$

with X' in $\mathcal{C}$. Then $g\phi : (X, f) \to (X', f')$ and $g\phi(x) = 0$, so x represents the zero element in the colimit, that is, $\ell = 0$. $\square$

Proposition. *A module X is finitely presented if and only if* $\mathrm{Hom}(X, -)$ *commutes with filtered colimits, that is, for any filtered category $\mathcal{I}$ and $\mathcal{I}$-diagram M, the map*

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}(X, M_i) \to \mathrm{Hom}(X, \operatorname*{colim}_{i \in \mathcal{I}} M_i)$$

is bijective.

Proof. Given a presentation $R^m \to R^n \to X \to 0$, since filtered colimits preserve exact sequences we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(X, \operatorname{colim} M_i) & \longrightarrow & \mathrm{Hom}(R^n, \operatorname{colim} M_i) & \longrightarrow & \mathrm{Hom}(R^m, \operatorname{colim} M_i) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{colim} \mathrm{Hom}(X, M_i) & \longrightarrow & \operatorname{colim} \mathrm{Hom}(R^n, M_i) & \longrightarrow & \operatorname{colim} \mathrm{Hom}(R^m, M_i)
\end{array}$$

Now the right hand vertical maps are isomorphisms (this follows easily from first lemma in this section), hence so is the left hand vertical map by the Five Lemma.

Conversely, suppose that $\mathrm{Hom}(X, -)$ commutes with filtered colimits. Write $X = \operatorname{colim}_{i \in \mathcal{I}} M_i$, a filtered colimit of f.p. modules. Then

$$\mathrm{Id}_X \in \mathrm{Hom}(X, X) = \mathrm{Hom}(X, \operatorname*{colim}_{i \in \mathcal{I}} M_i) = \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}(X, M_i).$$

This is a colimit of $\mathbb{Z}$-modules, so Id_X is represented by some element of $\mathrm{Hom}(X, M_i)$. This means that Id_X can be factored as the composition $X \to M_i \to \operatorname{colim}_{i \in \mathcal{I}} M_i = X$. This means that X is a direct summand of M_i. Now M_i is f.p. and hence so is X. $\square$

2 Projective, injective and flat modules

2.1 Projective modules

Proposition/Definition. *An object P in an abelian category is* projective *if it satisfies the following equivalent conditions.*
(i) $\mathrm{Hom}(P, -)$ *is an exact functor.*
(ii) Any short exact sequence $0 \to X \to Y \to P \to 0$ *is split.*
(iii) Given an epimorphism $\theta : Y \twoheadrightarrow Z$*, any morphism* $P \to Z$ *factors through* θ*.*

Proof. (i)⇒(ii) $\mathrm{Hom}(P, Y) \to \mathrm{Hom}(P, P)$ is onto. A lift of Id_P is a section.
(ii)⇒(iii) Take the pullback along the map $P \to Z$. The resulting exact sequence has P as third term, so is split. This gives a map from P to the pullback. Composing with the map to Y gives the map $P \to Y$.
(iii)⇒(i) Clear. □

Proposition. *A coproduct* $\coprod_i M_i$ *is projective* ⇔ *all* M_i *are projective.*

Proof. $\coprod_i M_i$ is projective
⇔ the functor $\mathrm{Hom}(\coprod_i M_i, -) = \prod_i \mathrm{Hom}(M_i, -)$ is exact
⇔ $0 \to \mathrm{Hom}(\coprod_i M_i, X) \to \mathrm{Hom}(\coprod_i M_i, Y) \to \mathrm{Hom}(\coprod_i M_i, Z) \to 0$ exact for all exact sequences $0 \to X \to Y \to Z \to 0$
⇔ $0 \to \prod_i \mathrm{Hom}(M_i, X) \to \prod_i \mathrm{Hom}(M_i, Y) \to \prod_i \mathrm{Hom}(M_i, Z) \to 0$ exact
⇔ $0 \to \mathrm{Hom}(M_i, X) \to \mathrm{Hom}(M_i, Y) \to \mathrm{Hom}(M_i, Z) \to 0$ are exact. (Recall that in the category of additive groups, or R-Mod, products of exact sequences are exact. The reverse implication is easy.)
⇔ all M_i are projective. □

Theorem. *Let R be a ring. An R-module is projective if and only if it is a direct summand of a free module. A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module R^n, for some n. Any f.g. projective module is f.p.*

Proof. $\mathrm{Hom}_R(R, X) \cong X$, so R is a projective module, hence so is any direct sum of copies of R. If $F \to P$ is onto with F free and P projective, then P is isomorphic to a summand of F. □

We write R-proj for the category of finitely generated projective left R-modules.

Lemma. *The functor* $\mathrm{Hom}_R(-, R)$ *defines an antiequivalence between* R-proj *and* R^{op}-proj.

Proof. Observe that if M is a left R-module, then $\mathrm{Hom}_R(M, R)$ is naturally a right module, and if M is free of rank n, then

$$\mathrm{Hom}_R(M, R) \cong \mathrm{Hom}_R(R^n, R) \cong \mathrm{Hom}_R(R, R)^n \cong (R_R)^n$$

is a free right R-module of rank n. If P is f.g. projective, then there is Q with $P \oplus Q \cong R^n$. Then

$$\mathrm{Hom}_R(P, R) \oplus \mathrm{Hom}_R(Q, R) \cong (R_R)^n$$

so $\mathrm{Hom}_R(P, R)$ is f.g. projective. The inverse equivalence is given by the same construction, but for right R-modules. There is a natural transformation

$$X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, R), R), \quad x \mapsto (\theta \mapsto \theta(x)).$$

It is an isomorphism for $X = R$, so for finite direct sums of copies of R, so for f.g. projective modules. □

Examples. (i) Every R-module is projective $\Leftrightarrow$ Every short exact sequence is split $\Leftrightarrow$ every submodule of a module has a complement $\Leftrightarrow$ Every module is semisimple $\Leftrightarrow$ R is a semisimple (artinian) ring R $\Leftrightarrow$ (the Artin-Wedderburn Theorem) R is a finite direct sum of matrix rings over division rings.

(ii) If R is a principal ideal domain, a standard theorem says that any f.g. module is a finite direct sum of cyclic modules. Now if $0 \neq a \in R$, then

$$\mathrm{Hom}_R(R/Ra, R) = \{r \in R : ra = 0\} = 0$$

so R/Ra cannot be projective (unless it is 0). Thus every f.g. projective module is a direct sum of copies of R, so it is free.

(iii) If $e \in R$ is an idempotent (that is, $e^2 = e$), then $R = Re \oplus R(1-e)$, so Re is a direct summand of ${}_RR$, so it is projective. Conversely any direct sum decomposition of $R = P \oplus Q$ arises in this way from an idempotent element of R, since the projection onto P is an idempotent $e \in \mathrm{End}_R(R) \cong R^{op}$.

In $R = M_n(K)$ the idempotents E^{ii} give the decomposition

$$R = RE^{11} \oplus \dots RE^{nn} = C_1 \oplus \dots \oplus C_n$$

where C_i is the matrices living in the ith column. Since $E^{ij}E^{ji} = E^{ii}$ and $E^{ji}E^{ij} = E^{jj}$, right multiplication by E^{ij} gives an isomorphism $C_i \to C_j$.

Let R be the ring of 2×2 upper triangular matrices with entries in K. Then

$$R = RE^{11} \oplus RE^{22} = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}.$$

Now the two summands are not isomorphic.

(iv) Let

$$R = \{\text{continuous } f : [0,\pi] \to \mathbb{R} : f(0) = f(\pi)\}.$$

If $f \in R$ is idempotent, then $f(x)^2 = f(x)$ for all x, so $f(x) \in \{0,1\}$. So by continuity $f = 0$ or 1. Let

$$P = \{\text{continuous } f : [0,\pi] \to \mathbb{R} : f(0) = -f(\pi)\}.$$

It is naturally an R-module. Now $R \ncong P$ since if there is an isomorphism sending $1 \in R$ to $g \in P$, then it sends any f to fg. By the Intermediate Value Theorem $g(x) = 0$ for some $0 < x < \pi$. But then every element of P vanishes at x, which is nonsense. On the other hand, there are inverse isomorphisms between R^2 and P^2 given by

$$\begin{pmatrix} f & g \end{pmatrix} \mapsto \begin{pmatrix} f & g \end{pmatrix} \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}$$

(See page 28 of T.-Y. Lam, Lectures on Modules and Rings, Springer 1999.) This is an example of the following theorem, with X the circle and the vector bundle given by the Möbius band.

Swan's Theorem (1962). The global section functor gives an equivalence between the category of topological vector bundles on a compact Hausdorff topological space X and the category of f.g. projective modules for its ring of continuous functions $C(X)$.

(v) Earlier was:

Serre's Theorem (1955). The global section functor gives an equivalence between the category of vector bundles on an affine variety X and the category of f.g. projective modules for its coordinate ring $K[X]$.

(vi) Quillen-Suslin Theorem (1976). Every f.g. projective module for a polynomial ring $K[X_1, \dots, X_n]$ with K a field is free (so every vector bundle on affine n-space is trivial).

(vii) Beginnings of K-Theory. The Grothendieck group $K_0(R)$ of a ring R is the $\mathbb{Z}$-module generated by the isomorphism classes $[P]$ of f.g. projective R-modules P, subject to the relations $[P \oplus Q] = [P] + [Q]$ for all P, Q.

(viii) Suppose R is an integral domain (commutative) with field of fractions K. A *fractional ideal* is a nonzero R-submodule I of K such that $I \subseteq d^{-1}R$ for some nonzero $d \in R$. For example any nonzero ideal in R is a fractional ideal. If I and J are fractional ideals, then

$$IJ := \{\sum_{i=1}^{n} x_i y_i : n \geq 0, x_i \in I, y_i \in J\}$$

is another fractional ideal and so is

$$I^{-1} := \{y \in K : Iy \subseteq R\}.$$

For example in $R = \mathbb{Z}[\sqrt{-5}]$ consider the ideal

$$I = (2, 1+\sqrt{-5}) = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}, a \equiv b (\mathrm{mod} 2)\}.$$

We have

$$\begin{aligned} I^{-1} &= \{x = a + b\sqrt{-5} : a, b \in \mathbb{Q}, 2x, (1+\sqrt{-5})x \in R\} \\ &= \{x = a + b\sqrt{-5} : 2a, 2b, a - 5b, a + b \in \mathbb{Z}\} \\ &= \frac{1}{2} I = R\,1 + R\,\frac{1+\sqrt{-5}}{2} \end{aligned}$$

and

$$1 = (1+\sqrt{-5}) \cdot (1 - \frac{1+\sqrt{-5}}{2}) - 2 \cdot 1 \in II^{-1}.$$

So $II^{-1} = R$. The next lemma shows that I is a projective module, but it is not a principal ideal, and in fact not a free module.

Lemma. *For a nonzero ideal I, the following are equivalent:*
(a) I is invertible, *meaning that $II^{-1} = R$.*
(b) I is f.g. projective
(c) I is projective

Proof. For (a)⇒(b), write $1 = \sum_{i=1}^n x_i y_i$ with $x_i \in I$ and $y_i \in I^{-1}$. Then the composition

$$I \xrightarrow{a \mapsto (ay_i)} R^n \xrightarrow{(r_i) \mapsto \sum_{i=1}^n r_i x_i} I$$

is Id_I.
(b)⇒(c) is trivial, and for (*c*) ⇒(a), for some indexing set Λ there are maps

$$I \xrightarrow{a \mapsto (f_\lambda(a))} R^{(\Lambda)} \xrightarrow{(r_\lambda) \mapsto \sum_{\lambda \in \Lambda} r_\lambda x_\lambda} I$$

with composition 1. Choose $0 \neq a \in I$. Only finitely many $f_\lambda(a)$ are nonzero. Let $y_\lambda = a^{-1} f_\lambda(a) \in K$. For any $b \in I$ we have

$$y_\lambda b = a^{-1} f_\lambda(a) b = a^{-1} f_\lambda(ab) = a^{-1} a f_\lambda(b) = f_\lambda(b) \in R.$$

Thus $y_\lambda \in I^{-1}$. Also

$$\sum_{\lambda \in \Lambda} x_\lambda y_\lambda = \sum_{\lambda \in \Lambda} x_\lambda a^{-1} f_\lambda(a) = a^{-1} a = 1.$$

so $II^{-1} = R$. □

Remark. A *Dedekind domain* is an integral domain with Krull dimension ≤ 1 (that is, all non-zero prime ideals are maximal) and integrally closed in its field of fractions. For example the ring of integers of a number field.

One can show that an integral domain is a Dedekind domain if and only if all nonzero ideals are invertible, or equivalently all fractional ideals are invertible.

In this case $K_0(R) \cong \mathbb{Z} \oplus Cl(R)$, where $Cl(R)$ is the *ideal class group*, the group of fractional ideals modulo the subgroup of principal fractional ideals.

2.2 Tensor products

If X is a right R-module and Y is a left R-module, the tensor product $X \otimes_R Y$ is a $\mathbb{Z}$-module $X \otimes_R Y$ equipped with a mapping

$$X \times Y \to X \otimes_R Y, \quad (x, y) \mapsto x \otimes y$$

such that the mapping is a homomorphism of additive groups in each argument, and R-balanced, meaning that

$$xr \otimes y = x \otimes ry$$

for all $x \in X$, $y \in Y$ and $r \in R$, and such that it is universal for this property, that is, if

$$f : X \times Y \to M$$

is additive in each argument and R-balanced, then there is a unique $\mathbb{Z}$-module homomorphism $\alpha : X \otimes_R Y \to M$ such that $f(x, y) = \alpha(x \otimes y)$.

Theorem. *(i) The tensor product exists and it is unique up to isomorphism.*
(ii) Any element can be written (non-uniquely) as a finite sum

$$x_1 \otimes y_1 + \cdots + x_n \otimes y_n.$$

(iii) $X \otimes_R R \cong X$ *and* $R \otimes_R Y \cong Y$ *via the maps* $x \otimes r \mapsto xr$ *and* $r \otimes y \mapsto ry$.
(iv) If $\theta : X \to X'$ *and* $\phi : Y \to Y'$ *are module homomorphisms, then there is a unique* $\mathbb{Z}$*-module homomorphism*

$$\theta \otimes \phi : X \otimes_R Y \to X' \otimes_R Y'$$

with $(\theta \otimes \phi)(x \otimes y) = \theta(x) \otimes \phi(y)$.
(v) We can identify $X \otimes_R Y$ *with* $Y \otimes_{R^{op}} X$.

For a proof see my Algebra II notes.

Definition. If S and R are rings, an *S-R-bimodule* X is given by a left S-module and a right R-module with the same underlying additive group, and such that the actions commute: $(sx)r = s(xr)$.

Theorem. *Let X be an S-R-bimodule. If Y is a left R-module, then $X \otimes_R Y$ becomes an S-module via $s(x \otimes y) = (sx) \otimes y$. This gives a tensor product functor*

$$X \otimes_R - : R\text{-Mod} \to S\text{-Mod}.$$

If Z is an S-module, then $\mathrm{Hom}_S(X, Z)$ becomes an R-module via $(r\theta)(x) = \theta(xr)$. This gives a functor

$$\mathrm{Hom}_S(X, -) : S\text{-Mod} \to R\text{-Mod}.$$

Moreover there is an isomorphism of additive groups

$$\mathrm{Hom}_S(X \otimes_R Y, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z))$$

which is natural in Y and Z. Thus $(X \otimes_R -, \mathrm{Hom}_S(X, -))$ is an adjoint pair of functors.

Proof. The first parts are straightforward. Given $\theta \in \mathrm{Hom}_S(X \otimes_R Y, Z)$ we get $\phi \in \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z))$ by $\phi(y)(x) = \theta(x \otimes y)$, and given ϕ we get θ by the same formula. □

After the results about adjoint functors, we get.

Corollary. *If X is an S-R-bimodule, then the tensor product functor $X \otimes_R - :$ R-Mod $\to$ S-Mod preserves colimits, so it is right exact and commutes with direct sums (coproducts):*

$$X \otimes_R \left(\bigoplus_{i \in I} Y_i \right) \cong \bigoplus_{i \in I} (X \otimes_R Y_i) .$$

(Similarly for the functor $- \otimes_R Y : \text{Mod-}R \to \text{Mod-}T$ for an R-T-bimodule Y.)

Theorem (Eilenberg, Watts)**.** *Any functor $F : R\text{-Mod} \to S\text{-Mod}$ which preserves colimits, that is, is right exact and commutes with direct sums, is naturally isomorphic to a tensor product functor $X \otimes_R -$ for some S-R-bimodule X.*

Proof. $F(R)$ is an S-module, and it becomes an S-R-bimodule via the map

$$R^{op} \xrightarrow{\cong} \mathrm{End}_R(R) \xrightarrow{F} \mathrm{End}_S(F(R)).$$

Now for any R-module Y there is a R-module map

$$Y \xrightarrow{\cong} \mathrm{Hom}_R(R, Y) \xrightarrow{F} \mathrm{Hom}_S(F(R), F(Y)).$$

By Hom-Tensor adjointness this corresponds to an S-module map $F(R) \otimes_R Y \to F(Y)$. This is natural in Y, so it is Φ_Y for some natural transformation $\Phi : F(R) \otimes_R - \to F$. Clearly Φ_R is an isomorphism. Then for any free module $R^{(I)}$ we have $F(R^{(I)}) = F(R)^{(I)} \cong F(R) \otimes R^{(I)}$, so $\Phi_{R^{(I)}}$ is an isomorphism. Then for any module Y there is a presentation $R^{(I)} \to R^{(J)} \to Y \to 0$ and the first two vertical maps in the diagram

$$\begin{array}{ccccccc} F(R) \otimes R^{(I)} & \longrightarrow & F(R) \otimes R^{(J)} & \longrightarrow & F(R) \otimes Y & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \Phi_{R^{(I)}}} & & \Big\downarrow{\scriptstyle \Phi_{R^{(J)}}} & & \Big\downarrow{\scriptstyle \Phi_Y} & & \\ F(R^{(I)}) & \longrightarrow & F(R^{(J)}) & \longrightarrow & F(Y) & \longrightarrow & 0 \end{array}$$

are isomorphisms. Also the rows are exact. Hence the third vertical map is an isomorphism. Thus Φ is a natural isomorphism. $\square$

Lemma. *If X is an S-R-bimodule, then there is homomorphism of additive groups*

$$\phi_{U,Y} : \mathrm{Hom}_S(U, X) \otimes_R Y \to \mathrm{Hom}_S(U, X \otimes_R Y), \quad \theta \otimes y \mapsto (u \mapsto \theta(u) \otimes y)$$

for U an S-module and Y an R-module, which is a natural transformation in U and Y. It is an isomorphism if U is f.g. projective. Conversely, taking $X = R = S$, if Id_U is in the image of the map

$$\phi_{U,U} : \mathrm{Hom}_S(U, S) \otimes_S U \to \mathrm{Hom}_S(U, S \otimes_S U) \cong \mathrm{End}_S(U),$$

then U is finitely generated projective.

Proof. The first part is clear. The map $\phi_{S,Y}$ is an isomorphism since it is identified with the identity map since $\mathrm{Hom}_S(S, X) \otimes_R Y$ and $\mathrm{Hom}(S, X \otimes_R Y)$ can both be identified with $X \otimes_R Y$. Now given a direct sum $U = U_1 \oplus \cdots \oplus U_n$ we get

$$\mathrm{Hom}_S(U, X) \otimes_R Y \cong \bigoplus_{i=1}^{n} (\mathrm{Hom}_S(U_i, X) \otimes_R Y)$$

and

$$\mathrm{Hom}_S(U, X \otimes_R Y) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_S(U_i, X \otimes_R Y)$$

so $\phi_{U,Y}$ corresponds to the mapping whose components are $\phi_{U_i,Y}$, so $\phi_{U,Y}$ is a bijection if and only if all $\phi_{U_i,Y}$ are bijections. Thus $\mathrm{Hom}(S^n, Y)$ is an isomorphism, and hence so is $\phi_{U,Y}$ for any direct summand U of a f.g. free module S^n.

Say Id_U comes from $\sum_i \theta_i \otimes u_i$, then the composition of the maps

$$U \xrightarrow{(\theta_i)} S^n \xrightarrow{(u_i)} U$$

is the identity, so U is a direct summand of S^n, so f.g. projective. $\square$

2.3 Morita equivalence

Recall that an R-module P is a generator if for any module M there is an epi from a direct sum of copies of P to M, $P^{(I)} \to M$.

Theorem (Morita equivalence). *Let R and S be rings. The following are equivalent.*
(i) The categories R-Mod and S-Mod are equivalent
(ii) There is an S-R-bimodule X giving an equivalence $X \otimes_R - : R\text{-Mod} \to S\text{-Mod}$.
(iii) $R \cong \mathrm{End}_S(X)^{op}$ for some f.g. projective generator X in S-Mod.

Proof. (i)⇒(iii) Let $F : R\text{-Mod} \to S\text{-Mod}$ be an equivalence and let $X = F(R)$. Since F is full and faithful we have $R \cong \mathrm{End}_S(X)^{op}$. Since F is an equivalence, X is projective. Now $\mathrm{Hom}_R(R, -)$ commutes with coproducts. Thus $\mathrm{Hom}_S(X, -)$ commutes with coproducts. Since X is projective, it is a summand of a free module $S^{(I)}$. The inclusion is in $\mathrm{Hom}_S(X, S^{(I)}) \cong \mathrm{Hom}(X, S)^{(I)}$, so only finitely many of the components $X \to S$ are nonzero. It follows that X is a summand of a f.g. free S-module, so it is a f.g. S-module.

(iii)⇒(ii) For any S-module T we have a mapping

$$X \otimes_R \mathrm{Hom}_S(X, T) \to T, \quad x \otimes \theta \mapsto \theta(x)$$

This is natural in T, and it is an isomorphism for $T = X$. Thus it is an isomorphism for $T = X^n$. Thus it is an isomorphism for T any summand of X^n. Now X is a generator as an S-module, and ${}_SS$ is finitely generated, so there is an epimorphism $X^n \to S$ for some n. Then since ${}_SS$ is projective, S is isomorphic to a summand of X^n. Thus we get an isomorphism

$$X \otimes_R \mathrm{Hom}_S(X, S) \to S, \quad x \otimes \theta \mapsto \theta(x)$$

This is an isomorphism of S-S-bimodules. Also, by the lemma above applied to the S-S-bimodule S, we have an isomorphism

$$\mathrm{Hom}_S(X, S) \otimes_S X \mapsto \mathrm{Hom}_S(X, S \otimes_S X) \cong R$$

and this is an isomorphism of R-R-bimodules. Thus the functors $X \otimes_R - : R\text{-Mod} \to S\text{-Mod}$ and $\mathrm{Hom}_S(X, S) \otimes_S - : S\text{-Mod} \to R\text{-Mod}$ are inverses (up to natural isomorphisms) so they are equivalences.

(ii)⇒(i) is trivial □

Examples. (i) R is Morita equivalent to $M_n(R)$ for $n \geq 1$. Namely the module R^n is a finitely generated projective generator in R-Mod with $\mathrm{End}_R(R^n)^{op} \cong M_n(R)$.

(ii) If $e \in R$ is idempotent, and $ReR = R$, then R is Morita equivalent to eRe. Namely, the condition ensures that the multiplication map $Re \otimes_{eRe} eR \to R$ is onto. Taking a map from a free eRe-module onto eR, say $eRe^{(I)} \to eR$, we get a map $Re^{(I)} \to R$, so Re is a generator. Then $\mathrm{End}_R(Re)^{op} \cong eRe$.

2.4 Injective modules

Proposition/Definition. *An object E in an abelian category is* injective *if it satisfies the following equivalent conditions.*
(i) $\mathrm{Hom}(-, E)$ *is an exact (contravariant) functor.*
(ii) Any short exact sequence $0 \to E \to Y \to Z \to 0$ *is split.*
(iii) Given an injective map $\theta : X \hookrightarrow Y$*, any map* $X \to E$ *factors through* θ*.*

Proposition. *A product* $\prod_{i \in I} M_i$ *is injective* $\Leftrightarrow$ *all* M_i *are injective. Thus a finite direct sum is injective if and only if each term is injective.*

Proof. This is the opposite category version of the result for projectives. Then a finite direct sum is the same as a finite product. □

Definition. An inclusion of R-modules $M \subseteq N$ is an *essential extension* of M if every non-zero submodule S of N has $S \cap M \neq 0$.

Theorem. *For an R-module M, following conditions are equivalent.*
(a) M is injective.
(b) (Baer's criterion) Every homomorphism $f : I \to M$ from a left ideal I of R can be extended to a homomorphism $R \to M$.
(c) M has no non-trivial essential extensions

Proof. (a)⇒(b) is trivial.

(b)⇒(c) Let $M \subseteq N$ be a non-trivial essential extension and fix $x \in N \setminus M$. We consider the pullback

$$\begin{array}{ccc} I & \longrightarrow & R \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

where $R \to N$ is the map $r \mapsto rx$. Then $I \to R$ is injective, so I is identified with a left ideal in R. By (b), the map $I \to M$ lifts to a map $R \to M$, say sending 1 to m.

Suppose $r \in R$ satisfies $r(x - m) \in M$. Then $rx \in M$, and it follows that $r \in I$. Then $rx = rm$, so $r(x - m) = 0$. Thus $M \cap R(x - m) = 0$ and $R(x - m) \neq 0$, contradicting that $M \subseteq N$ is an essential extension.

(c)⇒(a). Given an inclusion $M \subseteq N$, we need to show that M is a direct summand of N. By Zorn's Lemma, the set of submodules in N with zero intersection with M has a maximal element C. If $M + C = N$, then C is a complement. Otherwise, $M \cong (M + C)/C \subseteq N/C$ is a non-trivial extension. By (c) it cannot be an essential extension, so there is a non-zero submodule U/C with zero intersection with $(M + C)/C$. Then $U \cap (M + C) = C$, so $U \cap M \subseteq C \cap M = 0$. This contradicts the maximality of C. □

Definition. If R is an integral domain and M is an R-module, then

- M is *divisible* if for all $m \in M$ and $0 \neq a \in R$, there is $m' \in M$ with $m = am'$. For example the field of fractions of R is divisible.

- M is *torsion-free* if $am \neq 0$ for all nonzero $a \in R$ and $m \in M$. For example R and its field of fractions are torsion-free.

Lemma. *If R is an integral domain, then any injective module is divisible. If R is a principal ideal domain, then any divisible module is injective.*

Proof. Divisibility says that any map $Ra \to M$ lifts to a map $R \to M$. If R is a pid these are all ideals in R. □

Definition. For the rest of this section we assume that R is a K-algebra, where K is a field or a principal ideal domain. In particular, we can consider any ring R as a K-algebra with $K = \mathbb{Z}$.

We define $(-)^* = \mathrm{Hom}_K(-, E_K)$, where

$$E_K = \begin{cases} K & \text{(if } K \text{ is a field)} \\ F/K & \text{(if } K \text{ is a pid with fraction field } F \neq K) \end{cases}$$

For example $E_{\mathbb{Z}} = \mathbb{Q}/\mathbb{Z}$.

Lemma. *(i) E_K is an injective K-module, and $(-)^*$ defines an exact functor from R-modules on one side to R-modules on the other side.*

(ii) If M is an R-module, the map $M \to M^{**}$, $m \mapsto (\theta \mapsto \theta(m))$ is an injective map of R-modules. (It is an isomorphism if K is a field and M is a finite-dimensional K-vector space).

Proof. (i) Any R-module M also gets an action of K via $\lambda m = (\lambda 1_R)m$, and these two actions commute, so $M^* = \mathrm{Hom}_K(M, E_K)$ becomes an R-module on the other side.

Now any quotient of a divisible module is clearly divisible, so E_K is a divisible K-module, so an injective K-module, so $(-)^*$ is an exact functor.

(ii) Given $0 \neq m \in M$, let Km be the cyclic K-submodule of M generated by m. It suffices to find a K-module map $f : Km \to E_K$ with $f(m) \neq 0$, for then since E_K is injective, f lifts to a map $\theta : M \to E_K$.

If K is a field there is an isomorphism $Km \to E_K$.

If K is a principal ideal domain and not a field, choose a maximal ideal Ka containing $ann(m) = \{x \in K : xm = 0\}$. Since K is not a field, $a \neq 0$. Then there is a map $Km \to E_K$ sending xm to $K + a^{-1}x \in F/K$. This is well-defined since if $xm = x'm$, then $x - x' \in ann(m)$, so $x - x' = ba$ for some $b \in K$, and then $a^{-1}x - a^{-1}x' = b \in K$. It is clearly a K-module homomorphism. Now it sends m to $K + a^{-1}$. If this is zero, then $a^{-1} \in K$, so a is invertible in K, so $Ka = K$, contradicting that Ka is a maximal ideal.

If K is a field, and M is K-vector space of dimension d, then so is M^*, and so also M^{**}, so the map $M \to M^{**}$ must be an isomorphism. $\square$

Theorem. *Any R-module embeds in a product of copies of R^*, and such a product is an injective R-module. An R-module is injective if and only if it is isomorphic to a direct summand of such a product.*

Proof. We have natural isomorphisms of functors R-Mod $\to$ Ab,

$$\mathrm{Hom}_R(-, R^*) = \mathrm{Hom}_R(-, \mathrm{Hom}_K(R, E_K)) \cong \mathrm{Hom}_K(R \otimes_R -, E_K)$$

$$\cong \mathrm{Hom}_K(-, E_K)^* = (-)^*,$$

which is exact, so R^* injective. Thus any product of copies of R^* is injective.

