Fakultät für Mathematik
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Masters course: Representations of Algebras 2

I cover a number of key topics in the representation theory of finite-dimensional associative algebras. Specifically:

- Correspondences given by faithfully balanced modules, and applications to Auslander algebras and homological conjectures.
- Tilting theory and the beginnings of tau-tilting theory.
- Geometric methods for studying representations of algebras, including an introduction to varieties and schemes.
- Matrix reductions in the sense of Roiter and Kleiner, and Drozd's Tame and Wild Theorem.

Some relevant books:

- I. Assem and F. U. Coelho, Basic representation theory of algebras, Springer 2020.
- I. Assem, D. Simson and A. Skowroński, Elements of the representation theory of associative algebras. Volume 1, Techniques of representation theory, CUP 2006.
- H. Derksen and J. Weyman, An introduction to quiver representations, American Mathematical Society 2017.
- P. Gabriel and A. V. Roiter, Representations of finite dimensional algebras, Springer 1977.
- A. Kirillov Jr., Quiver Representations and Quiver Varieties, American Mathematical Society 2016.
- A. Skowroński and K. Yamagata, Frobenius algebras 2 Tilted and Hochschild extension algebras, European Mathematical Society 2017.

The section numbering continues from the previous lecture course.

Contents

4 Homological topics

In this section I want to discuss

- Some of the many homological conjectures for f.d. algebras

- Auslander's correspondence between algebras A of finite representation type and algebras B with $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$, and Iyama's generalization of this with cluster tilting objects.

The unifying feature is what I call endomorphism correspondence for faithfully balanced modules.

4.1 Higher generation and cogeneration

We are interested in finite-dimensional algebras A over a field K (but most things generalize easily to Artin algebras).

Except where explicitly stated, all modules are f.d., and we write A-mod for the category of finite-dimensional left A-modules.

We write D for the duality $\operatorname{Hom}_K(-, K)$ between A-mod and A^{op}-mod.

Recall that a *module class* in A-mod is a full subcategory closed under isomorphisms, direct sums and direct summands. Given any module M, $\operatorname{add}(M)$ is the smallest module class containing M. It is given by the modules isomorphic to a direct summand of M^n for some n.

Definition. Given a module M, $\operatorname{gen}(M)$ denotes the module class consisting of quotients of direct sums of copies of M and $\operatorname{cogen}(M)$ the module class of submodules of a direct sum of copies of M.

We say M is a *generator* if $\operatorname{gen}(M) = A$-mod. It is equivalent that $A \in \operatorname{gen}(A)$, or that $A \in \operatorname{add}(M)$. We say M is a *cogenerator* if $\operatorname{cogen}(M) = A$-mod. It is equivalent that $DA \in \operatorname{cogen}(M)$, or $DA \in \operatorname{add}(M)$.

There are higher versions as follows. Here

Proposition (1). *Let M be an A-module and $n \geq 0$ and consider M also as a B-module, where $B = \operatorname{End}_A(M)$. For an A-module X, the following are equivalent.*

(a) There is an exact sequence

$$M_n \xrightarrow{f_n} M_{n-1} \to \cdots \to M_0 \xrightarrow{f_0} X \to 0$$

with $M_i \in \operatorname{add} M$, such that the sequence

$$\operatorname{Hom}_A(M, M_{n-1}) \to \cdots \to \operatorname{Hom}_A(M, M_0) \to \operatorname{Hom}_A(M, X) \to 0$$

is exact (note that this is automatic if M is projective).

(b) The natural map $\operatorname{Hom}_A(M, X) \otimes_B M \to X$ is surjective (in case $n = 0$) or an isomorphism (in case $n > 0$) and $\operatorname{Tor}_i^B(\operatorname{Hom}_A(M, X), M) = 0$ for $0 < i < n$.

Definition. We define $\text{gen}_n(M)$ to be the full subcategory of A-mod given by the modules X satisfying these conditions. Using condition (b) it is easy to see that it is a module class. Clearly

$$\text{add}(M) \subseteq \cdots \subseteq \text{gen}_2(M) \subseteq \text{gen}_1(M) \subseteq \text{gen}_0(M) = \text{gen}(M).$$

Proof. (a)⇒(b). First note that we may assume that the sequence

$$\text{Hom}_A(M, M_n) \to \text{Hom}_A(M, M_{n-1}) \to \cdots \to \text{Hom}_A(M, M_0) \to \text{Hom}_A(M, X) \to 0$$

is exact. By assumption it is exact except possibly at $\text{Hom}_A(M, M_{n-1})$. Recall from section 1.9, that $\text{add}\, M$ is functorially finite in A-mod. Thus the module $\text{Im}(f_n)$ has a right $\text{add}\, M$-approximation, say $f' : M' \to \text{Im}(f_n)$. Since it is an approximation, we can factorize $f_n = f'g$ for some $g : M_n \to M'$. Thus the map f' has image $\text{Im}(f_n)$. Thus the sequence

$$M' \xrightarrow{f'} M_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \to 0$$

is exact. Also, the sequence

$$\text{Hom}_A(M, M') \to \text{Hom}_A(M, M_{n-1}) \to \cdots \to \text{Hom}_A(M, M_0) \to \text{Hom}_A(M, X) \to 0$$

is exact, since any morphism in $\text{Hom}(M, M_{n-1})$ which is sent to zero in $\text{Hom}(M, M_{n-2})$ has image contained in $\text{Ker}(f_{n-1}) = \text{Im}(f_n)$, and hence factors through the approximation f'. Thus replacing M_n by M' and f_n by f' if necessary, we have the claimed exactness.

Now we have a commutative diagram

$$\begin{array}{ccccccccc}
\text{Hom}(M, M_n) \otimes M & \longrightarrow & \ldots & \longrightarrow & \text{Hom}(M, M_0) \otimes M & \longrightarrow & \text{Hom}(M, X) \otimes M & \longrightarrow & 0 \\
\phi_n \downarrow & & & & \phi_0 \downarrow & & \theta \downarrow & & \\
M_n & \xrightarrow{f_n} & \ldots & \xrightarrow{f_1} & M_0 & \xrightarrow{f_0} & X & \longrightarrow & 0
\end{array}$$

For any $M' \in \text{add}\, M$, the natural map $\text{Hom}(M, M') \otimes_B M \to M'$ is an isomorphism, since it is for $M' = M$. Thus the ϕ_i are isomorphisms.

Since ϕ_0 and f_0 are surjective, so is θ. If $n > 0$, then since tensor products are right exact, the part of the diagram below and to the right of $\text{Hom}(M, M_1) \otimes M$ has exact rows, so implies that θ is an isomorphism.

Since $M_i \in \text{add}(M)$, as a right B-module, we have

$$\text{Hom}_A(M, M_i) \in \text{add}(\text{Hom}_A(M, M)) = \text{add}(B_B),$$

so the exact sequence

$$\text{Hom}_A(M, M_n) \to \text{Hom}_A(M, M_{n-1}) \to \cdots \to \text{Hom}_A(M, M_0) \to \text{Hom}_A(M, X) \to 0$$

is part of a projective resolution of $\mathrm{Hom}_A(M,X)$ as a right B-module. We can use it to compute $\mathrm{Tor}_i^B(\mathrm{Hom}_A(M,X),M)$ for $i<n$ as the homology of the complex

$$\mathrm{Hom}(M,M_n)\otimes_B M\to\cdots\to\mathrm{Hom}(M,M_0)\otimes_B M\to 0.$$

But by the commutative diagram above, this is isomorphic to the complex

$$M_n\to\cdots\to M_0\to 0$$

This is exact at M_i for $0<i<n$, giving the Tor vanishing.

(b)⇒(a). Take the start of a projective resolution of $\mathrm{Hom}_A(M,X)$ as a right B-module, say

$$P_n\xrightarrow{g_n}\cdots\to P_0\xrightarrow{g_0}\mathrm{Hom}_A(M,X)\to 0$$

Applying $-\otimes_B M$ gives a complex, which by the hypotheses is exact:

$$M_n\xrightarrow{f_n}\cdots\to M_0\xrightarrow{f_0}X\to 0,$$

where $M_i=P_i\otimes_B M\in\operatorname{add}M$. Applying $\mathrm{Hom}_A(M,-)$ to this, gives a complex

$$\mathrm{Hom}_A(M,M_n)\to\cdots\to\mathrm{Hom}_A(M,M_0)\to\mathrm{Hom}_A(M,X)\to 0.$$

Identifying $\mathrm{Hom}_A(M,M_i)=\mathrm{Hom}_A(M,P_i\otimes_B M)\cong P_i$, we see that this is the projective resolution we started with, so it is exact. Thus (a) holds.

Remark: if we took the projective resolution to be minimal, then the maps g_i would all be right minimal in the sense of section 1.6. It follows that the maps f_i are right minimal, for otherwise there is a decomposition $M_i=M_i'\oplus M_i''$ with $M_i''\neq 0$ and $f_i(M_i'')=0$. But then we get

$$P_i\cong\mathrm{Hom}_A(M,M_i)\cong\mathrm{Hom}_A(M,M_i')\oplus\mathrm{Hom}_A(M,M_i'')$$

and g_i is zero on the summand corresponding to $\mathrm{Hom}_A(M,M_i'')$, contradicting the minimality of g_i. □

Dually we have the following.

Proposition (2). *Let M be an A-module and $n\geq 0$ and consider M also as a B-module, where $B=\mathrm{End}_A(M)$. For an A-module X, the following are equivalent.*

(a') There is an exact sequence $0\to X\to M^0\to\cdots\to M^n$ with $M^i\in\operatorname{add}M$ such that the sequence

$$\mathrm{Hom}(M^{n-1},M)\to\cdots\to\mathrm{Hom}(M^0,M)\to\mathrm{Hom}(X,M)\to 0$$

is exact (this is automatic if M is injective).

(b') The natural map $X\to\mathrm{Hom}_B(\mathrm{Hom}_A(X,M),M)$ is a monomorphism (in case $n=0$) or an isomorphism (in case $n>0$) and $\mathrm{Ext}_B^i(\mathrm{Hom}_A(X,M),M)=0$ for $0<i<n$.

Definition. We define $\operatorname{cogen}^n(M)$ to be the full subcategory of A-mod given by the modules X satisfying these conditions. By the second condition it is a module class. Clearly

$$\operatorname{add}(M) \subseteq \cdots \subseteq \operatorname{cogen}^2(M) \subseteq \operatorname{cogen}^1(M) \subseteq \operatorname{cogen}^0(M) = \operatorname{cogen}(M).$$

It is clear from conditions (a) and (a') that $X \in \operatorname{cogen}^n({}_AM) \Leftrightarrow DX \in \operatorname{gen}_n({}_{A^{op}}DM)$.

4.2 Faithfully balanced modules and endomorphism correspondence

Definition. Let M be an A-module, and let $B = \operatorname{End}_A(M)$. Then M can be considered as a B-module, and there is a natural map

$$A \to \operatorname{End}_B(M).$$

Clearly M is faithful iff this map is injective. We say that M is a *balanced* A-module or that M has the *double centralizer property* if this map is onto, and that M is *faithfully balanced* (f.b.) if this map is an isomorphism.

Clearly M is a f.b. A-module iff DM is a f.b. A^{op}-module.

Lemma. *Let M be an A-module.*
(i) M is f.b. iff $A \in \operatorname{cogen}^1(M)$ iff $DA \in \operatorname{gen}_1(M)$.
(ii) If M is a generator or cogenerator, it is f.b.

Proof. (i) Apply the second proposition in the last section with $X = A$ and $n = 1$. Now M is f.b. iff DM is f.b. iff $A^{op} \in \operatorname{cogen}^1({}_{A^{op}}DM)$ iff $DA \in \operatorname{gen}_1({}_AM)$.

(ii) If M is a generator, then $A \in \operatorname{add}(M) \subseteq \operatorname{cogen}^1(M)$. If M is a cogenerator, then DM is a generator, so f.b., hence so is M. □

Definition. By an *f.b. pair* we mean a pair (A, M) consisting of an algebra and a f.b. A-module.

Given an f.b. pair, we construct a new f.b. pair (B, M), its *endomorphism correspondent*, where $B = \operatorname{End}_A(M)$ and M is considered in the natural way as a B-module.

Repeating the construction twice, one recovers essentially the original pair.

We say that f.b. pairs (A, M) and (A', M') are *equivalent* if there is an equivalence A-mod $\to A'$-mod sending $\operatorname{add}(M)$ to $\operatorname{add}(M')$.

One can show that equivalent pairs have equivalent endomorphism correspondents.

Theorem. *If* (A, M) *and* (B, M) *are f.b. pairs which are endomorphism correspondents, then* $\mathrm{Hom}_A(-, M)$ *and* $\mathrm{Hom}_B(-, M)$ *give inverse antiequivalences between* $\mathrm{cogen}^1({}_AM)$ *and* $\mathrm{cogen}^1({}_BM)$.

Proof. In view of (b') in the second proposition of the last section, and the symmetrical role of A and B, it suffices to show that if $X \in \mathrm{cogen}^1({}_AM)$, then $\mathrm{Hom}_A(X, M) \in \mathrm{cogen}^1({}_BM)$. Take a free presentation of ${}_AX$, say $A^m \to A^n \to X \to 0$. Applying $\mathrm{Hom}_A(-, M)$ gives an exact sequence

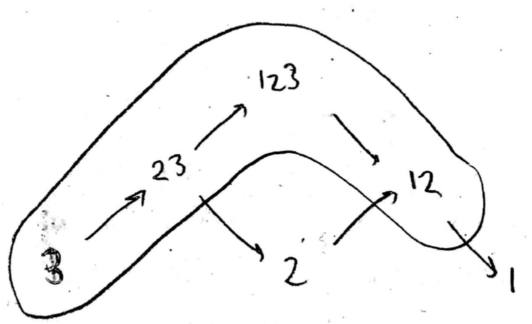
$$0 \to \mathrm{Hom}_A(X, M) \to M^n \to M^m.$$

Applying $\mathrm{Hom}_B(-, M)$ to this gives

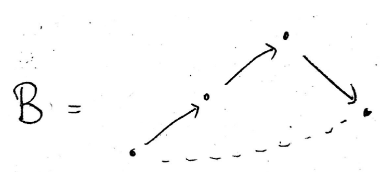
$$A^m \to A^n \to \mathrm{Hom}_B(\mathrm{Hom}_A(X, M), M) \to 0$$

which is isomorphic to the original exact sequence, so exact. Thus $\mathrm{Hom}_A(X, M) \in \mathrm{cogen}^1({}_BM)$. □

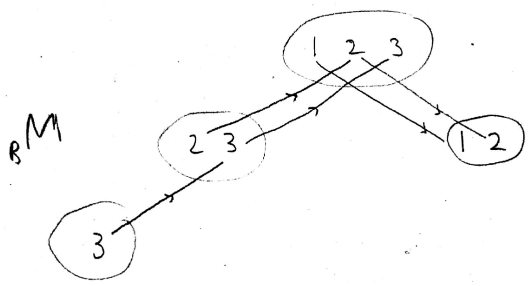
Example. Let A be the path algebra of the linear quiver $Q = 1 \to 2 \to 3$. We display its AR quiver below. Let ${}_AM$ be the direct sum of the circled indecomposables.



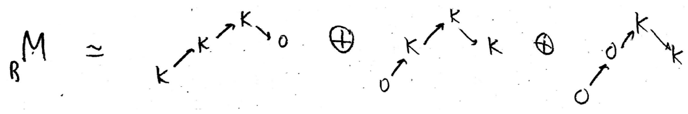
The endomorphism algebra of ${}_AM$ is



Considering M as a B-module, means to consider it as a representation of this quiver. The vector space at each vertex is the corresponding indecomposable A-module. In this example, the indecomposable A-modules are at most one-dimensional at each vertex of Q. In the following diagram we write i for the natural basis element at vertex i of Q. The arrows in the quiver for B correspond to homomorphisms of the indecomposable A-modules, and act on the basis elements as indicated below.



Thus



Observe that $_AM$ has all of the projective A-modules as summands, but not all injectives, so $_AM$ is a generator but not a cogenerator. On the other hand all of the summands of $_BM$ are projective, and one summand is not injective.

Proposition. *Let (A, M) and (B, M) be f.b. pairs which are endomorphism correspondents. Then:*

(a) $_AM$ is a generator iff $_BM$ is projective.

(b) $_AM$ is a cogenerator iff $_BM$ is injective.

(c) $A \in \operatorname{cogen}^n(_AM)$ iff $\operatorname{Ext}^i_B(M, M) = 0$ for $0 < i < n$.

Proof. (a) If $_AM$ is a generator, then $A \in \operatorname{add}(_AM)$, so $_BM \cong \operatorname{Hom}_A(A, M) \in \operatorname{add}(\operatorname{Hom}_A(M, M)) = \operatorname{add}(B)$, so $_BM$ is projective.

Conversely if $_BM$ is projective, then $_BM \in \operatorname{add}(B)$, so $A \cong \operatorname{Hom}_B(M, M) \in \operatorname{add}(\operatorname{Hom}_B(B, M)) = \operatorname{add}(_AM)$.

(b) Apply (a) to DM.

(c) Second proposition in last section with $X = A$. □

For (a), see Azumaya, Completely faithful modules and self-injective rings, Nagoya Math. J. 1966. Also (a) is similar to the Wedderburn correspondence introduced by Auslander, Representation theory of Artin algebras I, Comm. Algebra 1974.

For things similar to (b), see T. Kato, Rings of U-dominant dimension ≥ 1, Tohoku Math. J. 1969.

(c) is essentially Müller, The classification of algebras by dominant dimension, Canad. J. Math 1968.

See also:

B. Ma and J. Sauter, On faithfully balanced modules, F-cotilting and F-Auslander algebras, Journal of Algebra 2020.

M. Pressland and J. Sauter, On quiver Grassmannians and orbit closures for gen-finite modules, Algebras and Representation Theory 2022

4.3 Dominant dimension and Auslander correspondence

Definition. Given an algebra A, we take the minimal injective resolution

$$0 \to A \to I^0 \to I^1 \to \dots$$

of the module ${}_A A$. We say that A has *dominant dimension* $\geq n$ if $I^0, \dots, I^{n-1}$ are projective. This defines $\operatorname{dom.dim} A \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

Recall that an algebra A is *QF-3* if it A has a faithful projective-injective module M. If so, then $\operatorname{add}(M) = \mathcal{P}_A \cap \mathcal{I}_A$, since any indecomposable projective-injective module embeds in A, so in some M^n, so is in $\operatorname{add}(M)$. Thus M is unique, up to multiplicities, since it is the direct sum of all indecomposable projective-injective modules, each with some non-zero multiplicity.

Proposition. *(i)* $\operatorname{dom.dim} A \geq 1$ *iff A is QF-3.*
(ii) $\operatorname{dom.dim} A \geq 2$ *iff A has a f.b. projective-injective M.*

Proof. (i) If A is QF-3, with faithful projective-injective module M, then there is an embedding $A \to M^n$, and then the injective envelope of A is a direct summand of M^n, so it is projective.

(ii) If $\operatorname{dom.dim} A \geq 2$, there is an exact sequence $0 \to A \to I^0 \to I^1$ with I^0, I^1 projective-injective. Let M be the direct sum of the indecomposable projective-injectives, then $A \in \operatorname{cogen}^1(M)$ by condition (a) in the characterization of $\operatorname{cogen}^1(M)$. Thus M is f.b.

Conversely suppose A has a f.b. projective-injective M. Since it is f.b., $A \in \operatorname{cogen}^1(M)$. By the characterization of this means that there is an exact sequence

$$0 \to A \xrightarrow{\theta} M^0 \to M^1$$

with $M^0, M^1 \in \operatorname{add}(M)$. Moreover by the dual result to the remark at the end of the proof of (b)⇒(a) in Proposition (1) in the first subsection, we may suppose that the maps in this exact sequence are left minimal. Now the M^i are projective-injective, so they are injective, so this is the start of the injective resolution of A. Thus $I^0 \cong M^0$ and $I^1 \cong M^1$ are projective. Thus $\operatorname{dom.dim} B \geq 2$. □

For the following, see C. M. Ringel, Artin algebras of dominant dimension at least 2, manuscript 2007, available from his Bielefeld homepage.

Theorem (Morita-Tachikawa correspondence)**.** *Endomorphism correspondence gives a 1:1 correspondence between equivalence classes of pairs* (A, M) *where* ${}_AM$ *is a generator-cogenerator and Morita equivalence classes of algebras* B *with* $\operatorname{dom.dim} B \geq 2$.

The correspondence sends (A, M) *to* $B = \operatorname{End}_A(M)$*, and it sends* B *to* $A = \operatorname{End}_B(M)$ *where* ${}_BM$ *is the faithful projective-injective* B*-module.*

Proof. By endomorphism correspondence, the pairs (A, M) are in 1:1 correspondence with f.b. pairs (B, M) with M projective-injective. By the discussion above, these are in 1:1 correspondence with the Morita equivalence classes of algebras B with $\operatorname{dom.dim} B \geq 2$. □

The following correspondence comes from Auslander, Representation dimension of Artin algebras, Queen Mary College Lecture Notes, 1971. See also Auslander, Representation theory of Artin algebras II, Comm. Algebra 1974.

Theorem (Auslander correspondence)**.** *There is a 1-1 correspondence between algebras* A *of finite representation type up to Morita equivalence and algebras* B *with* $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$ *up to Morita equivalence.*

The correspondence sends A *to* $B = \operatorname{End}_A(M)$ *where* ${}_AM$ *is the direct sum of all the indecomposable* A*-modules, and it sends* B *to* $A = \operatorname{End}_B(M)$ *where* ${}_BM$ *is the faithful projective-injective* B*-module.*

The algebra B *is called the* Auslander algebra *of* A*.*

Proof. We show that under endomorphism correspondence, pairs (A, M) where $\operatorname{add}(M) = A\text{-mod}$ correspond to pairs (B, M) where $\operatorname{gl.dim} B \leq 2 \leq \operatorname{dom.dim} B$ and ${}_BM$ is the faithful projective-injective.

Suppose $\operatorname{add}(M) = A\text{-mod}$. Given a B-module Z, choose a projective presentation

$$P_1 \xrightarrow{f} P_0 \to Z \to 0.$$

Applying $\operatorname{Hom}_B(-, M)$ gives an exact sequence

$$0 \to \operatorname{Hom}_B(Z, M) \to \operatorname{Hom}_B(P_0, M) \xrightarrow{g} \operatorname{Hom}_B(P_1, M) \to \operatorname{Coker}(g) \to 0.$$

Applying $\operatorname{Hom}_A(-, M)$ we get a commutative diagram with bottom row exact

$$\begin{array}{ccccccc} & & & & P_1 & \xrightarrow{f} & P_0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Hom}_A(\operatorname{Coker}(g), M) & \longrightarrow & \operatorname{Hom}_A(\operatorname{Hom}_B(P_1, M), M) & \longrightarrow & \operatorname{Hom}_A(\operatorname{Hom}_B(P_0, M), M) \end{array}$$

The two vertical maps are isomorphisms, so $\operatorname{Ker}(f) \cong \operatorname{Hom}_A(\operatorname{Coker}(g), M)$. Now $\operatorname{Coker}(g)$ is an A-module, so in $\operatorname{add}(M)$, so as a B-module, we have

$$\operatorname{Hom}_A(\operatorname{Coker}(g), M) \in \operatorname{add}(\operatorname{Hom}_A(M, M)) = \operatorname{add}(B),$$

so it is projective. Thus $\operatorname{proj.dim} Z \le 2$. Thus $\operatorname{gl.dim} B \le 2$.

Conversely suppose $\operatorname{gl.dim} B \le 2 \le \operatorname{dom.dim} B$. If Y is an A-module, it has a projective resolution starting

$$P_1 \to P_0 \to Y \to 0.$$

Applying $\operatorname{Hom}_A(-, M)$ we get an exact sequence of B-modules

$$0 \to \operatorname{Hom}_A(Y, M) \to \operatorname{Hom}_A(P_0, M) \to \operatorname{Hom}_A(P_1, M).$$

The $\operatorname{Hom}_A(P_i, M)$ are projective B-modules since they are in $\operatorname{add}({}_B M)$, and ${}_B M$ is projective. Thus, since $\operatorname{gl.dim} B \le 2$, by dimension shifting we see that $\operatorname{Hom}_A(Y, M)$ is a projective B-module. Now ${}_B M$ is injective, so applying $\operatorname{Hom}_B(-, {}_B M)$ gives an exact sequence

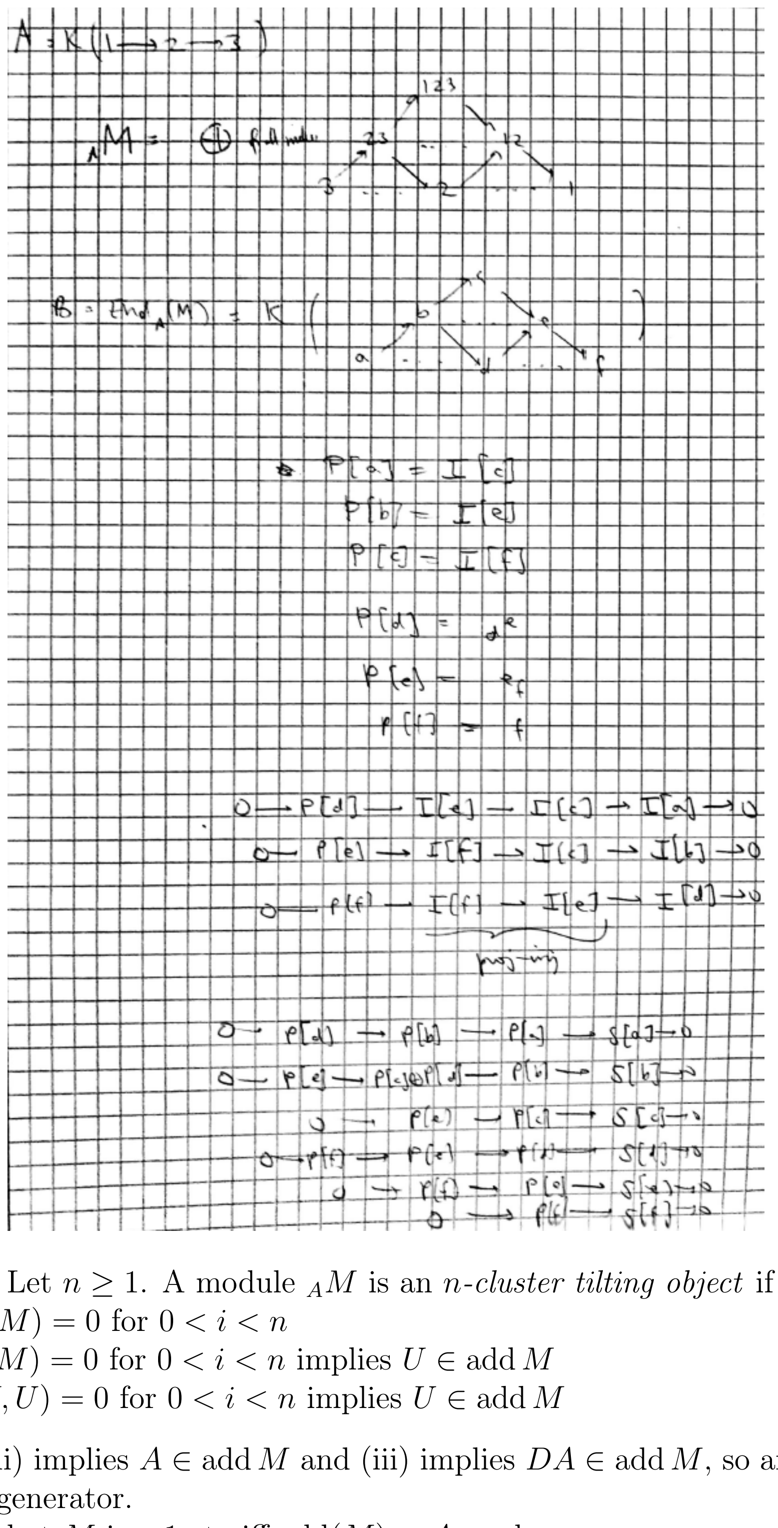
$$\operatorname{Hom}_B(\operatorname{Hom}_A(P_1, M), M) \to \operatorname{Hom}_B(\operatorname{Hom}_A(P_1, M), M) \to \operatorname{Hom}_B(\operatorname{Hom}_A(Y, M), M) \to 0.$$

For any A-module X there is a natural transformation from X to $\operatorname{Hom}_B(\operatorname{Hom}_A(X, M), M)$, and this is an isomorphism for X projective. We deduce that

$$Y \cong \operatorname{Hom}_B(\operatorname{Hom}_A(Y, M), M) \in \operatorname{add}(\operatorname{Hom}_B(B, M)) = \operatorname{add}({}_A M),$$

so $\operatorname{add}(M) = A\text{-mod}$. □

Example. We can check $\operatorname{gl.dim} B = 2 = \operatorname{dom.dim} B$ for the Auslander algebra of the linear quiver with three vertices.



Definition. Let $n \geq 1$. A module ${}_AM$ is an *n-cluster tilting object* if
(i) $\mathrm{Ext}^i_A(M, M) = 0$ for $0 < i < n$
(ii) $\mathrm{Ext}^i_A(U, M) = 0$ for $0 < i < n$ implies $U \in \operatorname{add} M$
(iii) $\mathrm{Ext}^i_A(M, U) = 0$ for $0 < i < n$ implies $U \in \operatorname{add} M$

Clearly (ii) implies $A \in \operatorname{add} M$ and (iii) implies $DA \in \operatorname{add} M$, so any n-cto is a generator-cogenerator.

Observe that M is a 1-cto iff $\operatorname{add}(M) = A$-mod.

Example. For the algebra with quiver

$$0 \to 1 \to 2 \to \cdots \to n$$

with all paths of length 2 zero, the module $S[0]$ has projective resolution

$$0 \to P[n] \to P[n-1] \to \cdots \to P[1] \to P[0] \to S[0] \to 0$$

so $\dim \operatorname{Ext}^i(S[0], S[j]) = \delta_{ij}$. It follows that

$$M = S[0] \oplus P[0] \oplus \cdots \oplus P[n-1] \oplus P[n] \cong I[0] \oplus I[1] \oplus \cdots \oplus I[n] \oplus S[n]$$

is an n-cto. It's endomorphism algebra B is the path algebra of the quiver

$$n \to \cdots \to 1 \to 0 \to *$$

with all paths of length 2 zero. It has global dimension $n+1$. The projectives $P[n], \dots, P[0]$ are injective, and $P[*]$ has injective resolution

$$0 \to P[*] \to I[*] \to I[0] \to \cdots \to I[n-1] \to I[n] \to 0.$$

Now $I[*] \cong P[0]$, $I[0] \cong P[1]$, ..., $I[n-1] \cong P[n]$ and $I[n] \cong S[n]$ is not projective, so $\operatorname{dom.dim} B = n+1$.

The following generalization of Auslander correspondence is due to Iyama, Auslander correspondence, Advances in Math. 2007.

Theorem (Iyama)**.** *There is a 1:1 correspondence between equivalence classes of pairs (A, M) where ${}_AM$ is an n-cto and Morita equivalence classes of algebras B with* $\operatorname{gl.dim} B \leq n+1 \leq \operatorname{dom.dim} B$.

Proof. **(To be omitted.)** We are in the setting of Morita-Tachikawa correspondence.

Now $\operatorname{Ext}^i_A(M, M) = 0$ for $1 < i < n$ corresponds to $B \in \operatorname{cogen}^n({}_BM)$, and since ${}_BM$ is the faithful projective-injective, this corresponds to $\operatorname{dom.dim} B \geq n+1$.

Suppose $\operatorname{gl.dim} B \leq n+1$.

We show that if $\operatorname{Ext}^i_A(U, M) = 0$ for $0 < i < n$ then $U \in \operatorname{add} M$. Take the start of a projective resolution of U, say

$$P_n \to \cdots \to P_0 \to U \to 0.$$

Applying $\operatorname{Hom}_A(-, M)$ gives a complex

$$0 \to \operatorname{Hom}_A(U, M) \to \operatorname{Hom}_A(P_0, M) \to \cdots \to \operatorname{Hom}_A(P_n, M)$$

which is exact because the Exts vanish. Since ${}_BM$ is injective, applying $\mathrm{Hom}_B(-,M)$ gives an exact sequence

$$\mathrm{Hom}_B(\mathrm{Hom}_A(P_n,M),M) \to \cdots \to \mathrm{Hom}_B(\mathrm{Hom}_A(P_0,M),M) \to \mathrm{Hom}_B(\mathrm{Hom}_A(U,M),M) \to 0.$$

Now the maps $P_i \to \mathrm{Hom}_B(\mathrm{Hom}_A(P_i,M),M)$ are isomorphisms since $P_i \in \mathrm{add}\, M$. Thus the map $U \to \mathrm{Hom}_B(\mathrm{Hom}_A(U,M),M)$ is an iso (so $U \in \mathrm{cogen}^1({}_AM)$). Also $\mathrm{Hom}_A(P_i,M) \in \mathrm{add}(\mathrm{Hom}_A(A,M)) = \mathrm{add}({}_BM)$. Thus, since $\mathrm{gl.dim}\, B \leq n+1$, the B-module $\mathrm{Hom}_A(U,M)$ must be projective, so it is in $\mathrm{add}({}_BB)$, and then $U \cong \mathrm{Hom}_B(\mathrm{Hom}_A(U,M),M) \in \mathrm{add}(\mathrm{Hom}_B(B,M)) = \mathrm{add}({}_AM)$.

Next we show that if $\mathrm{Ext}^i_A(M,U) = 0$ for $0 < i < n$ then $U \in \mathrm{add}\, M$. Take the start of an injective resolution of U, say

$$0 \to U \to I^0 \to \cdots \to I^n.$$

Applying $\mathrm{Hom}_A(M,-)$ gives a complex

$$0 \to \mathrm{Hom}_A(M,U) \to \mathrm{Hom}_A(M,I^0) \to \cdots \to \mathrm{Hom}_A(M,I^n)$$

which is exact because the Exts vanish. Since ${}_BM$ is projective, applying $- \otimes_B M$ gives an exact sequence

$$0 \to \mathrm{Hom}_A(M,U) \otimes_B M \to \mathrm{Hom}_A(M,I^0) \otimes_B M \to \cdots \to \mathrm{Hom}_A(M,I^n) \otimes_B M.$$

Now the maps $I^i \to \mathrm{Hom}_A(M,I^i) \otimes_B M$ are isomorphisms since $I^i \in \mathrm{add}\, M$. Thus the map $U \to \mathrm{Hom}_A(M,U) \otimes_B M$ is an iso. Also $\mathrm{Hom}_A(M,I^i) \in \mathrm{add}(\mathrm{Hom}_A(M,M)) = \mathrm{add}(B_B)$. Thus, since $\mathrm{gl.dim}\, B \leq n+1$, the right B-module $\mathrm{Hom}_A(M,U)$ must be projective, so it is in $\mathrm{add}(B_B)$, and then $U \cong \mathrm{Hom}_A(M,U) \otimes_B M \in \mathrm{add}(B \otimes_B M) = \mathrm{add}({}_AM)$.

Now suppose that M is an n-cto. Given a B-module Z, choose a projective presentation

$$P_1 \xrightarrow{f} P_0 \to Z \to 0.$$

Applying $\mathrm{Hom}_B(-,M)$ gives an exact sequence

$$0 \to \mathrm{Hom}_B(Z,M) \to \mathrm{Hom}_B(P_0,M) \xrightarrow{g} \mathrm{Hom}_B(P_1,M) \to \mathrm{Coker}(g) \to 0.$$

Let $C^0 = \mathrm{Coker}(g)$. Applying $\mathrm{Hom}_A(-,M)$ we get a commutative diagram with bottom row exact

$$\begin{array}{ccccccc} & & & & P_1 & \xrightarrow{f} & P_0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_A(C^0,M) & \longrightarrow & \mathrm{Hom}_A(\mathrm{Hom}_B(P_1,M),M) & \longrightarrow & \mathrm{Hom}_A(\mathrm{Hom}_B(P_0,M),M) \end{array}$$

The two vertical maps are isomorphisms, so $\mathrm{Ker}(f) \cong \mathrm{Hom}_A(C^0, M)$.

Now since M is a cogenerator, by repeatedly taking left M-approximations we can get an exact sequence

$$0 \to C^0 \to M^0 \to \cdots \to M^{n-2}$$

such that the sequence

$$\mathrm{Hom}_A(M^{n-2}, M) \to \cdots \to \mathrm{Hom}_A(M^0, M) \to \mathrm{Hom}_A(C^0, M) \to 0$$

is exact. Let C^i be the cosyzygies for this sequence, so

$$0 \to C^i \to M^i \to C^{i+1} \to 0.$$

Then

$$\mathrm{Hom}(M^i, M) \twoheadrightarrow \mathrm{Hom}(C^i, M) \to \mathrm{Ext}^1(C^{i+1}, M) \to \mathrm{Ext}^1(M^i, M) = 0 \to \dots,$$

so by dimension shifting

$$\mathrm{Ext}^{n-1}(C^{n-1}, M) \cong \mathrm{Ext}^{n-2}(C^{n-1}, M) \cong \ldots \cong \mathrm{Ext}^1(C^1, M) = 0$$

and similarly $\mathrm{Ext}^i(C^{n-1}, M) = 0$ for $0 < i < n$. Thus $C^{n-1} \in \operatorname{add} M$. Thus Z has projective resolution

$$0 \to \mathrm{Hom}_A(C^{n-1}, M) \to \mathrm{Hom}_A(M^{n-2}, M) \to \cdots \to \mathrm{Hom}_A(M^0, M) \to P_1 \to P_0 \to Z \to 0.$$

Thus $\operatorname{proj.dim} Z \leq n+1$. Thus $\operatorname{gl.dim} B \leq n+1$. □

4.4 Homological conjectures for f.d. algebras

Let $0 \to A \to I^0 \to I^1 \to \dots$ be the minimal injective resolution of a f.d. algebra A. Recall that A has dominant dimension $\geq n$ if $I^0, \dots, I^{n-1}$ are all projective.

Conjecture (Nakayama conjecture 1958)**.** *If all I^n are projective, i.e.* $\operatorname{dom.dim} A = \infty$*, then A is self-injective.*

Proposition. *The following are equivalent.*
(i) The Nakayama conjecture (if $\operatorname{dom.dim} B = \infty$ *then B is self-injective).*
(ii) If ${}_AM$ is a generator-cogenerator and $\mathrm{Ext}^i_A(M, M) = 0$ *for all $i > 0$ then M is projective.*

Proof. (i) implies (ii). Say ${}_AM$ satisfies the hypotheses. Let (B, M) be the endomorphism correspondent. Then ${}_BM$ is projective-injective and $B \in \operatorname{cogen}^n(M)$ for all n. Thus for all n there is an exact sequence

$$0 \to B \to I^0 \to \cdots \to I^n$$

with the I^i projective-injective. Thus $\operatorname{dom.dim} B = \infty$. Thus B is self-injective, so $\operatorname{add}(M) = \operatorname{add}(B)$, so ${}_BM$ is a generator, so ${}_AM$ is projective.

(ii) implies (i). Say $\operatorname{dom.dim} B = \infty$. Thus B is QF-3 and let ${}_BM$ be the faithful projective-injective module. Let ${}_AM$ be the endomorphism correspondent. It is a generator-cogenerator. Then $B \in \operatorname{cogen}^n(M)$ for all n, so $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$. Thus by (ii), ${}_AM$ is projective, so ${}_BM$ is a generator. Thus $B \in \operatorname{add}(M)$ is injective. □

Conjecture (Generalized Nakayama conjecture, Auslander and Reiten 1975)**.** *For any f.d. algebra A, every indecomposable injective occur as a summand of some I^n.*

It clearly implies the Nakayama conjecture, for if the I^n are projective, and each indecomposable injective occurs as a summand of some I^n, then the indecomposable injectives are projective.

Example. For the commutative square, vertices 1(source),2,3,4(sink). There are injective resolutions

$$\begin{aligned}
0 \to &P[1] \to I[4] \to 0, \\
0 \to &P[2] \to I[4] \to I[3] \to 0, \\
0 \to &P[3] \to I[4] \to I[2] \to 0, \\
0 \to &P[4] \to I[4] \to I[2] \oplus I[3] \to I[1] \to 0,
\end{aligned}$$

so

$$0 \to A \to I[4]^4 \to I[2]^2 \oplus I[3]^2 \to I[1] \to 0,$$

so all indecomposable injectives occur.

Proposition. *The following are equivalent.*
(i) The generalized Nakayama conjecture (every indecomposable injective occurs as a summand of some I^i in the minimal injective resolution of B).
(ii) If ${}_AM$ is a cogenerator and $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$ then M is injective.

Proof. (i) implies (ii). Suppose ${}_AM$ satisfies the conditions. Then there is corresponding ${}_BM$ which is injective, and $B \in \operatorname{cogen}^n(M)$ for all n. Thus by (i) every indecomposable injective is a summand of ${}_BM$. Thus ${}_BM$ is a cogenerator. Thus ${}_AM$ is injective.

