Fakultät für Mathematik
Universität Bielefeld

William Crawley-Boevey
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Masters course: Representations of Algebras

I plan to discuss the representation theory of algebras and quivers, including Auslander-Reiten theory, correspondences given by faithfully balanced modules, homological conjectures, representations of Dynkin and extended Dynkin quivers, tame and wild algebras, etc.

Students are expected to already have some familiarity with rings and modules, and topics such as categories, projective and injective modules, and Ext groups.

Here are some relevant books. The book by Erdmann and Holm is a good introduction, aimed at bachelor students. The book by Assem, Simson and Skowronski is a comprehensive introduction.

- I. Assem and F. U. Coelho, Basic representation theory of algebras, Springer 2020.
- I. Assem, D. Simson and A. Skowroński, Elements of the representation theory of associative algebras. Volume 1, Techniques of representation theory, CUP 2006.
- M. Auslander, I. Reiten and S. O. Smalø, Representation theory of Artin algebras, CUP 1997.
- M. Barot, Introduction to the Representation Theory of Algebras, Springer 2015.
- H. Derksen and J. Weyman, An introduction to quiver representations, American Mathematical Society 2017.
- K. Erdmann and T. Holm, Algebras and Representation Theory, Springer 2018.
- P. Etingof et al., Introduction to representation theory, American Mathematical Society 2011.
- P. Gabriel and A. V. Roiter, Representations of finite dimensional algebras, Springer 1977.

- R. Schiffler, Quiver Representations, Springer 2014.
- A. Skowroński and K. Yamagata, Frobenius algebras 1. Basic representation theory, European Mathematical Society 2011.
- A. Skowroński and K. Yamagata, Frobenius algebras 2. Tilted and Hochschild extension algebras, European Mathematical Society 2017.

Contents

1 Algebras, quivers and representations

1.1 Algebras

Definition. Let K be a commutative ring. By an *algebra* over K or K-*algebra* we mean a K-module R which is also a ring, such that the multiplication

$$R \times R \to R$$

is K-bilinear. Rings and algebras always have a one, denoted 1 or 1_R.

A *homomorphism of algebras* $\theta : R \to S$ is a K-module homomorphism which is also a ring homomorphism. In particular, $\theta(1_R) = 1_S$.

A *subalgebra* S of an algebra R is a K-submodule which is also a subring. In particular, $1_R \in S$.

Remarks. (1) Any ring is a $\mathbb{Z}$-algebra in a unique way.

(2) To specify a K-algebra, it is equivalent to give a ring R and a ring homomorphism $K \to Z(R)$, where $Z(R)$ is the centre of R.

(3) If R is a K-algebra, then any left R-module M becomes a K-module by restriction, that is, $\lambda m = (\lambda 1_R)m$ for $\lambda \in K$ and $m \in M$.

(4) If M is a K-module, then $\mathrm{End}_K(M)$ is a K-algebra in the natural way. A *representation* of an algebra R is given by a K-module M and a K-algebra homomorphism

$$\theta : R \to \mathrm{End}_K(M).$$

Using the formula

$$\theta(r)(m) = rm$$

we see that a representation of R is exactly the same thing as a left R-module.

(5) The category R-Mod of left R-modules is naturally a K-category, that is, the spaces $\mathrm{Hom}_R(X, Y)$ are naturally K-modules, and composition is K-bilinear.

Remark (Conventions)**.** Because this course is mainly about representations of finite-dimensional algebras over a field, from now on I shall assume that K is a field, unless stated otherwise. But many definitions work for K an arbitrary ring.

I shall not yet assume that all algebra are finite-dimensional. If R is a K-algebra, I write R-mod for the category of finite-dimensional R-module. Warning: this is not the same as the category of finitely generated R-modules, unless R is finite-dimensional.

Remark (Semisimplicity)**.** Recall that a module M is *semisimple* if it satisfies the following equivalent conditions.

(i) M is the sum of its simple submodules,

(ii) M is isomorphic to a direct sum of simple modules,

(iii) every submodule of M is a direct summand, that is, for every submodule N of M there is a submodule C with $N \oplus C = M$.

It follows that any submodule or quotient of a semisimple module is semisimple, and any direct sum of a family of semisimple modules is semisimple.

A ring R is *semisimple* if R is a semisimple R-module. It follows that every module is semisimple. According to the Artin-Wedderburn Theorem, it is equivalent that

$$R \cong M_{r_1}(D_1) \times \cdots \times M_{r_n}(D_n)$$

with the D_i division rings (i.e. all nonzero elements are invertible).

Many natural f.d. algebras are semisimple, but once one has determined the simple modules, the representation theory of such algebras is trivial, and so we are mainly interested in non-semisimple algebras.

Examples (For motivation, without proofs)**.** (1) The f.d. division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and the quaternions $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$.

(2) If G is a group then the group algebra is

$$KG = \{\sum_{g \in G} a_g g : a_g \in K, \text{ all but finitely many zero}\}.$$

Representations of KG correspond to representations of the group

$$\rho : G \to \mathrm{GL}(V).$$

Maschke's theorem: if G is finite and its order is invertible in K, then KG is semisimple.

(3) The polynomial ring $K[x_1, \dots, x_n]$. If K is algebraically closed, f.d. $K[x]$-modules are classified by Jordan normal form.

(4) The free algebra $K\langle x_1, \dots, x_n\rangle$. It has basis the words in the x_i. For example $K\langle x, y\rangle$ has basis

$$1, x, y, x^2, xy, yx, y^2, x^3, x^2y, xyx, xy^2, yx^2, yxy, \dots$$

A f.d. $K\langle x, y\rangle$-module with vector space K^n is given by two $n \times n$ matrices X, Y, and a homomorphism $(K^n, X, Y) \to (K^m, X', Y')$ is given by an $m \times n$ matrix A with $AX = X'A$ and $AY = Y'A$, so isomorphism is given by simultaneous conjugacy.

This is the basic *wild* problem. The 1-dimensional representations are given by a pair of elements of K. One can classify 2-dimensional representations, and with enough work also n-dimensional representations for small n, but there is no classification known, or expected, which works for all n.

(5) Let V be a vector space. The *tensor powers* are

$$T^n(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n},$$

where tensor products are over K and $T^0(V) = K$. The *tensor algebra* is the graded algebra

$$T(V) = \bigoplus_{n \in N} T^n(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

with the multiplication given by $T^n(V) \otimes_K T^m(V) \cong T^{n+m}(V)$. If V is has basis $x_1, \dots, x_n$, then $T(V) \cong K\langle x_1, \dots, x_n \rangle$.

(6) The *exterior algebra*

$$\Lambda(V) = T(V)/(v^2 : v \in V).$$

If V has basis $x_1, \dots, x_n$ then in $\Lambda(V)$ we have

$$0 = (x_i + x_j)^2 = x_i^2 + x_i x_j + x_j x_i + x_j^2 = x_i x_j + x_j x_i$$

and in fact

$$\Lambda(V) \cong K\langle x_1, \dots, x_n \rangle / (x_i^2, x_i x_j + x_j x_i).$$

More generally, suppose that $q : V \to K$ is a *quadratic form*, meaning that

(a) $q(\lambda x) = \lambda^2 q(x)$ for $\lambda \in K$ and $x \in V$, and

(b) the map $V \times V \to K$, $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is a bilinear form in x and y.

