Fakultät für Mathematik
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# Masters course: Representations of Algebras 2

I want to cover a number of key topics in the representation theory of finite-dimensional associative algebras. Specifically:

- Correspondences given by faithfully balanced modules, and applications to Auslander algebras and homological conjectures. (Originally planned for the previous semester, but carried over, since there was not enough time.)
- Tilting and tau-tilting theory, including equivalences of derived categories.
- Geometric methods for studying representations of algebras, including relevant facts about varieties and schemes without proofs. (There will be less time for this than originally planned.)
- If time, possibly preprojective algebras and Kleinian singularities.

Some relevant books:

- I. Assem and F. U. Coelho, Basic representation theory of algebras, Springer 2020.
- I. Assem, D. Simson and A. Skowroński, Elements of the representation theory of associative algebras. Volume 1, Techniques of representation theory, CUP 2006.
- H. Derksen and J. Weyman, An introduction to quiver representations, American Mathematical Society 2017.
- P. Gabriel and A. V. Roiter, Representations of finite dimensional algebras, Springer 1977.
- A. Kirillov Jr., Quiver Representations and Quiver Varieties, American Mathematical Society 2016.
- A. Skowroński and K. Yamagata, Frobenius algebras 2 Tilted and Hochschild extension algebras, European Mathematical Society 2017.

The section numbering continues from the previous lecture course.

# Contents

# 4 Homological topics

In this section I want to discuss

- Some of the many homological conjectures for f.d. algebras

- Auslander's correspondence between algebras $A$ of finite representation type and algebras $B$ with $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$, and Iyama's generalization of this with cluster tilting objects.

The unifying feature is what I call endomorphism correspondence for faithfully balanced modules.

## 4.1 Higher generation and cogeneration

We are interested in finite-dimensional algebras $A$ over a field $K$ (but most things generalize easily to Artin algebras).

Except where explicitly stated, all modules are f.d., and we write $A$-mod for the category of finite-dimensional left $A$-modules.

We write $D$ for the duality $\operatorname{Hom}_K(-, K)$ between $A$-mod and $A^{op}$-mod.

Recall that a *module class* in $A$-mod is a full subcategory closed under isomorphisms, direct sums and direct summands. Given any module $M$, $\operatorname{add}(M)$ is the smallest module class containing $M$. It is given by the modules isomorphic to a direct summand of $M^n$ for some $n$.

**Definition.** Given a module $M$, $\operatorname{gen}(M)$ denotes the module class consisting of quotients of direct sums of copies of $M$ and $\operatorname{cogen}(M)$ the module class of submodules of a direct sum of copies of $M$.

We say $M$ is a *generator* if $\operatorname{gen}(M) = A$-mod. It is equivalent that $A \in \operatorname{gen}(A)$, or that $A \in \operatorname{add}(M)$. We say $M$ is a *cogenerator* if $\operatorname{cogen}(M) = A$-mod. It is equivalent that $DA \in \operatorname{cogen}(M)$, or $DA \in \operatorname{add}(M)$.

There are higher versions as follows. Here

**Proposition** (1). *Let $M$ be an $A$-module and $n \geq 0$ and consider $M$ also as a $B$-module, where $B = \operatorname{End}_A(M)$. For an $A$-module $X$, the following are equivalent.*

*(a) There is an exact sequence*

$$M_n \xrightarrow{f_n} M_{n-1} \to \cdots \to M_0 \xrightarrow{f_0} X \to 0$$

*with $M_i \in \operatorname{add} M$, such that the sequence*

$$\operatorname{Hom}_A(M, M_{n-1}) \to \cdots \to \operatorname{Hom}_A(M, M_0) \to \operatorname{Hom}_A(M, X) \to 0$$

*is exact (note that this is automatic if $M$ is projective).*

*(b) The natural map $\operatorname{Hom}_A(M, X) \otimes_B M \to X$ is surjective (in case $n = 0$) or an isomorphism (in case $n > 0$) and $\operatorname{Tor}_i^B(\operatorname{Hom}_A(M, X), M) = 0$ for $0 < i < n$.*

**Definition.** We define $\mathrm{gen}_n(M)$ to be the full subcategory of $A$-mod given by the modules $X$ satisfying these conditions. Using condition (b) it is easy to see that it is a module class. Clearly

$$\mathrm{add}(M) \subseteq \cdots \subseteq \mathrm{gen}_2(M) \subseteq \mathrm{gen}_1(M) \subseteq \mathrm{gen}_0(M) = \mathrm{gen}(M).$$

*Proof.* (a)$\Rightarrow$(b). First note that we may assume that the sequence

$$\mathrm{Hom}_A(M, M_n) \to \mathrm{Hom}_A(M, M_{n-1}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

is exact. By assumption it is exact except possibly at $\mathrm{Hom}_A(M, M_{n-1})$. Recall from section 1.9, that $\mathrm{add}\, M$ is functorially finite in $A$-mod. Thus the module $\mathrm{Im}(f_n)$ has a right $\mathrm{add}\, M$-approximation, say $f' : M' \to \mathrm{Im}(f_n)$. Since it is an approximation, we can factorize $f_n = f'g$ for some $g : M_n \to M'$. Thus the map $f'$ has image $\mathrm{Im}(f_n)$. Thus the sequence

$$M' \xrightarrow{f'} M_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \to 0$$

is exact. Also, the sequence

$$\mathrm{Hom}_A(M, M') \to \mathrm{Hom}_A(M, M_{n-1}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

is exact, since any morphism in $\mathrm{Hom}(M, M_{n-1})$ which is sent to zero in $\mathrm{Hom}(M, M_{n-2})$ has image contained in $\mathrm{Ker}(f_{n-1}) = \mathrm{Im}(f_n)$, and hence factors through the approximation $f'$. Thus replacing $M_n$ by $M'$ and $f_n$ by $f'$ if necessary, we have the claimed exactness.

Now we have a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{Hom}(M, M_n) \otimes M & \longrightarrow & \ldots & \longrightarrow & \mathrm{Hom}(M, M_0) \otimes M & \longrightarrow & \mathrm{Hom}(M, X) \otimes M & \longrightarrow & 0 \\
{\scriptstyle \phi_n}\big\downarrow & & & & {\scriptstyle \phi_0}\big\downarrow & & {\scriptstyle \theta}\big\downarrow & & \\
M_n & \xrightarrow{f_n} & \ldots & \xrightarrow{f_1} & M_0 & \xrightarrow{f_0} & X & \longrightarrow & 0
\end{array}$$

For any $M' \in \mathrm{add}\, M$, the natural map $\mathrm{Hom}(M, M') \otimes_B M \to M'$ is an isomorphism, since it is for $M' = M$. Thus the $\phi_i$ are isomorphisms.

Since $\phi_0$ and $f_0$ are surjective, so is $\theta$. If $n > 0$, then since tensor products are right exact, the part of the diagram below and to the right of $\mathrm{Hom}(M, M_1) \otimes M$ has exact rows, so implies that $\theta$ is an isomorphism.

Since $M_i \in \mathrm{add}(M)$, as a right $B$-module, we have

$$\mathrm{Hom}_A(M, M_i) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}(B_B),$$

so the exact sequence

$$\mathrm{Hom}_A(M, M_n) \to \mathrm{Hom}_A(M, M_{n-1}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$$

is part of a projective resolution of $\mathrm{Hom}_A(M,X)$ as a right $B$-module. We can use it to compute $\mathrm{Tor}_i^B(\mathrm{Hom}_A(M,X),M)$ for $i<n$ as the homology of the complex

$$\mathrm{Hom}(M,M_n)\otimes_B M \to \cdots \to \mathrm{Hom}(M,M_0)\otimes_B M \to 0.$$

But by the commutative diagram above, this is isomorphic to the complex

$$M_n \to \cdots \to M_0 \to 0$$

This is exact at $M_i$ for $0<i<n$, giving the Tor vanishing.

(b)$\Rightarrow$(a). Take the start of a projective resolution of $\mathrm{Hom}_A(M,X)$ as a right $B$-module, say

$$P_n \xrightarrow{g_n} \cdots \to P_0 \xrightarrow{g_0} \mathrm{Hom}_A(M,X) \to 0$$

Applying $-\otimes_B M$ gives a complex, which by the hypotheses is exact:

$$M_n \xrightarrow{f_n} \cdots \to M_0 \xrightarrow{f_0} X \to 0,$$

where $M_i = P_i \otimes_B M \in \mathrm{add}\, M$. Applying $\mathrm{Hom}_A(M,-)$ to this, gives a complex

$$\mathrm{Hom}_A(M,M_n) \to \cdots \to \mathrm{Hom}_A(M,M_0) \to \mathrm{Hom}_A(M,X) \to 0.$$

Identifying $\mathrm{Hom}_A(M,M_i) = \mathrm{Hom}_A(M,P_i\otimes_B M) \cong P_i$, we see that this is the projective resolution we started with, so it is exact. Thus (a) holds.

Remark: if we took the projective resolution to be minimal, then the maps $g_i$ would all be right minimal in the sense of section 1.6. It follows that the maps $f_i$ are right minimal, for otherwise there is a decomposition $M_i = M_i' \oplus M_i''$ with $M_i'' \neq 0$ and $f_i(M_i'')=0$. But then we get

$$P_i \cong \mathrm{Hom}_A(M,M_i) \cong \mathrm{Hom}_A(M,M_i') \oplus \mathrm{Hom}_A(M,M_i'')$$

and $g_i$ is zero on the summand corresponding to $\mathrm{Hom}_A(M,M_i'')$, contradicting the minimality of $g_i$. $\square$

Dually we have the following.

**Proposition** (2). *Let $M$ be an $A$-module and $n\geq 0$ and consider $M$ also as a $B$-module, where $B=\mathrm{End}_A(M)$. For an $A$-module $X$, the following are equivalent.*

*(a') There is an exact sequence $0\to X\to M^0 \to \cdots \to M^n$ with $M^i\in \mathrm{add}\, M$ such that the sequence*

$$\mathrm{Hom}(M^{n-1},M) \to \cdots \to \mathrm{Hom}(M^0,M) \to \mathrm{Hom}(X,M) \to 0$$

*is exact (this is automatic if $M$ is injective).*

*(b') The natural map $X \to \mathrm{Hom}_B(\mathrm{Hom}_A(X,M),M)$ is a monomorphism (in case $n=0$) or an isomorphism (in case $n>0$) and $\mathrm{Ext}_B^i(\mathrm{Hom}_A(X,M),M)=0$ for $0<i<n$.*

**Definition.** We define $\mathrm{cogen}^n(M)$ to be the full subcategory of $A$-mod given by the modules $X$ satisfying these conditions. By the second condition it is a module class. Clearly

$$\mathrm{add}(M) \subseteq \cdots \subseteq \mathrm{cogen}^2(M) \subseteq \mathrm{cogen}^1(M) \subseteq \mathrm{cogen}^0(M) = \mathrm{cogen}(M).$$

It is clear from conditions (a) and (a') that $X \in \mathrm{cogen}^n({}_AM) \Leftrightarrow DX \in \mathrm{gen}_n({}_{A^{op}}DM)$.

## 4.2 Faithfully balanced modules and endomorphism correspondence

**Definition.** Let $M$ be an $A$-module, and let $B = \mathrm{End}_A(M)$. Then $M$ can be considered as a $B$-module, and there is a natural map

$$A \to \mathrm{End}_B(M).$$

Clearly $M$ is faithful iff this map is injective. We say that $M$ is a *balanced* $A$-module or that $M$ has the *double centralizer property* if this map is onto, and that $M$ is *faithfully balanced* (f.b.) if this map is an isomorphism.