(ii) implies (i). Let ${}_BM$ be the sum of all indecomposable injectives occuring in the I^i. Then $B \in \mathrm{cogen}^n(M)$ for all n. Let ${}_AM$ be the endomorphism correspondent. Then ${}_AM$ is a cogenerator and $\mathrm{Ext}^i_A(M,M) = 0$ for all $i > 0$. Thus by (ii) ${}_AM$ is injective. Thus ${}_BM$ is a cogenerator. Thus all indecomposable injectives occur as a summand of ${}_BM$. □

For the next conjecture, see Happel, Selforthogonal modules, 1995.

Conjecture (Boundedness Conjecture)**.** *If M is an A-module with $\mathrm{Ext}^i_A(M,M) = 0$ for all $i > 0$ then $\#M \leq \#A$, where $\#M$ denotes the number of non-isomorphic indecomposable summands of M.*

Since a cogenerator has all indecomposable injectives as summands, the boundedness conjecture implies the generalized Nakayama conjecture.

Definition. An algebra A is *(Iwanaga) Gorenstein* if both $\mathrm{inj.dim}\, {}_AA < \infty$ and $\mathrm{inj.dim}\, A_A < \infty$.

In Auslander and Reiten, Applications of contravariantly finite subcategories, Adv. Math 1991, one finds:

Conjecture (Gorenstein Symmetry Conjecture)**.** *If one of* $\mathrm{inj.dim}\, {}_AA$ *and* $\mathrm{inj.dim}\, A_A$ *is finite, so is the other.*

Lemma. *(i) If* $\mathrm{inj.dim}\, {}_AA = n < \infty$, *any A-module has* $\mathrm{proj.dim}\, M \leq n$ *or* ∞.
(ii) If $\mathrm{inj.dim}\, {}_AA = n$ *and* $\mathrm{inj.dim}\, A_A = m$ *are both finite, they are equal.*

For example, by (i) every non-projective module for a self-injective algebra has infinite projective dimension.

Proof. (i) Say $\mathrm{proj.dim}\, M = i < \infty$. There is some N with $\mathrm{Ext}^i(M,N) \neq 0$. Choose $0 \to L \to P \to N \to 0$ with P projective. The long exact sequence for $\mathrm{Hom}(M,-)$ gives

$$\dots \to \mathrm{Ext}^i(M,P) \to \mathrm{Ext}^i(M,N) \to \mathrm{Ext}^{i+1}(M,L) \to \dots$$

Now $\mathrm{Ext}^{i+1}(M,L) = 0$, so $\mathrm{Ext}^i(M,P) \neq 0$, so $\mathrm{Ext}^i(M,A) \neq 0$, so $i \leq n$.
(ii) $\mathrm{proj.dim}\, {}_ADA = \mathrm{inj.dim}\, A_A = m$, so $m \leq n$ by (i). Dually $m \geq n$. □

This also holds for noetherian rings, see Zaks, Injective dimension of semiprimary rings, J. Alg. 1969.

For the following, see H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 1960.

Conjecture (Finitistic Dimension Conjecture). *For any f.d. algebra A,*

$$\text{fin.}\dim A = \sup\{\text{proj.}\dim M \mid \text{proj.}\dim M < \infty\}$$

is finite.

For example if A is Gorenstein, with $\text{inj.}\dim {}_A A = n = \text{inj.}\dim A_A$, then $\text{fin.}\dim A = n$. For the lemma implies that any A-module M has $\text{proj.}\dim M \leq n$ or ∞, and $\text{proj.}\dim D(A_A) = n$.

Note that $\text{fin.}\dim A$ is not necessarily the same as the maximum of the projective dimensions of the simple modules of finite projective dimension.

There is also a big finitistic dimension, where the modules need not be finite-dimensional, and this may also always be finite.

Proposition. *The finitistic dimension conjecture implies the Gorenstein symmetry conjecture.*

Proof. Assuming $\text{inj.}\dim A_A = n < \infty$, we want to prove that $\text{inj.}\dim {}_A A < \infty$. We have $\text{proj.}\dim {}_A DA = n < \infty$. Thus any injective module has projective dimension $< \infty$. Take a minimal injective resolution $0 \to {}_A A \to I^0 \to \dots$. We show by induction on i that $\text{proj.}\dim \Omega^i A < \infty$. There is an exact sequence

$$0 \to \Omega^{i-1} A \to I^{i-1} \to \Omega^i A \to 0.$$

Applying $\text{Hom}_A(-, X)$ for a module X gives a long exact sequence

$$\dots \to \text{Ext}^m(\Omega^{i-1} A, X) \to \text{Ext}^{m+1}(\Omega^i A, X) \to \text{Ext}^{m+1}(I^{i-1}, X) \to \dots$$

For m sufficiently large, independent of X, the outside terms are zero, hence so is the middle.

Let $i > 0$. If $\Omega^i A = 0$, or is injective, then $\text{inj.}\dim {}_A A < \infty$, as desired, so suppose otherwise. Let $f : \Omega^i A \to I^i$ be the inclusion. Then f belongs to the middle term in the complex

$$\text{Hom}(\Omega^i A, I^{i-1}) \to \text{Hom}(\Omega^i A, I^i) \to \text{Hom}(\Omega^i A, I^{i+1})$$

and it is sent to zero in the third term. Now f is not in the image of the map from the first term, for otherwise the map $I^{i-1} \to \Omega^i A$ is a split epimorphism, so $\Omega^i A$ is injective. Thus the homology of this complex at the middle term is non-zero. Thus $\text{Ext}^i(\Omega^i A, A) \neq 0$. Thus $\text{proj.}\dim \Omega^i A \geq i$. This contradicts that $\text{fin.}\dim A < \infty$. $\square$

Proposition. *The finitistic dimension conjecture implies the generalized Nakayama conjecture.*

Proof. Assume the FDC. We show that if ${}_AM$ is a module and $\mathrm{Ext}^n(M, A) = 0$ for all $n \geq 0$ then $M = 0$ (the *strong Nakayama conjecture*).

If $S[i]$ is a simple A-module and $I[i]$ is its injective envelope, recall from section 1.10, that $\dim \mathrm{Ext}^n(S[i], A)$ is $\dim \mathrm{End}(S[i])$ times the multiplicity of $I[i]$ as a direct summand of I^n. Thus taking $M = S[i]$, the strong Nakayama conjecture gives the generalized Nakayama conjecture.

Take a minimal projective resolution $\to P_1 \to P_0 \to M \to 0$. By assumption the sequence

$$0 \to \mathrm{Hom}_A(P_0, A) \xrightarrow{f_0} \mathrm{Hom}_A(P_1, A) \xrightarrow{f_1} \mathrm{Hom}(P_2, A) \to \dots$$

of right A-modules is exact. Let $\mathrm{fin.dim}\, A^{op} = n < \infty$. Then $\mathrm{Coker}(f_n)$ has projective resolution

$$0 \to \mathrm{Hom}_A(P_0, A) \xrightarrow{f_0} \mathrm{Hom}_A(P_1, A) \to \dots \to \mathrm{Hom}_A(P_{n+1}, A) \to \mathrm{Coker}(f_n) \to 0$$

so it has finite projective dimension, so projective dimension $\leq n$, so by dimension shifting $\mathrm{Im}\, f_1$ is projective, so f_0 must be a split mono. But $\mathrm{Hom}_A(-, A)$ is an antiequivalence from $\mathcal{P}_A$ to $\mathcal{P}_{A^{op}}$. Thus the map $P_1 \to P_0$ must be a split epi, so $M = 0$. □

4.5 No loops conjecture

It is nice to see that some homological conjecture has been proved. In this section we do not assume that K is algebraically closed, but we do assume that $A = KQ/I$ with I admissible. The following conjecture was proved by Igusa, Notes on the no loops conjecture, J. Pure Appl. Algebra 1990.

Theorem (No loops conjecture)**.** *If* $\mathrm{gl.dim}\, A < \infty$ *then Q has no loops (that is,* $\mathrm{Ext}^1(S[i], S[i]) = 0$ *for all i).*

Proof. We use the trace function of Hattori and Stallings. I only sketch the proof of its properties.

(1) For any matrix $\theta \in M_n(A)$ we consider its trace $\mathrm{tr}(\theta) \in A/[A, A]$, where $[A, A]$ is the subspace of A spanned by the commutators $ab - ba$. This ensures that $\mathrm{tr}(\theta\phi) = \mathrm{tr}(\phi\theta)$. This equality holds also for $\theta \in M_{m\times n}(A)$ and $\phi \in M_{n\times m}(A)$.

(2) If P is a f.g. projective A-module it is a direct summand of a f.g. free module $F = A^n$. Let $p : F \to P$ and $i : P \to F$ be the projection and inclusion. One defines $\mathrm{tr}(\theta)$ for $\theta \in \mathrm{End}(P)$ to be $\mathrm{tr}(i\theta p)$. This is well defined, for if

$$A^n = F \underset{i}{\overset{p}{\rightleftarrows}} P \underset{p'}{\overset{i'}{\rightleftarrows}} F' = A^m$$

with $pi = 1_P = p'i'$, then $\operatorname{tr}(i\theta p) = \operatorname{tr}((ip')(i'\theta p)) = \operatorname{tr}((i'\theta p)(ip')) = \operatorname{tr}(i'\theta p')$.

(3) Any module M has a finite projective resolution $P_* \to M$, and an endomorphism θ of M lifts to a map between the projective resolutions

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_n} & & & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Define $\operatorname{tr}(\theta) = \sum_i (-1)^i \operatorname{tr}(\theta_i)$. One can show that does not depend on the projective resolution or the lift of θ, see section 4 of Lenzing, Nilpotente Elemente in Ringen von endlicher globaler Dimension, Math. Z. 1969.

(4) One can show that given a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta''} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

one has $\operatorname{tr}(\theta) = \operatorname{tr}(\theta') + \operatorname{tr}(\theta'')$.

(5) It follows that any nilpotent endomorphism has trace 0, since

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}\theta & \longrightarrow & M & \longrightarrow & M/\operatorname{Im}\theta & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta|_{\operatorname{Im}\theta}} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \operatorname{Im}\theta & \longrightarrow & M & \longrightarrow & M/\operatorname{Im}\theta & \longrightarrow & 0 \end{array}$$

so $\operatorname{tr}(\theta) = \operatorname{tr}(\theta|_{\operatorname{Im}\theta}) = \operatorname{tr}(\theta|_{\operatorname{Im}(\theta^2)}) = \dots = 0$.

(6) Thus any element of $J(A)$ as a map $A \to A$ has trace 0, so $J(A) \subseteq [A, A]$. Thus $(KQ)_+ \subseteq I + [KQ, KQ]$.

(7) Any loop of Q gives an element of $(KQ)_+$. But it is easy to see that

$$I + [KQ, KQ] \subseteq \text{span of arrows which are not loops} + (KQ)_+^2,$$

for example if p, q are paths then $[p, q] \in (KQ)_+^2$ unless they are trivial paths or one is trivial and the other is an arrow. Thus there are no loops. □

A strengthening (proved by Igusa, Liu and Paquette, A proof of the strong no loop conjecture, Adv. Math. 2011). If S is a 1-dimensional simple module for a f.d. algebra and S has finite injective or projective dimension, then $\operatorname{Ext}^1(S, S) = 0$.

An open problem (stated by Liu and Morin, The strong no loop conjecture for special biserial algebras, Proc. Amer. Math. Soc. 2004). The extension conjecture: if S is simple module for a f.d. algebra and $\operatorname{Ext}^1(S, S) \neq 0$ then $\operatorname{Ext}^n(S, S) \neq 0$ for infinitely many n.

5 Tilting theory

In order to give their proof of Gabriel's theorem, Bernstein, Gelfand and Ponomarev introduced some reflection functors.

If Q is a quiver and i is a sink (no arrows out), so that $P[i] = S[i]$, let Q' be the quiver obtained by reversing all arrows incident at i. Then reflection functors are functors

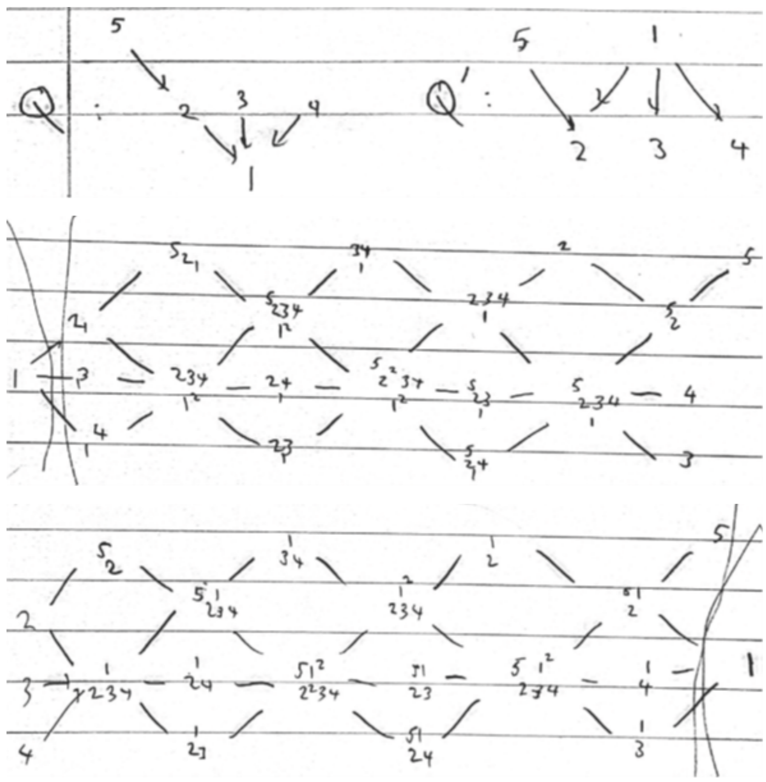
$$KQ\text{-mod} \rightleftarrows KQ'\text{-mod}$$

sending a representation X of Q to the representation X' of Q' which is the same, except that

$$X'_i = \mathrm{Ker}(\bigoplus_{a:j\to i} X_j \to X_i)$$

and the linear map $X'_i \to X_j$ is the canonical map.

This gives an equivalence between the module classes in KQ-mod and KQ'-mod given by the modules with no summand $S[i]$. For example.

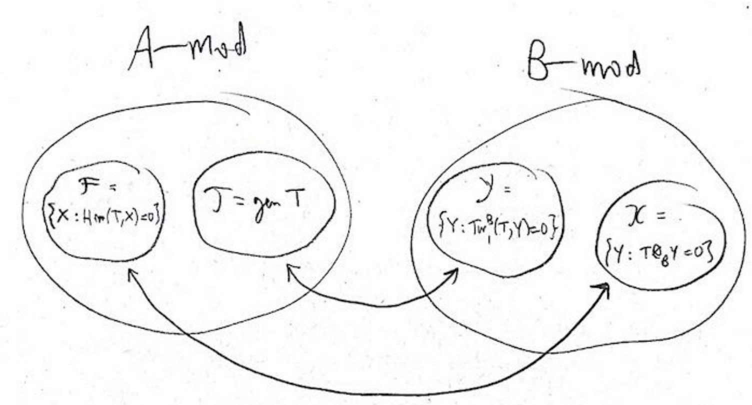


Brenner and Butler generalized this with the notion of a tilting module. Let A be an algebra. An A-module T is a *tilting module* if

- $\mathrm{proj.\,dim}\, T \leq 1$.
- $\mathrm{Ext}^1_A(T, T) = 0$.

- $\#T = \#A$, the number of non-isomorphic summands of T is the number of simple A-modules.

Let $B = \mathrm{End}_A(T)^{op}$, so T becomes an A-B-bimodule. The Brenner-Butler theorem gives equivalences between the following parts of the module categories.



5.1 Torsion theories and tau-rigid modules

The notion of a torsion theory comes from Dickson, A torsion theory for abelian categories, Trans. Amer. Math. Soc. 1966.

Definition. A *torsion theory* in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$, the *torsion* and *torsion-free* classes, such that
(i) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$.
(ii) Any object X has a subobject $t_{\mathcal{T}}X \in \mathcal{T}$ with $X/t_{\mathcal{T}}X \in \mathcal{F}$ (so it fits in an exact sequence $0 \to t_{\mathcal{T}}X \to X \to X/t_{\mathcal{T}}X \to 0$ with first term in $\mathcal{T}$ and last term in $\mathcal{F}$).

Examples. (1) The torsion and torsion-free modules give a torsion theory in the category of $\mathbb{Z}$-modules.

(2) For A the path algebra of the quiver $1 \to 2$, A-mod has torsion theory $(\mathrm{add}\, S[2], \mathrm{add}\, S[1])$.

Notation. For an a set $\mathcal{C}$ of modules in A-mod or more generally of objects in an abelian category

$$\mathcal{C}^{\perp i,j,\dots} = \{X : \mathrm{Ext}^n(M, X) = 0 \text{ for all } M \in \mathcal{C} \text{ and } n = i, j, \dots\},$$

$$^{\perp i,j,\dots}\mathcal{C} = \{X : \mathrm{Ext}^n(X, M) = 0 \text{ for all } M \in \mathcal{C} \text{ and } n = i, j, \dots\}.$$

Recall that $\mathrm{Ext}^0 = \mathrm{Hom}$.

Properties. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory.

(i) $\mathcal{T} = {}^{\perp 0}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp 0}$ so either of the classes determines the other.

(ii) $\mathcal{T}$ is closed under quotients and extensions; $\mathcal{F}$ is closed under subobjects and extensions.

(iii) The subobject $t_{\mathcal{T}}X$ is uniquely determined, and the assignment sending X to $t_{\mathcal{T}}X$ defines a functor $\mathcal{A} \to \mathcal{T}$ which is a right adjoint to the inclusion $\mathcal{T}$ in $\mathcal{A}$. The assignment sending X to $X/t_{\mathcal{T}}X$ defines a functor $\mathcal{A} \to \mathcal{F}$ which is a left adjoint to the inclusion $\mathcal{F}$ in $\mathcal{A}$.

Proof. (i) If $X \in \mathcal{T}^{\perp 0}$, then $\mathrm{Hom}(\mathcal{T}, X) = 0$, so we must have $t_{\mathcal{T}}X = 0$, so $X \cong X/t_{\mathcal{T}}X \in \mathcal{F}$. If $X \in {}^{\perp 0}\mathcal{F}$, then $\mathrm{Hom}(X, \mathcal{F}) = 0$, so we must have $X = t_{\mathcal{T}}X \in \mathcal{T}$.

For (ii), for $\mathcal{T}$ given an exact sequence $0 \to X \to Y \to Z \to 0$, apply $\mathrm{Hom}(-, F)$ for $F \in \mathcal{F}$ to get an exact sequence

$$0 \to \mathrm{Hom}(Z, F) \to \mathrm{Hom}(Y, F) \to \mathrm{Hom}(X, F).$$

Now if $X, Z \in \mathcal{T}$, then $\mathrm{Hom}(X, F) = \mathrm{Hom}(Z, F) = 0$, so $\mathrm{Hom}(Y, F) = 0$, so $Y \in \mathcal{T}$. Also, if $Y \in \mathcal{T}$, then $\mathrm{Hom}(Y, F) = 0$, so $\mathrm{Hom}(Z, F) = 0$, so $Z \in \mathcal{T}$.

For (iii) observe that any map $\theta : X \to Y$ induces a map $t_{\mathcal{T}}X \to t_{\mathcal{T}}Y$ since the composition $t_{\mathcal{T}}X \to X \to Y \to Y/t_{\mathcal{T}}Y$ must be zero. □

Remark. A *splitting torsion theory* is one in which the sequence $0 \to t_{\mathcal{T}}X \to X \to X/t_{\mathcal{T}}X \to 0$ is always split exact.

If A is a f.d. algebra, a torsion theory in A-mod is splitting if and only if every indecomposable module is either torsion or torsion-free.

A splitting torsion theory is thus given by a partition of the indecomposable modules into two sets T, F with $\mathrm{Hom}(T, F) = 0$. Then $(\mathrm{add}\, T, \mathrm{add}\, F)$ is a splitting torsion theory in A-mod.

This is very easy to do if A is an algebra whose AR quiver is obtained by knitting, so A is of finite representation type and all of its indecomposable modules are directing. We want there to be no irreducible maps from T to F.

Proposition. *If A is a f.d. algebra, for a module class $\mathcal{T}$ in A-mod the following are equivalent.*

(i) $\mathcal{T}$ is a torsion class for some torsion theory in A-mod.

(ii) $\mathcal{T} = {}^{\perp 0}(\mathcal{T}^{\perp 0})$.

(iii) $\mathcal{T} = {}^{\perp 0}\mathcal{C}$ for some module class $\mathcal{C}$.

(iv) $\mathcal{T}$ is closed under quotients and extensions.

Proof. (i) implies (ii) implies (iii) implies (iv). Straightforward.

(iv) implies (i). Define $\mathcal{F} = \mathcal{T}^{\perp 0}$. Given any module X, let T be a submodule of X in $\mathcal{T}$ of maximal dimension. Then $\mathrm{Hom}(\mathcal{T}, X/T) = 0$, for if T'/T is the image of such a map, then T'/T is in $\mathcal{T}$, hence so is T', contradicting maximality. Thus $X/T \in \mathcal{F}$. □

Thus a pair of module classes $(\mathcal{T}, \mathcal{F})$ is a torsion theory in A-mod if and only if $\mathcal{T} = {}^{\perp 0}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp 0}$.

Lemma (Auslander-Smalø, 1981)**.** *For modules M, N, the following are equivalent:*
(i) $\mathrm{Hom}(N, \tau M) = 0$.
(ii) $\mathrm{Ext}^1(M, \mathrm{gen}\, N) = 0$ *(that is,* $\mathrm{Ext}^1(M, G) = 0$ *for all* $G \in \mathrm{gen}\, N$*).*

Proof. (i)⇒(ii). If $\mathrm{Hom}(N, \tau M) = 0$, then $\mathrm{Hom}(G, \tau M) = 0$ for all $G \in \mathrm{gen}\, N$), so $\overline{\mathrm{Hom}}(G, \tau M) = 0$, so $\mathrm{Ext}^1(M, G) = 0$ by the Auslander-Reiten formula.

(ii)⇒(i). Say $f : N \to \tau M$ is a non-zero map. Factorize it as a surjection $g : N \to G$ followed by a mono $h : G \to \tau M$. Suppose that h factors through an injective. Then it factors through the injective envelope $E(G)$ of G. Since τM has no injective summand, the induced map $E(G) \to \tau M$ cannot be injective, so its kernel is non-zero. Since G is essential in $E(G)$, the kernel meets G. Thus $G \to \tau M$ has non-zero kernel. Contradiction. Thus $\overline{\mathrm{Hom}}(G, \tau M) \neq 0$, so $\mathrm{Ext}^1(M, G) \neq 0$. □

Definition. Given a module class $\mathcal{C}$ in A-mod and $X \in \mathcal{C}$, we say that
(i) X is *Ext-projective* in $\mathcal{C}$ if $\mathrm{Ext}^1(X, \mathcal{C}) = 0$.
(ii) X is *Ext-injective* in $\mathcal{C}$ if $\mathrm{Ext}^1(\mathcal{C}, X) = 0$.

Lemma. *If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in A-mod, then*
(i) $X \in \mathcal{T}$ is Ext-projective for $\mathcal{T}$ iff $\tau X \in \mathcal{F}$.
(ii) $X \in \mathcal{F}$ is Ext-injective for $\mathcal{F}$ iff $\tau^- X \in \mathcal{T}$.
(iii) There are bijections

$$\text{Non-proj indec Ext-projs in } \mathcal{T} \text{ up to iso} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \text{Non-inj indec Ext-injs in } \mathcal{F} \text{ up to iso}$$

Proof. (i) Say $X \in \mathcal{T}$. Then $\tau X \in \mathcal{F} \Leftrightarrow \mathrm{Hom}(T, \tau X) = 0$ for all $T \in \mathcal{T} \Leftrightarrow \mathrm{Ext}^1(X, \mathrm{gen}\, T) = 0$ for all $T \in \mathcal{T} \Leftrightarrow X$ is Ext-projective.
(ii) is dual.
(iii) follows. □

Lemma. *If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in A-mod, then*

(i) The Ext-injectives for $\mathcal{T}$ are the modules $t_{\mathcal{T}} I$ with I injective. The indecomposable Ext-injectives are the modules $t_{\mathcal{T}} I[i]$ with $I[i] \notin \mathcal{F}$.

(ii) The Ext-projectives for $\mathcal{F}$ are the modules $P/t_{\mathcal{T}} P$ with P projective. The indecomposable Ext-projectives are the modules $P[i]/t_{\mathcal{T}} P[i]$ with $P[i] \notin \mathcal{T}$.

Proof. (i) $t_{\mathcal{T}} I$ is in $\mathcal{T}$, and it is Ext-injective since if $T \in \mathcal{T}$ and $0 \to t_{\mathcal{T}} I \to E \to T \to 0$ is an exact sequence, then the pushout along $t_{\mathcal{T}} I \to I$ splits, giving a map $E \to I$. But $E \in \mathcal{T}$, so it gives a map $E \to t_{\mathcal{T}} I$, which is a retraction for the given sequence.

Conversely suppose X is Ext-injective in $\mathcal{T}$ and $X \to I$ is its injective envelope. Then we have an injection $X \to t_{\mathcal{T}} I$. Since $\mathcal{T}$ is closed under quotients, all terms in the exact sequence $0 \to X \to t_{\mathcal{T}} I \to t_{\mathcal{T}} I / X \to 0$ are in $\mathcal{T}$. Thus this sequence splits, so X is a direct summand of $t_{\mathcal{T}} I$, and we have equality since X is essential in I.

Also, if $I[i] \notin \mathcal{F}$, then $t_{\mathcal{T}} I[i]$ is non-zero and contained in $I[i]$, so it has simple socle, so it is indecomposable.

(ii) is dual. □

The following definition comes from Adachi, Iyama and Reiten, τ-tilting theory, 2014.

Definition. A module M is *τ-rigid* if $\mathrm{Hom}(M, \tau M) = 0$. Dually, it is *$\tau^-$-rigid* if $\mathrm{Hom}(\tau^- M, M) = 0$

Note that M is τ-rigid iff DM is τ^--rigid, since

$$\mathrm{Hom}(M, \tau M) = \mathrm{Hom}(M, D\,\mathrm{Tr}\,M) \cong \mathrm{Hom}(\mathrm{Tr}\,M, DM)$$
$$\cong \mathrm{Hom}(\mathrm{Tr}\,DDM, DM) = \mathrm{Hom}(\tau^- DM, DM).$$

Proposition. *The following are equivalent*

(i) M is τ-rigid.

(ii) $\mathrm{Ext}^1(M, \mathrm{gen}\,M) = 0$.

(iii) $\mathrm{gen}\,M$ *is a torsion class and M is Ext-projective in* $\mathrm{gen}\,M$.

(iv) M is Ext-projective in some torsion class.

Proof. (i)⇔(ii). The lemma of Auslander and Smalø.

(ii)⇒(iii). Suppose M is τ-rigid. To show that $\mathrm{gen}\,M$ is a torsion class, it suffices to show that if $0 \to X \to Y \to Z \to 0$ is exact and $X, Z \in \mathrm{gen}\,M$, then so is Y. Choose a surjection $M^n \to Z$. By (ii) The pullback sequence splits, so the middle term of it is in $\mathrm{gen}\,M$, and hence so is Y. Now $\mathrm{Ext}^1(M, \mathrm{gen}\,M) = 0$, so M is Ext-projective.

(iii)⇒(iv). Trivial.

(iv)⇒(ii). If M is Ext-projective in $\mathcal{T}$, then $\mathrm{Ext}^1(M, \mathrm{gen}\,M) = 0$ since $\mathrm{gen}\,M \subseteq \mathcal{T}$. □

Note that the torsion theory given by a τ-rigid module M is $(\mathrm{gen}\,M, M^{\perp 0})$.

Example. Let A be the path algebra of $1 \to 2 \to 3$. Let $M = 2 \oplus 123$. It is τ-rigid. Then $\mathcal{T} = \mathrm{gen}\,M$ contains $123, 12, 2, 1$. The torsion-free class is $\mathcal{F} = \mathcal{T}^{\perp 0} = M^{\perp 0}$. It contains 3 and 23.

The Ext-projectives in $\mathcal{T}$ are $2, 12, 123$.

The Ext-injectives in $\mathcal{T}$ are $1, 12, 123$.

The Ext-projectives in $\mathcal{F}$ are $3, 23$.

The Ext-injectives in $\mathcal{F}$ are $3, 23$.

The next result is dual to Theorem 4.1(c) of Auslander and Smalø, Almost split sequences in subcategories, J. Algebra 1981.

Theorem. *Let $\mathcal{T}$ be a torsion class which is functorially finite and let*

$$A \xrightarrow{f} M^0 \xrightarrow{c} M^1 \to 0$$

be an exact sequence with f be a minimal left $\mathcal{T}$-approximation of A. Then

(i) $\mathcal{T} = \operatorname{gen} M^0 = \operatorname{gen}(M^0 \oplus M^1)$.

(ii) M^0 is a splitting projective *for $\mathcal{T}$, meaning that any epimorphism $\theta : T \twoheadrightarrow M^0$ with $T \in \mathcal{T}$ must be a split epi.*

(iii) M^0 and $M^0 \oplus M^1$ are Ext-projective in $\mathcal{T}$, so they are τ-rigid.

(iv) Any module $T \in \mathcal{T}$ is a quotient of a module in $\operatorname{add}(M^0 \oplus M^1)$ by a submodule in $\mathcal{T}$.

(v) Any Ext-projective in $\mathcal{T}$ is in $\operatorname{add}(M^0 \oplus M^1)$, so there are only finitely many indecomposable Ext-projectives in $\mathcal{T}$.

Proof. (i) Clearly $\operatorname{gen} M^0 = \operatorname{gen}(M^0 \oplus M^1) \subseteq \mathcal{T}$. If $T \in \mathcal{T}$, then there is a map $A^n \twoheadrightarrow T$, and each component factors through M, giving an epimorphism $M^n \twoheadrightarrow T$.

(ii) Since A is projective, the map $f : A \to M^0$ lifts to a map $A \to T$. By the approximation property, this factors as $A \to M^0 \to T$. Now the composition $M^0 \to T \to M^0$ must be an isomorphism by minimality.

(iii) Let $T \in \mathcal{T}$. Any exact sequence $0 \to T \to E \to M^0 \to 0$ splits by (ii). Thus M^0 is Ext-projective.

Since f is a $\mathcal{T}$-approximation, the induced map $\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(A, T)$ is surjective. This is a composition $\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(\operatorname{Im} f, T) \to \operatorname{Hom}(A, T)$ and the second map is injective, so actually the second map is a bijection and the first map $\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(\operatorname{Im} f, T)$ is surjective.

Now the exact sequence $0 \to \operatorname{Im} f \xrightarrow{i} M^0 \xrightarrow{c} M^1 \to 0$ gives

$$\operatorname{Hom}(M^0, T) \to \operatorname{Hom}(\operatorname{Im} f, T) \to \operatorname{Ext}^1(M^1, T) \to \operatorname{Ext}^1(M^0, T) = 0.$$

so $\operatorname{Ext}^1(M^1, T) = 0$.

(iv) (My thanks to Andrew Hubery for this argument). Take a right $\operatorname{add}(M^0 \oplus M^1)$-approximation $\phi : W \to T$ for T. Since $T \in \operatorname{gen} M^0$, the map ϕ is surjective, so it gives an exact sequence

$$0 \to U \xrightarrow{\theta} W \xrightarrow{\phi} T \to 0.$$

Given $u \in U$ there is a map $r : A \to U$, $a \mapsto au$. Since $A \to M^0$ is a $\mathcal{T}$-approximation and $W \in \mathcal{T}$, there is a map p, and hence a map q giving a commutative diagram

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & M^0 & \xrightarrow{c} & M^1 & \longrightarrow & 0 \\ & & {\scriptstyle r}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & \\ 0 & \longrightarrow & U & \xrightarrow{\theta} & W & \xrightarrow{\phi} & T & \longrightarrow & 0. \end{array}$$

Since ϕ is an approximation, $q = \phi h$ for some $h : M^1 \to W$. Then $\phi(p - hc) = 0$. Thus $p - hc = \theta\ell$ for some $\ell : M^0 \to U$. Then $\theta(r - \ell f) = 0$, so since θ is mono, $r = \ell f$. Thus $u \in \operatorname{Im}(\ell)$. Repeating for a basis of U, we get a map from a direct sum of copies of M^0 onto U, so $U \in \mathcal{T}$.

(v) Follows. □

Corollary. *If M is a τ-rigid module, then* $\operatorname{gen} M$ *is a functorially finite torsion class. Conversely, any functorially finite torsion class $\mathcal{T}$ is of the form* $\operatorname{gen} M$ *for some τ-rigid module M, which we can take to be the direct sum of the indecomposable Ext-projectives in $\mathcal{T}$.*

Proof. Any torsion class in A-mod is contravariantly finite, since the inclusion has a right adjoint. Recall also that if M is a module, then $\operatorname{gen} M$ is always covariantly finite by the proposition at the end of section 1.9. In particular, if M is τ-rigid, then $\operatorname{gen} M$ is a functorially finite torsion class.

The last part follows from the theorem, since up to multiplicities, $M^0 \oplus M^1$ is the direct sum of the indecomposable Ext-projectives in $\mathcal{T}$. □

There is a better description of the Ext-injectives in a torsion class.

Remark. If $\mathcal{C}$ is a module class in A-mod, we write $I = \operatorname{ann}(\mathcal{C})$ for the ideal of all $a \in A$ annihilating all modules in $\mathcal{C}$. Then we can consider $\mathcal{C}$ as a module class in A/I-mod. Since A is finite-dimensional, some module in $\mathcal{C}$ is a faithful module for A/I. Thus A/I embeds in some module in $\mathcal{C}$.

Lemma. *If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in* A-mod, *then*

(i) The Ext-injectives for $\mathcal{T}$ are the injective $A/\operatorname{ann}(\mathcal{T})$*-modules.*

(ii) The Ext-projectives for $\mathcal{F}$ are the projective $A/\operatorname{ann}(\mathcal{F})$*-modules.*

Proof. (i) Let $I = \operatorname{ann}(\mathcal{T})$. Any injective A/I-module E has an epi $(A/I)^n \to E$. Now A/I embeds in some module $T \in \mathcal{T}$, and by the injective property the epi extends to an epi $T^n \to E$. Thus $E \in \mathcal{T}$.

Now if U is an Ext-injective, it embeds in an injective A/I-module, say $0 \to U \to E \to E/U \to 0$. Then $E/U \in \mathcal{T}$, so this sequence splits, so U is injective as an A/I-module.

(ii) is dual. □

5.2 Tilting modules

Definition. Let M be an A-module.

M is a *partial tilting module* if $\operatorname{proj.dim} M \leq 1$ and $\operatorname{Ext}^1(M, M) = 0$.

A partial tilting module M is a *tilting module* if there is an exact sequence $0 \to A \to M^0 \to M^1 \to 0$ with $M^i \in \operatorname{add} M$. (Later we will see that it is equivalent that $\#M = \#A$.)

M is a *partial cotilting module* if $\operatorname{inj.dim} M \leq 1$ and $\operatorname{Ext}^1(M, M) = 0$.

A partial cotilting module is a *cotilting module* if there is an exact sequence $0 \to M_1 \to M_0 \to DA \to 0$ with $M_i \in \operatorname{add} M$. (Again, it is equivalent that $\#M = \#A$.)

Clearly M is a (partial) tilting A-module iff DM is a (partial) cotilting A^{op}-module.

Note that we deal only with *classical* tilting theory. There is a version allowing higher projective dimension.

Lemma. *If M is a partial tilting module, then M is τ-rigid. Conversely if M is τ-rigid, then it is a partial tilting module for $A/\operatorname{ann}(M)$.*

Proof. Use the AR formula $D \operatorname{Ext}^1(M, N) \cong \overline{\operatorname{Hom}}(N, \tau M)$. If $\operatorname{proj.dim} M \leq 1$ then $\operatorname{Hom}(DA, \tau M) = 0$ by Lemma (2) in §2.2, so the AR formula takes the form $D \operatorname{Ext}^1(M, N) \cong \operatorname{Hom}(N, \tau M)$. The converse is the special case $\mathcal{T} = \operatorname{gen} M$ of (i) in the next lemma. □

Lemma. *(i) If $\mathcal{T}$ is a torsion class in A-mod, then any Ext-projective M in $\mathcal{T}$ is a partial tilting module for $A/\operatorname{ann}(\mathcal{T})$.*

(ii) If furthermore $\mathcal{T}$ is functorially finite, then the direct sum of all indecomposable Ext-projectives is a tilting module for $A/\operatorname{ann}(\mathcal{T})$.

Proof. (i) Consider $\mathcal{T}$ as a module class in A/I-mod, where $I = \operatorname{ann}(\mathcal{T})$. Clearly $\operatorname{Ext}^1_{A/I}(M, M) = 0$. Also the injective A/I-modules are in $\mathcal{T}$, and $\tau_{A/I} M$ is in the corresponding torsion-free class, so $\operatorname{Hom}(D(A/I), \tau_{A/I} M) = 0$. Thus by Lemma (2) in §2.2, $\operatorname{proj.dim}_{A/I} M \leq 1$.

(ii) If $\mathcal{T}$ is functorially finite, in the theorem of Auslander-Smalø, the map $f : A \to M^0$ induces an injection $A/I \to M^0$, so $M^0 \oplus M^1$ is a tilting module for A/I. □

Proposition (Bongartz)**.** *Let M be a partial tilting module. Take a basis of $\xi_1, \ldots, \xi_n$ of $\operatorname{Ext}^1(M, A)$, consider the tuple $(\xi_1, \ldots, \xi_n)$ as an element of $\operatorname{Ext}^1(M^n, A)$, and let*

$$0 \to A \to E \to M^n \to 0.$$

be the corresponding universal *extension. Then $T = E \oplus M$ is a tilting module. Thus every partial tilting module is a direct summand of a tilting module, and by duality every partial cotilting module is a direct summand of a cotilting module.*

Proof. The long exact sequence for $\mathrm{Hom}(M,-)$ gives

$$\mathrm{Hom}(M,M^n) \xrightarrow{\xi} \mathrm{Ext}^1(M,A) \to \mathrm{Ext}^1(M,E) \to \mathrm{Ext}^1(M,M^n),$$

the map ξ is onto, and $\mathrm{Ext}^1(M,M^n)=0$, so $\mathrm{Ext}^1(M,E)=0$. From the long exact sequence for $\mathrm{Hom}(-,M)$ one gets $\mathrm{Ext}^1(E,M)=0$, from the long exact sequence for $\mathrm{Hom}(-,E)$ one gets $\mathrm{Ext}^1(E,E)=0$. Also A and M^n have projective dimension ≤ 1, hence so does E. □

A partial tilting module M is τ-rigid, so gives a torsion theory $(\mathrm{gen}\, M, M^{\perp 0})$. Moreover $\mathrm{gen}_1 M \subseteq \mathrm{gen}\, M \subseteq M^{\perp 1}$.

Proposition (1)**.** *For a partial tilting module M, the following are equivalent:*
(i) M is a tilting module.
(ii) $M^{\perp 0,1}=0$.
(iii) $\mathrm{gen}\, M = M^{\perp 1}$.
(iv) $\mathrm{gen}_1 M = M^{\perp 1}$.
(v) X is Ext-projective in $M^{\perp 1} \Leftrightarrow X \in \mathrm{add}\, M$.

Proof. (i) $\Rightarrow$ (ii). If $X \in M^{\perp 0,1}$, apply $\mathrm{Hom}(-,X)$ to the exact sequence $0 \to A \to M^0 \to M^1 \to 0$, to deduce that $\mathrm{Hom}(A,X)=0$.

(ii) $\Rightarrow$ (iii). Suppose $X \in M^{\perp 1}$. Take a basis of $\mathrm{Hom}(M,X)$ and use it to form the universal map $f : M^n \to X$. Then $\mathrm{Im}\, f \in \mathrm{gen}\, M$. Consider the exact sequence $0 \to \mathrm{Im}\, f \to X \to X/\mathrm{Im}\, f \to 0$. Apply $\mathrm{Hom}(M,-)$ giving an exact sequence

$$0 \to \mathrm{Hom}(M,\mathrm{Im}\, f) \to \mathrm{Hom}(M,X) \to \mathrm{Hom}(M,X/\mathrm{Im}\, f) \to \mathrm{Ext}^1(M,\mathrm{Im}\, f).$$

By construction the map $\mathrm{Hom}(M,M^n) \to \mathrm{Hom}(M,X)$ is onto, hence so is the map $\mathrm{Hom}(M,\mathrm{Im}\, f) \to \mathrm{Hom}(M,X)$. Also $\mathrm{Ext}^1(M,\mathrm{Im}\, f)=0$ since M is τ-rigid. Thus $\mathrm{Hom}(M,X/\mathrm{Im}\, f)=0$. Also $\mathrm{Ext}^1(M,X/\mathrm{Im}\, f)=0$. Thus $X/\mathrm{Im}\, f \in M^{\perp 0,1}$. Thus $X/\mathrm{Im}\, f = 0$, so f is onto, so $X \in \mathrm{gen}\, M$.