The associated *Clifford algebra* is

$$C(V, q) = T(V)/(v^2 - q(v)1 : v \in V).$$

Now suppose that V has basis $x_1, \dots, x_n$ and there are $c_i \in K$ with

$$q(\lambda_1 x_1 + \cdots + \lambda_n x_n) = c_1 \lambda_1^2 + \cdots + c_n \lambda_n^2$$

for $\lambda_1, \dots, \lambda_n \in K$, then for $i \neq j$ we have

$$c_i + c_j = q(x_i + x_j) = (x_i + x_j)^2 = x_i^2 + x_i x_j + x_j x_i + x_j^2 = q(x_i) + q(x_j) = c_i + x_i x_j + x_j x_i + c_j$$

and in fact

$$C(V, q) \cong K\langle x_1, \dots, x_n \rangle / (x_i^2 - c_i, x_i x_j + x_j x_i).$$

One can show that $\Lambda(V)$ and $C(V,q)$ have basis the products $x_{i_1}\dots x_{i_r}$ with $i_1 < \cdots < i_r$.

For example the algebra of 3-d Euclidean space is given by $V = \mathbb{R}^3$ with

$$q(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

The Clifford algebra has basis

$$1, x_1, x_2, x_3, i = x_1x_2, j = x_2x_3, k = x_1x_3, \ell = i_1x_2x_3.$$

Then $i^2 = x_1x_2x_1x_2 = -x_1^2x_2^2 = -1$ and $ij = x_1x_2x_2x_3 = k$, etc, so $1, i, j, k$ span a subalgebra isomorphic to $\mathbb{H}$. Also $\ell^2 = -1$, so $1, \ell$ span a copy of $\mathbb{C}$.

If $\operatorname{char} K \neq 2$ and the bilinear form associated to q is non-degenerate, then $C(V,q)$ semisimple. In physics *spinors* are important—they are elements of a representation of a Clifford algebra.

(7) If G is a Lie group, one is usually interested in the representations

$$\rho : G \to \mathrm{GL}_N(\mathbb{C})$$

which are continuous or smooth. As an algebraic version, one can take $G = \mathrm{GL}_n(K)$ and then one is interesed in the representations

$$\rho : \mathrm{GL}_n(K) \to \mathrm{GL}_N(K)$$

such that each entry of $\rho(g)$ is a rational function of the components of g. For example the natural representaion of $\mathrm{GL}_2(K)$ is

$$\mathrm{GL}_2(K) \to \mathrm{GL}_2(K), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the determinant representation is

$$\mathrm{GL}_2(K) \to \mathrm{GL}_1(K), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc,$$

and the dual of the natural representation is

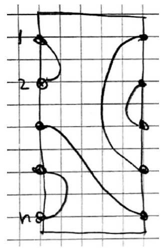
$$\mathrm{GL}_2(K) \to \mathrm{GL}_2(K), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (A^{-1})^T = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

To study such representations, it suffices to understand the representations in which all entries are homogeneous polynomials of fixed degree r. Such representations correspond to representations of a f.d. algebra $S(n,r)$ called the *Schur algebra.* In fact the symmetric group S_r acts on $T^r(V)$ where letting $V = K^n$ by permuting the terms in a tensor, and $S(n,r)$ can be defined as

$$S(n,r) := \mathrm{End}_{KS_r}(T^r(V)).$$

For K of characteristic zero it is a semisimple algebra, but for K of positive characteristic it need not be. The canonical reference for the Schur algebra is J.A. Green, Polynomial representations of GL_n, second edition, Springer 2007.

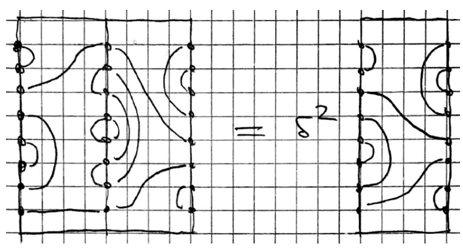
(8) The *Temperley-Lieb algebra* $TL_n(\delta)$ for $n \geq 1$ and $\delta \in K$ was invented to help make computations in Statistical Mechanics. It has basis the diagrams with two vertical rows of n dots, connected by n nonintersecting curves. For example



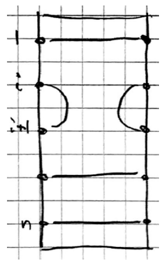
Two diagrams are considered equal if the same vertices are connected. The product is defined by

$$ab = \delta^r c$$

where c is obtained by concatenating a and b and deleting any loops, and r is the number of loops removed. For example



The algebra $TL_n(\delta)$ is f.d., with dimension the nth Catalan number. Let u_i be the diagram



Then $u_i^2 = \delta u_i$, $u_i u_{i\pm1} u_i = u_i$ and $u_i u_j = u_j u_i$ if $|i-j| > 1$

One can show that

$$TL_n(\delta) \cong K\langle u_1, \ldots, u_{n-1}\rangle / I$$

where I is generated by these relations. For generic δ, $TL_n(\delta)$ is semisimple, but for some δ it is not.

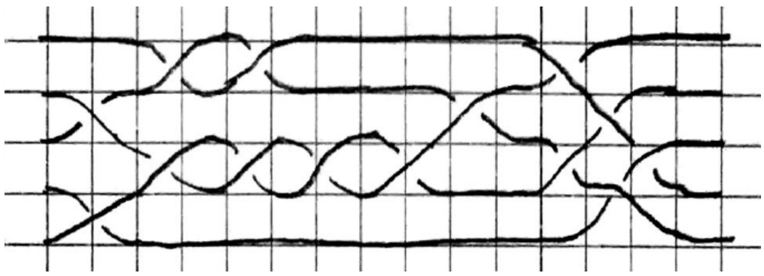
The Temperley-Lieb algebra is also important in Knot Theory.

The *Markov trace* is the linear map $\mathrm{tr} : TL_n(\delta) \to K$ sending a diagram to δ^{r-n} where r is the number of cycles in the diagram obtained by joining vertices at opposite ends.

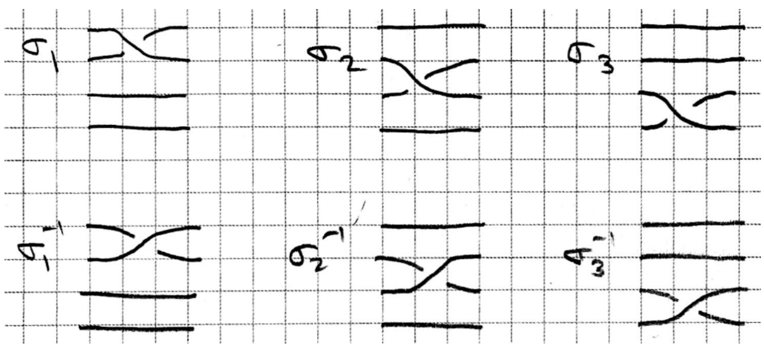
The *(Artin) braid group* B_n is the group generated by $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j| > 1), \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

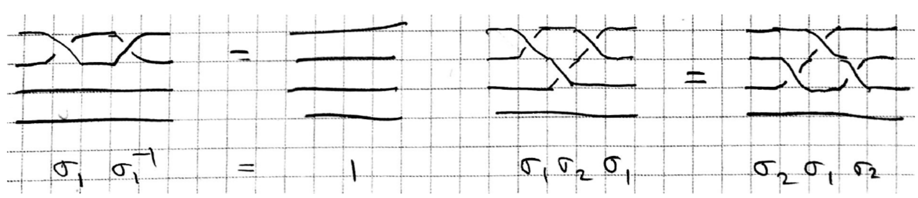
One can show that the elements of B_n can be identified with braids



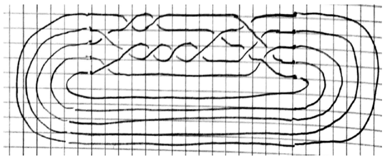
identifying two such braids if they are *isotopic*. The generators correspond to the braids



and the relations are as follows



By joining the ends of a braid, one gets a knot (or a link if it is not connected), for example



Moreover any knot arises from some braid (for some n).

