Algebra II
9. Übungsblatt

William Crawley-Boevey

Abgabe: Bis zum 14.06.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Let K be a field and let $a = (a_1, \dots, a_n) \in K^n$. The Clifford algebra $C(K^n, q_a)$ can be identified with quotient algebra $K\langle b_1, \dots, b_n\rangle / I(a)$ where $K\langle b_1, \dots, b_n\rangle$ is the free algebra on $b_1, \dots, b_n$ and $I(a)$ is the ideal generated by the elements $b_i^2 - a_i 1$ and $b_i b_j + b_j b_i$ for $i \neq j$. Moreover $C(K^n, q_a)$ has a K-basis given by the elements $b_{i_1} \dots b_{i_d}$ with $d \geq 0$ and $i_1 < \dots < i_d$.

Aufgabe 9.1. (4) Recall that the centre of an algebra R is

$$Z(R) = \{r \in R : rs = sr \ \forall \ s \in R\}.$$

Let K be a field with $\operatorname{char} K \neq 2$, and let $a = (a_1, a_2) \in K^2$. Show that

$$Z(C(K^2, q_a)) = \begin{cases} \{\lambda 1 + \mu b_1 b_2 : \lambda, \mu \in K\} & (a = 0) \\ \{\lambda 1 : \lambda \in K\} & (a \neq 0) \end{cases}$$

Aufgabe 9.2. (2+2) Let K be a field with $\operatorname{char} K \neq 2$ and let $0 \neq c \in K$.

(i) By considering the basis

$$e_{11} = \tfrac{1}{2}(1 - b_1),\ e_{22} = \tfrac{1}{2}(1 + b_1),\ e_{12} = \tfrac{1}{2}(b_2 - b_1 b_2),\ e_{21} = \frac{1}{2c}(b_2 + b_1 b_2),$$

show that $C(K^2, q_{(1,c)}) \cong M_2(K)$.

(ii) Find the matrices A_1 and A_2 in $M_2(K)$ corresponding to the elements b_1 and b_2, and check that they satisfy $A_1^2 = 1$, $A_2^2 = c1$ and $A_1 A_2 = -A_2 A_1$.

Aufgabe 9.3. (4) By considering the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix},$$

show that the so-called Space-Time Algebra, $C(\mathbb{R}^4, q_{(1,-1,-1,-1)})$, is isomorphic to $M_2(\mathbb{H})$, where $\mathbb{H}$ is the quaternion algebra.

Mehr...

Aufgabe 9.4. (1+1+2) Let K be a commutative ring and let V be a K-module. The *symmetric algebra* of V is the algebra

$$S(V) = T(V)/I$$

where I is the ideal in $T(V)$ generated by the elements $v \otimes w - w \otimes v$ for $v, w \in V$.

(i) Given an element $v_1 \otimes v_2 \otimes \cdots \otimes v_d \in T^d(V)$, we denote its image in $S(V)$ by $v_1 v_2 \ldots v_d$. Show that any two elements of this form commute, and deduce that $S(V)$ is commutative.

(ii) Show that if R is a commutative K-algebra, and $\theta : V \to R$ is a K-module homomorphism, then there is a unique algebra homomorphism $\tilde{\theta} : S(V) \to R$ with $\tilde{\theta}(v) = v$ for all $v \in V$. [Hint: use the corresponding statement for tensor algebras in §3.3 of the lecture notes.]

(iii) Show that if V is a free K-module with basis $(b_1, \ldots, b_n)$, then there is an isomorphism of algebras $K[X_1, \ldots, X_n] \to S(V)$ sending X_i to b_i. [Hint. Use Lemma (3) in §3.2 of Algebra I to find a homomorphism, and use (ii) to find an inverse.]