(iii) $\Rightarrow$ (iv). Suppose $X \in M^{\perp 1}$. Then it is in $\mathrm{gen}\, M$. Let L be the kernel of the universal map $M^n \to X$. Then applying $\mathrm{Hom}(M,-)$ we see that $L \in M^{\perp 1}$, so $L \in \mathrm{gen}\, M$. Say $M'' \twoheadrightarrow L$. Now the sequence $M'' \to M^n \to X \to 0$ shows that $X \in \mathrm{gen}_1 M$.

(iv) $\Rightarrow$ (v). Clearly M and so any $X \in \mathrm{add}(M)$ is in $M^{\perp 1}$ and Ext-projective. Conversely if X is in $M^{\perp 1}$ and Ext-projective, then by (iv) there is an exact sequence $M'' \xrightarrow{f} M' \to X \to 0$. This gives an exact sequence $0 \to \mathrm{Im}\, f \to M' \to X \to 0$ with $\mathrm{Im}\, f \in \mathrm{gen}\, M \subseteq M^{\perp 1}$. By assumption this sequence splits, so $X \in \mathrm{add}\, M$.

(v) $\Rightarrow$ (i). It suffices to show that E in Bongartz's sequence is in $\mathrm{add}\, M$, and for this it suffices to show it is Ext-projective in $M^{\perp 1}$. We know it is in $M^{\perp 1}$. If $Y \in M^{\perp 1}$, apply $\mathrm{Hom}(-,Y)$ to the Bongartz sequence to get $\mathrm{Ext}^1(M^n,Y) \to \mathrm{Ext}^1(E,Y) \to \mathrm{Ext}^1(A,Y)$, so $\mathrm{Ext}^1(E,Y)=0$. □

Dually, a partial cotilting module M is τ^--rigid, so gives a torsion theory $({}^{\perp 0}M, \operatorname{cogen} M)$. Moreover $\operatorname{cogen}^1 M \subseteq \operatorname{cogen} M \subseteq {}^{\perp 1}M$. The following is dual to the last proposition.

Proposition (2)**.** *For a partial cotilting module M, the following are equivalent:*
(i') M is a cotilting module.
(ii') ${}^{\perp 0,1}M = 0$.
(iii') $\operatorname{cogen} M = {}^{\perp 1}M$*.*
(iv') $\operatorname{cogen}^1 M = {}^{\perp 1}M$*.*
(v') X is Ext-injective in ${}^{\perp 1}M \Leftrightarrow X \in \operatorname{add} M$.

Proposition (3)**.** *If ${}_AM$ is a (co)tilting module, then it is f.b. and if $B = \operatorname{End}_A(M)$, then ${}_BM$ is also a (co)tilting module.*

Proof. If ${}_AM$ is tilting, then $\operatorname{gen}_1 M = M^{\perp 1}$, which contains DA, so ${}_AM$ is f.b.

(i) Applying $\operatorname{Hom}_A(-, M)$ to the exact sequence $0 \to A \to M^0 \to M^1 \to 0$ gives

$$0 \to \operatorname{Hom}_A(M^1, M) \to \operatorname{Hom}_A(M^0, M) \to M \to 0$$

and $\operatorname{Hom}_A(M^i, M) \in \operatorname{add}(\operatorname{Hom}_A(M, M)) = \operatorname{add}({}_BB)$, so $\operatorname{proj.dim} {}_BM \leq 1$.

(ii) The tilting sequence $0 \to A \to M^0 \to M^1 \to 0$ stays exact on applying $\operatorname{Hom}(-, M)$. Thus $A \in \operatorname{cogen}^2({}_AM)$. Thus $\operatorname{Ext}^1_B(M, M) = 0$ by the proposition about endomorphism correspondents.

(iii) Applying $\operatorname{Hom}_A(-, M)$ to a projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of M gives an exact sequence

$$0 \to B \to M^0 \to M^1 \to 0$$

where $M^i = \operatorname{Hom}_A(P_i, M) \in \operatorname{add}({}_BM)$. Thus ${}_BM$ is a tilting module.

Dually for cotilting. □

5.3 The Brenner-Butler Theorem

Let ${}_AM$ be a cotilting module and $B = \operatorname{End}_A(M)$, so ${}_BM$ is also cotilting.

In A-mod we have a torsion theory $(\mathcal{T}_A, \mathcal{F}_A) = ({}^{\perp 0}{}_AM, \operatorname{cogen} {}_AM)$. Since ${}_AM$ is cotilting we have

$$\mathcal{F}_A = \operatorname{cogen}({}_AM) = \operatorname{cogen}^1({}_AM) = {}^{\perp 1}{}_AM = \{X \in A\text{-mod} : \operatorname{Ext}^1_A(X, M) = 0\}.$$

In B-mod we have a torsion theory $(\mathcal{T}_B, \mathcal{F}_B) = ({}^{\perp 0}{}_BM, \operatorname{cogen} {}_BM)$. Since ${}_BM$ is cotilting we have the equivalent alternative descriptions of $\mathcal{F}_B$.

Theorem (Brenner-Butler Theorem, 1st version). *There are antiequivalences*

$$\mathcal{F}_A \underset{\mathrm{Hom}_B(-,M)}{\overset{\mathrm{Hom}_A(-,M)}{\rightleftarrows}} \mathcal{F}_B \quad \textit{and} \quad \mathcal{T}_A \underset{\mathrm{Ext}^1_B(-,M)}{\overset{\mathrm{Ext}^1_A(-,M)}{\rightleftarrows}} \mathcal{T}_B.$$

Proof. Since $\mathcal{F}_A = \mathrm{cogen}^1({}_AM)$ and $\mathcal{F}_B = \mathrm{cogen}^1({}_BM)$, the first antiequivalence is given by endomorphism correspondence.

Given a module ${}_AX$ in $\mathcal{T}_A$, so with $\mathrm{Hom}_A(X, M) = 0$, we show that

$$\mathrm{Hom}_B(\mathrm{Ext}^1_A(X, M), M) = 0$$

and construct a natural isomorphism

$$X \to \mathrm{Ext}^1_B(\mathrm{Ext}^1(X, M), M).$$

Indeed, take a projective cover of X to get a sequence $0 \to L \to P \to X \to 0$. It gives an exact sequence of B-modules

$$0 \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(L, M) \to \mathrm{Ext}^1_A(X, M) \to 0$$

Now $P, L \in \mathrm{cogen}\, M = \mathrm{cogen}^1 M$, so the natural maps $P \to \mathrm{Hom}_B(\mathrm{Hom}_A(P, M), M)$ and $L \to \mathrm{Hom}_B(\mathrm{Hom}_A(L, M), M)$ are isomorphisms. Also

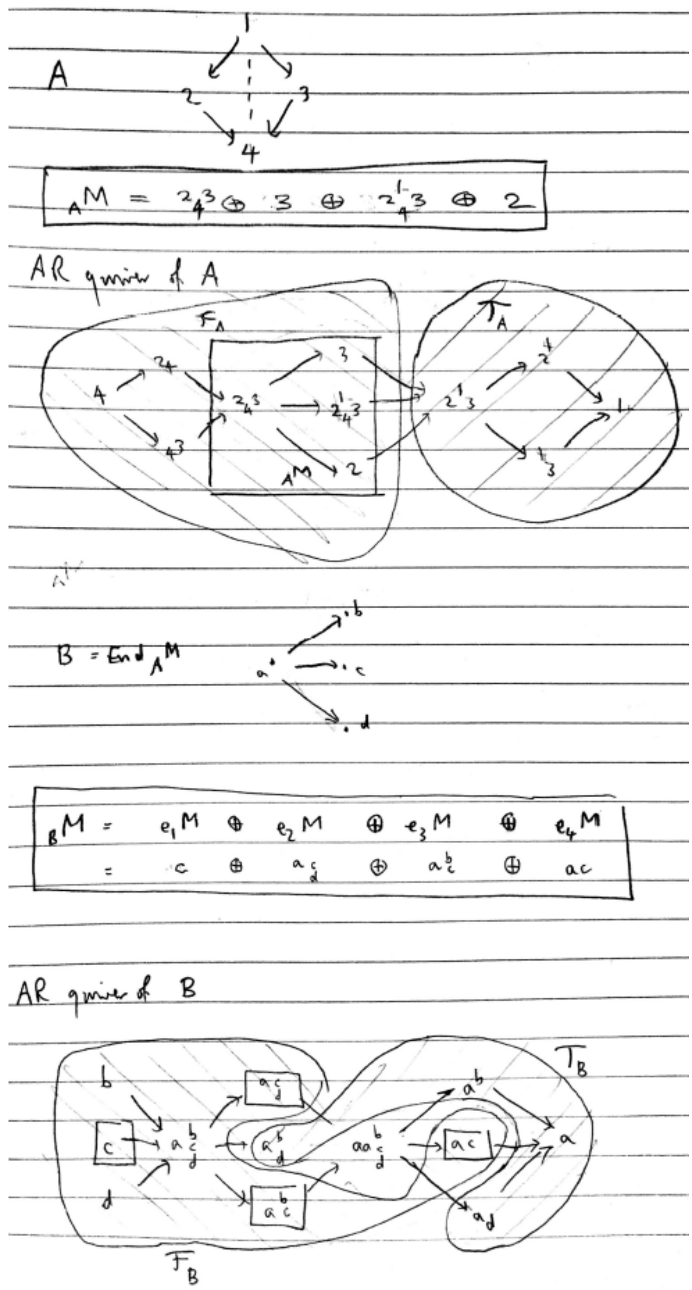
$$\mathrm{Hom}_A(L, M) \in \mathrm{cogen}^1({}_BM) = {}^{\perp 1}({}_BM),$$

so $\mathrm{Ext}^1_B(\mathrm{Hom}(L, M), M) = 0$. Thus we get a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & X & \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & ({}^1(X, M), M) & \longrightarrow & ((L, M), M) & \longrightarrow & ((P, M), M) & \longrightarrow & {}^1({}^1(X, M), M) & \longrightarrow 0 \end{array}$$

(where we omit the words Hom and Ext) with exact rows and in which the vertical maps are isomorphisms. Thus $\mathrm{Hom}_B(\mathrm{Ext}^1_A(X, M), M) = 0$ and there is an induced isomorphism $X \to \mathrm{Ext}^1_B(\mathrm{Ext}^1_A(X, M), M)$. One also needs to show that this is a natural isomorphism, but we omit the proof of this. $\square$

Example.



Theorem. *If B is hereditary, then the torsion theory $(\mathcal{T}_A, \mathcal{F}_A)$ is split (and by symmetry, if A is hereditary, then $(\mathcal{T}_B, \mathcal{F}_B)$ is split).*

Proof. We want to show that $\mathrm{Ext}^1_A(U, V) = 0$ for all $U \in \mathcal{F}_A$ and $V \in \mathcal{T}_A$. Now we have $V = \mathrm{Ext}^1_B(Y, M)$ for some $Y \in \mathcal{T}_B$. Taking a projective A-module Q mapping onto U, gives an exact sequence

$$0 \to \Omega_1 U \to Q \to U \to 0$$

and applying $\mathrm{Hom}_A(-, M)$, we get an isomorphism $\mathrm{Ext}^2(U, M) \cong \mathrm{Ext}^1(\Omega_1 U, M)$ (dimension shifting). Also $Q \in \mathrm{cogen}\, M$ since M is faithful, so $\Omega_1 U \in \mathrm{cogen}\, M = \mathcal{F}_A$, so $\mathrm{Ext}^1(\Omega_1 U, M) = 0$, so $\mathrm{Ext}^2(U, M) = 0$. We also have $\mathrm{Ext}^1(U, M) = 0$.

Now take a projective resolution

$$0 \to P_1 \to P_0 \to Y \to 0.$$

Since $\mathrm{Hom}_B(Y, M) = 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_B(P_0, M) \to \mathrm{Hom}_B(P_1, M) \to \mathrm{Ext}^1_B(Y, M) \to 0.$$

Thus

$$\dots \to \mathrm{Ext}^1_A(U, \mathrm{Hom}_B(P_1, M)) \to \mathrm{Ext}^1_A(U, \mathrm{Ext}^1_B(Y, M)) \to \mathrm{Ext}^2_A(U, \mathrm{Hom}_B(P_0, M)) \to \dots$$

Now $\mathrm{Hom}_B(P_i, M) \in \mathrm{add}({}_A M)$, so the outer terms are zero, giving the result. □

We now give another version of the Brenner-Butler theorem. Let A be an algebra and ${}_A T$ a tilting module. Let $B = \mathrm{End}(T)^{op}$, so T becomes an A-B-bimodule, and T_B is right B-module which is a tilting module. Thus DT is a left B-module which is cotilting.

The tilting module ${}_A T$ gives a torsion theory $(\mathcal{T}, \mathcal{F})$ in A-mod via

$$\mathcal{T} = \mathrm{gen}\, {}_A T = ({}_A T)^{\perp 1}$$
$$\mathcal{F} = ({}_A T)^{\perp 0}.$$

The cotilting left B-module DT gives a torsion theory $(\mathcal{X}, \mathcal{Y})$ in B-mod where

$$\mathcal{X} = {}^{\perp 0}({}_B DT)$$

and

$$\mathcal{Y} = \mathrm{cogen}\, {}_B DT = {}^{\perp 1}({}_B DT).$$

Note that if

$$\dots \to P_1 \to P_0 \to T \to 0$$

is a projective resolution of T as a right B-module, then

$$0 \to DT \to DP_0 \to DP_1 \to \dots$$

is an injective resolution of DT as a left B-module. Now if Y is a left B-module, then $\mathrm{Tor}^B_n(T, Y)$ is the homology of the complex $P_* \otimes_B Y$, so $D(\mathrm{Tor}^B_n(T, Y))$ is the cohomology of the complex $D(P_* \otimes_B Y) \cong \mathrm{Hom}_B(Y, DP_*)$, so

$$D(\mathrm{Tor}^B_n(T, Y)) \cong \mathrm{Ext}^n_B(Y, DT).$$

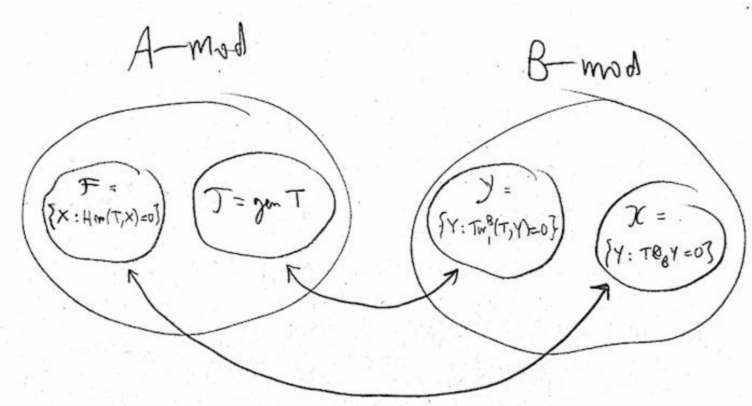
Thus $\mathcal{X} = \{Y : T \otimes_B Y = 0\}$ and $\mathcal{Y} = \{Y : \mathrm{Tor}^B_1(T, Y) = 0\}$.

Theorem (Brenner-Butler theorem, 2nd version). *We have inverse equivalences*

$$\mathcal{T} \underset{T\otimes_B -}{\overset{\mathrm{Hom}_A(T,-)}{\rightleftarrows}} \mathcal{Y}$$

and

$$\mathcal{F} \underset{\mathrm{Tor}_1^B(T,-)}{\overset{\mathrm{Ext}^1_A(T,-)}{\rightleftarrows}} \mathcal{X}.$$



For the proof, we consider DT as a cotilting right A-module, so as a cotilting left A^{op}-module, and $B = \mathrm{End}_{A^{op}}(DT)$. Use this in 1st version, and compose with duality.

Examples. (1) The Bernstein-Gelfand-Ponomarev reflection functors fit this picture. If i is a sink in Q, the tilting module is

$$T = \tau^{-1}P[i] \oplus \bigoplus_{j\neq i} P[j].$$

In fact, for any algebra A, if $P[i]$ is a simple projective (and not injective), this construction gives a tilting module, called an APR tilting module after Auslander, Platzeck and Reiten, Coxeter functors without diagrams, 1979.

(2) A *tilted algebra* is one of the form $B = \mathrm{End}_A(T)$ where A is hereditary and ${}_AT$ is a tilting module. Then the torsion theory $(\mathcal{X}, \mathcal{Y})$ is split.

(3) A *concealed algebra* is a tilted algebra of the form $B = \mathrm{End}_A(T)$ where A is representation-infinite connected hereditary and ${}_AT$ is a preprojective tilting module.

There is some $n > 0$ with $\tau^n T = 0$. If X is a module with $X \cong \tau^{-(n-1)}\tau^{n-1}X$, for example if X is indecomposable and not preprojective, or not near the start of the preprojective component, then since A is hereditary,

$$\mathrm{Ext}^1(T, X) \cong D\,\mathrm{Hom}(X, \tau T) \cong D\,\mathrm{Hom}(\tau^{-(n-1)}\tau^{n-1}X, \tau T)$$

$$\cong D\operatorname{Hom}(\tau^{n-1}X, \tau^n T) = 0$$

so $X \in \mathcal{T}$. Thus $\mathcal{T}$ contains all but finitely many indecomposables and $\mathcal{F}$ contains only finitely many indecomposables.

Then B-mod is obtained by reassembling these two pieces as $\mathcal{X}$ and $\mathcal{Y}$.

There is an example worked out in detail on p336 of Assem, Simson and Skowronski, Elements of the representation theory of associative algebras I.

A theorem of Happel and Vossieck, Minimal algebras of infinite representation type with preprojective component, Manuscripta Math. 1983: If B is an algebra with a preprojective component and B is minimal of infinite representation type, meaning that B/BeB of finite representation type for all nonzero idempotents e, then either B is Morita equivalent to the path algebra of an r-arrow Kronecker quiver with $r \geq 2$, or B is tame concealed, and there is a classification of all such algebras.

5.4 Derived equivalences

I promised to talk about how tilting theory is related to derived categories, but to do this properly would be too much of a digression. So I will only sketch things briefly.

Definition. An A-module T is a *generalized (or Miyashita) tilting module* if

(i) $\operatorname{proj.dim} T < \infty$, so there is a projective resolution $0 \to P_r \to \cdots \to P_0 \to T \to 0$

(ii) $\operatorname{Ext}^i(T, T) = 0$ for all $i > 0$

(iii) There is an exact sequence $0 \to A \to T^0 \to \cdots \to T^r \to 0$ with $T^i \in \operatorname{add}(T)$.

The following was proved by Happel for $\operatorname{gl.dim} A < \infty$ and in general by Cline, Parshall and Scott, Derived categories and Morita theory, J. Algebra 1986.

Theorem. *Let T be a generalized tilting A-module. Then T is faithfully balanced, and letting $B = \operatorname{End}(T)^{op}$, the module T_B is a generalized tilting right B-module. Moreover T induces inverse equivalences of triangulated categories*

$$D^b(A\text{-mod}) \rightleftarrows D^b(B\text{-mod})$$

The functor to the right is $\mathbf{R}\operatorname{Hom}(T, -)$, the right derived functor of $\operatorname{Hom}(T, -)$. This can be defined abstractly, but to show it exists and compute it, one uses the isomorphisms

$$D^b(A\text{-mod}) \cong D^{+,b}(A\text{-mod}) \cong K^{+,b}(A\text{-inj}).$$

Then $\operatorname{Hom}(T, -)$ can be applied to a complex of injectives $I^\cdot$, giving a complex $\operatorname{Hom}(T, I^\cdot)$ in $D^+(B\text{-mod})$.

Now if X is an A-module in degree 0, then $\mathbf{R}\operatorname{Hom}(T, X)$ is computed by taking an injective resolution of X, so its n-th cohomology is $\operatorname{Ext}^n(T, X)$. This is nonzero only for finitely many n, so $\mathbf{R}\operatorname{Hom}(T, X) \in D^{+,b}(B\text{-mod})$. Now any complex X in $D^b(A\text{-mod})$ can be built from modules in a finite number of degrees. Thus $\mathbf{R}\operatorname{Hom}(T, X) \in D^{+,b}(B\text{-mod}) \cong D^b(B\text{-mod})$.

Similarly the functor to the left is $\mathbf{L}T \otimes_B -$, constructed using

$$D^b(A\text{-mod}) \cong D^{-,b}(A\text{-mod}) \cong K^{-,b}(A\text{-proj}).$$

Using that $T \otimes_B -$ is left adjoint to $\operatorname{Hom}_A(T, -)$ one can show that $\mathbf{L}T \otimes_B -$ is left adjoint to $\mathbf{R}\operatorname{Hom}_A(T, -)$. Then one can show that they are inverse equivalences.

Now suppose A is hereditary. Then every object in $D^b(A\text{-mod})$ is a direct sum of stalk complexes - living in only one degree.

The shift $X[n]$ of a complex X is given by $X[n]^i = X^{i+n}$ and it multiplies the differential by $(-1)^i$.

Thus if X is an A-module considered as a complex in degree 0, then $X[n]$ is a module in degree $-n$.

Also $\operatorname{Hom}(X[i], Y[j]) \cong \operatorname{Ext}^{j-i}(X, Y)$ which is zero for $j < i$.

Thus we can picture $D^b(A\text{-mod})$ as below.

Now suppose in addition that T is a classical tilting module, so B is tilted.

If X is an A-module in degree 0, then it is isomorphic in the derived category to its injective resolution, and $\mathbf{R}\operatorname{Hom}(T, X)$ is the complex

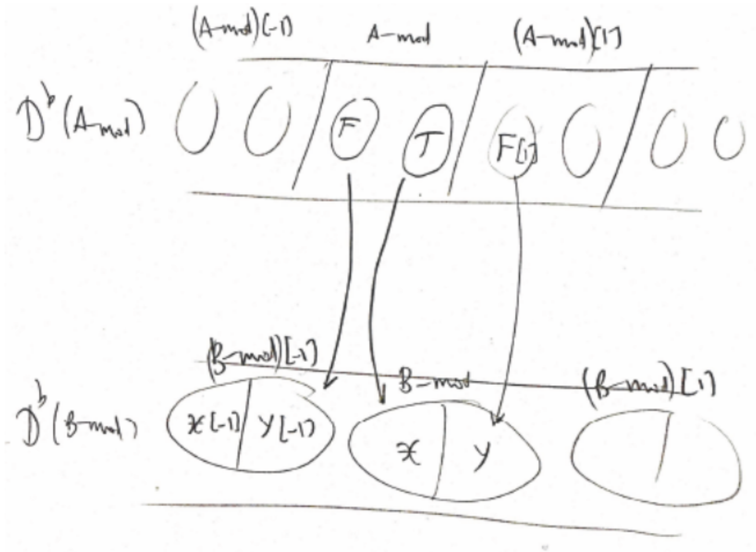
$$\cdots \to 0 \to \operatorname{Hom}(T, I^0) \to \operatorname{Hom}(T, I^1) \to 0 \to \dots$$

The cohomology in degree i is $\operatorname{Ext}^i(T, X)$.

If $X \in \mathcal{T} = ({}_AT)^{\perp 1}$ then $\mathbf{R}\operatorname{Hom}(T, X)$ is in B-mod. It is in the class $\mathcal{X} = \{Y : \operatorname{Tor}_1^B(T, Y) = 0\}$.

If $X \in \mathcal{F} = ({}_AT)^{\perp 0}$ then $\mathbf{R}\operatorname{Hom}(T, X)$ is a module in degree 1, so it is in $B\text{-mod}[-1]$. It is in the shift of $\mathcal{Y} = \{Y : T \otimes_B Y = 0\}$.

Since the torsion theory $(\mathcal{X}, \mathcal{Y})$ is splitting, we can picture $D^b(B\text{-mod})$ as follows.



5.5 Consequences for Grothendieck groups

Definition. We consider two types of *Grothendieck groups.*

If $\mathcal{A}$ is an abelian category, the Grothendieck group $G_0(\mathcal{A})$ is the additive group generated by symbols $[X]$ for each object X in $\mathcal{A}$, modulo the relations $[Y] = [X] + [Z]$ for any short exact sequence $0 \to X \to Y \to Z \to 0$.

If $\mathcal{C}$ is an additive category, the Grothendieck group $K_0(\mathcal{C})$ is the additive group generated by symbols $[X]$ for each object X in $\mathcal{C}$, modulo the relations $[Y] = [X] + [Z]$ whenever $Y \cong X \oplus Z$.

Lemma. *If A is a f.d. algebra with simples $S[i]$ $(i = 1, \dots, n)$ and indecomposable projectives $P[i]$, then:*

(i) The map sending a module X to its dimension vector gives an isomorphism $G_0(A\text{-mod}) \cong \mathbb{Z}^n$, $[X] \mapsto \underline{\dim}\, X$, so $G_0(A\text{-mod})$ is the free $\mathbb{Z}$-module on the symbols $[S[i]]$.

(ii) $K_0(A\text{-proj})$ is also isomorphic to $\mathbb{Z}^n$ since it is the free $\mathbb{Z}$-module on the symbols $[P[i]]$.

Proof. (i) is the Jordan-Hölder theorem and (ii) is Krull-Remak-Schmidt. □

Theorem. *If ${}_AM$ is a cotilting module and $B = \mathrm{End}_A(M)$, then there is an isomorphism*

$$\theta : G_0(A\text{-mod}) \to G_0(B\text{-mod}), \quad [X] \mapsto [\mathrm{Hom}_A(X, M)] - [\mathrm{Ext}^1_A(X, M)].$$

Thus the canonical basis of $G_0(A\text{-mod})$ gives a new basis of $G_0(B\text{-mod})$, hence the name "tilting".

Proof. If we apply $\operatorname{Hom}_A(-, M)$ to a short exact sequence of A-modules, say $0 \to X \to Y \to Z \to 0$ we get a long exact sequence of B-modules

$$0 \to \operatorname{Hom}(Z, M) \to \operatorname{Hom}(Y, M) \to \operatorname{Hom}(X, M) \to$$

$$\operatorname{Ext}^1(Z, M) \to \operatorname{Ext}^1(Y, M) \to \operatorname{Ext}^1(X, M) \to 0.$$

Now the relations for $G_0(B\text{-mod})$ imply that

$$\theta([Y]) = [\operatorname{Hom}_A(Y, M)] - [\operatorname{Ext}^1_A(Y, M)]$$

$$= [\operatorname{Hom}_A(X, M)] - [\operatorname{Ext}^1_A(X, M)] + [\operatorname{Hom}_A(Z, M)] - [\operatorname{Ext}^1_A(Z, M)]$$

$$= \theta([X]) + \theta([Z])$$

so that θ is well-defined.

Swapping the roles of A and B there is a map ϕ in the reverse direction.

If $X \in \operatorname{cogen} M$ or $X \in {}^{\perp 0}M$, then $\phi(\theta([X])) = [X]$. Because any X belongs to a short exact sequence whose ends are torsion and torsion-free, it follows that $\phi\theta = 1$. Similarly $\theta\phi = 1$. □

Recall that we write $\#M$ for the number of isomorphism classes of indecomposable summands of M. Thus $\#A$ is the number of isomorphism classes of indecomposable projective A-modules, so the number of isomorphism classes of simple A-modules.

Corollary. *Any partial (co)tilting module M has $\#M \leq \#A$, with equality if and only if M is (co)tilting.*

Proof. If ${}_AM$ is a cotilting module and $B = \operatorname{End}_A(M)$, then $\operatorname{Hom}(-, M)$ gives an antiequivalence between $\operatorname{add} M$ and B-proj, so $\#M$ is the rank of $G_0(B\text{-mod})$, which is the rank of $G_0(B\text{-mod})$, which is $\#A$.

By duality any tilting module has $\#A$ summands. By Bongartz, any partial tilting module is a summand of a tilting module. □

Theorem (Smalø, 1984)**.** *If $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then $\mathcal{T}$ is functorially finite iff $\mathcal{F}$ is functorially finite.*

Proof. By symmetry, it suffices to prove that if $\mathcal{T}$ is functorially finite, then so is $\mathcal{F}$.

The number of indec Ext-injectives in $\mathcal{F}$

= number of indec injectives in $\mathcal{F}$ + number of non-injective indec Ext-injectives in $\mathcal{F}$

= number of indec injectives in $\mathcal{F}$ + number of non-projective indec Ext-projectives in $\mathcal{T}$

= number of indec injectives in $\mathcal{F}$ + number of indec Ext-projectives in $\mathcal{T}$ - number of indec projectives in $\mathcal{T}$.

Since $\mathcal{T}$ is functorially finite, the direct sum of the indecomposable Ext-projectives in $\mathcal{T}$ is a tilting module for $A/\operatorname{ann}(\mathcal{T})$. Thus we get

= number of indec injectives in $\mathcal{F}$ + $\#A/\operatorname{ann}(\mathcal{T})$ - number of indec projectives in $\mathcal{T}$.

Now the number of indecomposable injectives $I[i]$ not in $\mathcal{F}$ is the number of indecomposable Ext-injectives in $\mathcal{T}$, which is the number of indecomposable injective $A/\operatorname{ann}(\mathcal{T})$-modules, so it is $\#A/\operatorname{ann}(\mathcal{T})$.

Similarly the number of indecomposable projectives not in $\mathcal{T}$ is $\#A/\operatorname{ann}(\mathcal{F})$.

So we get

$= (\#A - \#A/\operatorname{ann}(\mathcal{T})) + \#A/\operatorname{ann}(\mathcal{T}) - (\#A - \#A/\operatorname{ann}(\mathcal{F}))$.

$= \#A/\operatorname{ann}(\mathcal{F})$.

Now by the dual of an earlier result, any Ext-injective in $\mathcal{F}$ is a partial cotilting module for $A/\operatorname{ann}(\mathcal{F})$. Thus the direct sum M of all indecomposable Ext-injectives in $\mathcal{F}$ is a cotilting module for $A/\operatorname{ann}(\mathcal{F})$.

Thus working in $A/\operatorname{ann}(\mathcal{F})$-mod, we have $\operatorname{cogen} M = {}^{\perp 1}M$. Now since M is Ext-injective in $\mathcal{F}$, $\operatorname{Ext}^1(\mathcal{F}, M) = 0$, so $\mathcal{F} \subseteq {}^{\perp 1}M = \operatorname{cogen} M \subseteq \mathcal{F}$.

Thus also $\mathcal{F} = \operatorname{cogen} M$ as a module class in A-mod. Thus $\mathcal{F}$ is contravariantly finite by the proposition at the end of §1.9, and it is covariantly finite since the inclusion has a left adjoint. □

5.6 Some tau-tilting theory

It was started by Adachi, Iyama and Reiten, τ-tilting theory, 2014, although there was earlier work, see Derksen and Fei, General Presentations of Algebras and Foundations of tau-tilting Theory, arxiv 2409.12743. It has led to a lot of other work. We shall only do a little.

We have done all the necessary prerequisites in our theorems about functorially finite torsion and torsion-free classes.

Lemma. *(i) If M is an A-module, then $\#A/\operatorname{ann}(M)$ is the number of different simple composition factors involved in M.*

(ii) If M is τ-rigid, then the number of indecomposable Ext-projectives in $\operatorname{gen} M$ *is $\#A/\operatorname{ann}(M)$ and $\#M \leq \#A/\operatorname{ann}(M)$.*

Proof. (i) If S is involved in M, then S must be an $A/\operatorname{ann}(M)$-module. On the other hand, M is faithful as an $A/\operatorname{ann}(M)$-module, so $A/\operatorname{ann}(M)$ embeds in a direct sum of copies of M, so if S is a simple for $A/\operatorname{ann}(M)$, then it must be a composition factor of M.

(ii) M is a partial tilting module for $A/\operatorname{ann}(M)$, and $\operatorname{gen} M$ is a functorially finite torsion class, so the direct sum of the indecomposable Ext-projectives is a tilting module for $A/\operatorname{ann}(M)$. □

Definition. Let M be a τ-rigid A-module.

(i) M is a *support τ-tilting module* if $\#M = \#A/\operatorname{ann}(M)$, or equivalently M is the direct sum of the indecomposable Ext-projectives in $\operatorname{gen} M$, each with non-zero multiplicity.

(ii) M is a *τ-tilting module* if it is a sincere support τ-tilting module, or equivalently $\#M = \#A$. (Recall that *sincere* means that every simple module occurs as a composition factor.)

Lemma. *If M is τ-rigid, then $\mathcal{T} = {}^{\perp 0}(\tau M)$ is a sincere functorially finite torsion class. If T is the direct sum of the indecompsable Ext-projectives in $\mathcal{T}$, then T is a τ-tilting module, $M \in \operatorname{add}(T)$ and ${}^{\perp 0}(\tau T) = \operatorname{gen} T$.*

The module T is called the *Bongartz completion* of M.

Proof. τM is a τ^{-}-rigid module, so we get a torsion theory $({}^{\perp 0}(\tau M), \operatorname{cogen} \tau M)$. The torsion class is functorially finite by Smalø's theorem. The torsion class is sincere, since no injective $I[i]$ embeds in τM, so $I[i]$ is not in the torsion-free class, so its torsion submodule is non-zero, and this has $S[i]$ as a submodule. Clearly $M \in \mathcal{T}$ and it is Ext-projective since if $X \in \mathcal{T}$, then

$$\operatorname{Ext}^1(M, X) \cong D\overline{\operatorname{Hom}}(X, \tau M)$$

and $\operatorname{Hom}(X, \tau M) = 0$. Now

$${}^{\perp 0}(\tau M) = \mathcal{T} = \operatorname{gen} T \subseteq {}^{\perp 0}(\tau T) \subseteq {}^{\perp 0}(\tau M)$$

where the second equality holds by the Auslander-Smalø theorem of functorially finite torsion classes, the first inclusion since T is τ-rigid, and the second since $M \in \operatorname{add}(T)$. □

The following is an analogue of a result known as Wakamatsu's lemma.

Lemma. *If M is τ-rigid and $f : M' \to X$ is a right $\operatorname{add}(M)$-approximation of a module X, then $\operatorname{Hom}(\operatorname{Ker}(f), \tau M) = 0$.*

Proof. Replacing X by $\operatorname{Im}(f)$, we may suppose that f is surjective. Applying $\operatorname{Hom}(-, \tau M)$ gives an exact sequence

$$\operatorname{Hom}(M', \tau M) \to \operatorname{Hom}(\operatorname{Ker}(f), M) \to \operatorname{Ext}^1(X, \tau M) \to \operatorname{Ext}^1(M', \tau M).$$

The first hom space is zero since M is τ-rigid. Now the map $\operatorname{Hom}(M, M') \to \operatorname{Hom}(M, X)$ induced by f is surjective, hence so is the map on $\overline{\operatorname{Hom}}$, hence by the Auslander-Reiten formula, the map on Ext^1 is injective. □

Theorem. *A τ-rigid module M is τ-tilting iff* $\operatorname{gen} M = {}^{\perp 0}(\tau M)$.

Proof. If M is τ-tilting, then its Bongartz completion T can have no new indecomposable summands, so $\operatorname{add}(M) = \operatorname{add}(T)$, and we get the result from the lemma.

Suppose $\operatorname{gen} M = {}^{\perp 0}(\tau M)$. Let T be the Bongartz completion of M. Then

$$\operatorname{gen} M \subseteq \operatorname{gen} T = {}^{\perp 0}\tau T \subseteq {}^{\perp 0}\tau M = \operatorname{gen} M$$

so all are equal. Take a minimal right $\operatorname{add}(M)$-approximation of T, say $f : M' \to T$. It is surjective since $T \in \operatorname{gen} M$, so we get an exact sequence

$$0 \to \operatorname{Ker}(f) \to M' \to T \to 0.$$

By the Wakamatsu-type lemma $\operatorname{Hom}(\operatorname{Ker}(f), \tau M) = 0$. Since ${}^{\perp 0}(\tau M) = {}^{\perp 0}(\tau T)$ we get $\operatorname{Hom}(\operatorname{Ker}(f), \tau T) = 0$. Thus $\operatorname{Ext}^1(T, \operatorname{Ker}(f)) = 0$. Thus the sequence $0 \to \operatorname{Ker}(f) \to M' \to T \to 0$ splits. Thus $T \in \operatorname{add}(M)$, so M is τ-tilting. □

Corollary. *Any basic τ-rigid module M which is not τ-tilting, is a direct summand of at least two basic support τ-tilting modules.*

Proof. $\operatorname{gen} M$ and ${}^{\perp 0}(\tau M)$ are different functorially finite torsion classes containing M, and we can take the direct sum of the indecomposable Ext-projectives in either. □

Remark. It is useful to consider pairs (M, P) where M is a module, P is a projective module and $\operatorname{Hom}(P, M) = 0$, so that P is a direct sum of $P[i]$ such that $S[i]$ is not a composition factor of M.

We call it a *τ-rigid pair* if M is τ-rigid.

We call it a *support τ-tilting pair* if $\#M + \#P = \#A$. Note that we always have $\leq$ for a τ-rigid pair. Also M is necessarily support τ-tilting.

We call a pair *basic* if M and P are basic.

One can show that any basic τ-rigid pair (M, P), can be extended to a basic support τ-tilting pair $(M \oplus M', P \oplus P')$, and if $\#M + \#P = \#A - 1$, there are exactly two ways to do it.

Thus we get mutations of support τ-tilting pairs where we remove any one indecomposable summand, and replace it by the other possible extension of that pair.

Such mutations are related to the mutations in cluster algebras.

Remark. There is a natural homomorphism

$$\theta : K_0(A\text{-proj}) \to G_0(A\text{-mod}), \quad \theta([X]_K) = [X]_G.$$

It is an isomorphism if $\operatorname{gl.dim} A < \infty$. For this, it suffices to see that it is surjective. Now a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to X \to 0$$

gives

$$[X]_G = \sum_{i=0}^{n} (-1)^i [P_i]_G \in \operatorname{Im} \theta.$$

In general, however, it is not an isomorphism.

Instead there is a bilinear map

$$\langle -, - \rangle : K_0(A\text{-proj}) \times G_0(A\text{-mod}) \to \mathbb{Z}, \quad ([P],[X]) \mapsto \dim \operatorname{Hom}(P, X).$$

and

$$\langle [P[i]], [S[j]] \rangle = \dim \operatorname{Hom}(P[i], S[j]) = \delta_{ij} \dim D_i.$$

where $D_i = \operatorname{End}(S[i])^{op}$. The matrix is invertible over $\mathbb{Q}$, so gives a perfect pairing between $K_0(A\text{-proj}) \otimes_Z \mathbb{Q}$ and $G_0(A\text{-mod}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition. The *g-vector* of a module M is

$$g(M) = [P_0] - [P_1] \in K_0(A\text{-proj})$$

where $P_1 \to P_0 \to M \to 0$ is the minimal projective presentation.

Lemma. *If M and X are modules, then*

$$\langle g(M), [X] \rangle = \dim \operatorname{Hom}(M, X) - \dim \operatorname{Hom}(X, \tau M).$$

Proof. We have exact sequences

$$0 \to \tau M \to \nu P_1 \to \nu P_0$$

and

$$0 \to \operatorname{Hom}(M, X) \to \operatorname{Hom}(P_0, X) \to \operatorname{Hom}(P_1, X)$$

so we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}(X, \tau M) & \longrightarrow & \operatorname{Hom}(X, \nu P_1) & \longrightarrow & \operatorname{Hom}(X, \nu P_0) & & & & \\ & & & & \| & & \| & & & & \\ & & & & D\operatorname{Hom}(P_1, X) & \longrightarrow & D\operatorname{Hom}(P_0, X) & \longrightarrow & D\operatorname{Hom}(M, X) & \longrightarrow & 0 \end{array}$$

□

Lemma. *If M is τ-rigid, and $P_1 \xrightarrow{\theta} P_0 \xrightarrow{\phi} M \to 0$ is a minimal projective presentation, then P_0 and P_1 have no direct summand in common.*

Proof. The map $\mathrm{Hom}(P_0, M) \to \mathrm{Hom}(P_1, M)$ is surjective since its dual can be identified with the map $\mathrm{Hom}(M, \nu P_1) \to \mathrm{Hom}(M, \nu P_0)$ and the kernel of this map is $\mathrm{Hom}(M, \tau M) = 0$.

It suffices to show that any map $s : P_1 \to P_0$ is in the radical. The composition $\phi s \in \mathrm{Hom}(P_1, M)$, so by the surjectivity, $\phi s = t\theta$ for some $t : P_0 \to X$.

Since ϕ is surjective and P_0 is projective, we have $t = \phi u$ for some $u : P_0 \to P_0$.

