Algebra II
7. Übungsblatt

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Abgabe: Bis zum 31.05.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

The first exercise on this sheet generalizes the construction of the quotient field of an integral domain in Algebra I §4.1. If R is an integral domain and S is the set of non-zero elements of R, then $S^{-1}R$ is the quotient field of R. For example if $R = \mathbb{Z}$ then $S^{-1}R = \mathbb{Q}$.

Aufgabe 7.1. (1+1+1+1) Let R be a commutative ring and let $S \subseteq R$ be a *multiplicative subset* of R, which means that $1 \in S$ and $st \in S$ for all $s, t \in S$. Let M be an R-module.

(i) Show that the relation $\sim$ on $S \times M$ defined by
$$(s_1, m_1) \sim (s_2, m_2) \;\Leftrightarrow\; t(s_1 m_2 - s_2 m_1) = 0 \text{ for some } t \in S$$
is an equivalence relation.

(ii) We denote the equivalence class containing $(s, m) \in S \times M$ by m/s, and we denote by $S^{-1}M$ the set of all equivalence classes. We call $S^{-1}M$ the *localization of* M *with respect to* S. Show that $m/s = (um)/(us)$ for all $m \in M$ and $s, u \in S$.

(iii) Show that the operation
$$m/s + m'/s' = (s'm + sm')/(ss')$$
for $m, m' \in M$ and $s, s' \in S$ is well-defined and turns $S^{-1}M$ into an additive group.

(iii) Now consider the case $M = R$. Show that the operation
$$(r/s)(r'/s') = (rr')/(ss')$$
for $r, r' \in R$, $s, s' \in S$ is well-defined and turns $S^{-1}R$ into a ring.

Mehr...

Aufgabe 7.2. (1+1+1+1) As before, let R be a commutative ring, let $S \subseteq R$ be a multiplicative subset and let M be an R-module.

(i) Show that there is a well-defined mapping $\tau : S^{-1}R \times M \to S^{-1}M$ with $\tau(r/s, m) = (rm)/s$.

(ii) We consider $S^{-1}R$ as a right R-module via $(r/s)r' = (rr')/s$. Show that τ is a homomorphism of additive groups in each argument and R-balanced.

(iii) Let N be an additive group and let $f : S^{-1}R \times M \to N$ be a mapping which is R-balanced. Show that there is a well-defined mapping $\alpha : S^{-1}M \to N$, with $\alpha(m/s) = f(1/s, m)$. [Hint. If $m/s = m'/s'$, then $t(s'm - sm') = 0$ for some $t \in S$. Now write $1/s$ in the form $s't/ss't$.]

(iv) Show that $S^{-1}M$ together with τ satisfies the universal property defining a tensor product, and hence deduce that

$$S^{-1}R \otimes_R M \cong S^{-1}M$$

with $(r/s) \otimes m$ corresponding to $(rm)/s$.

Aufgabe 7.3. (2+2) As before, let R be a commutative ring, let $S \subseteq R$ be a multiplicative subset and let M be an R-module.

(i) Show that the mapping $i : M \to S^{-1}R \otimes_R M$, $i(m) = 1 \otimes m$, has kernel

$$\operatorname{Ker} i = \{m \in M : sm = 0 \text{ for some } s \in S\}.$$

(ii) Show that if $\theta : M \to N$ is an injective homomorphism of R-modules, then $1 \otimes \theta : S^{-1}R \otimes_R M \to S^{-1}R \otimes_R N$ is injective.

[Hint. Use Aufgabe 7.2 (iv) in both cases to turn it into a problem about $S^{-1}M$. Here 1 denotes the one for the ring $S^{-1}R$, which is 1/1.]

Aufgabe 7.4. (2+2) (i) For $n \geq 1$, let X_n be the additive group $\mathbb{Z}/\mathbb{Z}n$ of order n. Show that the additive group

$$\prod_{n=1}^{\infty} X_n$$

contains an element of infinite order, and hence that it has a subgroup isomorphic to $\mathbb{Z}$.

(ii) Deduce that

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\prod_{n=1}^{\infty} X_n) \not\cong \prod_{n=1}^{\infty} (\mathbb{Q} \otimes_{\mathbb{Z}} X_n).$$