If M is any R-module, then M embeds in M^{**}. Now M^* is a right R-module, so can be written as a quotient of a free right R-module, say $R^{(X)}$. Then

$$M \hookrightarrow M^{**} \hookrightarrow (R^{(X)})^* = \mathrm{Hom}_K(R^{(X)}, E_K) \cong \mathrm{Hom}_K(R, E_K)^X = (R^*)^X.$$

The last part is clear. $\square$

Corollary. *Any module over any ring embeds in an injective module.*

Remark. More generally one can show that any object in a Grothendieck category has a monomorphism to an injective object.

Theorem (Bass, Papp)**.** *For a ring R the following are equivalent*
(i) R is left noetherian
(ii) Any filtered colimit of injective left R-modules is injective
(iii) Any direct sum of injective left R-modules is injective.

Proof. (i)$\Rightarrow$(ii). Let $M = \mathrm{colim}_{i \in \mathcal{I}} M_i$ be a filtered colimit of injective modules. Suppose I is a left ideal in R. It gives an exact sequence $0 \to I \to R \to R/I \to 0$. Since the M_i are injective, we get exact sequences

$$0 \to \mathrm{Hom}(R/I, M_i) \to \mathrm{Hom}(R, M_i) \to \mathrm{Hom}(I, M_i) \to 0.$$

A colimit of exact sequences is exact, so

$$0 \to \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}(R/I, M_i) \to \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}(R, M_i) \to \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}(I, M_i) \to 0$$

is exact. Since the modules R/I, R and I are finitely presented, this is isomorphic to

$$0 \to \mathrm{Hom}(R/I, M) \to \mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M) \to 0.$$

Thus by Baer's criterion M is injective.

(ii)⇒(iii). We have

$$\bigoplus_{i \in I} M_i \cong \operatorname*{colim}_{J \subseteq I} \bigoplus_{j \in J} M_j \cong \operatorname*{colim}_{J \subseteq I} \prod_{j \in J} M_j$$

where J runs over the finite subsets of I, a filtered colimit of injective modules.

(iii)⇒(i). Consider an ascending chain of left ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq R.$$

Let I be their union. For each $n \geq 1$, choose an embedding $\phi_n : R/I_n \to E_n$ with E_n injective. We have a well-defined map

$$\theta : I \to E := \bigoplus_{n=1}^{\infty} E_n, \quad \theta(a)_n = \phi_n(I_n + a)$$

Since E is injective, θ extends to a map $R \to E$. Let that map send 1 to $e \in E$. Then $\theta(a) = ae$ for $a \in R$. But e only has finitely many non-zero components, so there is some n such that $e_n = 0$. Then $\theta(a)_n = 0$ for all $a \in I$, so $a \in I_n$. Thus $I = I_n$, so the chain of ideals stabilizes. Thus R is left noetherian. □

2.5 Flat modules

In this section R is a K-algebra and K is a field or pid, for example R is a ring and $K = \mathbb{Z}$.

Definition. A right R-module X is *flat* if $X \otimes_R -$ is an exact functor, either considered as a functor R-Mod $\to \mathbb{Z}$-Mod or equivalently as a functor R-Mod $\to$ K-Mod.

Remark. (i) A direct sum of modules is flat if and only if each summand is flat, since if X_i are right R-modules and $0 \to L \to M \to N \to 0$ is an exact sequence of left R-modules, then

$$0 \to \bigoplus_{i \in I} X_i \otimes_R L \to \bigoplus_{i \in I} X_i \otimes_R M \to \bigoplus_{i \in I} X_i \otimes_R N \to 0$$

is exact if and only if it is exact for each sequence

$$0 \to X_i \otimes_R L \to X_i \otimes_R M \to X_i \otimes_R N \to 0$$

is exact.

(ii) Any projective module is flat, for $R \otimes_R X \cong X$, so R is flat.

(iii) Any filtered colimit of flat modules is flat. If $\mathcal{I}$ is a small filtered category, X is an $\mathcal{I}$-diagram of flat right R-modules and $0 \to L \to M \to N \to 0$ is an exact sequence of left R-modules, then since the X_i are flat, we get exact sequences

$$0 \to X_i \otimes_R L \to X_i \otimes_R M \to X_i \otimes_R N \to 0.$$

Since R-Mod has exact filtered colimits, the sequence

$$0 \to \operatorname*{colim}_{i \in \mathcal{I}}(X_i \otimes_R L) \to \operatorname*{colim}_{i \in \mathcal{I}}(X_i \otimes_R M) \to \operatorname*{colim}_{i \in \mathcal{I}}(X_i \otimes_R N) \to 0$$

is exact. Since tensor products commute with colimits, this is

$$0 \to (\operatorname*{colim}_{i \in \mathcal{I}} X_i) \otimes_R L \to (\operatorname*{colim}_{i \in \mathcal{I}} X_i) \otimes_R M \to (\operatorname*{colim}_{i \in \mathcal{I}} X_i) \otimes_R N \to 0$$

so $\operatorname{colim}_{i \in \mathcal{I}} X_i$ is flat.

Proposition. *A right R-module X is flat if and only if X^* is injective.*

Proof. If Y is a left R-module, then $\operatorname{Hom}_R(Y, X^*) \cong (X \otimes_R Y)^*$. If X is flat, then this is exact as a functor of Y, so X^* is injective. Conversely, if X^* is injective then again this is exact as a functor of Y. Suppose X is not flat. Given an exact sequence of left R-modules

$$0 \to L \to M \to N \to 0$$

we get

$$0 \to H \to X \otimes_R L \to X \otimes_R M \to X \otimes_R N \to 0.$$

Then we get

$$(X \otimes_R M)^* \to (X \otimes_R L)^* \to H^* \to 0$$

Thus $H^* = 0$. But H embeds in H^{**}, so $H = 0$. □

Proposition. *A module X_R is flat if and only if the multiplication map $X \otimes_R I \to X$ is injective for every left ideal I in R.*

Proof. If X is flat, tensoring it with the exact sequence $0 \to I \to R \to R/I \to 0$ shows that the map is injective.

If the map is injective, then the map $X^* \to (X \otimes_R I)^*$ is surjective. We can write this as $\operatorname{Hom}_R(R, X^*) \to \operatorname{Hom}_R(I, X^*)$. By Baer's criterion X^* is injective. Thus X is flat. □

Example. If R is an integral domain, then any flat right R-module X is torsion-free, that is, if $x \in X$ and $a \in R$ and $xa = 0$, then $x = 0$ or $a = 0$. Namely, if $I = Ra$ with $0 \neq a \in R$, then the map $X \otimes_R I \to X$ is the identified with the map $X \to X$ of multiplication by a.

If R is a pid, then a right module X is flat if and only if it is torsion-free, since any non-zero ideal in R is of this form.

Thus $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. This also follows from the next construction.

Example. Let R be a commutative ring. A subset $S \subseteq R$ is *multiplicative* if $1 \in S$ and $st \in S$ for all $s, t \in S$. The *localization* of an R-module M with respect to S is

$$S^{-1}M = S \times M/\sim$$

where $\sim$ is the equivalence relation given by

$$(s, m) \sim (s', m') \Leftrightarrow t(sm' - s'm) = 0 \text{ for some } t \in S$$

It is equivalent that $um = u'm'$ for some $u, u' \in S$ with $us = u's'$. To see this, take $t = us$ or $u = ts'$ and $u' = ts$.

The equivalence class containing (s, m) is denoted $s^{-1}m$. Now $S^{-1}M$ has an addition given by the usual formula for adding fractions

$$s^{-1}m + t^{-1}n = (st)^{-1}(tm + sn).$$

Moreover $S^{-1}R$ becomes a ring and $S^{-1}M$ becomes an $S^{-1}R$-module with the usual formula for multiplication

$$(s^{-1}a)(t^{-1}b) = (st)^{-1}(ab).$$

This was all on an exercise sheet for Algebra II. It was also shown on the exercise sheet that $S^{-1}M \cong S^{-1}R \otimes_R M$. We can deduce this here from Eilenberg-Watts. The construction gives a localization functor

$$R\text{-Mod} \to S^{-1}R\text{-Mod}, \quad M \mapsto S^{-1}M.$$

and it is easy to see that this is an exact functor. It is easy to see that an exact sequence $0 \to L \to M \to N \to 0$ of R-modules gives an exact sequence

$$0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$$

so this functor is exact. It also commutes with arbitrary direct sums. Thus by the Eilenberg-Watts theorem,

$$S^{-1}M \cong X \otimes_R M$$

for all M, for some bimodule X. Then $X \cong S^{-1}R$ considered as a left $S^{-1}R$ module in the usual way, and as a right R-module via $(s^{-1}r)r' = s^{-1}(rr')$. Thus

$$S^{-1}M \cong S^{-1}R \otimes_R M.$$

Since the localization functor is exact, $S^{-1}R$ is a flat R-module. Here is another way to see this. Consider S as a the set of objects in a category, with

$$\mathrm{Hom}(s,t) = \{u \in S : us = t\}$$

It is filtered since it has object 1, if $s, s' \in S$ then they both have morphisms to ss', and if $u, u' : s \to t$, then $t = us = u's$. Thus considering s as a morphism $t \to st$, the compositions with u and u' are equal.

Consider the functor $S \to R$-Mod sending all $s \in S$ to $M_s = M$ and $u \in \mathrm{Hom}(s,t)$ to multiplication by u. Then our description of the colimit gives

$$\operatorname*{colim}_{s\in S} M_s = \dot{\bigcup}_{s\in S} M/\sim \; = \; (S \times M)/\sim$$

where $(s,m) \sim (s',m') \Leftrightarrow$ there are morphisms $u : s \to v$ and $u' : s' \to v$ with $um = u'm'$. Thus

$$S^{-1}M = \operatorname*{colim}_{s\in S} M_s$$

Thus if M is a flat R-module, so is $S^{-1}M$. In particular $S^{-1}R$ is flat.

Lemma. *Let X be an S-R-bimodule. If U is a f.p. left S-module and Y is a flat left R-module, then the natural map*

$$\mathrm{Hom}_S(U,X) \otimes_R Y \to \mathrm{Hom}_S(U, X \otimes_R Y)$$

is an isomorphism

Proof. It is clear for $U = S$. Then it follows for $U = S^n$. In general there is an exact sequence $S^m \to S^n \to U \to 0$, and in the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_S(U,X) \otimes_R Y & \longrightarrow & \mathrm{Hom}_S(S^n,X) \otimes_R Y & \longrightarrow & \mathrm{Hom}_S(S^m,X) \otimes_R Y \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_S(U,X \otimes_R Y) & \longrightarrow & \mathrm{Hom}_S(S^n,X \otimes_R Y) & \longrightarrow & \mathrm{Hom}_S(S^m,X \otimes_R Y)
\end{array}$$

the rows are exact and the right two vertical maps are isomorphisms, hence so is the first. $\square$

Recall that any f.g. projective module is finitely presented.

Theorem. *(i) A finitely presented flat module is projective.*
(ii) (Lazard, Govorov) Any flat module is a filtered colimit of finitely generated projective (even free) modules.

Proof. (i) If Y is a f.p. flat left R-module, then the natural map $\mathrm{Hom}_R(Y,R)\otimes_R Y \to \mathrm{End}_R(Y)$ is an isomorphism by the last lemma. Thus by the last lemma in the section on projective modules, Y is f.g. projective.

(ii) If M is a flat left R-module and X is f.p., then the map $\mathrm{Hom}(X,R)\otimes_R M \to \mathrm{Hom}(X,M)$ is an isomorphism. It follows that any map $f : X \to M$ can be factored as

$$X \xrightarrow{\theta} R^n \xrightarrow{g} M.$$

Now use the result in the section on filtered colimits. □

2.6 Envelopes and covers

Definition. Suppose $\mathcal{C}$ is a full subcategory of R-Mod. If M is an R-module, a homomorphism $\theta : M \to C$ with C in $\mathcal{C}$ is a $\mathcal{C}$-*envelope* if

- θ is a $\mathcal{C}$-*preenvelope*, meaning that any $\theta' : M \to C'$ with C' in $\mathcal{C}$ factors as $\theta' = \phi\theta$ for some $\phi : C \to C'$, and
- θ is *left minimal*, meaning that if $\phi \in \mathrm{End}_R(C)$ and $\phi\theta = \theta$, then ϕ is an automorphism.

Dually, a homomorphism $\theta : C \to M$ with C in $\mathcal{C}$ is a $\mathcal{C}$-*cover* if

- θ is a $\mathcal{C}$-*precover*, meaning that any $\theta' : C' \to M$ with C' in $\mathcal{C}$ factors as $\theta' = \theta\phi$ for some $\phi : C' \to C$, and
- θ is *right minimal*, meaning that if $\phi \in \mathrm{End}_R(C)$ and $\theta\phi = \theta$, then ϕ is an automorphism.

Note that if M has an envelope or cover, it is unique up to a (non-unique) isomorphism.

Theorem. *Take $\mathcal{C}$ to be the category of injective modules.*
(i) A morphism $\theta : M \to E$ with E injective is an injective preenvelope if and only if θ is injective. If so, identifying M as a submodule of E, it is an injective envelope if and only if the inclusion $M \subseteq E$ is an essential extension.
(ii) Every module has an injective envelope $M \to E(M)$.

Proof. (i) If E is injective and θ is injective, then clearly it is a preenvelope. Conversely if θ is a preenvelope then E is injective and, since there is an embedding $M \to E'$ with E' injective, we must have θ injective.

Now suppose $M \subseteq E$ is an essential extension. Suppose $\phi \in \mathrm{End}_R(E)$ satisfies $\phi\theta = \theta$. That is, $\phi(m) = m$ for all $m \in M$. Then $M \cap \mathrm{Ker}\,\phi = 0$. Thus $\mathrm{Ker}\,\phi = 0$. Thus ϕ is injective. Now $\mathrm{Im}\,\phi \cong E$ is injective, so it is a direct summand of E, so $E = \mathrm{Im}\,\phi \oplus C$ for some complement C. But $M \subseteq \mathrm{Im}\,\phi$, so $M \cap C = 0$, so $C = 0$, so ϕ is an automorphism.

We do the other direction later.

(ii) Any module M embeds as a submodule of an injective module F and Zorn's Lemma implies that the set of submodules of F which are essential extensions of M has a maximal element E.

Suppose that $E \subset N$ is a non-trivial essential extension (with N not necessarily contained in F). Since F is injective the inclusion $E \to F$ can be extended to a map $g : N \to F$.

Since $M \subset E$ and $E \subset N$ are essential extensions, so is $M \subset N$. Clearly $M \cap \mathrm{Ker}\,g = 0$, so since M is essential in N it follows that $\mathrm{Ker}\,g = 0$. Thus we can identify N with $g(N)$. But then M is essential in N, contradicting the maximality of E.

Thus E has no non-trivial essential extensions, so E is injective. Thus by (i) $\theta : M \to E$ is an injective envelope.

Completion of (i). By uniqueness, any injective envelope $M \to E$ is isomorphic to the one we just constructed, so E is an essential extension of M. □

Theorem. *Suppose R is a f.d. algebra over a field.*
(1) The injective envelopes of simples are f.g., and every injective module is a direct sum of injective envelopes of simples.
(2) Every module M has a projective cover $P(M)$, the projective covers of simples are f.g., and every projective modules is a direct sum of projective covers of simples.
(3) Every flat module is projective.

Proof. We will need properties of the *Jacobson radical* of R

$$J = \{r \in R : rS = 0 \text{ for all simple } R\text{-modules } S\}.$$

Equivalently it is the intersection of all maximal left ideals in R. It is a two sided ideal in R. Since R is f.d. we have

(a) R/J is semisimple.

(b) J is nilpotent, $J^n = 0$ for some n.

(c) If M is any module with $JM = M$, then $M = 0$.

Namely, since R is f.d., $J(R)$ is a finite intersection of maximal left ideals, say

$$J = I_1 \cap \cdots \cap I_k$$

and then R/J embeds in $R/I_1 \oplus \cdots \oplus R/I_k$, so R/J is a semisimple R-module, so a semisimple R/J-module, so R/J is a semisimple ring. Also, since R is f.d., we can choose a chain of submodules

$$0 = R_0 \subset R_1 \subset \cdots \subset R_n = R$$

as long as possible. Then each R_i/R_{i-1} is simple. Thus $JR_i \subseteq R_{i-1}$. Thus $J^n R = 0$ so $J^n = 0$. Now if M is a module with $M = JM$, then by induction $M = J^n M$, so $M = 0$.

Clearly f.g. R-modules are the same as f.d. modules. Duality $(-)^*$ gives an antiequivalence between R-mod and mod-R.

(1) If M is a f.g. module, then M^* is f.g, so a quotient of R^n, so M is a submodule of $(R^*)^n$. Thus M embeds in a f.g. injective module, so its injective envelope is f.g.

The *socle* of an arbitrary module M is the sum of its simple submodules. For R f.d., we have

$$\operatorname{soc} M = \{m \in M : Jm = 0\}$$

and it is an essential submodule of M since any nonzero submodule contains a nonzero f.g. submodule, and any submodule of this of minimal dimension is simple, so meets $\operatorname{soc} M$. Thus $E(M) = E(\operatorname{soc} M)$.

Now $\operatorname{soc} M$ is semisimple, so a direct sum of simples, so

$$M = \bigoplus_\lambda S_\lambda \hookrightarrow \bigoplus_\lambda E(S_\lambda) =: E.$$

and

$$\begin{aligned} \operatorname{soc} E &= \{x \in \bigoplus_\lambda E(S_\lambda) : Jx = 0\} \\ &= \bigoplus_\lambda \{x \in E(S_\lambda) : Jx = 0\} \\ &= \bigoplus_\lambda S_\lambda = M, \end{aligned}$$

so $E = E(M)$. Taking M to be injective gives $M \cong E$.

(2) If M is a f.g. module, then so is M^*, so it has an injective envelope $M^* \to E(M^*)$ with $E(M^*)$ f.g. Now $P(M) = E(M^*)^*$ is flat since its dual is injective, so it is projective, since it is f.p.. Now the map $P(M) \to M$ is surjective and a projective cover.

If M is any R-module then M/JM is an R/J-module, so semisimple. Write $M/JM = \bigoplus_i S_i$, a direct sum of simples, and let $P = \bigoplus_i P(S_i)$. The map $P \to M/JM$ lifts to a map $\theta : P \to M$, and it must be surjective, since $J(M/\operatorname{Im}(\theta)) = M/\operatorname{Im}(\theta)$. Now by construction of P, the map $P/JP \to M/JM$ is an isomorphism. It follows that if $\alpha \in \operatorname{End}(P)$ is a map with $\theta\alpha = \theta$, then $\overline{\alpha} = 1$ in $\operatorname{End}(P/JP)$. Then $\phi = \alpha - 1 \in \operatorname{End}(P)$ has image contained in JP. Thus ϕ is nilpotent, so $\alpha = 1 + \phi$ is invertible. Thus $\theta : P \to M$ is a projective cover.

Apply this with M projective, and we see that M is isomorphic to a direct sum of projective covers of simples.

(3) If F is a flat left R-module, take a projective cover $\theta : P \to F$ and let L be the kernel. Thus we have an exact sequence

$$0 \to L \to P \to F \to 0. \qquad (\dagger)$$

Dualizing, we get an exact sequence of right R-modules

$$0 \to F^* \to P^* \to L^* \to 0.$$

Since F is flat, F^* is injective, so this sequence splits, so considering R/J as a right R-module, the sequence

$$0 \to \operatorname{Hom}_R(R/J, F^*) \to \operatorname{Hom}_R(R/J, P^*) \to \operatorname{Hom}_R(R/J, L^*) \to 0$$

is exact. By Hom-tensor adjointness, for any right R-module, we get

$$\operatorname{Hom}_R(R/J, M^*) \cong ((R/J) \otimes_R M)^* \cong (M/JM)^*.$$

Thus we have an exact sequence

$$0 \to (F/JF)^* \to (P/JP)^* \to (L/JL)^* \to 0.$$

But by the construction above, the map $P/JP \to F/JF$ is an isomorphism. Thus $(L/JL)^* = 0$, so $L/JL = 0$, so $L = 0$. Thus $F \cong P$ is projective. (Alternatively, using that F is flat, the long exact sequence for Tor, which comes later, implies that the sequence obtained by tensoring $(\dagger)$ with R/J is exact, so $0 \to L/JL \to P/JP \to F/JF \to 0$ is exact. But $P/JP \to F/JF$ is an isomorphism, so $L/JL = 0$, so $L = 0$.) $\square$

Remark. The rings for which all left modules have projective covers are the 'left perfect rings'. They are also the rings for which flat = projective, so the best generalization of the last theorem is a theorem of Bican, El Bashir and Enochs 2001, that every module has a flat cover. The proof is much harder.

3 Complexes

3.1 Chain and cochain complexes

Definition. Let R be a ring. A *chain complex* C (or $C.$ or C_*) of R-modules consists of modules and homomorphisms

$$\ldots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \to \ldots$$

satisfying $d_{n-1}d_n = 0$ for all n. The elements of C_n are called *chains* of degree n or *n-chains.* The morphisms d_n are the *differential*, also denoted d or d_n^C.

If C is a chain complex, then its *homology* is defined by

$$H_n(C) = \operatorname{Ker}(d_n)/\operatorname{Im}(d_{n+1}) = Z_n(C)/B_n(C).$$

The elements of $B_n(C)$ are *n-boundaries.* The elements of $Z_n(C)$ are *n-cycles.*

A chain complex C is *non-negative* if $C_n = 0$ for $n < 0$. It is *bounded* if there are only finitely many nonzero C_n. It is *acyclic* if $H_n(C) = 0$ for all n, that is, if it is an exact sequence.

Definition. A *cochain complex* C (or $C^{\cdot}$ or C^*) consists of R-modules and homomorphisms

$$\ldots \to C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \ldots$$

satisfying $d^{n+1}d^n = 0$ for all n. The elements of C^n are called *cochains* of degree n or *n-cochains.* The differential is also denoted d_C^n.

The *cohomology* of a cochain complex is defined by

$$H^n(C) = \operatorname{Ker}(d^n)/\operatorname{Im}(d^{n-1}) = Z^n(C)/B^n(C).$$

The elements of $B^n(C)$ are *n-coboundaries.* The elements of $Z^n(C)$ are *n-cocycles.*

A cochain complex C is *non-negative* if $C^n = 0$ for $n < 0$. It is *bounded* if there are only finitely many nonzero C^n. It is *acyclic* if $H^n(C) = 0$ for all n.

Remarks. (i) There is no difference between chain and cohain complexes, apart from numbering. Pass between them by setting $C^n = C_{-n}$, $d^n = d_{-n}$.

(ii) Many complexes are zero to the right, so naturally thought of as non-negative chain complexes, or zero to the left, so naturally thought of as non-negative cochain complexes.

Definition. If C is a chain complex of right R-modules and M is a left R-module, the *homology of C with coefficients in M* is

$$H_n(C; M) := H_n(C \otimes_R M)$$

where $C \otimes_R M$ is the chain complex with

$$(C \otimes_R M)_n = C_n \otimes_R M, \quad d_n^{C \otimes_R M} = d_n^C \otimes \mathrm{Id}_M.$$

If C is a chain complex of left R-modules and M is a left R-module M, the *cohomology of C with coefficients in M* is

$$H^n(C; M) := H^n(\mathrm{Hom}(C, M))$$

where $\mathrm{Hom}(C, M)$ is the cochain complex of $\mathbb{Z}$-modules (or R-modules if R is commutative, or K-modules if R is a K-algebra) with

$$\mathrm{Hom}(C, M)^n = \mathrm{Hom}_R(C_n, M)$$

and differential

$$d^n_{\mathrm{Hom}(C,M)} : \mathrm{Hom}_R(C_n, M) \to \mathrm{Hom}_R(C_{n+1}, M), \quad d^n_{\mathrm{Hom}(C,M)}(\theta) = \theta\, d^C_{n+1}.$$

Note that other conventions are possible, for example

$$d^n_{\mathrm{Hom}(C,M)}(\theta) = (-1)^{n+1}\theta\, d^C_{n+1}.$$

Example. If C is the acyclic chain complex

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/\mathbb{Z}2 \to 0$$

with $\mathbb{Z}/\mathbb{Z}2$ in degree 0, then $C \otimes_{\mathbb{Z}} (\mathbb{Z}/\mathbb{Z}2)$ is the chain complex

$$0 \to \mathbb{Z}/\mathbb{Z}2 \xrightarrow{0} \mathbb{Z}/\mathbb{Z}2 \xrightarrow{1} \mathbb{Z}/\mathbb{Z}2 \to 0$$

so $H_2(C, \mathbb{Z}/\mathbb{Z}2) \cong \mathbb{Z}/\mathbb{Z}2$, and $\mathrm{Hom}(C, \mathbb{Z})$ is the cochain complex

$$0 \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$$

with $\mathbb{Z}$ in degrees 1 and 2, so $H^2(C; \mathbb{Z}) \cong \mathbb{Z}/\mathbb{Z}2$.

3.2 Examples from algebraic and differential topology

Example (Simplicial homology)**.** If $v_0, \dots, v_n$ are $n+1$ points in $\mathbb{R}^N$, and the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent, then the *n-simplex* with vertices $v_0, \dots, v_n$ is

$$[v_0, \dots, v_n] := \{\text{convex span of the } v_i\} = \left\{ \sum_{i=0}^{n} \lambda_i v_i : \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

It is a closed subset of $\mathbb{R}^N$ and its vertices are uniquely determined as the extremal points. A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc.

A *face* of a simplex is a simplex given by a subset of its vertices. A *simplicial complex* in $\mathbb{R}^N$ is a finite set K of simplices, satisfying

(1) If $s \in K$ then so is every face of s.

(2) If $s, t \in K$, then their intersection is either empty or it is a face of s and t.

An *oriented simplicial complex* is a simplicial complex together with a total ordering on its vertices. We can do this by labelling its vertices $1, 2, 3, \dots$. If K is an oriented simplicial complex, its chain complex $C = C(K)$ is

$$C_n = \text{free } \mathbb{Z}\text{-module on the } n\text{-simplices in } K.$$

with differential

$$d_n([v_0, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

for $v_0 < \cdots < v_n$, where the hat means to omit that term. This is a chain complex, for example

$$\begin{aligned} d_2 d_3 [v_0, v_1, v_2, v_3] = d_2[v_1, v_2, v_3] - d_2[v_0, v_2, v_3] + d_2[v_0, v_1, d_3] - d_2[v_0, v_1, v_2] \\ = ([v_2, v_3] - [v_1, v_3] + [v_1, v_2]) \\ -([v_2, v_3] - [v_0, v_3] + [v_0, v_2]) \\ +([v_1, v_3] - [v_0, v_3] + [v_0, v_1]) \\ -([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \\ = 0. \end{aligned}$$

The *simplicial homology* of K is $H_n(C(K))$.

The naming of cycles and boundaries can be explained as follows. Let K be a oriented simplicial complex, for simplicity in $\mathbb{R}^2$. A path along the edges gives an element of C_1. The path is a *cycle* if it returns to its starting point. The path is a boundary if you can fill its interior with 2-simplices.

For example suppose K is given by vertices 1,2,3,4, with edges 1–2–3–4–1–3, and a triangle 1–2–3–1. Then C_0 free on $[1], [2], [3], [4]$, C_1 is free on $[12], [13], [14], [23], [34]$, C_2 is free on $[123]$. We have $d([123]) = [23] - [13] + [12]$, $d([12]) = [2] - [1]$, $d([13]) = [3] - [1]$, $d([14]) = [4] - [1]$, $d([23]) = [3] - [2]$, $d([34]) = [4] - [3]$. We have $Z_0(C) = C_0$ and

$$\begin{aligned} B_0(C) &= \mathbb{Z}\text{-span}([2] - [1], [3] - [1], [4] - [1], [3] - [2], [4] - [3]) \\ &= \{\alpha[1] + \beta[2] + \gamma[3] + \delta[4] : \alpha + \beta + \gamma + \delta = 0\}. \end{aligned}$$

so $H_0(C) \cong \mathbb{Z}$. Now $Z_1(C)$ is the set of $\alpha[12] + \beta[13] + \gamma[14] + \delta[23] + \epsilon[34]$ such that

$$(-\alpha - \beta - \gamma)[1] + (\alpha - \delta)[2] + (\beta + \delta - \epsilon)[3] + (\gamma + \epsilon)[4] = 0$$

and $B_1(C) = \mathbb{Z}([23] - [13] + [12])$. Then

$$Z_1(C) = B_1(C) \oplus \mathbb{Z}([13] - [14] + [34])$$

so $H_1(C) \cong \mathbb{Z}$. Also $Z_2(C) = 0$, so $H_2(C) = 0$.