Then $\phi(s - u\theta) = 0$. Thus since P_1 is projective, $s - u\theta = \theta v$ for some $v : P_1 \to P_1$. Now θ is in the radical, hence so is s. □

Theorem. *Two τ-rigid modules with the same g-vector must be isomorphic.*

By the lemma, the two modules have the same projectives in their minimal projective presentations. Thus we are dealing with two homomorphisms in $\mathrm{Hom}(P_1, P_0)$. Can reduce to the case of an algebra over an algebraically closed field. Then it is a simple geometric argument. Hopefully we will do it later.

Remark. There are nice connections with semibricks. See Asai, Semibricks, IMRN 2020 and Ringel, Brick chain filtrations, arxiv 2411.18427

Also Demonet, Iyama and Jasso, tau-tilting finite algebras, bricks, and g-vectors, IMRN 2019. For example the following are equivalent.

- A has only finitely many τ-tilting modules.
- Every torsion class in A-mod is functorially finite.
- A has only finitely many bricks.

6 Varieties and schemes of algebras and modules

6.1 Varieties of algebras

First we need to discuss varieties. We work over an algebraically closed field K, and follow Kempf, Algebraic varieties, 1993.

Definition. A *space with functions* consists of a topological space X and the assignment of a set $\mathcal{O}(U)$ of *regular functions* for each open set $U \subseteq X$, satisfying:

(a) $\mathcal{O}(U)$ is a K-subalgebra of the algebra of all functions $U \to K$, with pointwise operations.

(b) If U is a union of open sets, $U = \bigcup U_\alpha$, then $f \in \mathcal{O}(U)$ iff $f|_{U_\alpha} \in \mathcal{O}(U_\alpha)$ for all α.

(c) If $f \in \mathcal{O}(U)$, then $D(f) = \{u \in U \mid f(u) \neq 0\}$ is open in U and $1/f \in \mathcal{O}(D(f))$.

A *morphism* of spaces with functions is a continuous map $\theta : X \to Y$ with the property that for any open subset U of Y, and any $f \in \mathcal{O}(U)$, the composition

$$\theta^{-1}(U) \xrightarrow{\theta} U \xrightarrow{f} K$$

is in $\mathcal{O}(\theta^{-1}(U))$. In this way one gets a category of spaces with functions.

Properties. (1) If X is a space with functions and $V \subseteq X$ is any subset, one defines $\mathcal{O}(V)$ to be the set of functions $f : V \to K$ such that each $v \in V$ has an open neighbourhood U in X such that $f|_{V\cap U} = g|_{V\cap U}$ for some $g \in \mathcal{O}(U)$.

(2) Any subset V of a space with functions X becomes a space with functions with the induced topology and induced functions, and the inclusion $V \to X$ is a morphism. (Kempf, Exercise 1.5.3.)

(3) An *embedding* is a morphism $\theta : X \to Y$ which induces an isomorphism $X \to \mathrm{Im}(\theta)$. If so, then for any Z is a space with functions, a mapping $\phi : Z \to X$ is a morphism if and only if $\theta\phi : Z \to Y$ is a morphism.

(4) If X and Y are spaces with functions, then the set $X \times Y$ can be given the structure of a space with functions, so that it becomes a product of X and Y in the category of spaces with functions. See Kempf, Lemma 3.1.1. The topology is not the usual product topology. Instead a basis of open sets is given by the sets

$$\{(u, v) \in U \times V : f(u, v) \neq 0\}$$

where U is open in X, V is open in Y and $f(x, y) = \sum_{i=1}^{n} g_i(x)h_i(y)$ with $g_i \in \mathcal{O}(U)$ and $h_i \in \mathcal{O}(V)$.

(5) The diagonal map $X \to X \times X$ is an embedding, since if Δ_X is its image, then there is a morphism $\Delta_X \to X$ given by the composition $\Delta_X \to X \times X \xrightarrow{p_1} X$.

(6) The projection morphism $p : X \times Y \to X$ is an *open morphism*, that is, the image of any open set U is open. Namely, for $y \in Y$, the identity morphism $X \to X$ and the morphism $X \to Y$ sending every element to y induce a morphism $i_y : X \to X \times Y$ with $i_y(x) = (x, y)$. Now if $U \subseteq X \times Y$, then $p(U) = \bigcup_{y \in Y} i_y^{-1}(U)$, which is open.

Definition. *Affine n-space* is $\mathbb{A}^n = K^n$ considered as a space with functions with:

- The topology is the *Zariski topology*, so closed sets are of the form

$$V(S) = \{(x_1, \dots, x_n) \in K^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in S\}$$

where S is a subset of the polynomial ring $K[X_1, \dots, X_n]$. Equivalently, the sets

$$D(f) = \{(x_1, \dots, x_n) \in K^n \mid f(x_1, \dots, x_n) \neq 0\}$$

with $f \in K[X_1, \dots, X_n]$ are a base of open subsets, and by noetherianness, any open set is a finite union of $D(f)$.

- If U is an open subset of $\mathbb{A}^n$, then the set of regular functions $\mathcal{O}(U)$ consists of the functions $f : U \to K$ such that each point $u \in U$ has an open neighbourhood $W \subseteq U$ such that $f|_W = p/q$ with $p, q \in K[X_1, \dots, X_n]$ and $q(x_1, \dots, x_n) \neq 0$ for all $(x_1, \dots, x_n) \in W$.

Properties. (a) If X is a space with functions, then a mapping

$$\theta : X \to \mathbb{A}^n, \quad \theta(x) = (\theta_1(x), \dots, \theta_n(x))$$

is a morphism of spaces with functions iff the θ_i are regular functions on X. If θ is a morphism, then since the ith projection $\pi_i : \mathbb{A}^n \to K$ is regular, so it $\theta_i = \pi_i\theta$ is regular. Conversely suppose that $\theta_1, \dots, \theta_n$ are regular. Let U be an open subset of $\mathbb{A}^n$ and $f = p/q \in \mathcal{O}(U)$ with $q(u) \neq 0$ for $u \in U$. We need to show that $f\theta$ is regular on $\theta^{-1}(U)$. Now by assumption $p\theta = p(\theta_1(x), \dots, \theta_n(x))$ and $q\theta$ are regular on U. Also $q\theta$ is non-vanishing on $\theta^{-1}(U)$. Thus $p\theta/q\theta$ is regular on $\theta^{-1}(U)$.

(b) It follows that $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$.

(c) An n-dimensional vector space V can be considered as a space with functions isomorphic to $\mathbb{A}^n$ by choosing any basis. Any linear map $\mathbb{A}^n \to \mathbb{A}^m$ is a morphism of spaces with functions, and an invertible linear map is an isomorphism, so a different basis gives the same space with functions.

(d) $X = \mathbb{A}^n$ is *separated*, meaning that the diagonal

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in $X \times X$, since

$$\Delta_{\mathbb{A}^n} = \{(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{A}^{2n} : x_1 = y_1, \dots, x_n = y_n\},$$

so it is closed. Note that if X is a topological space and $X \times X$ is considered with the product topology, then Δ_X is closed if and only if X is Hausdorff.

Definition. An *affine variety* is a space with functions which is, or is isomorphic to, a closed subset of $\mathbb{A}^n$. If X is an affine variety, its *coordinate ring* is $K[X] := \mathcal{O}(X)$. An (abstract) *variety* is a space with functions X which is *separated* and with a finite open covering by affine varieties.

Note that an affine variety is a variety, since separatedness passes to subsets of a space with functions equipped with the induced structure, for if Y is a subset of X, then $\Delta_Y = (Y \times Y) \cap \Delta_X$ in $X \times X$.

Example. *Determinantal varieties.* If V and W are f.d. vector spaces then the space $\mathrm{Hom}(V, W)_{\leq r}$ of linear maps of rank $\leq r$ is closed in $\mathrm{Hom}(V, W)$, so an affine variety. Namely, identifying this with $M_{n\times m}(K)$, it is defined by the vanishing of all minors of size $r + 1$.

Recall that the *radical* of an ideal I in a commutative ring A is

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n > 0\}$$

It is an ideal. The ideal I is *radical* if $I = \sqrt{I}$. Equivalently, if the factor ring A/I is *reduced*, that is, it has no nonzero nilpotent elements. Since $K[X_1, \ldots, X_n]$ is a UFD, if f is an irreducible polynomial in $K[X_1, \ldots, X_n]$, then (f) is a prime ideal, so $K[X_1, \ldots, X_n]/(f)$ is a domain, so (f) is a radical ideal.

Theorem. *Let X be a closed subset of $\mathbb{A}^n$, say $X = V(S)$ with S is a subset of $K[X_1, \ldots, X_n]$. Then there is a canonical isomorphism*

$$K[X] \cong K[X_1, \ldots, X_n]/\sqrt{I}$$

where I is the ideal generated by S and $\sqrt{I}$ is its radical.

The kernel of the canonical map $K[X_1, \ldots, X_n] \to K[X]$ is $\sqrt{I}$ by Hilbert's Nullstellensatz. The fact that is is surjective is proved in Kempf §1.5.

Corollary. *The assignment $X \mapsto K[X]$ gives an anti-equivalence between the categories of affine varieties and finitely generated reduced commutative K-algebras. Moreover if Z is any space with functions, we get a bijection*

$$\mathrm{Hom}_{\textit{spaces with functions}}(Z, X) \to \mathrm{Hom}_{K\textit{-algebras}}(K[X], \mathcal{O}(Z))$$

sending $\theta : Z \to X$ to the composition map $f \mapsto f\theta$.

Proof. The theorem shows that $K[X]$ is a f.g. reduced commutative algebra, and any such occurs. The statement about morphisms follows from our observation about morphisms $Z \to \mathbb{A}^n$. □

Theorem. *A product $X \times Y$ of varieties is a variety. If X and Y are affine varieties, so is $X \times Y$, and $K[X \times Y] \cong K[X] \otimes_K K[Y]$.*

Proof. Recall that the product $X \times Y$ exists for any two spaces with functions.

If X is closed in $\mathbb{A}^n$ and Y is closed in $\mathbb{A}^m$ then $X \times Y$ is closed in $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$, so affine. Clearly $K[X] \otimes_K K[Y]$ is a f.g. commutative algebra, and with commutative algebra (using that K is algebraically closed) one can show it is reduced. Now the categorical property shows that $X \times Y$ has coordinate ring $K[X] \otimes_K K[Y]$.

In general, it is straightforward that if $U \subseteq X$ and $V \subseteq Y$ are open (resp. closed) subsets, then $U \times V$ is open (resp. closed) in $X \times Y$. Moreover with the induced structure as a space with functions it is a categorical product.

Assuming that X and Y are separated, $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$ which is closed in $(X \times X) \times (Y \times Y)$. □

Properties. (i) If X is a variety and $x \in X$, then the singleton set $\{x\}$ is closed in X. This is easy to see for affine space, it follows immediately for X an affine variety, and then for X an arbitrary variety.

(ii) Any variety is a *noetherian* topological space, that is it has the ascending chain condition on open subsets. The noetherian property of polynomial rings proves this for affine space, and then it follows for affine varieties and then for arbitrary varieties.

(iii) In particular, any variety is *quasi-compact*, meaning that any open covering has a finite subcovering. (Usually this is just called compactness, but in this context it is called quasi-compactness, apparently to make clear that the topological spaces needn't be Hausdorff.)

(iv) For a subset Y of a topological space, the following are equivalent, and then Y is called *locally closed.*
(1) Y is an open subset of a closed subset of X
(2) Y is open in its closure
(3) Y is the intersection of an open and a closed subset of X.

Definition. A *subvariety* Y of a variety X is a locally closed subset equipped with the induced structure as a space with functions.

Clearly a closed subvariety of an affine variety is affine.

Proposition. *If X is an affine variety and $f \in K[X]$, then the open subset $D(f) = \{x \in X : f(x) \neq 0\}$ is an affine variety and $K[D(f)] \cong K[X]_f$ (the localization, inverting f).*

Proof. We have an isomorphisms $D(f) \rightleftarrows \{(x,t) \in X \times \mathbb{A}^1 : f(x)t = 1\}$, where the map to the left sends (x,t) to x and the map to the right sends x to $(x, 1/f(x))$. It is a morphism since $1/f \in \mathcal{O}(D(f))$. □

Corollary. *Any subvariety of a variety is a variety.*

Proof. Suppose $Y \subseteq X$. We need to show that Y is a finite union of affine open subsets. Since X is a finite union of affine opens, we may reduce to the case when X is affine. We may also assume that Y is open in X. But then $Y = X \cap U$ with $U = D(f_1) \cup \dots \cup D(f_m)$, and then $Y = V_1 \cup \dots \cup V_m$ with $V_i = X \cap D(f_i)$ a closed subset of the affine variety $D(f_i)$, hence affine. □

Example. If V and W are vector spaces, the set of injective linear maps $\mathrm{Inj}(V,W)$ is an open subvariety in $\mathrm{Hom}(V,W)$, since the complement is $\mathrm{Hom}_{\leq r}(V,W)$ where $r = \dim V - 1$.

Remark. The example of $D(f)$ shows that some open subvarieties of affine varieties quasi-affine varieties are again affine. But this is not always true. For example $U = \mathbb{A}^2 \setminus \{0\} = D(X_1) \cup D(X_2)$ is not affine.

To see this, we show first that $\mathcal{O}(U) = K[X_1, X_2]$. A function $f \in \mathcal{O}(U)$ is determined by its restrictions f_i to $D(X_i)$ $(i = 1, 2)$. Now $f_i \in \mathcal{O}(D(X_i)) = K[X_1, X_2, X_i^{-1}]$. Moreover the restrictions of f_1 and f_2 to $D(X_1) \cap D(X_2) = D(X_1X_2)$ are equal, so f_1 and f_2 are equal as elements of $K[X_1, X_2, 1/X_1X_2]$. But this is only possible if they are both in $K[X_1, X_2]$, and equal. Thus $f \in K[X_1, X_2]$.

Now the inclusion morphism $\theta : U \to \mathbb{A}^2$ induces a homomorphism $\mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(U)$ which is actually an isomorphism. Now the category of affine varieties is anti-equivalent to the category of finitely generated reduced K-algebras. If U were affine, then since the map on coordinate rings is an isomorphism, θ would have to be an isomorphism. But is isn't.

Definition. A (non-empty) topological space X is *irreducible* if it cannot be written as a union of two proper closed subsets.

Properties. (1) X is irreducible iff every non-empty open subset U is dense in X. Thus any non-empty open subset of an irreducible space is irreducible.

(2) An affine variety X is irreducible iff $K[X]$ is a domain. (Kempf, Lemma 2.3.1.) In particular $\mathbb{A}^n$ is irreducible.

(3) Any variety is a finite union of maximal irreducible closed subvarieties, its *irreducible components.*

(4) A product of irreducible varieties is irreducible. Indeed if $X \times Y = Z_1 \cup Z_2$ with the Z_i closed, then for all $x \in X$ we have

$$Y = i_x^{-1}(Z_1) \cup i_x^{-1}(Z_2),$$

so by irreduciblity, one of the sets on the right is Y. Thus $\{x\} \times Y$ is contained in Z_1 or Z_2. Thus $X = X_1 \cup X_2$ where

$$X_i = \bigcap_{y \in Y} i_y^{-1}(Z_i)$$

Thus by irreducibility, we have $X = X_i$ for some i, so $Z_i = X \times Y$.

Definition. An *algebraic group* is a group which is also a variety, such that multiplication $G \times G \to G$ and inversion $G \to G$ are morphisms of varieties.

A morphism of algebraic groups is a map which is a group homomorphism and a morphism of varieties.

When considering an action of an algebraic group on a variety X we shall suppose that the map $G \times X \to X$ is a morphism of varieties.

The *general linear group* $\mathrm{GL}_n(K)$ is the open subset $D(\det)$ of $M_n(K)$, so an affine variety. It is an algebraic group thanks to the formula $g^{-1} = \operatorname{adj} g / \det g$. It acts by left multiplication or by conjugation on $M_n(K)$.

A *linear algebraic group* is an algebraic group which is isomorphic to a closed subgroup of $\mathrm{GL}_n(K)$. For example the special linear group, orthogonal group or any finite group. The additive and multiplicative groups of the field are

$$G_a = (K, +) \cong \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in K \}, \quad G_m = (K \setminus \{0\}, \times) = \mathrm{GL}_1(K).$$

Any finite product of linear algebraic groups is a linear algebraic group, using that $\mathrm{GL}_n(K) \times \mathrm{GL}_m(K)$ embeds in $\mathrm{GL}_{n+m}(K)$.

Remark. Any linear algebraic group is an affine variety, and conversely one can show that any affine algebraic group is linear, see for example Humphreys, Linear algebraic groups, section 8.6. An elliptic curve is an example of an algebraic group which is a projective variety, so not linear.

Lemma. *A connected algebraic group is an irreducible variety.*

Proof. Write the group as a union of irreducible components $G = G_1 \cup \cdots \cup G_n$. Since G_1 is not a subset of the union of the other components, some element $g \in G_1$ does not lie in any other component. Now any two elements of an algebraic group look the same, since multiplication by any $h \in G$ defines an isomorphism $G \to G$. It follows that every element of G lies in only one irreducible component. Thus G is the disjoint union of its irreducible components. But then the components are open and closed, and since G is connected, there is only one component. □

Let V be a vector space of dimension n, with basis $e_1, \dots, e_n$. We write $\mathrm{Bil}(n)$ for the set of bilinear maps $V \times V \to V$. A map $\mu \in \mathrm{Bil}(n)$ is given by its structure constants $(c_{ij}^k) \in K^{n^3}$ with

$$\mu(e_i, e_j) = \sum_k c_{ij}^k e_k.$$

Equivalently $\mathrm{Bil}(n) \cong \mathrm{Hom}(V \otimes V, V)$, Thus it is affine space $\mathbb{A}^{n^3}$.

We write $\mathrm{Ass}(n)$ for the subset consisting of associative multiplications. This is a closed subset of $\mathrm{Bil}(n)$, hence an affine variety, since it is defined by the equations

$$\mu(\mu(e_i, e_j), e_k) = \mu(e_i, \mu(e_j, e_k)),$$

that is

$$\sum_p c_{ij}^p c_{pk}^s = \sum_q c_{iq}^s c_{jk}^q$$

for all s.

We write $\mathrm{Alg}(n)$ for the subset of associative unital multiplications, so algebra structures on V.

Theorem. $\mathrm{Alg}(n)$ *is an affine open subset of* $\mathrm{Ass}(n)$*, hence an affine variety. The algebraic group* $\mathrm{GL}(V)$ *acts by basis change, and the orbits correspond to isomorphism classes of* n*-dimensional algebras.*

Proof. (i) We use that a vector space A with an associative multiplication has a 1 if and only if there is some $a \in A$ for which the maps $\ell_a, r_a : A \to A$ of left and right multiplication by a are invertible.

Namely, if $u = \ell_a^{-1}(a)$, then $au = a$. Thus $aub = ab$ for all b, so since ℓ_a is invertible, $ub = b$. Thus u is a left 1. Similarly there is a right 1, and they must be equal.

(ii) For the algebra V with multiplication μ, write ℓ_a^μ and r_a^μ for left and right multiplication by $a \in V$. Then $\mathrm{Alg}(n) = \bigcup_{a \in V} D(f_a)$ where $f_a(\mu) = \det(\ell_a^\mu)\det(r_a^\mu)$. Thus $\mathrm{Alg}(n)$ is open in $\mathrm{Ass}(n)$.

(iii) The map

$$\mathrm{Alg}(n) \to V, \quad \mu \mapsto \text{the 1 for } \mu$$

is a morphism of varieties, since on $D(f_a)$ it is given by $(\ell_a^\mu)^{-1}(a)$, whose components are rational functions, with $\det(\ell_a^\mu)$ in the denominator.

(iv) $\mathrm{Alg}(n)$ is affine. In fact

$$\mathrm{Alg}(n) \cong \{(\mu, u) \in \mathrm{Ass}(n) \times V \mid u \text{ is a 1 for } \mu\}.$$

The right hand side is a closed subset, hence it is affine. Certainly there is a bijection, and the maps both ways are morphisms.

(v) Last statement is clear. □

Example. The structure of $\mathrm{Alg}(n)$ is known for small n. For example $\mathrm{Alg}(4)$ has 5 irreducible components, of dimensions 15, 13, 12, 12, 9. See P. Gabriel, Finite representation type is open, 1974.

6.2 Schemes and varieties of modules

More general than varieties are schemes. I only discuss affine schemes, using representable functors rather than sheaves. See:

- M. Demazure and P. Gabriel, Groupes Algébriques, 1970. Partial English translation, Introduction to Algebraic Geometry and Algebraic Groups, 1980.
- W. C. Waterhouse, Affine group schemes, 1979.
- D. Eisenbud and J. Harris, The geometry of schemes, 2000. (Chapter VI)

Let K be a commutative ring. We write K-comm for the category of commutative K-algebras, or equivalently commutative rings R equipped with a homomorphism $K \to R$.

Definition. The category of *affine (K-)schemes* is the category of representable (covariant) functors

$$F : K\text{-comm} \to \mathrm{Sets}$$

with morphisms given by natural transformations. (These are not additive categories.)

Recall that a functor F is said to be *representable* if there is an object A in the category (a commutative K-algebra) such that

$$F(-) \cong \mathrm{Hom}_{K\text{-comm}}(A, -).$$

By Yoneda's lemma, the functor $A \mapsto \mathrm{Hom}_{K\text{-comm}}(A, -)$ defines an anti-equivalence from K-comm to the category of affine schemes.

Examples. (i) $\mathbf{A}^n$ is the affine scheme with $\mathbf{A}^n(R) = R^n$. It is represented by the polynomial ring $K[X_1, \ldots, X_n]$, since

$$\mathrm{Hom}_{K\text{-comm}}(K[X_1, \ldots, X_n], R) = R^n.$$

(ii) Any subset S of $K[X_1, \ldots, X_n]$ defines a functor $\mathbf{V}(S)$ by

$$\mathbf{V}(S)(R) = \{(x_1, \ldots, x_n) \in R^n : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in S\}.$$

It is an affine scheme, represented by the algebra $K[X_1, \ldots, X_n]/(S)$.

Definition. The affine scheme represented by A is

- *algebraic* if A is f.g. as a K-algebra (and K is a noetherian ring).
- *reduced* if A is reduced.

We immediately get:

Proposition. *If K is an algebraically closed field, then there is an equivalence*

$$\text{Cat. of affine varieties} \to \text{Cat. of reduced affine algebraic schemes}$$

$$X \mapsto \mathrm{Hom}_{K\text{-comm}}(K[X], -).$$

We usually identify an affine variety with the reduced affine algebraic scheme. Note that if X is the functor, then the underlying set for the variety is $X(K)$.

Lemma. *Given an affine (algebraic) scheme F, there is a reduced affine (algebraic) scheme F_{red} and a morphism $F_{\mathrm{red}} \to F$ such that for all R the map*

$$F_{\mathrm{red}}(R) \to F(R)$$

is injective, and a bijection for R reduced. This defines a functor $F \mapsto F_{\mathrm{red}}$ which is right adjoint to the inclusion of reduced affine (algebraic) schemes into affine (algebraic) schemes.

Proof. If $F(-) = \mathrm{Hom}(A, -)$ we set $F_{\mathrm{red}}(-) = \mathrm{Hom}(A_{\mathrm{red}}, -)$. The natural map $A \to A_{\mathrm{red}}$ gives a morphism $F_{\mathrm{red}} \to F$. □

For example $\mathbf{V}(S)$ is algebraic. It is reduced if and only if $K[X_1, \dots, X_n]/(S)$ is reduced. The scheme $\mathbf{V}(S)_{\mathrm{red}}$ is represented by $K[X_1, \dots, X_n]/\sqrt{(S)}$

Remark. If K is any commutative ring, then an *affine group scheme* over K is a representable functor $F : K\text{-comm} \to \text{Groups}$. If A is the commutative K-algebra representing F, then A becomes a Hopf algebra, see Waterhouse §1.4. For example $\mathbf{GL}_n$ is the affine group scheme with $\mathbf{GL}_n(R) = \mathrm{GL}_n(R)$ for all R. It is represented by the algebra $K[X_{ij}, 1/\det]$, so reduced.

Let A be a f.g. K-algebra (possibly non-commutative). A d-dimensional A-module V can be considered as a homomorphism $A \to \mathrm{End}_K(V)$, or choosing a basis of V, as a homomorphism $A \to M_d(K)$.

Definition. Let A be a f.g. K-algebra and d a natural number. We define the scheme $\mathbf{Rep}(A, d)$ (or $\mathbf{Mod}(A, d)$) of d-dimensional A-modules to be the functor

$$K\text{-comm} \to \text{Sets}, \quad R \mapsto \mathrm{Hom}_{K\text{-algebra}}(A, M_d(R)).$$

Lemma. $\mathbf{Rep}(A, d)$ *is an affine algebraic K-scheme.*

Proof. Write A as a quotient of a free algebra, say $A = K\langle X_1, \dots, X_k\rangle/I$. Then

$$\mathbf{Rep}(A,d)(R) = \{(A_1,\dots,A_k) \in M_d(R)^k : p(A_1,\dots,A_k) = 0 \text{ for all } p \in I\}.$$

Consider the polynomial ring $S = K[X_{rij} : 1 \le r \le k, 1 \le i,j \le d]$ and let $U_r \in M_d(S)$ be the matrix with (i,j) entry X_{rij}. If $p \in K\langle X_1,\dots,X_k\rangle$, then considering it as a noncommutative polynomial, we obtain $p(U_1,\dots,U_k) \in M_d(S)$. Then $\mathbf{Rep}(A,d)(R)$ is in bijection with

$$\mathrm{Hom}_{K\text{-algebra}}(S/J, R).$$

where J is the ideal generated by all entries of $p(U_1,\dots,U_k)$ with $p \in I$. □

Definition. If K is algebraically closed, the variety corresponding to the reduced scheme is denoted $\mathrm{Rep}(A,d)$. Thus

$$\mathrm{Rep}(A,d) = \mathrm{Hom}_{K\text{-algebra}}(A, M_d(K)),$$

and if $A = K\langle X_1,\dots,X_k\rangle/I$, we have

$$\mathrm{Rep}(A,d) = \{(A_1,\dots,A_k) \in M_d(K)^k : p(A_1,\dots,A_k) = 0 \text{ for all } p \in I\}.$$

There is an action of $\mathrm{GL}_d(K)$ on $\mathrm{Rep}(A,d)$ by conjugation, so given by $(g\cdot\theta)(a) = g\theta(a)g^{-1}$. The orbits correspond to isomorphism classes of d-dimensional modules.

Examples. (1) $\mathbf{Rep}(A,1)$ is the affine algebraic scheme given by the largest commutative quotient of A, which is $A/([A,A])$, where $[A,A] = \{ab - ba : a,b \in A\}$. Then the variety has coordinate ring

$$K[\mathrm{Rep}(A,1)] = (A/([A,A]))/\sqrt{0}.$$

(2) The *nilpotent variety* consists of the $d \times d$ nilpotent matrices over K. In fact that dth power of such a matrix must be zero, so the nilpotent variety is

$$N_d = \{A \in M_d(K) : A^d = 0\} = \mathrm{Rep}(K[x]/(x^d), d)$$

(3) The *commuting variety* consists of the pairs of commuting matrices

$$C_d = \{(A,B) \in M_d(K)^2 : AB = BA\} = \mathrm{Rep}(K[x,y], d).$$

Definition. We can do the same thing with quivers and dimension vectors. For an algebra $A = KQ/I$ and a dimension vector α with $d = \sum_i \alpha_i$, we define an affine scheme $\mathbf{Rep}(A,\alpha)$ with

$$\mathbf{Rep}(A,\alpha)(R) = \{\theta \in \mathrm{Hom}_{K\text{-algebra}}(A, M_d(R)) : \theta(e_i) = I_i\}$$

where I_i is the block matrix with blocks of size α_j, the ith diagonal block the identity matrix and all other blocks zero. We can identify

$$\mathbf{Rep}(A,\alpha)(R) \subseteq \mathbf{Rep}(KQ,\alpha)(R) \cong \prod_{a:i\to j} \mathrm{Hom}_R(R^{\alpha_i}, R^{\alpha_j}).$$

The linear algebraic group

$$\mathrm{GL}(\alpha) = \prod_{i\in Q_0} \mathrm{GL}_{\alpha_i}(K)$$

embedded diagonally in $\mathrm{GL}_d(K)$ acts by conjugation on the variety $\mathrm{Rep}(A,\alpha)$ and the orbits correspond to the isomorphism classes of representations of A of dimension vector α.

Example. Let Q be the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ and I the ideal generated by ba.

$$\mathrm{Rep}(KQ/I,(2,2,1)) = \{(a,b) \in M_{2\times 2}(K) \times M_{1\times 2}(K) : ba = 0\}.$$

6.3 Geometric quotients and projective space

Definition. Suppose that a linear algebraic group G acts on a space with functions X.

Let X/G be the set of orbits Gx and let $\pi : X \to X/G$ be the quotient map. We can turn X/G into a space with functions via

- A subset U of X/G is open iff $\pi^{-1}(U)$ is open in X. (Thus also U is closed iff $\pi^{-1}(U)$ is closed in X.)

- A function $f : U \to K$ is in $\mathcal{O}(U)$ iff $f\pi \in \mathcal{O}(\pi^{-1}(U))$.

Lemma. *X/G is a space with functions, π is a morphism, and it is a categorical quotient in the category of spaces with functions. That is, any morphism $\phi : X \to Z$ which is constant on G-orbits factors uniquely as $\psi\pi$ for some morphism $\psi : X/G \to Z$.*

The proof is easy.

Definition. If X is a variety and X/G is also variety, we call X/G or π a *geometric quotient.*

Remark. (i) A necessary condition to have a geometric quotient is that the orbits of G must be closed in X, since Gx is the inverse image of a point in X/G, and any point in a variety is closed.

The multiplicative group $G_m = \mathrm{GL}_1(K)$ acts on a vector space V by rescaling. But the only closed orbit is $\{0\}$, so V/G is not a geometric quotient.

(ii) If X/G is a geometric quotient, then it is a categorical quotient in the category of varieties. Categorical quotients can exist more generally, but they may not contain interesting information.

The closure of the G_m-orbit of $x \in V$ is the subspace spanned by x, so it contains 0. Suppose $\phi : X \to Z$ is a morphism to a variety which is constant on orbits and $x \in X$. Then $\phi(gx) = \phi(x)$ for all $g \in G$. Thus $G_m x \subseteq \phi^{-1}(\phi(x))$. Since the singleton sets on Z are closed, so is this, so $0 \in \phi^{-1}(\phi(x))$, so $\phi(x) = \phi(0)$. Thus ϕ is constant. It follows that the map $V \to \{pt\}$ is a categorical quotient in the category of varieties. Thus everything interesting is lost.

(iii) If the orbits aren't closed, a better approach is 'geometric invariant theory'. More later, maybe. Even if the orbits of G are closed, there may not be a geometric quotient. See for example H. Derksen, Quotients of algebraic group actions, in: Automorphisms of affine spaces, 1995. Maybe you need to work with algebraic spaces rather than varieties. See for example J. Kollár, Quotient spaces modulo algebraic groups, Ann. of Math. 1997.

Lemma. *If Y is a variety and G acts on $G \times Y$ by $g(g', y) = (gg', y)$, then the projection morphism $p : G \times Y \to Y$ is a geometric quotient, i.e. $(G \times Y)/G \cong Y$.*

Proof. The set of orbits is in bijection with Y. To check that they are the same spaces with functions, we need to see

(i) A set U is open in $Y \Leftrightarrow p^{-1}(U)$ is open. The implication $\Rightarrow$ holds because p is a morphism. The other implication holds because $U = p(p^{-1}(U))$ and any projection morphism is open.

(ii) A function f on an open subset U of Y is regular $\Leftrightarrow fp$ is regular on $G \times U$. The implication $\Rightarrow$ is because p is a morphism. The other implication holds since f is the composition of fp and the morphism $U \to G \times U$, $x \mapsto (1, x)$. □

Definition. Given an action of a linear algebraic group G on a variety X and a morphism of varieties $\pi : X \to Y$ which is constant on G-orbits, we say that π is a *Zariski-locally-trivial principal G-bundle* if each point in Y has an open neighbourhood U, such that there is an isomorphism

$$\phi : G \times U \to \pi^{-1}(U)$$

with $\pi\phi$ the projection onto U and such that ϕ commutes with the natural G-action, $\phi(g'g, u) = g'\phi(g, u)$ for $g, g' \in G$ and $u \in U$.

Remark. A basic reference for fibre bundles in algebraic geometry is J.-P. Serre, Espaces fibrés algébriques, Séminaire Claude Chevalley, 1958. In general a principal G-bundle need not be Zariski-locally-trivial, but only locally trivial for the 'étale topology'; but be warned, this is a 'Grothendieck topology', which is not a topology in the usual sense. However, Serre showed that $\mathrm{SL}_n(K)$ and $\mathrm{GL}_n(K)$ are 'special' groups, meaning that any principal bundle for these groups is automatically Zariski-locally-trivial.

Lemma. *Let* $\pi : X \to Y$ *be a Zariski-locally-trivial principal* G*-bundle.*
(i) π *is surjective and each fibre* $\pi^{-1}(y)$ *is isomorphic to* G*.*
(ii) π *induces an isomorphism* $X/G \cong Y$*, so* X/G *is a geometric quotient.*
(iii) π *is* universally open*, that is, if* Z *is a variety, and* U *is an open subset of* $X \times Z$*, then its image in* $Y \times Z$ *is open.*
(iv) π *is* universally submersive*, that is, if* Z *is a variety, and* V *is a subset of* $Y \times Z$*, then* V *is open if and only if its inverse image in* $X \times Z$ *is open.*

Proof. Straightforward, using previous results. □

Remark. The book Mumford, Fogarty and Kirwan, Geometric Invariant Theory, 3rd edition, 1994, claims in remark (4) on page 6 that any geometric quotient is universally open. But this does not seem to be true. In the first edition universally submersive was included as part of the definition of a geometric quotient. The definition was changed in the second edition, but the remark was not.

Definition. If V is a vector space, then G_m acts on $V_* := V \setminus \{0\}$ by rescaling, and we define *projective space* to be

$$\mathbb{P}(V) = V_*/G_m.$$

We can identify $\mathbb{P}(V)$ with the set of 1-dimensional subspaces of V. Working with coordinates, we define $\mathbb{P}^n = \mathbb{P}(K^{n+1})$ and denote the G_m-orbit of $(x_0, \dots, x_n)$ by $[x_0 : \dots : x_n]$.

Proposition. $\mathbb{P}(V)$ *is a variety and the projection* $p : V_* \to \mathbb{P}(V)$ *is a Zariski-locally-trivial principal* G_m*-bundle, so a geometric quotient. In fact*

$$\mathbb{P}^n = U_0 \cup \dots \cup U_n$$

where $U_i = \{[x_0 : \dots : x_n] : x_i \neq 0\}$ *is an open subset of* $\mathbb{P}^n$ *isomorphic to* $\mathbb{A}^n$*.*

Proof. U_i is open, since it lifts to the open set

$$W_i = \{(x_0, \dots, x_n) \in K^{n+1} : x_i \neq 0\}$$

of K_*^{n+1}. We have an isomorphism

$$W_i \to G_m \times \mathbb{A}^n, \quad (x_0, \dots, x_n) \mapsto (x_i, (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i)).$$

The action of G_m on W_i corresponds to the multiplication action on the first factor of $G_m \times \mathbb{A}^n$, so $U_i = W_i/G_m \cong \mathbb{A}^n$.

To see that $\mathbb{P}^n$ is separated, it suffices to show that

$$D_{ij} = \Delta_{\mathbb{P}^n} \cap (U_i \times U_j)$$

is closed in $U_i \times U_j$ for all i, j. Identify $U_i \cong \{x \in K^{n+1} : x_i = 1\}$. Then

$$U_i \times U_j \cong \{(x, y) \in K^{n+1} \times K^{n+1} : x_i = y_j = 1\},$$

and

$$D_{ij} \cong \{(x, y) : x_r y_s = x_s y_r \text{ for all } r, s\},$$

so it is closed. □

Properties. (1) $\mathbb{P}^n$ is a disjoint union $U_0 \cup V_0$ where

$$V_0 = \{[x_0 : \cdots : x_n] \mid x_0 = 0\}$$

is a closed subvariety isomorphic to $\mathbb{P}^{n-1}$. Repeating, we can write $\mathbb{P}^n$ as a disjoint union of copies of $\mathbb{A}^n$, $\mathbb{A}^{n-1}$, ..., $\mathbb{A}^0 = \{pt\}$.

(2) $\mathcal{O}(\mathbb{P}^n) = K$, so $\mathbb{P}^n$ is not affine for $n > 0$. For example a regular function f on $\mathbb{P}^1$ induces regular functions on $U_i \cong \mathbb{A}^1$, so there are polynomials $p, q \in K[X]$ with $f([x_:x_1]) = p(x_1/x_0)$ for $x_0 \neq 0$ and $f([x_0 : x_1])) = q(x_0/x_1)$ for $x_1 \neq 0$. Thus $p(t) = q(1/t)$ for $t \neq 0$. Thus both p and q are constant polynomials, so f is constant.

Definition. A *(quasi)projective variety* is a variety which is, or is isomorphic to, a (locally) closed subset of $\mathbb{P}^n$.

Example. A curve in $\mathbb{A}^2$, for example

$$\{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + x\},$$

can be homogenized to give a curve in $\mathbb{P}^2$

$$\{[w : x : y] \in \mathbb{P}^2 : y^2 w = x^3 + xw^2\}.$$

Recall that $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. On the affine space part $w \neq 0$, we recover the original curve. On the line at infinity $w = 0$ the equation is $x^3 = 0$, which has solution $x = 0$, giving rise to one point at infinity $[w : x : y] = [0 : 0 : 1]$.

For the curve $y^3 = x^3 + x$, the points at infinity are $[0 : 1 : \epsilon]$ where $\epsilon^3 = 1$.

Theorem (Segre)**.** *There is a closed embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$, given by*

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \mapsto [x_0 y_0 : \cdots : x_i y_j : \cdots : x_n y_m].$$

Thus a product of (quasi-)projective varieties is (quasi-)projective.

For a proof see Kempf, Theorem 3.2.1.

6.4 Grassmannians

Definition. If V is a vector space of dimension n, the *Grassmannian* $\mathrm{Gr}(V,d)$ is the set of subspaces of V of dimension d.

We write $\mathrm{Inj}(K^d,V)$ for the set of injective linear maps $K^d \to V$. It is open in $\mathrm{Hom}(K^d,V)$, so a variety. The group $\mathrm{GL}_d(K)$ act on $\mathrm{Inj}(K^d,V)$ by $g\cdot\theta = \theta g^{-1}$. The map

$$\pi : \mathrm{Inj}(K^d,V) \to \mathrm{Gr}(V,d), \quad \theta \mapsto \mathrm{Im}\,\theta$$

is surjective and the fibres are the orbits of $\mathrm{GL}_d(K)$, so it identifies $\mathrm{Gr}(V,d)$ with $\mathrm{Inj}(K^d,V)/\mathrm{GL}_d(K)$. Thus $\mathrm{Gr}(V,d)$ becomes a space with functions and π a morphism.

Theorem. *(i) There is a closed embedding called the* Plücker map *of of* $\mathrm{Gr}(V,d)$ *in* $\mathbb{P}^N$, *where* $N = \binom{n}{d} - 1$. *Thus the Grassmannian* $\mathrm{Gr}(V,d)$ *is a projective variety.*

(ii) $\pi : \mathrm{Inj}(K^d,V) \to \mathrm{Gr}(V,d)$ *is a Zariski-locally-trivial principal bundle.*

We use the following facts.

Lemma (1)**.** *Given a mapping* $\theta : X \to Y$ *between spaces with functions and an open covering* $Y = \bigcup U_\lambda$, *the map* θ *is a closed embedding if and only if its restrictions* $\theta_\lambda : \theta^{-1}(U_\lambda) \to U_\lambda$ *are closed embeddings.*

Proof. Suppose the θ_λ are closed embeddings. Then $Y \setminus \mathrm{Im}\,\theta$ is the union of the sets $U_\lambda \setminus \mathrm{Im}\,\theta_\lambda$, so it is open in Y, hence $\mathrm{Im}\,\theta$ is closed.

Clearly θ is 1-1, so it defines a bijective morphism $X \to \mathrm{Im}\,\theta$. We need to show that the inverse map $g : \mathrm{Im}\,\theta \to X$ is a morphism. But $\mathrm{Im}\,\theta$ has an open covering by sets of the form $U_\lambda \cap \mathrm{Im}\,\theta$, and the restriction of g to each of these sets is a morphism, hence so is g. □

Lemma (2)**.** *If* $g : X \to Y$ *is a morphism of spaces with functions and* Y *is separated, then the map* $X \to X \times Y$, $x \mapsto (x, g(x))$ *is a closed embedding.*

Proof. Its image is the inverse image of the diagonal Δ_Y under the map $X \times Y \to Y \times Y$, $(x,y) \mapsto (g(x), y)$. Since Y is separated, this is closed. Now the projection from $X \times Y \to X$ gives an inverse map from the image to X. □

Sketch proof of the theorem. Fixing a basis $e_1,\dots,e_n$ of V, we identify $\mathrm{Inj}(K^d,V)$ with the set of $n \times d$ matrices of rank d.