Given $0 \neq A \in K$, there is a homomorphism $\theta : KB_n \to TL_n(\delta)$ where $\delta = -A^2 - 1/A^2$, with $\theta(\sigma_i) = Au_i + (1/A)$, $\theta(\sigma_i^{-1}) = (1/A)u_i + A$. Composing with the Markov trace, this gives a map $\operatorname{tr}\theta : KB_n \to K$. One can show that the image of an element of B_n only depends on the knot obtained by joining the ends of the braid, and it is a Laurent polynomial in A. It is essentially the Jones polynomial of the knot, see Lemma 2.18 in D. Aharonov, V. Jones and Z. Landau, A polynomial quantum algorithm for approximating the Jones polynomial, Algorithmica 2009.

(9) Suppose that G is a group, R is an algebra, and we have an action

$$G \times R \to R, \quad (g, r) \mapsto {}^g r$$

of G on R by algebra automorphisms. To be an action means that

$${}^g({}^h r) = {}^{(gh)} r, \quad {}^1 r = r,$$

and we want also that for all $g \in G$ the map $R \to R$, $r \mapsto {}^g r$ is an algebra homomorphism (necessarily an automorphism).

One can form the algebra of invariants

$$R^G = \{r \in R : {}^g r = r \text{ for all } g \in G\}.$$

We can also form the skew group algebra

$$R * G = \{\sum_{g \in G} a_g * g : a_g \in R, \text{ all but finitely many zero}\}$$

with the multiplication given by

$$(a * g)(b * h) = (a\, {}^g b) * (gh).$$

1.2 Idempotents and catalgebras

Definition. Let R be a ring.

(a) An element $e \in R$ is *idempotent* if $e^2 = e$.

(b) Idempotents $e_1, \dots, e_n$ are *orthogonal* if $e_i e_j = 0$ for $i \neq j$.

(c) A family of orthogonal idempotents $e_1, \dots, e_n$ is *complete* if $e_1 + \cdots + e_n = 1_R$.

Lemma. *Let R be a ring and M an R-module.*

(a) If $e \in R$ is an idempotent, then

$$eM = \{m \in M : em = m\},$$

and if R is a K-algebra, then eM is a K-subspace of M.

(b) If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents, then

$$M = e_1 M \oplus \cdots \oplus e_n M.$$

Proof. Straightforward. □

Proposition (Peirce decomposition)**.** *If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents in R, then*

$$R = \bigoplus_{i,j=1}^{n} e_i R e_j.$$

Displaying this as a matrix

$$R = \begin{pmatrix} e_1Re_1 & e_1Re_2 & \dots & e_1Re_n \\ e_2Re_1 & e_2Re_2 & \dots & e_2Re_n \\ \dots & & & \\ e_nRe_1 & e_nRe_2 & \dots & e_nRe_n \end{pmatrix},$$

multiplication in R corresponds to matrix multiplication.

Proof. Straightforward. □

Definition. Recall that an R-module P is *projective* if it satisfies the following equivalent conditions.

(i) $\mathrm{Hom}(P, -)$ is an exact functor $R\text{-Mod} \to \mathrm{Ab}$.

(ii) Any short exact sequence $0 \to X \to Y \to P \to 0$ is split.

(iii) Given an epimorphism $\theta : Y \twoheadrightarrow Z$, any morphism $P \to Z$ factors through θ.

(iv) P is a direct summand of a free R-module.

Lemma. *(i) If e is idempotent in R, then Re is a left ideal which is a direct summand of R, so a projective left R-module, and if M is an R-module, then* $\mathrm{Hom}_R(Re, M) \cong eM$.

(ii) Any left ideal of R which is a direct summand of R is equal to Re for some idempotent e.

Proof. (i) Send θ to $\theta(e)$ or $m \in eM$ to the map $r \mapsto re$,

(ii) If I is a direct summand, then the projection onto it is an idempotent element of $\mathrm{End}_R(R) \cong R^{op}$. □

Sometimes it is useful to consider non-unital rings and algebras, but usually one wants some weaker form of unital condition, and there are many possibilities. One possibility is rings "with enough idempotents". In categorical language, this is the theory of "rings with several objects". I call the algebra version "catalgebras", since they correspond exactly to small K-categories.

Definition. By a *catalgebra* we mean a K-vector space R with a multiplication $R \times R \to R$ which is associative and K-bilinear, such that there exists a (possibly infinite) family $(e_i)_{i\in I}$ of orthogonal idempotents which is *complete* in the sense that

$$R = \bigoplus_{i,j\in I} e_i R e_j.$$

If R is a catalgebra, then an R-module M is given by an additive group M and an action

$$R \times M \to M, (r, m) \mapsto rm$$

which is distributive over addition, satisfies $(rr')m = r(r'm)$ and is unital in the sense that

$$M = \bigoplus_{i\in I} e_i M.$$

This last condition doesn't depend on the choice of the idempotents, since it is equivalent that $RM = M$. For example if $m \in M$ then $RM = M$ implies $m = \sum_{s=1}^{t} r_s m_s$. Now each $r_s = \sum_{i\in I} e_i r_{si}$. Thus $m = \sum_i e_i(\sum_s r_{si} m_s) \in \sum_{i\in I} e_i M$.

Observe that R is itself an R-module, but not in general finitely generated! Also any subgroup L of M which is closed under the action is itself a module, for if $x \in L$ then $x = \sum_{i\in I} e_i x \in RL$.

Examples. (1) Any algebra is a catalgebra with 1_R being a complete family of orthogonal idempotents. Conversely, a catalgebra with a finite complete family of orthogonal idempotents $e_1, \dots, e_n$ is an algebra with $1_R = e_1 + \dots + e_n$.

(2) The Temperley-Lieb algebras $TL_n(\delta)$ sit inside a catalgebra, with K-basis given by the diagrams with a possibly different number of dots on each side, with the composition of two diagrams being zero if they do not have a compatible number of dots.

(3) There is a 1:1 correspondence

small K-categories $\mathcal{C} \leftrightarrow$ catalgebras R equipped with with a complete family of orthogonal idempotents $(e_i)_{i\in I}$

given as follows. Given $\mathcal{C}$ we set

$$R = \bigoplus_{X,Y\in \mathrm{ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

with multiplication given by composition, or zero if two morphisms are not composable. The identity morphisms $(1_X)_{X\in \mathrm{ob}(\mathcal{C})}$ are a complete family of orthogonal idempotents. Conversely if R is a catalgebra and $(e_i)_{i\in I}$ is a complete family of orthogonal idempotents, then one obtains a small category $\mathcal{C}$ with objects $\mathrm{ob}(\mathcal{C}) = I$, morphisms $\mathrm{Hom}(i, j) = e_j R e_i$ and composition given by multiplication. Under this correspondence there is an equivalence

$$R\text{-Mod} \simeq \text{Category of additive functors } \mathcal{C} \to \text{Ab}.$$

P. Gabriel, Des categories abeliennes, Bull. Soc. Math. France 1962, Chapter 2, section 1, prop 2, p347.

(4) Whereas any product of algebras is an algebra, any direct sum of catalgebras

$$\bigoplus_{j\in J} R_j$$

is a catalgebra. If I is a set and R an algebra or catalgebra, then the set $R^{(I\times I)}$ of matrices with entries in R, with rows and columns indexed by I, and only finitely many non-zero entries is a catalgebra under matrix multiplication. The analogue of Artin-Wedderburn for catalgebras is that the semisimple catalgbras are those of the form

$$\bigoplus_{j\in J} D_j^{(I_j\times I_j)}$$

for some sets J, I_j and division algebras D_j.