Example (de Rham cohomology)**.** Let M be a smooth manifold. The *de Rham complex* is the cochain complex

$$\cdots \to 0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \dots$$

where $\Omega^0(M)$ is the set of smooth functions $M \to \mathbb{R}$ (that is all partial derivatives exist and are continuous), $\Omega^n(M)$ is the space of differential n-forms and d is the exterior derivative. The *de Rham cohomology* is $H^n_{DR}(M) = H^n(\Omega(M))$.

For example if M is an open subset of $\mathbb{R}^2$ then:

$$\begin{aligned} \Omega^1(M) =& \{\omega = p\,dx + q\,dy : \ p, q \text{ smooth functions on } M\}, \\ \Omega^2(M) =& \{h\,dx\,dy : h \text{ a smooth function on } M.\} \end{aligned}$$

For $f \in \Omega^0(M)$ we have

$$d(f) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

For $\omega = p\,dx + q\,dy \in \Omega^1(M)$ we have

$$d\omega = (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})\,dx\,dy.$$

We have $d^2 = 0$ since $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

The spaces of cocycles and coboundaries are:
$Z^1 = \{\omega \in \Omega^1(M) : d\omega = 0\}$, the set of closed 1-forms.
$B^1 = \{df : f \in \Omega^0(M)\}$, the set of exact 1-forms.
Thus $H^1_{DR}(M) = \{\text{closed 1-forms}\}/\{\text{exact 1-forms}\}$.

The Poincaré Lemma implies that $H^1_{DR}(M) = 0$ if M is an open disc in $\mathbb{R}^2$, or more generally simply connected. On the other hand $H^1_{DR}(\mathbb{R}^2 \setminus \{0\}) \neq 0$ since one can show that the 1-form

$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$$

is closed but not exact.

Example (Singular homology and cohomology)**.** Let X be a topological space. For each n, let $\Delta^n = [v_0, \ldots, v_n]$ be an n-simplex. Let C_n be the free $\mathbb{Z}$-module with basis the set of continuous maps $\sigma : \Delta^n \to X$. The image of the map might look like a deformed simplex, but it might be singular, hence the name. We can make the C_n into a chain complex via

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

where we must consider the restriction $\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$ as a mapping $\Delta^{n-1} \to X$, so an element of C_{n-1}. We get *singular homology* and *cohomology.*

(1) Suppose K is an ordered simplicial complex and $|K|$ is the union of its simplices. Then simplicial homology of K and singular homology of $|K|$ coincide.

(2) Suppose M is a manifold, then singular cohomology of M with coefficients in $\mathbb{R}$ and de Rham cohomology of M coincide (de Rham's theorem).

The proofs use results from topology and also about complexes. We shall develop the latter only.

3.3 The category of complexes

We shall work with cochain complexes. The definitions work for an abelian category $\mathcal{A}$, and some more generally for $\mathcal{A}$ an additive or preadditive category. But we do most proofs only for R-Mod.

Definition. Let $\mathcal{A}$ be a preadditive category (usually abelian, or at least additive). The *category of complexes* $C(\mathcal{A})$ has as objects the cochain complexes of objects and morphisms in $\mathcal{A}$

$$\ldots \to C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \ldots$$

satisfying $d^n d^{n-1} = 0$ for all n. We denote the differential also by d^n_C or just d. A morphism $f : C \to D$ is given by morphisms $f^n : C^n \to D^n$ for $n \in \mathbb{Z}$ such that each square in the diagram commutes

$$\begin{tikzcd}
\ldots \arrow[r] & C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d, "f^{n-1}"'] & C^n \arrow[r, "d_C^n"] \arrow[d, "f^n"'] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] & \ldots \\
\ldots \arrow[r] & D^{n-1} \arrow[r, "d_D^{n-1}"] & D^n \arrow[r, "d_C^n"] & D^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Composition of morphisms is done degreewise.

For $i \in \mathbb{Z}$ there is a *shift functor* $\Sigma^i : C(\mathcal{A}) \to C(\mathcal{A})$ defined on objects by

$$(\Sigma^i C)^n = C^{n+i}, \quad d^n_{\Sigma^i C} = (-1)^i d_C^{n+i}$$

and on morphisms $f : C \to D$ by

$$(\Sigma^i f)^n = f^{n+i}.$$

This is an automorphism of the category. It is the ith power of the functor $\Sigma = \Sigma^1$. Other notation is $C[i]$ or $T^i C$. Other names are *suspension* and *translation*.

If $\mathcal{A}$ is an abelian category, then we get the cohomology

$$H^n(C) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}) = Z^n(C)/B^n(C) \in \mathrm{ob}(\mathcal{A}).$$

Lemma. *$C(\mathcal{A})$ is a preadditive category. If $\mathcal{A}$ is additive or abelian, so is $C(\mathcal{A})$. In the last case, a sequence of complexes*

$$0 \to C \to D \to E \to 0$$

is exact if and only if the sequence in each degree

$$0 \to C^n \to D^n \to E^n \to 0$$

is exact.

Proof. Sums of morphisms, direct sums, kernels and cokernels are computed 'degreewise':

$$(f+g)^n = f^n + g^n, \quad (C \oplus D)^n = C^n \oplus D^n, \quad (\operatorname{Ker} f)^n = \operatorname{Ker}(f^n)$$

etc. $\square$

Lemma. *A morphism of complexes $f : C \to D$ for an abelian category $\mathcal{A}$ induces morphisms on cohomology $H^n(f) : H^n(C) \to H^n(D)$, giving a functor $H^n : C(\mathcal{A}) \to \mathcal{A}$.*

Proof. We do it for $\mathcal{A} = R$-Mod. An arbitrary element of $H^n(C)$ is of the form $[x]$ with $x \in Z^n(C) = \operatorname{Ker} d_C^n$. Then $f^n(x) \operatorname{Ker} d_D^n = Z^n(D)$, so induces an element $[f^n(x)] \in H^n(D)$. This is well-defined, for if $x \in B^n(C) = \operatorname{Im} d_C^{n-1}$, then $x = d_C^{n-1}(y)$ for some $y \in C^{n-1}$, but then $f^n(x) = f^n d_C^{n-1}(y) = d_D^{n-1} f^{n-1}(y) \in \operatorname{Im} d_D^{n-1} = B^n(D)$. Thus we get a mapping $H^n(f) : H^n(C) \to H^n(D)$. It is easy to see that this defines a functor. $\square$

Definition. A morphism of complexes $f : C \to D$ for an abelian category $\mathcal{A}$ is a *quasi-isomorphism* if the morphism $H^n(C) \to H^n(D)$ is an isomorphism for all n.

Example. For $0 \neq a \in \mathbb{Z}$, there is a quasi-isomorphism of complexes of $\mathbb{Z}$-modules

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "a"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"'] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/\mathbb{Z}a \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

where π is the projection.

Theorem. *A short exact sequence of complexes $0 \to C \to D \to E \to 0$ for an abelian category $\mathcal{A}$ induces a long exact sequence on cohomology*

$$\dots \to H^{n-1}(E) \to H^n(C) \to H^n(D) \to H^n(E) \to H^{n+1}(C) \to H^{n+1}(D) \to \dots$$

for suitable connecting morphisms $c^n : H^n(E) \to H^{n+1}(C)$.

Proof. For all n we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & C^n \arrow[r] \arrow[d, "d_C^n"'] & D^n \arrow[r] \arrow[d, "d_D^n"'] & E^n \arrow[r] \arrow[d, "d_E^n"'] & 0 \\
0 \arrow[r] & C^{n+1} \arrow[r] & D^{n+1} \arrow[r] & E^{n+1} \arrow[r] & 0
\end{tikzcd}$$

and the easy part of the Snake Lemma gives exact sequences on kernels of the vertical maps

$$0 \to Z^n(C) \to Z^n(D) \to Z^n(E)$$

and on cokernels

$$C^{n+1}/B^{n+1}(C) \to D^{n+1}/B^{n+1}(D) \to E^{n+1}/B^{n+1}(E) \to 0$$

This holds for all n, so shows that the rows in the following diagram are exact

$$\begin{array}{ccccccc} & & C^n/B^n(C) & \longrightarrow & D^n/B^n(D) & \longrightarrow & E^n/B^n(E) & \longrightarrow & 0 \\ & & {\scriptstyle \overline{d}^n_C}\Big\downarrow & & {\scriptstyle \overline{d}^n_C}\Big\downarrow & & {\scriptstyle \overline{d}^n_E}\Big\downarrow & & \\ 0 & \longrightarrow & Z^{n+1}(C) & \longrightarrow & Z^{n+1}(D) & \longrightarrow & Z^{n+1}(E). & & \end{array}$$

Here the vertical maps are induced by d^n_C, d^n_D and d^n_E, so the diagram commutes. Thus by the snake lemma one gets an exact sequence

$$\mathrm{Ker}(\overline{d}^n_C) \to \mathrm{Ker}(\overline{d}^n_D) \to \mathrm{Ker}(\overline{d}^n_E) \to \mathrm{Coker}(\overline{d}^n_C) \to \mathrm{Coker}(\overline{d}^n_D) \to \mathrm{Coker}(\overline{d}^n_E)$$

That is,

$$H^n(C) \to H^n(D) \to H^n(E) \to H^{n+1}(C) \to H^{n+1}(D) \to H^{n+1}(E)$$

as required. □

3.4 Mapping cones

Definition. The *mapping cone* of a morphism of complexes $f : B \to C$ in $C(\mathcal{A})$, with $\mathcal{A}$ an additive category, is the complex $\mathrm{cone}(f)$ with

$$\mathrm{cone}(f)^n = (\Sigma B)^n \oplus C^n = B^{n+1} \oplus C^n,$$

$$d^n = \begin{pmatrix} d^n_{\Sigma B} & 0 \\ f^{n+1} & d^n_C \end{pmatrix} = \begin{pmatrix} -d^{n+1}_B & 0 \\ f^{n+1} & d^n_C \end{pmatrix} : B^{n+1} \oplus C^n \to B^{n+2} \oplus C^{n+1}$$

That is, for complexes of R-modules,

$$d^n(b, c) = (-d^{n+1}_B(b), f^{n+1}(b) + d^n_C(c)).$$

Observe that $\mathrm{cone}(0 \to C) \cong C$ and $\mathrm{cone}(B \to 0) \cong \Sigma B$.

Proposition. *There is a sequence of complexes*

$$0 \to C \to \mathrm{cone}(f) \to \Sigma B \to 0$$

which in degree n is the split exact sequence

$$0 \to C^n \xrightarrow{i_{C^n}} B^{n+1} \oplus C^n \xrightarrow{p_{B^{n+1}}} B^{n+1} \to 0$$

Thus if $\mathcal{A}$ is abelian, it is an exact sequence of complexes. In the corresponding long exact sequence on cohomology, the connecting morphism

$$H^n(B) = H^{n-1}(\Sigma B) \to H^n(C)$$

is equal to $H^n(f)$.

Proof. The first part is straightforward. We do the second part for R-Mod. The connecting map $H^{n+1}(B) \to H^{n+1}(C)$ is given by the Snake Lemma from the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & H^{n+1}(B) \arrow[d] & \\
 & C^n/B^n(C) \arrow[r] \arrow[d] & \mathrm{cone}(f)^n/B^n(\mathrm{cone}(f)) \arrow[r] \arrow[d] & B^{n+1}/B^{n+1}(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{n+1}(C) \arrow[r] \arrow[d] & Z^{n+1}(\mathrm{cone}(f)) \arrow[r] & Z^{n+2}(B) & \\
 & H^{n+1}(C) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

Now an element $[b]$ of $H^{n+1}(B)$ lifts to an element $[(b,0)]$ of $\mathrm{cone}(f)^n/B^n(\mathrm{cone}(f))$, and applying the differential of $\mathrm{cone}(f)$ it gives $[(0, f^{n+1}(b))] \in Z^{n+1}(\mathrm{cone}(f))$, which comes from $[f^{n+1}(b)]$ in $H^{n+1}(C)$. □

Corollary. *If $\mathcal{A}$ is abelian, then a morphism $f : B \to C$ of complexes is a quasi-isomorphism if and only if* $\mathrm{cone}(f)$ *is acyclic.*

Proof. Follows from the long exact sequence on cohomology

$$\cdots \to H^{n-1}(\mathrm{cone}(f)) \to H^n(B) \to H^n(C) \to H^n(\mathrm{cone}(f)) \to H^{n+1}(B) \to H^{n+1}(C) \to \ldots$$

□

3.5 The homotopy category

Definition. An *ideal* in a preadditive category $\mathcal{A}$ is a class of morphisms I in $\mathcal{A}$ such that

- $I(X,Y) := I \cap \mathrm{Hom}_{\mathcal{A}}(X,Y)$ is an additive subgroup of $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ for all X, Y, and
- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in $\mathcal{A}$, and f or g is in I, then so is gf.

If I is an ideal in $\mathcal{A}$, then there is a *quotient category* $\mathcal{A}/I$, with the same objects as $\mathcal{A}$ and

$$\operatorname{Hom}_{\mathcal{A}/I}(X,Y) = \operatorname{Hom}_{\mathcal{A}}(X,Y)/I(X,Y).$$

If $\mathcal{A}$ is an additive category, so is $\mathcal{A}/I$.

Definition. A morphisms $f : C \to D$ of complexes is *null-homotopic* if there are morphisms $h^n : C^n \to D^{n-1}$ for all $n \in \mathbb{Z}$ such that

$$f^n = h^{n+1} d_C^n + d_D^{n-1} h^n$$

for all $n \in \mathbb{Z}$. Two morphisms $f, g : C \to D$ of complexes are *homotopic* if $f - g$ is null-homotopic.

Proposition/Definition. *The null-homotopic morphisms form an ideal in the category of $C(\mathcal{A})$, so we get the quotient category*

$$K(\mathcal{A}) := \mathcal{A}/\{\textit{null-homotopic morphisms}\}$$

is called the homotopy category *of $\mathcal{A}$. The morphisms are the homotopy classes of morphisms in $C(\mathcal{A})$. If $\mathcal{A}$ is additive, so is $K(\mathcal{A})$.*

Proof. We need to show that if $C \xrightarrow{f} D \xrightarrow{g} E$ are morphisms of complexes and f or g is null-homotopic, then so is gf. If f is null-homotopic with morphisms h^n, then

$$\begin{aligned}(gf)^n = g^n f^n &= g^n(h^{n+1} d_C^n + d_D^{n-1} h^n) \\ &= g^n h^{n+1}) d_C^n + g^n d_D^{n-1} h^n \\ &= (g^n h^{n+1}) d_C^n + d_E^{n-1} (g^{n-1} h^n).\end{aligned}$$

Similarly if g is null-homotopic. $\square$

Lemma. *If $\mathcal{A}$ is abelian, then homotopic morphisms $f, g : C \to D$ induce the same morphism on cohomology $H^n(f) = H^n(g) : H^n(C) \to H^n(D)$. Thus cohomology induces a functor on the homotopy category*

$$H^n : K(\mathcal{A}) \to \mathcal{A}$$

also denoted by H^n.

Proof. It suffices to show that null-homotopic morphisms induce the zero morphism on cohomology. We do it for R-modules. Thus suppose

$$f^n = h^{n+1} d_C^n + d_D^{n-1} h^n$$

An element of $H^n(C)$ is represented by an element $x \in Z^n(C) = \operatorname{Ker} d_C^n$. Then $f^n(x) = d_D^{n-1} h^n(x) \in \operatorname{Im}(d_D^{n-1}) = B^n(D)$, so $H^n(f)([x]) = [f^n(x)]$ is zero in $H^n(D)$. $\square$

Definition. A morphism $f : C \to D$ is a *homotopy equivalence* if its image in $K(\mathcal{A})$ is an isomorphism. Equivalently, there is a morphism $g : D \to C$ such that fg is homotopic to Id_D and gf is homotopic to Id_C.

Lemma. *If $\mathcal{A}$ is abelian, then a homotopy equivalence is a quasi-isomorphism.*

Proof. A homotopy equivalence gives an isomorphism in $K(\mathcal{A})$, so it is sent to an isomorphism by H^n. $\square$

Lemma. *An additive functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor*

$$F : C(\mathcal{A}) \to C(\mathcal{B}), \quad F(C)^n = F(C^n),\ d^n_{F(C)} = F(d^n_C),$$

and this induces a functor $\overline{F} : K(\mathcal{A}) \to K(\mathcal{B})$. Similarly an additive functor $F : \mathcal{A}^{op} \to \mathcal{B}$ induces a functor

$$F : C(\mathcal{A})^{op} \to C(\mathcal{B}), \quad F(C)^n = F(C^{-n}),\ d^n_{F(C)} = F(d^{-n-1}_C),$$

and this induces a functor $\overline{F} : K(\mathcal{A})^{op} \to K(\mathcal{B})$.

Proof. If $f : C \to D$ is null-homotopic, then

$$f^n = h^{n+1} d^n_C + d^{n-1}_D h^n$$

so

$$F(f^n) = F(h^{n+1}) d^n_{F(C)} + d^{n-1}_{F(D)} F(h^n),$$

so $F(f)$ is null-homotopic. $\square$

Corollary. *If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor and $f : C \to D$ is a homotopy equivalence in $C(\mathcal{A})$, then $F(f) : F(C) \to F(D)$ is a homotopy equivalence. In particular, if $\mathcal{B}$ is abelian, then $F(f)$ is a quasi-isomorphism. Similarly for a contravariant functor.*

Proof. f becomes an isomorphism in $K(\mathcal{A})$, so $\overline{F}(f)$ is an isomorphism, but this is the image of $F(f)$ in $K(\mathcal{B})$. $\square$

Remark. If M is a left R-module and $f : C \to D$ is a homotopy equivalence of complexes of right R-modules, taking $F = - \otimes_R M$, one gets an isomorphism

$$H_n(C; M) \to H_n(D; M)$$

on homology with coefficients. Similarly if $C \to D$ is a homotopy equivalence of complexes of left R-modules, one get an isomorphism

$$H^n(D; M) \to H^n(C; M)$$

on cohomology with coefficients.

Definition. A complex C is *contractible* if it is homotopy equivalent to the zero complex, or equivalently if Id_C is null-homotopic.

Proposition. *If $\mathcal{A}$ is abelian, a complex is contractible if and only if it is acyclic (i.e. exact) and all of the short exact sequences*

$$0 \to Z^n(C) \xrightarrow{i_n} C^n \xrightarrow{d_C^n} B^{n+1}(C) \to 0$$

are split.

Proof. We do it for R-Mod. If C is contractible, then it is quasi-isomorphic to the zero complex, so it is acyclic. Now Id_C is null-homotopic, so there are $h^n : C^n \to C^{n-1}$ with

$$\mathrm{Id}_{C^n} = h^{n+1} d_C^n + d_C^{n-1} h^n$$

Let $s^{n+1} : B^{n+1}(C) = Z^{n+1}(C) \to C^n$ be the restriction of h^{n+1}. If $x \in B^{n+1}(C)$, then

$$x = \mathrm{Id}_{C^{n+1}}(x) = (h^{n+2} d_C^{n+1} + d_C^n h^{n+1})(x) = d_C^n h^{n+1}(x) = d_C^n s^{n+1}(x)$$

so s^{n+1} is a section for the short exact sequence $0 \to Z^n(C) \to C^n \to B^{n+1}(C) \to 0$.

Now suppose that C is acyclic and the short exact sequences are all split, with sections $s^{n+1} : B^{n+1}(C) \to C^n$. If $x \in C^n$, then $x - s^{n+1} d_C^n(x) \in Z^n(C) = B^n(C)$, so we can define a homomorphism $h^n : C^n \to C^{n-1}$ by

$$h^n(x) = s^n(x - s^{n+1} d_C^n(x)).$$

Then

$$\begin{aligned}(h^{n+1} d_C^n + d_C^{n-1} h^n)(x) &= s^{n+1}(d_C^n(x) - s^{n+2} d_C^{n+1}(d_C^n(x))) + d_C^{n-1} s^n(x - s^{n+1} d_C^n(x)) \\ &= s^{n+1} d_C^n(x) + (x - s^{n+1} d_C^n(x)) = x.\end{aligned}$$

Thus C is contractible. □

4 Resolutions, Ext and Tor

4.1 Projective and injective resolutions

Definition. We suppose that $\mathcal{A}$ is an abelian category which has *enough projectives*, meaning that for every object M there is an epimorphism from a projective object to M, for example R-Mod.
A *projective resolution* of M is an exact sequence

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

with the P_i projective. It is equivalent to give a non-negative chain complex P of projectives and a quasi-isomorphism $P \to M$ (with M considered as a chain complex in degree 0),

$$\begin{tikzcd}
\dots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\epsilon"'] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

The *syzygies* of M with respect to this projective resolution are the objects

$$\Omega_n M = \mathrm{Im}(d_n : P_n \to P_{n-1}) = \begin{cases} \mathrm{Ker}(\epsilon : P_0 \to M) & (n = 1) \\ \mathrm{Ker}(d_{n-1} : P_{n-1} \to P_{n-2}) & (n > 1) \end{cases}$$

and $\Omega_0 M = M$. Thus there are exact sequences

$$0 \to \Omega_{n+1} M \to P_n \to \Omega_n M \to 0.$$

Note that object module has many different projective resolutions. Choose any epimorphism $\epsilon : P_0 \to M$. This gives $\Omega_1 M$. Then choose any epimorphism $d_1 : P_1 \to \mathrm{Ker}\,\Omega_1 M$, then any epimorphism $d_2 : P_2 \to \Omega_2 M$, etc.

Dually, suppose that $\mathcal{A}$ has *enough injectives*, meaning that every object has a monomorphism to an injective object. An *injective resolution* of an object X is an exact sequence

$$0 \to X \to I^0 \to I^1 \to I^2 \to \dots$$

with the I^n injective. The *cosyzygies* are $\Omega^n X = \mathrm{Im}(I^{n-1} \to I^n)$ (and), so

$$0 \to \Omega^n X \to I^n \to \Omega^{n+1} X \to 0.$$

Injective resolutions in $\mathcal{A}$ are exactly the same as projective resolutions in $\mathcal{A}^{op}$.

Theorem (Comparison Theorem). *Given projective resolutions $\epsilon : P \to M$ and $\epsilon' : P' \to M'$, any morphism $f : M \to M'$ can be lifted to a morphism of projective resolutions*

$$\begin{array}{ccccccccc}
\dots \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle g_2} & & \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle g_0} & & \Big\downarrow{\scriptstyle f} & \\
\dots \longrightarrow & P_2' & \xrightarrow{d_2'} & P_1' & \xrightarrow{d_1'} & P_0' & \xrightarrow{\epsilon'} & M' & \longrightarrow 0
\end{array}$$

or equivalently to a morphism of complexes $g : P \to P'$ with $\epsilon' g_0 = f\epsilon$. Moreover g is unique up to homotopy.

Proof. Consider the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_1 M & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \Omega_1 g} & & \Big\downarrow{\scriptstyle g_0} & & \Big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \Omega_1 M' & \longrightarrow & P_0' & \longrightarrow & M' & \longrightarrow & 0.
\end{array}$$

Since P_0 is projective and $P_0' \to M'$ is an epimorphism, there is a morphism g_0 making the right hand square commute. Then there is an induced morphism $\Omega_1 g$ making the left hand square commute.

Now the same argument gets g_1 and $\Omega_2 g$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_2 M & \longrightarrow & P_1 & \longrightarrow & \Omega_1 M & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \Omega_2 g} & & \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle \Omega_1 g} & & \\
0 & \longrightarrow & \Omega_2 M' & \longrightarrow & P_1' & \longrightarrow & \Omega_1 M' & \longrightarrow & 0
\end{array}$$

etc.

To show that any two lifts are homotopic, it is equivalent to show that any lift g of the zero morphism $M \to M'$ is null-homotopic. Say

$$\begin{array}{ccccccccc}
\dots \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle g_2} & & \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle g_0} & & \Big\downarrow{\scriptstyle 0} & \\
\dots \longrightarrow & P_2' & \xrightarrow{d_2'} & P_1' & \xrightarrow{d_1'} & P_0' & \xrightarrow{\epsilon'} & M' & \longrightarrow 0
\end{array}$$

Since $\epsilon' g_0 = 0$ we have $\mathrm{Im}(g_0) \subseteq \mathrm{Ker}(\epsilon') = \Omega_1 M'$, then since $P_1' \to \Omega_1 M'$ is an epimorphism, g_0 lifts to a morphism $h_0 : P_0 \to P_1'$ with $d_1' h_0 = g_0$. By induction we find morphisms $h_n : P_n \to P_{n+1}'$ for $n > 0$ with $g_n = d_{n+1}' h_n + h_{n-1} d_n$. Having found $h_1, \dots, h_{n-1}$, we have

$$d_n'(g_n - h_{n-1} d_n) = g_{n-1} d_n - d_n' h_{n-1} d_n = (g_{n-1} - d_n' h_{n-1}) d_n.$$

This is zero both if $n = 1$ or $n > 1$. Thus $\mathrm{Im}(g_n - h_{n-1}d_n) \subseteq \Omega_{n+1}M'$. Thus it lifts to a morphism $g_n : P_n \to P'_{n+1}$. □

Corollary. *If $\epsilon : P \to M$ and $\epsilon' : P' \to M$ are projective resolutions of M, then there is a homotopy equivalence $g : P \to P'$ with $\epsilon' g_0 = \epsilon$. Moreover g is unique up to homotopy.*

Proof. The identity Id_M lifts to a morphism $g : P \to P'$ and to a morphism $g' : P' \to P$. Now $g'g - \mathrm{Id}_P$ is a lift of the zero morphism $M \to M$, so is null-homotopic, and so is $gg' - \mathrm{Id}_{P'}$. □

Lemma (Horseshoe Lemma)**.** *Given a short exact sequence*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

and projective resolutions $P' \to M'$ and $P'' \to M''$, we can find a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & P'_1 \arrow[r] \arrow[d, "\binom{1}{0}"'] & P'_0 \arrow[r, "\epsilon'"] \arrow[d, "\binom{1}{0}"'] & M' \arrow[r] \arrow[d, "f"'] & 0 \\
\dots \arrow[r] & P'_1 \oplus P''_1 \arrow[r] \arrow[d, "(0\ 1)"'] & P'_0 \oplus P''_0 \arrow[r] \arrow[d, "(0\ 1)"'] & M \arrow[r] \arrow[d, "g"'] & 0 \\
\dots \arrow[r] & P''_1 \arrow[r] \arrow[d] & P''_0 \arrow[r, "\epsilon''"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the middle row is a projective resolution of M.

Proof. Since g is an epimorphism and P''_0 is projective, we have $\epsilon'' = gh$ for some

$h : P_0'' \to M$. Then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
P_0' \arrow[r, "\epsilon'"] \arrow[d, "\binom{1}{0}"'] & M' \arrow[r] \arrow[d, "f"'] & 0 \\
P_0' \oplus P_0'' \arrow[r, "(f\epsilon' \ h)"] \arrow[d, "(0 \ 1)"'] & M \arrow[r] \arrow[d, "g"'] & 0 \\
P_0'' \arrow[r, "\epsilon''"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

By the snake lemma, the sequence of syzygies $0 \to \Omega_1 M' \to \Omega_1 M \to \Omega_1 M' \to 0$ is exact. Now iterate. $\square$

4.2 Derived functors

Definition. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor between abelian categories and $\mathcal{A}$ has enough projective. For any $M \in \mathrm{ob}(\mathcal{A})$, we fix a projective resolution $P \to M$. For $n \geq 0$, the *nth left derived functor* of F is the functor $L_n F : \mathcal{A} \to \mathcal{B}$ given by

$$L_n F(M) = H_n(F(P)),$$

the nth homology of the chain complex

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0) \to 0$$

A morphism $f : M \to M'$ lifts to a morphism of projective resolutions $g : P \to P'$, unique up to homotopy. Then $F(g)$ is a morphism : $F(P) \to F(P')$, unique up to homotopy, so it induces unique morphisms $H_n(F(P)) \to H_n(F(P'))$, that is $L_n F(M) \to L_n F(M')$. This makes $L_n F$ a functor.

Proposition. *(i) $L_n F(M)$ is independent of the projective resolution of M.*
(ii) $L_n F(M) = 0$ for $n < 0$ and $L_0 F(M) \cong F(M)$.
(iii) $L_n F(M) = 0$ for M projective and $n > 0$.
(iv) Any short exact sequence $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence

$$\cdots \to L_2 F(M'') \to L_1 F(M') \to L_1 F(M) \to L_1 F(M'') \to F(M') \to F(M) \to F(M'') \to 0.$$

Proof. (i) Any two projective resolutions P, P' of M have a homotopy equivalence $P \to P'$. Thus $F(P) \to F(P')$ is a homotopy equivalence, so a quasi-isomorphism. Thus $H_n(F(P)) \cong H_n(F(P'))$.

(ii) Since P is a non-negative chain complex, so is $F(P)$, so $L_n F(M) = 0$ for $n < 0$. Since F is right exact, the exact sequence $P_1 \to P_0 \to M \to 0$ gives an exact sequence

$$F(P_1) \to F(P_0) \to F(M) \to 0,$$

so $H_0(F(P)) \cong F(M)$.

(iii) If M is projective it has a projective resolution with $P_0 = M$ and $P_i = 0$ for $i > 0$.