Let I be a subset of $\{1,\dots,n\}$ with d elements. If $A \in \mathrm{Inj}(K^d,V)$, we write A_I for the square matrix obtained by selecting the rows of A in I. Then $\det(A_I)$ is a minor of A. We write A'_I for the $(n-d)\times d$ matrix obtained by deleting the rows in I.

We write elements of $\mathbb{P}^N$ in the form $[x_I]$ with $x_I \in K$, not all zero, where I runs through the subsets of $\{1,\dots,n\}$ of size d.

We consider the morphism

$$f : \mathrm{Inj}(K^d, V) \to \mathbb{P}^N, \quad A \mapsto [\det(A_I)].$$

The action of $g \in \mathrm{GL}_d(K)$ on $\mathrm{Inj}(K^d, V)$ sends A to Ag^{-1}, and $\det((Ag^{-1})_I) = \det(A_I)\det(g)^{-1}$, so f is constant on the orbits of $\mathrm{GL}_d(K)$. Thus it induces a morphism $\overline{f} : \mathrm{Gr}(V, d) \to \mathbb{P}^N$.

Now $\mathbb{P}^N$ has an affine open covering by the sets $U_J = \{[x_I] : x_J \neq 0\}$ for J a subset of $\{1, \dots, n\}$ with d elements. Then $X_J = f^{-1}(U_J)$ is an open subset of $\mathrm{Inj}(K^d, V)$ and $Y_J = \overline{f}^{-1}(U_J)$ is an open subset of $\mathrm{Gr}(V, d)$, and we get a commutative diagram

$$\begin{array}{ccccccc} X_J & \longrightarrow & X_J/\mathrm{GL}_d(K) & \xrightarrow{\cong} & Y_J & \longrightarrow & U_J \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Inj}(K^d, V) & \longrightarrow & \mathrm{Inj}(K^d, V)/\mathrm{GL}_d(K) & \xrightarrow{\cong} & \mathrm{Gr}(V, d) & \xrightarrow{\overline{f}} & \mathbb{P}^N \end{array}$$

(i) By Lemma 1 it suffices to show that $Y_J \to U_J$ is a closed embedding for all J. Now X_J consists of the matrices A such that A_J is invertible. Thus there is an isomorphism of varieties

$$\phi_J : \mathrm{GL}_d(K) \times M_{(n-d)\times d}(K) \to X_J, \quad \phi_J(g, B) = \hat{B}g^{-1},$$

where $\hat{B} \in M_{n\times d}(K)$ denotes the matrix A with $A_J = I_d$ and $A'_J = B$. This ensures that $\phi_J(g'g, B) = g' \cdot \phi_J(g, B)$ (where we recall that the action of $\mathrm{GL}_d(K)$ on $\mathrm{Inj}(K^d, V)$ is given by $g' \cdot A = A(g')^{-1}$). Thus

$$Y_J \cong X_J/\mathrm{GL}_d(K) \cong M_{(n-d)\times d}(K),$$

so it is an affine variety. We can identify U_J with $\mathbb{A}^N = \{(x_I)_I : I \neq J\}$, and the map $Y_J \to U_J$ with the map

$$M_{(n-d)\times d}(K) \to \mathbb{A}^N, \quad B \mapsto (\det \hat{B}_I)_I.$$

Now observe that if we take I to be equal to J, except that we omit the j element, and instead insert the ith element of $\{1, \dots, n\} \setminus J$, then $\det(\hat{B}_I) = \pm b_{ij}$. For example if $J = \{1, 2, \dots, d\}$, then

$$\hat{B} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ b_{11} & b_{12} & \dots & b_{1d} \\ b_{21} & b_{22} & \dots & b_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-d,1} & b_{n-d,2} & \dots & b_{n-d,d} \end{pmatrix}$$

and if $I = \{1, \ldots, j-1, j+1, \ldots, d, d+i\}$, then

$$\hat{B}_I = \begin{pmatrix} 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\ b_{i1} & \ldots & b_{i,j-1} & b_{ij} & b_{i,j+1} & \ldots & b_{id} \end{pmatrix}$$

so $\det(\hat{B}_I) = (-1)^{d-j} b_{ij}$. Thus, up to sign, the map $Y_J \to U_J$ is of the form $Y_J \to Y_J \times W$ for some morphism $Y_J \to W$, and by Lemma 2 this is a closed embedding.

(ii) The Y_J give an open cover of $Gr(V, d)$, and the isomorphisms ϕ_J shows that the map $\pi : \mathrm{Inj}(K^d, V) \to \mathrm{Gr}(V, d)$ is locally a projection. □

Lemma (3)**.** *If* $\mathrm{Surj}(V, K^c) \subseteq \mathrm{Hom}(V, K^c)$ *denotes the variety of surjective linear maps, where* $c + d = \dim V$*, then the map* $\mathrm{Surj}(V, K^c) \to \mathrm{Gr}(V, d)$*,* $\phi \mapsto \mathrm{Ker}\,\phi$ *is a morphism of varieties.*

Sketch. We check this locally. Identify $\mathrm{Surj}(V, K^c)$ with the set of matrices $C \in M_{c\times n}(K)$ of rank c.

Given a subset I of $\{1, \ldots, n\}$ of size d, let C_I be the $c \times c$ matrix obtained by deleting the columns in I and C'_I the $c \times d$ matrix obtained by keeping only the columns in I.

Let W_I be the open subset of $\mathrm{Surj}(V, K^c)$ consisting of the matrices C with C_I invertible. As I varies, this gives an open cover of $\mathrm{Surj}(V, K^c)$. Thus it suffices to show that the restriction to W_I is a morphism.

Now we have a map of varieties

$$W_I \xrightarrow{f} \mathrm{Inj}(K^d, V)$$

where $f(C)$ is the $n \times d$ matrix A with $A_I = I_d$ and $A'_I = -(C_I)^{-1}(C'_I)$. Observe that we have an exact sequence

$$0 \to K^d \xrightarrow{A} K^n \xrightarrow{C} K^c \to 0.$$

The composition is zero since it is $C_I A'_I + C'_I A_I = 0$. Thus the composition of f and the map $\mathrm{Inj}(K^d, V) \to \mathrm{Gr}(V, d)$ is the required map $W_I \to \mathrm{Gr}(V, d)$, and it is a morphism of varieties. □

Remark. We have turned $\mathrm{Gr}(V,d)$ into a space with functions by realizing it as a quotient $\mathrm{Inj}(K^d,V)/\mathrm{GL}_d(K)$, using the map sending an injective map to its image. Using Lemma 3 and similar results, two other possibilities give the same structure.

(i) $\mathrm{Surj}(V,K^c)/\mathrm{GL}_c(K)$, and

(ii) $\mathrm{Exact}(K^d,V,K^c)/(\mathrm{GL}_d(K)\times\mathrm{GL}_c(K))$ where $\mathrm{Exact}(K^d,V,K^c)$ is the set of exact sequences $0\to K^d\to V\to K^c\to 0$.

Lemma (4). *The set $S=\{(U,U',\theta):\theta(U)\subseteq U'\}$ is a closed subset of the product*

$$\mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')\times\mathrm{Hom}(V,V').$$

Thus, fixing θ, the subset $\{(U,U'):\theta(U)\subseteq U'\}$ is closed in $\mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')$.

Proof. We realise

$$\mathrm{Gr}(V,d)=\mathrm{Inj}(K^d,V')/\mathrm{GL}(d),\quad \mathrm{Gr}(V',d')=\mathrm{Surj}(V',K^c)/\mathrm{GL}_c(K),$$

where $c=\dim V'-d'$. Then we have a closed subset

$$C=\{(f,g,\theta)\in\mathrm{Inj}(K^d,V)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V'):g\theta f=0\}$$

whose complement C' is sent under the map

$$\pi:\mathrm{Inj}(K^d,V)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V')\to\mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')\times\mathrm{Hom}(V,V')$$

to the complement S' of S. To show this is open, we factorize π as

$$\mathrm{Inj}(K^d,V)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V')\xrightarrow{\pi_1}\mathrm{Gr}(V,d)\times\mathrm{Surj}(V',K^c)\times\mathrm{Hom}(V,V')$$

$$\xrightarrow{\pi_2}\mathrm{Gr}(V,d)\times\mathrm{Gr}(V',d')\times\mathrm{Hom}(V,V').$$

Since $\mathrm{Inj}(K^d,V)\to\mathrm{Gr}(V,d)$ is a Zariski-locally-trivial principal bundle it is universally open, so $\pi_1(C')$ is open, and then since $\mathrm{Surj}(V',K^c)\to\mathrm{Gr}(V',d')$ is a Zariski-locally-trivial principal bundle it too is universally open, so $S'=\pi(C')=\pi_2(\pi_1(C'))$ is open. □

Remark. Let G be a linear algebraic group and let H be a closed subgroup. We consider the action of H on G by left multiplication (or by the formula $h\cdot g=gh^{-1}$), the set of orbits G/H is then the set of right (respectively left) cosets of H in G.

(i) Fix $\theta_0\in\mathrm{Inj}(K^d,V)$, say with image W. It is easy to see that the map

$$\mathrm{GL}(V)\to\mathrm{Inj}(K^d,V),\quad g\mapsto g\theta_0$$

is a Zariski-locally-trivial principal S-bundle, where

$$S=\{s\in\mathrm{GL}(V):s\theta_0=\theta_0\},$$

the pointwise stabilizer of W, so $\mathrm{Inj}(K^d, V) \cong \mathrm{GL}(V)/S$.

(ii) The map

$$\mathrm{GL}(V) \to \mathrm{Gr}(V, d), \quad g \mapsto g(W)(= \mathrm{Im}\, g\theta_0)$$

is a Zariski-locally-trivial principal H-bundle, where

$$H = \{g \in \mathrm{GL}(V) : g(W) = W\},$$

the setwise stabilizer of W, so $\mathrm{GL}(V)/H \cong \mathrm{Gr}(V, d)$.

(iii) Fix $0 \leq d_1 \leq \cdots \leq d_k \leq \dim V$. Using the lemma, the flag variety

$$\mathrm{Flag}(V, d_1, \dots, d_k) = \{0 \subseteq W_1 \subseteq \cdots \subseteq W_k \subseteq V : \dim W_i = d_i\}$$

is a closed subset of $\prod_i \mathrm{Gr}(V, d_i)$, hence a projective variety. It is isomorphic to $\mathrm{GL}(V)/P$ where P is the stabilizer of a given flag.

In fact quotients G/H are well-understood. It is known that:

- G/H is always a quasi-projective variety, so a geometric quotient. See T. A. Springer, Linear Algebraic Groups, Second edition, 1998, Corollary 5.5.6.

- If H is a normal subgroup, G/H is an affine variety, so a linear algebraic group. Springer, Proposition 5.5.10.

- G/H is a projective variety if and only if H contains a Borel subgroup (a maximal closed connected soluble subgroup of G). Springer, Theorem 6.2.7. In this case H is called a *parabolic subgroup*.

Definition. Let $A = KQ/I$ and let M be a finite dimensional A-module. Recall that its dimension vector is $\alpha \in \mathbb{N}^{Q_0}$ defined by $\alpha_i = \dim e_i M$. Let β be another dimension vector and let $d = \sum_{i=1}^n \beta_i$. We define

$$\mathrm{Gr}_A(M, \beta) = \{U \in \mathrm{Gr}(M, d) : U \text{ is an } A\text{-submodule of } M \text{ of dim. vector } \beta\}.$$

This is called a *Quiver Grassmannian*. This name is used even if A is not a path algebra, because we can always reduce to this case, since we can consider M as a KQ-module and $\mathrm{Gr}_A(M, \beta) = \mathrm{Gr}_{KQ}(M, \beta)$.

Proposition. *$\mathrm{Gr}_A(M, \beta)$ is a closed subset of $\mathrm{Gr}(M, d)$, so a projective variety.*

Proof. Being a submodule is a closed condition. Namely, given $a \in A$ we need $\hat{a}(U) \subseteq U$, where $\hat{a} : M \to M$ is the homothety $\hat{a}(m) = am$, and this is a closed condition by Lemma 4.

Amongst the submodules U of dimension d, the ones of dimension vector β are those with $\hat{e}_i$ having rank $\leq \beta_i$. This is also a closed condition. □

Alternatively, a submodule U is determined by the subspaces $e_i U \subseteq e_i M$, and so $\mathrm{Gr}_A(M, \beta)$ could be defined as a closed subset of $\prod_{i=1}^n \mathrm{Gr}(e_i M, \beta_i)$.

Remark. It is a theorem of M. Reineke that every projective variety is isomorphic to a quiver Grassmannian for an indecomposable representation of a quiver. See M. Reineke, Every projective variety is a quiver Grassmannian, Algebr. Represent. Theory 2013. It turned out that the result could have been known earlier, see for example the discussion in C. M. Ringel, Quiver Grassmannians and Auslander varieties for wild algebras, J. Algebra 2014.

Remark. We can vary the module M at the same time. Given A as before and dimension vectors α and β with $\sum \alpha_i = n$ and $\sum \beta_i = d$, we define

$$\operatorname{Rep}\operatorname{Gr}(A,\alpha,\beta) = \{(x,U) \in \operatorname{Rep}(A,\alpha) \times \operatorname{Gr}(K^n,d) : U \in \operatorname{Gr}_A(K_x,\beta)\}.$$

It is a closed subset, so a variety. To see this, for simplicity we do it without dimension vectors.

$$\operatorname{Rep}\operatorname{Gr}(A,n,d) = \{(x,U) \in \operatorname{Rep}(A,n) \times \operatorname{Gr}(K^n,d) : U \in \operatorname{Gr}_A(K_x,d)\}.$$

Let $c = n - d$. Then we have a Zariski-locally-trivial principal $\mathrm{GL}_d(K) \times \mathrm{GL}_c(K)$-bundle

$$\operatorname{Exact}(K^d, K^n, K^c) \to \operatorname{Gr}(K^n, d).$$

The set lifts to

$$\{(x,(\theta,\phi)) \in \operatorname{Rep}(A,n) \times \operatorname{Exact}(K^d,K^n,K^c) : \phi x(a)\theta = 0 \text{ for all } a \in A\}.$$

which is closed. Then using that the bundle is universally open, we get that our subset is closed.

Now there is a morphism $\pi : \operatorname{Rep}\operatorname{Gr}(A,\alpha,\beta) \to \operatorname{Rep}(A,\alpha)$ whose fibres are $\pi^{-1}(x) \cong \operatorname{Gr}_A(K_x,\beta)$.

7 Tools of algebraic geometry

7.1 Dimension

Definition. The *dimension* of a variety is the supremum of the n such that there is a chain of distinct (non-empty) irreducible closed subsets $X_0 \subset X_1 \subset \cdots \subset X_n$ in X. ($\dim \emptyset = -\infty$.)

If X is an affine variety, $\dim X$ is the Krull dimension of $K[X]$, the maximal length of a chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$.

Any irreducible variety X has a function field

$$K(X) = \operatorname*{colim}_U \mathcal{O}(U)$$

where U runs through the nonempty open subsets of X. If X is an irreducible affine variety, then $K(X)$ is the field of fractions of $K[X]$.

Lemma (1)**.** *If X is an irreducible affine variety, then* $\dim X$ *is the transcendence degree of the field extension $K(X)/K$.*

The proof is commutative algebra. As a consequence we get the following.

Lemma (2)**.** *(i)* $\dim \mathbb{A}^n = n$.

(ii) Any variety has finite dimension.

(iii) If $X \subseteq Y$ is a locally closed subset, then $\dim X \leq \dim Y$, *strict if Y is irreducible and X is a proper closed subset.*

(iv) If X is irreducible then $\dim X$ *=transcendence degree of $K(X)/K$. Thus if U is nonempty open in X,* $\dim U = \dim X$.

(v) If $X = Y_1 \cup \cdots \cup Y_n$, with the Y_i locally closed in X, then $\dim X = \max\{\dim Y_i\}$.

Proof. (i) By commutative algebra.

(iii) If X_i is a chain of irreducible closed subsets in X, then $\overline{X_i}$ is a chain of irreducible closed subsets of Y, and if $\overline{X_i} = \overline{X_{i+1}}$ then X_i is open in $\overline{X_i}$, so

$$X_{i+1} = X_i \cup (X_{i+1} \cap (\overline{X_i} \setminus X_i))$$

a union of two closed subsets, so $X_{i+1} = X_i$.

(v) for the special case when the Y_i are open in X. Take a chain $X_0 \subset X_1 \subset \cdots \subset X_n$ in X. Then X_0 meets some Y_i. Consider the chain $Y_i \cap X_0 \subset Y_i \cap X_1 \subset \cdots \subset Y_i \cap X_n$ in Y_i. Now $Y_i \cap X_j$ is nonempty and open in X_j, hence irreducible. The terms are distinct, for if $Y_i \cap X_j = Y_i \cap X_{j+1}$ then $X_{j+1} = X_j \cup (X_{j+1} \setminus Y_i)$ is a proper decomposition. Thus $\dim Y_i \geq n$.

(ii) Combine (i), (iii) and the special case of (v).

(iv) X is a union of affine opens, and these all have function field $K(X)$, so the dimension is given by the transcendence degree.

(v) in general. Suppose F is an irreducible closed subset of X. Then F is the union of the sets $\overline{F \cap Y_i}$. By irreducibility, some $\overline{F \cap Y_i} = F$. Thus $F \cap Y_i$ is open in F. Thus $\dim F = \dim F \cap Y_i \leq \dim Y_i$. □

Definition. A morphism $\theta : X \to Y$ of varieties, with X and Y irreducible, is *dominant* if its image is dense in Y.

Lemma (3)**.** *If* $\theta : X \to Y$ *is a morphism of varieties and* X *is irreducible, then* $Z = \overline{\operatorname{Im}\theta}$ *is irreducible, the restricted map* $\theta' : X \to Z$ *is dominant and it induces an injection* $K(Z) \to K(X)$*. Thus* $\dim Z \leq \dim X$*.*

The proof is straightforward.

Lemma (Main Lemma)**.** *If* $\pi : X \to Y$ *is a dominant morphism of irreducible varieties then any irreducible component of a fibre* $\pi^{-1}(y)$ *has dimension at least* $\dim X - \dim Y$*. Moreover, there is a nonempty open subset* $U \subseteq Y$ *with* $\dim \pi^{-1}(u) = \dim X - \dim Y$ *for all* $u \in U$*.*

See §I.8 of D. Mumford, The red book of varieties and schemes, 2nd edition, 1999.

Examples. (1) $\dim X \times Y = \dim X + \dim Y$. Reduce to the case of irreducible varieties, and then consider the projection $X \times Y \to Y$.

(2) A hypersurface in $\mathbb{A}^n$ is an irreducible closed subset of $\mathbb{A}^n$ of dimension $n-1$. They are exactly the zero sets $V(f)$ of irreducible polynomials $f \in K[X_1, \dots, X_n]$.

Namely, if f is irreducible then $V(f)$ is irreducible, a proper closed subset of $\mathbb{A}^n$, so dimension $< n$, but a fibre of $f : \mathbb{A}^n \to K$, so of dimension $\geq n-1$.

Conversely if $X \subseteq \mathbb{A}^n$ is an irreducible closed subset of dimension $n-1$ then $X = V(I)$, so $X \subseteq V(g)$ for some non-zero $g \in I$. But then $X \subseteq V(f)$ for some irreducible factor f of g, and these are equal by dimensions.

(3) The commuting variety C_d is irreducible of dimension $d^2 + d$. (Theorem of Motzkin and Taussky, 1955.) We follow R. M. Guralnick, A note on commuting pairs of matrices, 1992.

A $d \times d$ matrix A is *regular* or *non-derogatory* if it satisfies the following equivalent conditions

- in it's Jordan normal form, each Jordan block has a different eigenvalue,
- its minimal polynomial is equal to its characteristic polynomial,

- the matrices $I, A, A^2, \ldots, A^{d-1}$ are linearly independent.
- it turns K^d into a cyclic $K[X]$-module,
- all eigenspaces are at most one-dimensional,
- the only matrices which commute with A are polynomials in A,

The set of regular matrices is an open subset U of $M_d(K)$.

Suppose B is any matrix and R is regular. Consider the map

$$f : \mathbb{A}^1 \to M_d(K), \quad f(\lambda) = R + \lambda B.$$

The image meets U. Thus $f^{-1}(M_d(K) \setminus U)$ is a proper closed subset of $\mathbb{A}^1$, so finite. Thus $R + \lambda B$ is regular for all but finitely many λ. Thus $B + \nu R$ is regular for all but finitely many $\nu \in K$.

Every matrix A commutes with a regular matrix R. To see this we may suppose that A is in Jordan normal form. Now if A has diagonal blocks $J_{n_i}(\lambda_i)$ with the λ_i not necessarily distinct, then it commutes with the matrix with diagonal blocks $J_{n_i}(\mu_i)$, with the μ_i distinct.

Suppose $(A, B) \in C_d$ and there is an open set W of C_d containing (A, B) but not meeting $C'_d = C_d \cap (M_d \times U)$. Consider the map $g : \mathbb{A}^1 \to C_d$, $g(\nu) = (A, B + \nu R)$. Then $g^{-1}(C'_d)$ and $g^{-1}(W)$ are non-empty open subsets of $\mathbb{A}^1$ which don't meet. Impossible. Thus C'_d is dense in C_d.

Let P be the set of polynomials of degree $\leq d-1$. Now the map $h : P \times U \to C_d$, $(f(t), B) \mapsto (f(B), B))$ has image C'_d. Thus $C_d = \overline{\operatorname{Im} h}$, and since $P \times U$ is irreducible, so is C_d. Also this map is injective, so $\dim C_d = \dim U + \dim P = d^2 + d$.

7.2 Constructibility, upper semicontinuity and completeness

We give three important applications of the main lemma.

Definition. A subset of a variety is *constructible* if it is a finite union of locally closed subsets.

Example. The punctured x-axis $\{(x, 0) : x \neq 0\}$ is locally closed in $\mathbb{A}^2$. Its complement C is not locally closed, but it is constructible, the union of the plane minus the x-axis, and the origin. Clearly C is the image of the map $\mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (xy, y)$.

Lemma. *(i) The class of constructible subsets is closed under finite unions and intersections, complements, and inverse images.*

(ii) If V is a constructible subset of X and $\overline{V}$ is irreducible, then there is a nonempty open subset U of $\overline{V}$ with $U \subset V$.

Proof. (i) Exercise.

(ii) Write V as a finite union of locally closed subsets V_i. Then $\overline{V} = \bigcup_i \overline{V_i}$. Thus some $\overline{V_i} = \overline{V}$. Then V_i is open in $\overline{V}$. □

Theorem (Chevalley's Constructibility Theorem)**.** *The image of a morphism of varieties $\theta : X \to Y$ is constructible. More generally, the image of any constructible set is constructible.*

Proof. Sketch. We may assume that X is irreducible and then that $Y = \overline{\mathrm{Im}(\theta)}$. The main lemma says that $\mathrm{Im}(\theta)$ contains a dense open subset U of Y. Thus it suffices to prove that the image under θ of $X \setminus \theta^{-1}(U)$ is constructible. Now work by induction on dimension. □

Example. Let $A = KQ/I$. The set $\{x \in \mathrm{Rep}(A, \alpha) : K_x \text{ is indecomposable }\}$ is constructible in $\mathrm{Rep}(A, \alpha)$. Here K_x denotes the A-module of dimension vector α corresponding to x.

If $\alpha = \beta + \gamma$, then there is a direct sum map

$$f : \mathrm{Rep}(A, \beta) \times \mathrm{Rep}(A, \gamma) \to \mathrm{Rep}(A, \alpha)$$

sending (x, y) to the representation which has x and y as diagonal blocks. It is a morphism of varieties. Thus the map

$$\mathrm{GL}(\alpha) \times \mathrm{Rep}(A, \beta) \times \mathrm{Rep}(A, \gamma) \to \mathrm{Rep}(A, \alpha), \quad (g, x, y) \mapsto g.f(x, y)$$

has as image all modules which can be written as a direct sum of modules of dimensions β and γ. This is constructible. Thus so is the union of these sets over all non-trivial decompositions $\alpha = \beta + \gamma$. Hence so is its complement, the set of indecomposables.

Definition. A function $f : X \to \mathbb{Z}$ is *upper semicontinuous* if $\{x \in X : f(x) < n\}$ is open for all $n \in \mathbb{Z}$. Equivalently $\{x \in X : f(x) \geq n\}$ closed for all n.

Clearly a composition of a morphism and an upper semicontinuous function is upper semicontinuous.

Examples. (1) The map $\mathrm{Hom}(V, W) \to \mathbb{Z}$, $\theta \mapsto \dim \mathrm{Ker}\, \theta$ is upper semicontinuous, since the set where it is $\geq t$ is the set of maps of rank $\leq r = \dim V - t$, so identifying with matrices, the set where all minors of size $r + 1$ are zero.

(2) On the variety $\{(\theta, \phi) \in \mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) : \phi\theta = 0\}$, the map $(\theta, \phi) \mapsto \dim(\mathrm{Ker}\, \phi / \mathrm{Im}\, \theta)$ is upper semicontinuous, since it is equal to $\dim \mathrm{Ker}\, \theta + \dim \mathrm{Ker}\, \phi - \dim U$.

Definition. The *local dimension* of a variety X at a point $x \in X$, denoted $\dim_x X$ is the infemum of the dimensions of neighbourhoods of x. Equivalently it is the maximal dimension of an irreducible component containing x.

Any point $x \in X$ has a local ring

$$\mathcal{O}_{X,x} = \operatorname*{colim}_{x \in U} \mathcal{O}(U)$$

where the colimit is over all open neighbourhoods U of x, and then $\dim_x X$ is the Krull dimension of this local ring.

Theorem (Upper Semicontinuity Theorem). *If $\theta : X \to Y$ is a morphism, then the function $X \to \mathbb{Z}$, $x \mapsto \dim_x \theta^{-1}(\theta(x))$ is upper semicontinuous.*

Proof. Sketch. We may assume that X is irreducible, and then that $Y = \overline{\mathrm{Im}(\theta)}$. By the Main Lemma, the minimal value of the function is $\dim X - \dim Y$, and it takes this value on an open subset $\theta^{-1}(U)$ of X. Thus need to know for the morphism $X \setminus \theta^{-1}(U) \to Y \setminus U$. Now use induction. □

Definition. A *cone* in a vector space is a subset which contains 0 and is closed under multiplication by $\lambda \in K$. In particular any subspace is a cone.

Corollary. *Suppose X is a variety, V is a vector space, and for each x we have a cone V_x in V in such a way that $Y = \{(x, v) \in X \times V : v \in V_x\}$ is a closed subset of $X \times V$. Then the function $X \to \mathbb{Z}$, $x \mapsto \dim V_x$ is upper semicontinuous.*

Proof. Note that if C is a closed cone in V, then every irreducible component of C contains 0, so $\dim_0 C = \dim C$. Namely, let D be an irreducible component of C, there is a scaling map $f : \mathbb{A}^1 \times D \to C$, so $D \subseteq \overline{\mathrm{Im}\, f} \subseteq C$. Now $\overline{\mathrm{Im}\, f}$ is irreducible, so equal to D, and it contains 0.

Now if $i_x : V \to X \times V$ is the map $i_x(v) = (x, v)$, then $V_x = i_x^{-1}(Y)$, so it is closed in V, and if $\theta : Y \to X$ is the projection, then $\theta^{-1}(x) \cong V_x$.

Composing the upper semicontinuous function $Y \to \mathbb{Z}$, $(x, v) \mapsto \dim_{(x,v)} \theta^{-1}(\theta(x))$ with the zero section $\phi : X \to Y$, $x \mapsto (x, 0)$ gives an upper semicontinuous function

$$X \to \mathbb{Z}, \quad x \mapsto \dim_{(x,0)} \theta^{-1}(\theta(x)) = \dim_0 V_x = \dim V_x$$

since V_x is a cone. □

Example. The function $\mathrm{Rep}(A, d) \to \mathbb{Z}$, $x \mapsto \dim \mathrm{End}_A(K_x)$ is upper semicontinuous. An element of $\mathrm{Rep}(A, d)$ is a homomorphism $x : A \to M_d(K)$, and K_x is K^d considered as an A-module using x. We can identify $\mathrm{End}_A(K_x)$ as a subspace of $M_d(K)$, so it is a cone, and

$$Y = \{(x, B) \in \mathrm{Rep}(A, d) \times M_d(K) : B \in \mathrm{End}_A(K_x))\}$$

$$= \{(x, B) \in \mathrm{Rep}(A, d) \times M_d(K) : Bx(a) = x(a)B \text{ for all } a \in A\}$$

is a closed subset of $\mathrm{Rep}(A, \alpha) \times \mathrm{End}_K(K^d)$.

A variation: for a fixed finite-dimensional module M, the maps $\mathrm{Rep}(A,d) \to \mathbb{Z}$, $x \mapsto \dim \mathrm{Hom}_A(M, K_x)$ and $\dim \mathrm{Hom}_A(M, K_x)$ are upper semicontinuous.

Another variation: the map $\mathrm{Rep}(A,d) \times \mathrm{Rep}(A,e) \to \mathbb{Z}$ given by $(x,y) \mapsto \dim \mathrm{Hom}_A(K_x, K_y)$ is upper semicontinuous.

Definition. A variety X is *complete* or *proper over K* if it is universally closed, that is, for any variety Y, the projection $X \times Y \to Y$ is a closed map. (Image of a closed set is closed.)

Properties. (1) A closed subvariety of a complete variety is complete.

(2) A product of complete varieties is complete

(3) If X is complete and $\theta : X \to Y$ is a morphism, then the image is closed and complete. (The image is the projection of the graph, hence closed using separatedness.)

(4) A complete affine or quasi-projective variety is projective, since there is an embedding $X \to \mathbb{P}^n$.

Theorem. *Projective varieties are complete.*

Proof. It suffices to prove this for $\mathbb{P}^n$. Let $V = K^{n+1}$, let $V_* = V \setminus \{0\}$ and let $p : V_* \to \mathbb{P}^n$ be the morphism sending a nonzero vector $(x_0, \dots, x_n)$ to $[x_0 : \dots : x_n]$.

Let C be closed in $\mathbb{P}^n \times Y$. We need to show that its image under the projection to Y is closed.

If $y \in Y$ then $V_y = \{0\} \cup \{v \in V_* : (p(v), y) \in C\}$ is a cone in V. Also $Z = \{(v, y) : v \in V_y\}$ is closed in $V \times Y$. Namely, p gives a morphism $(p, 1) : V_* \times Y \to \mathbb{P}^n \times Y$. Then $(p,1)^{-1}(C)$ is closed in $V_* \times Y = (V \times Y) \setminus (\{0\} \times Y)$, so $Z = (p,1)^{-1}(C) \cup (\{0\} \times Y)$ is closed in $V \times Y$.

Thus the function $y \mapsto \dim V_y$ is upper semicontinuous. Thus $\{y \in Y : \dim V_y = 0\}$ is open. This is the complement of the image of C. □

Example. Given A and dimension vectors α and β, we have a closed subset

$$\mathrm{Rep}\,\mathrm{Gr}(A, \alpha, \beta) \subseteq \mathrm{Rep}(A, \alpha) \times \mathrm{Gr}(K^n, d)$$

where $n = \sum_i \alpha_i$ and $d = \sum_i \beta_i$. Since Grassmannians are projective varieties, and projective varieties are complete, we get that

$$\{x \in \mathrm{Rep}(A, \alpha) : K_x \text{ has a submodule of dimension } \beta\}$$

which is the image of the projection

$$\mathrm{Rep}\,\mathrm{Gr}(A, \alpha, \beta) \to \mathrm{Rep}(A, \alpha)$$

is closed. Taking the union over all $\beta \neq 0, \alpha$, and then the complement, we get that the set

$$\mathrm{Simple}(A, \alpha) = \{x \in \mathrm{Rep}(A, \alpha) : K_x \text{ is a simple module}\}$$

is open in $\mathrm{Rep}(A, \alpha)$.

7.3 Orbits

Let G be a (linear) algebraic group. For simplicity we assume G is connected. Suppose G acts on a variety X. We are interested in the orbits Gx for $x \in X$.

Properties. (i) The orbit $Gx = \{gx : g \in G\}$ is a locally closed subset of X.

The map $G \to X$, $g \mapsto gx$ is a morphism, so its image Gx is constructible. Since G is connected, it is an irreducible variety, so $\overline{Gx}$ is irreducible. Thus Gx contains a nonempty open subset U of $\overline{Gx}$. Left multiplication by $g \in G$ induces an isomorphism $X \to X$, so gU is an open subset of $g\overline{Gx} = \overline{Gx}$. Thus $Gx = \bigcup_{g \in G} gU$ is an open subset of $\overline{Gx}$. Thus Gx is locally closed.

(ii) Gx and $\overline{Gx}$ are irreducible varieties.

We know $\overline{Gx}$ is irreducible, and Gx is non-empty dense open subset of it, so also irreducible.

(iii) The stabilizer $\mathrm{Stab}_G(x) = \{g \in G : gx = x\}$ is a closed subgroup of G, and $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$.

Clearly the stabilizer is closed. The morphism $G \to Gx$, $g \mapsto gx$ is surjective. Its fibres are cosets of $\mathrm{Stab}_G(x)$, so all are isomorphic as varieties to $\mathrm{Stab}_G(x)$, so they have the same dimension. Then the Main Lemma gives $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$.

(iv) The closure $\overline{Gx}$ is the union of Gx with orbits of smaller dimension.

Clearly $\overline{Gx}$ is G-stable, so a union of orbits. If Gy is one of them and $\dim Gy \not< \dim Gx$, then $\overline{Gy} = \overline{Gx}$, so Gy is open in $\overline{Gx}$, so $C = \overline{Gx} \setminus Gy$ is closed in X. If $Gy \neq Gx$ then C contains Gx, which is nonsense.

(v) The closure $\overline{Gx}$ contains a closed orbit.

An orbit of minimal dimension contained in $\overline{Gx}$ must be closed.

(vi) Any irreducible component Y of X is G-stable.

If $\theta : G \times Y \to X$ is the action, then $\overline{\mathrm{Im}(\theta)}$ is irreducible and contains Y, so equals Y But it also contains gy for all $g \in G$ and $y \in Y$.

(vii) The orbit Gx is open in X if and only if $\dim Gx = \dim_x X$.

If it is open, then $\dim_x X = \dim_x Gx = \dim Gx$, since all points of the orbit look the same. Let Y be a irreducible component of X containing x, then Y contains Gx, so also $\overline{Gx}$, so $Y = \overline{Gx}$ by dimensions. Thus Gx is open in Y. If Z is the union of all other irreducible components, then it is disjoint from Gx. Thus $Gx \subseteq X \setminus Z \subseteq Y$. Thus Gx is open in $X \setminus Z$, and $X \setminus Z$ is open in X.

Proposition. *The map $X \to \mathbb{Z}$, $x \mapsto \dim \mathrm{Stab}_G(x)$ is upper semicontinuous. Thus the set*

$$X_{\leq s} = \{x \in X : \dim \mathrm{Stab}_G(x) \leq s\} = \{x \in X : \dim Gx \geq \dim G - s\}$$

is open and the set

$$X_s = \{x \in X : \dim \mathrm{Stab}_G(x) = s\} = \{x \in X : \dim Gx = \dim G - s\}$$

is locally closed.

Proof. Let $Z = \{(g, x) \in G \times X : gx = x\}$ and let $\pi : Z \to X$ be the projection. Now

$$\dim_{(1,x)} \pi^{-1}\pi(1, x) = \dim_1 \mathrm{Stab}_G(x) = \dim \mathrm{Stab}_G(x)$$

since $\mathrm{Stab}_G(x)$ is a group, so every point looks the same. □

Example. We show that τ-rigid modules for a f.d. algebra A are determined by their g-vectors.

Let P_0 and P_1 be projective A-modules.

The group $G = \mathrm{Aut}(P_0) \times \mathrm{Aut}(P_1)$ acts on $\mathrm{Hom}(P_1, P_0)$ via $(g, h) \cdot \theta = g\theta h^{-1}$.

Now G is open in $\mathrm{End}(P_0) \times \mathrm{End}(P_1)$, so they have the same dimension.

Fix an exact sequence

$$P_1 \xrightarrow{\theta} P_0 \xrightarrow{\phi} M \to 0.$$

Then

$$\mathrm{Stab}_G(\theta) = \{(g, h) \in \mathrm{Aut}(P_0) \times \mathrm{Aut}(P_1) : g\theta = \theta h\}$$
$$\subseteq W := \{(g, h) \in \mathrm{End}(P_0) \times \mathrm{End}(P_1) : g\theta = \theta h\}.$$

Letting

$$V = \{(k, h) \in \mathrm{Hom}(P_0, P_1) \times \mathrm{End}(P_1) : \theta(h - k\theta) = 0\}$$

we get an exact sequence

$$0 \to \mathrm{Hom}(P_0, \mathrm{Ker}\,\theta) \xrightarrow{\alpha} V \xrightarrow{\beta} W \xrightarrow{\gamma} \mathrm{End}(M) \to 0$$

where $\gamma(g, h)$ is the induced unique $a \in \mathrm{End}(M)$ with $a\phi = \phi g$, $\beta(k, g) = (\theta k, h)$, and $\alpha(b) = (b, 0)$. Also there is an exact sequence

$$0 \to \mathrm{Hom}(P_0, P_1) \xrightarrow{k \mapsto (k, k\theta))} V \xrightarrow{(k,h) \mapsto h - k\theta} \mathrm{Hom}(P_1, \mathrm{Ker}\,\theta) \to 0.$$

Thus

$$\dim W = \dim \mathrm{End}(M) + \dim V - \dim \mathrm{Hom}(P_0, \mathrm{Ker}\,\theta)$$
$$= \dim \mathrm{End}(M) + \dim \mathrm{Hom}(P_0, P_1) + \dim \mathrm{Hom}(P_1, \mathrm{Ker}\,\theta) - \dim \mathrm{Hom}(P_0, \mathrm{Ker}\,\theta)$$

Applying the exact functor $\mathrm{Hom}(P_i, -)$ to the exact sequence $0 \to \mathrm{Ker}\,\theta \to P_1 \to P_0 \to M \to 0$, this becomes

$$= \dim \mathrm{End}(M) + \dim \mathrm{Hom}(P_0, P_1) + \dim \mathrm{End}(P_1) - \dim \mathrm{Hom}(P_1, P_0) + \dim \mathrm{Hom}(P_1, M)$$

$$-\dim \operatorname{Hom}(P_0, P_1) + \dim \operatorname{End}(P_0) - \dim \operatorname{Hom}(P_0, M)$$

Now suppose that $P_1 \to P_0 \to M \to 0$ is a minimal projective presentation of M. Recall that the g-vector of M is $g(M) = [P_0] - [P_1] \in K_0(A\text{-proj})$ and

$$\dim \operatorname{Hom}(P_0, X) - \dim \operatorname{Hom}(P_1, X) = \langle g(M), [X] \rangle = \dim \operatorname{Hom}(M, X) - \dim \operatorname{Hom}(X, \tau M).$$

Thus

$$\dim W = -\dim \operatorname{Hom}(P_1, P_0) - \dim \operatorname{End}(P_0) + \dim \operatorname{End}(P_1) + \dim \operatorname{Hom}(M, \tau M)$$

Thus dimension of the orbit of θ is

$$\begin{aligned} \dim G - \dim \operatorname{Stab}_G(\theta) &= \dim \operatorname{End}(P_0) + \dim \operatorname{End}(P_1) - \dim W \\ &= \dim \operatorname{Hom}(P_1, P_0) - \dim \operatorname{Hom}(M, \tau M). \end{aligned}$$

Since also $\operatorname{Hom}(P_1, P_0)$ is affine space, its dimension at any point is equal to its dimension. It follows that the orbit is open if and only if M is τ-rigid.

As mentioned in the section on τ-tilting theory, the projectives occuring in the minimal projective presentation of a τ-rigid module M have no indecomposable summand in common. Thus the projectives are uniquely determined by $g(M)$. Thus if M' is another τ-rigid module with $g(M') = g(M)$, its minimal projective presentation is given by an element θ' of $\operatorname{Hom}(P_1, P_0)$. Now the orbits of θ and θ' are open, so by irreducibility the must intersect, so they are the same. It follows that $M' \cong M$.

Let G be a connected algebraic group acting on a variety X. If the set of orbits X/G was a variety, we could study its dimension and its irreducible components. Unfortunately it is usually not a variety, so we will do the best we can. The basic idea is that if all orbits have dimension e, then the number of parameters for the action should be $\dim X - d$. Actually we want to be able to define the number of parameters for any G-stable constructible subset of X.

