Algebra II
4. Übungsblatt

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Abgabe: Bis zum 10.05.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 4.1. (2+2) Let K be a field. Recall that K-modules are the same thing as vector spaces over K.

(i) Show that if $0 \to X \to Y \to Z \to 0$ is an exact sequence of finite-dimensional vector spaces over K, then $\dim Y = \dim X + \dim Z$. [Hint: Rank Theorem from Linear Algebra I §5.3]

(ii) Show that if $0 \to X_1 \to X_2 \to \cdots \to X_n \to 0$ is an exact sequence of finite-dimensional vector spaces over K, then

$$\sum_{i=1}^{n} (-1)^i \dim X_i = 0.$$

Aufgabe 4.2. (4) Let I be a set and for each $i \in I$, let M_i be an R-module.

Show that $M = \bigoplus_{i \in I} M_i$ is finitely generated if and only if the M_i are all finitely generated, and all but finitely many of the M_i are zero.

[Hint. Use that if $j \in I$, then the projection homomorphism $\bigoplus_{i \in I} M_i \to M_j$ is surjective.]

Aufgabe 4.3. (1+2+1) Let R be a ring. An element $e \in R$ is called an *idempotent* if $e^2 = e$.

(i) Show that $e \in R$ is idempotent if and only if $1 - e$ is idempotent.

(ii) Show that if $e \in R$ is idempotent, then $R = Re \oplus R(1 - e)$. Show similarly, that if M is a left R-module, then as an additive group we have a decomposition $M = eM \oplus (1 - e)M$.

(iii) Show that if $e \in R$ is idempotent and M is a left R-module, then $\operatorname{Hom}_R(Re, M) \cong eM$ as additive groups.

Mehr...

Aufgabe 4.4. (1+1+1+1) Let R be a ring and let I be a left ideal in R. By definition

$$I^2 = \{a_1b_1 + \cdots + a_nb_n : n \geq 0, a_1, \ldots, a_n, b_1, \ldots, b_n \in I\}.$$

We suppose that $I^2 \neq 0$ and that I is simple as a left R-module.

(i) Show that if $b \in I$, then $Ib = \{xb : x \in I\}$ is a submodule of I.

(ii) Show that there is some $b \in I$ with $Ib = I$.

(iii) With b as in (ii), show that the set $N = \{x \in I : xb = 0\}$ is a submodule of I and that $N \neq I$. Hence deduce that $N = 0$.

(iv) With b as in (ii), show that there is an element $e \in I$ with $eb = b$. By considering $(e^2 - e)b$, show that e is an idempotent and that $I = Re$.