Algebra II
12. Übungsblatt

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Abgabe: Bis zum 05.07.24 um 10:00h im Postfach Ihres Tutors
[Sarah Meier: 129]

Aufgabe 12.1. (2+2) Let K be a field. Recall that the *centre* of a K-algebra R is the subalgebra

$$Z(R) = \{a \in R : ab = ba \text{ for all } b \in R\}.$$

Show that:

(i) $Z(R \times S) = Z(R) \times Z(S)$ for K-algebras R, S.

(ii) $Z(M_n(K)) = KI$ where I is the identity matrix.

Aufgabe 12.2. (2+2) Let G be a group and K a field.

(i) Show that the mapping $\epsilon : KG \to K$ defined by

$$\epsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$$

is an algebra homomorphism. Deduce that the set

$$I = \left\{\sum_{g\in G} a_g g \in KG : \sum_{g\in G} a_g = 0\right\}$$

is an ideal in KG. (The homomorphism is called the *augmentation*, and the ideal is called the *augmentation ideal*.)

(ii) Given a representation V of G, the fixed point set is

$$V^G = \{v \in V : gv = v \text{ for all } g \in G\}.$$

Considering KG as a KG-module in the usual way (the *regular representation*), show that

$$(KG)^G = \left\{\sum_{g\in G} a_g g : a_g = a_1 \text{ for all } g \in G\right\},$$

so if G is finite, then $(KG)^G$ is spanned by the element $\sum_{g\in G} g$. What happens if G is infinite?

More...

Aufgabe 12.3. (2+2) Let K be a field with $\operatorname{char} K \neq 2$, let V be a K-vector space and let $T^2(V) = V \otimes_K V$ be its tensor square. Recall from Aufgabe 8.4 that the subspaces of symmetric and antisymmetric tensors are

$$T^2(V)_s = \{\xi \in T^2(V) : \tau(\xi) = \xi\} \quad \text{and} \quad T^2(V)_a = \{\xi \in T^2(V) : \tau(\xi) = -\xi\}.$$

where $\tau : T^2(V) \to T^2(V)$ is the linear map with $\tau(x \otimes y) = y \otimes x$ for all $x, y \in V$.

(i) Show that if V has basis $(v_1, \dots, v_n)$, then

$$(v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n)$$

is a basis of $T^2(V)_s$, and

$$(v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n)$$

is a basis of $T^2(V)_a$.

(ii) Recall that the exterior square of V is

$$\Lambda^2(V) = T^2(V)/J_2,$$

where J_2 is the subspace of $T^2(V)$ spanned by elements of the form $x \otimes x$ with $x \in V$. The *symmetric square* is

$$S^2(V) = T^2(V)/U,$$

where U is the subspace spanned by the elements of the form $x \otimes y - y \otimes x$ with $x, y \in V$. Show that there are natural isomorphism of vector spaces

$$S^2(V) \cong T^2(V)_s \quad \text{and} \quad \Lambda^2(V) \cong T^2(V)_a.$$

Aufgabe 12.4. (1+1+1+1) Let V be a representation of a group G over the field $K = \mathbb{C}$.

(i) Let G act on $T^2(V) = V \otimes V$ by $g(x \otimes y) = (gx) \otimes (gy)$. Show that the linear mapping

$$\tau : T^2(V) \to T^2(V), \quad \tau(x \otimes y) = y \otimes x$$

is a homomorphism of representations.

(ii) Show the subspaces $T^2(V)_s$ and $T^2(V)_a$ are subrepresentations of $T^2(V)$.

(iii) Let $g \in G$ and let $(v_1, \dots, v_n)$ be a basis of V with respect to which the action of g is diagonal, say $gv_i = \lambda_i v_i$ with $\lambda_i \in \mathbb{C}$. Using that

$$g(v_i \otimes v_j \pm v_j \otimes v_i) = \lambda_i \lambda_j (v_i \otimes v_j \pm v_j \otimes v_i),$$

show that the characters χ_s and χ_a of $T^2(V)_s$ and $T^2(V)_a$ are given by

$$\chi_s(g) = \sum_{i \leq j} \lambda_i \lambda_j \quad \text{and} \quad \chi_a(g) = \sum_{i < j} \lambda_i \lambda_j.$$

(iv) Hence show that

$$\chi_s(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right) \quad \text{and} \quad \chi_a(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$$

where χ is the character of V.