Clearly $M$ is a f.b. $A$-module iff $DM$ is a f.b. $A^{op}$-module.

**Lemma.** *Let $M$ be an $A$-module.*
*(i) $M$ is f.b. iff $A \in \mathrm{cogen}^1(M)$ iff $DA \in \mathrm{gen}_1(M)$.*
*(ii) If $M$ is a generator or cogenerator, it is f.b.*

*Proof.* (i) Apply the second proposition in the last section with $X = A$ and $n = 1$. Now $M$ is f.b. iff $DM$ is f.b. iff $A^{op} \in \mathrm{cogen}^1({}_{A^{op}}DM)$ iff $DA \in \mathrm{gen}_1({}_AM)$.

(ii) If $M$ is a generator, then $A \in \mathrm{add}(M) \subseteq \mathrm{cogen}^1(M)$. If $M$ is a cogenerator, then $DM$ is a generator, so f.b., hence so is $M$. $\square$

**Definition.** By an *f.b. pair* we mean a pair $(A, M)$ consisting of an algebra and a f.b. $A$-module.

Given an f.b. pair, we construct a new f.b. pair $(B, M)$, its *endomorphism correspondent*, where $B = \mathrm{End}_A(M)$ and $M$ is considered in the natural way as a $B$-module.

Repeating the construction twice, one recovers essentially the original pair.

We say that f.b. pairs $(A, M)$ and $(A', M')$ are *equivalent* if there is an equivalence $A$-mod $\to A'$-mod sending $\mathrm{add}(M)$ to $\mathrm{add}(M')$.

One can show that equivalent pairs have equivalent endomorphism correspondents.

**Theorem.** *If* $(A, M)$ *and* $(B, M)$ *are f.b. pairs which are endomorphism correspondents, then* $\mathrm{Hom}_A(-, M)$ *and* $\mathrm{Hom}_B(-, M)$ *give inverse antiequivalences between* $\mathrm{cogen}^1({}_A M)$ *and* $\mathrm{cogen}^1({}_B M)$.

*Proof.* In view of (b') in the second proposition of the last section, and the symmetrical role of $A$ and $B$, it suffices to show that if $X \in \mathrm{cogen}^1({}_A M)$, then $\mathrm{Hom}_A(X, M) \in \mathrm{cogen}^1({}_B M)$. Take a free presentation of ${}_A X$, say $A^m \to A^n \to X \to 0$. Applying $\mathrm{Hom}_A(-, M)$ gives an exact sequence

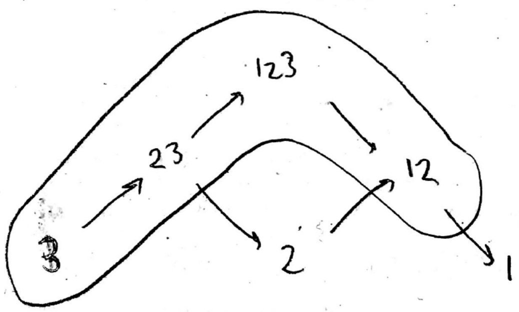
$$0 \to \mathrm{Hom}_A(X, M) \to M^n \to M^m.$$

Applying $\mathrm{Hom}_B(-, M)$ to this gives

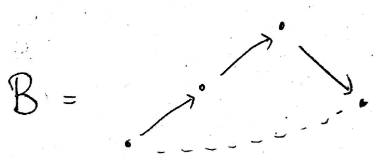
$$A^m \to A^n \to \mathrm{Hom}_B(\mathrm{Hom}_A(X, M), M) \to 0$$

which is isomorphic to the original exact sequence, so exact. Thus $\mathrm{Hom}_A(X, M) \in \mathrm{cogen}^1({}_B M)$. $\square$

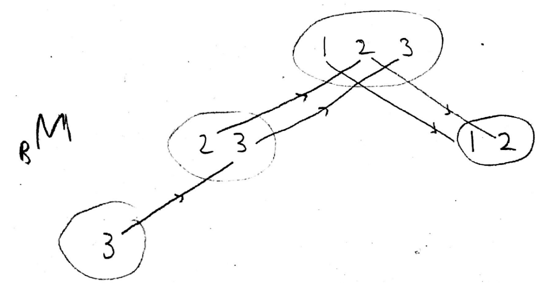
**Example.** Let $A$ be the path algebra of the linear quiver $Q = 1 \to 2 \to 3$. We display its AR quiver below. Let ${}_A M$ be the direct sum of the circled indecomposables.



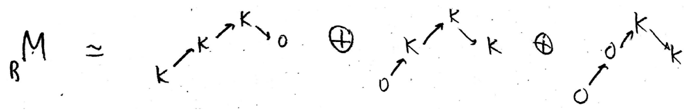

The endomorphism algebra of ${}_A M$ is




Considering $M$ as a $B$-module, means to consider it as a representation of this quiver. The vector space at each vertex is the corresponding indecomposable $A$-module. In this example, the indecomposable $A$-modules are at most one-dimensional at each vertex of $Q$. In the following diagram we write $i$ for the natural basis element at vertex $i$ of $Q$. The arrows in the quiver for $B$ correspond to homomorphisms of the indecomposable $A$-modules, and act on the basis elements as indicated below.




Thus



Observe that ${}_AM$ has all of the projective $A$-modules as summands, but not all injectives, so ${}_AM$ is a generator but not a cogenerator. On the other hand all of the summands of ${}_BM$ are projective, and one summand is not injective.

**Proposition.** *Let $(A, M)$ and $(B, M)$ be f.b. pairs which are endomorphism correspondents. Then:*

*(a) ${}_AM$ is a generator iff ${}_BM$ is projective.*

*(b) ${}_AM$ is a cogenerator iff ${}_BM$ is injective.*

*(c) $A \in \mathrm{cogen}^n({}_AM)$ iff $\mathrm{Ext}^i_B(M, M) = 0$ for $0 < i < n$.*

*Proof.* (a) If ${}_AM$ is a generator, then $A \in \mathrm{add}({}_AM)$, so ${}_BM \cong \mathrm{Hom}_A(A, M) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}(B)$, so ${}_BM$ is projective.

Conversely if ${}_BM$ is projective, then ${}_BM \in \mathrm{add}(B)$, so $A \cong \mathrm{Hom}_B(M, M) \in \mathrm{add}(\mathrm{Hom}_B(B, M)) = \mathrm{add}({}_AM)$.

(b) Apply (a) to $DM$.

(c) Second proposition in last section with $X = A$. □

For (a), see Azumaya, Completely faithful modules and self-injective rings, Nagoya Math. J. 1966. Also (a) is similar to the Wedderburn correspondence introduced by Auslander, Representation theory of Artin algebras I, Comm. Algebra 1974.

For things similar to (b), see T. Kato, Rings of U-dominant dimension $\geq 1$, Tohoku Math. J. 1969.

(c) is essentially Müller, The classification of algebras by dominant dimension, Canad. J. Math 1968.

See also:

B. Ma and J. Sauter, On faithfully balanced modules, F-cotilting and F-Auslander algebras, Journal of Algebra 2020.

M. Pressland and J. Sauter, On quiver Grassmannians and orbit closures for gen-finite modules, Algebras and Representation Theory 2022

## 4.3 Dominant dimension and Auslander correspondence

**Definition.** Given an algebra $A$, we take the minimal injective resolution

$$0 \to A \to I^0 \to I^1 \to \dots$$

of the module ${}_AA$. We say that $A$ has *dominant dimension* $\geq n$ if $I^0, \dots, I^{n-1}$ are projective. This defines $\operatorname{dom.dim} A \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

Recall that an algebra $A$ is *QF-3* if it $A$ has a faithful projective-injective module $M$. If so, then $\operatorname{add}(M) = \mathcal{P}_A \cap \mathcal{I}_A$, since any indecomposable projective-injective module embeds in $A$, so in some $M^n$, so is in $\operatorname{add}(M)$. Thus $M$ is unique, up to multiplicities, since it is the direct sum of all indecomposable projective-injective modules, each with some non-zero multiplicity.

**Proposition.** *(i)* $\operatorname{dom.dim} A \geq 1$ *iff* $A$ *is QF-3.*

*(ii)* $\operatorname{dom.dim} A \geq 2$ *iff* $A$ *has a f.b. projective-injective* $M$*.*

*Proof.* (i) If $A$ is QF-3, with faithful projective-injective module $M$, then there is an embedding $A \to M^n$, and then the injective envelope of $A$ is a direct summand of $M^n$, so it is projective.

(ii) If $\operatorname{dom.dim} A \geq 2$, there is an exact sequence $0 \to A \to I^0 \to I^1$ with $I^0, I^1$ projective-injective. Let $M$ be the direct sum of the indecomposable projective-injectives, then $A \in \operatorname{cogen}^1(M)$ by condition (a) in the characterization of $\operatorname{cogen}^1(M)$. Thus $M$ is f.b.

Conversely suppose $A$ has a f.b. projective-injective $M$. Since it is f.b., $A \in \operatorname{cogen}^1(M)$. By the characterization of this means that there is an exact sequence

$$0 \to A \xrightarrow{\theta} M^0 \to M^1$$

with $M^0, M^1 \in \operatorname{add}(M)$. Moreover by the dual result to the remark at the end of the proof of (b)⇒(a) in Proposition (1) in the first subsection, we may suppose that the maps in this exact sequence are left minimal. Now the $M^i$ are projective-injective, so they are injective, so this is the start of the injective resolution of $A$. Thus $I^0 \cong M^0$ and $I^1 \cong M^1$ are projective. Thus $\operatorname{dom.dim} B \geq 2$. □

For the following, see C. M. Ringel, Artin algebras of dominant dimension at least 2, manuscript 2007, available from his Bielefeld homepage.

**Theorem** (Morita-Tachikawa correspondence). *Endomorphism correspondence gives a 1:1 correspondence between equivalence classes of pairs $(A, M)$ where ${}_AM$ is a generator-cogenerator and Morita equivalence classes of algebras $B$ with* $\operatorname{dom.dim} B \geq 2$.

*The correspondence sends $(A, M)$ to $B = \operatorname{End}_A(M)$, and it sends $B$ to $A = \operatorname{End}_B(M)$ where ${}_BM$ is the faithful projective-injective $B$-module.*

*Proof.* By endomorphism correspondence, the pairs $(A, M)$ are in 1:1 correspondence with f.b. pairs $(B, M)$ with $M$ projective-injective. By the discussion above, these are in 1:1 correspondence with the Morita equivalence classes of algebras $B$ with $\operatorname{dom.dim} B \geq 2$. □

The following correspondence comes from Auslander, Representation dimension of Artin algebras, Queen Mary College Lecture Notes, 1971. See also Auslander, Representation theory of Artin algebras II, Comm. Algebra 1974.

**Theorem** (Auslander correspondence). *There is a 1-1 correspondence between algebras $A$ of finite representation type up to Morita equivalence and algebras $B$ with* $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$ *up to Morita equivalence.*

*The correspondence sends $A$ to $B = \operatorname{End}_A(M)$ where ${}_AM$ is the direct sum of all the indecomposable $A$-modules, and it sends $B$ to $A = \operatorname{End}_B(M)$ where ${}_BM$ is the faithful projective-injective $B$-module.*

*The algebra $B$ is called the* Auslander algebra *of $A$.*

*Proof.* We show that under endomorphism correspondence, pairs $(A, M)$ where $\operatorname{add}(M) = A\text{-mod}$ correspond to pairs $(B, M)$ where $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$ and ${}_BM$ is the faithful projective-injective.

