Algebra II
8. Übungsblatt

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Abgabe: Bis zum 07.06.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 8.1. (4) Give an example of injective map of additive groups $\theta : X_1 \to X_2$ and an additive group Y, such that the map of tensor products $\theta \otimes \mathrm{Id}_Y : X_1 \otimes_{\mathbb{Z}} Y \to X_2 \otimes_{\mathbb{Z}} Y$ is not injective.

Aufgabe 8.2. (2+2) Let K be a field. Recall that the *dual* of a K-vector space V is the vector space $V^* = \mathrm{Hom}_K(V, K)$.

(i) For any vector spaces V, W over K, show how to define a natural linear map

$$V^* \otimes_K W \to \mathrm{Hom}_K(V, W).$$

(ii) If V is finite-dimensional, show that this mapping is an isomorphism.

Aufgabe 8.3. (2+2) Let U, V, W be vector spaces over a field K.

(i) Show that $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(V, \mathrm{Hom}_K(U, W))$. [Hint. To use Hom-Tensor adjointness from the end of §3.2 in the notes, explain how to consider U as a bimodule.]

(ii) Show that if V is finite-dimensional, then $(U \otimes_K V)^* \cong V^* \otimes_K U^*$.

Aufgabe 8.4. (1+1+1+1) Let V be a finite dimensional vector space over a field K and let $T^2V = V \otimes_K V$ be the tensor square of V. We call elements of T^2V *tensors*. Assume that the characteristic of the field satisfies $\operatorname{char} K \neq 2$.

(i) Show that there is a linear mapping $\tau : T^2V \to T^2V$ with $\tau(x \otimes y) = y \otimes x$ for $x, y \in V$.

(ii) We say that a tensor ξ is *antisymmetric* if $\tau(\xi) = -\xi$. Show that the set $(T^2V)_a$ of antisymmetric tensors is spanned by the elements of the form $x \otimes y - y \otimes x$ with $x, y \in V$.

(iii) We say that a tensor ξ is *symmetric* if $\tau(\xi) = \xi$. Show that the set $(T^2V)_s$ of symmetric tensors is spanned by the elements of the form $x \otimes x$ with $x \in V$. [Hint: consider $(x + y) \otimes (x + y) - x \otimes x - y \otimes y$.]

(iv) Show that $T^2V = (T^2V)_s \oplus (T^2V)_a$.