Algebra II
11. Übungsblatt

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Abgabe: Bis zum 28.06.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Let K be a field and G a group, written multiplicatively with neutral element 1.

The *group algebra* KG consists of the formal sums $\sum_{g\in G} a_g g$ with $a_g \in K$, all but finitely many zero, and with multiplication given by that in G.

A *representation* (V, ρ) of G over K is given by a vector space V and a group homomorphism $\rho : G \to \mathrm{GL}(V)$. For $g \in G$ and $v \in V$, we write gv instead of $\rho(g)(v)$, and in this way representations correspond to KG-modules.

A *subrepresentation* of (V, ρ) is a KG-submodule, so a subspace U of V with $gu \in U$ for all $u \in U$ and $g \in G$.

A representation is *irreducible* if it is simple as a KG-module, and *semisimple* if it is semisimple as a KG-module.

Aufgabe 11.1. (2+1+1) Let G be an abelian group and V an irreducible finite-dimensional complex representation of G (that is, the base field is $K = \mathbb{C}$).

(i) If $g \in G$ and $\lambda \in \mathbb{C}$, show that $\{v \in V : gv = \lambda v\}$ is a subrepresentation of V.

(ii) Deduce that g acts on V as multiplication by a scalar. (Hint. Let λ be an eigenvalue of $\rho(g)$.)

(iii) Conclude that V is one dimensional.

Aufgabe 11.2. (4) Determine all irreducible complex representations of finite cyclic groups C_n.

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Aufgabe 11.3. (2+1+1) Let $G = C_4 = \{1, g, g^2, g^3\}$, the cyclic group of order 4.

(i) Let U be the representation of G over $\mathbb{R}$ corresponding to the matrix representation

$$A : G \to \mathrm{GL}_2(\mathbb{R}), \quad A(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

That is, $U = \mathbb{R}^2$ and g^n acts as left multiplication by $A(g)^n$. Decompose U as a direct sum of two 1-dimensional subrepresentations. (Hint. Consider the eigenvectors of the matrix $A(g)$.)

(ii) Let V be the representation of G over $\mathbb{R}$ corresponding to the matrix representation

$$B : G \to \mathrm{GL}_2(\mathbb{R}), \quad B(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that V is irreducible.

(iii) Let W be the representation of G over $\mathbb{C}$ corresponding to the matrix representation $B : G \to \mathrm{GL}_2(\mathbb{C})$ with $B(g)$ as in (ii). Decompose W as a direct sum of two 1-dimensional subrepresentations.

Aufgabe 11.4. (2+1+1) Let V be a complex representation of a group G. We say that V is a *unitary* representation of there is a scalar product $\langle -, - \rangle$ on V satisfying

$$\langle gv, gw \rangle = \langle v, w \rangle$$

for all $v, w \in V$ and $g \in G$. (For scalar products, see Linear Algebra II, §7.1.)

(i) Show that if V is unitary and U is a subrepresentation of V, then the orthogonal subspace

$$U^{\perp} = \{v \in V : \langle v, u \rangle = 0 \text{ for all } u \in U\}$$

is also a subrepresentation of V.

(ii) Show that any finite-dimensional unitary representation is semisimple. (Hint. See Linear Algebra II, §7.4.)

(iii) Show that if G is a finite group, then any complex representation is unitary. (Hint. Consider $\langle -, - \rangle$ defined by

$$\langle v, w \rangle = \sum_{g \in G} \langle gv, gw \rangle_0$$

where $\langle -, - \rangle_0$ is an arbitrary scalar product.)