Algebra II
1. Übungsblatt

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Abgabe: Bis zum 19.04.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 1.1. (4) Let $\mathbb{N} = \{0, 1, 2, \dots\}$. We define a relation $\leq$ on $\mathbb{N}^2$ by

$$(a_1, a_2) \leq (b_1, b_2) \;\Leftrightarrow\; a_1 < b_1 \text{ or } (a_1 = b_1 \text{ and } a_2 \leq b_2).$$

Show that with this relation $\mathbb{N}^2$ becomes a well-ordered set. [This is called the lexicographic ordering. You need to show first that it is a partial ordering.]

Aufgabe 1.2. (4) Let G be a group with neutral element e and let A be a subset of G such that $e \notin A$. Let S be the set of subgroups $H \leq G$ such that $H \cap A = \emptyset$. We partially order S by inclusion. Use Zorn's lemma to show that S has a maximal element.

Aufgabe 1.3. (1+1+1+1) An object X in a category $\mathcal{C}$ is called an initial object if there is a unique morphism $X \to Y$ for every object Y. Identify the initial objects in the following categories.

(i) Grp, the category of groups
(ii) Set, the category of sets.
(iii) Set^{op}, the opposite of the category of sets. [An initial object in $\mathcal{C}^{op}$ is called a final object in $\mathcal{C}$. It is an object X such that for every object Y in $\mathcal{C}$ there is a unique morphism $Y \to X$.]
(iv) The category associated with a partially ordered set S.

Aufgabe 1.4. (1+1+1+1) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in a category $\mathcal{C}$ and let $F : \mathcal{C} \to \mathcal{D}$ be a faithful functor. Show the following.

(i) If f and g are monomorphisms, then so is gf.
(ii) If gf is a monomorphism, then so is f.
(iii) If $F(f)$ is a monomorphism, then so is f.
(iv) The monomorphisms in the category Grp of groups are exactly the injective group homomorphisms.