# Algebra II
# 10. Übungsblatt

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Abgabe: Bis zum 21.06.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

**Aufgabe 10.1.** (2+1+1) Let $V = \mathbb{R}^3$ with the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Recall that there is a scalar product

$$V \times V \to \mathbb{R}, \quad (v,w) \mapsto \langle v,w\rangle = v_1w_1 + v_2w_2 + v_3w_3$$

for $v = (v_1,v_2,v_3)$ and $w = (w_1,w_2,w_3)$, and there is a vector product

$$V \times V \to V, \quad (v,w) \mapsto v \times w = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$

Let $A$ be the set of expressions of the form

$$a1 + v + wi + bi \in \mathbb{R}1 \oplus V \oplus Vi \oplus \mathbb{R}i$$

where $1, i$ are symbols, $v, w \in V$ and $a, b \in \mathbb{R}$. We define a multiplication on $A$ by the rules

$$vw = \langle v,w\rangle 1 + (v \times w)i$$

together with $i^2 = -1$, $iv = vi$ and with 1 being a one.

(i) Show that in the algebra $A$ we have $u(vw) = (uv)w$ for $u, v, w \in V$. (Hint. Use properties of the vector product in the Proposition in §7.2 of Linear Algebra II.)

(ii) Show that the elements $e_1, e_2, e_3$ in $A$ satisfy the relations for the Clifford algebra $C(\mathbb{R}^3, q_{(1,1,1)})$, and hence give an algebra homomorphism $\theta : C(\mathbb{R}^3, q_{(1,1,1)}) \to A$. (You may use that $A$ is associative without further proof.)

(iii) Find the images under $\theta$ of the usual basis of $C(\mathbb{R}^3, q_{(1,1,1)})$, and hence show that $\theta$ is an isomorphism.

**Aufgabe 10.2.** (2+2) Let $V$ be a vector space over a field $K$, let $(v_1, \dots, v_d) \in V^d$ and consider the element

$$v_1 \wedge \dots \wedge v_d \in \Lambda^d(V).$$

(i) Show that if the tuple $(v_1, \dots, v_d)$ is linearly dependent, then $v_1 \wedge \dots \wedge v_d = 0$. (Hint: if it is linearly dependent, then some $v_i$ is a linear combination of the others.)

(ii) Show that if the tuple $(v_1, \dots, v_d)$ is linearly independent, then $v_1 \wedge \dots \wedge v_d \neq 0$. (You may suppose for simplicity that $V$ is finite-dimensional, though this isn't necessary. Now extend $(v_1, \dots, v_d)$ to a basis of $V$.)

Mehr...

**Aufgabe 10.3.** (2+2) Show the following:

(i) $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2})$.

(ii) $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}(\sqrt{2}, \sqrt{3})$. (You may use without proof that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$.)

**Aufgabe 10.4.** (2+2)

(i) Let $R$, $S$ and $T$ be $K$-algebras. Show that there is an isomorphism of algebras

$$R \otimes_K (S \times T) \cong (R \otimes_K S) \times (R \otimes_K T).$$

(ii) Use the theorem about tensor products of Clifford algebras to show that

$$C(\mathbb{R}^3, q_{(-1,-1,-1)}) \cong \mathbb{H} \times \mathbb{H}.$$