Suppose $\operatorname{add}(M) = A\text{-mod}$. Given a $B$-module $Z$, choose a projective presentation

$$P_1 \xrightarrow{f} P_0 \to Z \to 0.$$

Applying $\operatorname{Hom}_B(-, M)$ gives an exact sequence

$$0 \to \operatorname{Hom}_B(Z, M) \to \operatorname{Hom}_B(P_0, M) \xrightarrow{g} \operatorname{Hom}_B(P_1, M) \to \operatorname{Coker}(g) \to 0.$$

Applying $\operatorname{Hom}_A(-, M)$ we get a commutative diagram with bottom row exact

$$\begin{array}{ccccccc} & & & & P_1 & \xrightarrow{f} & P_0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Hom}_A(\operatorname{Coker}(g), M) & \longrightarrow & \operatorname{Hom}_A(\operatorname{Hom}_B(P_1, M), M) & \longrightarrow & \operatorname{Hom}_A(\operatorname{Hom}_B(P_0, M), M) \end{array}$$

The two vertical maps are isomorphisms, so $\operatorname{Ker}(f) \cong \operatorname{Hom}_A(\operatorname{Coker}(g), M)$. Now $\operatorname{Coker}(g)$ is an $A$-module, so in $\operatorname{add}(M)$, so as a $B$-module, we have

$$\operatorname{Hom}_A(\operatorname{Coker}(g), M) \in \operatorname{add}(\operatorname{Hom}_A(M, M)) = \operatorname{add}(B),$$

so it is projective. Thus $\operatorname{proj.dim} Z \leq 2$. Thus $\operatorname{gl.dim} B \leq 2$.

Conversely suppose $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$. If $Y$ is an $A$-module, it has a projective resolution starting

$$P_1 \to P_0 \to Y \to 0.$$

Applying $\operatorname{Hom}_A(-, M)$ we get an exact sequence of $B$-modules

$$0 \to \operatorname{Hom}_A(Y, M) \to \operatorname{Hom}_A(P_0, M) \to \operatorname{Hom}_A(P_1, M).$$

The $\operatorname{Hom}_A(P_i, M)$ are projective $B$-modules since they are in $\operatorname{add}({}_BM)$, and ${}_BM$ is projective. Thus, since $\operatorname{gl.dim} B \leq 2$, by dimension shifting we see that $\operatorname{Hom}_A(Y, M)$ is a projective $B$-module. Now ${}_BM$ is injective, so applying $\operatorname{Hom}_B(-, {}_BM)$ gives an exact sequence

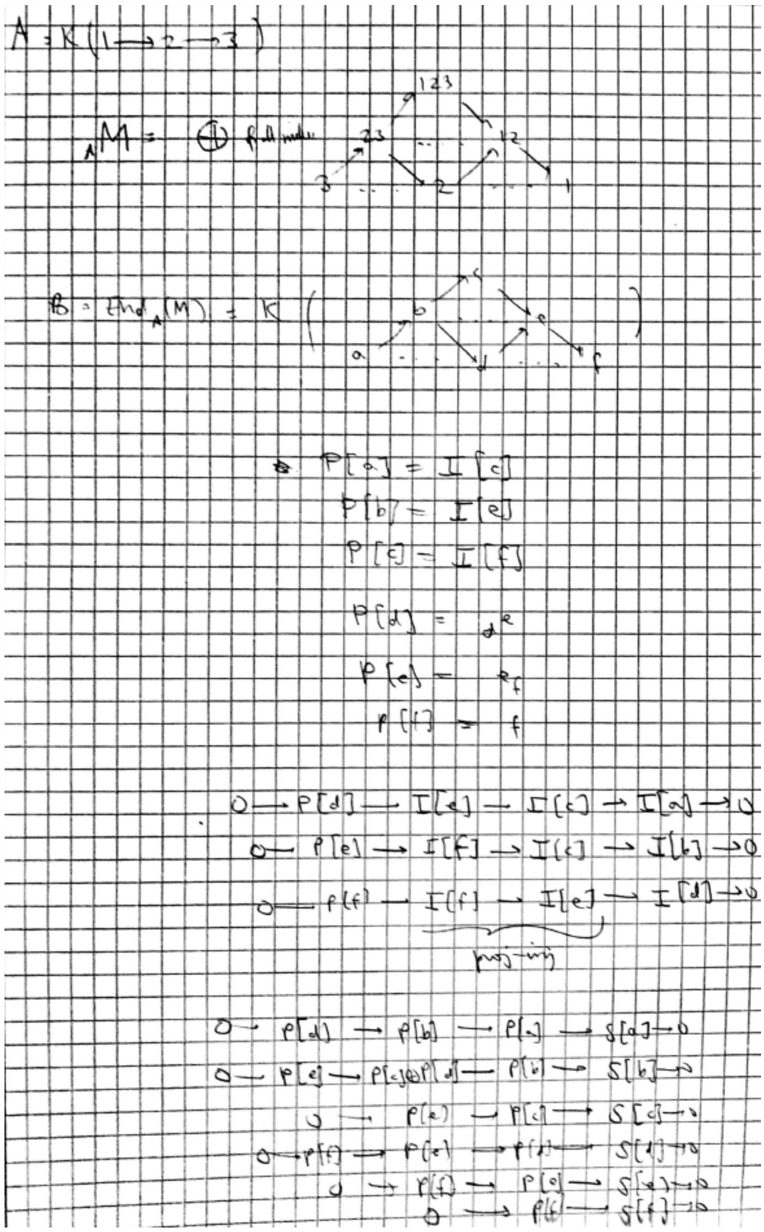
$$\operatorname{Hom}_B(\operatorname{Hom}_A(P_1, M), M) \to \operatorname{Hom}_B(\operatorname{Hom}_A(P_1, M), M) \to \operatorname{Hom}_B(\operatorname{Hom}_A(Y, M), M) \to 0.$$

For any $A$-module $X$ there is a natural transformation from $X$ to $\operatorname{Hom}_B(\operatorname{Hom}_A(X, M), M)$, and this is an isomorphism for $X$ projective. We deduce that

$$Y \cong \operatorname{Hom}_B(\operatorname{Hom}_A(Y, M), M) \in \operatorname{add}(\operatorname{Hom}_B(B, M)) = \operatorname{add}({}_AM),$$

so $\operatorname{add}(M) = A$-mod. $\square$

**Example.** We can check $\operatorname{gl.dim} B = 2 = \operatorname{dom.dim} B$ for the Auslander algebra of the linear quiver with three vertices.




**Definition.** Let $n \geq 1$. A module ${}_AM$ is an *n-cluster tilting object* if
(i) $\mathrm{Ext}^i_A(M, M) = 0$ for $0 < i < n$
(ii) $\mathrm{Ext}^i_A(U, M) = 0$ for $0 < i < n$ implies $U \in \mathrm{add}\, M$
(iii) $\mathrm{Ext}^i_A(M, U) = 0$ for $0 < i < n$ implies $U \in \mathrm{add}\, M$

Clearly (ii) implies $A \in \mathrm{add}\, M$ and (iii) implies $DA \in \mathrm{add}\, M$, so any $n$-cto is a generator-cogenerator.

Observe that $M$ is a 1-cto iff $\mathrm{add}(M) = A$-mod.

**Example.** For the algebra with quiver

$$0 \to 1 \to 2 \to \cdots \to n$$

with all paths of length 2 zero, the module $S[0]$ has projective resolution

$$0 \to P[n] \to P[n-1] \to \cdots \to P[1] \to P[0] \to S[0] \to 0$$

so $\dim \operatorname{Ext}^i(S[0], S[j]) = \delta_{ij}$. It follows that

$$M = S[0] \oplus P[0] \oplus \cdots \oplus P[n-1] \oplus P[n] \cong I[0] \oplus I[1] \oplus \cdots \oplus I[n] \oplus S[n]$$

is an $n$-cto. It's endomorphism algebra $B$ is the path algebra of the quiver

$$n \to \cdots \to 1 \to 0 \to *$$

with all paths of length 2 zero. It has global dimension $n+1$. The projectives $P[n], \ldots, P[0]$ are injective, and $P[*]$ has injective resolution

$$0 \to P[*] \to I[*] \to I[0] \to \cdots \to I[n-1] \to I[n] \to 0.$$

Now $I[*] \cong P[0]$, $I[0] \cong P[1]$, $\ldots$, $I[n-1] \cong P[n]$ and $I[n] \cong S[n]$ is not projective, so $\operatorname{dom.dim} B = n+1$.

The following generalization of Auslander correspondence is due to Iyama, Auslander correspondence, Advances in Math. 2007.

**Theorem** (Iyama)**.** *There is a 1:1 correspondence between equivalence classes of pairs $(A, M)$ where ${}_AM$ is an $n$-cto and Morita equivalence classes of algebras $B$ with* $\operatorname{gl.dim} B \leq n+1 \leq \operatorname{dom.dim} B$.

*Proof.* **(To be omitted.)** We are in the setting of Morita-Tachikawa correspondence.

Now $\operatorname{Ext}^i_A(M, M) = 0$ for $1 < i < n$ corresponds to $B \in \operatorname{cogen}^n({}_BM)$, and since ${}_BM$ is the faithful projective-injective, this corresponds to $\operatorname{dom.dim} B \geq n+1$.

Suppose $\operatorname{gl.dim} B \leq n+1$.

We show that if $\operatorname{Ext}^i_A(U, M) = 0$ for $0 < i < n$ then $U \in \operatorname{add} M$. Take the start of a projective resolution of $U$, say

$$P_n \to \cdots \to P_0 \to U \to 0.$$

Applying $\operatorname{Hom}_A(-, M)$ gives a complex

$$0 \to \operatorname{Hom}_A(U, M) \to \operatorname{Hom}_A(P_0, M) \to \cdots \to \operatorname{Hom}_A(P_n, M)$$

which is exact because the Exts vanish. Since ${}_BM$ is injective, applying $\mathrm{Hom}_B(-,M)$ gives an exact sequence

$$\mathrm{Hom}_B(\mathrm{Hom}_A(P_n,M),M) \to \cdots \to \mathrm{Hom}_B(\mathrm{Hom}_A(P_0,M),M) \to \mathrm{Hom}_B(\mathrm{Hom}_A(U,M),M) \to 0.$$

Now the maps $P_i \to \mathrm{Hom}_B(\mathrm{Hom}_A(P_i,M),M)$ are isomorphisms since $P_i \in \mathrm{add}\, M$. Thus the map $U \to \mathrm{Hom}_B(\mathrm{Hom}_A(U,M),M)$ is an iso (so $U \in \mathrm{cogen}^1({}_AM)$). Also $\mathrm{Hom}_A(P_i,M) \in \mathrm{add}(\mathrm{Hom}_A(A,M)) = \mathrm{add}({}_BM)$. Thus, since $\mathrm{gl.dim}\, B \leq n+1$, the $B$-module $\mathrm{Hom}_A(U,M)$ must be projective, so it is in $\mathrm{add}({}_BB)$, and then $U \cong \mathrm{Hom}_B(\mathrm{Hom}_A(U,M),M) \in \mathrm{add}(\mathrm{Hom}_B(B,M)) = \mathrm{add}({}_AM)$.