First without a group action. Given a constructible subset Y of a variety X

Proposition/Definition. *If $Y \subseteq X$ is a constructible subset of X, then it can be written as a disjoint union*

$$Y = Z_1 \cup \cdots \cup Z_n$$

with the Z_i being irreducible locally closed subsets of X. Moreover

$$\max\{\dim Z_i\} = \dim \overline{Y}$$

and we denote this $\dim Y$*, and*

$$\#\{i : \dim Z_i = \dim \overline{Z}\}$$

is the number of top-dimensional irreducible components of $\overline{Z}$. We denote this $\operatorname{top} Z$.

The proof is an exercise. Now suppose that G acts on X. We define

$$X_{(d)} := \{x \in X : \dim Gx = d\}.$$

It is the same as $X_{\dim G - d}$ using the notation before, so a locally closed G-stable subset of X. Similarly we define

$$X_{(\leq d)} := \{x \in X : \dim Gx \leq d\}.$$

This is the complement of $X_{\leq \dim G - d - 1}$, so a closed G-stable subset of X.

Definition. Suppose Y is a G-stable constructible subset of X. We define the number of parameters and number of top-dimensional families by

$$\dim_G Y = \max\{\dim(Y \cap X_{(d)}) - d : d \geq 0\},$$

$$\mathrm{top}_G Y = \sum \{\mathrm{top}(Y \cap X_{(d)}) : d \geq 0, \dim(Y \cap X_{(d)}) - d = \dim_G Y\}.$$

The following properties are easy.

Properties. (i) If Y_1, Y_2 are G-stable subsets then $\dim_G(Y_1 \cup Y_2) = \max\{\dim_G Y_1, \dim_G Y_2\}$.

(ii) $\dim_G Y = 0$ if and only if Y contains only finitely many orbits, and if so, $\mathrm{top}_G Y$ is the number of orbits.

(iii) If Y contains a constructible subset Z meeting every orbit, then $\dim_G Y \leq \dim Z$.

(iv) If $f : Z \to X$ is a morphism and the inverse image of each orbit has dimension $\leq d$, then $\dim_G X \geq \dim Z - d$.

(v) $\dim_G Y = \max\{\dim(Y \cap X_{(\leq d)}) - d : d \geq 0\}$.

Lemma. *Suppose G acts on X and that $\pi : X \to Y$ is constant on orbits. Suppose that the image of any closed G-stable subset of X is a closed subset of Y. Then the function $\pi(X) \to \mathbb{Z}$, $y \mapsto \dim_G(\pi^{-1}(y))$ is upper semicontinuous.*

Proof. We show first that for the function dim and for any r, the set

$$\{y \in Y : \dim \pi^{-1}(y) \geq r\}$$

is closed in Y. By the Upper Semicontinuity Theorem, the set

$$C_r = \{x \in X : \dim_x \pi^{-1}(\pi(x)) \geq r\}$$

is closed in X. It is also a G-stable subset, so by hypothesis $\pi(C_r)$ is closed. Now if $y \in Y$ then $\dim \pi^{-1}(y) = \max\{\dim_x \pi^{-1}(y) : x \in \pi^{-1}(y)\}$. Thus

$$\{y \in Y : \dim \pi^{-1}(y) \geq r\} = \pi(C_r),$$

so it is closed in Y.

Now $X_{(\leq d)} = \{x \in X : \dim Gx \leq d\}$ is closed in X, and π_d, which is the restriction of π to this set, sends closed G-stable subsets to closed subsets, so

$$\{y \in Y : \dim \pi_d^{-1}(y) \geq r\}$$

is closed in Y. Then

$$\{y \in Y : \dim_G \pi^{-1}(y) \geq r\} = \bigcup_d \{y \in Y : \dim \pi_d^{-1}(y) \geq d + r\}$$

which is closed in Y. This the function is upper semicontinuous. □

7.4 Tangent spaces

Definition. Given an algebra A, its *enveloping algebra* is $A^e = A \otimes_K A^{op}$. To give an A-A-bimodule L (on which the actions of K on the right and left are the same) is the same as giving a left A^e-module, where $A = A \otimes_K A^{op}$. In particular we can consider A as an A^e-module. Also the bimodule $A \otimes_K A$ corresponds to A^e as a left A^e-module.

The *Hochschild cohomology* of a bimodule L can be defined to be

$$H^n(A, L) = \mathrm{Ext}^n_{AA}(A, L).$$

Here the subscript AA means we're working with bimodules, so with A^e-modules.

If L is an A-A-bimodule, then the set of derivations $R \to L$ is

$$\mathrm{Der}(A, L) = \{d \in \mathrm{Hom}_K(A, L) : d(ab) = ad(b) + d(a)b \text{ for all } a, b \in A\}$$

Observe that $d(1) = 0$ since $d(1) = d(1 \cdot 1) = 1d(1) + d(1)1 = 2d(1)$. An *inner derivation* is one of the form $d(a) = a\ell - \ell a$ for some $\ell \in L$. This defines a subspace $\mathrm{Inn}(A, L) \subseteq \mathrm{Der}(A, L)$

Lemma. *(i)* $H^0(A, L) \cong \{x \in L : ax = xa \textit{ for all } a \in A\}$

(ii) $H^1(A, L) \cong \mathrm{Der}(A, L)/\mathrm{Inn}(A, L)$.

Proof. The *bimodule of non-commutative 1-forms* for A is the kernel of the multiplication map $A \otimes_K A \to A$, so

$$0 \to \Omega^1 A \to A \otimes_K A \to A \to 0.$$

Now $\mathrm{Der}(A, L)$ is isomorphic to $\mathrm{Hom}_{AA}(\Omega^1 A, L)$ as a vector space via the maps sending a derivation d to the map θ with $\theta(\sum_i a_i \otimes a_i') = \sum_i a_i d(a_i')$ and sending a map θ to the derivation d with $d(a) = \theta(a \otimes 1 - 1 \otimes a)$.

We get an exact sequence

$$0 \to \mathrm{Hom}_{AA}(A, L) \to \mathrm{Hom}_{AA}(A \otimes_K A, L) \to \mathrm{Hom}(\Omega^1 A, L) \to \mathrm{Ext}^1_{AA}(A, L) \to 0$$

now the middle two terms are isomorphic to L and $\mathrm{Der}(A, L)$, and the map sends $\ell \in L$ to the corresponding inner derivation. □

Lemma. *If M and N are A-modules, then considering $\mathrm{Hom}_K(M, N)$ as an A-A-bimodule, we have $H^1(A, \mathrm{Hom}_K(M, N)) \cong \mathrm{Ext}^1_A(M, N)$.*

Proof. The exact sequence for $\Omega^1 A$ is split as a sequence of right A-modules, so tensoring with M we get an exact sequence of A-modules

$$0 \to \Omega^1 A \otimes_A M \to A \otimes_K M \to M \to 0$$

Thus $\mathrm{Ext}^1_A(M, N)$ is isomorphic to the cokernel of the map

$$\mathrm{Hom}_A(A \otimes_K M, N) \to \mathrm{Hom}_A(\Omega^1 A \otimes_A M, N).$$

We can identify the right hand term with $\mathrm{Hom}_{AA}(\Omega^1 A, \mathrm{Hom}_K(M, N))$ so with $\mathrm{Der}(A, \mathrm{Hom}_K(M, N))$, and then the image of the map is $\mathrm{Inn}(A, \mathrm{Hom}_K(M, N))$. □

In the special case when A is commutative and L is an A-module, considered as a bimodule with the same action on each side, the inner derivations are all zero.

Definition. If X is a variety and $p \in X$, then there is a local ring $\mathcal{O}_{X,p}$ of germs of functions at p. There is a homomorphism $\mathcal{O}_{X,p} \to K$, $f \mapsto f(p)$. Its kernel is the maximal ideal $\mathfrak{m}_p$. Now p makes K into an $\mathcal{O}_{X,p}$-module, denoted ${}_pK$, and also into a bimodule, denoted ${}_pK_p$. The *tangent space* of X at $p \in X$ is the set of point derivations

$$\begin{aligned} T_p(X) &= \mathrm{Der}(\mathcal{O}_{X,p}, {}_pK_p) \\ &= \{\xi \in \mathcal{O}^*_{X,p} : \xi(fg) = f(p)\xi(g) + \xi(f)g(p) \text{ for all } f, g \in \mathcal{O}_{X,p}\} \\ &\cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \\ &\cong \mathrm{Ext}^1_{\mathcal{O}_{X,p}}({}_pK, {}_pK). \end{aligned}$$

where $*$ is duality into the field K.

If $\theta : X \to Y$ is a morphism of varieties, then one gets a homomorphism of algebras $\theta^* : \mathcal{O}_{Y,\theta(p)} \to \mathcal{O}_{X,p}$, and this induces a linear map

$$d\theta_p : T_pX \to T_{\theta(p)}Y, \quad \xi \mapsto \xi \circ \theta^*.$$

If $X \xrightarrow{\theta} Y \xrightarrow{\phi} Z$, then $d(\phi\theta)_p$ is the composition

$$T_pX \xrightarrow{d\theta_p} T_{\theta(p)}Y \xrightarrow{d\phi_{\theta(p)}} T_{\phi\theta(p)}Z.$$

Definition. Given an affine scheme $\mathbf{X} = \mathrm{Hom}_{K\text{-comm}}(A, -)$ and a point $p \in \mathbf{X}(K)$ corresponding to a K-algebra homomorphism $A \to K$, we define

$$T_p\mathbf{X} = \mathrm{Der}(A, {}_pK_p) \cong \mathrm{Ext}^1_A({}_pK, {}_pK).$$

If $\mathbf{X}$ is reduced and algebraic, this corresponds to the definition for varieties.

Again, a morphism $\theta : \mathbf{X} \to \mathbf{Y}$ induces a linear map

$$d\theta_p : T_p\mathbf{X} \to T_{\theta(p)}\mathbf{Y}.$$

Proposition. *Let $\mathbf{X} = \mathrm{Hom}_{K\text{-comm}}(A, -)$ and $p \in \mathbf{X}(K)$, so p is a K-alggebra map $A \to K$.*

(i) Let $K[\epsilon]/(\epsilon^2)$ be the algebra of dual numbers and $\pi : K[\epsilon]/(\epsilon^2) \to K$ the projection. Then we have a mapping

$$\mathbf{X}(\pi) : \mathbf{X}(K[\epsilon]/(\epsilon^2)) \to \mathbf{X}(K)$$

and we can identify

$$T_p\mathbf{X} = \{\phi \in \mathbf{X}(K[\epsilon]/(\epsilon^2)) : \mathbf{X}(\pi)(\phi) = p\}.$$

(ii) Suppose $A = K[X_1, \dots, X_n]/I$, so we can identify

$$\mathbf{X}(K) = \{p \in K^n : f(p) = 0 \textit{ for all } f \in I\ \}.$$

Then we have an isomorphism

$$T_p\mathbf{X} \to \{v \in K^n : \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p) = 0\ \forall f \in I\}, \quad \xi \mapsto (\xi(X_1), \dots, \xi(X_n)).$$

Proof. (i) A linear map $\phi : A \to K[\epsilon]/(\epsilon^2)$ whose composition with π is p can be written in the form $\phi(a) = p(a) + \epsilon\xi(a)$ for some linear map $\xi \in \mathrm{Hom}_K(A, K)$, and then ϕ is an algebra homomorphism if and only if ξ is a derivation.

(ii) Considering K as a bimodule over $K[X_1, \dots, X_n]$ using p, we have an isomorphism

$$\mathrm{Der}(K[X_1, \dots, X_n], {}_pK_p) \to K^n, \quad \xi \mapsto (\xi(X_1), \dots, \xi(X_n))$$

with inverse sending $v \in K^n$ to the derivation ξ given by

$$\xi(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p).$$

Now

$$\mathrm{Der}(A, K) = \{\xi \in \mathrm{Der}(K[X_1, \dots, X_n], K) : \xi(f) = 0 \text{ for all } f \in I\ \}$$

□

Lemma. *If U is an open subset of X and $p \in U$, the induced map $T_{U,p} \to T_{X,p}$ is an isomorphism. If U is locally closed in X, the induced map is injective.*

Proof. If U is open, the local rings are the same. More generally, we may assume that X is affine and U is closed in X. Then use that the map $K[X] \to K[U]$ is surjective. $\square$

Examples. (i) Any point $p \in \mathbb{A}^n$ has $T_p\mathbb{A}^n \cong K^n$. In coordinate free terms, if V is a f.d. vector space, $T_pV \cong V$.

(ii) Since $\mathrm{GL}_n(K)$ is open in $M_n(K)$, we have $T_g \mathrm{GL}_n(K) \cong M_n(K)$ for all g. Explicitly we have an isomorphism

$$M_n(K) \to T_g \mathrm{GL}_n(K) = \{\phi \in \mathrm{GL}_n(K[\epsilon]/(\epsilon^2)) : \pi(\phi) = g\}, \quad v \mapsto g + \epsilon v$$

There is a morphism of schemes $\theta : \mathrm{GL}_n \to \mathrm{GL}_n$ given by inversion. For any $g \in \mathrm{GL}_n(K)$, it induces a linear map

$$d\theta_g : M_n(K) \cong T_g \mathrm{GL}_n(K) \to T_{g^{-1}} \mathrm{GL}_n(K) \cong M_n(K)$$

via

$$g^{-1} + \epsilon\, d\theta_g(v) = (g + \epsilon v)^{-1} \in \mathrm{GL}_n(K[\epsilon]/(\epsilon^2)).$$

Then

$$\begin{aligned} 1 &= (g + \epsilon v)(g + \epsilon v)^{-1} \\ &= (g + \epsilon v)(g^{-1} + \epsilon\, d\theta_g(v)) \\ &= 1 + \epsilon(vg^{-1} + g\, d\theta_g(v)). \end{aligned}$$

so $d\theta_g(v) = -g^{-1}vg^{-1}$. In particular $d\theta_1(v) = -v$.

(iii) The *Lie algebra* of a linear algebraic group G is $\mathfrak{g} = T_1G$. If $g \in G$, there is a map $c^g : G \to G$, $x \mapsto gxg^{-1}$, and hence $d(c^g)_1 : \mathfrak{g} \to \mathfrak{g}$. This defines an action of G on $\mathfrak{g}$, the *adjoint action*

$$Ad : G \to \mathrm{GL}(\mathfrak{g}), g \mapsto d(c^g)_1.$$

Taking the tangent space map gives a linear map

$$ad = d(Ad)_1 : \mathfrak{g} \to \mathrm{End}_K(\mathfrak{g}).$$

Defining $[u, v] = ad(u)(v)$ turns $\mathfrak{g}$ into a Lie algebra.

(iv) Consider $G = \mathrm{GL}_n(K)$ again. For $v \in M_n(K)$, we have $1+\epsilon v \in \mathrm{GL}_n(K[\epsilon]/(\epsilon^2))$. Then

$$c^g(1 + \epsilon v) = g(1 + \epsilon v)g^{-1} = 1 + \epsilon gvg^{-1},$$

so $d(c^g)_1(v) = gvg^{-1}$ for $v \in M_n(K)$. Then $Ad(g)(v) = gvg^{-1}$, so working with the scheme, if $u \in M_n(K)$, then

$$\begin{aligned} Ad(1+\epsilon u)(v) &= (1+\epsilon u)v(1+\epsilon u)^{-1} \\ &= (1+\epsilon u)v(1-\epsilon u) \\ &= v + \epsilon(uv - vu). \end{aligned}$$

Thus $[u, v] = ad(u)(v) = uv - vu$.

The next definition and theorem were skipped in the lecture, but part (iv) is needed below.

Definition. A variety X is *smooth* (or *nonsingular*, or *regular*) at $p \in X$ if $\dim T_pX = \dim_p X$, or equivalently if the local ring $\mathcal{O}_{X,p}$ is a 'regular' local ring, which means that $\dim \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim \mathcal{O}_{X,p}$. The variety X is *smooth* if it is smooth at all points. Similarly for a scheme. A smooth scheme must be reduced.

Clearly $\mathbb{A}^n$ and $\mathbb{P}^n$ are smooth.

Theorem. *For a variety X we have:*
(i) The function $X \to \mathbb{Z}$, $p \mapsto \dim T_pX$ is upper semicontinuous;
(ii) If X is irreducible, then $\dim T_pX = \dim X$ for all p in a nonempty open subset of X;
(iii) The set of smooth points of X is a dense open subset of X;
(iv) $\dim T_pX \geq \dim_p X$ for all $p \in X$.
(v) Any point in an intersection of irreducible components cannot be smooth.

Proof. (i) Follows from upper semicontinuity for cones.

(ii) We use that any irreducible variety of dimension $n-1$ is birational to a hypersurface in $\mathbb{A}^n$ (see Hartshorne, Algebraic Geometry, Proposition I.4.9). Thus we only need to prove the statement for a hypersurface. Say $X = V(f)$ for $f \in K[X_1, \dots, X_n]$ an irreducible polynomial. For $p \in X$ we have

$$T_pX = \{(v_1, \dots, v_n) \in K^n \mid \sum_{i=1}^n v_i \frac{\partial f}{\partial X_i}(p) = 0\}.$$

which has the right dimension if some $\partial f/\partial X_i(p) \neq 0$.

In characteristic 0, if all partial derivatives $\partial f/\partial X_i$ are identically zero then f is constant. In characteristic ℓ this is not true, for example $a + bX_1^\ell + cX_2^\ell X_3^{2\ell}$, but all exponents must be multiples of ℓ, and choosing an ℓ-th root of each coefficient, one gets that f is an ℓ-th power, here

$$f = \left(\sqrt[\ell]{a} + \sqrt[\ell]{b}X_1 + \sqrt[\ell]{c}X_2X_3^2\right)^\ell,$$

contradicting irreducibility of f.

Thus some partial derivative $\partial f/\partial X_i$ is not identically zero. If it vanishes on X, then it is in (f), which is impossible by degrees. Thus $X \cap D(\partial f/\partial X_i)$ is a dense open subset of X with the right property.

(iii) Reduce to the irreducible case, which is (ii).

(iv) Reduce to the irreducible case, when it follows from (i), (ii).

(v) Regular local rings are domains, so have a unique minimal prime ideal. □

Theorem. *If* $\theta \in \mathrm{Rep}(A,d)$ *and the corresponding module* $M = {}_\theta K^d$ *satisfies* $\mathrm{Ext}^1_A(M,M) = 0$*, then the corresponding orbit* $\mathcal{O}_M = \mathrm{GL}_d(K)\theta$ *is open. The converse holds if the scheme* $\mathbf{Rep}(A,d)$ *is reduced (at* θ*).*

Proof. We identify $\mathrm{End}_K(M)$ with $M_d(K)$. Then θ is the action $A \to \mathrm{End}_K(M)$. Also $M_d(K)$ becomes an A-A bimodule.

Now θ is a K-point of the scheme $\mathbf{Rep}(A,d)$, and $T_\theta\,\mathbf{Rep}(A,d)$ is the set of K-algebra homomorphisms $A \to M_d(K[\epsilon]/(\epsilon^2))$ such that the composition to $M_d(K)$ is θ. Such homomorphisms can be written in the form $\theta + \epsilon d$ where $d : A \to M_d(K)$ is a derivation.

Thus $T_\theta\,\mathbf{Rep}(A,d) \cong \mathrm{Der}(A, \mathrm{End}_K(M)))$.

Then $T_\theta\,\mathrm{Rep}(A,d)$ is a subspace of this, equal if the scheme is reduced.

The action of $\mathrm{GL}_d(K)$ on $\mathrm{Rep}(A,d)$ defines a morphism

$$m : \mathrm{GL}_d(K) \to \mathrm{Rep}(A,d), \quad m(g) = {}^g\theta$$

where $({}^g\theta)(a) = g\theta(a)g^{-1}$ for $a \in A$. We can consider this as the map on K-points of a morphism of schemes $\mathrm{GL}_d \to \mathbf{Rep}(A,d)$ which on R-valued points is given by the same formula. To compute the map on tangent spaces for these schemes we compute

$$\begin{aligned}({}^{1+\epsilon v}\theta)(a) &= (1+\epsilon v)\theta(a)(1+\epsilon v)^{-1} \\ &= (1+\epsilon v)\theta(a)(1-\epsilon v) \\ &= \theta(a) + \epsilon(v\theta(a) - \theta(a)v)\end{aligned}$$

so $dm_1 : M_d(K) \to T_\theta\,\mathbf{Rep}(A,d)$ is given by $dm_1(v) = (a \mapsto v\theta(a) - \theta(a)v)$. Thus the image of dm_1 is the set of inner derivations from A to $\mathrm{End}_K(M)$. Since GL_d is reduced, we can factor the morphism on schemes as

$$\mathrm{GL}_d \to \mathrm{Rep}(A,d) \to \mathbf{Rep}(A,d)$$

so the image of dm_1 is contained in $T_\theta\,\mathrm{Rep}(A,d)$. Thus

$$\begin{aligned}\frac{T_\theta\,\mathrm{Rep}(A,d)}{\mathrm{Im}(dm_1)} \subseteq \frac{T_\theta\,\mathbf{Rep}(A,d)}{\mathrm{Im}(dm_1)} &\cong \frac{\mathrm{Der}(A,\mathrm{End}_K(M))}{\mathrm{Inn}(A,\mathrm{End}_K(M))} \\ &\cong H^1(A,\mathrm{End}_K(M)) \cong \mathrm{Ext}^1_A(M,M).\end{aligned}$$

Now if $\operatorname{Ext}^1_A(M, M) = 0$, then the map on tangent spaces

$$dm_1 : \mathrm{GL}_d(K) \to T_\theta \operatorname{Rep}(A, d)$$

is surjective and has kernel $\operatorname{End}_A(M)$. Thus

$$\begin{aligned} \dim \mathcal{O}_M &= \dim_\theta \mathcal{O}_M \le \dim_\theta \operatorname{Rep}(A, d) \le \dim T_\theta \operatorname{Rep}(A, d) \\ &= \dim T_1 \mathrm{GL}_d(K) - \dim \operatorname{End}_A(M) = \dim \mathcal{O}_M. \end{aligned}$$

Thus $\dim \mathcal{O}_M = \dim_\theta \operatorname{Rep}(A, d)$, so the orbit is open.

Conversely, if $\mathcal{O}_M$ is open in $\operatorname{Rep}(A, d)$, then $T_\theta \mathcal{O}_M = T_\theta \operatorname{Rep}(A, d)$. It follows that the map $T_1 \mathrm{GL}(d) \to T_\theta \operatorname{Rep}(A, d)$ is onto. If also $\mathbf{Rep}(A, d)$ is reduced at θ, then the map $T_1 \mathrm{GL}(\alpha) \to T_\theta \mathbf{Rep}(A, d)$ is onto. Thus $\operatorname{Ext}^1_A(M, M) = 0$. □

Remark. The analogue of the theorem hold with dimension vectors in case $A = KQ/I$. In particular if $A = KQ$, then $\mathbf{Rep}(KQ, \alpha)$ is an affine space, so reduced and smooth.

One can show that if $\theta \in \operatorname{Rep}(A, d)$ and $\operatorname{Ext}^2({}_\theta K^d, {}_\theta K^d) = 0$, then $\mathbf{Rep}(A, d)$ is smooth at θ, so reduced, so $T_\theta \operatorname{Rep}(A, d) = T_\theta \mathbf{Rep}(A, d)$ in this case. For details see section 6.4 of Crawley-Boevey and Sauter, On quiver Grassmannians and orbit closures for representation-finite algebras, 2016, or the work of Geiß cited there. The proof uses $H^2(A, \operatorname{End}_K(M))$.

8 Applications to representations of algebras

8.1 Degenerations of modules

We consider the variety $\mathrm{Rep}(A,d)$ of A-module structures on K^d. The elements are K-algebra homomorphisms $x : A \to M_d(K)$, and the corresponding A-module is $K_x := {}_xK^d$.

The group $\mathrm{GL}_d(K)$ acts by conjugation, for $x \in \mathrm{Rep}(A,d)$ and $g \in \mathrm{GL}_d(K)$ we have $(g.x)(a) = gx(a)g^{-1} \in M_d(K)$.

Th orbits are correspond to the isomorphism classes of d-dimensional A-modules. We write $\mathcal{O}_M$ for the orbit corresponding to a module M, so if $M \cong K_x$, then $\mathcal{O}_M = \mathrm{GL}_d(K)x$. Also

$$\dim \mathcal{O}_M = \dim \mathrm{GL}_d(K) - \dim \mathrm{Stab}_{\mathrm{GL}_d(K)}(x) = d^2 - \dim \mathrm{End}_A(M)$$

since $\mathrm{Stab}_{\mathrm{GL}_d(K)}(x) \cong \mathrm{Aut}_A(M)$ is non-empty open in $\mathrm{End}_A(M)$.

Definition. If M and N are modules of the same dimension d, we say that M *degenerates to* N if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$ in $\mathrm{Rep}(A,d)$. To see that this is a partial order, use that the orbit closure $\overline{\mathcal{O}_M}$ is the union of $\mathcal{O}_M$ and orbits of strictly smaller dimension.

Proposition. *Given A-modules M and N of the same dimension d, consider the following conditions:*

(i) There is a finite sequence of modules $M = M_0, M_1, \dots, M_n = N$ and exact sequences $0 \to L_i \to M_i \to L'_i \to 0$ with $M_{i+1} \cong L_i \oplus L'_i$.

(ii) M degenerates to N.

(iii) $\dim \mathrm{Hom}(X,M) \le \dim \mathrm{Hom}(X,N)$ for all X.

Then we have (i) ⇒ (ii) ⇒ (iii).

The relations given by (i), (ii) and (iii) can be denotes $\le_{\mathrm{ext}}$, $\le_{\mathrm{deg}}$ and $\le_{\mathrm{hom}}$.

Proof. (i) ⇒ (ii). Since the relation of degeneration is transitive, it suffices to prove this for one exact sequence $0 \to L \to M \to L' \to 0$, say with L has dimension e, and $N \cong L \oplus L'$. Taking a basis of L and extending it to a basis of M, we have $M \cong K_x$ for some $x \in \mathrm{Rep}(A,d)$ where

$$x(a) = \begin{pmatrix} y(a) & w(a) \\ 0 & z(a) \end{pmatrix}$$

for $a \in A$, for suitable matrix-valued linear maps y, z, w on A. Clearly then $y \in \mathrm{Rep}(A,e)$ and $z \in \mathrm{Rep}(A,d-e)$ with $L \cong K_y$ and $L' \cong K_z$. For $t \in K$, consider the map $x^t : A \to M_d(K)$ given by

$$x^t(a) = \begin{pmatrix} y(a) & tw(a) \\ 0 & z(a) \end{pmatrix}.$$

For $t \neq 0$, this can be written as

$$x^t(a) = \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y(a) & w(a) \\ 0 & z(a) \end{pmatrix} \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}^{-1}$$

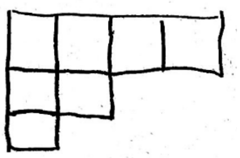
so x^t is a K-algebra homomorphism $A \to M_d(K)$, and in the same orbit as x. For $t = 0$ it is also clear that x^0 is a K-algebra homomorphism, corresponding to $L \oplus L'$. Thus M degenerates to $L \oplus L'$. (We have a map $f : \mathbb{A}^1 \to \operatorname{Rep}(A, d)$, $t \mapsto x^t$. Now $f^{-1}(\overline{\mathcal{O}_M})$ is closed and contains all $t \neq 0$, so it contains 0, so $x^0 \in \overline{\mathcal{O}_M}$, so $\mathcal{O}_{L \oplus L'} \subseteq \overline{\mathcal{O}_M}$.)

(ii) $\Rightarrow$ (iii). Use that $\dim \operatorname{Hom}_A(X, -)$ is upper semicontinuous. $\square$

Example. Recall that the nilpotent variety is

$$N_d = \{A \in M_d(K) : A^d = 0\} \cong \operatorname{Rep}(K[T]/(T^d), d),$$

and the group $\mathrm{GL}_d(K)$ acts by conjugation. The orbits are classified by partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of d (that is, decreasing sequences of non-negative integers with sum d). The Young diagram of shape λ has rows of length λ_i. For example for $(4, 2, 1)$ it is



The corresponding orbit is the conjugacy class of the nilpotent matrix A in Jordan normal with Jordan blocks of sizes given by the lengths of the columns. In the example,

$$A = \left(\begin{array}{ccc|cc|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

In terms of modules, the orbit is $\mathcal{O}_{M(\lambda)}$ where $M(\lambda)$ is the $K[T]/(T^d)$-module with vector space K^d and T acting as A.

Observe that

$$\operatorname{Hom}(K[T]/(T^i), M(\lambda)) \cong \operatorname{Ker} A^i$$

which has dimension $\lambda_1 + \lambda_2 + \dots + \lambda_i$.

Suppose condition (iii) holds for $M(\lambda)$ and $M(\mu)$.

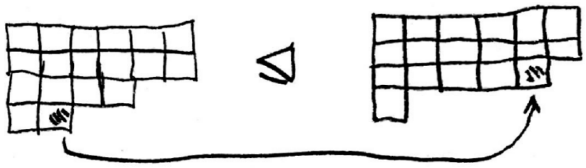
Thus $\dim \operatorname{Hom}(X, M(\lambda)) \leq \dim \operatorname{Hom}(X, M(\mu))$ for all X.

Thus $\dim \operatorname{Hom}(K[T]/(T^i), M(\lambda)) \leq \dim \operatorname{Hom}(K[T]/(T^i), M(\mu))$ for all i.

Thus $\dim \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i.

This is called the *dominance ordering* of partitions, denoted $\lambda \trianglelefteq \mu$.

Now the dominance order is generated by the following move: $\lambda \trianglelefteq \mu$ if μ is obtained from λ by moving a corner block from a column of length j to a column further to the right to make it of length $i < j$, for example $(6, 6, 4, 2) \trianglelefteq (6, 6, 5, 1)$ since



(See for example I. G. Macdonald, Symmetric functions and Hall polynomials, I, (1.16).) We want to show in this case that there is an exact sequence

$$0 \to L \to M(\lambda) \to L' \to 0$$

with $M(\mu) \cong L \oplus L'$. Now $M(\lambda) = K[T]/(T^j) \oplus K[T]/(T^{i-1}) \oplus C$ and $M(\mu) = K[T]/(T^{j-1}) \oplus K[T]/(T^i) \oplus C$, so the exact sequence

$$0 \to K[T]/(T^i) \xrightarrow{\begin{pmatrix} -1 \\ T^{j-i} \end{pmatrix}} K[T]/(T^{i-1}) \oplus K[T]/(T^j) \xrightarrow{\begin{pmatrix} T^{j-i} & 1 \end{pmatrix}} K[T]/(T^{j-1}) \to 0$$

will do.

It follows that (iii) implies (i). Thus all three conditions are equivalent, and $M(\lambda)$ degenerates to $M(\mu)$ if and only if $\lambda \trianglelefteq \mu$.

This is the Gerstenhaber-Hesselink Theorem.

For the partition $(1^d) := (1, 1, \dots)$ the corresponding module is $M(1^d) \cong K[T]/(T^d)$, and the matrix is the Jordan block of size d.

We have $(1^d) \trianglelefteq \mu$ for all μ. Thus $M(1^d)$ degenerates to any other module. Thus $N_d = \overline{\mathcal{O}_{M(1^d)}}$. Thus N_d is irreducible of dimension

$$\dim \mathrm{GL}_d(K) - \dim \operatorname{End}(M(1^d)) = d^2 - d.$$

Remark. Two beautiful but difficult results:
- M degenerates to $N \Leftrightarrow \exists$ an exact sequence $0 \to Z \to Z \oplus M \to N \to 0$ for some module Z [G. Zwara, Degenerations of finite-dimensional modules are given by extensions, 2000].
- If A has finite representation type, then (ii) and (iii) in the theorem are equivalent. If also all indecomposable modules X have $\operatorname{Ext}^1(X, X) = 0$, for example A is the path algebra of a Dynkin quiver, then (i) is also equivalent [G. Zwara, Degenerations for modules over representation-finite algebras, 1999].

Lemma. *If C is a finite-dimensional algebra, then the variety $N(C)$ of nilpotent elements in C is irreducible of dimension* $\dim C - s$*, where s is the sum of the dimensions of the simple C-modules.*

Proof. Since K is algebraically closed, we can write $C = S \oplus J(C)$, where S is semisimple, so $S \cong M_{d_1}(K) \oplus \cdots \oplus M_{d_r}(K)$. Then $N(C) \cong N_{d_1} \times \dots N_{d_r} \times J(C)$, so it is irreducible of dimension

$$\dim N(C) = \sum_i (d_i^2 - d_i) + \dim J(C) = \dim C - \sum_i d_i.$$

□

Proposition. *If A is a finitely generated algebra, $d > 0$, and $r \in \mathbb{N}$, then the set*

$$\mathrm{Ind}(A, d)_r = \{x \in \mathrm{Rep}(A, d) : K_x \text{ is indecomposable and } \dim \mathrm{End}_A(K_x) = r\}$$

is a closed subset of

$$\mathrm{Rep}(A, d)_{\le r} = \{x \in \mathrm{Rep}(A, d) : \dim \mathrm{End}_A(K_x) \le r\},$$

which is an open subset of $\mathrm{Rep}(A, d)$.

Proof. By the upper semicontinuity result for cones, the function

$$\mathrm{Rep}(A, d) \to \mathbb{Z}, \quad x \mapsto \dim N(\mathrm{End}_A(K_x))$$

is upper semicontinuous. Now by the lemma $\mathrm{Ind}(A, d)_r$ is equal to

$$\{x \in \mathrm{Rep}(A, d) : \dim \mathrm{End}_A(K_x) \le r\} \cap \{x \in \mathrm{Rep}(A, d) : \dim N(\mathrm{End}_A(K_x)) \ge r-1\}.$$

□

Finally, we consider closed orbits in module varieties. Given a module M, we write $\mathrm{gr}\, M$ for the semisimple module with the same composition factors as M. Clearly it can be obtained by a sequence of short exact sequences, so M degenerates to $\mathrm{gr}\, M$.

Recall that a complex representation V of a finite group is determined by its character. This can be generalized to modules for an arbitrary algebra, but can only determine the semisimple modules up to isomorphism. Moreover for fields of positive characteristic, one also needs to consider the other coefficients of the characteristic polynomial.

Let θ be $d \times d$ matrix or an endomorphism of a d-dimensional vector space. Its characteristic polynomial is

$$\chi_\theta(t) = \det(t1 - \theta) = t^d - c_1(\theta)t^{d-1} + c_2(\theta)t^{d-2} + \cdots + (-1)^d c_d(\theta)$$

Thus $c_1(\theta) = \mathrm{tr}(\theta)$ and $c_d(\theta) = \det(\theta)$.

Let M be a d-dimensional module for an algebra A. For $a \in A$ we write $\hat{a}_M$ for the homothety, $M \to M$, $m \mapsto am$.

Lemma. *Given a f.d. A-module M and a simple module S, the multiplicity of S in M is given by*

$$[M:S] = \frac{1}{\dim S} \min_{a \in \mathrm{Ann}(S)} \{\textit{Order of zero at } t=0 \textit{ of } \chi_{\hat{a}_M}(t)\}.$$

Proof. Given an exact sequence $0 \to X \to Y \to Z \to 0$ of A-modules, the endomorphism $\hat{a}_Y$ has upper triangular block form, so

$$\chi_{\hat{a}_Y}(t) = \chi_{\hat{a}_X}(t)\chi_{\hat{a}_Z}(t) = \chi_{\hat{a}_{X\oplus Z}}(t).$$

Thus we may assume that M is semisimple. Next we may assume that $M \oplus S$ is faithful. Thus A is semisimple. Now if $M \cong S^k \oplus N$ with $[N:S]=0$, then the smallest order we could hope to get is if a acts on S as 0 and invertibly on N. This is possible, for writing A as a product of matrix algebras we can take a to correspond to 0 in the block for S and 1 in the other blocks. With this order, the formula holds. □

Recall that $\mathrm{Rep}(A,d) = \mathrm{Hom}_{K\text{-alg}}(A, M_d(K))$, so if $a \in A$ and $x \in \mathrm{Rep}(A,d)$, then $x(a) \in M_d(K)$.

Theorem. *(i) Given $a \in A$, the map $c_i^a : \mathrm{Rep}(A,d) \to K$ sending x to $c_i(x(a))$ is a regular map which is constant on the orbits of $\mathrm{GL}_d(K)$.*

(ii) If $x, y \in \mathrm{Rep}(A,d)$, then $\mathrm{gr}\, K_x \cong \mathrm{gr}\, K_y$ if and only if $c_i^a(x) = c_i^a(y)$ for all i and all a.

(iii) Any orbit closure $\overline{\mathcal{O}_M}$ contains a unique orbit of semisimple modules, namely $\mathcal{O}_{\mathrm{gr}\, M}$.

(iv) An orbit $\mathcal{O}_M$ is closed if and only if M is semisimple.

Proof. (i) Clear.

(ii) If $c_i^a(x) = c_i^a(y)$ for all i and a, then for all a, the characteristic polynomials of $\hat{a}_{K_x}$ and $\hat{a}_{K_y}$ are equal so by the lemma $\mathrm{gr}\, K_x \cong \mathrm{gr}\, K_y$.

Conversely for any a, i, the the function c_i^a is constant on $\mathcal{O}_M$, so it takes the same constant value on $\overline{\mathcal{O}_M}$, so on $\mathcal{O}_{\mathrm{gr}\, M}$. Thus if $\mathrm{gr}\, K_x \cong \mathrm{gr}\, K_y$, then c_i^a takes the same values on the orbits of x and y.

(iii) We know that $\overline{\mathcal{O}_M}$ contains $\mathcal{O}_{\mathrm{gr}\, M}$. If $\overline{\mathcal{O}_M}$ contains another orbit of semisimple modules N, then by continuity the functions c_i^a are equal on $\mathcal{O}_M$ and on the orbit $\mathcal{O}_N$, and then $N \cong \mathrm{gr}\, M$ by (ii).

(iv) Clear. □

8.2 The variety AlgRep and global dimension

For a finite-dimensional algebra A, $\mathrm{Rep}(A,d)$ is the set of K-algebra maps $A \to M_d(K)$. We set

$$\mathrm{Alg\,Rep}(r,d) = \{(a,x) \in \mathrm{Alg}(r) \times \mathrm{Hom}_K(K^r, M_d(K)) : x \in \mathrm{Rep}(K_a, d)\}$$

where K_a denotes the algebra structure on K^r given by a. This is a closed subset, so an affine variety. The group $\mathrm{GL}_d(K)$ acts by conjugation on the second factor.

The following is a reformulation of Lemma 3.2 in P. Gabriel, Finite representation type is open, 1975. This reformulation is mentioned in C. Geiss, On degenerations of tame and wild algebras, 1995.

Theorem (Gabriel). *The projection $\pi : \mathrm{Alg\,Rep}(r,d) \to \mathrm{Alg}(r)$ sends $\mathrm{GL}_d(K)$-stable closed subsets to closed subsets.*

Before the proof we need a lemma.

Lemma (1). *If X is a variety, then the projection $X \times \mathrm{Inj}(K^d, V) \to X$ sends $\mathrm{GL}_d(K)$-stable closed subsets to closed subsets. Similarly for the projection $X \times \mathrm{Surj}(V, K^c) \to X$.*

Proof. We factor it as

$$X \times \mathrm{Inj}(K^d, V) \to X \times \mathrm{Gr}(V,d) \to X$$

Now the map $\mathrm{Inj}(K^d, V) \to \mathrm{Gr}(V,d)$ is universally submersive, so it sends open subsets of $X \times \mathrm{Inj}(K^d, V)$ to open subsets of $X \times \mathrm{Gr}(V,d)$. Thus it sends $\mathrm{GL}_d(K)$-stable closed subsets of $X \times \mathrm{Inj}(K^d, V)$ to closed subsets of $X \times \mathrm{Gr}(V,d)$. Now use that $\mathrm{Gr}(V,d)$ is complete. □

Proof of the theorem. Choose $N \geq d$ and let $V = (K^r)^N$, a vector space of dimension rN. An element $a \in \mathrm{Alg}(r)$ turns K^r into an algebra K_a, and it turns V into the free K_a-module $(K_a)^N$ of rank N. Let

$$W = \{(a,\theta) \in \mathrm{Alg}(r) \times \mathrm{Surj}(V, K^d) : \mathrm{Ker}\,\theta \text{ is a } K_a\text{-submodule of } (K_a)^N\}.$$

Let $e = N - d$. By Lemma (3) in section 6.4 we know that the map $\mathrm{Surj}(V, K^d) \to \mathrm{Gr}(V,e)$ sending θ to $\mathrm{Ker}\,\theta$ is a morphism of varieties, and using Lemma (4) in section 6.4. we know that the set $\{(U,\phi) \in \mathrm{Gr}(V,e) \times \mathrm{End}_K(V) : \phi(U) \subseteq U\}$ is a closed subset of the product. Using these, it can be checked that W is a closed subset of the product. We have a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & \mathrm{Alg}(r) \times \mathrm{Surj}(V, K^d) \\ {\scriptstyle g}\downarrow & & {\scriptstyle p}\downarrow \\ \mathrm{Alg\,Rep}(r,d) & \xrightarrow{\pi} & \mathrm{Alg}(r) \end{array}$$

where p is the projection and g sends (a, θ) to the pair consisting of a and the induced K_a-module structure on K^d. Now g is onto since any d-dimensional K_a-module is a quotient of a free module of rank N.