(iv) Given an exact sequence $0 \to M' \to M \to M'' \to 0$, by the Horseshoe Lemma we get an exact sequence of projective resolutions $0 \to P' \to P \to P'' \to 0$. Since the sequences $0 \to P'_n \to P_n \to P''_n \to 0$ are split, they stay exact under F, so we get an exact sequence of complexes

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

and hence the long exact sequence on homology. $\square$

Remark. Variations. Replacing $\mathcal{A}$ by $\mathcal{A}^{op}$ and/or $\mathcal{B}$ by $\mathcal{B}^{op}$, and noting that a functor $\mathcal{A}^{op} \to \mathcal{B}^{op}$ is the same thing as a functor $\mathcal{A} \to \mathcal{B}$, we get the following variants.

	Right exact	Left exact
Covariant, $F : \mathcal{A} \to \mathcal{B}$	done	(b)
Contravariant, $F : \mathcal{A}^{op} \to \mathcal{B}$	(a)	(c)

(a) If $F : \mathcal{A}^{op} \to \mathcal{B}$ is a right exact functor and $\mathcal{A}$ has enough injectives, then the nth left derived functor $L_n F : \mathcal{A}^{op} \to \mathcal{B}$ is defined by $L_n F(M) = H_n(F(I))$ where $M \to I$ is a fixed injective resolution of X. A short exact sequence $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence

$$\cdots \to L_2F(M') \to L_1F(M'') \to L_1F(M) \to L_1F(M') \to F(M'') \to F(M) \to F(M') \to 0.$$

(b) If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor and $\mathcal{A}$ has enough injectives, then the nth right derived functor $R^n F : \mathcal{A} \to \mathcal{B}$ is defined by $R^n F(M) = H^n(F(I))$ where $M \to I$ is a fixed injective resolution of X. A short exact sequence $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence

$$0 \to F(M') \to F(M) \to F(M'') \to R^1F(M') \to R^1F(M) \to R^1F(M'') \to R^2F(M') \to \dots$$

(c) If $F : \mathcal{A}^{op} \to \mathcal{B}$ is a left exact functor and $\mathcal{A}$ has enough projectives, then the nth right derived functor $R^nF : \mathcal{A}^{op} \to \mathcal{B}$ is defined by $R^nF(M) = H^n(F(P))$ where $P \to M$ is a fixed projective resolution of M. A short exact sequence $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence

$$0 \to F(M'') \to F(M) \to F(M') \to R^1F(M'') \to R^1F(M) \to R^1F(M') \to R^2F(M'') \to \dots$$

Example. If X is topological space, there is a category $Sh(X)$ of sheaves of abelian groups on X. It is a Grothendieck category, so has enough injectives. The global section functor

$$\Gamma(X, -) : Sh(X) \to \mathrm{Ab}$$

is left exact, so it has right derived functors $H^n(X, -) = R^n\Gamma(X, -)$. This is *sheaf cohomology.* For a nice enough topological space (locally contractible), one has

$$H^n(X, \mathbb{Z}_X) \cong H^n_{sing}(X; \mathbb{Z}),$$

where $\mathbb{Z}_X$ is the constant sheaf on X and the right hand side is singular cohomology with coefficients in $\mathbb{Z}$.

4.3 Ext

Definition. For each R-module M, fix a projective resolution $P \to M$. Given an R-module X, we define $\mathrm{Ext}^n_R(M, X)$ to be the cohomology of the cochain complex

$$\dots \to 0 \to \mathrm{Hom}_R(P_0, X) \to \mathrm{Hom}_R(P_1, X) \to \mathrm{Hom}_R(P_2, X) \to \dots$$

That is,

$$\mathrm{Ext}^n_R(M, X) = H^n(P; X) = H^n(\mathrm{Hom}_R(P, X)) = (R^n \mathrm{Hom}_R(-, X))(M)$$

using the right derived functors of the left exact functor

$$\mathrm{Hom}_R(-, X) : R\text{-}\mathrm{Mod}^{op} \to \mathrm{Ab}\,.$$

The results about derived functors give:

Proposition. *(i)* $\mathrm{Ext}^n_R(M, X)$ *is independent of the projective resolution of* M.
(ii) $\mathrm{Ext}^0_R(M, X) \cong \mathrm{Hom}_R(M, X)$.
(iii) $\mathrm{Ext}^n_R(M, X) = 0$ *for* M *projective and* $n > 0$.
(iv) Any short exact sequence $0 \to M' \to M \to M'' \to 0$ *induces a long exact sequence*

$$0 \to \mathrm{Hom}_R(M'', X) \to \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M', X) \to$$

$$\mathrm{Ext}^1_R(M'', X) \to \mathrm{Ext}^1_R(M, X) \to \mathrm{Ext}^1_R(M', X) \to \mathrm{Ext}^2_R(M'', X) \to \dots$$

Further properties.

Proposition. *(i) By definition* $\mathrm{Ext}^n_R(M,X)$ *is a contravariant functor in* M*, but it is also a covariant functor in* X*. If* R *is a* K*-algebra (e.g.* $K=\mathbb{Z}$ *for a ring), we get a bifunctor*

$$\mathrm{Ext}^n_R(-,-) : R\text{-Mod}^{op} \times R\text{-Mod} \to K\text{-Mod}$$

which is K*-linear in each argument.*
(ii) $\mathrm{Ext}^n_R(M,X)=0$ *for* $n>0$ *and* X *injective.*
(iii) A short exact sequence $0 \to X' \to X \to X'' \to 0$ *induces a long exact sequence*

$$0 \to \mathrm{Hom}_R(M,X') \to \mathrm{Hom}_R(M,X) \to \mathrm{Hom}_R(M,X'') \to$$

$$\mathrm{Ext}^1_R(M,X') \to \mathrm{Ext}^1_R(M,X) \to \mathrm{Ext}^1_R(M,X'') \to \mathrm{Ext}^2_R(M,X') \to \dots$$

Proof. (i) Let $P \to M$ be a projective resolution of M. A morphism $X \to X'$ induces a morphism of complexes $\mathrm{Hom}_R(P,X) \to \mathrm{Hom}_R(P,X')$, and hence morphisms $\mathrm{Ext}^n_R(M,X) \to \mathrm{Ext}^n_R(M,X')$.

If R is a K-algebra, then any space $\mathrm{Hom}_R(M,X)$ is a K-module, and an morphism $M \to M'$ or $X \to X'$ induces a morphism of K-modules. Now we need that if $X \to X'$ and if $M \to M'$, then the square

$$\begin{array}{ccc}
\mathrm{Ext}^n(M',X) & \longrightarrow & \mathrm{Ext}^n(M,X) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^n(M',X') & \longrightarrow & \mathrm{Ext}^n(M,X')
\end{array}$$

commutes. This holds because if $P \to P'$ is a lift of $M \to M'$, then the square of complexes

$$\begin{array}{ccc}
\mathrm{Hom}(P',X) & \longrightarrow & \mathrm{Hom}(P,X) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(P',X') & \longrightarrow & \mathrm{Hom}(P,X')
\end{array}$$

commutes.

(ii) Holds since $\mathrm{Hom}_R(-,X)$ is exact.

(iii) If $P \to M$ is a projective resolution, then since each P_n is projective, one gets an exact sequence of complexes

$$0 \to \mathrm{Hom}_R(P,X') \to \mathrm{Hom}_R(P,X) \to \mathrm{Hom}_R(P,X'') \to 0.$$

This induces a long exact sequence on cohomology. $\square$

Theorem. *If* $0 \to X \to I^0 \to I^1 \to I^2 \to \dots$ *is an injective resolution of* X*, then one can compute* $\operatorname{Ext}^n_R(M,X)$ *as the nth cohomology of the complex* $\operatorname{Hom}_R(M,I)$*:*

$$0 \to \operatorname{Hom}_R(M,I^0) \to \operatorname{Hom}_R(M,I^1) \to \operatorname{Hom}_R(M,I^2) \to \dots$$

Proof. Break the injective resolution into exact sequences

$$0 \to \Omega^i X \to I^i \to \Omega^{i+1} X \to 0$$

for $i \geq 0$ where $\Omega^0 X = X$. One gets long exact sequences

$$0 \to \operatorname{Hom}_R(M,\Omega^i X) \to \operatorname{Hom}_R(M,I^i) \to \operatorname{Hom}_R(M,\Omega^{i+1} X)$$

$$\to \operatorname{Ext}^1_R(M,\Omega^i X) \to 0 \to \operatorname{Ext}^1_R(M,\Omega^{i+1} X)$$
$$\to \operatorname{Ext}^2_R(M,\Omega^i X) \to 0 \to \operatorname{Ext}^2_R(M,\Omega^{i+1} X) \dots$$

so

$$\operatorname{Ext}^1_R(M,\Omega^i X) \cong \operatorname{Coker}(\operatorname{Hom}_R(M,I^i) \to \operatorname{Hom}_R(M,\Omega^{i+1} X))$$

and

$$\operatorname{Ext}^j_R(M,\Omega^{i+1} X) \cong \operatorname{Ext}^{j+1}_R(M,\Omega^i X)$$

for $j \geq 1$. Thus (it is called *dimension shifting*)

$$\operatorname{Ext}^n_R(M,X) \cong \operatorname{Ext}^{n-1}_R(M,\Omega^1 X) \cong \dots \cong \operatorname{Ext}^1_R(M,\Omega^{n-1} X)$$

$$\cong \operatorname{Coker}\big(\operatorname{Hom}_R(M,I^{n-1}) \to \operatorname{Hom}_R(M,\Omega^n X)\big)$$
$$= \frac{\operatorname{Hom}_R(M,\Omega^n X)}{\operatorname{Im}(\operatorname{Hom}_R(M,I^{n-1}) \to \operatorname{Hom}_R(M,\Omega^n X))}$$

Now $0 \to \Omega^n X \to I^n \to I^{n+1}$ is exact, hence so is

$$0 \to \operatorname{Hom}_R(M,\Omega^n X) \to \operatorname{Hom}_R(M,I^n) \to \operatorname{Hom}_R(M,I^{n+1}),$$

so we can identify

$$\operatorname{Ext}^n_R(M,X) \cong \frac{\operatorname{Ker}(\operatorname{Hom}_R(M,I^n) \to \operatorname{Hom}_R(M,I^{n+1}))}{\operatorname{Im}(\operatorname{Hom}_R(M,I^{n-1}) \to \operatorname{Hom}_R(M,I^n))}$$

which is the cohomology in degree n of the complex $\operatorname{Hom}_R(M,I)$. □

Remark. We have defined $\operatorname{Ext}^n_R(M,X)$ as the right derived functor $R^n \operatorname{Hom}_R(-,X)$ applied to M. Instead one can consider the right derived functor $R^n \operatorname{Hom}_R(M,-)$ applied to X. The theorem shows that you get the same result. With more care one can show that the resulting bifunctors are isomorphic.

Examples. (1) If R is a semisimple (artinian) ring then all short exact sequences of R-modules are split exact, so all modules are projective and injective. Thus

$$\mathrm{Ext}^n_R(M,X) \cong \begin{cases} \mathrm{Hom}_R(M,X) & (n=0) \\ 0 & (n>0). \end{cases}$$

(2) If R is a pid and $0 \neq a \in R$ then R/Ra has projective resolution $0 \to R \xrightarrow{a} R \to R/Ra \to 0$. Thus $\mathrm{Ext}^n_R(R/Ra,X)$ is the cohomology of the complex

$$\dots \to 0 \to \mathrm{Hom}(R,X) \xrightarrow{a} \mathrm{Hom}(R,X) \to 0 \to \dots$$

that is,

$$\dots \to 0 \to X \xrightarrow{a} X \to 0 \to \dots$$

so $\mathrm{Ext}^0_R(R/Ra,X) = \mathrm{Hom}(R/Ra,X) \cong \{x \in X : ax = 0\}$, $\mathrm{Ext}^1_R(R/Ra,X) \cong X/aX$ and $\mathrm{Ext}^n_R(R/Ra,X) = 0$ for $n > 1$.

(3) Let $R = K[x]/(x^2)$ with K a field. Any finitely generated module is a direct sum of copies of K (with x acting as 0) and R. The module K has projective resolution

$$\to R \xrightarrow{x} R \xrightarrow{x} R \to K \to 0.$$

Now $\mathrm{Hom}_R(R,K) \cong K$, and we get $\mathrm{Ext}^n_R(K,K) \cong K$ for all $n \geq 0$.

4.4 Description of Ext^1 using short exact sequences

Definition. Two short exact sequences ξ, ξ' with the same end terms are *equivalent* if there is a map θ (necessarily an isomorphism) giving a commutative diagram

$$\begin{tikzcd}
\xi : 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\theta"'] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi' : 0 \arrow[r] & L \arrow[r] & M' \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

It is easy to see that the split exact sequences form one equivalence class.

Definition. For any short exact sequence of modules

$$\xi : 0 \to L \to M \to N \to 0$$

we define an element $\hat{\xi} \in \mathrm{Ext}^1(N,L)$ as follows. The long exact sequence for $\mathrm{Hom}(N,-)$ applied to ξ gives a connecting map $\mathrm{Hom}(N,N) \to \mathrm{Ext}^1(N,L)$, and $\hat{\xi}$ is the image of Id_N under this map.

One can show that $\hat{\xi}$ is also the image of Id_L under the connecting map $\mathrm{Hom}(L,L) \to \mathrm{Ext}^1(N,L)$ in the long exact sequence obtained by applying $\mathrm{Hom}(-,L)$ to ξ.

Theorem. *The assignment $\xi \mapsto \hat{\xi}$ gives a bijection between equivalence classes of short exact sequences $0 \to L \to M \to N \to 0$ and elements of $\operatorname{Ext}^1_R(N, L)$. The split exact sequences correspond to the element $0 \in \operatorname{Ext}^1_R(N, L)$.*

Proof. Fix a projective resolution of N, and hence an exact sequence

$$0 \to \Omega_1 N \xrightarrow{\theta} P_0 \xrightarrow{\epsilon} N \to 0.$$

An exact sequence ξ gives a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \operatorname{Hom}(N,N) \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}(P_0,L) \arrow[r] \arrow[d, "\theta^*"'] & \operatorname{Hom}(P_0,M) \arrow[r] \arrow[d] & \operatorname{Hom}(P_0,N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Hom}(\Omega_1 N,L) \arrow[r] \arrow[d] & \operatorname{Hom}(\Omega_1 N,M) \arrow[r] & \operatorname{Hom}(\Omega_1 N,N) & \\
 & \operatorname{Ext}^1(N,L) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

and the connecting map $\operatorname{Hom}(N, N) \to \operatorname{Ext}^1(N, L)$ is given by diagram chasing, so by the choice of maps α, β giving a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_1 N \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0.
\end{tikzcd}$$

Then $\hat{\xi} = [\alpha]$ where $[\dots]$ denotes the map $\operatorname{Hom}(\Omega_1 N, L) \to \operatorname{Ext}^1(N, L)$.

Any element of $\operatorname{Ext}^1(N, L)$ arises from some ξ. Namely, write it as $[\alpha]$ for some $\alpha \in \operatorname{Hom}(\Omega_1 N, L)$. Then take ξ to be the pushout

$$\begin{tikzcd}
0 \arrow[r] & \Omega_1 N \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Now if ξ, ξ' are equivalent exact sequences one gets a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_1 N & \longrightarrow & P_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & \big\| & & \\
\xi : 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow & & \big\| & & \\
\xi' : 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

so ξ and ξ' correspond to the same map α, so $\hat{\xi} = \hat{\xi}'$. If two short exact sequences ξ, ξ' give the same element of $\mathrm{Ext}^1(N, L)$ there are diagrams with maps α, β and α', β' and with $\alpha - \alpha'$ in the image of the map $\theta^* : \mathrm{Hom}(P_0, L) \to \mathrm{Hom}(\Omega_1 N, L)$. Say $\alpha - \alpha' = \phi\theta$ with $\phi : P_0 \to L$. Then there is also a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_1 N & \xrightarrow{\theta} & P_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha'}\big\downarrow & & {\scriptstyle\beta - f\phi}\big\downarrow & & \big\| & & \\
\xi : 0 & \longrightarrow & L & \xrightarrow{f} & M & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

This is a pushout, so by the uniqueness of pushouts, ξ and ξ' are equivalent. $\square$

Remark. Homomorphisms $L \to L'$ and $N'' \to N$ induce maps $\mathrm{Ext}^1(N, L) \to \mathrm{Ext}^1(N, L')$ and $\mathrm{Ext}^1(N, L) \to \mathrm{Ext}^1(N'', L)$. One can show that these maps correspond to pushouts and pullbacks of short exact sequences. For pushouts this follows directly from the definition. For pullbacks it needs more thought.

Theorem. *The following are equivalent for a module M.*
(i) M is projective
(ii) $\mathrm{Ext}^n(M, X) = 0$ *for all X and all $n > 0$.*
(iii) $\mathrm{Ext}^1(M, X) = 0$ *for all X.*

The following are equivalent for a module X.
(i) X is injective
(ii) $\mathrm{Ext}^n(M, X) = 0$ *for all M and all $n > 0$.*
(iii) $\mathrm{Ext}^1(M, X) = 0$ *for all cyclic modules M.*

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are clear.

(iii)$\Rightarrow$(i) using the characterization of a projective module as one for which all short exact sequences ending at the module split. If $\mathrm{Ext}^1(R/I, X) = 0$ for all left ideals R, then by the long exact for $\mathrm{Hom}(-, X)$ applied to $0 \to I \to R \to R/I \to 0$, we get a surjective map $\mathrm{Hom}(R, X) \to \mathrm{Hom}(I, X)$, so X is injective by Baer's criterion. $\square$

4.5 Projective, injective and global dimensions

Proposition/Definition. *Let M be a module and $n \geq 0$. The following are equivalent.*
(i) There is a projective resolution $0 \to P_n \to \dots \to P_0 \to M \to 0$
(ii) $\mathrm{Ext}^m(M, X) = 0$ *for all* $m > n$ *and all* X*.*
(iii) $\mathrm{Ext}^{n+1}(M, X) = 0$ *for all* X*.*
(iv) For any projective resolution of M*, we have* $\Omega_n M$ *projective.*
The projective dimension*,* $\mathrm{proj.\,dim}\, M$*, is the smallest* n *with this property (or* ∞ *if there is none).*

Let X be a module and $n \geq 0$. The following are equivalent.
(i) There is an injective resolution $0 \to X \to I^0 \to \dots \to I^n \to 0$
(ii) $\mathrm{Ext}^m(M, X) = 0$ *for all* $m > n$ *and all* X*.*
(iii) $\mathrm{Ext}^{n+1}(M, X) = 0$ *for all* ***cyclic*** M*.*
(iv) For any injective resolution of X*, we have* $\Omega^n X$ *injective.*
The injective dimension*,* $\mathrm{inj.\,dim}\, X$*, is the smallest* n *with this property (or* ∞ *if there is none).*

Proof. (i)⇒(ii)⇒(iii) are trivial. For (iii)⇒(iv) let $P \to M$ be a projective resolution. For any X, dimension shifting gives

$$0 = \mathrm{Ext}^{n+1}(M, X) \cong \mathrm{Ext}^n(\Omega_1 M, X) \cong \ldots \cong \mathrm{Ext}^1(\Omega_n M, X),$$

so $\Omega_n M$ is projective. Then

$$0 \to \Omega_n M \to P_{n-1} \to \dots \to P_0 \to M \to 0$$

is also a projective resolution of M, giving (i). □

Lemma. *If* $0 \to L \to M \to N \to 0$ *is exact, then*

$$\mathrm{proj.\,dim}\, M \leq \max\{\mathrm{proj.\,dim}\, L, \mathrm{proj.\,dim}\, N\},$$
$$\mathrm{inj.\,dim}\, M \leq \max\{\mathrm{inj.\,dim}\, L, \mathrm{inj.\,dim}\, N\}.$$

Proof. For any X the long exact sequence for $\mathrm{Hom}(-, X)$ gives an exact sequence

$$\dots \to \mathrm{Ext}^{n+1}(N, X) \to \mathrm{Ext}^{n+1}(M, X) \to \mathrm{Ext}^{n+1}(L, X) \to \ldots$$

and the outer terms are zero for $n = \max$. □

Definition. The *(left) global dimension* of R (in $\mathbb{N} \cup \{\infty\}$) is

$$\begin{aligned}
\mathrm{gl.\,dim}\, R &= \sup\{\mathrm{proj.\,dim}\, M : M \in R\text{-Mod}\} \\
&= \inf\{n \in \mathbb{N} : \mathrm{Ext}^{n+1}(M, X) = 0 \ \forall\ M, X\} \\
&= \sup\{\mathrm{inj.\,dim}\, X : X \in R\text{-Mod}\} \\
&= \inf\{n \in \mathbb{N} : \mathrm{Ext}^{n+1}(M, X) = 0 \ \forall\ M, X \text{with } M \text{ cyclic}\} \\
&= \sup\{\mathrm{proj.\,dim}\, M : M \text{ cyclic}\}.
\end{aligned}$$

Examples. (1) $\operatorname{gl.dim} R = 0 \Leftrightarrow$ all modules are projective $\Leftrightarrow$ all short exact sequences split $\Leftrightarrow$ every submodule has a complement $\Leftrightarrow$ R is a semisimple (artinian) ring.

(2) If R is a f.d. algebra over a field, then

$$\operatorname{gl.dim} R = \max\{\operatorname{proj.dim} S : S \text{ is a simple module}\}.$$

Namely, call the maximum m. Using the lemma and induction on $\dim M$ we get $\operatorname{proj.dim} M \leq m$ for any f.d. M. Thus $\operatorname{proj.dim} M \leq m$ for all cyclic M. Thus $\operatorname{gl.dim} R \leq m$. But clearly $m \leq \operatorname{gl.dim} R$.

(3) Let $R = K[x]/(x^2)$ with K a field. Then K becomes an R-module with x acting as 0, and we saw that $\operatorname{Ext}^n_R(K, K) \cong K$ for all $n \geq 0$. Thus $\operatorname{proj.dim} K = \infty$, so also $\operatorname{gl.dim} R = \infty$.

Proposition/Definition. *A ring R is said to be* (left) hereditary *if it satisfies the following equivalent conditions*
(i) $\operatorname{gl.dim} R \leq 1$ *(left global dimension).*
(ii) Every submodule of a projective (left) module is projective.
(iii) Every left ideal in R is projective.

Proof. (i)$\Rightarrow$(ii) If N is a submodule of P then for any X, by the long exact sequence, $\operatorname{Ext}^1(N, X) \cong \operatorname{Ext}^2(P/N, X) = 0$.

(ii)$\Rightarrow$(iii) Trivial.

(iii)$\Rightarrow$(i) For any cyclic module R/I we have $\operatorname{proj.dim} R/I \leq 1$. □

Example. A principal ideal domain is hereditary. As discussed in the section on projective modules, if R is an integral domain, then a non-zero ideal is projective if and only if it is invertible. Thus R is hereditary if and only if every nonzero ideal is invertible, and as mentioned before, this is if and only if R is a Dedekind domain.

Definition. Let $Q = (Q_0, Q_1, s, t)$ be a quiver with finite vertex set Q_0. If R is a commutative ring, the path algebra RQ is the free R-module with basis the paths in Q, including a trivial path e_i for each vertex. For example the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

has paths e_1, e_2, e_3, a, b, ba. It becomes an R-algebra with multiplication given by concatination of paths, or zero if they are not compatible. For example

$$b \cdot a = ba, \quad a \cdot a = 0,$$

$$b \cdot e_2 = b, \quad b \cdot e_1 = 0,$$

$$e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = 0.$$

Thus the e_i are orthogonal idempotents and the 1 is $\sum_{i \in Q_0} e_i$.

Note that if $i \in Q_0$ then RQe_i is an RQ-R-bimodule, and it is a projective left RQ-module.

Proposition. *If M is a left RQ-module, there is an exact sequence of RQ-modules*

$$0 \to \bigoplus_{a \in Q_1} RQe_{t(a)} \otimes_R e_{s(a)}M \xrightarrow{f} \bigoplus_{i \in Q_0} RQe_i \otimes_R e_iM \xrightarrow{g} M \to 0$$

where $x \otimes m \in RQe_i \otimes e_iM$ is sent by g to xm and where $x \otimes m \in RQe_{t(a)} \otimes e_{s(a)}M$ is sent by f to $xa \otimes m - x \otimes am$, where the summands are in $RQe_{s(a)} \otimes e_{s(a)}M$ and $RQe_{t(a)} \otimes e_{t(a)}M$.

In particular, if R is a field, this is a projective resolution of M, so $\operatorname{gl.dim} RQ \leq 1$.

Proof. Clearly $gf = 0$. We show that it is contractible as a complex of R-modules. Let the middle term be C_0 and the left hand term C_1. Consider the R-module maps $s : M \to C_0$ and $r : C_0 \to C_1$ given by

$$s(m) = \sum_{i \in Q_0} e_i \otimes e_i m$$

and for $m \in e_iM$ and a path starting at i, by $r(e_i \otimes m) = 0$ and

$$r(a_1 a_2 \dots a_n \otimes m) = \sum_{j=1}^{n} (a_1 \dots a_{j-1} \otimes a_{j+1} \dots a_n m)_{a_j},$$

where the jth term is an element of $RQe_{t(a_j)} \otimes_R e_{s(a_j)}M$. It is straightforward to check that $gs = \mathrm{Id}_M$, $fr + sg = \mathrm{Id}_{C_0}$ and $rf = \mathrm{Id}_{C_1}$. □

Lemma (a version of Shapiro's Lemma)**.** *If $R \to S$ is a ring homomorphism and S_R is flat, then for an R-module M and an S-module X we have*

$$\operatorname{Ext}^n_S(S \otimes_R M, X) \cong \operatorname{Ext}^n_R(M, {}_RX).$$

Proof. If P is a projective R-module, say a direct summand of $R^{(I)}$, then $S \otimes_R P$ is direct summand of $S \otimes_R R^{(I)} \cong S^{(I)}$, so $S \otimes_R P$ is a projective S-module.
Now if

$$\dots \to P_1 \to P_0 \to M \to 0$$

is an R-module projective resolution of M, then

$$\dots \to S \otimes_R P_1 \to S \otimes_R P_0 \to S \otimes_R M \to 0$$

is an S-module projective resolution of $S \otimes_R M$. Thus $\operatorname{Ext}^n_S(S \otimes_R M, X)$ is the nth cohomology of the complex

$$\operatorname{Hom}_S(S \otimes_R P, X) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_S(S, X)) \cong \operatorname{Hom}_R(P, {}_R X)$$

which is $\operatorname{Ext}^n_R(M, {}_R X)$. $\square$

Theorem (Hilbert's Syzygy Theorem)**.** *For any (commutative) ring R we have* $\operatorname{gl.dim} R[x] = \operatorname{gl.dim} R + 1$. *In particular, if K is a field,* $\operatorname{gl.dim} K[x_1, \dots, x_n] = n$.

Proof. (i) Let $S = R[x]$. For any S-module M there is an exact sequence

$$0 \to S \otimes_R M \xrightarrow{f} S \otimes_R M \xrightarrow{g} M \to 0$$

where g is multiplication and $f(s \otimes m) = sx \otimes m - s \otimes xm$. This is the case of a path algebra given by a loop.

(ii) $\operatorname{gl.dim} S \leq 1 + \operatorname{gl.dim} R$. If M and X are S-modules, the long exact sequence for $\operatorname{Hom}_S(-, X)$ applied to the exact sequence of (i) gives

$$\begin{tikzcd}
\operatorname{Ext}^n_S(S \otimes_R M, X) \arrow[r, "h"] \arrow[d, equal] & \operatorname{Ext}^n_S(S \otimes_R M, X) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^{n+1}_S(M, X) \arrow[r] & \operatorname{Ext}^{n+1}_S(S \otimes_R M, X) \arrow[d, equal] \\
\operatorname{Ext}^n_R(M, X) & \operatorname{Ext}^n_R(M, X) & & \operatorname{Ext}^{n+1}_R(M, X)
\end{tikzcd}$$

Thus $\operatorname{Ext}^{n+1}_S(M, X) = 0$ for $n > \operatorname{gl.dim} R$, so $\operatorname{gl.dim} S \leq 1 + \operatorname{gl.dim} R$.

