Algebra II
5. Übungsblatt

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Abgabe: Bis zum 17.05.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 5.1. (1+2+1) Let R and S be rings. By definition an *R-S-bimodule* is an additive group M which is both a left R-module and a right S-module, such that $(rm)s = r(ms)$ for all $m \in M$, $r \in R$ and $s \in S$.

(i) The *centre* of an R-R-bimodule M is the set $Z_R(M) = \{m \in M : rm = mr \ \forall \ r \in R\}$. Show that it is an additive subgroup of M.

(ii) If X is an R-module, we write ${}_{\mathbb{Z}}X$ for the underlying additive group (or equivalently, $\mathbb{Z}$-module). Suppose that X is a left S-module and Y is a left R-module. Show how to define actions of R and S so that $\mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}X, {}_{\mathbb{Z}}Y)$ becomes an R-S-bimodule.

(iii) Show that if X and Y are left R-modules, then $Z_R(\mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}X, {}_{\mathbb{Z}}Y)) = \mathrm{Hom}_R(M, N)$.

Aufgabe 5.2. (2+2) (i) Find all submodules of the $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Z}100$, and draw a diagram to show which submodules are contained in which others.

(ii) Show that $\mathbb{Z}/\mathbb{Z}100 \cong (\mathbb{Z}/\mathbb{Z}4) \oplus (\mathbb{Z}/\mathbb{Z}25)$ as $\mathbb{Z}$-modules.

[Hint. Recall that $R = \mathbb{Z}$ is a principal ideal domain (Hauptidealbereich, Algebra I, §4.2), and that $Ra \subseteq Rb$ if and only if b is a divisor of a and $Ra = Rb$ if and only if a and b are associates (Algebra I, §4.3). See also the Chinese Remainder Theorem (chinesische Restsatz, Algebra I, §3.3.]

Aufgabe 5.3. (2+2) An R-module is said to be *artinian*, or to have the *descending chain condition* on submodules, if any descending chain of submodules

$$M_1 \supseteq M_2 \supseteq \dots$$

of M breaks off, that is, there is some n such that $M_n = M_{n+1} = \dots$.

(i) Show that M is artinian if and only if any non-empty set of submodules of M has a minimal element.

(ii) Let N be a submodule of M. Show that M is artinian if and only if N and M/N are artinian.

Mehr...

Aufgabe 5.4. (2+2) (i) Show that if R is a principal ideal domain and $0 \neq a \in R$, then R/Ra is an artinian R-module.

(ii) Show that $\mathbb{Z}$ and $\mathbb{Q}$ are not artinian as $\mathbb{Z}$-modules.

[Hint. Use that a principal ideal domain is a unique factorization domain (Algebra I, §4.3).]