One can check using the affine open covering of $\mathrm{Surj}(V, K^d)$ that g is a morphism of varieties.

Suppose $Z \subseteq \mathrm{Alg\,Rep}(r, d)$ is $\mathrm{GL}_d(K)$-stable and closed. Then $g^{-1}(Z)$ is also. Thus it is a $\mathrm{GL}_d(K)$-stable closed subset of $\mathrm{Alg}(r) \times \mathrm{Surj}(V, K^d)$. Thus $\pi(Z) = p(g^{-1}(Z))$ is closed by the lemma. $\square$

Lemma (2). *Any algebra A has a projective resolution as an A-A-bimodule*

$$\to A \otimes A \otimes A \to A \otimes A \to A \to 0$$

(where tensor products are over the base field K). Here the maps are

$$b_n : A^{\otimes n+1} \to A^{\otimes n}, \quad a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n$$

Tensoring with a left A-module X, one gets a projective resolution of X,

$$\to A \otimes A \otimes X \to A \otimes X \to X \to 0.$$

The complex of bimodules is sometimes called the *standard complex*. In MacLane, Homology, the resolution of X is called the *un-normalized bar resolution of X*.

Proof. Define a map (of right A-modules) $h_n : A^{\otimes n} \to A^{\otimes n+1}$ by $h_n(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n$. One easily checks that $b_1 h_1 = 1$ and

$$b_{n+1} h_{n+1} + h_n b_n = 1 \quad (n \geq 1).$$

Also $b_1 b_2 = 0$ and then by induction $b_n b_{n+1} = 0$ for all $n \geq 1$ since

$$b_{n+1} b_{n+2} h_{n+2} = b_{n+1}(1 - h_{n+1} b_{n+1}) = b_{n+1} - b_{n+1} h_{n+1} b_{n+1} = b_{n+1} - (1 - h_n b_n) b_{n+1} = 0.$$

Now clearly $\mathrm{Im}(h_{n+2})$ generates $A^{\otimes n+3}$ as a left A-module, and the b_i are left A-module maps (in fact bimodule maps), so $b_{n+1} b_{n+2} = 0$. Finally if $x \in \mathrm{Ker}(b_n)$ then $x = (b_{n+1} h_{n+1} + h_n b_n)(x)$ implies $x \in \mathrm{Im}(b_{n+1})$, giving exactness.

Applying $- \otimes_A X$ with a left A-module X to the standard complex gives an exact sequence. This is because the terms in the standard complex are projective right A-modules, so if you break it into short exact sequences of right A-modules, all of them are split. $\square$

Proposition (Schofield, 1985). *For any i, the map*

$$\mathrm{Alg\,Rep}(r, d) \to \mathbb{Z}, \quad (a, x) \mapsto \dim \mathrm{Ext}^i_{K_a}(K_x, K_x)$$

is upper semicontinuous.

Proof. Applying $\mathrm{Hom}_A(-, Y)$ to the projective resolution of X given by the standard complex, with Y another A-module, and using that $\mathrm{Hom}_A(A \otimes M, Y) \cong \mathrm{Hom}_K(M, Y)$, we see that $\mathrm{Ext}^i_A(X, Y)$ is computed as the cohomology of a complex

$$0 \to \mathrm{Hom}_K(X, Y) \to \mathrm{Hom}_K(A \otimes X, Y) \to \mathrm{Hom}_K(A \otimes A \otimes X, Y) \to \dots$$

Taking $A = K_a$ and $X = Y = K_x$ for $(a, x) \in \mathrm{Alg\,Rep}(r, d)$, we see that the terms in this complex are fixed vector spaces V^i, and the maps are given by morphisms $f_i : \mathrm{Alg\,Rep}(r, d) \to \mathrm{Hom}_K(V^i, V^{i+1})$. Thus we get a morphism

$$\mathrm{Alg\,Rep}(r, d) \to \{(\theta, \phi) \in \mathrm{Hom}(V^{i-1}, V^i) \times \mathrm{Hom}(V^i, V^{i+1}) : \phi\theta = 0\}.$$

Now use that the map $(\theta, \phi) \mapsto \dim(\mathrm{Ker}\,\phi / \mathrm{Im}\,\theta)$ is upper semicontinuous. □

Corollary (Schofield)**.** *The algebras of global dimension $\leq g$ form an open subset of* $\mathrm{Alg}(r)$*, as do the algebras of finite global dimension. There is an integer N_r, depending on r, such that any algebra of dimension r of finite global dimension has global dimension $\leq N_r$.*

Proof. A has global dimension $\leq g$ if and only if $\mathrm{Ext}^{g+1}_A(M, N) = 0$ for all M, N. By the long exact sequences, it is equivalent that $\mathrm{Ext}^{g+1}_A(M, N) = 0$ for all simple M and N. Thus it is equivalent that $\mathrm{Ext}^{g+1}_A(M, M) = 0$ for $M = \mathrm{gr}\,A$. Consider the pairs $(a, x) \in \mathrm{Alg\,Rep}(r, r)$ such that $\mathrm{Ext}^{g+1}_{K_a}(K_x, K_x) \neq 0$. By upper semicontinuity this is a closed subset of $\mathrm{Alg\,Rep}(r, r)$. It is also stable under $\mathrm{GL}_r(K)$, so its image in $\mathrm{Alg}(r)$ is closed. This is the set of algebras of global dimension $> g$. Thus the algebras of global dimension $\leq g$ form an open subset D_g. Now since varieties are noetherian topological spaces, the chain of open sets

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$$

stabilizes. □

8.3 Tame and wild algebras

Let A and B be K-algebras and $d \in \mathbb{N}$.

There is a 1-1 correspondence between K-algebra homomorphisms $\theta : A \to M_d(B)$ up to conjugacy by an element of $\mathrm{GL}_d(B)$ and A-B-bimodules M which are free of rank d over B. (Recall that we always want K to act centrally on bimodules.)

Namely, given θ we send it to the bimodule M given by $M = B^n$ with the left and right actions given by

$$a \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \theta(a) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} b = \begin{pmatrix} b_1 b \\ \vdots \\ b_n b \end{pmatrix}.$$

Conversely given such a bimodule, we choose a basis of it as a right B-module, and send $a \in A$ to the matrix of the homothety $M \to M$, $m \mapsto am$ with respect to this basis.

If A and B are finitely generated and $s \in \mathbb{N}$, then a homomorphism $\theta : A \to M_d(B)$ induces a morphism of varieties

$$f : \mathrm{Rep}(B, s) \to \mathrm{Rep}(A, ds)$$

sending a K-algebra map $B \to M_s(K)$ to the composition $A \to M_d(B) \to M_d(M_s(K)) \cong M_{ds}(K)$. In terms of the corresponding A-B-bimodule M, we have $M \otimes_B K_x \cong K_{f(x)}$ for all $x \in \mathrm{Rep}(B, s)$.

If X is an affine variety, with coordinate ring $B = K[X]$, then there is a 1-1 correspondence between maps of varieties $X \to \mathrm{Rep}(A, d)$ and K-algebra homomorphisms $\theta : A \to M_d(K[X])$.

Namely, given θ we get a map $\mathrm{Rep}(K[X], 1) \to \mathrm{Rep}(A, d)$, and we can identify $\mathrm{Rep}(K[X], 1)$ with X. On the other hand, given a map $X \to \mathrm{Rep}(A, d)$, we consider X as a reduced scheme and $\mathrm{Rep}(A, d)$ as the reduced subscheme for the scheme $\mathbf{Rep}(A, d)$, and take the composition

$$X \to \mathrm{Rep}(A, d) \to \mathbf{Rep}(A, d)$$

Now $\mathbf{Rep}(A, d)$ is an affine scheme, so it is of the form $\mathrm{Hom}_{K\text{-comm}}(C, -)$ for some commutative K-algebra C. The category of affine schmemes is opposite to the category of commutative K-algebras, so the morphism corresponds to a homomorphism of K-algebras $C \to K[X]$, thus to an element of $\mathbf{Rep}(A, d)(K[X]) = \mathrm{Hom}_{K\text{-alg}}(A, M_d(K[X]))$, giving the homomorphism θ.

Definition. A f.d. algebra A is of *tame representation type* if, for any d, there are a finite number of A-$K[T]$-bimodules $M_1, \dots, M_N$, f.g. and free over $K[T]$, such that all but finitely many indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{K[T]} K[T]/(T - \lambda)$ for some i and λ.

Equivalently, for any d, there are a finite number of morphisms $\mathbb{A}^1 \to \mathrm{Rep}(A, d)$ such that the images meet all but finitely many orbits of indecomposables in $\mathrm{Rep}(A, d)$.

Remarks. (i) In the definition of tame can delete the "but finitely many" by including additional maps $\mathbb{A}^1 \to \mathrm{Rep}(A, d)$ which are constant. In terms of bimodules it means including bimodules of the form $M = X \otimes_K K[T]$ where X is a given left A-module.

(ii) By an observation of Drozd (mentioned in Lemma 3 in Dowbor and Skowronski, On the representation type of locally bounded categories, 1986) we can instead use A-R-bimodules, f.g. free over R, where R is a localization $K[T]_f$ with

$0 \neq f \in K[T]$. Namely, given such a bimodule M, take a finite subset which generates it as a right R-module, and let N_0 be the right $K[T]$-submodule generated by this subset. Then $N = AN_0$ is an A-$K[T]$-subbimodule of M, and since A is finite dimensional, N is f.g. over $K[T]$. It is also torsion-free, so free over $K[T]$. Now the inclusion of N in M gives an exact sequence of A-$K[T]$-bimodules.

$$0 \to N \to M \to C \to 0$$

Since R is a localization of $K[T]$, it is flat over $K[T]$, so the sequence

$$0 \to N \otimes_{K[T]} R \to M \otimes_{K[T]} R \to C \otimes_{K[T]} R \to 0$$

is exact. Also the middle term is isomorphic to the localization of M, so it is M itself, and then the first map is surjective since we started with a generating set of M as an R-module. Thus we have $N \otimes_{K[T]} R \cong M$, and so if $\lambda \in K$ and $f(\lambda) \neq 0$, so that $R/(T-\lambda)$ is a 1-dimensional R-module, we have

$$R/(T-\lambda) \cong K[T]/(T-\lambda)$$

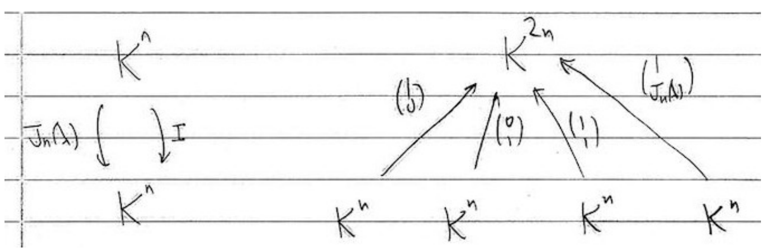
as $K[T]$-modules, so

$$M \otimes_R R/(T-\lambda) \cong N \otimes_{K[T]} R \otimes_R R/(T-\lambda) \cong N \otimes_{K[T]} K[T]/(T-\lambda).$$

(iii) We may also drop the requirement that the A-$K[T]$-bimodules are free over $K[T]$. For if M is an A-$K[T]$-bimodule which is f.g. over $K[T]$, then for a suitable localization $R = K[T]_f$ with $0 \neq f \in K[T]$, the A-R-bimodule $M \otimes_{K[T]} R$ is f.g. free over R. One just needs to take f so that it annihilates the torsion submodule of M considered as a $K[T]$-module.

Examples. (a) Any algebra of finite representation type is clearly tame by this definition (but this case is usually excluded).

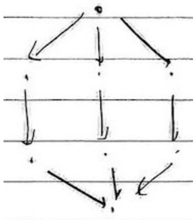
(b) Path algebras of extended Dynkin quivers are tame. For example for the Kronecker quiver and the four-subspace quiver, the 1-parameter families are given by the representations with matrices



with $\lambda \in K$. These define maps $\mathbb{A}^1 \to \mathrm{Rep}(A, d)$ for $d = 2n$ and $d = 6n$ respectively. For each dimension d not of the form $2n$ or $6n$ respectively, there are only finitely many isomorphism classes of indecomposables of dimension d. For $d = 2n$ or $6n$, there are only finitely many isomorphism classes of indecomposables not isomorphic to a module in the appropriate family.

(c) String algebras are tame, see Butler and Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras, 1987. They are the algebras KQ/I where:
- I is an admissible ideal generated by paths
- at most two arrows start (respectively end) at any vertex
- for any arrow a there is at most one arrow b such that ab (respectively ba) is a path not in I.

(d) Tubular algebras are tame. The easiest examples are the 'canonical algebras' associated to extended Dynkin diagram, for example associated to $\tilde{E}_6$ is the canonical algebra



with the relation that the sum of all paths from top to bottom is zero. See C. M. Ringel, Tame algebras and integral quadratic forms, 1984, §3.7 for the canonical algebras and Chapter 5 for the classification of the indecomposable modules.

Definition. We write A-mod for the category of f.d. A-modules. Let us say that a functor $F : B\text{-mod} \to A\text{-mod}$ is a *representation embedding* if
(i) F sends indecomposable modules to indecomposable modules.
(ii) If X and Y are B-modules and $F(X) \cong F(Y)$ then $X \cong Y$.
(iii) F is naturally isomorphic to a tensor product functor $M \otimes_B -$ for an A-B-bimodule which is finitely generated projective over B.

An algebra A is *wild* if there is a representation embedding from $K\langle X, Y\rangle$-modules to A-modules.

Remarks. (a) Note that (i) and (ii) hold if F is fully faithful, since if X is an indecomposable B-module, then $\mathrm{End}_A(F(X)) \cong \mathrm{End}_B(X)$ has no non-trivial idempotents, and if $F(X) \cong F(Y)$ then there are inverse isomorphisms, which since F is full come from morphisms between X and Y, and then since F is faithful, these must be inverse isomorphisms.

(b) One can also work with functors on the whole module categories F : B-Mod $\to$ A-Mod. This is necessary to prove results about 'generic' modules, see Crawley-Boevey, Tame algebras and generic modules, 1991.

Lemma. *(i) If I is an ideal in A then the natural functor A/I-mod $\to$ A-mod is a fully faithful representation embedding.*

(ii) For any n there is a fully faithful representation embedding

$$K\langle X_1,\dots,X_n\rangle\text{-mod} \to K\langle X,Y\rangle\text{-mod}.$$

Thus if A is wild there is a representation embedding B-mod $\to$ A-mod for any finitely generated algebra B.

Proof. (i) is the tensor product functor $A/I\otimes_{A/I}$ and the claim is trivial.

For (ii) Let $B = K\langle X_1,\dots,X_n\rangle$. Consider the A-B-bimodule $M = {}_\theta B^{n+2}$ given by the homomorphism $\theta : A \to M_{n+2}(B)$ sending X and Y to the matrices C and D,

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ X_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & X_2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & X_n & 1 & 0 \end{pmatrix}$$

These matrices are in S. Brenner, Decomposition properties of some small diagrams of modules, 1974. If U is a B-module, then $M \otimes_B U \cong {}_\theta U^{n+2}$. We want to show that if U, V are B-modules, then the map

$$\mathrm{Hom}_B(U,V) \to \mathrm{Hom}_{K\langle X,Y\rangle}({}_\theta U^{n+2}, {}_\theta V^{n+2}), \quad \theta \mapsto \mathrm{diag}(\theta,\dots,\theta)$$

is a bijection. Clearly it is injective, so we want it to be surjective. Now any morphism in $\mathrm{Hom}_{K\langle X,Y\rangle}({}_\theta U^{n+2}, {}_\theta V^{n+2})$ is given by an $(n+2)\times(n+2)$ matrix of linear maps $U \to V$, say $\Theta = (\theta_{ij})$ such that $C\Theta = \Theta C$ and $D\Theta = \Theta D$. The condition $C\Theta = \Theta C$ gives

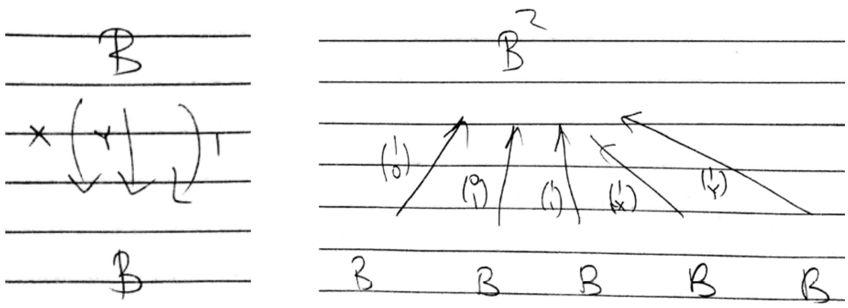
$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} & \dots \\ \theta_{21} & \theta_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} & \dots \\ \theta_{21} & \theta_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

so $\theta_{i+1,j} = \theta_{i,j-1}$ for $1 \le i,j \le n+2$, where the terms are zero if i or j are out of range. This forces Θ to be constant on diagonals, and zero below the main diagonal,

$$\Theta = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \dots & \theta_{n+1} & \theta_{n+2} \\ 0 & \theta_1 & \theta_2 & \dots & \theta_n & \theta_{n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \theta_1 & \theta_2 \\ 0 & 0 & 0 & \dots & 0 & \theta_1 \end{pmatrix}.$$

Now the condition $D\Theta = \Theta D$ gives $\theta_i = 0$ for $i > 1$ and $X_i\theta_1 = \theta_1 X_i$ for all i. Thus θ_1 is a B-module map $U \to V$. $\square$

Examples. (i) Path algebras of connected quivers which are not Dynkin or extended Dynkin are wild. For example, letting $B = K\langle X, Y\rangle$, for the path algebra A of the three arrow Kronecker quiver or five subspace quiver, consider the A-B-bimodule which is the direct sum of the indicated powers of B, with the natural action of B, and with the A-action given by the indicated matrices, acting as left multiplication.



(ii) The algebra $A = K[x, y, z]/(x, y, z)^2$ is wild. (This argument is taken from Ringel, The representation type of local algebras, 1975)

Let C be the full subcategory of A-Mod consisting of the modules M which are free over $K[z]/(z^2)$, or equivalently with $\operatorname{Ker}\hat{z}_M = \operatorname{Im}\hat{z}_M$, where $\hat{z}_M$ is the homothety $M \to M$, $m \mapsto zm$.

Given a $K\langle X, Y\rangle$-module V, we send it to the A-module V^2 with

$$x = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This is a tensor product functor, and it gives a functor $F : K\langle X, Y\rangle\text{-Mod} \to C$.

There is also functor $G : C \to K\langle X, Y\rangle$-Mod, sending M to $\operatorname{Im}\hat{z}_M$, with the action of X and Y given by

$$X(zm) = xm, \quad Y(zm) = ym.$$

These are well-defined, for if $zm = zm'$, then $m - m' \in \operatorname{Ker} \hat{z}_M = \operatorname{Im} \hat{z}_M$, so $m - m' = zm''$ for some m'', so $x(m - m') = xzm'' = 0$. Now the composition

$$K\langle X, Y\rangle\text{-Mod} \xrightarrow{F} C \xrightarrow{G} K\langle X, Y\rangle\text{-Mod}$$

is isomorphic to the identity functor. Also, if $G(M) = 0$ then $M = 0$. It follows that F is a representation embedding.

(iii) The algebra $K[x, y]$ is wild (Gelfand and Ponomarev), in fact even the algebra $K[x, y]/(x^2, xy^2, y^3)$ is wild (Drozd).

Theorem (Drozd)**.** *Any finite dimensional algebra is tame or wild, and not both.*

The proof of the first part is difficult and will be discussed later. The second part follows from the following.

Lemma. *If A is wild, then there is $r > 0$ with $\dim_{\mathrm{GL}_{rd}(K)} \operatorname{Rep}(A, rd) \geq d^2$ for all d, so $\dim_{\mathrm{GL}_e(K)} \operatorname{Rep}(A, e) > e$ for $e = 2r^2$. If A is tame, then $\dim_{\mathrm{GL}_d(K)} \operatorname{Rep}(A, d) \leq d$ for all d.*

Proof. If M is an A-B-bimodule, free of rank r over B, then choosing a free basis of M, one obtains a homomorphism $A \to M_r(B)$, and this gives a morphism of varieties

$$\operatorname{Mod}(B, d) \to \operatorname{Mod}(A, rd)$$

corresponding to the functor $M \otimes_B -$.

If A is wild we have a map

$$\operatorname{Rep}(K\langle X, Y\rangle, d) \to \operatorname{Rep}(A, rd).$$

The inverse image of any orbit is an orbit, so

$$\dim_{\mathrm{GL}_{rd}(K)} \operatorname{Rep}(A, rd) \geq \dim_{\mathrm{GL}_d(K)} \operatorname{Rep}(K\langle X, Y\rangle, d).$$

Now $\operatorname{Rep}(K\langle X, Y\rangle, d) \cong M_d(K)^2$, so $\dim \operatorname{Rep}(K\langle X, Y\rangle, d) = 2d^2$. Also every orbit of $\mathrm{GL}_d(K)$ has dimension $\leq \dim \mathrm{GL}_d(K) = d^2$. It follows that

$$\dim_{\mathrm{GL}_d(K)} \operatorname{Rep}(K\langle X, Y\rangle, d) \geq 2d^2 - d^2 = d^2$$

so $\dim_{\mathrm{GL}_{rd}(K)} \operatorname{Rep}(A, rd) \geq d^2$.

If A is tame, we can suppose that any d-dimensional module is isomorphic to a direct sum of

$$M_{i_1} \otimes K[T]/(T - \lambda_1) \oplus \cdots \oplus M_{i_m} \otimes K[T]/(T - \lambda_m)$$

where the sum of the ranks of the M_{i_j} is d. In particular $m \leq d$. This defines a map

$$\mathbb{A}^m \to \operatorname{Rep}(A, d).$$

The union of the images of these maps, over all possible choices is a constructible subset of $\operatorname{Rep}(A, d)$ of dimension $\leq d$ which meets every orbit, giving the claim. □

Theorem (Geiß)**.** *A degeneration of a wild algebra is not tame.*

Thus by Drozd's Theorem, a degeneration of a wild algebra is wild, and if an algebra degenerates to a tame algebra, it is tame.

Proof. We had the following lemma. Suppose a linear algebraic group G acts on a variety X and that $\pi : X \to Y$ is constant on orbits. Suppose that the image of any closed G-stable subset of X is a closed subset of Y. Then the function $\pi(X) \to \mathbb{Z}$, $y \mapsto \dim_G(\pi^{-1}(y))$ is upper semicontinuous.

Gabriel's theorem says that the projection $\pi : \mathrm{Alg}\,\mathrm{Rep}(r,d) \to \mathrm{Alg}(r)$ sends $\mathrm{GL}_d(K)$-stable closed subsets to closed subsets. It follows that

$$W_d = \{x \in \mathrm{Alg}(r) : \dim_{\mathrm{GL}_d(K)} \mathrm{Rep}(K_x, d) > d\}$$

is closed in $\mathrm{Alg}(r)$, and it is obviously $\mathrm{GL}_r(K)$-stable.

Suppose $x, y \in \mathrm{Alg}(r)$, K_x is wild and $y \in \overline{\mathrm{GL}_r(K)x}$. By the lemma, $x \in W_d$ for some d. Then the orbit of x is contained in W_d, and hence so is the orbit closure. Thus $y \in W_d$. Thus by the lemma, K_y is not tame. □

Examples. (a) The algebra

$$A = K\langle a, b\rangle/(a^2 - bab, b^2 - aba, (ab)^2, (ba)^2)$$

degenerates to

$$B = K\langle a, b\rangle/(a^2, b^2, (ab)^2, (ba)^2)$$

and B is a string algebra, so tame, hence A is tame.

The degeneration is given as follows. For $t \in K$, let $A_t = K\langle a, b\rangle/(a^2 - tbab, b^2 - taba, (ab)^2, (ba)^2)$. The elements $1, a, b, ab, ba, aba, bab$ are a basis, and writing the multiplication in terms of this basis gives an element $x^t \in \mathrm{Alg}(7)$ with $K_{x^t} \cong A_t$. Now the map $\mathbb{A}^1 \to \mathrm{Alg}(7)$, $t \mapsto x^t$ is a map of varieties, $A_0 \cong B$ and $A_t \cong A$ for $t \neq 0$.

The algebra A arises when one studies representations of the quaternion group G of order 8 over an algebraically closed field K of field of characteristic 2. Namely, KG is a local self-injective algebra. Thus it has simple socle S, which is an ideal in KG, and apart from the free module, all other indecomposable modules are annihilated by S, so are KG/S-modules. Now $KQ/S \cong A$.

[At the moment, no classification of the indecomposable modules for this algebra A seems to be known.]

(b) (**Omitted in the lecture**) A problem raised by I. M. Gelfand, The cohomology of infinite dimensional Lie algebras: some questions of integral geometry, 1971 is the classification of the representations of the quiver

$$\circ \; \underset{b_1}{\overset{a_1}{\leftrightarrows}} \; \circ \; \underset{b_2}{\overset{a_2}{\rightleftarrows}} \; \circ$$

with $b_1a_1 = b_2a_2$ nilpotent. To turn it into a f.d. algebra, we use the admissible relations $b_1a_1 = b_2a_2$ and $(b_1a_1)^n = 0$ for some n.

This algebra is isomorphic to the algebra given by the quiver with vertices $1, 2$, a loop c at 1 and arrows $a : 2 \to 1$ and $b : 1 \to 2$ and non-admissible relations $ba = 0$, $(bca)^n = 0$ and $c^2 - c = 0$. Namely, since c is idempotent, we can send it to the trivial path at left hand vertex and $e_1 - c$ to the trivial path at the right hand vertex.

For $t \in K$ we can change the last relation to $c^2 - tc = 0$. This gives a family of algebras A_t. In fact A_t has basis given by the paths which occur as a proper subpath of $(bca)^n$, so the dimension d doesn't depend on t. Also, using these bases for all t, we get a map $\mathbb{A}^1 \to \mathrm{Alg}(d)$.

The algebras A_t with $t \neq 0$ are all isomorphic. Thus A_0 is a degeneration of A_1. Now the algebra A_0 is a string algebra, so tame. Thus A_1 is tame. Thus the Gelfand problem is tame in a suitable sense.

The modules for A_1 and similar algebras (eg clannish or skew gentle) were classified independently by V. M. Bondarenko and by me.

Remark. An algebra A is of *strongly unbounded type* if there are infinitely many d, such that there are infinitely many non-isomorphic indecomposable A-modules of dimension d.

The second Brauer-Thrall conjecture states that an algebra of infinite representation type must be of strongly unbounded representation type. It is proved (for algebras over an algebraically closed field) by the efforts of Bautista, Bongartz, Gabriel, Nazarova, Roiter, Salmeron and others. See Bautista, On algebras of strongly unbounded representation type, 1985.

The method of Geiss's Theorem shows that a degeneration of an algebra of unbounded representation type cannot be of finite representation type. Gabriel used this, together with the second Brauer-Thrall conjecture to prove that the set of algebras of finite representation type is open in $\mathrm{Alg}(r)$.

9 Matrix reductions and Drozd's Theorem

This chapter is non-examinable. We discuss A. V. Roiter and M. M. Kleiner's formalism of 'matrix reductions' and their use in the proof of Yu. A. Drozd's 'Tame and Wild' Theorem. We work over an algebraically closed field K.

9.1 Bocses and corings

Roiter's eventual setting for his matrix reductions was with the notion of a 'bocs', a bimodule over a category with coalgebra structure. The references are:

A. V. Roiter, Matrix problems and representations of BOCS's, in: Representations and Quadratic Forms (Yu. A. Mitropol'skii, editor). Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1979, pp. 3-38;

A. V. Roiter, Matrix problems and representations of BOCSs, in: Representations of Algebras I, SLN 831, 1980.

It is easier to work with a non-categorical version. This is the notion of a 'coring'.

Definition. Let A be a ring. A *coring* over A is an A-A-bimodule C equipped with A-A-bimodule homomorphisms

$$\mu : C \to C \otimes_A C, \quad \epsilon : C \to A$$

the *comultiplication* and the *counit*, such that the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{\mu} & C \otimes_A C \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu\otimes 1} \\ C \otimes_A C & \xrightarrow{1\otimes\mu} & C \otimes_A \otimes_A C \end{array}$$

and the following compositions are the identity

$$C \xrightarrow{\mu} C \otimes_A C \xrightarrow{1\otimes\epsilon} C \otimes_A A \cong C$$

$$C \xrightarrow{\mu} C \otimes_A C \xrightarrow{\epsilon\otimes 1} A \otimes_A C \cong C.$$

We write (A, C) for the pair. By abuse of terminology, we might call this a *bocs.*

Definition. The category $\mathrm{Rep}(A, C)$ of representations of a bocs (A, C) has:

- Objects are A-modules V
- The set of morphisms from V to V' is $\mathrm{Hom}_A(C \otimes_A V, V')$.
- The composition of $f : C \otimes_A V \to V'$ and $f' : C \otimes_A V' \to V''$ is

$$C \otimes_A V \xrightarrow{\mu\otimes 1} C \otimes_A C \otimes_A V \xrightarrow{1\otimes f} C \otimes_A V' \xrightarrow{f'} V''.$$

- The identity morphism is $\epsilon \otimes 1 : C \otimes V \to V$, Observe that A is naturally an A-coring with the identity maps, and $\text{Rep}(A, A)$ is isomorphic to A-Mod.

Note that a morphism $V \to V'$ can also be given by an A-A-bimodule map $C \to \text{Hom}_K(V, V')$.

Remark. A *left C-comodule* is an A-module X equipped with an A-module homomorphisms $\nu : X \to C \otimes_A X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\nu} & C \otimes_A X \\ \nu \downarrow & & \downarrow \mu\otimes 1 \\ C \otimes_A X & \xrightarrow{1\otimes\nu} & C \otimes_A C \otimes_A X \end{array}$$

commutes and

$$X \xrightarrow{\nu} C \otimes_A X \xrightarrow{\epsilon\otimes 1} A \otimes_A X \cong X$$

is the identity map.

A homomorphism of C-comodules $\theta : X \to X'$ is an A-module map such that

$$\begin{array}{ccc} X & \xrightarrow{\nu} & C \otimes_A X \\ \theta \downarrow & & \downarrow 1\otimes\theta \\ X' & \xrightarrow{\nu'} & C \otimes_A X' \end{array}$$

commutes.

This gives a category of C-comodules. It is an additive category, but in general it does not have kernels if C_A is not flat, and we can't even in general talk about 'subcomodules'.

Given an A-module V, one can turn $C \otimes_A V$ into a C-comodule via the map

$$C \otimes_A V \xrightarrow{\mu\otimes 1} C \otimes_A (C \otimes_A V).$$

It is called an *induced comodule.*

Given A-modules V and V' there is a 1-1 correspondence between A-module homomorphisms $f : C \otimes_A V \to V'$ and C-comodule maps $\theta : C \otimes_A V \to C \otimes_A V'$ given by

- θ is the composition

$$C \otimes_A V \xrightarrow{\mu\otimes 1} C \otimes_A C \otimes_A V \xrightarrow{1\otimes f} C \otimes_A V'$$

- f is the composition

$$C \otimes_A V \xrightarrow{\theta} C \otimes_A V' \xrightarrow{\epsilon\otimes 1} A \otimes_A V \cong V.$$

Thus $\text{Rep}(A, C)$ is equivalent to the category of induced comodules.

The main purpose here is to turn tensor product functors into fully faithful functors. If M is an A-B-bimodule, then one get a tendsor product functor

$$M \otimes_B - : B\text{-Mod} \to A\text{-Mod},$$

but in general it is not fully faithful. We are interested in the case when M is f.g. projective over B. Given a f.g. projective right B-module P, we write $P^\vee := \mathrm{Hom}_B(P, B)$. It is a f.g. projective left B-module. Note that if X is any right B-module, then there is an isomorphism

$$X \otimes_B P^\vee \to \mathrm{Hom}_B(P, X), \quad m \otimes \phi \mapsto (p \mapsto x\phi(p)).$$

In particular there is an isomorphism $M \otimes_B M^\vee \to \mathrm{Hom}_B(M, M)$ as A-A-bimodules, so the homothety $A \to \mathrm{Hom}_B(M, M)$, $a \mapsto (m \mapsto am)$ gives A-A-bimodule map

$$\delta : A \to M \otimes_B M^\vee.$$

There is also an evaluation map

$$ev : M^\vee \otimes_A M \to B, \quad \phi \otimes m \mapsto \phi(m).$$

which is a B-B-bimodule map.

Proposition. *Let (A, C) be a bocs and M an A-B-bimodule which is f.g. projective as a B-module. Then $C^B := M^\vee \otimes_A C \otimes_A M$ becomes a coring over B via*

$$C^B = M^\vee \otimes C \otimes_A M \xrightarrow{1\otimes\mu\otimes 1} M^\vee \otimes_A C \otimes_A C \otimes_A M$$

$$\cong M^\vee \otimes_A C \otimes_A A \otimes_A C \otimes_A M$$

$$\xrightarrow{1\otimes 1\otimes\delta\otimes 1\otimes 1} M^\vee \otimes_A C \otimes_A M \otimes_B M^\vee \otimes_A C \otimes_A M \cong C^B \otimes_B C^B$$

and

$$C^B = M^\vee \otimes C \otimes_A M \xrightarrow{1\otimes\epsilon\otimes 1} M^\vee \otimes_A A \otimes_A M \cong M^\vee \otimes_A M \xrightarrow{ev} B.$$

Moreover there is a fully faithful functor

$$\mathrm{Rep}(B, C^B) \to \mathrm{Rep}(A, C)$$

sending a B-module V to $M \otimes_B V$.

Proof. Straightforward. □

Definition. A bocs (A, C) is *normal* if there is some $g \in C$ with $\mu(g) = g \otimes g$ and $\epsilon(g) = 1$. Thus g is a *grouplike* element of C.

9.2 Differential graded algebras and biquivers

Roiter and Kleiner first formalized matrix reductions using differential graded categories.

- A. V. Roĭter and M. M. Kleiner, Representations of differential graded categories, in: Representations of algebras (Proc. Internat. Conf., Carleton Univ., Ottawa, Ont., 1974), pp. 316-339, Lecture Notes in Math., Vol. 488, Springer-Verlag, Berlin-New York, 1975

- M. M. Kleĭner and A. V. Roĭter, Representations of differential graded categories (Russian), in: Matrix problems (Russian), pp. 5-70, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1977

I will explain a noncategorical version. Different versions can be found in:

- Crawley-Boevey, Matrix problems and Drozd's theorem, in: Topics in algebra, 1990.

- Bautista, Salmeron and Zuazua, Differential Tensor Algebras and Their Module Categories, 2009.

Definition. A *differential graded algebra (dga)* is a pair (Λ, d) where Λ as a graded K-algebra, so

$$\Lambda = \bigoplus_{n\in\mathbb{Z}} \Lambda^n$$

with $\Lambda^n\Lambda^m \subseteq \Lambda^{n+m}$, and $d : \Lambda \to \Lambda$ is a K-linear differential which raises degree by 1, so

$$d(\Lambda^n) \subseteq \Lambda^{n+1}$$

and $d \circ d = 0$, and which satisfies the the graded Leibnitz rule

$$d(ab) = d(a)b + (-1)^{\deg a} a d(b)$$

for a homogeneous.

Here we use cohomological numbering. One could instead use homological numbering with $\Lambda_n = \Lambda^{-n}$, so that d reduces degees by 1.

In any graded algebra Λ, one can show that $1 \in \Lambda^0$, so that Λ^0 is an algebra.

Proposition (Roiter correspondence)**.** *(i) Given a normal bocs (A, C) one gets a dga (Λ, d) as follows. Fix a grouplike element $g \in C$. Let $A = \Lambda^0$, let $\overline{C} = \operatorname{Ker}\epsilon$, and let*

$$\Lambda = T_A(\overline{C})$$

the tensor algebra of $\overline{C}$ over A, with $\overline{C}$ in degree 1. The differential is determined by

$$d(a) = ag - ga \in \overline{C}$$

for $a \in A$*, and for* $c \in \overline{C}$ *with* $\mu(c) = \sum_i c_i \otimes c_i'$*,*

$$d(c) = \mu(c) - g \otimes c - c \otimes g = \sum_i (c_i - g\epsilon(c_i)) \otimes (c_i' - \epsilon(c_i')g) \in \overline{C} \otimes_A \overline{C}.$$

(ii) Given a dga (Λ, d)*, such that* Λ *is the tensor algebra over* $A = \Lambda^0$ *of the* A*-*A*-bimodule* Λ^1*, one gets a normal bocs* (A, C) *as follows. Let*

$$C = A \oplus \Lambda^1$$

with the A*-*A*-bimodule structure given by*

$$a(b, x) = (ab, ax + d(a)b), \quad (a, x)b = (ab, xb),$$

the coring structure given by

$$\mu : C \to C \otimes_A C, \quad (a, x) \mapsto (a, x) \otimes (1, 0) + (1, 0) \otimes (0, x) + d(x),$$

$$\epsilon : C \to A, \quad (a, x) \mapsto a$$

and grouplike element $g = (1, 0)$*.*

(iii) These constructions are inverse, up to suitable equivalences.

For more, see T. Brzezinski, Flat connections and (co)modules, arxiv:math/0608170v2

Proof. Straightforward. Note that given a bocs (A, C) with grouplike g, the exact sequence

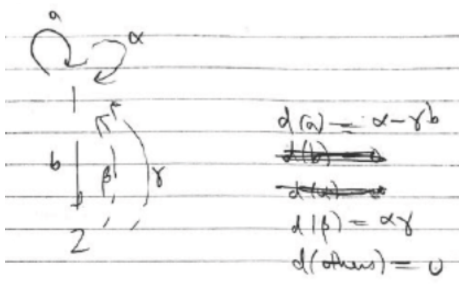
$$0 \to \overline{C} \to C \xrightarrow{\epsilon} A \to 0$$

is split as a sequence of left A-modules, with section $A \to C$, $a \mapsto ag$ and split as a sequence of right modules, with section $a \mapsto ga$. $\square$

Definition. A *biquiver* is a quiver with two types of arrows, solid of degree 0 and dotted of degree 1. Then KQ becomes a graded algebra with the degree of a path being the number of dotted arrows in the path.

A *differential biquiver* is a differential graded algebra of the form (KQ, d) where Q is a finite biquiver, and d is a differential with $d(e_i) = 0$ for each trivial path e_i.

For example



Clearly $A = KQ^0$ is the path algebra of the quiver Q^0 of solid arrows. There is an isomorphism

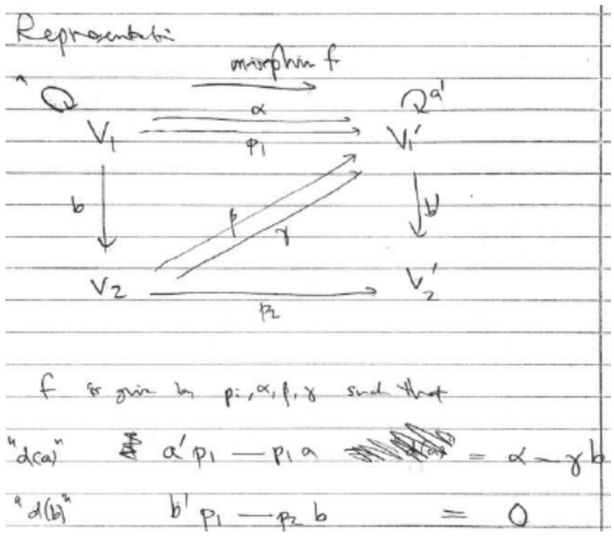
$$\bigoplus_{\alpha: i \dashrightarrow j} Ae_j \otimes e_i A \to KQ^1$$

sending $e_j \otimes e_i$ in the summand corresponding to α to α. Thus KQ^1 is a projective A-A-bimodule, and KQ is the tensor algebra of KQ^1 over A. Thus there is a bocs corresponding to the dga (KQ, d).

Definition. Given a dbq (Q, d), we define a category of representations $\mathrm{Rep}(Q, d)$ as follows.

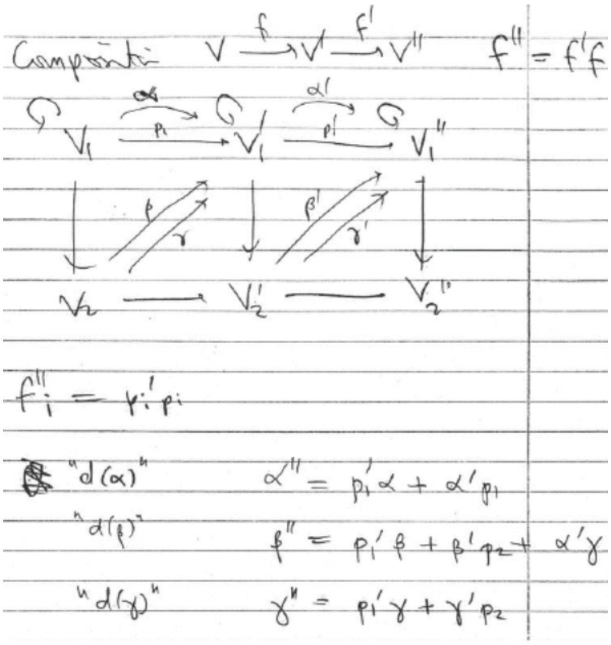
- Objects are representations V of Q^0, the quiver of solid arrows given by a vector space V_i for each vertex and a linear map $a : V_i \to V_j$ for each solid arrow $a : i \to j$.