Remark. If R is a catalgebra, then $R_1 = R \oplus K$ becomes an algebra with multiplication

$$(r, \lambda)(r', \lambda') = (rr' + \lambda r' + \lambda' r, \lambda\lambda').$$

and $1_{R_1} = (0, 1)$. We can identify R as an ideal in R_1, and R-Mod is isomorphic to the category of R_1-modules M satisfying $RM = M$. Moreover, if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of R_1-modules, then $RM = M$ if and only if $RL = L$ and $RN = N$.

1.3 Representations of quivers and path algebras

Recall that K is a field.

Definition. A *quiver* is a quadruple $Q = (Q_0, Q_1, h, t)$ where Q_0 is a set of vertices, Q_1 a set of arrows, and $h, t : Q_1 \to Q_0$ are mappings, specifying the head and tail vertices of each arrow,

$$\overset{t(a)}{\bullet} \xrightarrow{a} \overset{h(a)}{\bullet}.$$

Definition. The category of representations of Q over K is defined as follows.

A *representation* of Q is a tuple $V = (V_i, V_a)$ consisting of a K-vector space V_i for each vertex i and a K-linear map $V_a : V_i \to V_j$ for each arrow $a : i \to j$ in Q. If there is no risk of confusion, we write $a : V_i \to V_j$ instead of V_a.

A *homomorphism* of representations $\theta : V \to W$ is given by K-linear maps $\theta_i : V_i \to W_i$ for each vertex, such that $\theta_j V_a = W_a \theta_i$ for each arrow $a : i \to j$.

The composition of morphisms $\phi : U \to V$ and $\theta : V \to W$ is given by $(\theta\phi)_i = \theta_i \phi_i$.

If V is a finite-dimensional representation, its *dimension vector* is $\underline{\dim} V = (\dim V_i) \in \mathbb{N}^{Q_0}$.

Remark. A homomorphism $\theta : V \to W$ is an isomorphism if and only if θ_i is an isomorphism for each vertex i, for in the latter case, the maps $(\theta_i)^{-1}$ define a morphism $W \to V$ which is inverse to θ.

Example. Let us compute the endomorphisms of the representation V of the quiver with vertices $1, 2, 3, 4$ represented by K, K, K, K^2 and arrows $1 \to 4$, $2 \to 4$, $3 \to 4$ represented by the maps with matrices

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

An endomorphism is given by matrices

$$(a), \quad (b), \quad (c), \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

satisfying

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} (a) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} (b) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} (c) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solving gives that the matrices are

$$(a), \quad (a), \quad (a), \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

so $\mathrm{End}(V) = K$.

Definition. Let Q be a quiver. A *path* in Q of length $n > 0$ in Q is a sequence $p = a_1 a_2 \dots a_n$ of arrows satisfying $t(a_i) = h(a_{i+1})$ for all $1 \le i < n$,

$$\bullet \xleftarrow{a_1} \bullet \xleftarrow{a_2} \bullet \cdots \bullet \xleftarrow{a_n} \bullet.$$

The head and tail of p are $h(a_1)$ and $t(a_n)$. For each vertex $i \in Q_0$ there is also a *trivial path* e_i of length zero with head and tail i.

If Q has only finitely many vertices, the *path algebra* KQ is the free K-module with basis the paths in Q, equipped with the multiplication in which the product of two paths given by $p \cdot q = 0$ if the tail of p is not equal to the head of q, and otherwise $p \cdot q = pq$, the concatenation of p and q. The one for the algebra is

$$1 = \sum_{i \in Q_0} e_i.$$

More generally, if Q has infinitely many vertices, KQ exists and is a catalgebra.

We write $(KQ)_+$ for the ideal spanned by the non-trivial paths, or equivalently the ideal generated by the arrows. Clearly

$$KQ = (KQ)_+ \oplus \bigoplus_{i \in Q_0} Ke_i, \quad KQ/(KQ)_+ \cong \bigoplus_{i \in Q_0} Ke_i \cong K \times \cdots \times K$$

Examples. (i) The path algebra of the quiver $1 \xrightarrow{a} 2$ with loop b at 2 has basis $e_1, e_2, a, b, ba, b^2, b^2a, b^3, b^3a, \ldots$.

(ii) The algebra of lower triangular matrices in $M_n(K)$ is isomorphic to the path algebra of the quiver

$$1 \to 2 \to \cdots \to n$$

with the matrix unit e^{ij} corresponding to the path from j to i, since

$$e^{ij} e^{k\ell} = \begin{cases} e^{u\ell} & (j = k) \\ 0 & (j \neq k). \end{cases}$$

(iii) The free algebra $K\langle x_1, \ldots, x_n \rangle$ is the same as KQ where Q has one vertex and loops $x_1, \ldots, x_n$.

Properties. (i) KQ is finite-dimensional if and only if Q is finite and has no oriented cycles.

(ii) If $0 \neq a \in KQe_i$ and $0 \neq b \in e_iKQ$ then $ab \neq 0$. Namely, look at the longest paths p and q involved in a and b. In the product, the path pq must be involved.

(iii) e_iKQe_i is isomorphic to the free algebra on the set X of paths with head and tail at i, but which don't pass through i.

(iv) Let Q be the oriented cycle with vertices $1, \ldots, n$ and arrows $a_i : i \to i + 1$ for $i < n$ and $a_n : n \to 1$. Let T be the sum of all paths of length n,

$$T = a_n \ldots a_2 a_1 + a_1 a_n \ldots a_2 + a_2 a_1 a_n \ldots a_3 + \cdots + a_{n-1} \ldots a_1 a_n,$$

Then $Z(KQ) = K[T]$.

Proposition. *The category of representations of Q is equivalent to KQ*-Mod.

Proof. If V is a KQ-module, then $V = \bigoplus e_i V$. We get a representation, also denoted V, with $V_i = e_i V$, and, for any arrow $a : i \to j$, the map $V_a : V_i \to V_j$ is given by left multiplication by $a \in e_j KQ e_i$.

Conversely any representation V determines a KQ-module via $V = \bigoplus_{i \in Q_0} V_i$, with the action of KQ given as follows:

- The trivial path e_i acts on V as the projection onto V_i, and
- A nontrivial path $a_1 a_2 \dots a_n$ acts by

$$a_1 a_2 \dots a_n v = V_{a_1}(V_{a_2}(\dots(V_{a_n}(v_{t(a_n)}))\dots)) \in V_{h(a_1)} \subseteq V.$$

It is straightforward to extend these to functors, and then to check that they are inverse equivalences. □

Remark. (1) Under this correspondence, submodules correspond to subrepresentations. A *subrepresentation* W of a representation V is given by a subspace $W_i \subseteq V_i$ for each vertex i such that $V_a(W_i) \subseteq W_j$ for all arrows $a : i \to j$.

