Algebra II
3. Übungsblatt

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Abgabe: Bis zum 03.05.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 3.1. (2+1+1)

(i) Let R be a ring, let $a \in R$ and let M be a left R-module. Show that $\mathrm{Hom}_R(R/Ra, M)$ is isomorphic as an additive group to the subgroup $\{m \in M : am = 0\}$ of M.

(ii) Suppose $n > 0$. Show that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}n, \mathbb{Z}) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}n, \mathbb{Z}/\mathbb{Z}n) \cong \mathbb{Z}/\mathbb{Z}n$.

(iii) Let

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/\mathbb{Z}n \to 0$$

be the exact sequence, where f is multiplication by n and g is the canonical map. Show that the sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}n, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}n, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}n, \mathbb{Z}/\mathbb{Z}n) \to 0$$

obtained by applying the representable functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}n, -)$ is not exact at the term $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}n, \mathbb{Z}/\mathbb{Z}n)$.

Aufgabe 3.2. (2+2) A short exact sequence of R-modules

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is said to be *split* if g has a *section*, that is, there is a homomorphism $s : Z \to Y$ with the property that $gs = \mathrm{Id}_Z$.

(i) Show that if the sequence is split, then the mapping

$$\theta : X \oplus Z \to Y, \quad \theta(x, z) = f(x) + s(z)$$

is an isomorphism of R-modules.

(ii) Show that the sequence is split $\Leftrightarrow$ for all R-modules M, the sequence

$$0 \to \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, Y) \to \mathrm{Hom}_R(M, Z) \to 0$$

is exact. [Hint: for $\Leftarrow$, take $M = Z$. By a result in the lecture notes, for $\Rightarrow$ one just needs to show that the mapping $\mathrm{Hom}_R(M, Y) \to \mathrm{Hom}_R(M, Z)$ sending ϕ to $g\phi$ is surjective.]

Mehr...

Aufgabe 3.3. (2+2) Recall that if $\phi : S \to R$ is a ring homomorphism, then restriction gives a functor

$$F : R\text{-Mod} \to S\text{-Mod}, \quad M \mapsto {}_S M.$$

It is easy to see that F is always faithful.

(i) Show that if ϕ is surjective, then the functor F is full.

(ii) Show that for the inclusion $\phi : \mathbb{R} \to \mathbb{C}$, the functor F is not full.

Aufgabe 3.4. (4) Suppose we have a diagram of R-modules and homomorphisms

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & X_5 \\
{\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle h_5}\downarrow \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & Y_5
\end{array}$$

in which the rows are exact and the squares commute (that is, $g_i h_i = h_{i+1} f_i$ for $i = 1, 2, 3, 4$). Show that if h_1, h_2, h_4, h_5 are isomorphisms, then so is h_3.

[This is called the "Five lemma". You can easily find a proof for it in a book or online, but I hope you can do it on your own.]