Next we show that if $\mathrm{Ext}^i_A(M,U) = 0$ for $0 < i < n$ then $U \in \mathrm{add}\, M$. Take the start of an injective resolution of $U$, say

$$0 \to U \to I^0 \to \cdots \to I^n.$$

Applying $\mathrm{Hom}_A(M,-)$ gives a complex

$$0 \to \mathrm{Hom}_A(M,U) \to \mathrm{Hom}_A(M,I^0) \to \cdots \to \mathrm{Hom}_A(M,I^n)$$

which is exact because the Exts vanish. Since ${}_BM$ is projective, applying $- \otimes_B M$ gives an exact sequence

$$0 \to \mathrm{Hom}_A(M,U) \otimes_B M \to \mathrm{Hom}_A(M,I^0) \otimes_B M \to \cdots \to \mathrm{Hom}_A(M,I^n) \otimes_B M.$$

Now the maps $I^i \to \mathrm{Hom}_A(M,I^i) \otimes_B M$ are isomorphisms since $I^i \in \mathrm{add}\, M$. Thus the map $U \to \mathrm{Hom}_A(M,U) \otimes_B M$ is an iso. Also $\mathrm{Hom}_A(M,I^i) \in \mathrm{add}(\mathrm{Hom}_A(M,M)) = \mathrm{add}(B_B)$. Thus, since $\mathrm{gl.dim}\, B \leq n+1$, the right $B$-module $\mathrm{Hom}_A(M,U)$ must be projective, so it is in $\mathrm{add}(B_B)$, and then $U \cong \mathrm{Hom}_A(M,U) \otimes_B M \in \mathrm{add}(B \otimes_B M) = \mathrm{add}({}_AM)$.

Now suppose that $M$ is an $n$-cto. Given a $B$-module $Z$, choose a projective presentation

$$P_1 \xrightarrow{f} P_0 \to Z \to 0.$$

Applying $\mathrm{Hom}_B(-,M)$ gives an exact sequence

$$0 \to \mathrm{Hom}_B(Z,M) \to \mathrm{Hom}_B(P_0,M) \xrightarrow{g} \mathrm{Hom}_B(P_1,M) \to \mathrm{Coker}(g) \to 0.$$

Let $C^0 = \mathrm{Coker}(g)$. Applying $\mathrm{Hom}_A(-,M)$ we get a commutative diagram with bottom row exact

$$\begin{tikzcd}
 & & P_1 \arrow[r, "f"] \arrow[d] & P_0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_A(C^0,M) \arrow[r] & \mathrm{Hom}_A(\mathrm{Hom}_B(P_1,M),M) \arrow[r] & \mathrm{Hom}_A(\mathrm{Hom}_B(P_0,M),M)
\end{tikzcd}$$

The two vertical maps are isomorphisms, so $\mathrm{Ker}(f) \cong \mathrm{Hom}_A(C^0, M)$.

Now since $M$ is a cogenerator, by repeatedly taking left $M$-approximations we can get an exact sequence

$$0 \to C^0 \to M^0 \to \cdots \to M^{n-2}$$

such that the sequence

$$\mathrm{Hom}_A(M^{n-2}, M) \to \cdots \to \mathrm{Hom}_A(M^0, M) \to \mathrm{Hom}_A(C^0, M) \to 0$$

is exact. Let $C^i$ be the cosyzygies for this sequence, so

$$0 \to C^i \to M^i \to C^{i+1} \to 0.$$

Then

$$\mathrm{Hom}(M^i, M) \twoheadrightarrow \mathrm{Hom}(C^i, M) \to \mathrm{Ext}^1(C^{i+1}, M) \to \mathrm{Ext}^1(M^i, M) = 0 \to \dots,$$

so by dimension shifting

$$\mathrm{Ext}^{n-1}(C^{n-1}, M) \cong \mathrm{Ext}^{n-2}(C^{n-1}, M) \cong \ldots \cong \mathrm{Ext}^1(C^1, M) = 0$$

and similarly $\mathrm{Ext}^i(C^{n-1}, M) = 0$ for $0 < i < n$. Thus $C^{n-1} \in \operatorname{add} M$. Thus $Z$ has projective resolution

$$0 \to \mathrm{Hom}_A(C^{n-1}, M) \to \mathrm{Hom}_A(M^{n-2}, M) \to \cdots \to \mathrm{Hom}_A(M^0, M) \to P_1 \to P_0 \to Z \to 0.$$

Thus $\operatorname{proj.dim} Z \leq n+1$. Thus $\operatorname{gl.dim} B \leq n+1$. □

## 4.4 Homological conjectures for f.d. algebras

Let $0 \to A \to I^0 \to I^1 \to \dots$ be the minimal injective resolution of a f.d. algebra $A$. Recall that $A$ has dominant dimension $\geq n$ if $I^0, \dots, I^{n-1}$ are all projective.

**Conjecture** (Nakayama conjecture 1958)**.** *If all $I^n$ are projective, i.e.* $\operatorname{dom.dim} A = \infty$*, then $A$ is self-injective.*

**Proposition.** *The following are equivalent.*
*(i) The Nakayama conjecture (if* $\operatorname{dom.dim} B = \infty$ *then $B$ is self-injective).*
*(ii) If ${}_AM$ is a generator-cogenerator and* $\mathrm{Ext}^i_A(M, M) = 0$ *for all $i > 0$ then $M$ is projective.*

*Proof.* (i) implies (ii). Say ${}_AM$ satisfies the hypotheses. Let $(B, M)$ be the endomorphism correspondent. Then ${}_BM$ is projective-injective and $B \in \mathrm{cogen}^n(M)$ for all $n$. Thus for all $n$ there is an exact sequence

$$0 \to B \to I^0 \to \cdots \to I^n$$

with the $I^i$ projective-injective. Thus $\mathrm{dom.\,dim}\, B = \infty$. Thus $B$ is self-injective, so $\mathrm{add}(M) = \mathrm{add}(B)$, so ${}_BM$ is a generator, so ${}_AM$ is projective.

(ii) implies (i). Say $\mathrm{dom.\,dim}\, B = \infty$. Thus $B$ is QF-3 and let ${}_BM$ be the faithful projective-injective module. Let ${}_AM$ be the endomorphism correspondent. It is a generator-cogenerator. Then $B \in \mathrm{cogen}^n(M)$ for all $n$, so $\mathrm{Ext}^i_A(M, M) = 0$ for all $i > 0$. Thus by (ii), ${}_AM$ is projective, so ${}_BM$ is a generator. Thus $B \in \mathrm{add}(M)$ is injective. □

**Conjecture** (Generalized Nakayama conjecture, Auslander and Reiten 1975)**.** *For any f.d. algebra $A$, every indecomposable injective occur as a summand of some $I^n$.*

It clearly implies the Nakayama conjecture, for if the $I^n$ are projective, and each indecomposable injective occurs as a summand of some $I^n$, then the indecomposable injectives are projective.

**Example.** For the commutative square, vertices 1(source),2,3,4(sink). There are injective resolutions

$$\begin{aligned}
0 \to &P[1] \to I[4] \to 0,\\
0 \to &P[2] \to I[4] \to I[3] \to 0,\\
0 \to &P[3] \to I[4] \to I[2] \to 0,\\
0 \to &P[4] \to I[4] \to I[2] \oplus I[3] \to I[1] \to 0,
\end{aligned}$$

so

$$0 \to A \to I[4]^4 \to I[2]^2 \oplus I[3]^2 \to I[1] \to 0,$$

so all indecomposable injectives occur.

**Proposition.** *The following are equivalent.*
*(i) The generalized Nakayama conjecture (every indecomposable injective occurs as a summand of some $I^i$ in the minimal injective resolution of $B$).*
*(ii) If ${}_AM$ is a cogenerator and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i > 0$ then $M$ is injective.*

*Proof.* (i) implies (ii). Suppose ${}_AM$ satisfies the conditions. Then there is corresponding ${}_BM$ which is injective, and $B \in \mathrm{cogen}^n(M)$ for all $n$. Thus by (i) every indecomposable injective is a summand of ${}_BM$. Thus ${}_BM$ is a cogenerator. Thus ${}_AM$ is injective.

(ii) implies (i). Let ${}_BM$ be the sum of all indecomposable injectives occuring in the $I^i$. Then $B \in \mathrm{cogen}^n(M)$ for all $n$. Let ${}_AM$ be the endomorphism correspondent. Then ${}_AM$ is a cogenerator and $\mathrm{Ext}^i_A(M,M) = 0$ for all $i > 0$. Thus by (ii) ${}_AM$ is injective. Thus ${}_BM$ is a cogenerator. Thus all indecomposable injectives occur as a summand of ${}_BM$. □

For the next conjecture, see Happel, Selforthogonal modules, 1995.

**Conjecture** (Boundedness Conjecture)**.** *If $M$ is an $A$-module with $\mathrm{Ext}^i_A(M,M) = 0$ for all $i > 0$ then $\#M \leq \#A$, where $\#M$ denotes the number of non-isomorphic indecomposable summands of $M$.*

Since a cogenerator has all indecomposable injectives as summands, the boundedness conjecture implies the generalized Nakayama conjecture.

**Definition.** An algebra $A$ is *(Iwanaga) Gorenstein* if both $\operatorname{inj.dim} {}_AA < \infty$ and $\operatorname{inj.dim} A_A < \infty$.

In Auslander and Reiten, Applications of contravariantly finite subcategories, Adv. Math 1991, one finds:

**Conjecture** (Gorenstein Symmetry Conjecture)**.** *If one of* $\operatorname{inj.dim} {}_AA$ *and* $\operatorname{inj.dim} A_A$ *is finite, so is the other.*

**Lemma.** *(i) If* $\operatorname{inj.dim} {}_AA = n < \infty$*, any $A$-module has* $\operatorname{proj.dim} M \leq n$ *or* $\infty$*.*
*(ii) If* $\operatorname{inj.dim} {}_AA = n$ *and* $\operatorname{inj.dim} A_A = m$ *are both finite, they are equal.*

For example, by (i) every non-projective module for a self-injective algebra has infinite projective dimension.

*Proof.* (i) Say $\operatorname{proj.dim} M = i < \infty$. There is some $N$ with $\mathrm{Ext}^i(M,N) \neq 0$. Choose $0 \to L \to P \to N \to 0$ with $P$ projective. The long exact sequence for $\mathrm{Hom}(M,-)$ gives

$$\cdots \to \mathrm{Ext}^i(M,P) \to \mathrm{Ext}^i(M,N) \to \mathrm{Ext}^{i+1}(M,L) \to \dots$$

Now $\mathrm{Ext}^{i+1}(M,L) = 0$, so $\mathrm{Ext}^i(M,P) \neq 0$, so $\mathrm{Ext}^i(M,A) \neq 0$, so $i \leq n$.
(ii) $\operatorname{proj.dim} {}_ADA = \operatorname{inj.dim} A_A = m$, so $m \leq n$ by (i). Dually $m \geq n$. □

This also holds for noetherian rings, see Zaks, Injective dimension of semiprimary rings, J. Alg. 1969.

For the following, see H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 1960.

**Conjecture** (Finitistic Dimension Conjecture)**.** *For any f.d. algebra $A$,*

$$\operatorname{fin.dim} A = \sup\{\operatorname{proj.dim} M \mid \operatorname{proj.dim} M < \infty\}$$

*is finite.*

For example if $A$ is Gorenstein, with $\operatorname{inj.dim}{}_A A = n = \operatorname{inj.dim} A_A$, then $\operatorname{fin.dim} A = n$. For the lemma implies that any $A$-module $M$ has $\operatorname{proj.dim} M \leq n$ or $\infty$, and $\operatorname{proj.dim} D(A_A) = n$.

Note that $\operatorname{fin.dim} A$ is not necessarily the same as the maximum of the projective dimensions of the simple modules of finite projective dimension.