(2) The corresponding *quotient representation* V/W is given by the vector spaces V_i/W_i and the induced maps $\overline{V_a} : V_i/W_i \to V_j/W_j$ for $a : i \to j$.

(3) The *direct sum* $V \oplus W$ of two representations is given by the vector spaces $V_i \oplus W_i$ and maps

$$\begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix} : V_i \oplus W_i \to V_j \oplus W_j$$

for an arrow $a : i \to j$. Similarly for direct sums over any indexing set.

(4) A sequence of representations

$$\dots \to V \to V' \to V'' \to \dots$$

is exact if and only if for each vertex i, the sequence of vector spaces

$$\dots \to V_i \to V'_i \to V''_i \to \dots$$

is exact. The kernel, image and cokernel of a morphism can be computed vertex-wise.

Notation. Let i be a vertex.

(a) We write $S[i]$ for the representation with $S[i]_i = K$, $S[i]_j = 0$ for $i \neq j$ and all $S[i]_a = 0$. It is a simple representation, but there can be other simple representations, for example we only get one $K[x]$-module.

(b) We define $P[i] = KQe_i$. It is a projective KQ-module, and $KQ = \bigoplus_{i \in Q_0} P[i]$. Considered as a representation of Q, the vector space at vertex j has basis the paths from i to j. For $i \neq j$ we have $P[i] \ncong P[j]$, since

$$\mathrm{Hom}(P[i], S[j]) = \mathrm{Hom}(KQe_i, S[j]) \cong e_i S[j] \cong \begin{cases} K & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Example. For example for the quiver

$$1 \overset{a}{\underset{b}{\rightrightarrows}} 3 \overset{c}{\underset{d}{\rightrightarrows}} 3,$$

we have

$$P[1] \cong K \overset{a}{\underset{b}{\rightrightarrows}} K^2 \overset{c}{\underset{d}{\rightrightarrows}} K^4,$$

with bases e_1, and a, b and ca, da, cb, db, and linear maps given by $a(e_1) = a, b(e_1) = b, c(a) = ca, c(b) = cb, d(a) = da, d(b) = db$.

Example. Let Q be the quiver $1 \xrightarrow{a} 2$.

(i) $S[1]$ is the representation $K \to 0$, $S[2]$ is the representation $0 \to K$.

$P[1]$ is the representation $K \xrightarrow{1} K$ and $P[2] \cong S[2]$.

(ii) We have $\mathrm{Hom}(S[1], P[1]) = 0$ and $\mathrm{Hom}(S[2], P[1]) \cong K$.

(iii) The subspaces $(K \subseteq V_1, 0 \subseteq V_2)$ do not give a subrepresentation of $V = P[1]$, but the subspaces $(0 \subseteq V_1, K \subseteq V_2)$ do, and this subrepresentation is isomorphic to $S[2]$.

(iv) There is an exact sequence $0 \to S[2] \to P[1] \to S[1] \to 0$.

(v) $S[1] \oplus S[2] \cong K \xrightarrow{0} K$ and for $0 \neq \lambda \in K$ we have $K \xrightarrow{\lambda} K \cong P[1]$.

(vi) Every representation of Q is isomorphic to a direct sum of copies of $S[1]$, $S[2]$ and $P[1]$. Namely, given the representation $V_1 \xrightarrow{a} V_2$, take a basis $(x_i)_{i \in I}$ of $\mathrm{Ker}(V_a)$. Extend it to a basis of V_1 with elements $(y_j)_{j \in J}$. Then the elements $(V_a(y_j))_{j \in J}$ are linearly independent in V_2. Extend them to a basis of V_2 with elements $(z_\ell)_{\ell \in L}$. Then

$$V \cong S[1]^{(I)} \oplus P[1]^{(J)} \oplus S[2]^{(L)}.$$

1.4 Algebras given by quivers with relations

We are interested in algebras of the form KQ/I. For simplicity we take Q to be a finite quiver.

Any algebra R is a quotient of a free algebra $K\langle X \rangle / I$, and if R is finitely generated as an algebra we can take X to be finite. Similarly, if $e_1, \ldots, e_n$ is a complete set of orthogonal idempotents in an algebra R, then we can write

$$R \cong KQ/I$$

for some quiver Q with vertex set $\{1, \ldots, n\}$, in such a way that the e_i correspond to the trivial paths in KQ, and if R is finitely generated we can take Q to be finite.

Definition. By a *relation* for Q we mean an element $a \in e_j KQ e_i$ for some $i, j \in Q_0$, so a K-linear combination of paths in Q which all have the head j and tail i. A representation V of Q *satisfies the relation* a if the corresponding linear map $V_i \to V_j$ is zero. If $a, b \in e_j KQ e_i$, we say that V satisfies the relation $a = b$ if it satisfies the relation $a - b$.

Lemma. *Any ideal I in a path algebra KQ can generated by a set of relations, and then the category of KQ/I-modules is equivalent to the category of representations which satisfy these relations.*

Proof. If I is an ideal and $x \in I$, then $x = \sum_{i,j \in Q_0} e_j x e_i$ and $e_j x e_i \in I$. $\square$

Notation. Let $R = KQ/I$. If i is a vertex, we define $P[i] = Re_i$, so it is a projective R-module and

$$R = \bigoplus_{i \in Q_0} P[i].$$

In case $I = 0$ we already used this notation, but note that $P[i]$ depends in I. Considered as a representation of Q, the vector space $P[i]_j = e_j(KQ/I)e_i$, so it has basis given by the paths from i to j modulo I.

Recall that $(KQ)_+$ is the ideal in KQ spanned by the non-trivial paths. Clearly $(KQ)_+^n$ is the ideal spanned by paths of length $\geq n$, and $KQ/(KQ)_+ \cong K \times \cdots \times K$.

Definition. An ideal $I \subseteq KQ$ is *admissible* if
(1) $I \subseteq (KQ)_+^2$, and
(2) $(KQ)_+^n \subseteq I$ for some n.

Lemma. *Suppose I is admissible. Then*
(i) $R = KQ/I$ is finite-dimensional
(ii) The KQ-modules $S[i]$ are annihilated by I, so become simple R-modules.
(iii) The $S[i]$ are the only simple R-modules up to isomorphism.
(iv) The modules $P[i]$ are pairwise non-isomorphic.

Proof. (i) By (2), R is spanned by the paths of length $< n$.
(ii) This just needs $I \subseteq (KQ)_+$, which is weaker than (1).
(iii) Let S be a simple R-module, and consider it as a KQ-module. Now $(KQ)_+ S$ is a submodule of S, so by simplicity it is equal to 0 or S. But $IS = 0$, so $(KQ)_+^n S = 0$, so we must have $(KQ)_+ S = 0$. Thus S is a module for

$$KQ/(KQ)_+ \cong K \times \cdots \times K$$

so it is isomorphic to an $S[i]$.
(iv) $\mathrm{Hom}(P[i], S[j]) \cong \mathrm{Hom}(Re_i, S[j]) \cong e_i S[j]$, which is K if $i = j$, else 0. $\square$

Examples. (1) A finite complex of K-vector spaces is a representation of the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \ldots \xrightarrow{a_{n-1}} n$$

satisfying the admissible relations $a_{i+1}a_i = 0$ for $1 \leq i < n-1$.