(iii) $\operatorname{gl.dim} S = 1 + \operatorname{gl.dim} R$. Let M and X be R-modules, considered as S-modules with x acting as 0. Let $X \to I$ an R-module injective resolution. We get cosyzygies $0 \to \Omega^i X \to I^i \to \Omega^{i+1} X \to 0$. We consider these also as S-modules with x acting as 0. If U is an S-module, applying $\operatorname{Hom}_S(U, -)$ gives long exact sequences

$$0 \to \operatorname{Hom}_S(U, \Omega^i X) \to \operatorname{Hom}_S(U, I^i) \to \operatorname{Hom}_S(U, \Omega^{i+1} X)$$
$$\to \operatorname{Ext}^1_S(U, \Omega^i X) \to \operatorname{Ext}^1_S(U, I^i) \to \operatorname{Ext}^1_S(U, \Omega^{i+1} X)$$
$$\to \operatorname{Ext}^2_S(U, \Omega^i X) \to \operatorname{Ext}^2_S(U, I^i) \to \dots$$

Thus we get morphisms

$$\operatorname{Hom}_S(U, \Omega^n X) \to \operatorname{Ext}^1_S(U, \Omega^{n-1} X) \to \operatorname{Ext}^2_S(U, \Omega^{n-2} X) \to \cdots \to \operatorname{Ext}^n_S(U, X).$$

This gives a natural transformation of contravariant functors

$$\operatorname{Hom}_S(-, \Omega^n X) \to \operatorname{Ext}^n_S(-, X).$$

Now if $U = S \otimes_R M$ then $\mathrm{Ext}^j_S(U, I^i) \cong \mathrm{Ext}^j_R(M, I^i)$, which is zero for $j > 0$, and then the morphism $\mathrm{Hom}_S(U, \Omega^n X) \to \mathrm{Ext}^n_S(U, X)$ is surjective since, as in dimension shifting, it is a composition

$$\mathrm{Hom}_S(U, \Omega^n X) \twoheadrightarrow \mathrm{Ext}^1_S(U, \Omega^{n-1} X) \cong \mathrm{Ext}^2_S(U, \Omega^{n-2} X) \cong \ldots \cong \mathrm{Ext}^n_S(U, X).$$

Thus the map f gives a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_S(S \otimes M, \Omega^n X) & \longrightarrow & \mathrm{Ext}^n_S(S \otimes M, X) \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ \mathrm{Hom}_S(S \otimes M, \Omega^n X) & \longrightarrow & \mathrm{Ext}^n_S(S \otimes M, X) \end{array}$$

with surjective horizontal maps, where f' is composition with f and where h is the morphism in (ii). Now

$$f'(\phi)(s \otimes m) = \phi f(s \otimes m) = \phi(sx \otimes m - s \otimes xm) = \phi(xs \otimes m) = x\phi(s \otimes m) = 0$$

since x acts as zero on M and $\Omega^n X$. Thus f' is zero. Since the horizontal maps in the square are surjective, h is zero. Thus the exact sequence in (ii) gives an embedding

$$\mathrm{Ext}^n_R(M, X) \cong \mathrm{Ext}^n_S(S \otimes_R M, X) \hookrightarrow \mathrm{Ext}^{n+1}_S(M, X)$$

so if $\mathrm{gl.\,dim}\, S = n$, then $\mathrm{gl.\,dim}\, R \leq n - 1$, that is, $\mathrm{gl.\,dim}\, S \geq 1 + \mathrm{gl.\,dim}\, R$. $\square$

4.6 Tor

Definition. Given a right R-module M, a *flat resolution* of M is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

with the P_i flat, or equivalently a non-negative chain complex P of flat right R-modules and a quasi-isomorphism $P \to M$.

Given a flat resolution P of M and a left R-module X, we can consider $H_n(P; X) = H_n(P \otimes_R X)$, nth homology of the complex

$$\cdots \to P_2 \otimes_R X \to P_1 \otimes_R X \to P_0 \otimes_R X \to 0.$$

Since any projective module is flat, any projective resolution of M is a flat resolution. Fixing a projective resolution $P \to M$, we define

$$\mathrm{Tor}^R_n(M, X) := H_n(P; X) = H_n(P \otimes_R X) = (L_n(- \otimes_R X))(M),$$

the nth left derived functor of the functor

$$- \otimes_R X : \text{Mod-}R \to \text{Ab}$$

evaluated at M. In general $\operatorname{Tor}_n^R(M, X)$ is a $\mathbb{Z}$-module. If R is a K-algebra, it is a K-module. If R is a commutative ring, it is an R-module.

Remarks. (1) Since $\operatorname{Tor}_n^R(M, X)$ is a left derived functor, we know that it is functorial in M, so a morphism $M \to M'$ of right R-modules induces a morphism $\operatorname{Tor}_n^R(M, X) \to \operatorname{Tor}_n^R(M', X)$. Also $\operatorname{Tor}_0^R(M, X) \cong M \otimes_R X$, and a short exact sequence $0 \to M' \to M \to M'' \to 0$ of right R-modules gives a long exact sequence

$$\cdots \to \operatorname{Tor}_2^R(M'', X) \to \operatorname{Tor}_1^R(M', X) \to \operatorname{Tor}_1^R(M, X) \to \operatorname{Tor}_1^R(M'', X) \to$$

$$\to M' \otimes_R X \to M \otimes_R X \to M'' \otimes_R X \to 0.$$

(2) If P is the chosen projective resolution of M, then a homomorphism $X \to X'$ of left R-modules induces a morphism $P \otimes_R X \to P \otimes_R X'$ of complexes of $\mathbb{Z}$-modules, and hence a morphism $\operatorname{Tor}_n^R(M, X) \to \operatorname{Tor}_n^R(M, X')$, so $\operatorname{Tor}_n^R(M, X)$ is functorial in X. Also, a short exact sequence $0 \to X' \to X'' \to 0$ of left R-modules induces a short exact sequence

$$0 \to P \otimes_R X' \to P \otimes_R X \to P \otimes_R X'' \to 0$$

of complexes of $\mathbb{Z}$-modules, and hence a long exact sequence on homology

$$\cdots \to \operatorname{Tor}_2^R(M, X'') \to \operatorname{Tor}_1^R(M, X') \to \operatorname{Tor}_1^R(M, X) \to \operatorname{Tor}_1^R(M, X'') \to$$

$$\to M \otimes_R X' \to M \otimes_R X \to M \otimes_R X'' \to 0.$$

(3) If $Q \to X$ is a flat resolution of X, then analogous to the theorem in section 4.3 showing that Ext is a derived functor of its second argument, we get

$$\operatorname{Tor}_n^R(M, X) \cong H_n(M \otimes_R Q).$$

In particular, taking Q to be a projective resolution of X, this shows that

$$\operatorname{Tor}_n^R(M, X) = (L_n(M \otimes_R -))(X).$$

(4) Comparing (3) with the definition of Tor, it follows that Tor is symmetrical with respect to the two arguments. We can state this formally as a natural isomorphism

$$\operatorname{Tor}_n^R(M, X) \cong \operatorname{Tor}_n^{R^{op}}(X, M)$$

where X is a left R-module, or equivalently a right R^{op}-module and M is a right R-module, or equivalently a left R^{op}-module.

(5) It follows from (3) and (4) that

$$\operatorname{Tor}_n^R(M,X) = H_n(P \otimes_R X)$$

where P is any flat resolution of P of M.

Theorem. *The following are equivalent for a right R-module M.*
(i) M is flat
(ii) $\operatorname{Tor}_n^R(M,X) = 0$ *for all X and all $n > 0$.*
(iii) $\operatorname{Tor}_1^R(M,X) = 0$ *for all X.*

Proof. (i)⇒(ii) since M is its own flat resolution. (ii)⇒(iii) is trivial. (iii)⇒(i) The long exact sequence shows that M is flat. □

Proposition/Definition. *Let M be a right R-module and $n \geq 0$. The following are equivalent.*
(i) There is a flat resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$
(ii) $\operatorname{Tor}_m^R(M,X) = 0$ *for all X and $m > n$*
(iii) $\operatorname{Tor}_{n+1}^R(M,X) = 0$ *for all X.*
(iv) For any flat resolution of M, we have $\Omega_n M$ flat.
The flat dimension $\operatorname{flatdim} M$ *is the smallest n with this property (or ∞ if there is none).*

Proof. As for projective dimension. □

Definition. The *weak dimension* of R is

$$\operatorname{w.dim} R = \sup\{\operatorname{flatdim} M : \forall M\} = \inf\{n \in \mathbb{N} : \operatorname{Tor}_{n+1}^R(M,X) = 0 \ \forall\ M, X\}.$$

It is left/right symmetric.

Proposition. *(i) For M a left R-module,* $\operatorname{flatdim} M \leq \operatorname{proj.dim} M$*, with equality if M is finitely generated and R is left noetherian.*
(ii) $\operatorname{w.dim} R \leq \operatorname{gl.dim} R$*, with equality if R is left noetherian.*
(iii) (Auslander) If R is left and right noetherian, the left and right global dimensions of R are equal.

Proof. (i) The inequality holds since any projective resolution is also a flat resolution. If R is left noetherian and M is f.g., we have a projective resolution with all P_n finitely generated. Then $\operatorname{flatdim} M \leq n$ implies $\Omega_n M$ is flat. Since it is also finitely presented, it is projective. Thus $\operatorname{proj.dim} M \leq n$.

(ii) The inequality follows from the first part of (i). For equality use that

$$\operatorname{gl.dim} R = \sup\{\operatorname{proj.dim} M : M \text{ cyclic}\} = \sup\{\operatorname{flatdim} M : M \text{ cyclic}\} \leq \operatorname{w.dim} R$$

(iii) Clear. □

Recall that if R is an integral domain, then any flat R-module is torsion-free (and these are equivalent if R is a pid). We also have the following (possibly justifying the name "Tor").

Proposition. *If R is an integral domain, then $\mathrm{Tor}_1^R(M,X)$ is a torsion R-module for any R-modules M and X.*

Proof. Note that since R is commutative, $\mathrm{Tor}_1^R(M,X)$ is an R-module.

Recall that a left R-module T is *torsion* if there is some $0 \neq t \in T$ and some $0 \neq a \in R$ with $at = 0$. In fact T is torsion $\Leftrightarrow K \otimes_R T = 0$ where K is the field of fractions of R. Namely, suppose $0 \neq t \in T$. If $at = 0$ with $a \neq 0$, then for any $\lambda \in K$ we have $\lambda \otimes t = \lambda a^{-1} \otimes at = 0$. Conversely if there is no such a, then the map $R \to M$, $r \mapsto rt$ is injective. Since K is flat as an R-module we get an injection $K \cong K \otimes R \to K \otimes T$, so $K \otimes T \neq 0$.

Now let $0 \to L \to F \to X \to 0$ be exact with F flat. Then we get

$$0 \to \mathrm{Tor}_1^R(M,X) \to M \otimes_R L \to M \otimes_R F \to M \otimes_R X \to 0.$$

Thus since K is flat over R we get an exact sequence

$$0 \to K \otimes_R \mathrm{Tor}_1^R(M,X) \to K \otimes_R M \otimes_R L \to K \otimes_R M \otimes_R F \to K \otimes_R M \otimes_R X \to 0.$$

But $K \otimes_R M$ is K-module, with K a field, so it is isomorphic to a direct sum of copies of K, so flat over R, so the sequence

$$0 \to K \otimes_R M \otimes_R L \to K \otimes_R M \otimes_R F \to K \otimes_R M \otimes_R X \to 0.$$

is exact. □

4.7 Universal coefficient theorem

Lemma. *Every complex of projective modules for a hereditary ring is a direct sum of complexes of the form*

$$\dots 0 \to P \xrightarrow{\theta} Q \to 0 \to \dots$$

with P and Q projective and θ injective.

Proof. Given a complex C, the exact sequence

$$0 \to Z^n(C) \to C^n \xrightarrow{d^n} B^{n+1}(C) \to 0$$

splits since $B^{n+1}(C) \subseteq C^{n+1}$, so it is projective. Thus $C^n = Z^n(C) \oplus U^n$ for some complement U^n. Then C is the direct sum of the complexes

$$\dots \to 0 \to U^n \xrightarrow{d^n} Z^{n+1}(C) \to 0 \to \dots$$

□

Theorem. *If R is left hereditary, C is a chain complex of projective left R-modules and M is a left R-module, then there are split exact sequences*

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(C), M) \to H^n(C; M) \to \operatorname{Hom}_R(H_n(C), M) \to 0.$$

If R is right hereditary, C is a chain complex of projective right R-modules and M is a left R-module, then there are split exact sequences

$$0 \to H_n(C) \otimes_R M \to H_n(C; M) \to \operatorname{Tor}^R_1(H_{n-1}(C), M) \to 0.$$

Proof. We prove the result for homology with coefficients. We write the cycles, boundaries and homology of the chain complex C as Z_n, B_n and H_n. Tensor products are over R.

(1) There is a natural map $H_n \otimes M \to H_n(C \otimes M)$. The composition

$$Z_n \otimes M \xrightarrow{inc \otimes 1} C_n \otimes M \xrightarrow{d_n \otimes 1} C_{n-1} \otimes M$$

is zero, so we get a natural map

$$Z_n \otimes M \to Z_n(C \otimes M) \to H_n(C \otimes M), \quad \sum_i c_i \otimes m_i \mapsto [\sum_i c_i \otimes m_i].$$

Now we have an exact sequence

$$B_n \otimes M \to Z_n \otimes M \to H_n \otimes M \to 0$$

and the composition

$$B_n \otimes M \to Z_n \otimes M \to H_n(C \otimes M)$$

is zero, so we get a map

$$H_n(C) \otimes M \to H_n(C \otimes M), \quad \sum_i [c_i] \otimes m_i \mapsto [\sum_i c_i \otimes m_i].$$

(2) There is a natural map $H_n(C; M) \to \operatorname{Tor}^R_1(H_{n-1}, M)$. As in the lemma above, B_{n-2} is a submodule of C_{n-2}, so projective, so the exact sequence

$$0 \to Z_{n-1} \to C_{n-1} \xrightarrow{d_{n-1}} B_{n-2} \to 0$$

splits. Thus the map

$$Z_{n-1} \otimes M \to C_{n-1} \otimes M$$

is injective. Also Z_{n-1} is projective and

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1} \to 0$$

is exact, so we get an exact sequence

$$0 \to \mathrm{Tor}_1^R(H_{n-1}, M) \to B_{n-1} \otimes M \to Z_{n-1} \otimes M \to H_{n-1} \otimes M \to 0.$$

Thus we can identify $\mathrm{Tor}_1^R(H_{n-1}, M)$ with the kernel K of the natural map $B_{n-1} \otimes M \to Z_{n-1} \otimes M$. We get a map

$$Z_n(C \otimes M) \to K, \quad \sum_i c_i \otimes m_i \mapsto \sum_i d_n(c_i) \otimes m_i.$$

Namely $\sum_i d_n(c_i) \otimes m_i = 0$ in $C_{n-1} \otimes M$. Thus $\sum_i d_n(c_i) \otimes m_i = 0$ in $Z_{n-1} \otimes M$. Thus, considered as an element of $B_{n-1} \otimes M$, it is an element of K. This induces a map

$$H_n(C \otimes M) \to K, \quad [\sum_i c_i \otimes m] \to \sum_i d_n(c_i) \otimes m_i$$

since $B_n(C \otimes M)$ is spanned by elements of the form $d_{n+1}(c) \otimes m$, and this is sent to 0.

(3) By the lemma, any chain complex is a direct sum of two term complexes, so it suffices to prove the result for C of the form

$$\cdots \to 0 \to P \xrightarrow{\theta} Q \to 0 \to \ldots$$

with P and Q projective and θ injective. Say P is in degree i and Q in degree $i-1$.

We have an exact sequence

$$0 \to P \to Q \to H_{i-1} \to 0$$

so

$$0 \to \mathrm{Tor}_1^R(H_{i-1}, M) \to P \otimes M \to Q \otimes M \to H_{i-1} \otimes M \to 0.$$

Now the wanted sequence

$$0 \to H_n \otimes M \to H_n(C; M) \to \mathrm{Tor}_1^R(H_{n-1}, M) \to 0$$

is as follows: for $n = i$ it is

$$0 \to 0 \to \mathrm{Tor}_1^R(H_{i-1}, M) \to \mathrm{Tor}_1^R(H_{i-1}, M) \to 0.$$

for $n = i - 1$ it is

$$0 \to H_{i-1} \otimes M \to H_{i-1} \otimes M \to 0 \to 0$$

and for all other n it has all terms zero. This is split exact in each case. $\square$

5 Applications to commutative algebra and group actions

5.1 Some preliminary results on prime ideals

We begin with some generalities for a commutative ring R.

Lemma (Prime avoidance)**.** *If J and I_i are ideals with*

$$J \subseteq \bigcup_{i=1}^{n} I_i$$

and at least $n-2$ of the I_i are prime, then $J \subseteq I_i$ for some i.

For a proof, search the internet for "Stacks project Lemma 10.15.2".

Lemma (Support)**.** *Suppose M is a f.g. R-module and P is a prime ideal in R. Let R_P and M_P be the localizations with respect to the multiplicative set $S = R \setminus P$. Then $M_P \neq 0$ if and only if $\operatorname{Ann}(M) \subseteq P$.*

Proof. If $\operatorname{Ann}(M) \not\subseteq P$, there is some element $r \in \operatorname{Ann}(M) \setminus P$. Then r kills M_P, but it is invertible in R_P, so $M_P = 0$.

If $\operatorname{Ann}(M) \subseteq P$, then we have an epi $R/\operatorname{Ann}(M) \to R/P$. A generating set of M gives a mono $R/\operatorname{Ann}(M) \to M^n$. Localizing at P and using exactness, we get $M_P \neq 0$. □

Definition. Recall that the *height* of a prime ideal P in R is

$$\operatorname{ht} P = \sup\{d \geq 0 : \text{there are distinct prime ideals } P_0 \subset P_1 \subset \cdots \subset P_d = P\ \}$$

The *Krull dimension* of R is

$$\operatorname{Kdim} R = \sup\{\operatorname{ht} P : P \text{ a prime ideal in } R\}$$

The Krull dimension of a f.g. R-module M is defined to be

$$\operatorname{Kdim} M = \operatorname{Kdim}(R/\operatorname{Ann}(M)).$$

Theorem (Krull's height theorem)**.** *In a noetherian ring, any minimal prime over an ideal generated by n elements has height $\leq n$. Conversely any prime of height n is minimal over some ideal generated by n elements.*

Proof. The first part is proved in my Algebra II notes, or Stacks project Lemma 10.60.12. (Note that every prime ideal containing an ideal I contains a minimal prime over I, and in a noetherian ring there are only finitely many minimal primes over I, see my Algebra II notes, or Stacks project Lemma 10.17.2 and Lemma 10.31.6.)

For the second part, for $0 \le r \le n$ we find by induction an ideal $(x_1, \dots, x_r) \subseteq P$ such that any minimal prime over it has height r. This is clear for $r = 0$. Given $(x_1, \dots, x_{r-1})$, by prime avoidance there is an element $x_r \in P$ not contained in any of the (finitely many) minimal primes over $(x_1, \dots, x_{r-1})$. Then any minimal prime over $(x_1, \dots, x_r)$ has height $\le r$ by Krull's height theorem, and height $\ge r$ since it properly contains a minimal prime over $(x_1, \dots, x_{r-1})$. □

Lemma (Associated primes)**.** *If R is noetherian and M is a f.g. R-module, then the set of* associated primes

$$\mathrm{Ass}(M) := \{P \text{ prime ideal in } R : R/P \text{ is isomorphic to a submodule of } M\}$$

is finite and contains the minimal primes over $\mathrm{Ann}(M)$. *(In particular, if $M \neq 0$ then* $\mathrm{Ass}(M) \neq \emptyset$.*)*

For a proof see Stacks project Lemmas 10.63.5 and 10.63.8.

5.2 Regular sequences for local noetherian commutative rings

Throughout this section R is a local noetherian commutative ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Thus $\mathrm{Kdim}(R)$ is the height of $\mathfrak{m}$. Also every element of $R \setminus \mathfrak{m}$ is invertible, since the ideal it generates must be R.

Examples. (a) If S is a noetherian ring and P is a prime ideal in S, then the localization $R = S_P$ is a local noetherian ring with maximal ideal $\mathfrak{m} = PS_P$. The residue field $R/\mathfrak{m}$ is the field of fractions of S/P.

For example if K is a field, $S = K[x_1, \dots, x_n]$ and P is the maximal ideal $(x_1, \dots, x_n)$, then

$$R = S_P = \{f/g : f, g \in K[x_1, \dots, x_n], g(0, \dots, 0) \neq 0\}.$$

(b) A formal power series ring $K[[x_1, \dots, x_n]]$. The maximal ideal is $(x_1, \dots, x_n)$, the set of power series with constant term 0.

(c) The ring of p-adic integers $\hat{\mathbb{Z}}_p$. The maximal ideal is (p). The residue field is $\mathbb{Z}/\mathbb{Z}p$, the field with p elements.

(d) Any factor ring of a local noetherian ring is again a local noetherian ring.

Lemma (Nakayama)**.** *If M is a f.g. R-module and $\mathfrak{m}M = M$, then $M = 0$.*

Proof. In general Nakayama's Lemma says that if M is a f.g. R-module for any ring R and $JM = M$, where J is the Jacobson radical, then $M = 0$. Now J is the intersection of the maximal left ideals, so in this case it is $\mathfrak{m}$. The proof is easy. Suppose $M \neq 0$. Since M is f.g., by Zorn's lemma it has a maximal submodule N. Then M/N is simple, so $J(M/N) = 0$. Thus $JM \subseteq N$. □

Lemma (Projective covers). *(i) Any f.g. module M has a projective cover.*
(ii) Every f.g. projective module is free.
(iii) Any f.g. module M has a projective resolution

$$\cdots \to P_1 \to P_0 \to M \to 0$$

which is minimal, in the sense that each map $P_i \to \Omega_i M$ is a projective cover. It has the property that in the complex $k \otimes_R P$

$$\cdots \to P_1/\mathfrak{m}P_1 \to P_0/\mathfrak{m}P_0 \to 0$$

the maps are all zero.
(iv) If M is f.g. R-module, then $\operatorname{proj.dim} M \leq n \Leftrightarrow \operatorname{Tor}^R_{n+1}(k, M) = 0$.

Proof. (i) Take a basis of $M/\mathfrak{m}M$ as a k-vector space, and lift it to elements of M. They give a map $\theta : R^n \to M$ with the property that $\overline{\theta} : R^n/\mathfrak{m}R^n = k^n \to M/\mathfrak{m}M$ is an isomorphism. Thus $M = \operatorname{Im}(\theta) + \mathfrak{m}M$, so θ must be onto. Now suppose $\phi \in \operatorname{End}(R^n)$ satisfies $\theta\phi = \theta$. Tensoring with k we get $\overline{\phi} \in \operatorname{End}(k^n)$ with $\overline{\theta\phi} = \overline{\theta}$. This implies that $\overline{\phi} = 1$. Now $\det\phi \in R$ and $\overline{\det\phi} = \det\overline{\phi} = 1 \in k$, so $\det\phi \notin \mathfrak{m}$, so $\det\phi$ is invertible in R, so ϕ is invertible.

(ii) The projective cover of a projective module is itself, but by (i) it is free.

(iii) Construct the resolution iteratively, taking $P_i \to \Omega_i M$ to be a projective cover. Then the sequence

$$P_{i+1} \to P_i \to \Omega_i M \to 0$$

is exact, hence so is

$$k \otimes P_{i+1} \to k \otimes P_i \to k \otimes \Omega_i M \to 0$$

but the map $k \otimes P_i \to k \otimes \Omega_i M$ is an isomorphism, so $k \otimes P_{i+1} \to k \otimes P_i$ is zero.

(iv) If $\operatorname{proj.dim} M \leq n$, then there is a projective resloution of length n, so $\operatorname{Tor}^R_{n+1}(k, M) = 0$. Conversely, using a minimal projective resolution of M we see that

$$0 = \operatorname{Tor}^R_{n+1}(k, M) \cong P_{n+1}/\mathfrak{m}P_{n+1}$$

so $P_{n+1} = 0$, so $\operatorname{proj.dim} M \leq n$. □

Definition. Let M be a nonzero f.g. module. An element $x \in \mathfrak{m}$ is *regular* for M if it is not a zero divisor on M, that is, if $xm = 0$ with $m \in M$, then $m = 0$.

Note that since R is commutative, xM is a submodule of M.

A sequence $x_1, x_2, \ldots, x_n$ of elements of $\mathfrak{m}$ is a *regular sequence* for M if, for all i, x_i is regular for the module

$$M/(x_1M + \cdots + x_{i-1}M) = M/(x_1, \ldots, x_{i-1})M.$$

A *regular sequence* is one which is regular for R.

Definition. Given $x_1, \ldots, x_n$ in R the *Koszul complex* $C = K(x_1, \ldots, x_n)$ is given as follows. Let F be the free R-module with basis $b_1, \ldots, b_n$. Then $C_i = \Lambda^i F$, the ith exterior power of F, with basis

$$b_{j_1} \wedge \cdots \wedge b_{j_i}$$

for $j_1 < \cdots < j_i$, and the differential is given by

$$d(b_{j_1} \wedge \cdots \wedge b_{j_i}) = \sum_{r=1}^{i} (-1)^{r-1} x_{j_r} b_{j_1} \wedge \cdots \wedge \hat{b}_{j_r} \wedge \cdots \wedge b_{j_i}.$$

If M is an R-module, then clearly $H_0(C; M) = M/(x_1, ..., x_n)M$.

Theorem. *If $x_1, \ldots, x_n$ is a regular sequence on M, then $H_i(C; M) = 0$ for $i > 0$ (and the converse holds if M is f.g. nonzero and $x_i \in \mathfrak{m}$).*

Proof. We just do the cases $n = 1$ and $n = 2$. We won't need the result later. In case $n = 1$, the Koszul complex C is

$$0 \to Rb_1 \xrightarrow{x_1} R1 \to 0$$

so $C \otimes_R M$ is the complex

$$0 \to M \xrightarrow{x_1} M \to 0,$$

and the assertion is clear.

In case $n = 2$, the Koszul complex C is

$$0 \to R(b_1 \wedge b_2) \to Rb_1 \oplus Rb_2 \to R1 \to 0$$

with $b_1 \wedge b_2 \mapsto x_1 b_2 - x_2 b_1$, $b_1 \mapsto x_1 1$ and $b_2 \mapsto x_2 1$, so the complex $C \otimes_R M$ is

$$0 \to M \xrightarrow{f} M \oplus M \xrightarrow{g} M \to 0.$$

with $f(m) = (-x_2 m, x_1 m)$ and $g(m_1, m_2) = x_1 m_1 + x_2 m_2$.

Suppose that x_1, x_2 is a regular sequence for M. If $f(m) = 0$, then $x_1 m = 0$ so $m = 0$. If $g(m_1, m_2) = 0$, then $x_1 m_1 + x_2 m_2 = 0$. Thus $x_2(x_1 M + m_2) = 0$. Thus $m_2 \in x_1 M$. Thus $m_2 = x_1 m$ for some m. Then $x_1(m_1 + x_2 m) = 0$, so $m_1 = -x_2 m$, so $(m_1, m_2) = f(m)$, so the complex is acyclic.

Now suppose the complex is acyclic, M is f.g. and $x_i \in \mathfrak{m}$. Let $U = \{m \in M : x_1 m = 0\}$, a submodule of M, which by f.g. since R is noetherian. If $m \in U$, then $g(m, 0) = 0$, so $(m, 0) = f(m')$ for some m', that is, $m = -x_2 m'$ and $x_1 m' = 0$. Thus $m \in x_2 U$. Thus $U \subseteq x_2 U \subseteq \mathfrak{m} U$. Thus $U = 0$ by Nakayama's Lemma. Thus x_1 is regular on M.

Next suppose that $m \in M$ and $x_2(x_1 M + m) = x_1 M + 0$. Then $x_2 m \in x_1 M$. Thus $x_2 m = x_1 m'$ for some m'. Then $g(-m', m) = 0$. Thus $(-m', m) = f(m'')$ for some m''. Thus $m = x_1 m''$, so $x_1 M + m = x_1 M + 0$. Thus x_2 is regular on $M/x_1 M$. $\square$

Lemma (Existence of a regular element)**.** *Let M be a nonzero f.g. module.*
(i) If $x \in \mathfrak{m}$, then x is regular on M if and only if it is not contained in any associated prime of M.
(ii) There is some $x \in \mathfrak{m}$ which is regular on M if and only if $\mathfrak{m}$ is not an associated prime of M. (Equivalently M has no submodule isomorphic to k, or also $\mathrm{Hom}(k, M) = 0$.*)*

Proof. (i) If $x \in P$ and R/P is a submodule of M, then $x(R/P) = 0$, giving a nonzero element $m \in M$ with $xm = 0$. Conversely, if x is not regular on M, then $N = \{m \in M : xm = 0\}$ is a nonzero submodule of M, so has an associated prime P. But then P is an associated prime of M. Now $xN = 0$, so $x(R/P) = 0$ so $x \in P$.

(ii) Follows from prime avoidance. $\square$

Lemma. *If M is a nonzero f.g. R-module and $x_1, \dots, x_n$ is a regular sequence for M, then*

$$\mathrm{Hom}_R(k, M/(x_1, \dots, x_n)M) \cong \mathrm{Ext}^n_R(k, M).$$

Proof. We prove this for all M and all regular sequence by induction on n. The case $n = 0$ is empty, so suppose $n > 0$. By induction

$$\mathrm{Ext}^{n-1}_R(k, M) \cong \mathrm{Hom}_R(k, M/(x_1, \dots, x_{n-1})M)$$

and this is zero by the lemma, since x_n is regular on $M/(x_1, \dots, x_{n-1})M$. Thus the sequence

$$0 \to M \xrightarrow{x_1} M \to M/x_1 M \to 0$$

gives

$$0 = \mathrm{Ext}^{n-1}_R(k, M) \to \mathrm{Ext}^{n-1}_R(k, M/x_1M) \to \mathrm{Ext}^n_R(k, M) \to \mathrm{Ext}^n_R(k, M)$$

The last map is induced by multiplication by x_1 on M, but x_1 kills k, so it is zero. Thus $\mathrm{Ext}^n(k, M) \cong \mathrm{Ext}^{n-1}(k, M/x_1M)$. Again, by induction, since $x_2, \ldots, x_n$ is a regular sequence for M/x_1M, this is $\mathrm{Hom}(k, M/(x_1, \ldots, x_n)M)$. □

Theorem. *If $x \in \mathfrak{m}$ and M is a f.g. R-module, then*

$$\mathrm{Kdim}(M/xM) \geq \mathrm{Kdim}(M) - 1,$$

with equality if x is regular on M.