There is also a big finitistic dimension, where the modules need not be finite-dimensional, and this may also always be finite.

**Proposition.** *The finitistic dimension conjecture implies the Gorenstein symmetry conjecture.*

*Proof.* Assuming $\operatorname{inj.dim} A_A = n < \infty$, we want to prove that $\operatorname{inj.dim}{}_A A < \infty$. We have $\operatorname{proj.dim}{}_A DA = n < \infty$. Thus any injective module has projective dimension $< \infty$. Take a minimal injective resolution $0 \to {}_A A \to I^0 \to \dots$. We show by induction on $i$ that $\operatorname{proj.dim} \Omega^i A < \infty$. There is an exact sequence

$$0 \to \Omega^{i-1} A \to I^{i-1} \to \Omega^i A \to 0.$$

Applying $\operatorname{Hom}_A(-, X)$ for a module $X$ gives a long exact sequence

$$\dots \to \operatorname{Ext}^m(\Omega^{i-1} A, X) \to \operatorname{Ext}^{m+1}(\Omega^i A, X) \to \operatorname{Ext}^{m+1}(I^{i-1}, X) \to \dots$$

For $m$ sufficiently large, independent of $X$, the outside terms are zero, hence so is the middle.

Let $i > 0$. If $\Omega^i A = 0$, or is injective, then $\operatorname{inj.dim}{}_A A < \infty$, as desired, so suppose otherwise. Let $f : \Omega^i A \to I^i$ be the inclusion. Then $f$ belongs to the middle term in the complex

$$\operatorname{Hom}(\Omega^i A, I^{i-1}) \to \operatorname{Hom}(\Omega^i A, I^i) \to \operatorname{Hom}(\Omega^i A, I^{i+1})$$

and it is sent to zero in the third term. Now $f$ is not in the image of the map from the first term, for otherwise the map $I^{i-1} \to \Omega^i A$ is a split epimorphism, so $\Omega^i A$ is injective. Thus the homology of this complex at the middle term is non-zero. Thus $\operatorname{Ext}^i(\Omega^i A, A) \neq 0$. Thus $\operatorname{proj.dim} \Omega^i A \geq i$. This contradicts that $\operatorname{fin.dim} A < \infty$. $\square$

**Proposition.** *The finitistic dimension conjecture implies the generalized Nakayama conjecture.*

*Proof.* Assume the FDC. We show that if ${}_AM$ is a module and $\mathrm{Ext}^n(M,A)=0$ for all $n\geq 0$ then $M=0$ (the *strong Nakayama conjecture*).

If $S[i]$ is a simple $A$-module and $I[i]$ is its injective envelope, recall from section 1.10, that $\dim\mathrm{Ext}^n(S[i],A)$ is $\dim\mathrm{End}(S[i])$ times the multiplicity of $I[i]$ as a direct summand of $I^n$. Thus taking $M=S[i]$, the strong Nakayama conjecture gives the generalized Nakayama conjecture.

Take a minimal projective resolution $\to P_1\to P_0\to M\to 0$. By assumption the sequence

$$0\to\mathrm{Hom}_A(P_0,A)\xrightarrow{f_0}\mathrm{Hom}_A(P_1,A)\xrightarrow{f_1}\mathrm{Hom}(P_2,A)\to\dots$$

of right $A$-modules is exact. Let $\mathrm{fin.dim}\,A^{op}=n<\infty$. Then $\mathrm{Coker}(f_n)$ has projective resolution

$$0\to\mathrm{Hom}_A(P_0,A)\xrightarrow{f_0}\mathrm{Hom}_A(P_1,A)\to\dots\to\mathrm{Hom}_A(P_{n+1},A)\to\mathrm{Coker}(f_n)\to 0$$

so it has finite projective dimension, so projective dimension $\leq n$, so by dimension shifting $\mathrm{Im}\,f_1$ is projective, so $f_0$ must be a split mono. But $\mathrm{Hom}_A(-,A)$ is an antiequivalence from $\mathcal{P}_A$ to $\mathcal{P}_{A^{op}}$. Thus the map $P_1\to P_0$ must be a split epi, so $M=0$. □

## 4.5 No loops conjecture

It is nice to see that some homological conjecture has been proved. In this section we do not assume that $K$ is algebraically closed, but we do assume that $A=KQ/I$ with $I$ admissible. The following conjecture was proved by Igusa, Notes on the no loops conjecture, J. Pure Appl. Algebra 1990.

**Theorem** (No loops conjecture)**.** *If* $\mathrm{gl.dim}\,A<\infty$ *then* $Q$ *has no loops (that is,* $\mathrm{Ext}^1(S[i],S[i])=0$ *for all* $i$*).*

*Proof.* We use the trace function of Hattori and Stallings. I only sketch the proof of its properties.

(1) For any matrix $\theta\in M_n(A)$ we consider its trace $\mathrm{tr}(\theta)\in A/[A,A]$, where $[A,A]$ is the subspace of $A$ spanned by the commutators $ab-ba$. This ensures that $\mathrm{tr}(\theta\phi)=\mathrm{tr}(\phi\theta)$. This equality holds also for $\theta\in M_{m\times n}(A)$ and $\phi\in M_{n\times m}(A)$.

(2) If $P$ is a f.g. projective $A$-module it is a direct summand of a f.g. free module $F=A^n$. Let $p:F\to P$ and $i:P\to F$ be the projection and inclusion. One defines $\mathrm{tr}(\theta)$ for $\theta\in\mathrm{End}(P)$ to be $\mathrm{tr}(i\theta p)$. This is well defined, for if

$$A^n=F\;\underset{i}{\overset{p}{\rightleftarrows}}\;P\;\underset{p'}{\overset{i'}{\rightleftarrows}}\;F'=A^m$$

with $pi = 1_P = p'i'$, then $\mathrm{tr}(i\theta p) = \mathrm{tr}((ip')(i'\theta p)) = \mathrm{tr}((i'\theta p)(ip')) = \mathrm{tr}(i'\theta p')$.

(3) Any module $M$ has a finite projective resolution $P_* \to M$, and an endomorphism $\theta$ of $M$ lifts to a map between the projective resolutions

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle\theta_n}\big\downarrow & & & & {\scriptstyle\theta_1}\big\downarrow & & {\scriptstyle\theta_0}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & \\ 0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Define $\mathrm{tr}(\theta) = \sum_i (-1)^i \,\mathrm{tr}(\theta_i)$. One can show that does not depend on the projective resolution or the lift of $\theta$, see section 4 of Lenzing, Nilpotente Elemente in Ringen von endlicher globaler Dimension, Math. Z. 1969.

(4) One can show that given a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & {\scriptstyle\theta'}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & {\scriptstyle\theta''}\big\downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

one has $\mathrm{tr}(\theta) = \mathrm{tr}(\theta') + \mathrm{tr}(\theta'')$.

(5) It follows that any nilpotent endomorphism has trace 0, since

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\,\theta & \longrightarrow & M & \longrightarrow & M/\mathrm{Im}\,\theta & \longrightarrow & 0 \\ & & {\scriptstyle\theta|_{\mathrm{Im}\,\theta}}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}\,\theta & \longrightarrow & M & \longrightarrow & M/\mathrm{Im}\,\theta & \longrightarrow & 0 \end{array}$$

so $\mathrm{tr}(\theta) = \mathrm{tr}(\theta|_{\mathrm{Im}\,\theta}) = \mathrm{tr}(\theta|_{\mathrm{Im}(\theta^2)}) = \dots = 0$.

(6) Thus any element of $J(A)$ as a map $A \to A$ has trace 0, so $J(A) \subseteq [A, A]$. Thus $(KQ)_+ \subseteq I + [KQ, KQ]$.

(7) Any loop of $Q$ gives an element of $(KQ)_+$. But it is easy to see that

$$I + [KQ, KQ] \subseteq \text{span of arrows which are not loops} + (KQ)_+^2,$$

for example if $p, q$ are paths then $[p, q] \in (KQ)_+^2$ unless they are trivial paths or one is trivial and the other is an arrow. Thus there are no loops. $\square$

A strengthening (proved by Igusa, Liu and Paquette, A proof of the strong no loop conjecture, Adv. Math. 2011). If $S$ is a 1-dimensional simple module for a f.d. algebra and $S$ has finite injective or projective dimension, then $\mathrm{Ext}^1(S, S) = 0$.

An open problem (stated by Liu and Morin, The strong no loop conjecture for special biserial algebras, Proc. Amer. Math. Soc. 2004). The extension conjecture: if $S$ is simple module for a f.d. algebra and $\mathrm{Ext}^1(S, S) \neq 0$ then $\mathrm{Ext}^n(S, S) \neq 0$ for infinitely many $n$.

# 5 Tilting theory

In order to give their proof of Gabriel's theorem, Bernstein, Gelfand and Ponomarev introduced some reflection functors.

If $Q$ is a quiver and $i$ is a sink (no arrows out), so that $P[i] = S[i]$, let $Q'$ be the quiver obtained by reversing all arrows incident at $i$. Then reflection functors are functors

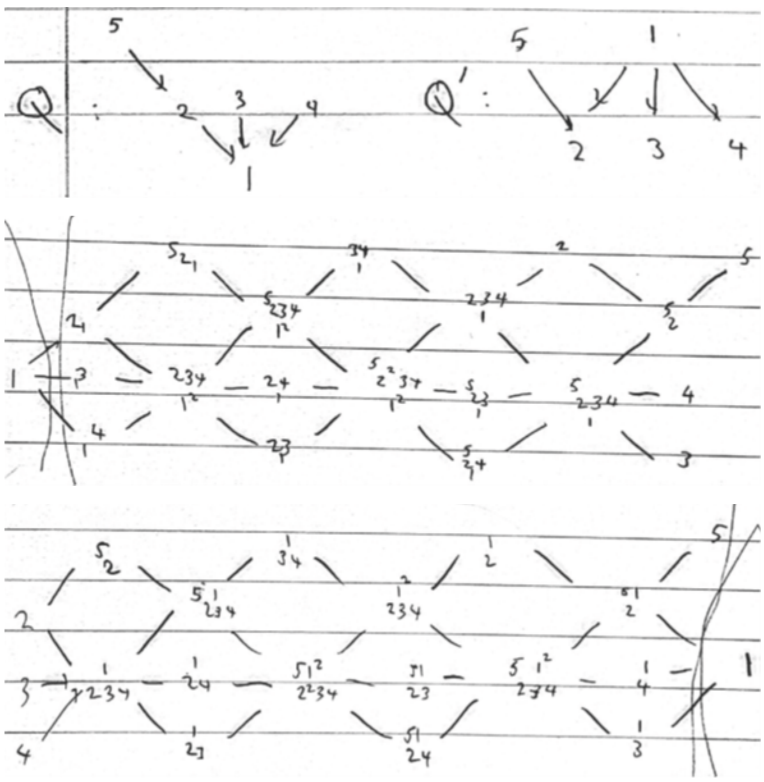
$$KQ\text{-mod} \rightleftarrows KQ'\text{-mod}$$

sending a representation $X$ of $Q$ to the representation $X'$ of $Q'$ which is the same, except that

$$X'_i = \mathrm{Ker}(\bigoplus_{a:j\to i} X_j \to X_i)$$

and the linear map $X'_i \to X_j$ is the canonical map.

This gives an equivalence between the module classes in $KQ$-mod and $KQ'$-mod given by the modules with no summand $S[i]$. For example.