For $n = 4$ the representations $P[i]$ are

$$P[1] = K \to K \to 0 \to 0, \quad P[2] = 0 \to K \to K \to 0,$$

$$P[3] = 0 \to 0 \to K \to K, \quad P[4] = 0 \to 0 \to 0 \to 0.$$

(2) A commutative square of K-vector spaces is a representation of the quiver

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle b}\big\downarrow & & {\scriptstyle c}\big\downarrow \\ 3 & \xrightarrow{d} & 4 \end{array}$$

satisfying the admissible relation $db = ca$. The projective $P[1]$ is

$$\begin{array}{ccc} K & \xrightarrow{1} & K \\ {\scriptstyle 1}\big\downarrow & & {\scriptstyle 1}\big\downarrow \\ K & \xrightarrow{1} & K. \end{array}$$

(3) A cyclically oriented square

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle d}\big\uparrow & & \big\downarrow{\scriptstyle b} \\ 4 & \xleftarrow[c]{} & 3 \end{array}$$

with admissible relations cba and dc, has

$$P[1] = \begin{array}{ccc} K & \xrightarrow{1} & K \\ \big\uparrow & & \big\downarrow{\scriptstyle 1} \\ 0 & \longleftarrow & K \end{array} \quad P[2] = \begin{array}{ccc} 0 & \longrightarrow & K \\ \big\uparrow & & \big\downarrow{\scriptstyle 1} \\ K & \xleftarrow[1]{} & K \end{array} \quad P[3] = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \big\uparrow & & \big\downarrow \\ K & \xleftarrow[1]{} & K \end{array} \quad P[4] = \begin{array}{ccc} K & \xrightarrow{1} & K \\ {\scriptstyle 1}\big\uparrow & & \big\downarrow{\scriptstyle 1} \\ K & \xleftarrow[0]{} & K \end{array}$$

For example in $P[4]$ the arrow c sends the basis element bad in the vector space at vertex 3 to $cbad = 0$, and not to e_4, which is the basis element of the vector space at vertex 4.

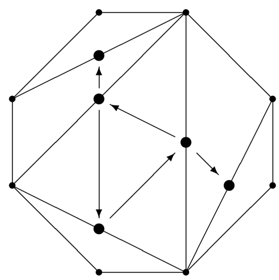
(4) [I. M. Gelfand and V. A. Ponomarev, Indecomposable representations of the Lorentz group, Russian Math. Surv. 1968.] To classify certain infinite-dimensional

representations, called Harish-Chandra representations of the (Lie algebra of the) group $\mathrm{SL}_2(\mathbb{C})$, they reduce the problem to linear algebra, and it corresponds to f.d. representations of the quiver

$$1 \underset{c}{\overset{a}{\rightleftarrows}} 2 \text{ loop } b$$

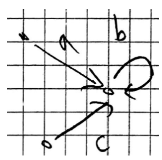
with relations $ba = 0$, $cb = 0$ and b and ac nilpotent. To write this as admissible relations we should impose $b^n = 0$ and $(ac)^n = 0$ for some large n.

(5) [I. Assem, T. Brustle, G. Charbonneau-Jodoin and P.-G. Plamondon, Gentle algebras arising from surface triangulations, Algebra Number Theory 2010]. A triangulation of an oriented surface with marked points on its boundary gives a quiver with relations. For example (taken from the paper)

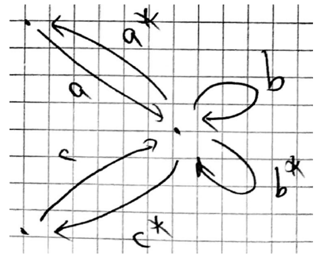


There is one vertex on each internal arc. Arrows go clockwise around the marked points. The relations are the length two paths in an internal triangle. This is related to Fukaya categories in symplectic geometry.

Example. The *double* $\overline{Q}$ of a quiver Q is obtained by adjoining an reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. For example if Q is the quiver



then $\overline{Q}$ is the quiver



The *preprojective algebra* for a finite quiver Q is

$$\Pi(Q) = K\overline{Q}/(\sum_{a\in Q}(aa^* - a^*a))$$

This ideal is not necessarily admissible. For example if Q is a loop x, then $\Pi(Q) = K\langle x, x^*\rangle/(xx^* - x^*x) \cong K[x, x^*]$.

Note that up to isomorphism, $\Pi(Q)$ does not depend on the orientation of Q, for if Q' is obtained from Q by replacing a by a reverse arrow a', then there is an isomorphism $\Pi(Q) \to \Pi(Q')$ sending a to $(a')^*$, a^* to $-a'$ and fixing all other arrows.

Observe that if $r = \sum_{a\in Q}(aa^* - a^*a)$ then $e_i r e_j = 0$ if $i \neq j$, so $\Pi(Q)$ is given by the relations

$$r_i = e_i r e_i = \sum_{a\in Q, h(a)=i} aa^* - \sum_{a\in Q, t(a)=i} a^*a$$

for $i \in Q_0$. For example if $Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ the relations are

$$a^*a = 0,\ aa^* = b^*b,\ bb^* = 0.$$

Later we will be able to determine the quivers Q whose preprojective algebra is finite dimensional. The preprojective algebra is useful for studying sums of matrices. This is illustrated by the following. See A. Mellit, Kleinian singularities and algebras generated by elements that have given spectra and satisfy a scalar sum relation, Algebra Discrete Math. 2004.

Theorem. *Given* $k, d_1, \dots, d_k > 0$*, we have*

$$K\langle x_1, \dots, x_k\rangle/(x_1 + \dots + x_k, x_1^{d_1}, \dots, x_k^{d_k}) \cong e_0\Pi(Q)e_0$$

where Q *is star-shaped with central vertex* 0 *and arms*

$$0 \xleftarrow{a_{i,1}} (i,1) \xleftarrow{a_{i,2}} \dots \xleftarrow{a_{i,d_i-1}} (i, d_i - 1)$$

for i=1,...,k.

Proof. Let the algebra on the left be A and the one on the right be $B = e_0\Pi(Q)e_0$. Now B is spanned by the paths in $\overline{Q}$ which start and end at vertex 0. If vertex (i,j) is the furthest out that a path reaches on arm i, then it must involve $a_{ij}a_{ij}^*$, and if $j > 1$, the relation

$$a_{ij}a_{ij}^* = a_{i,j-1}^* a_{i,j-1}$$

shows that this path is equal in B to a linear combination of paths which only reach $(i, j-1)$. Repeating, we see that B is spanned by paths which only reach out to vertices $(i,1)$. Thus we get a surjective map

$$K\langle x_1, \dots x_k\rangle \to B$$

sending each x_i to $a_{i1}a_{i1}^*$. It descends to a surjective map $\theta : A \to B$ since it sends $x_1 + \cdots + x_k$ to 0 and $x_i^{d_i}$ is sent to

$$\begin{aligned}
(a_{i1}a_{i1}^*)^{d_i} &= a_{i1}(a_{i1}^*a_{i1})^{d_i-1}a_{i1} \\
&= a_{i1}(a_{i2}a_{i2}^*)^{d_i-1}a_{i1}^* \\
&= a_{i1}a_{i2}(a_{i2}^*a_{i2})^{d_i-2}a_{i2}^*a^*i1 \\
&= \cdots = \\
&= a_{i1}a_{i2}\dots a_{i,d_i-1}(a_{i,d_i-1}^*a_{i,d_i-1})a_{i,d_i-1}^*\dots a_{i1}^* = 0
\end{aligned}$$

since $a_{i,d_i-1}^*a_{i,d_i-1} = 0$.