Proof. Let $I = \mathrm{Ann}(M) \subseteq \mathrm{Ann}(M/xM) = J$. We want

$$\mathrm{Kdim}(R/I) \leq \mathrm{Kdim}(R/J) + 1.$$

Clearly we have $(I, x) \subseteq J$, and in fact if P is a prime ideal containing (I, x), then P contains J, so that $\mathrm{Kdim}(R/J) = \mathrm{Kdim}(R/(I, x))$. Namely, P contains I, so $M_P \neq 0$. But x is an element of the maximal ideal of R_P, so $M_P/xM_P \neq 0$. Thus $(M/xM)_P \neq 0$, so $J \subseteq P$.

Let $\mathrm{Kdim}(R/J) = n$. Thus the ideal $\mathfrak{m}/(I, x)$ in $R/(I, x)$ has height n. Then by Krull's height theorem, it is minimal over some ideal $(\bar{y}_1, \ldots, \bar{y}_n)$ with $y_i \in R/I$ and $\bar{y}_i \in R/(I, x)$. Now the ideals of R/I containing $\bar{x} := I + x$ are in bijection with the ideals of $R/(I, x)$, so the ideal $\mathfrak{m}/I$ in R/I is minimal over $(\bar{x}, y_1, \ldots, y_n)$, so by Krull's height theorem again, $\mathfrak{m}/I$ has height $\leq n + 1$. Thus $\mathrm{Kdim}(R/I) \leq n + 1$.

Now suppose x is regular on M. Then x is not contained in any associated prime of M. Thus by the lemma about associated primes, x is not contained in any minimal prime over $I = \mathrm{Ann}(M)$. Thus $\mathrm{Kdim}(R/(I, x)) < \mathrm{Kdim}(R/I)$. □

Corollary. *Any regular sequence $x_1, \ldots, x_n$ for a nonzero f.g. module M has length $n \leq \mathrm{Kdim}\, M$.*

Definition. The *depth* of a nonzero f.g. module M, is the maximal length of a regular sequence for M.

Clearly M is an R/I-module for an ideal I, then $\mathrm{depth}_{R/I} M = \mathrm{depth}_R M$.

Theorem (Rees)**.** *If M is a nonzero f.g. R-module, then*

$$\mathrm{depth}\, M = \min\{i \geq 0 : \mathrm{Ext}^i(k, M) \neq 0\}.$$

Moreover any regular sequence $x_1, \ldots, x_i$ for M can be extended to a regular sequence $x_1, \ldots, x_n$ of length $n = \mathrm{depth}\, M$, so

$$\mathrm{depth}\, M/(x_1, \ldots, x_i)M = \mathrm{depth}\, M - i.$$

Proof. It suffices to show that if $x_1, \dots, x_n$ is a regular sequence for M which cannot be extended to one of length $n+1$, then n is given by the formula. By assumption, no element of $\mathfrak{m}$ is regular on $M/(x_1, \dots, x_n)M$, so by the lemma on existence of a regular element, $\operatorname{Hom}(k, M/(x_1, \dots, x_n)M) \neq 0$, and so $\operatorname{Ext}^n(k, M) \neq 0$. On the other hand, for $i < n$, the element x_{i+1} is regular on $M/(x_1, \dots, x_i)M$, so $\operatorname{Hom}(k, M/(x_1, \dots, x_i)M) = 0$, so $\operatorname{Ext}^i(k, M) = 0$. $\square$

Lemma (Regular elements preserve acyclicity)**.** *If C is an acyclic chain complex of R-modules and x is regular on each C_i, then the complex $R/(x) \otimes_R C$ is also acyclic.*

Proof. The complex is

$$\cdots \to C_{n+1}/xC_{n+1} \xrightarrow{\overline{d}_{n+1}} C_n/xC_n \xrightarrow{\overline{d}_n} C_{n-1}/xC_{n-1} \to \dots$$

Say $c \in C_n$ and $\overline{d}_n(\overline{c}) = 0$. Then $d(c) \in xC_{n-1}$, so $d(c) = xc'$ for some $c' \in C_{n-1}$. Then $xd(c') = d(xc') = d^2(c) = 0$, so since x is regular on C_{n-1} we have $d(c') = 0$. Thus since C is acyclic, $c' = d(c'')$ for some $c'' \in C_n$. Then $d(c - xc'') = 0$. Thus $c - xc'' = d(c''')$ for some $c''' \in C_{n+1}$. Then $\overline{c} = \overline{d}_{n+1}(\overline{c'''})$. $\square$

Theorem (Auslander-Buchsbaum formula)**.** *If M is nonzero f.g. R-module and* $\operatorname{proj.dim} M < \infty$*, then* $\operatorname{proj.dim} M + \operatorname{depth} M = \operatorname{depth} R$*.*

Proof. We prove this by induction on $\operatorname{depth} R$. Say $\operatorname{depth} R = 0$, so there is an embedding $i : k \to R$. If $\operatorname{proj.dim} M = n > 0$, then the last terms in a minimal projective resolution of M are a monomorphism $\theta : P_n \to P_{n-1}$ with $P_n \neq 0$. This gives a commutative square

$$\begin{array}{ccc} k \otimes P_n & \xrightarrow{\mathrm{Id}_k \otimes \theta} & k \otimes P_{n-1} \\ {\scriptstyle i\otimes 1}\downarrow & & {\scriptstyle i\otimes 1}\downarrow \\ R \otimes P_n & \xrightarrow{\mathrm{Id}_R \otimes \theta} & R \otimes P_{n-1} \end{array}$$

Now the bottom and the vertical maps are injective, hence so is the top map. But since we used a minimal projective resolution, the top map is zero (as in the proof of the lemma about projective covers). Also $k \otimes P_n \cong P_n/\mathfrak{m}P_n$ is nonzero by Nakayama's Lemma. Contradiction. Thus M is projective, so free, so also has depth 0, and the formula holds.

Now suppose $\operatorname{depth} R > 0$. Suppose $\operatorname{depth} M = 0$. Applying $\operatorname{Hom}(k, -)$ to the exact sequence $0 \to \Omega_1 M \to P_0 \to M \to 0$ gives an exact sequence

$$0 \to \operatorname{Hom}(k, \Omega_1 M) \to \operatorname{Hom}(k, P_0) \to \operatorname{Hom}(k, M) \to \operatorname{Ext}^1(k, \Omega_1 M)$$

and we have $\mathrm{Hom}(k, P_0) = 0$ and $\mathrm{Hom}(k, M) \neq 0$, so $\mathrm{Hom}(k, \Omega_1 M) = 0$ and $\mathrm{Ext}^1(k, \Omega_1 M) \neq 0$, so by Rees' Theorem, $\operatorname{depth} \Omega_1 M = 1$. Also M is not projective, and $\operatorname{proj.dim} \Omega_1 M = \operatorname{proj.dim} M - 1$. Thus it suffices to prove the result for $\Omega_1 M$.

Thus we may assume that $\operatorname{depth} M > 0$. Then $\mathrm{Hom}(k, R \oplus M) = 0$, so $\mathfrak{m}$ contains an element x which is regular for $R \oplus M$, so for R and for M. Take a minimal projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

with $P_n \neq 0$. Tensoring with $R/(x)$, it stays exact by the lemma, so gives a minimal projective resolution

$$0 \to P_n/xP_n \to \cdots \to P_0/xP_0 \to M/xM \to 0$$

of M/xM as an $R/(x)$-module. Thus

$$\operatorname{proj.dim}_{R/(x)}(M/xM) = n = \operatorname{proj.dim}_R M < \infty$$

Also

$$\operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_R M/xM = \operatorname{depth}_R M - 1$$

by Rees' Theorem, and

$$\operatorname{depth}_{R/(x)} R/(x) = \operatorname{depth}_R R/(x) = \operatorname{depth}_R R - 1.$$

Then by induction

$$\operatorname{proj.dim}_{R/(x)} M/xM + \operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_{R/(x)} R/(x)$$

so

$$\operatorname{proj.dim}_R M + \operatorname{depth}_R M - 1 = \operatorname{depth}_R R - 1$$

giving the result. □

5.3 Regular local rings

R is still a local noetherian commutative ring.

Proposition/Definition. *Elements $x_1, \dots, x_n$ generate $\mathfrak{m}$ as an ideal if and only if they span $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space, so*

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \textit{minimal number of generators of } \mathfrak{m} \geq \operatorname{Kdim} R.$$

If equality holds, R is said to be a regular local ring.

Proof. Let $I = (x_1, \ldots, x_n)$. If $I = \mathfrak{m}$, then the map $I \to \mathfrak{m}/\mathfrak{m}^2$ is surjective, and since it kills any multiple ax_i with $a \in \mathfrak{m}$, it follows that the x_i span. Conversely, if the x_i span $\mathfrak{m}/\mathfrak{m}^2$, then $\mathfrak{m}^2 + I = \mathfrak{m}$. But then $\mathfrak{m}(\mathfrak{m}/I) = \mathfrak{m}/I$, so $\mathfrak{m}/I = 0$ by Nakayama's Lemma. The last inequality is Krull's height theorem. $\square$

Example. The ring $K[[x_1, \ldots, x_n]]$ is regular since it has Krull dimension n and $\mathfrak{m} = (x_1, \ldots, x_n)$.

Lemma. *(i) Any regular local ring R is an integral domain.*
(ii) R is a regular local ring $\Leftrightarrow$ $\mathfrak{m}$ is generated by a regular sequence.

Proof. For (ii)($\Leftarrow$), suppose that $\mathfrak{m}$ is generated by a regular sequence of length n. Then $n \leq \operatorname{depth} R \leq \operatorname{Kdim} R$. On the other hand, $\operatorname{ht} \mathfrak{m} \leq n$ by Krull's height theorem. Thus $\operatorname{Kdim} R = n$ and R is a regular local ring.

We prove (i) and (ii)($\Rightarrow$) by induction on $n = \operatorname{Kdim} R$. If $n = 0$, then $\mathfrak{m} = 0$, and both are clear, so suppose $n > 0$. By Nakayama, $\mathfrak{m}^2 \neq \mathfrak{m}$, so by prime avoidance, there is some element $x \in \mathfrak{m}$ which is not contained in $\mathfrak{m}^2$ or any minimal prime of R. Then $\mathfrak{m} = (x, x_2, \ldots, x_n)$ for suitable $x_2, \ldots, x_n$. Then the maximal ideal $\mathfrak{m}/(x)$ of $R/(x)$ is generated by $\bar{x}_2, \ldots, \bar{x}_n$, so $\operatorname{Kdim} R/(x) \leq n - 1$. But by the theorem about the Krull dimension of M/xM, we have $\operatorname{Kdim} R/(x) \geq n-1$. Thus $\operatorname{Kdim} R/(x) = n - 1$ and $R/(x)$ is a regular local ring.

(i) Thus by induction $R/(x)$ is a domain, so (x) is a prime ideal, so it contains a minimal prime ideal P. Now $P = xP$, for if $y \in P$, then $y = ax$ for some $a \in R$, and then since $x \notin P$, we must have $a \in P$. Thus by Nakayama, $P = 0$, so R is a domain.

(ii) By induction $\mathfrak{m}/(x)$ is generated by a regular sequence $(\bar{y}_1, \ldots, \bar{y}_{n-1})$. Also x is regular on R by (i), so $\mathfrak{m}$ is generated by the regular sequence $(x, y_1, \ldots, y_{n-1})$. $\square$

Lemma. *If $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $\mathfrak{m}/(x)$ is isomorphic to a direct summand of $\mathfrak{m}/x\mathfrak{m}$.*

Proof. Let $x, y_1, \ldots, y_k$ give a basis of $\mathfrak{m}/\mathfrak{m}^2$ and let $I = (y_1, \ldots, y_k)$. Then $I + (x) = \mathfrak{m}$ and $x \notin I$. Observe that $(I + x\mathfrak{m}) \cap (x) = x\mathfrak{m}$, for if $rx = i + xa$ with $i \in I$ and $a \in \mathfrak{m}$, then $(r - a)x \in I$, so $r - a$ is not invertible, so $r - a \in \mathfrak{m}$ (otherwise the ideal it generates must be R), so also $r \in \mathfrak{m}$. Thus

$$\frac{\mathfrak{m}}{x\mathfrak{m}} = \frac{I + x\mathfrak{m}}{x\mathfrak{m}} \oplus \frac{(x)}{x\mathfrak{m}}.$$

Now

$$\frac{\mathfrak{m}}{(x)} = \frac{I + (x)}{(x)} \cong \frac{I}{I \cap (x)} = \frac{I}{I \cap x\mathfrak{m}} \cong \frac{I + x\mathfrak{m}}{x\mathfrak{m}}.$$

$\square$

Theorem ((Auslander-Buchsbaum-)Serre)**.** *The following are equivalent*
(i) R is a regular local ring
(ii) $\operatorname{proj.dim} k < \infty$.
(iii) $\operatorname{gl.dim} R < \infty$
If so, then $\operatorname{proj.dim} k = \operatorname{gl.dim} R = \operatorname{Kdim} R = \operatorname{depth} R$.

Proof. (i)⇒(ii) We could use the Koszul resolution. Alternatively use the following. If $x \in \mathfrak{m}$ is regular on M and Y is an R-module, applying $\operatorname{Hom}(-, Y)$ to the short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

gives an exact sequence

$$\dots \to \operatorname{Ext}^n(M,Y) \to \operatorname{Ext}^{n+1}(M/xM,Y) \to \operatorname{Ext}^{n+1}(M,Y) \to \dots$$

and if $n > \operatorname{proj.dim} M$, then the outer two terms vanish, hence so does the middle, so

$$\operatorname{proj.dim} M/xM \leq \operatorname{proj.dim} M + 1.$$

Now $\mathfrak{m}$ is generated by a regular sequence $x_1, \dots, x_n$. By induction on i, this shows that $\operatorname{proj.dim}_R R/(x_1, \dots, x_i) \leq i$. Thus $\operatorname{proj.dim} k = \operatorname{proj.dim} R/(x_1, \dots, x_n) \leq n$.

(ii)⇒(iii) If $\operatorname{proj.dim} k \leq n$, then for any module M we have $\operatorname{Tor}^R_{n+1}(M, k) = 0$, so by the lemma about projective covers, if M is f.g., then $\operatorname{proj.dim} M \leq n$. It follows that $\operatorname{gl.dim} R \leq n$, since it is the supremum of the projective dimensions of cyclic modules.

(iii)⇒(i) We prove this by induction on the Krull dimension of R. Let $n = \operatorname{gl.dim} R < \infty$. If $n = 0$, then R is semisimple, so a field (since it is commutative), so a regular local ring. Thus we may suppose that $n > 0$. Now there is a cyclic module M with $\operatorname{proj.dim} M = n$. By the lemma about projective covers, it follows that $\operatorname{Tor}^R_n(M, k) \neq 0$. Thus $\operatorname{proj.dim} k = n$, so by the same argument $\operatorname{Tor}^R_n(k, k) \neq 0$.

Now $\mathfrak{m}$ is not an associated prime of R, for if R has a submodule isomorphic to k, we have an exact sequence $0 \to k \to R \to R/k \to 0$. The long exact sequence gives

$$0 = \operatorname{Tor}_{n+1}(R,k) \to \operatorname{Tor}_{n+1}(R/k,k) \to \operatorname{Tor}_n(k,k) \to \operatorname{Tor}_n(R,k) = 0$$

But $\operatorname{Tor}_n(k, k) \neq 0$, so $\operatorname{Tor}_{n+1}(R/k, k) \neq 0$, and this is impossible since $\operatorname{gl.dim} R = n$. Contradiction.

By prime avoidance, there is some element $x \in \mathfrak{m}$ not contained in $\mathfrak{m}^2$ or in any associated prime of R. In particular x is regular on R. Let

$$0 \to P_n \to \dots \to P_0 \to \mathfrak{m} \to 0$$

be a projective resolution of $\mathfrak{m}$. Now x is regular on $\mathfrak{m}$ and on any projective R-module, so the sequence

$$0 \to R/(x) \otimes P_n \to \cdots \to R/(x) \otimes P_0 \to R/(x) \otimes \mathfrak{m} \to 0$$

is still exact by the lemma that regular elements preserve acyclicity. Thus it is a projective resolution of $\mathfrak{m}/x\mathfrak{m}$ as an $R/(x)$-module. Thus $\operatorname{proj.dim}_{R/(x)}(\mathfrak{m}/x\mathfrak{m}) < \infty$. By the lemma, $\mathfrak{m}/(x)$ is isomorphic to a direct summand of $\mathfrak{m}/x\mathfrak{m}$, and hence $\operatorname{proj.dim}_{R/(x)} \mathfrak{m}/(x) < \infty$. Now if

$$0 \to Q_m \to \cdots \to Q_0 \to \mathfrak{m}/(x) \to 0$$

is a projective resolution of $\mathfrak{m}/(x)$ as an $R/(x)$-module, then

$$0 \to Q_m \to \cdots \to Q_0 \to R/(x) \to k \to 0$$

is a projective resolution of k as an $R/(x)$-module. Thus as in (ii)⇒(iii) we have $\operatorname{gl.dim} R/(x) < \infty$. Now since x is regular on R, we have $\operatorname{Kdim} R/(x) = \operatorname{Kdim} R - 1$, so by induction $R/(x)$ is a regular local ring, say with maximal ideal generated by a regular sequence $\bar{y}_1, \dots, \bar{y}_{n-1}$. Then $\mathfrak{m}$ is generated by the regular sequence $x, y_1, \dots, y_{n-1}$, so R is a regular local ring.

Finally, if the conditions hold, the implication (i)⇒(ii) shows $\operatorname{proj.dim} k \le \operatorname{Kdim} R$, (ii)⇒(iii) shows $\operatorname{gl.dim} R = \operatorname{proj.dim} k$, and Rees' theorem shows that $\operatorname{depth} R \le \operatorname{gl.dim} R$. But since R is regular local, $\operatorname{depth} R = \operatorname{Kdim} R$, so all are equal. □

Corollary. *A localization R_P of a regular local ring is regular local.*

Proof. If M is an R_P-module, we can consider it as an R-module by restriction. Then it has a finite projective resolution. Applying localization, this stays exact, and it is an R_P-module projective resolution of $M_P \cong M$. □

Theorem (Auslander-Buchsbaum, 1959)**.** *Any regular local ring is a UFD.*

This was one of the early achievements of homological algebra. The statement does not involve homological algebra, but the proof does. Unfortunately the proof is too long for us.

5.4 Cohen-Macaulay rings (omitted)

Due to lack of time, this section will be omitted.

R is still a local noetherian commutative ring.

Definition. R is *Cohen-Macaulay (CM)* if $\operatorname{depth} R = \operatorname{Kdim} R$.

Examples. (i) Any regular local ring.

(ii) If R is an integral domain and $\operatorname{Kdim} R = 1$, then R is CM, since any nonzero element of $\mathfrak{m}$ is a regular sequence.

(iii) If R is CM of Krull dimension n and $x_1, \ldots, x_i$ is a regular sequence on R, then $R/(x_1, \ldots, x_i)$ is CM of dimension $n - i$, as in Rees' Theorem.

Lemma (Additional lemma to add at end of §5.1)**.** *If R is a commutative noetherian ring and M is a f.g. R-module, then there is a chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/P_i for some prime ideal P_i. Moreover $\operatorname{Kdim} M = \max\{\operatorname{Kdim} R/P_i : 1 \leq i \leq n\}$.

Proof. If $M = 0$ then this holds trivially. If not, then M has an associated prime, so a submodule $M_1 \cong R/P_1$. Now M/M_1 is either zero, or it has a submodule $M_2/M_1 \cong R/P_2$. This gives an ascending chain of submodules of M which must stabilize, so $M_n = M$ for some n. Now the primes P with $M_P \neq 0$ are those with $(R/P_i)_P \neq 0$ for some i, so with $P_i \subseteq P$. Thus a maximal chain of primes ideals P with $M_P \neq 0$ will start with some P_i and increase up to $\mathfrak{m}$. □

Using this lemma, we can prove another result about depth.

Lemma (Ischebeck)**.** *If M and N are nonzero f.g. R-modules then*

$$\operatorname{Ext}^i(N, M) = 0$$

if $i + \operatorname{Kdim} N < \operatorname{depth} M$.

Proof. We prove this by induction on $r = \operatorname{Kdim} N$. Now N has a filtration by modules of the form R/P_i, so we reduce to the case when $N = R/P$. If $r = 0$, this is Rees' Theorem, so suppose $r > 0$. Choose $x \in \mathfrak{m} \setminus P$. We get an exact sequence

$$0 \to N \xrightarrow{x} N \to N' \to 0.$$

where $N' = N/xN = R/(x, P)$, which has Krull dimension $< r$, so by induction $\operatorname{Ext}^j(N', M) = 0$ if $j + (r - 1) < \operatorname{depth} M$. Thus if $i + r < \operatorname{depth} M$, we get an exact sequence

$$\operatorname{Ext}^i(N, M) \xrightarrow{x} \operatorname{Ext}^i(N, M) \to \operatorname{Ext}^{i+1}(N', M) = 0$$

so by Nakayama's Lemma, $\operatorname{Ext}^i(N, M) = 0$. □

Theorem. *Suppose R is a CM ring.*
(i) If P is an associated prime of R, then $\operatorname{Kdim} R/P = \operatorname{Kdim} R$. *Thus the associated primes of R are the minimal primes.*
(ii) Any two maximal chains of distinct prime ideals in R have the same length. Equivalently, for any prime ideal P, we have

$$\operatorname{ht} P + \operatorname{Kdim} R/P = \operatorname{Kdim} R$$

Proof. (i) If P is an associated prime of R, then $\operatorname{Hom}(R/P, R) \neq 0$. Thus by Ischebeck's Lemma,

$$\operatorname{depth} R \leq 0 + \operatorname{Kdim} R/P \leq \operatorname{Kdim} R = \operatorname{depth} R$$

so $\operatorname{Kdim} R/P = \operatorname{Kdim} R$, and clearly P must be a minimal prime. Conversely We already know that, in general, any minimal prime is an associated prime of R.

(ii) We prove this by induction on $n = \operatorname{Kdim} R$. Suppose $\operatorname{ht} P = h$. If $h = 0$ this is (i), so suppose $h > 0$. Then there is a chain of distinct prime ideals

$$P_0 \subset \cdots \subset P_h = P$$

and P_0 is a minimal prime. By prime avoidance there is some $x \in P_1$ not contained in any minimal prime. Then x is regular on R, so $R/(x)$ is CM of Krull dimension $n-1$. Now $P_1/(x)$ is a minimal prime in $R/(x)$, so $P/(x)$ has height $h-1$. Thus by induction

$$n - 1 = \operatorname{Kdim} R/(x) = \operatorname{ht}(P/(x)) + \operatorname{Kdim}(R/(x))/(P/(x)) = h - 1 + \operatorname{Kdim} R/P$$

giving the result. □

Theorem. *The following are equivalent*
(i) R is a regular local ring
(ii) R is a CM ring and every nonzero f.g. module M with $\operatorname{depth} M = \operatorname{depth} R$ *is projective (so free).*

Proof. (i)⇒(ii) $\mathfrak{m}$ is generated by a regular sequence, so R is CM. Then use the Auslander-Buchsbaum formula.

(ii)⇒(i) Let M be nonzero and f.g. We show by descending induction on $d = \operatorname{depth}(M)$ that $\operatorname{proj.dim} M < \infty$. If $\operatorname{depth}(M) = \operatorname{depth}(R)$, then M is projective by hypothesis. If $\operatorname{depth}(M) < \operatorname{depth}(R)$, then the long exact sequence gives $\operatorname{depth}(\Omega_1 M) = \operatorname{depth}(M) + 1$, so by induction $\operatorname{proj.dim} \Omega_1 M < \infty$, so $\operatorname{proj.dim} M < \infty$. □

Remark. A local noetherian commutative ring R is said to be *Gorenstein* if $\operatorname{inj.dim} R < \infty$. Thus clearly a regular local ring is Gorenstein. It can be shown that if R has Krull dimension n, then R is Gorenstein if and only if it is CM and $\operatorname{Ext}^n(k, R) \cong k$. It follows that if $x_1, \ldots, x_i$ is a regular sequence for R, then R is Gorenstein if and only if $R/(x_1, \ldots, x_i)$ is Gorenstein. If you are interested, there is a reasonably accessible treatment in §18 of H. Matsumura, Commutative rings, CUP 1986.

5.5 Group homology and cohomology

Definition. Let K be a commutative ring and let G be a group. Recall that the group algebra KG has elements

$$\sum_{g\in G} a_g g$$

with $a_g \in K$, at most finitely many nonzero. It is a ring with

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g$$

$$(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = \sum_{g\in G} c_g g, \quad c_g = \sum_{hk=g} a_h b_k.$$

The *augmentation* is the ring homomorphism

$$\epsilon : KG \to K, \quad \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g.$$

The *augmentation ideal* is $\Delta(G) = \operatorname{Ker}(\epsilon)$.

Lemma. $\Delta(G) = \sum_{g\in G} K(g-1)$

Proof. Clearly the right hand side is contained in the left. Conversely if $x = \sum_{g\in G} a_g g \in \Delta(G)$, then

$$x = x - \epsilon(x)1 = \sum_{g\in G} (a_g g - a_g 1) = \sum_{g\in G} a_g (g-1).$$

□

Remarks. (a) To give a left KG-module M it is equivalent to give a K-module M and a group homomorphism $\theta : G \to \operatorname{Aut}_K(M)$. Namely, given θ, we define

$$(\sum_{g\in G} a_g g)m = \sum_{g\in G} a_g \theta(g)(m).$$

Alternatively, a KG-module is given by K-module M and a map

$$\rho : G \times M \to M$$

which:
- is an action of G on M, meaning that $\rho(gg', m) = \rho(g, \rho(g'm))$ and $\rho(1, m) = m$, and
- is K-linear for fixed $g \in G$, that is $\rho(g, am + bm') = a\rho(g, m) + b\rho(g, m')$ for all $a, b \in K$ and $m, m' \in M$.

(b) We can turn any K-module M into a KG-module by making G act trivially, so $gm = m$ for all $g \in G$ and $m \in M$. In particular *the trivial KG-module* is K with G acting trivially. We just denote it as K.

(c) Any left KG-module becomes a right KG-module via

$$m(\sum_{g \in G} a_g g) = (\sum_{g \in G} a_g g^{-1})m$$

and conversely. (You can't do this for rings in general!) Namely,

$$(mg)h = (g^{-1}m)h = h^{-1}g^{-1}m = (gh)^{-1}m = m(gh).$$

This gives an isomorphism of categories between KG-Mod and Mod-KG. Thus, for example, M is projective as a left KG-module if and only if it is projective as a right KG-module.

For most purposes we can actually take $K = \mathbb{Z}$.

Proposition/Definition. *Let M be a $\mathbb{Z}G$-module. The set of* invariants

$$M^G = \{m \in M : gm = m\}$$

is the unique largest submodule of M on which G acts trivially. The set of coinvariants

$$M_G = M/S_M, \quad S_M = \textit{submodule generated by } gm - m \textit{ for } g \in G \textit{ and } m \in M,$$

is the unique largest quotient module of M on which G acts trivially. They give functors $-^G$ and $-_G$ from $\mathbb{Z}G$-Mod *to* Ab.

Proof. The first part is clear. If S is a submodule of M, then G acts trivially on M/S if and only if $g(S + m) = S + m$ for all g, m, that is, $gm - m \in S$.

It is clear that any homomorphism $M \to N$ restricts to a map $M^G \to N^G$. Moreover it sends S_M into S_N, so induces a map $M_G \to N_G$. $\square$

Lemma. *(i) There is a natural isomorphism* $M^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$ *and* $-^G$ *is left exact.*

(ii) There is a natural isomorphism $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$ *and* $-_G$ *is right exact.*

Proof. (i) is clear. For (ii) the exact sequence

$$0 \to \Delta(G) \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

gives an exact sequence

$$\Delta(G) \otimes_{\mathbb{Z}G} M \to \mathbb{Z}G \otimes_{\mathbb{Z}G} M \to \mathbb{Z} \otimes_{\mathbb{Z}G} M \to 0$$

Now we can identify $\mathbb{Z}G \otimes_{\mathbb{Z}G} M$ with M, and then since $\Delta(G) = \sum_{g \in G} \mathbb{Z}(g-1)$, the image of the map from $\Delta(G) \otimes_{\mathbb{Z}G} M$ is identified with S_M. $\square$

Definition. The *homology* of G with coefficients in a $\mathbb{Z}G$-module M is

$$H_n(G; M) := L_n(-_G)(M) \cong \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M).$$

The *cohomology* is

$$H^n(G; M) := R^n(-^G)(M) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M).$$

Thus $H^0(G; M) = M^G$ and $H_0(G; M) = M_G$. A short exact sequence $0 \to L \to M \to N \to 0$ of $\mathbb{Z}G$-modules gives long exact sequences

$$\cdots \to H_2(G; N) \to H_1(G; L) \to H_1(G; M) \to H_1(G; N) \to L_G \to M_G \to N_G \to 0$$

$$0 \to L^G \to M^G \to N^G \to H^1(G; L) \to H^1(G; M) \to H^2(G; N) \to H^2(G; L) \to \dots$$

Example. (1) If $G = 1$ then $M^G = M = M_G$ so $-_G$ and $-^G$ are exact functors, so $H^n(G; M) = 0$ and $H_n(G; M) = 0$ for $n > 0$.