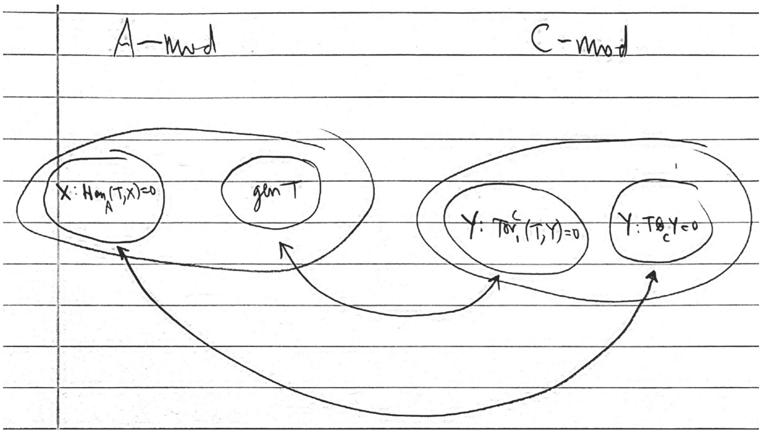



Brenner and Butler generalized this with the notion of a tilting module. Let $A$ be an algebra. An $A$-module $T$ is a *tilting module* if

- $\mathrm{proj.\,dim}\, T \leq 1$.
- $\mathrm{Ext}^1_A(T,T) = 0$.

- $\#T = \#A$, the number of non-isomorphic summands of $T$ is the number of simple $A$-modules.

Let $B = \mathrm{End}_A(T)^{op}$, so $T$ becomes an $A$-$B$-bimodule. The Brenner-Butler theorem gives equivalences between the following parts of the module categories ($C = B$ in the picture).




## 5.1 Torsion theories and tau-rigid modules

The notion of a torsion theory comes from Dickson, A torsion theory for abelian categories, Trans. Amer. Math. Soc. 1966.

**Definition.** A *torsion theory* in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$, the *torsion* and *torsion-free* classes, such that
(i) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$.
(ii) Any object $X$ has a subobject $t_\mathcal{T} X \in \mathcal{T}$ with $X/t_\mathcal{T} X \in \mathcal{F}$ (so it fits in an exact sequence $0 \to t_\mathcal{T} X \to X \to X/t_\mathcal{T} X \to 0$ with first term in $\mathcal{T}$ and last term in $\mathcal{F}$).

**Examples.** (1) The torsion and torsion-free modules give a torsion theory in the category of $\mathbb{Z}$-modules.

(2) For $A$ the path algebra of the quiver $1 \to 2$, $A$-mod has torsion theory $(\mathrm{add}\, S[2], \mathrm{add}\, S[1])$.

**Notation.** For an a set $\mathcal{C}$ of modules in $A$-mod or more generally of objects in an abelian category

$$\mathcal{C}^{\perp i,j,\dots} = \{X : \mathrm{Ext}^n(M, X) = 0 \text{ for all } M \in \mathcal{C} \text{ and } n = i, j, \dots\},$$

$$^{\perp i,j,\dots}\mathcal{C} = \{X : \mathrm{Ext}^n(X, M) = 0 \text{ for all } M \in \mathcal{C} \text{ and } n = i, j, \dots\}.$$

Recall that $\mathrm{Ext}^0 = \mathrm{Hom}$.

**Properties.** Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory.

(i) $\mathcal{T} = {}^{\perp 0}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp 0}$ so either of the classes determines the other.

(ii) $\mathcal{T}$ is closed under quotients and extensions; $\mathcal{F}$ is closed under subobjects and extensions.

(iii) The subobject $t_{\mathcal{T}}X$ is uniquely determined, and the assignment sending $X$ to $t_{\mathcal{T}}X$ defines a functor $\mathcal{A} \to \mathcal{T}$ which is a right adjoint to the inclusion $\mathcal{T}$ in $\mathcal{A}$. The assignment sending $X$ to $X/t_{\mathcal{T}}X$ defines a functor $\mathcal{A} \to \mathcal{F}$ which is a left adjoint to the inclusion $\mathcal{F}$ in $\mathcal{A}$.

*Proof.* (i) If $X \in \mathcal{T}^{\perp 0}$, then $\mathrm{Hom}(\mathcal{T}, X) = 0$, so we must have $t_{\mathcal{T}}X = 0$, so $X \cong X/t_{\mathcal{T}}X \in \mathcal{F}$. If $X \in {}^{\perp 0}\mathcal{F}$, then $\mathrm{Hom}(X, \mathcal{F}) = 0$, so we must have $X = t_{\mathcal{T}}X \in \mathcal{T}$.

For (ii), for $\mathcal{T}$ given an exact sequence $0 \to X \to Y \to Z \to 0$, apply $\mathrm{Hom}(-, F)$ for $F \in \mathcal{F}$ to get an exact sequence

$$0 \to \mathrm{Hom}(Z, F) \to \mathrm{Hom}(Y, F) \to \mathrm{Hom}(X, F).$$

Now if $X, Z \in \mathcal{T}$, then $\mathrm{Hom}(X, F) = \mathrm{Hom}(Z, F) = 0$, so $\mathrm{Hom}(Y, F) = 0$, so $Y \in \mathcal{T}$. Also, if $Y \in \mathcal{T}$, then $\mathrm{Hom}(Y, F) = 0$, so $\mathrm{Hom}(Z, F) = 0$, so $Z \in \mathcal{T}$.

For (iii) observe that any map $\theta : X \to Y$ induces a map $t_{\mathcal{T}}X \to t_{\mathcal{T}}Y$ since the composition $t_{\mathcal{T}}X \to X \to Y \to Y/t_{\mathcal{T}}Y$ must be zero. $\square$

**Remark.** A *splitting torsion theory* is one in which the sequence $0 \to t_{\mathcal{T}}X \to X \to X/t_{\mathcal{T}}X \to 0$ is always split exact.

If $A$ is a f.d. algebra, a torsion theory in $A$-mod is splitting if and only if every indecomposable module is either torsion or torsion-free.

A splitting torsion theory is thus given by a partition of the indecomposable modules into two sets $T, F$ with $\mathrm{Hom}(T, F) = 0$. Then $(\mathrm{add}\, T, \mathrm{add}\, F)$ is a splitting torsion theory in $A$-mod.

This is very easy to do if $A$ is an algebra whose AR quiver is obtained by knitting, so $A$ is of finite representation type and all of its indecomposable modules are directing. We want there to be no irreducible maps from $T$ to $F$.

**Proposition.** *If $A$ is a f.d. algebra, for a module class $\mathcal{T}$ in $A$-mod the following are equivalent.*

*(i) $\mathcal{T}$ is a torsion class for some torsion theory in $A$-mod.*

*(ii) $\mathcal{T} = {}^{\perp 0}(\mathcal{T}^{\perp 0})$.*

*(iii) $\mathcal{T} = {}^{\perp 0}\mathcal{C}$ for some module class $\mathcal{C}$.*

*(iv) $\mathcal{T}$ is closed under quotients and extensions.*

*Proof.* (i) implies (ii) implies (iii) implies (iv). Straightforward.

(iv) implies (i). Define $\mathcal{F} = \mathcal{T}^{\perp 0}$. Given any module $X$, let $T$ be a submodule of $X$ in $\mathcal{T}$ of maximal dimension. Then $\mathrm{Hom}(\mathcal{T}, X/T) = 0$, for if $T'/T$ is the image of such a map, then $T'/T$ is in $\mathcal{T}$, hence so is $T'$, contradicting maximality. Thus $X/T \in \mathcal{F}$. $\square$

Thus a pair of module classes $(\mathcal{T},\mathcal{F})$ is a torsion theory in $A$-mod if and only if $\mathcal{T} = {}^{\perp 0}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp 0}$.

**Lemma** (Auslander-Smalø, 1981)**.** *For modules $M, N$, the following are equivalent:*
*(i)* $\operatorname{Hom}(N, \tau M) = 0$.
*(ii)* $\operatorname{Ext}^1(M, \operatorname{gen} N) = 0$ *(that is,* $\operatorname{Ext}^1(M, G) = 0$ *for all* $G \in \operatorname{gen} N$*).*

*Proof.* (i)⇒(ii). If $\operatorname{Hom}(N, \tau M) = 0$, then $\operatorname{Hom}(G, \tau M) = 0$ for all $G \in \operatorname{gen} N$), so $\overline{\operatorname{Hom}}(G, \tau M) = 0$, so $\operatorname{Ext}^1(M, G) = 0$ by the Auslander-Reiten formula.

(ii)⇒(i). Say $f : N \to \tau M$ is a non-zero map. Factorize it as a surjection $g : N \to G$ followed by a mono $h : G \to \tau M$. Suppose that $h$ factors through an injective. Then it factors through the injective envelope $E(G)$ of $G$. Since $\tau M$ has no injective summand, the induced map $E(G) \to \tau M$ cannot be injective, so its kernel is non-zero. Since $G$ is essential in $E(G)$, the kernel meets $G$. Thus $G \to \tau M$ has non-zero kernel. Contradiction. Thus $\overline{\operatorname{Hom}}(G, \tau M) \neq 0$, so $\operatorname{Ext}^1(M, G) \neq 0$. □

**Definition.** Given a module class $\mathcal{C}$ in $A$-mod and $X \in \mathcal{C}$, we say that
(i) $X$ is *Ext-projective* in $\mathcal{C}$ if $\operatorname{Ext}^1(X, \mathcal{C}) = 0$.
(ii) $X$ is *Ext-injective* in $\mathcal{C}$ if $\operatorname{Ext}^1(\mathcal{C}, X) = 0$.

**Lemma.** *If $(\mathcal{T},\mathcal{F})$ is a torsion theory in $A$-*mod*, then*
*(i) $X \in \mathcal{T}$ is Ext-projective for $\mathcal{T}$ iff $\tau X \in \mathcal{F}$.*
*(ii) $X \in \mathcal{F}$ is Ext-injective for $\mathcal{F}$ iff $\tau^- X \in \mathcal{T}$.*
*(iii) There are bijections*

$$\text{Non-proj indec Ext-projs in } \mathcal{T} \text{ up to iso} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \text{Non-inj indec Ext-injs in } \mathcal{F} \text{ up to iso}$$

*Proof.* (i) Say $X \in \mathcal{T}$. Then $\tau X \in \mathcal{F} \Leftrightarrow \operatorname{Hom}(T, \tau X) = 0$ for all $T \in \mathcal{T} \Leftrightarrow \operatorname{Ext}^1(X, \operatorname{gen} T) = 0$ for all $T \in \mathcal{T} \Leftrightarrow X$ is Ext-projective.
(ii) is dual.
(iii) follows. □

**Lemma.** *If $(\mathcal{T},\mathcal{F})$ is a torsion theory in $A$-*mod*, then*

*(i) The Ext-injectives for $\mathcal{T}$ are the modules $t_{\mathcal{T}} I$ with $I$ injective. The indecomposable Ext-injectives are the modules $t_{\mathcal{T}} I[i]$ with $I[i] \notin \mathcal{F}$.*

*(ii) The Ext-projectives for $\mathcal{F}$ are the modules $P/t_{\mathcal{T}} P$ with $P$ projective. The indecomposable Ext-projectives are the modules $P[i]/t_{\mathcal{T}} P[i]$ with $P[i] \notin \mathcal{T}$.*

*Proof.* (i) $t_{\mathcal{T}} I$ is in $\mathcal{T}$, and it is Ext-injective since if $T \in \mathcal{T}$ and $0 \to t_{\mathcal{T}} I \to E \to T \to 0$ is an exact sequence, then the pushout along $t_{\mathcal{T}} I \to I$ splits, giving a map $E \to I$. But $E \in \mathcal{T}$, so it gives a map $E \to t_{\mathcal{T}} I$, which is a retraction for the given sequence.