To show that θ is an isomorphism it suffices to show that any A-module can be obtained by restriction from a B-module, for if $a \in \operatorname{Ker}\theta$ and $M = {}_\theta N$, then $aM = \theta(a)N = 0$. Thus if A can be obtained from a B-module by restriction, then $aA = 0$, so $a = 0$.

Thus take an A-module M. We construct a representation of $\overline{Q}$ by defining $V_0 = M$ and $V_{(i,j)} = x_i^j M$ with a_{ij} the inclusion map, and a_{ij}^* multiplication by x_i. This is easily seen to satisfy the preprojective relations, so it becomes a module for $\Pi(Q)$. Then $e_0V = M$ becomes a module for $e_0\Pi(Q)e_0 = B$. Clearly its restriction via θ is the original A-module M. □

The "Diamond Lemma" is due to Max Newman—see the exposition in P. M. Cohn, Further Algebra. There is a version for rings by G. M. Bergman, The diamond lemma for ring theory, Advances in Mathematics 1978. We formulate it for quivers with relations. (For further discussion, see D. Farkas, C. Feustel and E. Green, Synergy in the theories of Gröbner bases and path algebras, Canad. J. Math. 1993.)

Definition. We consider the following setup. Let $R = KQ/(S)$ for a quiver Q and a set S of relations. We fix a well-ordering on the set of paths, such that if w, w' have the same head and tail and $w < w'$, then $uwv < uw'v$ for all compatible products of paths. This can be done by choosing a total ordering on the vertices $1 < 2 < \cdots < n$ and on the arrows $a < b < \dots$ and using the *length-lexicographic* ordering on paths, so $w < w'$ if
- length $w <$ length w', or
- $w = e_i$ and $w' = e_j$ with $i < j$, or
- length $w =$ length $w' > 0$ and w comes before w' in the dictionary ordering.

We write the relations in S in the form

$$w_j = s_j \quad (j \in J)$$

where each w_j is a path and s_j is a linear combination of smaller paths with the same head and tail as w_j.

(i) Given a relation $w_j = s_j$ and paths u, v such that uw_jv is a path, the associated *reduction* is the linear map $KQ \to KQ$ sending uw_jv to us_jv and any other path to itself. We write $f \rightsquigarrow g$ to indicate that g is obtained from f by applying reduction with respect to some $w_j = s_j$ and u, v. Clearly $f - g \in (S)$.

(ii) We say that $f \in KQ$ is *irreducible* if $f \rightsquigarrow g$ implies $g = f$. It is equivalent that no path involved in f can be written as a product uw_jv.

(iii) We say that f is *reduction-unique* if there is a unique irreducible element which can be obtained from f by a sequence of reductions. If so, the irreducible element is denoted $r(f)$.

(iv) We say that two reductions of f, say $f \rightsquigarrow g$ and $f \rightsquigarrow h$, satisfy the *diamond condition* if there exist sequences of reductions starting with g and h, which lead to the same element, $g \rightsquigarrow \cdots \rightsquigarrow k$, $h \rightsquigarrow \cdots \rightsquigarrow k$. (You can draw this as a diamond.)

In particular we are interested in this in the following two cases:

An *overlap ambiguity* is a path w which can be written as w_iv and also as uw_j for some i, j and some non-trivial paths u, v, so that w_i and w_j overlap. There are reductions $w \rightsquigarrow s_iv$ and $w \rightsquigarrow us_j$.

An *inclusion ambiguity* is a path w which can be written as w_i and as uw_jv for some $i \neq j$ and some u, v. There are reductions $w \rightsquigarrow s_i$ and $w \rightsquigarrow us_jw$.

Lemma (Diamond Lemma). *$R = KQ/(S)$ is spanned by the irreducible paths, and the following conditions are equivalent:*

(a) The diamond condition holds for all overlap and inclusion ambiguities.

(b) Every element of KQ is reduction-unique.

(c) The irreducible paths give a basis of R.

In this case the algebra R has multiplication given by $\overline{f}.\overline{g} = \overline{r(fg)}$.

Example. Consider the algebra $R = K\langle x, y\rangle/(S)$ where S is given by

$$x^2 = x, \quad y^2 = 1, \quad yx = y - xy$$

and the alphabet ordering $x < y$. The ambiguities are:

$$\underline{x\overline{x}}\overline{x} \quad \underline{y\overline{y}}\overline{y} \quad \underline{y\overline{y}}\overline{x} \quad \underline{y\overline{x}}\overline{x}.$$

The diamond condition holds since
$\underline{xx}x \rightsquigarrow xx \rightsquigarrow x$ and $x\overline{xx} \rightsquigarrow xx \rightsquigarrow x$.
$\underline{yy}y \rightsquigarrow 1y = y$ and $y\overline{yy} \rightsquigarrow y1 = y$.
$\underline{yy}x \rightsquigarrow 1x = x$ and $y\overline{yx} \rightsquigarrow y(y - xy) = y^2 - yxy = y^2 - (yx)y \rightsquigarrow y^2 - (y - xy)y = xyy = x(yy) \rightsquigarrow x1 = x$.
$\underline{yx}x \rightsquigarrow (y-xy)x = yx-xyx \rightsquigarrow yx-x(y-xy) = yx-xy+xxy \rightsquigarrow yx-xy+xy = yx$ and $y\overline{xx} \rightsquigarrow yx$.

Thus the irreducible paths $1, x, y, xy$ induce a basis of R.

On the other hand, if the relations were

$$x^2 = x, \quad y^2 = 1, \quad yx = 1 - xy$$

Then yxx would not be reduction unique, since

$$(yx)x \rightsquigarrow (1 - xy)x = x - x(yx) \rightsquigarrow x - x(1 - xy) = x^2 y \rightsquigarrow xy$$

and

$$y(xx) \rightsquigarrow yx \rightsquigarrow 1 - xy.$$

Example. The preprojective algebra for the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

with $1 < 2 < 3$ and $a < b < a^* < b^*$. The relations are

$$a^* a = 0, \ b^* b = aa^*, \ bb^* = 0.$$

We have ambiguities

$$\underline{b^* b}\overline{b b^*} \quad \underline{b\overline{b^*}}\overline{b^* b}$$

but the diamond condition fails, since $b^* bb^*$ reduces to 0 or $aa^* b^*$ and $bb^* b$ reduces to 0 or baa^*. But we can add the relations

$$aa^* b^* = 0, \ baa^* = 0$$

and then the diamond condition holds, for example

$$b^*(baa^*) \rightsquigarrow b0 = 0, \quad (b^* b)aa^* \rightsquigarrow (aa^*)aa^* = a(a^* a)a^* \rightsquigarrow a0a^* = 0.$$

Thus the preprojective algebra has basis given induced by the irreducible paths

$$e_1, e_2, e_3, a, b, a^*, b^*, aa^*, ba, a^* b^*.$$

I shall omit the following proof of the Diamond Lemma in my lectures.