- A morphism $f : V \to V'$ is given by linear maps $p_i : V_i \to V'_i$ for each vertex and linear maps $\alpha : V_i \to V'_j$ for each dotted arrow $\alpha : i \to j$, satisfying one relation $a'p_i - p_j a =$ "$d(a)$" for each solid arrow $a : i \to j$, where "$d(a)$" means the map $V_i \to V'_j$ obtained from $d(a)$ by replacing dotted arrows by the corresponding linear maps and solid arrows by the linear maps defining V and V' appropriately. For example



- The composition $f'' = f'f$ of morphisms $f : V \to V'$ and $f' : V' \to V''$ is given by $p''_i = p'_i p_i$ and if $\alpha : i \to j$, then $\alpha'' = p'_j \alpha + \alpha' p_i +$ "$d(\alpha)$", where again

"$d(\alpha)$" is obtained from $d(\alpha)$ by appropriate substitutions. In the example



The identity morphism $1 : V \to V$ is given by the maps $p_i = 1_{V_i}$ and the zero map $\alpha : V_i \to V_i$ for all dotted arrows $\alpha : i \to j$.

Proposition. *For a dbq* (Q, d)*, the category* $\mathrm{Rep}(Q, d)$ *is equivalent to the category of representations of the corresponding bocs* (A, C)*.*

Proof. By construction $A = KQ^0$, so an A-module is given by a representation of Q^0. Thus objects in $\mathrm{Rep}(Q, d)$ correspond to objects in $\mathrm{Rep}(A, C)$.

To give a morphism $f : V \to V'$ in $\mathrm{Rep}(A, C)$, it is equivalent to give an A-A-bimodule map $f : C \to \mathrm{Hom}_K(V, V')$.

Now $C = A \oplus KQ^1$ with its natural structure as a right A-module and left A-module structure given by

$$a(b, x) = (ab, ax + d(a)b).$$

The bimodule map $f : C \to \mathrm{Hom}_K(V, V')$ restricts to give a map $KQ^1 \to \mathrm{Hom}_K(V, V')$, and since

$$\bigoplus_{\alpha : i \dashrightarrow j} Ae_j \otimes e_i A \cong KQ^1$$

an A-A-bimodule map $KQ^1 \to \mathrm{Hom}_K(V, V')$ is determined by elements in

$$e_j \,\mathrm{Hom}_K(V, V')e_i \cong \mathrm{Hom}_K(e_i V, e_j V') = \mathrm{Hom}(V_i, V_j')$$

for each dotted arrow $\alpha : i \dashrightarrow j$. Call the elements just α. To extend this to a map $f : C \to \mathrm{Hom}_K(V, V')$ we need to give the image of the element $(1,0) \in C$. It is a K-linear map $p : V \to V'$. This must be compatible with the A-A-bimodule structure, so we need that $f(a,0) = f((1,0)a) = f(1,0)a = pa$, which is the map $V \to V'$, $v \mapsto p(av)$, and $f(a + d(a)) = f(a(1,0)) = af(1,0) = ap$, which is the map $v \mapsto ap(v)$. Thus we need $ap(v) - p(av) = f(d(a)) =$ "$d(a)$". Since d is zero on the trivial paths e_i, it follows that $p(e_i v) = e_i p(v)$, so p restricts to linear maps $p_i : V_i \to V_i$, and then if $a : i \to j$ is an arrow, $a' p_i - p_j a =$ "$d(a)$", etc. □

Definition. A dbq (Q, d) is:

- *Linear* if for any arrow a, $d(a)$ is a linear combination of paths of length at most 2.

- *Triangular* if the arrows can be ordered so that $d(a)$ only involves smaller arrows than a.

More generally one can ask for triangularity with respect to the solid arrows, or with respect to the dotted arrows, meaning that these arrows can be ordered so that the differential of one of these arrow only involves smaller arrows of this type, together with arbitrary arrows of the other type.

Lemma. *If (Q, d) is triangular, then*

(a) A morphism $f : V \to V'$ is invertible if and only if all p_i are invertible. (This only needs triangularity of the dotted arrows)

(b) Given a representation V, vector spaces V_i', invertible maps $p_i : V_i \to V_i'$ and maps $\alpha_i : V_i \to V_i'$ for dotted arrows $\alpha : i \to j$, there is a unique representation V' such that this data gives a homomorphism $V \to V'$. (This only needs triangularity of the solid arrows)

Proof. (a) Clearly if f is invertible, then the p_i must be invertible. Conversely, given a morphism f with the p_i invertible, we can construct a morphism f' with $f'f = 1$ by taking $p_i' = p_i^{-1}$ and working up the ordering of the dotted arrows $\alpha : i \to j$, choosing α' so that

$$p_j \alpha + \alpha' p_i + \text{“}d(\alpha)\text{”}$$

is zero. Similarly we can construct f'' with $ff'' = 1$. Thus $f' = f''$ is an inverse for f.

(b) Working up the ordering of the solid arrows $a : i \to j$ we construct linear maps $a' : V_i' \to V_j'$ so that

$$a' p_i - p_j a = \text{“}d(a)\text{”}.$$

□

9.3 Bimodule matrix problems

Definition. A *bimodule matrix problem* (of the most general sort) is given by a tuple (R, E, π) where R is a finite-dimensional algebra, E is a finite dimensional R-R-bimodule, and $\pi : E \to R$ is a surjective bimodule map. We define a category of representations $\mathrm{Rep}(R, E, \pi)$ via:

- Objects are pairs (P, e) where P is a finitely generated projective R-module and $e : P \to E \otimes_R P$ is a section of the surjective map $\pi \otimes 1 : E \otimes_R P \to R \otimes_R P \cong P$, meaning that $(\pi \otimes 1)e = 1_P$.

- Morphisms $(P, e) \to (P', e')$ are homomorphisms $\theta : P \to P'$ such that the following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{e} & E \otimes_R P \\ {\scriptstyle\theta}\big\downarrow & & {\scriptstyle 1\otimes\theta}\big\downarrow \\ P' & \xrightarrow{e'} & E \otimes_R P'. \end{array}$$

This type of category appeared with the name 'lift category' in Crawley-Boevey, Matrix reductions for Artinian rings and an application to rings of finite representation type, J. Alg. 1993. The following special cases were already considered by Roiter, Matrix problems and representations of bisystems, J. Soviet Math. 1975 and Drozd, Matrix problems and categories of matrices, J. Soviet Math. 1975.

Remark. Special cases.

(1) Given (R, M) where R is a f.d. algebra and M is an R-R-bimodule, let $E = R \oplus M$ and let π be the projection onto R. In this case we can identify $\mathrm{Rep}(R, E, \pi)$ as the category $\mathrm{Rep}(R, M)$ with

- objects are pairs (P, m) where P is a finitely generated projective R-module and $m : P \to E \otimes_R P$ is an R-module map

- morphisms $(P, m) \to (P', m')$ are homomorphisms $\theta : P \to P'$ such that the following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{m} & M \otimes_R P \\ {\scriptstyle\theta}\big\downarrow & & {\scriptstyle 1\otimes\theta}\big\downarrow \\ P' & \xrightarrow{m'} & M \otimes_R P'. \end{array}$$

(2) Given (A, B, N) where A and B are f.d. algebras and N is an A-B-bimodule, let $R = A \oplus B$ and let M be N considered as an R-R-bimodule by restriction on each side. In this case we can identify $\mathrm{Rep}(R, M)$ as the category $\mathrm{Rep}(A, B, N)$ with

- Objects are triples (P, Q, n) where P is a f.g. projective A-module, Q is a f.g. projective B-module and $n : P \to N \otimes_B Q$ is an A-module map

- Morphisms $(P, Q, n) \to (P', Q', n')$ are pairs (θ, ϕ) with $\theta : P \to P'$ and $\phi : Q \to Q'$ homomorphisms, such that the following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{n} & N \otimes_B Q \\ {\scriptstyle\theta}\big\downarrow & & {\scriptstyle 1\otimes\phi}\big\downarrow \\ P' & \xrightarrow{n'} & N \otimes_B Q'. \end{array}$$

(3) Let A be a f.d. algebra. Then A is naturally an A-A-bimodule, and $\mathrm{Rep}(A, A, A)$ is the category with:

- Objects are morphisms between f.g. projective A-modules $\alpha : P \to Q$, or equivalently 2-term complexes of projectives, with P in degree -1 and Q in degree 0.

- Morphisms are pairs of morphisms (θ, ϕ) giving a commutative square, or equivalently morphisms of complexes.

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\theta}\big\downarrow \\ P' & \xrightarrow{\alpha'} & Q' \end{array}$$

This category is closely related to A-mod. This correspondence is what is used in τ-tilting theory. Let $\mathbf{C}$ be the full subcategory of $\mathrm{Rep}(A, A, A)$ given by the maps $\alpha : P \to Q$ such that

$$P \overset{\alpha}{\to} Q \to \mathrm{Coker}\,\alpha \to 0$$

is a minimal projective presentation of $\mathrm{Coker}\,\alpha$. Equivalently $\mathrm{Im}\,\alpha \subseteq \mathrm{rad}\,Q$ and $\mathrm{Ker}\,\alpha \subseteq \mathrm{rad}\,P$. By properties of projective covers, any object in $\mathrm{Rep}(A, A, A)$ can be written as a direct sum

$$(P, 0, 0) \oplus (Q, Q, 1_Q) \oplus (P', Q', \alpha)$$

where the last term is in $\mathbf{C}$.

Thus the indecomposable objects are $(P[i], 0, 0)$, $(P[i], P[i], 1)$ and the indecomposable objects of $\mathbf{C}$.

Also there is a functor $\mathrm{Rep}(A, A, A) \to A$-mod sending (X, Y, α) to $\mathrm{Coker}\,\alpha$. This restricts to a functor $F : \mathbf{C} \to A$-mod which is a *representation equivalence*, meaning that F is full, dense, and it reflects isomorphisms (which means that if a morphism is sent by F to an isomorphism, then it is an isomorphism).

It follows that F gives a bijection between the isomorphism classes of indecomposable objects in $\mathbf{C}$ and in A-mod.

Definition. An additive category is said to have *split idempotents* if for every idempotent endomorphism θ of an object X there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $gf = \theta$ and $fg = 1_Y$.

Applying this to θ and $\theta' = 1 - \theta$ gives objects and morphisms

$$Y \underset{g}{\overset{f}{\leftrightarrows}} X \underset{g'}{\overset{f'}{\rightleftarrows}} Y'$$

with $gf + g'f' = 1_X$ as well, showing that $X \cong Y \oplus Y'$ and θ corresponds to the projection onto Y.

Lemma. $\mathrm{Rep}(A, E, \pi)$ *is an additive category with split idempotents.*

Proof. It is straightforward that it is additive. Let $\theta : (P, e) \to (P, e)$ be an idempotent endomorphism. Then $P = \mathrm{Im}\,\theta \oplus \mathrm{Ker}\,\theta$. Let $f : P \to \mathrm{Im}\,\theta$ be the map induced by θ and $g : \mathrm{Im}\,\theta \to P$ the inclusion. Then $gf = \theta$ and $fg = 1_{\mathrm{Im}\,\theta}$ (since θ is idempotent). We need to find a map $e' : \mathrm{Im}\,\theta \to E \otimes \mathrm{Im}\,\theta$ such that f and g give morphisms between (P, e) and $(\mathrm{Im}\,\theta, e')$.

The map $1 \otimes g : E \otimes \mathrm{Im}\,\theta \to E \otimes P$ is a kernel for $1 \otimes (1 - \theta) : E \otimes P \to E \otimes P$. Now $(1 - \theta)g = 0$, so

$$0 = e(1 - \theta)g = (1 \otimes (1 - \theta))eg$$

so eg factors as $(1 \otimes g)e'$ for some $e' : \mathrm{Im}\,\theta \to E \otimes \mathrm{Im}\,\theta$, and this works. □

Theorem. *There is a correspondence between bimodule matrix problems* (R, E, π) *and triangular linear dbqs* (Q, d)*, such that they have equivalent categories of representations.*

Thus the category of maps between projectives arises as $\mathrm{Rep}(Q, d)$ for some triangular linear dbq (Q, d). Also $\mathrm{Rep}(Q, d)$ has split idempotents. Also, the matrix reductions which we shall study for dbqs in the next section can be done purely in the context of bimodule matrix problems of the form (R, E, π). This is done in the paper of Crawley-Boevey mentioned above.

This theorem is mentioned, without an explicit statement, in the introduction to the paper of Crawley-Boevey cited at the start of this section.

Proof. (Sketch) Given (R, E, π), we may suppose that R is basic. Choose a complete set of primitive orthogonal idempotents $e_1, \ldots, e_n$. Since R is basic and we are working over an algebraically closed field, we have $R = J \oplus S$ where J is the Jacobson radical and S is the semisimple subalgebra spanned by the e_i.

There is some $h \in E$ with $\pi(h) = 1$, and replacing h with

$$\sum_{i=1}^{n} e_i h e_i$$

we may suppose that $sh = hs$ for all $s \in S$. The assignment $\delta(r) = rh - hr$ defines a derivation $\delta : R \to M$ which vanishes on S, where $M = \operatorname{Ker} \pi$.

We write D for the duality $\operatorname{Hom}_K(-, K)$. If V is a right S-module and W is a left S-module, then we can identify

$$V \otimes_S W = \bigoplus_{i=1}^{n} V e_i \otimes_K e_i W$$

and if V and W are finite-dimensional, we get a natural isomorphism

$$DW \otimes_S DV \to D(V \otimes W).$$

Also, if V and W are left S-modules, and X is a f.d. S-S-bimodule, then there is a natural isomophism

$$\operatorname{Hom}_S(V, X \otimes_S W) \to \operatorname{Hom}_S(DX \otimes_S V, W).$$

When choosing a basis of an S-S-bimodule V, because of the decomposition

$$V = \bigoplus_{i,j=1}^{n} e_j V e_i,$$

we can ensure that each basis element is in $e_j V e_i$ for some i, j.

We construct a differential biquiver as follows. The vertex set for Q is $\{1, \ldots, n\}$. The solid arrows $a : i \to j$ correspond to basis elements of DM in $e_j DM e_i$. The dotted arrows $\alpha : i \to j$ correspond to basis elements of DJ in $e_j DJ e_i$.

The multiplication map $m : J \otimes_S J \to J$ gives a map $Dm : DJ \to DJ \otimes_S DJ$. We define the differential of the dotted arrows by $d(\alpha) = Dm(\alpha)$, a linear combination of compositions of two dotted arrows.

The left action $\ell : J \otimes_S M \to M$, the right action $r : M \otimes_S J \to M$ and the restriction of the derivation $\delta : R \to M$ to J dualize to give maps

$$D\ell : DM \to DM \otimes_S DJ, \quad Dr : DM \to DJ \otimes_S DM, \quad D\delta : DM \to DJ$$

We define the differential of the solid arrows by $d(a) = D\delta(a) + D\ell(a) - Dr(a)$, a linear combination of paths of degree 1 and length at most 2.

Using the graded Leibnitz rule we can define d on an arbitrary path in Q. We have $d^2 = 0$, so it is a differential. This is essentially associativity in R and for the action on M and the derivation property.

Thus we get a dbq. By construction it is linear, and by choosing bases working up the dual of the radical series

$$(J^2)^\perp \subseteq (J^3)^\perp \subseteq \cdots \subseteq DJ$$

$$(M_1)^\perp \subseteq (M_2)^\perp \subseteq \cdots \subseteq DM, \quad M_0 = M,\ M_{i+1} = JM_i + M_iJ$$

we can ensure that it is triangular.

The reverse construction is clear. Given a linear dbq (Q, d), let S be the semisimple subalgebra of KQ spanned by the trivial paths, J the dual of the subspace spanned by the dotted arrows, M the dual of the subspace spanned by the solid arrows, etc.

We define an equivalence $\mathrm{Rep}(Q, d) \to \mathrm{Rep}(R, E, \pi)$ as follows. An object in $\mathrm{Rep}(Q, d)$ is a KQ^0-module, so given by an S-module V and an S-module map $a : DM \otimes_S V \to V$. Let $P = R \otimes_S V$. As V runs through all S-modules up to isomorphism, P runs through the projective R-modules up to isomorphism. Now the maps a correspond 1:1 with S-module maps $\overline{a} : V \to M \otimes_S V$. These correspond 1:1 with the R-module sections e of $\pi \otimes 1 : E \otimes_R P \to P$ via

$$e : R \otimes_S V = P \to E \otimes_R P = E \otimes_R R \otimes_S V \cong E \otimes_S V, \quad e(r \otimes v) = r\overline{a}(v) + rh \otimes v.$$

A morphism $f : V \to V'$ in $\mathrm{Rep}(Q, d)$, where V and V' are given by maps $a : DM \otimes_S V \to V$ and $a' : DM \otimes_S V' \to V'$, is given by an S-module map $p : V \to V'$ and an S-module map $\alpha : DJ \otimes_S V \to V'$ satisfying a condition corresponding to $a'p_i - p_j a =$ "$d(a)$" for all solid arrows $a : i \to j$. This is

$$p\ a - a'\ (1 \otimes p) = \alpha\ (D\delta \otimes 1) - \alpha\ (1 \otimes a)\ (Dr \otimes 1) + a'\ (1 \otimes \alpha)\ (D\ell \otimes 1).$$

as a map $DM \otimes_S V \to V'$. The corresponding equation for p and $\overline{\alpha} : V \to J \otimes_S V'$ is

$$(1 \otimes p)\ \overline{a} - \overline{a'}\ (1 \otimes p) = (\delta \otimes 1)\ \overline{\alpha} - (r \otimes 1)\ (1 \otimes \overline{\alpha})\ \overline{a} + (\ell \otimes 1)\ (1 \otimes \overline{a'})\ \overline{\alpha}$$

as maps $V \to M \otimes_S V'$.

Consider the objects (P, e) and (P', e') given by V, a and V', a'. Any R-module map $\theta : P = R \otimes_S V \to R \otimes_S V' = P'$ corresponds to an S-module map

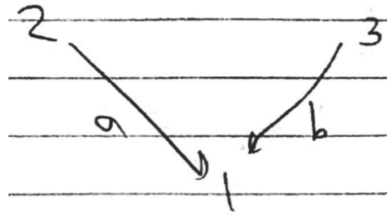
$$V \to R \otimes_S V' = (S \oplus J) \otimes V' \cong V' \oplus (J \otimes_S V').$$

so is given by a pair of maps $p, \overline{\alpha}$. One can check that the condition for θ to be a morphism between the objects (P, e) and (P', e') is exactly the condition above on p and $\overline{\alpha}$.

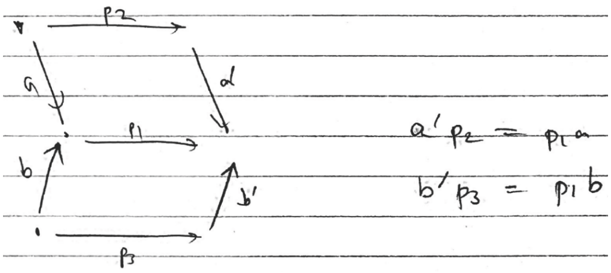
One also needs to check that composition corresponds, but this is omitted. □

9.4 Matrix reductions

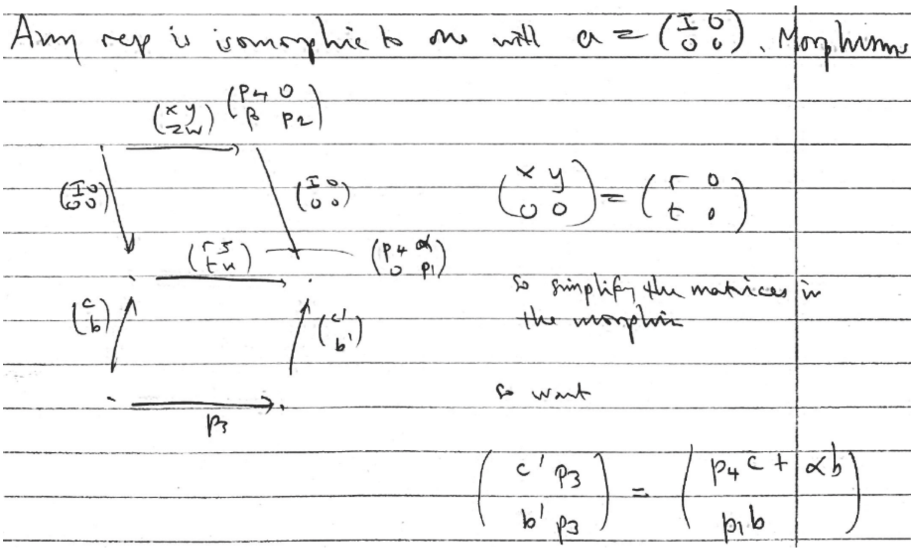
We do an example. We start with representations of the quiver (or dbq) $1 \xrightarrow{a} 2 \xleftarrow{b} 3$.



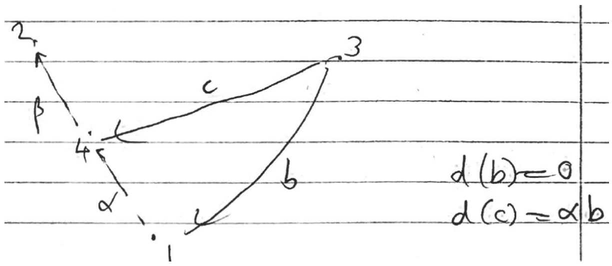
Homomorphisms are given by intertwining matrices



Using the normal form for rectangular matrices under row and column operations, and part (b) of the lemma in §9.2 we get that any representation is isomorphic to one in which the matrix a has block form $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Morphisms between representations of this form are given as follows:



This corresponds to representations of a new dbq (Q', d').



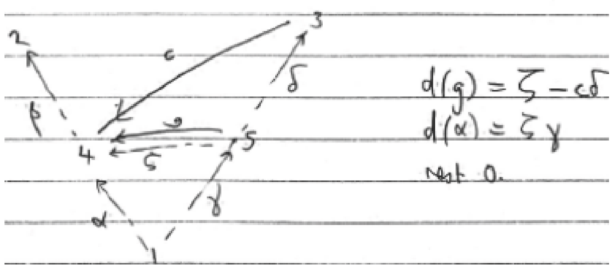
This easily generalizes to the following.

Theorem (Edge reduction). *Let (Q, d) be a dbq with a solid edge a which is not a loop and with $d(a) = 0$. Then there is a new dbq (Q', d') and an equivalence* $\mathrm{Rep}(Q, d) \leftarrow \mathrm{Rep}(Q', d')$. *The quiver Q' has one more vertex than Q.*

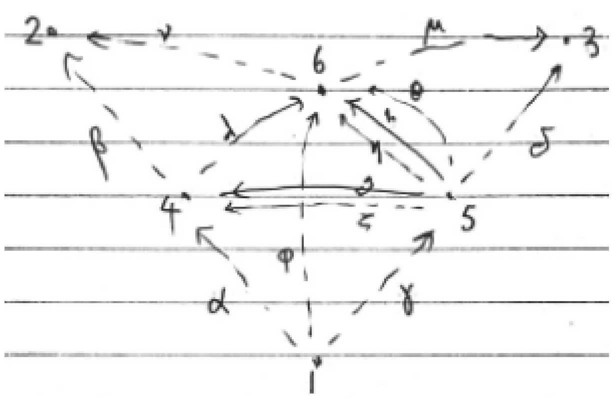
The new dbq is constructed from the old one, and we indicate this by writing $(Q, d) \to (Q', d')$.

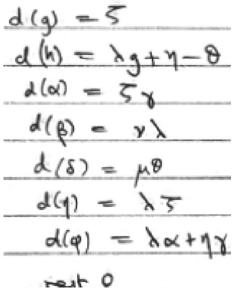
Note that if there is a loop a at a vertex i with $d(a) = 0$, then the representations with $V_i = K$, a a scalar, and all other vector spaces and arrows zero, are nonisomorphic and indecomposable, so this can't happen if (Q, d) has finite representation type.

Now reducing edge b in the same way gives



Then reducing c gives

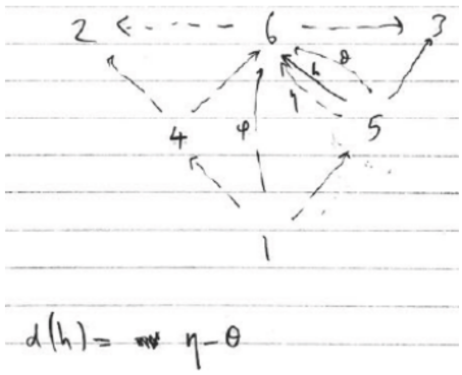




Since $d(g) = \zeta$, a morphism $f : V \to V'$ satisfies $g'p_5 - p_4 g = \zeta$.

Taking $p_i = 1_{V_i}$, $\zeta = -g$, $\alpha = 0$, etc. we get that any representation is isomorphic to one with $g = 0$.

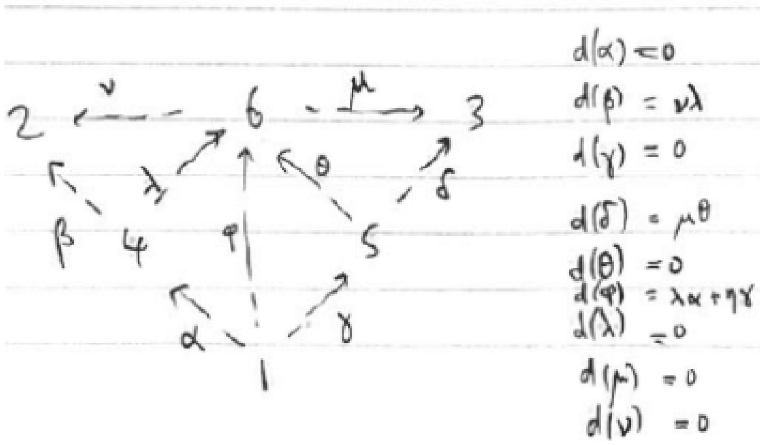
Then any morphism between such representations has $\zeta = 0$. This corresponds to the following dbq.



In general we get:

Theorem (Regularization of an arrow)**.** *If a is a solid arrow with $d(a)$ a dotted arrow, or more generally a nonzero linear combination of dotted arrows, then there is a reduction $(Q, d) \to (Q', d')$ and an equivalence* $\mathrm{Rep}(Q, d) \leftarrow \mathrm{Rep}(Q', d')$. *The number of arrows decreases.*

Now in order to regularize h we make a substitution, replacing η by $\eta' = \eta - \theta$. Thus the dbq has $d(h) = \eta'$ and $d(\varphi) = \lambda\alpha + (\eta' + \theta)\gamma$. Now we can regularize h to get



(There is an error in the picture. The differential of φ should be $\lambda\alpha + \theta\gamma$.)

Now there are no solid arrows, so the indecomposable representations are exactly the representations given by a one-dimensional vector space at a vertex, zero elsewhere. The dotted arrows give dual bases of the Hom spaces between indecomposables. The differential encodes the composition of homomorphisms. Because of triangularity, the dotted arrows with differential zero give dual bases of the spaces of irreducible maps between indecomposables.

Note that any triangular dbq of finite representation type must reduce after a finite number of steps of edge reduction and regularization to a dbq like this, with no solid arrows. Namely, each step of edge reduction increases the number of vertices, but this is bounded by the number of indecomposable representations, and between any two steps of edge reduction, only finitely many steps of regularization are possible, as each reduces the number of arrows.

End of lecture on 2026-01-29. Provisional script for the next lecture follows (subject to change).

9.5 Layered dbqs and Drozd's theorem

Theorem (Drozd's Tame and Wild Theorem)**.** *Any f.d. algebra (over an algebraically closed field) is either tame or wild.*

We need a generalization of triangular dbqs.

- The original paper, Drozd, Tame and wild matrix problems, Amer. Math. Soc. Transl. 1986, used 'almost free bocses'.

- When I wrote my paper Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc. 1988, I found it necessary to make some changes, and used 'layered bocses'.

- A more recent treatment is given by Bautista, Salmeron and Zuazua, Differential Tensor Algebras and Their Module Categories, 2009.

Here I will sketch a proof, essentially as in my paper, but without the categorical setup.

Definition. A *layered dbq* is given by the following:

- A biquiver Q with solid and dotted arrows of degree 0 and 1.
- A set of solid loops in Q, called *minimal loops*, at most one at each vertex.
- A nonzero polynomial $g(x) \in K[x]$ for each minimal loop a.

Given this data, let $\widetilde{KQ}$ be the algebra obtained from KQ by inverting the $g(a)$. It is the path algebra of a quiver with, for each minimal loop a, a new loop ℓ_a of degree 0 at the same vertex i, modulo relations $g(a)\ell_a = \ell_a g(a) = e_i$. These relations are homogeneous of degree 0, so $\widetilde{KQ}$ is still graded,

$$\widetilde{KQ} = \bigoplus_{n \geq 0} \widetilde{KQ}^n.$$

and it is still the tensor algebra of $\widetilde{KQ}^1$ over $\widetilde{KQ}^0$.

- A differential d making $\widetilde{KQ}$ into a dga, with $d(e_i) = 0$ and $d(a) = 0$ and $d(\ell_a) = 0$ for all minimal loops a.

We get a category of representations $\mathrm{Rep}(Q, d)$ as before.

- A representation is a $\widetilde{KQ}^0$-module, so a representation of the quiver of solid arrows such that $g_a(a)$ is invertible for each minimal loop a.
- A homomorphism $f : V \to V'$ is given by linear maps $p_i : V_i \to V_i'$ for each vertex and $\alpha : V_i \to V_j'$ for each dotted arrow $\alpha_i : i \dashrightarrow j$ such that $a'p_i - p_j a =$ "$d(a)$" for each solid arrow $a : i \to j$.
- We assume triangularity for the solid arrows.
- Instead of triangularity for dotted arrows, which we cannot preserve, we assume that a morphism $f : V \to V'$ is invertible if and only if all p_i are invertible.

The *dimension vector* of a representation V has components $\dim V_i$. The *total dimension* is the sum of the components.

Theorem. *Let (Q, d) be layered dbq. Then either:*

(Tame) For any n, there are finitely many $\widetilde{KQ}^0$-$K[x, g_i(x)^{-1}]$-bimodules B_i, f.g. free over $K[x, g_i(x)^{-1}]$, such that

- The functor $B_i \otimes - : K[x, g_i(x)^{-1}]$-mod $\to \mathrm{Rep}(Q, d)$ preserves indecomposability and sends non-isomorphic modules to non-isomorphic representations, and

- All but finitely many f.d. indecomposable representations of (Q, d) of total dimension n are isomorphic to $B_i \otimes N$ for some i and some finite dimensional indecomposable $K[x, g_i(x)^{-1}]$-module N.

or

(Wild) There is a $\widetilde{KQ}^0$*-*$K\langle x,y\rangle$*-bimodule, f.g. free over* $K\langle x,y\rangle$*, such that the tensor product functor* $K\langle x,y\rangle\text{-mod} \to \mathrm{Rep}(Q,d)$

- sends indecomposable modules to indecomposable representations

- sends non-isomorphic modules to non-isomorphic representations.

Given a f.d. algebra A, we have seen that there is a dbq (Q,d) whose representations correspond to 2-terms complexes of projectives A-modules. The theorem above applies to this dbq, and shows that the algebra A is tame or wild. One complication is that it gives A-B-bimodules which are not necessarily free over B. For tameness, I explained how to get around this in the remarks after the definition of tameness. For wildness, a similar trick is possible. There is an A-$K\langle X,Y\rangle$-bimodule, f.g. over $K\langle X,Y\rangle$. Tensoring with $K[X,Y]$ there is an A-$K[X,Y]$-bimodule, f.g. over $K[X,Y]$. Then tensoring up to $K[X,Y]_f$ for some nonzero polynomial $f \in K[X,Y]$ we get an A-$K[X,Y]_f$-bimodule, f.g. free over $K[X,Y]_f$. Then $K[X,Y]_f$ is known to be wild, so we can compose this with a representation embedding $K\langle X,Y\rangle\text{-mod} \to K[X,Y]_f\text{-mod}$.

Theorem (Wild configurations)**.** *If* $a : i \to j$ *is a solid arrow, not a minimal loop, then* (Q,d) *is wild if*

(i) $d(a) = 0$ *and there is a minimal loop at* i *or* j*, or*

(ii) There are minimal loops b *and* c *at* i *and* j*, say with polynomials* $f(x)$ *and* $g(x)$*, and*

$$d(a) = h(b,c)\alpha$$

with α *a dotted arrow and* $h(x,y)$ *non-invertible in* $K[x,y,f(x)^{-1},g(y)^{-1}]$*.*

Proof. For example if a is a loop at i and b is the minimal loop at i, with polynomial $f(x)$, then in case (i) one takes the representation of $\widetilde{KQ}^0$ given by $K\langle x,y\rangle^3$ with b acting as an upper triangular Jordan block with eigenvalue λ not a root of $f(x)$ and a acting as

$$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$$

We omit details and the other cases. □

We need new versions of the reductions.

(a) Edge reduction. If $a : i \to j$ is a solid edge and $d(a) = 0$ and there is no minimal loop at i or j, there is a reduction $(Q,d) \to (Q',d')$ replacing a by $i \dashleftarrow k \dashleftarrow j$, giving an equivalence $R(Q,d) \leftarrow R(Q',d')$.

What has improved? The *norm* of a representation V is

$$\sum_{\text{solid arrows } a\,:\,i\to j} \dim V_i \dim V_j.$$

It is the number of entries in the matrices for a representation. When we reduce an edge $a : i \to j$, if V_i and V_j are nonzero, then V comes from a representation of (Q', d') of smaller norm.

(b) Regularization. If $a : i \to j$ is a solid arrow of degree 0 and $d(a)$ is a nonzero linear combination of dotted arrows, there is a reduction $(Q, d) \to (Q', d')$ giving an equivalence $R(Q, d) \leftarrow R(Q', d')$. This has similar behaviour for the norm.

(c) Theorem (Unravelling a loop). Suppose a is a minimal loop at vertex i with polynomial $f(x)$. Let $g(x)$ be a polynomial and $n \geq 0$. There is a reduction $(Q, d) \to (Q', d')$ and a fully faithful functor $R(Q, d) \leftarrow R(Q', d')$ such that any representation of total dimension $\leq n$ with $g(a)$ not invertible comes from one of smaller norm.

Sketch. Let $\lambda_1, \ldots, \lambda_k$ be the roots of $g(x)$ which are not roots of $f(x)$. By Fitting's Lemma and Jordan normal form, the matrix for a is similar to a diagonal block matrix where the first block a' does not have any of the λ_i as an eigenvalue, and the other blocks are Jordan blocks $J_r(\lambda_j)$ with $r \leq n$, say with n_{jr} such blocks of this form. Collecting identical Jordan blocks gives a diagonal block matrix where the first block is a' and the other blocks are of the form

$$\begin{pmatrix} \lambda_j I & I & & \\ & \lambda_j I & \ddots & \\ & & \ddots & I \\ & & & \lambda_j I \end{pmatrix}$$

with r copies of $\lambda_j I$ on the diagonal, and where I is an identity matrix of size n_{jr}.

Now, as in the case of edge reduction, we need to consider a homomorphism between two representations of this form, and see that the matrices take block form. We then encode the blocks using a new dbq Q'. The loop at vertex i is now marked with the polynomial $f(x)g(x)$ and there are new vertices indexed by pairs (j, r) with $1 \leq j \leq k$ and $1 \leq r \leq n$ picking out a block matrix as displayed above. There are also many new arrows. For example an arrow b from some other vertex ℓ to i still gives an arrow $\ell \to i$, but it also gives r arrows $\ell \to (j, r)$ for each (j, r).

Two silly reductions:

(d) Localization. If a is minimal loop with polynomial $f(x)$, replace it with a polynomial $f(x)g(x)$. One gets a reduction $(Q, d) \to (Q', d')$ with a fully faithful functor $R(Q,d) \leftarrow R(Q',d')$ whose image is the representations with $g(a)$ invertible.

(e) Deleting a vertex gives a reduction $(Q,d) \to (Q',d')$ and a fully faithful functor $R(Q,d) \leftarrow R(Q',d')$ whose image is the representations which are zero at the vertex.

We now come to the proof of tame and wild theorem. Suppose (Q,d) is not wild. We show that for any n, there are a finite number of sequences of reductions

$$(Q,d) \begin{cases} \to \cdots \to (Q_1,d_1) \\ \dots \\ \to \cdots \to (Q_N,d_N) \end{cases}$$

leading to minimal dbqs (meaning that there are no solid arrows except minimal loops), and such that every representation of Q of total dimension $\leq n$ is in the image of one of the compositions of functors $R(Q,d) \leftarrow \cdots \leftarrow R(Q_i,d_i)$. This gives tameness.

First consider reductions deleting any set of vertices. Then we only need to worry about sincere representations, and we can be sure that the norm will reduce.

Let a be a solid arrow, not a minimal loop, but otherwise minimal in the ordering of the solid arrows. There are the following possibilities.

(1) If $d(a) = 0$ and a is an edge, then by the wild configurations, there are no minimal loops at either end, and we can do edge reduction.

(2) If $d(a) = 0$ and a is a loop at vertex i, then by the wild configurations there is no minimal loop at i, and we can nominate a as a new minimal loop with the constant polynomial $f(x) = 1$.

(3) If $d(a) \neq 0$ and there is no minimal loop at either end, then we can regularize.

(4) If $d(a) \neq 0$ and there is a minimal loop at one end, say loop b at the head. Then we have

$$d(a) = f_1(b)\alpha_1 + \cdots + f_k(b)\alpha_k$$

for distinct dotted arrows α_j and non-zero polynomials $f_j(x)$. The representations with $f_1(b)$ not invertible come from unravelling the loop b with the polynomial $f_1(x)$. The representations with $f_1(b)$ invertible come from the dbq with the polynomial $g(x)f_1(x)$ for b, which is obtained by localization. But for this dbq, since $f_1(b)$ is invertible, we can make a substitution, replacing α_1 by

$$\alpha_1' = f_1(b)\alpha_1 + \cdots + f_k(b)\alpha_k,$$

so that $d(a) = \alpha_1'$, and then regularize a.

(5) If $d(a) \neq 0$ and there minimal loops b and c at the start and end of a (possibly $b = c$ if a is a loop). Let the polynomials for b and c be $f(x)$ and $g(x)$.

We can write

$$d(a) = h_1(b,c)\alpha_1 + \cdots + h_k(b,c)\alpha_k$$

for distinct dotted arrows α_j and polynomials $h_j(x,y) \in K[x,y,f(x)^{-1},g(y)^{-1}]$. Replacing α_j with $\alpha'_j = g(c)^{-N}\alpha_j f(b)^{-N}$ for N sufficiently large, we may assume that $h_j(x,y) \in K[x,y]$ for all j.

Using that $K(x)[y]$ is a principal ideal domain, there is a localization $\tilde{R} = K[x,y,\phi(x)^{-1}]$ such that the ideal in $\tilde{R}$ generated by $h_1,\dots,h_k$ is principal, so of the form (h_0) with $h_0 \in K[x,y]$. Then h_0 divides the h_j, and the ideal generated by the quotients h_j/h_0 is $\tilde{R}$.

We use loop unravelling to get the representations in which $\phi(b)$ is not invertible. For the remaining representations we can localize, so that the minimal loop b has associated polynomial $f(x)\phi(x)$. This ensures that $\phi(b)$ is invertible.

By a theorem of Seshadri, f.g. projective modules for a polynomial ring over a principal ideal domain are free, see C. S. Seshadri, Triviality of vector bundles over the affine space K^2, Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 456-458.

This applies to $\tilde{R}$. It follows that it is a Hermite ring, which for a commutative ring means that any tuple of elements which generates the whole ring as an ideal can occur as the first row of an invertible matrix.

Thus there is an invertible $k\times k$ matrix over $\tilde{R}$ with first row $(h_1/h_0,\dots,h_k/h_0)$. Making a substitution among the α_j using this matrix, we may suppose that

$$d(a) = h_0(b,c)\alpha_1.$$

Note that because of this substitution we would not be able to preserve triangularity of the dotted arrows. Now because of the wild configurations, $h_0(x,y)$ must be invertible in the ring $K[x,y,(f(x)\phi(x))^{-1},g(y)^{-1}]$. Thus after another substitution, we can regularize.