Conversely suppose $X$ is Ext-injective in $\mathcal{T}$ and $X \to I$ is its injective envelope. Then we have an injection $X \to t_{\mathcal{T}} I$. Since $\mathcal{T}$ is closed under quotients, all terms in the exact sequence $0 \to X \to t_{\mathcal{T}} I \to t_{\mathcal{T}} I/X \to 0$ are in $\mathcal{T}$. Thus this sequence splits, so $X$ is a direct summand of $t_{\mathcal{T}} I$, and we have equality since $X$ is essential in $I$.

Also, if $I[i] \notin \mathcal{F}$, then $t_{\mathcal{T}} I[i]$ is non-zero and contained in $I[i]$, so it has simple socle, so it is indecomposable.

(ii) is dual. $\square$

The following definition comes from Adachi, Iyama and Reiten, $\tau$-tilting theory, 2014.

**Definition.** A module $M$ is *$\tau$-rigid* if $\operatorname{Hom}(M, \tau M) = 0$. Dually, it is *$\tau^-$-rigid* if $\operatorname{Hom}(\tau^- M, M) = 0$

Note that $M$ is $\tau$-rigid iff $DM$ is $\tau^-$-rigid, since

$$\operatorname{Hom}(M, \tau M) = \operatorname{Hom}(M, D \operatorname{Tr} M) \cong \operatorname{Hom}(\operatorname{Tr} M, DM)$$
$$\cong \operatorname{Hom}(\operatorname{Tr} DDM, DM) = \operatorname{Hom}(\tau^- DM, DM).$$

**Proposition.** *The following are equivalent*

*(i) $M$ is $\tau$-rigid.*

*(ii)* $\operatorname{Ext}^1(M, \operatorname{gen} M) = 0$.

*(iii)* $\operatorname{gen} M$ *is a torsion class and $M$ is Ext-projective in* $\operatorname{gen} M$.

*(iv) $M$ is Ext-projective in some torsion class.*

*Proof.* (i)⇔(ii). The lemma of Auslander and Smalø.

(ii)⇒(iii). Suppose $M$ is $\tau$-rigid. To show that $\operatorname{gen} M$ is a torsion class, it suffices to show that if $0 \to X \to Y \to Z \to 0$ is exact and $X, Z \in \operatorname{gen} M$, then so is $Y$. Choose a surjection $M^n \to Z$. By (ii) The pullback sequence splits, so the middle term of it is in $\operatorname{gen} M$, and hence so is $Y$. Now $\operatorname{Ext}^1(M, \operatorname{gen} M) = 0$, so $M$ is Ext-projective.

(iii)⇒(iv). Trivial.

(iv)⇒(ii). If $M$ is Ext-projective in $\mathcal{T}$, then $\operatorname{Ext}^1(M, \operatorname{gen} M) = 0$ since $\operatorname{gen} M \subseteq \mathcal{T}$. $\square$

Note that the torsion theory given by a $\tau$-rigid module $M$ is $(\operatorname{gen} M, M^{\perp 0})$.

**Example.** Let $A$ be the path algebra of $1 \to 2 \to 3$. Let $M = 2 \oplus 123$. It is $\tau$-rigid. Then $\mathcal{T} = \operatorname{gen} M$ contains $123, 12, 2, 1$. The torsion-free class is $\mathcal{F} = \mathcal{T}^{\perp 0} = M^{\perp 0}$. It contains 3 and 23.

The Ext-projectives in $\mathcal{T}$ are $2, 12, 123$.

The Ext-injectives in $\mathcal{T}$ are $1, 12, 123$.

The Ext-projectives in $\mathcal{F}$ are $3, 23$.

The Ext-injectives in $\mathcal{F}$ are $3, 23$.

The next result is dual to Theorem 4.1(c) of Auslander and Smalø, Almost split sequences in subcategories, J. Algebra 1981.

**Theorem.** *Let $\mathcal{T}$ be a torsion class which is functorially finite and let*

$$A \xrightarrow{f} M^0 \xrightarrow{c} M^1 \to 0$$

*be an exact sequence with $f$ be a minimal left $\mathcal{T}$-approximation of $A$. Then*

*(i) $\mathcal{T} = \operatorname{gen} M^0 = \operatorname{gen}(M^0 \oplus M^1)$.*

*(ii) $M^0$ is a* splitting projective *for $\mathcal{T}$, meaning that any epimorphism $\theta : T \twoheadrightarrow M^0$ with $T \in \mathcal{T}$ must be a split epi.*

*(iii) $M^0$ and $M^0 \oplus M^1$ are Ext-projective in $\mathcal{T}$, so they are $\tau$-rigid.*

*(iv) Any module $T \in \mathcal{T}$ is a quotient of a module in* $\operatorname{add}(M^0 \oplus M^1)$ *by a submodule in $\mathcal{T}$.*

*(v) Any Ext-projective in $\mathcal{T}$ is in* $\operatorname{add}(M^0 \oplus M^1)$*, so there are only finitely many indecomposable Ext-projectives in $\mathcal{T}$.*

*Proof.* (i) Clearly $\operatorname{gen} M^0 = \operatorname{gen}(M^0 \oplus M^1) \subseteq \mathcal{T}$. If $T \in \mathcal{T}$, then there is a map $A^n \twoheadrightarrow T$, and each component factors through $M$, giving an epimorphism $M^n \twoheadrightarrow T$.

(ii) Since $A$ is projective, the map $f : A \to M^0$ lifts to a map $A \to T$. By the approximation property, this factors as $A \to M^0 \to T$. Now the composition $M^0 \to T \to M^0$ must be an isomorphism by minimality.

(iii) Let $T \in \mathcal{T}$. Any exact sequence $0 \to T \to E \to M^0 \to 0$ splits by (ii). Thus $M^0$ is Ext-projective.

Since $f$ is a $\mathcal{T}$-approximation, the induced map $\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(A, T)$ is surjective. This is a composition $\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(\operatorname{Im} f, T) \to \operatorname{Hom}(A, T)$ and the second map is injective, so actually the second map is a bijection and the first map $\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(\operatorname{Im} f, T)$ is surjective.

Now the exact sequence $0 \to \operatorname{Im} f \xrightarrow{i} M^0 \xrightarrow{c} M^1 \to 0$ gives

$$\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(\operatorname{Im} f, T) \to \operatorname{Ext}^1(M^1, T) \to \operatorname{Ext}^1(M^0, T) = 0.$$

so $\operatorname{Ext}^1(M^1, T) = 0$.

(iv) (My thanks to Andrew Hubery for this argument). Take a right $\operatorname{add}(M^0 \oplus M^1)$-approximation $\phi : W \to T$ for $T$. Since $T \in \operatorname{gen} M^0$, the map $\phi$ is surjective, so it gives an exact sequence

$$0 \to U \xrightarrow{\theta} W \xrightarrow{\phi} T \to 0.$$

Given $u \in U$ there is a map $r : A \to U$, $a \mapsto au$. Since $A \to M^0$ is a $\mathcal{T}$-approximation and $W \in \mathcal{T}$, there is a map $p$, and hence a map $q$ giving a commutative diagram

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & M^0 & \xrightarrow{c} & M^1 & \longrightarrow & 0 \\ & & {\scriptstyle r}\big\downarrow & & {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow & & \\ 0 & \longrightarrow & U & \xrightarrow{\theta} & W & \xrightarrow{\phi} & T & \longrightarrow & 0. \end{array}$$

Since $\phi$ is an approximation, $q = \phi h$ for some $h : M^1 \to W$. Then $\phi(p - hc) = 0$. Thus $p - hc = \theta\ell$ for some $\ell : M^0 \to U$. Then $\theta(r - \ell f) = 0$, so since $\theta$ is mono, $r = \ell f$. Thus $u \in \mathrm{Im}(\ell)$. Repeating for a basis of $U$, we get a map from a direct sum of copies of $M$ onto $U$, so $U \in \mathcal{T}$.

(v) Follows. □

End of lecture on 2025-10-30. Provisional script for the next lecture follows (subject to change).

**Corollary.** *If $M$ is a $\tau$-rigid module, then* $\operatorname{gen} M$ *is a functorially finite torsion class. Conversely, any functorially finite torsion class $\mathcal{T}$ is of the form* $\operatorname{gen} M$ *for some $\tau$-rigid module $M$, which we can take to be the direct sum of the indecomposable Ext-projectives in $\mathcal{T}$.*

*Proof.* Any torsion class in $A$-mod is contravariantly finite, since the inclusion has a right adjoint. Recall also that if $M$ is a module, then $\operatorname{gen} M$ is always covariantly finite by the proposition at the end of section 1.9. In particular, if $M$ is $\tau$-rigid, then $\operatorname{gen} M$ is a functorially finite torsion class.

The last part follows from the theorem, since up to multiplicities, $M^0 \oplus M^1$ is the direct sum of the indecomposable Ext-projectives in $\mathcal{T}$. □

## 5.2 Tilting modules

**Definition.** Let $M$ be an $A$-module.

$M$ is a *partial tilting module* if $\operatorname{proj.dim} M \leq 1$ and $\operatorname{Ext}^1(M, M) = 0$.

A partial tilting module $M$ is a *tilting module* if there is an exact sequence $0 \to A \to M^0 \to M^1 \to 0$ with $M^i \in \operatorname{add} M$. (Later we will see that it is equivalent that $\#M = \#A$.)

$M$ is a *partial cotilting module* if $\operatorname{inj.dim} M \leq 1$ and $\operatorname{Ext}^1(M, M) = 0$.

A partial cotilting module is a *cotilting module* if there is an exact sequence $0 \to M_1 \to M_0 \to DA \to 0$ with $M_i \in \operatorname{add} M$. (Again, it is equivalent that $\#M = \#A$.)

Clearly $M$ is a (partial) tilting $A$-module iff $DM$ is a (partial) cotilting $A^{op}$-module.

Note that we deal only with *classical* tilting theory. There is a version allowing higher projective dimension.

**Lemma.** *If $M$ is a partial tilting module, then $M$ is $\tau$-rigid. Conversely if $M$ is $\tau$-rigid, then* $\operatorname{Ext}^1(M,M)=0$*, and if $M$ is in addition faithful, then* $\operatorname{proj.dim} M \le 1$ *so it is a partial tilting module.*

*Proof.* Use the AR formula $D\operatorname{Ext}^1(M,N) \cong \overline{\operatorname{Hom}}(N,\tau M)$.

If $\operatorname{proj.dim} M \le 1$ then $\operatorname{Hom}(DA,\tau M)=0$, so the AR formula takes the form $D\operatorname{Ext}^1(M,N) \cong \operatorname{Hom}(N,\tau M)$.