(2) If $G = C_\infty$, an infinite cyclic group with generator σ, we have a projective resolution

$$0 \to \mathbb{Z}G \xrightarrow{\sigma - 1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

so

$$H^0(G; M) = M^G \cong H_1(G; M), \qquad H_0(G; M) = M_G \cong H^1(G; M),$$

and $H^n(G; M) = H_n(G; M) = 0$ for $n > 1$.

(3) If $G = C_m$, a finite cyclic group of order m with generator σ, we have a periodic projective resolution

$$\cdots \to \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma - 1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma - 1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1}$. Thus $H^n(G; M)$ is the cohomology of the complex

$$0 \to M \xrightarrow{\sigma-1} M \xrightarrow{N} M \xrightarrow{\sigma-1} M \to \dots$$

so

$$H^n(G; M) = \begin{cases} M^G = \{m \in M : \sigma m = m\} & (n = 0) \\ \{m \in M : Nm = 0\}/(\sigma - 1)M & (n = 1, 3, 5, \dots) \\ M^G/NM & (n = 2, 4, 6, \dots). \end{cases}$$

Proposition. *If K is a commutative ring and M is a KG-module, then*

$$\mathrm{Tor}_n^{KG}(K, M) \cong \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M) \quad \text{and} \quad \mathrm{Ext}^n_{KG}(K, M) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

where K is the trivial KG-module.

Proof. Observe that if X is a $\mathbb{Z}G$-module, then $K \otimes_{\mathbb{Z}} X$ is naturally a KG-module, with the action of K on K and the action of G on X. Let $\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ be a resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules. Clearly each P_i is free as a $\mathbb{Z}$-module. Since $\Omega_0\mathbb{Z} \cong \mathbb{Z}$ is projective as a $\mathbb{Z}$-module, an induction on n shows that $\Omega_n\mathbb{Z}$ is projective as a $\mathbb{Z}$-module and the sequence

$$0 \to \Omega_{n+1}\mathbb{Z} \to P_n \to \Omega_n\mathbb{Z} \to 0$$

splits as a sequence of $\mathbb{Z}$-modules. It follows that the tensor product sequence

$$\cdots \to K \otimes_{\mathbb{Z}} P_1 \to K \otimes_{\mathbb{Z}} P_0 \to K \to 0$$

is exact, so it is a resolution of the trivial KG-module by free KG-modules. Now if M is a KG-module, we have a natural isomorphism of additive groups

$$\mathrm{Hom}_{KG}(K \otimes_{\mathbb{Z}} P_n, M) \cong \mathrm{Hom}_{\mathbb{Z}G}(P_n, M)$$

so taking cohomology we get $\mathrm{Ext}^n_{KG}(K, M) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$. Similarly for Tor. $\square$

Corollary. *If G is a finite group and $|G|$ acts invertibly on M, then $H^n(G; M) = H_n(G; M) = 0$ for $n > 0$.*

Proof. Let K be obtained from $\mathbb{Z}$ by inverting the multiplicative set

$$S = \{1, |G|, |G|^2, |G|^3, \dots\}.$$

We write $K = \mathbb{Z}[1/|G|]$. We can consider M as a KG-module. Now the map

$$K \to KG, \quad 1 \mapsto \frac{1}{|G|} \sum_{g \in G} g$$

is a KG-module map, and a section for the augmentation ϵ, so K is a projective KG-module. (cf. Maschke's Theorem.) $\square$

Definition. The *bar resolution* for G is the sequence

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where:
- P_n is the free $\mathbb{Z}$-module with basis the elements $[g_0|g_1|\dots|g_n]$ with $g_0,\dots,g_n \in G$, considered as a $\mathbb{Z}G$-module with the action given by

$$g[g_0|g_1|\dots|g_n] = [gg_0|gg_1|\dots|gg_n]$$

- ϵ is the homomorphism sending each basis element $[g_0]$ to 1.
- $d_n : P_n \to P_{n-1}$ is given by

$$d_n([g_0|g_1|\dots|g_n]) = \sum_{i=0}^{n}(-1)^i[g_0|\dots|\hat{g}_i|\dots|g_n].$$

Proposition. *The bar resolution is a projective resolution for $\mathbb{Z}$ as a $\mathbb{Z}G$-module.*

Proof. Clearly P_n is a free $\mathbb{Z}G$-module with basis the elements $[1|g_1|\dots|g_n]$. It is easy to check that ϵ and the d_n are homomorphisms and give a complex. For exactness, we set $P_{-1} = \mathbb{Z}$ and $d_0 = \epsilon$ and use that the resulting complex is contractible as a complex of $\mathbb{Z}$-modules, using the homotopy h given by

$$h_{-1} : \mathbb{Z} = P_{-1} \to P_0, \quad h_{-1}(1) = [1]$$

and for $n \geq 0$,

$$h_n : P_n \to P_{n+1}, \quad [g_0|\dots|g_n] \mapsto [1|g_0|\dots|g_n].$$

$\square$

Definition. Given a $\mathbb{Z}G$-module M, we denote by $C(G, M)$ the cochain complex C of additive groups

$$\to 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \dots,$$

where

$$C^n = \{\text{functions } f : G^n \to M\}$$

considered as an additive group by pointwise addition, and

$$d^n(f)(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1})$$
$$+(-1)^{n+1} f(g_1,\dots,g_n).$$

The elements of C^n are *n-cochains of G with values in M*
$Z^n(G, M) = \mathrm{Ker}(d^n) =$ *n-cocycles of G with values in M*
$B^n(G, M) = \mathrm{Im}(d^{n-1}) =$ *n-coboundaries of G with values in M.*

Theorem. *If M is a $\mathbb{Z}G$-module, then $C(G, M) \cong \operatorname{Hom}(P, M)$ where P is the bar resolution, so*

$$H^n(G, M) \cong H^n(C(G, M)) = \frac{Z^n(G, M)}{B^n(G, M)}.$$

Proof. P_n is a free $\mathbb{Z}G$-module with basis the elements $[1|g_1|\dots|g_n]$ with $g_i \in G$. Thus it also has basis the elements $[1|h_1|h_1h_2|\dots|h_1\dots h_n]$ with $h_i \in G$. We get an isomorphism

$$\operatorname{Hom}_{\mathbb{Z}G}(P_n, M) \to C^n, \quad \theta \mapsto f$$

where

$$f(h_1, \dots, h_n) = \theta([1|h_1|h_1h_2|\dots|h_1\dots h_n]).$$

Now if f corresponds to θ, then $d^n(f)$ corresponds to θd_{n+1} where $d_{n+1} : P_{n+1} \to P_n$ is in the bar resolution. Thus

$$\begin{aligned} d^n(f)(h_1, \dots, h_{n+1}) &= \theta d_{n+1}([1|h_1|h_1h_2|\dots|h_1\dots h_{n+1}]) \\ &= \theta\Big([h_1|h_1h_2|\dots|h_1\dots h_{n+1}] \\ &\quad + \sum_{i=1}^{n} (-1)^i [1|h_1|\dots|\widehat{h_1\dots h_i}|\dots|h_1\dots h_{n+1}] \\ &\quad + (-1)^{n+1}[1|h_1|\dots|h_1\dots h_n]\Big) \end{aligned}$$

$$= h_1 f(h_2, \dots, h_{n+1}) + \sum_{i=1}^{n} (-1)^i f(h_1, \dots, h_ih_{i+1}, \dots, h_{n+1}) + (-1)^{n+1} f(h_1, \dots, h_n).$$

□

Example. (i) A 1-cocycle $f : G \to M$ is called a *crossed homomorphism*. It must satisfy $f(g_1g_2) = g_1f(g_2) + f(g_1)$. A 1-coboundary is called a *principal crossed homomorphism*. It is of the form $f(g) = gm - m$ for some $m \in M$. Thus

$$H^1(G; M) = \frac{Z^1(G, M)}{B^1(G, M)} = \frac{\text{crossed homomorphisms}}{\text{principal crossed homomorphisms}}.$$

For example if M has trivial G-action, then $H^1(G; M)$ is identified with the set of group homomorphisms $G \to M$. Thus if G is finite, then $H^1(G; \mathbb{Z}) = 0$.

(ii) A 2-cocycle $f : G \times G \to M$ must satisfy

$$g_1 f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0.$$

The 2-coboundaries are those of the form

$$f(g_1, g_2) = (d^1\alpha)(g_1, g_2) = g_1\alpha(g_2) - \alpha(g_1g_2) + \alpha(g_1)$$

for some function $\alpha : G \to M$. Then

$$H^2(G; M) = \frac{Z^2(G, M)}{B^2(G, M)} = \frac{\text{2-cocycles}}{\text{2-coboundaries}}.$$

5.6 Second cohomology classifies group extensions

Definition. If M and G are groups, then a group extension

$$1 \to M \xrightarrow{\theta} E \xrightarrow{\phi} G \to 1$$

is given by a group E, an injective group homomorphism θ and a surjective group homomorphism ϕ with $\mathrm{Im}(\theta) = \mathrm{Ker}(\phi)$. Two extensions of M and G are equivalent if there is an isomorphism τ giving a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r, "\theta"] \arrow[d, equal] & E \arrow[r, "\phi"] \arrow[d, "\tau"'] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & M \arrow[r, "\theta'"] & E' \arrow[r, "\phi'"] & G \arrow[r] & 1
\end{tikzcd}$$

Now suppose that M is an additive group. It becomes a $\mathbb{Z}G$-module as follows. Let $m \in M$ and $g \in G$. Choose $e \in E$ with $\phi(e) = g$. Then gm is the unique element of M with $\theta(gm) = e\theta(m)e^{-1}$. This does not depend on the choice of e since M is abelian. Also, equivalent extensions induce the same $\mathbb{Z}G$-module structure.

For example a *central extension* is one with $\mathrm{Im}(\theta) \subseteq Z(E)$, or equivalently G acts trivially on M.

Theorem. *If M is a $\mathbb{Z}G$-module, then $H^2(G; M)$ classifies the group extensions*

$$1 \to M \xrightarrow{\theta} E \xrightarrow{\phi} G \to 1$$

inducing the given $\mathbb{Z}G$-module structure on M, up to equivalence. The zero element corresponds to the semidirect product

$$E = M \rtimes G, \quad (m, g)(m', g') = (m + gm', gg'), \theta(m) = (m, 1), \phi(m, g) = g.$$

Proof. (1) Given an extension, choose a map of sets $s : G \to E$ which is a section for ϕ. If $g, h \in G$, then $\phi(s(g)s(h)) = gh = \phi(s(gh))$, so there is a uniquely determined $f : G \times G \to M$ with

$$\theta(f(g, h)) = s(g)s(h)s(gh)^{-1}.$$

It is called the *factor set* associated to the extension E and s. It is a 2-cocycle. It follows that $f(g,1) = gf(1,1)$ and $f(1,h) = f(1,1)$ for all $g, h \in G$. (Usually one chooses s with $s(1) = 1$. Then f is normalized, meaning that $f(1,1) = 0$.)

(2) Any other section s' is given by

$$s'(g) = \theta(\alpha(g))s(g)$$

for some $\alpha : G \to M$. Let f, f' be the factor sets given by s, s'. Then

$$\theta(f'(g,h)) = s'(g)s'(h)s'(gh)^{-1}$$

$$= \theta(\alpha(g))s(g)\theta(\alpha(h))s(h)s(gh)^{-1}\theta(-\alpha(gh))$$
$$= \theta(\alpha(g))\theta(g\alpha(h))s(g)s(h)s(gh)^{-1}\theta(-\alpha(gh))$$
$$= \theta(\alpha(g)\theta(g\alpha(h))\theta(f(g,h))\theta(-\alpha(gh))$$
$$= \theta(\alpha(g) + g\alpha(h) + f(g,h) - \alpha(gh))$$
$$= \theta(f(g,h) + (d^1\alpha)(g,h)).$$

so $f' = f + d^1\alpha$, so f and f' give the same element of $H^2(G;M)$. Thus the extension gives a well-defined element of $H^2(G;M)$.

(3) Suppose two extensions, give the same element of $H^2(G;M)$. The element is given by factor sets f, f' associated to sections $s : G \to E$ and $s' : G \to E'$, and by modifying one of them as in (2), we may assume that $f = f'$.

Any element of E can be written uniquely in the form $\theta(m)s(g)$ with $m \in M$ and $g \in G$. Define $\tau : E \to E'$ be

$$\tau(\theta(m)s(g)) = \theta'(m)s'(g).$$

Provided τ is a group homomorphism, it shows that the extensions are equivalent. Now the product of two elements of E is

$$\theta(m)s(g)\ \theta(m')s(g') = \theta(m)s(g)\theta(m')s(g)^{-1}s(g)s(g')$$

$$= \theta(m)\theta(gm')\theta(f(g,g'))s(gg')$$
$$= \theta(m + gm' + f(g,g'))s(gg')$$

Then

$$\tau(\theta(m)s(g)\ \theta(m')s(g')) = \tau(\theta(m + gm' + f(g,g'))s(gg'))$$
$$= \theta'(m + gm' + f(g,g'))s'(gg') = \theta'(m)s'(g)\ \theta'(m')s'(g')$$
$$= \tau(\theta(m)s(g))\ \tau(\theta(m')s(g')).$$

(4) If $f : G \times G \to M$ is 2-cocycle, then $E = M \times G$ becomes a group with the operation

$$(m, g) \cdot (m', g') = (m + gm' + f(g, g'), gg'),$$

identity element $(-f(1, 1), 1)$, and

$$(m, g)^{-1} = (-g^{-1}(m + f(1, 1) + f(g, g^{-1})), g^{-1}).$$

We get an extension with $\theta(m) = (m - f(1, 1), 1)$ and $\phi(m, g) = g$. Using the section $s(g) = (0, g)$ one recovers f, since

$$(0, g)(0, h)(0, gh)^{-1} = (f(g, h), gh)(0, gh)^{-1}$$

$$= (f(g, h) - f(1, 1), 1)(0, gh)(0, gh)^{-1} = (f(g, h) - f(1, 1), 1) = \theta(f(g, h)).$$

□

5.7 Cohomology with nonabelian coefficients

Some low degree cohomology can be generalized to the nonabelian case.

Definition. Let G be a group. A *multiplicative G-module* is a group M, written multiplicatively, together with a homomorphism $\rho : G \to \mathrm{Aut}(M)$. If $g \in G$ and $x \in M$ we write gx for $\rho(g)(x)$. Thus $g(xy) = (gx)(gy)$ and $g(x^{-1}) = (gx)^{-1}$.

A homomorphism of multiplicative G-module $\theta : M \to M'$ is a group homomorphism with $\theta(gm) = g\theta(m)$ for all g, m.

An abelian multiplicative G-module is the same as a $\mathbb{Z}G$-module, just written multiplicatively.

If M is a multiplicative G-module, we define:
- $M^G = \{m \in M : gm = m \ \forall \ g \in G\}$. It is a subgroup of M.
- A mapping $f : G \to M$ is a *crossed homomorphism* if $f(g_1 g_2) = f(g_1)(g_1 f(g_2))$ for all $g_1, g_2 \in G$.
- Two crossed homomorphisms f, f' are *equivalent* if there is some $m \in M$ with $f'(g) = m^{-1} f(g)(gm)$ for all $g \in G$.
- A crossed homomorphism f is *principal* if there is $m \in M$ with $f(g) = m^{-1}(gm)$ for all $g \in G$. The principal crossed homomorphisms form one equivalence class.
- Let $H^1(G; M)$ be the set of equivalence classes of crossed homomorphisms. It is a *pointed set*, that is, a set with a distinguished element, corresponding to the principal crossed homomorphisms.

Let L and M be pointed sets with distinguished element $*_L$ and $*_M$. A morphism of pointed sets $f : L \to M$ is a mapping with $f(*_L) = *_M$. A sequence

$$L \xrightarrow{f} M \xrightarrow{g} N$$

is exact at M if $\mathrm{Im}(f) = g^{-1}(*_N)$. A group is a pointed set with distinguished element the identity element.

The long exact sequence in cohomology extends to multiplicative G-modules.

Theorem. *Let*

$$1 \to L \xrightarrow{\theta} M \xrightarrow{\phi} N \to 1$$

be a central extension of multiplicative G-modules (so L is abelian, so a $\mathbb{Z}G$-module). Then there is a natural exact sequence of pointed sets

$$1 \to L^G \to M^G \to N^G \to H^1(G; L) \to H^1(G; M) \to H^1(G; N) \to H^2(G; L).$$

Proof. I'll define the maps. The exactness is straightforward.

The maps $L^G \to M^G \to N^G$ are the restrictions of the homomorphisms θ and ϕ.

The maps $H^1(G; L) \to H^1(G; M) \to H^1(G; N)$ are given by composing a crossed homomorphism with θ or ϕ.

The connecting map $N^G \to H^1(G; L)$ is given as follows. If $x \in N^G$, choose $m_x \in \phi^{-1}(x)$. If $g \in G$, then $\phi(m_x^{-1}\ (gm_x)) = x^{-1}\ (gx) = 1$, so there is a unique mapping $f_x : G \to L$ with $\theta(f_x(g)) = m_x^{-1}\ (gm_x)$. Now f_x is a crossed homomorphism, since

$$\theta(f_x(g_1)g_1 f_x(g_2)) = \theta(f_x(g_1))g_1\theta(f_x(g_2)) = m_x^{-1}(g_1 m_x)g_1(m_x^{-1}(g_2 m_x))$$

$$= m_x^{-1}(g_1 m_x)(g_1 m_x)^{-1}(g_1 g_2 m_x) = m_x^{-1}(g_1 g_2 m_x) = \theta(f_x(g_1 g_2))$$

and the image of x is the corresponding equivalence class in $H^1(G; L)$. This doesn't depend on the choice of m_x.

The connecting map $H^1(G; N) \to H^2(G; L)$ is given as follows. An element of $H^1(G; N)$ is represented by a crossed homomorphism $f : G \to N$. For each $g \in G$, choose $m_g \in \phi^{-1}(f(g))$. For $g, h \in G$, note that

$$\phi(m_g\ (gm_h)\ m_{gh}^{-1}) = f(g)\ (gf(h))\ f(gh)^{-1} = 1$$

since f is a crossed homomorphism. Thus there is a mapping $\alpha : G \times G \to L$ with

$$\theta(\alpha(g, h)) = m_g\ (gm_h)\ m_{gh}^{-1}.$$

Then θ is a 2-cocycle, so induces an element of $H^2(G; L)$. $\square$

5.8 Projective representations of groups

Definition. Let K be a field and $n \geq 1$. It is easy to see that

$$Z(\mathrm{GL}_n(K)) = \{\lambda I : \lambda \in K^\times\}.$$

The quotient group

$$\mathrm{PGL}_n(K) := \mathrm{GL}_n(K)/Z(\mathrm{GL}_n(K))$$

is called the *projective linear group*.

Remark. Projective space $\mathbb{P}^{n-1}(K)$ is the set of equivalence classes of n-tuples $(x_1, \dots, x_n)$ with $x_i \in K$, not all zero, under the equivalence relation $\sim$ with

$$(x_1, \dots, x_n) \sim (\lambda x_1, \dots, \lambda x_n)$$

for $\lambda \in K^\times$. The group $\mathrm{GL}_n(K)$ acts on K^n and induces an action on $\mathbb{P}^{n-1}(K)$. If $A \in \mathrm{GL}_n(K)$, then

$$\begin{aligned} A \text{ acts trivially on } \mathbb{P}^{n-1} &\Leftrightarrow Ax \text{ is a multiple of } x \text{ for all } x \in K^n \setminus 0 \\ &\Leftrightarrow \text{Every } x \text{ is an eigenvector for } A \\ &\Leftrightarrow A = \lambda I \text{ for some } \lambda \in K^\times \end{aligned}$$

so $\mathrm{PGL}_n(K)$ acts faithfully on $\mathbb{P}^{n-1}(K)$.

Definition. Let G be a group. An *(ordinary) matrix representation* of G of degree n is a group homomorphism

$$\rho : G \to \mathrm{GL}_n(K).$$

Two representations ρ, ρ' are equivalent if there is a matrix $A \in \mathrm{GL}_n(K)$ with

$$\rho'(g) = A^{-1}\rho(g)A$$

for all $g \in G$. Matrix representations of degree n up to equivalence correspond to isomorphism classes of n-dimensional KG-modules.

A *projective representation* of G is a group homomorphism

$$\sigma : G \to \mathrm{PGL}_n(K).$$

This is in the sense of Schur, it has nothing to do with projective modules! Two projective representations σ, σ' are equivalent if there is $A \in \mathrm{PGL}_n(K)$ with

$$\sigma'(g) = A^{-1}\sigma(g)A$$

for all $g \in G$.

Any ordinary representation ρ gives a projective representation as the composition

$$G \xrightarrow{\rho} \mathrm{GL}_n(K) \to \mathrm{PGL}_n(K).$$

Which projective representations lift to ordinary representations?

Example. Suppose $\rho : \mathrm{SU}_2 \to \mathrm{GL}_n(\mathbb{C})$ is an ordinary representation which is irreducible, so corresponds to a simple module for $\mathbb{C}\,\mathrm{SU}_2$. Now $Z(\mathrm{SU}_2) = \{I, -I\}$, and any eigenspace of $\rho(-I)$, say

$$V_\lambda = \{v \in \mathbb{C}^n : \rho(-I)v = \lambda v\},$$

is a subrepresentation of ρ, since if $v \in V_\lambda$, then

$$\rho(-I)\rho(g)v = \rho(g)\rho(-I)v = \lambda\rho(g)v$$

so $\rho(g)v \in V_\lambda$. Thus by irreducibility $V_\lambda = C^n$ for some λ. Thus $\rho(-I) = \lambda I$. Thus ρ induces a projective representation

$$\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2 / \{I, -I\} \to \mathrm{PGL}_n(\mathbb{C}).$$

Now if ρ is the natural representation (sending any $g \in \mathrm{SU}_2$ to itself in $\mathrm{GL}_2(\mathbb{C})$), then $\rho(-I) = -I$, and I don't think that the corresponding projective representation lifts to an ordinary representation of $\mathrm{SO}_3(\mathbb{R})$. (Using the representation theory of Lie groups, one can see that it doesn't lift to a representation of $\mathrm{SO}_3(\mathbb{R})$ as a Lie group.)

Theorem. *(i) Given a projective representation $\sigma : G \to \mathrm{PGL}_n(K)$, there is a natural way to define an element $c(\sigma) \in H^2(G; K^\times)$, the* obstruction*, so that G lifts to an ordinary representation if and only if $c(\sigma) = 0$.*

(ii) If K is algebraically closed and the Schur muultiplier *$H_2(G; \mathbb{Z})$ of G is zero, then every projective representation lifts.*

Proof. (i) If we consider $\mathrm{GL}_n(K)$ and $\mathrm{PGL}_n(K)$ as multiplicative G-modules, with G acting trivially, then clearly

$$H^1(G; \mathrm{GL}_n(K)) = \frac{\text{matrix representations } G \to \mathrm{GL}_n(K)}{\text{equivalence}}$$

$$H^1(G; \mathrm{PGL}_n(K)) = \frac{\text{projective representations } G \to \mathrm{PGL}_n(K)}{\text{equivalence}}$$

Now the central extension of groups

$$1 \to K^\times \to \mathrm{GL}_n(K) \to \mathrm{PGL}_n(K) \to 1,$$

gives an exact sequence

$$H^1(G; \mathrm{GL}_n(K)) \xrightarrow{b} H^1(G; \mathrm{PGL}_n(K)) \xrightarrow{c} H^2(G; K^\times).$$

Thus a projective representation σ lifts to an ordinary representation if and only if $c(\sigma)$ is zero.

(ii) If P is a projective resolution of the trivial $\mathbb{Z}G$-module, then

$$\begin{aligned}
H^2(G; K^\times) &= H^2(\mathrm{Hom}_{\mathbb{Z}G}(P, K^\times)) && \text{by definition of cohomology with coefficients} \\
&\cong H^2(\mathrm{Hom}_{\mathbb{Z}}(P_G, K^\times)) && \text{since } G \text{ acts trivially on } K^\times \\
&\cong H^2(\mathrm{Hom}_{\mathbb{Z}}(P \otimes_{\mathbb{Z}G} \mathbb{Z}, K^\times)) && \text{since } M_G \cong M \otimes_{\mathbb{Z}G} \mathbb{Z} \\
&= H^2(P \otimes_{\mathbb{Z}G} \mathbb{Z}; K^\times) && \text{by definition of cohomology with coefficients.}
\end{aligned}$$

Since K is algebraically closed, $K^\times$ is divisible, hence injective, as a $\mathbb{Z}$-module. Also $P \otimes_{\mathbb{Z}G} \mathbb{Z}$ is a complex of projective $\mathbb{Z}$-modules, so by the Universal Coefficient Theorem

$$H^2(P \otimes_{\mathbb{Z}G} \mathbb{Z}; K^\times) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(P \otimes_{\mathbb{Z}G} \mathbb{Z}), K^\times) = \mathrm{Hom}_{\mathbb{Z}}(H_2(G; \mathbb{Z}), K^\times).$$

Thus if $H_2(G; \mathbb{Z}) = 0$, then $H^2(G; K^\times) = 0$, so $c(\sigma) = 0$ for all σ. $\square$

5.9 Galois descent

Due to lack of time, the first theorem and the proof of the second theorem will by omitted. A possible reference for this section is §2.3 of P. Gille and T. Szamuely, Central simple algebras and Galois cohomology, CUP 2006.

Definition. Let K be a field. We consider the category whose objects are pairs (V, ϕ) where V is a K-vector space and ϕ is some additional structure, for example:
- An associative multiplication $V \otimes_K V \to V$
- A bilinear form $V \otimes_K V \to K$
- An A-module structure $A \otimes_K V \to V$ where A is a fixed K-algebra.

A morphism $(V, \phi) \to (V', \phi')$ is a K-linear map $\theta : V \to V'$ compatible with the additional structures ϕ and ϕ'.

$\mathrm{Aut}(V, \phi)$ is the group of automorphisms of (V, ϕ).

If L/K is a field extension, then there is an induced L-vector space $V^L = L \otimes_K V$, and there is an induced structure ϕ^L on V^L. For example
- A multiplication $V \otimes_K V \to V$ gives a multiplication $V^L \otimes_L V^L \to V^L$.
- A bilinear form $V \otimes_K V \to K$ gives a bilinear form $V^L \otimes_L V^L \to L$.
- An A-module structure $A \otimes_K V \to V$ gives an A^L-module structure $A^L \otimes_L V^L \to V^L$.

If (W, ψ) is an L-vector space with additional structure and K is subfield of L, a K-*form* of (W, ψ) is a K-vector space with additional structure (V, ϕ) with $(V^L, \phi^L) \cong (W, \psi)$.

If (V, ϕ) and (V', ϕ') are K-vector spaces with additional structure, we say that (V', ϕ') is a *twisted form* of (V, ϕ) *split by* a field extension L/K if $((V')^L, (\phi')^L) \cong (V^L, \phi^L)$.

Theorem. *Let L/K be a Galois field extension with group G and let W be an L-vector space. There is a bijection between*
(i) K-subspaces V of W such that the multiplication map $m : L \otimes_K V \to W$ is an isomorphism, and
(ii) group homomorphisms $\alpha : G \to \mathrm{Aut}_K(W)$ such that $\alpha(g)$ is a g-semilinear map for each $g \in G$, meaning that $\alpha(g)(\lambda w) = g(\lambda)\alpha(g)(w)$ for all $\lambda \in L$ and $w \in W$.

Proof. Given $V \subseteq W$ as in (i), define α by $\alpha(g)(w) = m(g \otimes 1)m^{-1}(w)$. It has the properties in (ii). Conversely, given α as in (ii), define

$$V = \{w \in W : \alpha(g)(w) = w \ \forall \ g \in G\}.$$

This is a K-subspace of W. Let $m : L \otimes_K V \to W$ be the multiplication map.

Let $g_1, \ldots, g_n$ be the elements of G and $\lambda_1, \ldots, \lambda_n$ a basis of L over K (same n, since L/K is Galois). Suppose $\sum_{i=1}^n \lambda_i \otimes v_i$ is in the kernel of m. Then $\sum_{i=1}^n \lambda_i v_i = 0$. Applying $\alpha(g_j)$ we get $\sum_{i=1}^n g_j(\lambda_i)v_i = 0$. By Dedekind's Independence Theorem, the matrix $(g_j(\lambda_i)) \in M_n(L)$ is invertible. Thus $v_i = 0$ for all i. Thus m is injective.