Suppose $M$ is $\tau$-rigid. If $M$ is faithful, then so is $DM$, so $A^{op} \hookrightarrow DM^n$, for some $n$, so $M^n \twoheadrightarrow DA$. Applying $\operatorname{Hom}(-,\tau M)$ we get $\operatorname{Hom}(DA,\tau M) \hookrightarrow \operatorname{Hom}(M^n,\tau M)=0$. Thus $\operatorname{Hom}(DA,\tau M)=0$, so $\operatorname{proj.dim} M \le 1$. □

**Proposition** (Bongartz)**.** *Let $M$ be a partial tilting module. Take a basis of $\xi_1,\dots,\xi_n$ of* $\operatorname{Ext}^1(M,A)$*, consider the tuple* $(\xi_1,\dots,\xi_n)$ *as an element of* $\operatorname{Ext}^1(M^n,A)$*, and let*

$$0 \to A \to E \to M^n \to 0.$$

*be the corresponding* universal *extension. Then $T = E \oplus M$ is a tilting module. Thus every partial tilting module is a direct summand of a tilting module, and by duality every partial cotilting module is a direct summand of a cotilting module.*

*Proof.* The long exact sequence for $\operatorname{Hom}(M,-)$ gives

$$\operatorname{Hom}(M,M^n) \xrightarrow{\xi} \operatorname{Ext}^1(M,A) \to \operatorname{Ext}^1(M,E) \to \operatorname{Ext}^1(M,M^n),$$

the map $\xi$ is onto, and $\operatorname{Ext}^1(M,M^n)=0$, so $\operatorname{Ext}^1(M,E)=0$. From the long exact sequence for $\operatorname{Hom}(-,M)$ one gets $\operatorname{Ext}^1(E,M)=0$, from the long exact sequence for $\operatorname{Hom}(-,E)$ one gets $\operatorname{Ext}^1(E,E)=0$. Also $A$ and $M^n$ have projective dimension $\le 1$, hence so does $E$. □

A partial tilting module $M$ is $\tau$-rigid, so gives a torsion theory $(\operatorname{gen} M, M^{\perp 0})$. Moreover $\operatorname{gen}_1 M \subseteq \operatorname{gen} M \subseteq M^{\perp 1}$.

**Proposition** (1)**.** *For a partial tilting module $M$, the following are equivalent:*

*(i) $M$ is a tilting module.*

*(ii) $M^{\perp 0,1}=0$.*

*(iii)* $\operatorname{gen} M = M^{\perp 1}$.

*(iv)* $\operatorname{gen}_1 M = M^{\perp 1}$.

*(v) $X$ is Ext-projective in $M^{\perp 1} \Leftrightarrow X \in \operatorname{add} M$.*

*Proof.* (i) $\Rightarrow$ (ii). If $X \in M^{\perp 0,1}$, apply $\mathrm{Hom}(-, X)$ to the exact sequence $0 \to A \to M^0 \to M^1 \to 0$, to deduce that $\mathrm{Hom}(A, X) = 0$.

(ii) $\Rightarrow$ (iii). Suppose $X \in M^{\perp 1}$. Take a basis of $\mathrm{Hom}(M, X)$ and use it to form the universal map $f : M^n \to X$. Then $\mathrm{Im}\, f \in \mathrm{gen}\, M$. Consider the exact sequence $0 \to \mathrm{Im}\, f \to X \to X/\mathrm{Im}\, f \to 0$. Apply $\mathrm{Hom}(M, -)$ giving an exact sequence

$$0 \to \mathrm{Hom}(M, \mathrm{Im}\, f) \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, X/\mathrm{Im}\, f) \to \mathrm{Ext}^1(M, \mathrm{Im}\, f).$$

By construction the map $\mathrm{Hom}(M, M^n) \to \mathrm{Hom}(M, X)$ is onto, hence so is the map $\mathrm{Hom}(M, \mathrm{Im}\, f) \to \mathrm{Hom}(M, X)$. Also $\mathrm{Ext}^1(M, \mathrm{Im}\, f) = 0$ since $M$ is $\tau$-rigid. Thus $\mathrm{Hom}(M, X/\mathrm{Im}\, f) = 0$. Also $\mathrm{Ext}^1(M, X/\mathrm{Im}\, f) = 0$. Thus $X/\mathrm{Im}\, f \in M^{\perp 0,1}$. Thus $X/\mathrm{Im}\, f = 0$, so $f$ is onto, so $X \in \mathrm{gen}\, M$.

(iii) $\Rightarrow$ (iv). Suppose $X \in M^{\perp 1}$. Then it is in $\mathrm{gen}\, M$. Let $L$ be the kernel of the universal map $M^n \to X$. Then applying $\mathrm{Hom}(M, -)$ we see that $L \in M^{\perp 1}$, so $L \in \mathrm{gen}\, M$. Say $M'' \twoheadrightarrow L$. Now the sequence $M'' \to M^n \to X \to 0$ shows that $X \in \mathrm{gen}_1\, M$.

(iv) $\Rightarrow$ (v). Clearly $M$ and so any $X \in \mathrm{add}(M)$ is in $M^{\perp 1}$ and Ext-projective. Conversely if $X$ is in $M^{\perp 1}$ and Ext-projective, then by (iv) there is an exact sequence $M'' \xrightarrow{f} M' \to X \to 0$. This gives an exact sequence $0 \to \mathrm{Im}\, f \to M' \to X \to 0$ with $\mathrm{Im}\, f \in \mathrm{gen}\, M \subseteq M^{\perp 1}$. By assumption this sequence splits, so $X \in \mathrm{add}\, M$.

(v) $\Rightarrow$ (i). It suffices to show that $E$ in Bongartz's sequence is in $\mathrm{add}\, M$, and for this it suffices to show it is Ext-projective in $M^{\perp 1}$. We know it is in $M^{\perp 1}$. If $Y \in M^{\perp 1}$, apply $\mathrm{Hom}(-, Y)$ to the Bongartz sequence to get $\mathrm{Ext}^1(M^n, Y) \to \mathrm{Ext}^1(E, Y) \to \mathrm{Ext}^1(A, Y)$, so $\mathrm{Ext}^1(E, Y) = 0$. □

Dually, a partial cotilting module $M$ gives a torsion theory $({}^{\perp 0}M, \mathrm{cogen}\, M)$. Moreover $\mathrm{cogen}^1\, M \subseteq \mathrm{cogen}\, M \subseteq {}^{\perp 1}M$. The following is dual to the last proposition.

**Proposition** (2)**.** *For a partial cotilting module $M$, the following are equivalent:*
*(i') $M$ is a cotilting module.*
*(ii') ${}^{\perp 0,1}M = 0$.*
*(iii')* $\mathrm{cogen}\, M = {}^{\perp 1}M$.
*(iv')* $\mathrm{cogen}^1\, M = {}^{\perp 1}M$.
*(v') $X$ is Ext-injective in ${}^{\perp 1}M \Leftrightarrow X \in \mathrm{add}\, M$.*

**Proposition** (3)**.** *If ${}_AM$ is a (co)tilting module, then it is f.b. and if $B = \mathrm{End}_A(M)$, then ${}_BM$ is also a (co)tilting module.*

*Proof.* If ${}_AM$ is tilting, then $\mathrm{gen}_1\, M = M^{\perp 1}$, which contains $DA$, so ${}_AM$ is f.b.

(i) Applying $\mathrm{Hom}_A(-, M)$ to the exact sequence $0 \to A \to M^0 \to M^1 \to 0$ gives

$$0 \to \mathrm{Hom}_A(M^1, M) \to \mathrm{Hom}_A(M^0, M) \to M \to 0$$

and $\mathrm{Hom}_A(M^i, M) \in \mathrm{add}(\mathrm{Hom}_A(M, M)) = \mathrm{add}({}_BB)$, so $\mathrm{proj.dim}\,{}_BM \leq 1$.

(ii) The tilting sequence $0 \to A \to M^0 \to M^1 \to 0$ stays exact on applying $\mathrm{Hom}(-, M)$. Thus $A \in \mathrm{cogen}^2({}_AM)$. Thus $\mathrm{Ext}^1_B(M, M) = 0$ by the proposition about endomorphism correspondents.

(iii) Applying $\mathrm{Hom}_A(-, M)$ to a projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of $M$ gives an exact sequence

$$0 \to B \to M^0 \to M^1 \to 0$$

where $M^i = \mathrm{Hom}_A(P_i, M) \in \mathrm{add}({}_BM)$. Thus ${}_BM$ is a tilting module.

Dually for cotilting. $\square$

## 5.3 The Brenner-Butler Theorem

Let ${}_AM$ be a cotilting module and $B = \mathrm{End}_A(M)$, so ${}_BM$ is also cotilting.

In $A$-mod we have a torsion theory $(\mathcal{T}_A, \mathcal{F}_A) = ({}^{\perp 0}{}_AM, \mathrm{cogen}\,{}_AM)$. Since ${}_AM$ is cotilting we have

$$\mathcal{F}_A = \mathrm{cogen}({}_AM) = \mathrm{cogen}^1({}_AM) = {}^{\perp 1}{}_AM = \{X \in A\text{-mod} : \mathrm{Ext}^1_A(X, M) = 0\}.$$

In $B$-mod we have a torsion theory $(\mathcal{T}_B, \mathcal{F}_B) = ({}^{\perp 0}{}_BM, \mathrm{cogen}\,{}_BM)$. Since ${}_BM$ is cotilting we have the equivalent alternative descriptions of $\mathcal{F}_B$.

**Theorem** (Brenner-Butler Theorem, 1st version)**.** *There are antiequivalences*

$$\mathcal{F}_A \underset{\mathrm{Hom}_B(-,M)}{\overset{\mathrm{Hom}_A(-,M)}{\rightleftarrows}} \mathcal{F}_B \quad \textit{and} \quad \mathcal{T}_A \underset{\mathrm{Ext}^1_B(-,M)}{\overset{\mathrm{Ext}^1_A(-,M)}{\rightleftarrows}} \mathcal{T}_B.$$

*Proof.* Since $\mathcal{F}_A = \mathrm{cogen}^1({}_AM)$ and $\mathcal{F}_B = \mathrm{cogen}^1({}_BM)$, the first antiequivalence is given by endomorphism correspondence.

Given a module ${}_AX$ in $\mathcal{T}_A$, so with $\mathrm{Hom}_A(X, M) = 0$, we show that

$$\mathrm{Hom}_B(\mathrm{Ext}^1_A(X, M), M) = 0$$

and construct a natural isomorphism

$$X \to \mathrm{Ext}^1_B(\mathrm{Ext}^1(X, M), M).$$

Indeed, take a projective cover of $X$ to get a sequence $0 \to L \to P \to X \to 0$. It gives an exact sequence of $B$-modules

$$0 \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(L, M) \to \mathrm{Ext}^1_A(X, M) \to 0$$

Now $P, L \in \operatorname{cogen} M = \operatorname{cogen}^1 M$, so the natural maps $P \to \operatorname{Hom}_B(\operatorname{Hom}_A(P, M), M)$ and $L \to \operatorname{Hom}_B(\operatorname{Hom}_A(L, M), M)$ are isomorphisms. Also

$$\operatorname{Hom}_A(L, M) \in \operatorname{cogen}^1({}_BM) = {}^{\perp 1}({}_BM),$$

so $\operatorname{Ext}^1_B(\operatorname{Hom}(L, M), M) = 0$. Thus we get a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & X & \longrightarrow 0 \\ & & & & \big\downarrow & & \big\downarrow & & & \\ 0 & \longrightarrow & ({}^1(X, M), M) & \longrightarrow & ((L, M), M) & \longrightarrow & ((P, M), M) & \longrightarrow & {}^1({}^1(X, M), M) & \longrightarrow 0 \end{array}$$

(where we omit the words Hom and Ext) with exact rows and in which the vertical maps are isomorphisms. Thus $\operatorname{Hom}_B(\operatorname{Ext}^1_A(X, M), M) = 0$ and there is an induced isomorphism $X \to \operatorname{Ext}^1_B(\operatorname{Ext}^1_A(X, M), M)$. One also needs to show that this is a natural isomorphism, but we omit the proof of this. $\square$