If $w \in W$, then clearly $\sum_{j=1}^n \alpha(g_j)(w) \in V$. Applying this to the elements $\lambda_i w$, we obtain elements $v_i \in V$ with

$$v_i = \sum_j \alpha(g_j)(\lambda_i w) = \sum_i g_j(\lambda_i)\alpha(g_j)(w)$$

Now if $(b_{ij}) \in M_n(L)$ is the inverse of the matrix $(g_j(\lambda_i))$, then

$$\alpha(g_j)(w) = \sum_i b_{ji} v_i \in \mathrm{Im}(m),$$

so in particular $w = \alpha(1)(w) \in \mathrm{Im}(m)$. Thus m is an isomorphism.

Now it is easy to see that the constructions are inverse. $\square$

Theorem. *Let L/K be a Galois field extension with group G. Given a K-vector space with additional structure (V, ϕ), the group $\mathrm{Aut}(V^L, \phi^L)$ is naturally a multiplicative G-module and the twisted forms of (V, ϕ) split by L/K are in bijection with $H^1(G; \mathrm{Aut}(V^L, \phi^L))$.*

Proof. We consider $\mathrm{Aut}(V^L, \phi^L)$ as a multiplicative G-module as follows. If $\theta \in \mathrm{Aut}(V^L, \phi^L)$ and $g \in G$, let $g\theta$ be the composition

$$L \otimes_K V \xrightarrow{g^{-1}\otimes 1} L \otimes_K V \xrightarrow{\theta} L \otimes_K V \xrightarrow{g\otimes 1} L \otimes_K V.$$

By construction it is a K-linear map, and in fact it is L-linear, since the twists by g^{-1} and g cancel out. Moreover $g\theta$ preserves the additional structure. For example if the extra structure is a multiplication $\phi : V \otimes_K V \to V$, then

$$\phi^L : V^L \otimes_L V^L \to V^L, \quad (\lambda \otimes v) \otimes (\lambda \otimes v') \mapsto \lambda\lambda' \otimes \phi(v \otimes v'),$$

and

$$\begin{aligned}
\phi^L((g\theta)(\lambda \otimes v) \otimes (g\theta)(\lambda' \otimes v')) &= (g \otimes 1)\phi^L(\theta(g^{-1}(\lambda) \otimes v) \otimes \theta(g^{-1}(\lambda') \otimes v'))) \\
&= (g \otimes 1)\theta(g^{-1}(\lambda)g^{-1}(\lambda')) \otimes \phi(v \otimes v') \\
&= (g\theta)(\lambda\lambda' \otimes \phi(v \otimes v')) \\
&= (g\theta)\phi^L((\lambda \otimes v) \otimes (\lambda' \otimes v')).
\end{aligned}$$

A twisted form (V', ϕ') of (V, ϕ) gives a crossed homomorphism as follows. Choose an isomorphism $f : ((V')^L, (\phi')^L) \to (V^L, \phi^L)$. If $g \in G$, let $\rho_f(g)$ be the composition

$$L \otimes_K V \xrightarrow{g^{-1}\otimes 1} L \otimes V \xrightarrow{f^{-1}} L \otimes_K V' \xrightarrow{g\otimes 1} L \otimes_K V' \xrightarrow{f} L \otimes_K V.$$

Then $\rho_f(g)$ is L-linear and belongs to $\mathrm{Aut}(V^L, \phi^L)$. Also

$$\begin{aligned}
\rho_f(gg') &= f(gg' \otimes 1)f^{-1}((g')^{-1}g^{-1} \otimes 1) \\
&= f(g \otimes 1)(g' \otimes 1)f^{-1}((g')^{-1} \otimes 1)(g^{-1} \otimes 1) \\
&= f(g \otimes 1)f^{-1}(g^{-1} \otimes 1)(g \otimes 1)f(g' \otimes 1)f^{-1}((g')^{-1} \otimes 1)(g^{-1} \otimes 1) \\
&= \rho_f(g)(g \otimes 1)\rho_f(g')(g^{-1} \otimes 1) \\
&= \rho_f(g)(g\rho_f(g'))
\end{aligned}$$

so ρ_f is a crossed homomorphism $G \to \mathrm{Aut}(V^L, \phi^L)$.

Now if $f' : ((V')^L, (\phi')^L) \to (V^L, \phi^L)$ is another isomorphism, then $\theta = f(f')^{-1} \in \mathrm{Aut}(V^L, \phi^L)$, and

$$\rho_{f'}(g) = \theta^{-1}\rho_f(g)(g\theta)$$

so ρ_f and $\rho_{f'}$ are equivalent, so determine one element in $H^1(G, \mathrm{Aut}(V^L, \phi^L))$.

Conversely a crossed homomorphism $\rho : G \to \mathrm{Aut}(V^L, \phi^L)$ gives a twisted form V_ρ as follows. The map $\alpha : G \to \mathrm{Aut}_K(V^L)$ given by

$$\alpha(g) = \rho(g)(g \otimes 1)$$

satisfies the conditions of the previous theorem, so

$$V_\rho = \{w \in V^L : \rho(g)((g \otimes 1)w) = w \ \forall \ g \in G\}$$

is a K-form for V^L as a vector space. Moreover the additional structure on V^L restricts to an additional structure ϕ_ρ on V_ρ. For example if ϕ is a multiplication and $w, w' \in L \otimes V$, then

$$\begin{aligned}\rho(g)((g \otimes 1)\phi^L(w \otimes w')) &= \rho(g)\phi^L((g \otimes 1)w \otimes (g \otimes 1)w') \\ &= \phi^L(\rho(g)(g \otimes 1)w \otimes \rho(g)((g \otimes 1)w'))\end{aligned}$$

since $\rho(g) \in \operatorname{Aut}(V^L, \phi^L)$. Thus if $w, w' \in V_\rho$, so is $\phi^L(w \otimes w')$.

Now it is easy to check that if ρ and ρ' are equivalent crossed homomorphisms, then $(X_\rho, \phi_\rho) \cong (V_{\rho'}, \phi_{\rho'})$ and that the constructions (X_ρ, ϕ_ρ) and ρ_f are inverse. □

Corollary. *Let L/K be a Galois field extension with group G. Considering $\mathrm{GL}_n(L)$ as a multiplicative G-module with action*

$$g\,(a_{ij}) = (g(a_{ij})), \quad g \in G, \ (a_{ij}) \in \mathrm{GL}_n(L),$$

the set $H^1(G, \mathrm{GL}_n(L)) = 0$ has only one-element; in particular $H^1(G, L^\times) = 0$.

Proof. Take $V = K^n$ with no additional structure, we have $\operatorname{Aut}(V^L) = \mathrm{GL}_n(L)$, and the G-module structure is as indicated. Since a vector space is determined up to isomorphism by its dimension, all twisted forms of V are isomorphic to V. □

Theorem (Hilbert's Theorem 90)**.** *Suppose L/K is a Galois field extension whose group G is cyclic of order n, say generated by σ. Let N be the norm for L, so*

$$N(x) = x\,\sigma(x)\,\sigma^2(x)\,\ldots\,\sigma^{n-1}(x).$$

Then $x \in L^\times$ is of the form $y^{-1}\sigma(y)$ for some $y \in L$ if and only if $N(x) = 1$.

Proof. Observe that $N(xx') = N(x)N(x')$ and $N(\sigma(x)) = N(x)$. It follows that if x has the indicated form, then $N(x) = 1$. Now suppose that $N(x) = 1$. Define a map $\rho : G \to L^\times$ by

$$\rho(\sigma^i) = x\,\sigma(x)\,\sigma^2(x)\,\ldots\,\sigma^{i-1}(x).$$

for $i \geq 0$. This is well-defined since $N(x) = 1$. It is a crossed homomorphism since

$$\rho(\sigma^{i+j}) = x\,\sigma(x)\,\sigma^2(x)\,\ldots\,\sigma^{i+j-1} = \rho(\sigma^i) \cdot \sigma^i\rho(\sigma^j).$$

Thus it is principal, so of the form

$$\rho(\sigma^i) = y^{-1}\sigma^i(y)$$

for some $y \in L^\times$. Taking $i = 1$ gives $x = y^{-1}\sigma(y)$. □

6 Triangulated categories and derived categories

Unfortunately there is no time to do this properly. I just discuss a few basics, without proofs. A good reference (but in French) is P.-P. Grivel, Catégorie dérivées et Foncteurs dérives, chapter I of A. Borel et. al., Algebraic D-modules, Academic Press, 1987.

6.1 Triangulated categories

We consider an additive category $\mathcal{C}$ equipped with an additive functor Σ which is an automorphism. A *triangle* is a collection of objects and morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

It is sometimes written with X, Y, Z at the vertices of a triangle, and the edges given by arrows $u : X \to Y$, $v : Y \to Z$, and w represented as a arrow $Z \to X$ labelled with Σ.

A morphism of triangles from $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ to $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ is given by a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

A *triangulated category* is given by an additive category $\mathcal{C}$ with an additive automorphism Σ and a collection of *distinguished triangles* satisfying the following.

TR1. (a) Every triangle isomorphic to a distinguished triangle is distinguished.
(b) Every morphism $u : X \to Y$ can be included in a distinguished triangle.
(c) The triangle $X \xrightarrow{\mathrm{Id}_X} X \to 0 \to \Sigma X$ is distinguished.

TR2. The triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is distinguished if and only if its *rotation*

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is distinguished.

TR3. Given a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

in which the rows are distinguished triangles, there is some $h : Z \to Z'$ (not necessarily unique) turning it into a morphism of triangles.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

TR4. (Octahedral axiom) Given three distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \to \Sigma X$$

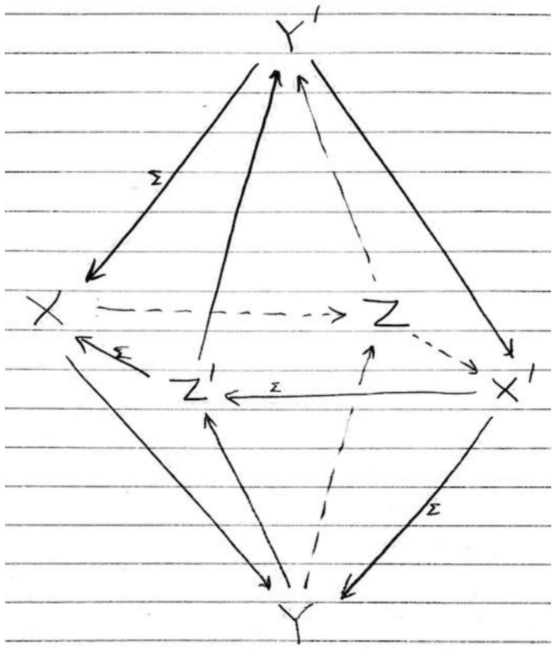
$$Y \xrightarrow{v} Z \to X' \to \Sigma Y$$

$$X \xrightarrow{vu} Z \to Y' \to \Sigma X$$

there is a distinguished triangle

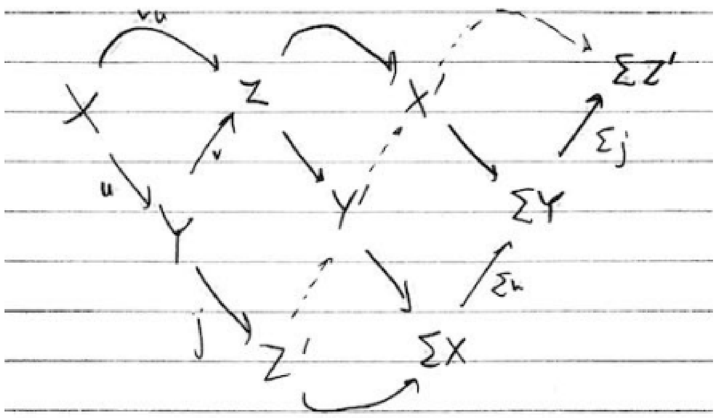
$$Z' \to Y' \to X' \to \Sigma Z'$$

such that on the octahedron, whose top front and back, and bottom left and right faces are given by the triangles,
- the other four faces commute,
- the compositions from bottom to top via Z and Z' give the same map $Y \to Y'$.
- the compositions from top to bottom via X and X' give the same map $Y' \to \Sigma Y$.



Remarks. (i) In fact TR3 follows from the other axioms, and one only needs to show one direction of TR2, see J. P. May, The additivity of traces in triangulated categories, Adv. Math. 163 (2001), 34–-73.

(ii) As suggested in May's article, another way to draw the octahedral axiom is as a commutative "braid" or "sine wave diagram":



(iii) Pretending that we are in an abelian category and that X and Y are subobjects of Z with $X \subseteq Y \subseteq Z$, we might label the three given triangles as

$$X \to Y \to Y/X \to \Sigma X$$
$$Y \to Z \to Z/Y \to \Sigma Y$$
$$X \to Z \to Z/X \to \Sigma X$$

and then the last one is

$$Y/X \to Z/X \to Z/Y \to \Sigma(Y/X)$$

corresponding to the isomorphism $Z/Y \cong (Z/X)/(Y/X)$ we would have in the abelian category.

(iv) Note that in TR3, the morphism h is not unique. A. Neeman, Some new axioms for triangulated categories, J. Algebra 139 (1991), 221–255, has pointed out that some choices of h are better than others. This leads to problems with higher K-theory for triangulated categories. (As a preprint, Neeman's paper was called 'Triangulated categories are all wrong'). In modern work, the problems are tackled by working with 'algebraic triangulated categories', or DG-categories or ∞-categories.

Definition. Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated categories with automorphisms Σ and Σ'. A functor $F : \mathcal{C} \to \mathcal{C}'$ is a *triangle functor* if it is additive, $\Sigma' F = F\Sigma$, and it sends distinguished triangles to distinguished triangles.

If $\mathcal{C}$ is a triangulated category, a full additive subcategory $\mathcal{B}$ is a *triangulated subcategory* if
- $X \in \mathrm{ob}(\mathcal{B}) \Leftrightarrow \Sigma X \in \mathrm{ob}(\mathcal{B})$.
- If $X \to Y \to Z \to \Sigma X$ is a distinguished triangle in $\mathcal{C}$ and $X, Y \in \mathrm{ob}(\mathcal{B})$, then $Z \in \mathrm{ob}(\mathcal{B})$.

In this case $\mathcal{B}$ is a triangulated category and the inclusion is a triangle functor.

$\mathcal{B}$ is a *thick* or *épaisse* subcategory if in addition it is closed under direct summands, that is, if X is isomorphic to a direct summand of Y in $\mathcal{C}$ and $Y \in \mathrm{ob}(\mathcal{B})$, then $X \in \mathrm{ob}(\mathcal{B})$.

Proposition. *(i) The composition of two morphisms in a distinguished triangle is zero.*
(ii) If M is an object, a distinguished triangle gives a long exact sequence

$$\cdots \to \mathrm{Hom}(M, \Sigma^{-1}Z) \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y) \to \mathrm{Hom}(M, Z) \to$$

$$\mathrm{Hom}(M, \Sigma X) \to \mathrm{Hom}(M, \Sigma Y) \to \mathrm{Hom}(M, \Sigma Z) \to \mathrm{Hom}(M, \Sigma^2 X) \to \dots$$

and similarly for $\mathrm{Hom}(-, M)$.
(iii) Given a morphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

if two of f, g, h are isomorphisms, so is the third.

Proof. (i) We have a commutative diagram whose rows are distinguished triangles

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[r, "0"] \arrow[d, "u"'] & 0 \arrow[r, "0"] & \Sigma X \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & \Sigma X \end{tikzcd}$$

By TR3 this extends to a morphism of triangles, via a morphism $h : 0 \to Z$ which must be zero. Since the middle square commutes, we get $vu = 0$. Using TR2 one gets that also $wv = 0$ and $\Sigma u \circ w = 0$.

(ii) Suppose $f \in \mathrm{Hom}(M, Y)$ is sent to 0 in $\mathrm{Hom}(M, Z)$, that is, $vf = 0$. Then we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "0"] \arrow[d, "f"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & \Sigma M \arrow[r, "-\mathrm{Id}"] & \Sigma M \\ Y \arrow[r, "v"] & Z \arrow[r, "w"] & \Sigma X \arrow[r, "-\Sigma u"] & \Sigma Y \end{tikzcd}$$

whose rows are rotations of distinguished triangles, so distinguished. Thus it can be completed to a morphism of triangles with a morphism $h : \Sigma M \to \Sigma X$. Since the right hand square is commutative, we have $\Sigma f \circ (-\mathrm{Id}) = (-\Sigma u) \circ h$, so $f = u \circ \Sigma^{-1} h$, so f is in the image of the map $\mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y)$. Thus the sequence is exact at $\mathrm{Hom}(M, Y)$. By considering rotations, we get exactness at all places.

(iii) By considering rotations, we may suppose that f and g are isomorphisms, and want to show that h is an isomorphism. For any M, by (ii) we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathrm{Hom}(M,X) & \longrightarrow & \mathrm{Hom}(M,Y) & \longrightarrow & \mathrm{Hom}(M,Z) & \longrightarrow & \mathrm{Hom}(M,\Sigma X) & \longrightarrow & \mathrm{Hom}(M,\Sigma Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(M,X') & \longrightarrow & \mathrm{Hom}(M,Y') & \longrightarrow & \mathrm{Hom}(M,Z') & \longrightarrow & \mathrm{Hom}(M,\Sigma X') & \longrightarrow & \mathrm{Hom}(M,\Sigma Y')
\end{array}$$

Now the 1st, 2nd, 4th and 5th vertical maps are induced by $f, g, \Sigma f, \Sigma g$, so they are isomorphisms. Thus by the five lemma, the middle vertical map is an isomorphism. Thus h induces an isomorphism of representable functors $\mathrm{Hom}(-, Z) \to \mathrm{Hom}(-, Z')$. Thus by Yoneda's Lemma, h is an isomorphism. □

Recall that if $\mathcal{A}$ is an additive category, we have a category of complexes $C(\mathcal{A})$ and the homotopy category $K(\mathcal{A})$. Both have a shift automorphism Σ. Recall that if $f : B \to C$ is a morphism in $C(\mathcal{A})$, it has a mapping cone $\mathrm{cone}(f)$ and there is a sequence of complexes

$$0 \to C \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{p} \Sigma B \to 0$$

which is split exact in each degree.

Theorem. *$K(\mathcal{A})$ becomes a triangulated category, where the the distinguished triangles are those isomorphic to one of the form*

$$B \xrightarrow{f} C \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{p} \Sigma B.$$

The proof is omitted. For example we need to know that the rotation

$$C \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{p} \Sigma B \xrightarrow{-\Sigma f} \Sigma C$$

is isomorphic to the mapping cone triangle

$$C \xrightarrow{i} \mathrm{cone}(f) \to \mathrm{cone}(i) \to \Sigma C.$$

Of course this is not true in $C(\mathcal{A})$, only in $K(\mathcal{A})$. (It is related to the "mapping cylinder" construction.)

Proposition. *If $\mathcal{A}$ is an abelian category, the cohomology functor $H^0 : K(\mathcal{A}) \to \mathcal{A}$ sends triangles to long exact sequences*

$$\dots \to H^0(\Sigma^{-1}Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to$$

$$H^0(\Sigma X) \to H^0(\Sigma Y) \to H^0(\Sigma Z) \to H^0(\Sigma^2 X) \to \dots$$

Since $H^0(\Sigma^n(X) = H^n(X)$, we can also write this as

$$\dots \to H^{-1}(Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to$$

$$H^1(X) \to H^1(Y) \to H^1(Z) \to H^2(X) \to \dots$$

Example. The full subcategory $K^b(\mathcal{A})$ of $K(\mathcal{A})$ consisting of the bounded complexes is a triangulated subcategory, so triangulated itself. Similarly $K^-(\mathcal{A})$, consisting of the bounded above complexes. Also, if $\mathcal{A}$ is abelian, the category $K^{-,b}(\mathcal{A})$ of the bounded above complexes X which have bounded cohomology, that is, with $H^i(X) = 0$ for all but finitely many i. There are many other variations.

6.2 Localization of categories

In the homotopy category $K(R\text{-Mod})$, we can consider any module M as a stalk complex, and a projective resolution gives a quasi-isomorphism $P \to M$ from a complex of projectives. We would like to construct a 'derived category' $D(R\text{-Mod})$ from $K(R\text{-Mod})$ in which the quasi-isomorphisms become isomorphisms. We do this by explicitly inverting them.

Definition. Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$. One says that S has a *calculus of left fractions* if
(i) S contains all identity morphisms and is closed under composition.
(ii) Any diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle s}\big\downarrow & & \\ X' & & \end{array}$$

with $s \in S$ can be completed to a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle s}\big\downarrow & & {\scriptstyle t}\big\downarrow \\ X' & \longrightarrow & Y' \end{array}$$

with $t \in S$.
(iii) If $f, g \in \mathrm{Hom}(X, Y)$ and $fs = gs$ for some $s \in S$, then $tf = tg$ for some $t \in S$.

Remark. If R is a ring, considered as a category with one object, and S is a subset of R, then (i) is the condition for S to be a multiplicative subset of R. If R is a commutative ring, this is enough to construct a localization $S^{-1}R$. If R is noncommutative, it is not enough. One also needs the *left Ore condition* (ii) and *left reversibility* (iii).

Proposition. *If S admits a calculus of left fraction, then there is a (BIG) category $S^{-1}\mathcal{C}$ with*
- objects as in $\mathcal{C}$,
- $\mathrm{Hom}(X,Y) = \{(f,s) : X \xrightarrow{f} Y' \xleftarrow{s} Y$ *and* $s \in S\}/\sim$ *where*

$$(f_1,s_1) \sim (f_2,s_2) \Leftrightarrow \exists \textit{commutative diagram } \begin{array}{ccccc} X & \xrightarrow{f_1} & Y_1' & \xleftarrow{s_1} & Y \\ \| & & \downarrow & & \| \\ X & \longrightarrow & Y_3' & \xleftarrow{s_3} & Y \\ \| & & \uparrow & & \| \\ X & \xrightarrow{f_2} & Y_2' & \xleftarrow{s_2} & Y \end{array} \textit{with } s_3 \in S$$

The equivalence class of (f,s) is denoted $s^{-1}f$.
- The composition of $s^{-1}f : X \to Y$ and $(s')^{-1}(f') : Y \to Z$ is $(ts')^{-1}(gf)$ where

$$\begin{tikzcd} & & Z \arrow[d, "s'"'] \\ & Y \arrow[r, "f'"] \arrow[d, "s"'] & Z' \arrow[d, "t"'] \\ X \arrow[r, "f"] & Y' \arrow[r, "g"] & Z'' \end{tikzcd}$$

where the square is given by part (ii) of the definition

There is a natural functor $i : \mathcal{C} \to S^{-1}\mathcal{C}$ sending a morphism $f : X \to Y$ to $1_Y^{-1}f$. If $f \in S$, then $i(f)$ is invertible with inverse $f^{-1}1_Y$, and it sends elements of S to invertible morphisms.

Moreover any functor $F : \mathcal{C} \to \mathcal{D}$ sending the elements of S to invertible morphisms factors as Gi for a unique functor $G : S^{-1}\mathcal{C} \to \mathcal{D}$.

Remark. We say that S has a *calculus of right fractions* in $\mathcal{C}$ if it has a calculus of left fractions in $\mathcal{C}^{op}$. If so, we define $\mathcal{C}S^{-1} = (S^{-1}\mathcal{C}^{op})^{op}$. If S has calculi of left and right fractions, then we can identify $S^{-1}\mathcal{C}$ and $\mathcal{C}S^{-1}$.

Definition. Suppose $\mathcal{C}$ is a triangulated category and S has calculi of left and right fractions. We say that S is *compatible with the triangulated structure of* $\mathcal{C}$ if
- $s \in S \Leftrightarrow \Sigma(s) \in S$, and
- In the situation of axiom TR3, if $f, g \in S$, there is some $h \in S$ giving a morphism of triangles.

Theorem. *Suppose $\mathcal{C}$ is a triangulated category, S has calculi of left and right fractions and it is compatible with the triangulated structure, then $S^{-1}\mathcal{C}$ has the structure of a triangulated category in which the distinguished triangles are those isomorphic to the image of a distinguished triangle in $\mathcal{C}$. Moreover the functor $i : \mathcal{C} \to S^{-1}\mathcal{C}$ is a triangle functor.*

Theorem. *Suppose $\mathcal{C}$ is a triangulated category and $\mathcal{B}$ is a (full additive) triangulated subcategory of $\mathcal{C}$. Let S be the collection of morphisms $u : X \to Y$ such that there is a distinguished triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

in $\mathcal{C}$ with Z isomorphic to an object in $\mathcal{B}$. Then S has calculi of left and right fractions and it is compatible with the triangulated structure.

The category $S^{-1}\mathcal{C}$ in this case is denoted $\mathcal{C}/\mathcal{B}$ and called the *Verdier quotient.*

6.3 Derived categories

Definition. Let $\mathcal{A}$ be an abelian category. If

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is a distinguished triangle in $K(\mathcal{A})$, then the sequence

$$\cdots \to H^{-1}(Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to$$

$$H^1(X) \to H^1(Y) \to H^1(Z) \to H^2(X) \to \dots$$

is exact. It follows that the full subcategory $K(\mathcal{A})_{ac}$ of $K(\mathcal{A})$ consisting of acyclic complexes is a triangulated subcategory.

Let S be the collection of all quasi-isomorphisms in $K(\mathcal{A})$. The long exact sequence also shows that in a triangle as above, $u \in S$ if and only if Z is acyclic. Thus S is the collection of morphisms arising from the subcategory $K(\mathcal{A})_{ac}$, so it has calculi of left and right fractions and is compatible with the triangulated structure.

The *derived category* is the triangulated category

$$D(\mathcal{A}) = K(\mathcal{A})/K(\mathcal{A})_{ac} = S^{-1}K(\mathcal{A}).$$

Remark. The *bounded derived category* $D^b(\mathcal{A})$ can be defined in several ways:
- $K^b(\mathcal{A})/(K^b(\mathcal{A}) \cap K(\mathcal{A})_{ac}) = (S \cap K^b(\mathcal{A}))^{-1}K^b(\mathcal{A})$.
- The full subcategory of $D(\mathcal{A})$ given by the bounded complexes.
- The full subcategory of $D(\mathcal{A})$ given by the complexes with bounded cohomology.
These give triangle equivalent categories, see Stacks Project Lemma 13.11.6. Similarly for $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$.

This uses that if X is any complex and $n \in \mathbb{Z}$, then there is a morphism from a truncation $\tau_{\leq n}X \to X$

$$\begin{tikzcd}
\dots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & \mathrm{Ker}(d^n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & X^{n-1} \arrow[r] & X^n \arrow[r] & X^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

which is an isomorphism on cohomology H^i with $i \leq n$, and a morphism to a truncation $X \to \tau_{\geq n}X$

$$\begin{tikzcd}
\dots \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & X^n/\,\mathrm{Im}(d^{n-1}) \arrow[r] & X^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

which is an isomorphism on cohomology H^i with $i \geq n$.

Example. Consider the derived category $D(R\text{-Mod})$. We can consider any module M as a complex in degree 0. If

$$0 \to M \to I^0 \to I^1 \to \dots$$

is an injective, then M is isomorphic in the derived category to the complex I:

$$\dots \to 0 \to I^0 \to I^1 \to \dots$$

Any complex I of injectives which is bounded below is *K-injective*, meaning that

$$\mathrm{Hom}_{K(R\text{-Mod})}(X, I) = 0$$

for any acyclic complex X, see Stacks project Lemma 13.31.4. Moreover, I is K-injective if and only if the natural map

$$\mathrm{Hom}_{K(R\text{-Mod})}(X, I) \to \mathrm{Hom}_{D(R\text{-Mod})}(X, I)$$

is an isomorphism for all X, see Stacks project Lemma 13.31.2.

Now if N is an R-module, considered as a stalk complex in degree 0, then a homomorphism $N \to \Sigma^n I$ is given by a diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \longrightarrow & I^{n+1} & \longrightarrow & \dots
\end{array}$$

and taking into account homotopies, we see that

$$\mathrm{Hom}_{K(R\text{-Mod})}(N, I) \cong H^n(\mathrm{Hom}(N, I)) \cong \mathrm{Ext}^n(N, M).$$

Thus

$$\mathrm{Hom}_{D(R\text{-Mod})}(N, M) \cong \mathrm{Hom}_{D(R\text{-Mod})}(N, I) \cong \mathrm{Hom}_{K(R\text{-Mod})}(N, I) \cong \mathrm{Ext}^n(N, M).$$

In general any complex X which is bounded below has a quasi-isomorphism $X \to I$ with I a complex of injectives which is bounded below, see Stacks project Lemma 13.19.3. If X is not bounded below, by Spaltenstein's work, there is still a quasi-isomorphism to a K-injective complex, see Stacks project Lemma 13.34.6.

Example. If R is hereditary, then we saw in the section on the universal coefficient theorem that any complex of projectives is isomorphic to a direct sum of two term complexes $0 \to P \xrightarrow{\theta} Q \to 0$ with θ injective, and this is quasi-isomorphic to the stalk complex $\mathrm{Coker}(\theta)$ in the same degree as Q. Now any bounded above complex is quasi-isomorphic to a bounded above complex of projectives (this is dual to the statement about bounded below complexes and injectives), so isomorphic in $D(R\text{-Mod})$ to a direct sum shifts of stalk complexes.