Lemma (1)**.** *If $f \rightsquigarrow g$ and u', v' are paths, then either $u' f v' = u' g v'$ or $u' f v' \rightsquigarrow u' g v'$.*

Proof. Suppose g is the reduction of f with respect to u, v and the relation $w_j = s_j$. If $u'u$ or vv' are not paths, then $u' f v' = u' g v'$. Else $u' g v'$ is the reduction of $u' f v'$ with respect to $u'u$, vv' and the relation $w_j = s_j$. □

Lemma (2). *Any $f \in KQ$ can be reduced by a finite sequence of reductions to an irreducible element, so the irreducible paths span R.*

Proof. Any $f \in KQ$ which is not irreducible involves paths of the form uw_jv. Among all paths of this form involved in f, let $\text{tip}(f)$ be the maximal one. Consider the set of tips of elements which cannot be reduced to an irreducible element. For a contradiction assume this set is non-empty. Then by well-ordering it contains a minimal element. Say it is $\text{tip}(f) = w = uw_jv$. Writing $f = \lambda uw_jv + f'$ where $\lambda \in K$ and f' only involving paths different from uw_jv, we have $f \rightsquigarrow g$ where $g = \lambda us_jv + f'$. By the properties of the ordering, us_jv only involves paths which are less than $uw_jv = w$, so $\text{tip}(g) < w$. Thus by minimality, g can be reduced to an irreducible element, hence so can f. Contradiction. $\square$

Lemma (3). *The set of reduction-unique elements is a subspace of KQ, and the assignment $f \mapsto r(f)$ is an endomorphism of it.*

Proof. Consider a linear combination $\lambda f + \mu g$ where f, g are reduction-unique and $\lambda, \mu \in K$. Suppose there is a sequence of reductions (labelled (1))

$$\lambda f + \mu g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} h$$

with h irreducible. Let a be the element obtained by applying the same reductions to f. By Lemma 2, a can be reduced by some sequence of reductions (labelled (2)) to an irreducible element. Since f is reduction-unique, this irreducible element must be $r(f)$.

$$f \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} a \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} r(f).$$

Applying all these reductions to g we obtain elements b and c, and after applying more reductions (labelled (3)) we obtain an irreducible element, which must be $r(g)$.

$$g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} b \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} c \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} r(g).$$

But $h, r(f)$ are irreducible, so these extra reductions don't change them:

$$\lambda f + \mu g \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} h \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} h \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} h,$$

$$f \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(1)} a \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(2)} r(f) \overbrace{\rightsquigarrow \cdots \rightsquigarrow}^{(3)} r(f).$$

Now the reductions are linear maps, hence so is a composition of reductions, so $h = \lambda r(f) + \mu r(g)$. Thus $\lambda f + \mu g$ is reduction-unique and $r(\lambda f + \mu g) = \lambda r(f) + \mu r(g)$. $\square$

Proof of the Diamond Lemma. The implications (c)⇒(b)⇒(a) are trivial.

(a)⇒(b). Since the reduction-unique elements form a subspace, it suffices to show that every *path* is reduction-unique. For a contradiction, suppose not. Then there is a minimal path w which is not reduction-unique. Let $f = w$. Suppose that f reduces under some sequence of reductions to g, and under another sequence of reductions to h, with g, h irreducible. We want to prove that $g = h$, giving a contradiction. Let the elements obtained in each case by applying one reduction be f_1 and g_1. Thus

$$f \rightsquigarrow g_1 \rightsquigarrow \cdots \rightsquigarrow g, \qquad f \rightsquigarrow h_1 \rightsquigarrow \cdots \rightsquigarrow h.$$

By the properties of the ordering, g_1 and h_1 are linear combinations of paths which are less than w, so by minimality they are reduction-unique. Thus $g = r(g_1)$ and $h = r(h_1)$. It suffices to prove that the reductions $f \rightsquigarrow g_1$ and $f \rightsquigarrow h_1$ satisfy the diamond condition, for if there are sequences of reductions $g_1 \rightsquigarrow \cdots \rightsquigarrow k$ and $h_1 \rightsquigarrow \cdots \rightsquigarrow k$, combining them with a sequence of reductions $k \rightsquigarrow \cdots \rightsquigarrow r(k)$, we have $g = r(g_1) = r(k) = r(h_1) = h$.

Thus we need to check the diamond condition for $f \rightsquigarrow g_1$ and $f \rightsquigarrow h_1$. Recall that $f = w$, so these reductions are given by subpaths of w of the form w_i and w_j. There are two cases:

(i) If these paths overlap, or one contains the other, the diamond condition follows from the corresponding overlap or inclusion ambiguity. For example w might be of the form $u'w_ivv' = u'uw_jv'$ where $w_iv = uw_j$ is an overlap ambiguity and u', v' are paths. Now condition (a) says that the reductions $w_iv \rightsquigarrow s_iv$ and $uw_j \rightsquigarrow us_j$ can be completed to a diamond, say by sequences of reductions $s_iv \rightsquigarrow \cdots \rightsquigarrow k$ and $us_j \rightsquigarrow \cdots \rightsquigarrow k$. Then Lemma 1 shows that the two reductions of w, which are $w = u'w_ivv' \rightsquigarrow u's_ivv'$ and $w = u'uw_jv' \rightsquigarrow u'vs_jv'$, can be completed to a diamond by reductions leading to $u'kv'$.

(ii) Otherwise w is of the form uw_ivw_jz for some paths u, v, z, and $g_1 = us_ivw_jz$ and $h_1 = uw_ivs_jz$ (or vice versa). Writing s_i as a linear combination of paths, $s_i = \lambda t + \lambda' t' + \dots$, we have

$$r(g_1) = r(us_ivw_jz) = \lambda r(utvw_jz) + \lambda' r(ut'vw_jz) + \dots \ .$$

Reducing each path on the right hand side using the relation $w_j = s_j$, we have $utvw_jz \rightsquigarrow utvs_jz$, and $ut'vw_jz \rightsquigarrow ut'vs_jz$, and so on, so

$$r(g_1) = \lambda r(utvs_jz) + \lambda' r(ut'vs_jz) + \dots \ .$$

Collecting terms, this gives $r(g_1) = r(us_ivs_jz)$. Similarly, writing s_j as a linear combination of paths, we have $r(h_1) = r(us_ivs_jz)$. Thus $r(h_1) = r(g_1)$, so the diamond condition holds.

(b)⇒(c) The ideal (S) is spanned by expressions of the form $u(w_j - s_j)v$, and $uw_jv \rightsquigarrow us_jv$ so $r(uw_jv) = r(us_jv)$, so $r(u(w_j - s_j)v) = 0$. By linearity, any element $f \in (S)$ satisfies $r(f) = 0$. In particular, if a linear combination f of irreducible paths is zero in R, then $f \in (S)$, so $f = r(f) = 0$. □

1.5 Radical and socle

Definition. Let M be a module for a ring R. The *socle* of M is the sum of its simple submodules,

$$\operatorname{soc} M = \sum_{S \subseteq M \text{ simple}} S.$$

The *radical* of M is the intersection of its maximal submodules.

$$\operatorname{rad} M = \bigcap_{U \subseteq M,\ M/U \text{ simple}} U$$

$$= \{x \in M : \phi(x) = 0 \text{ for any homomorphism } \phi : M \to S \text{ with } S \text{ simple}\}$$

The quotient $\operatorname{top} M = M/\operatorname{rad} M$ is called the *top* of M.

Properties. (i) $\operatorname{soc} M$ is the unique largest semisimple submodule of M.

(ii) If $\theta : M \to N$ then $\theta(\operatorname{soc} M) \subseteq \operatorname{soc} N$ and $\theta(\operatorname{rad} M) \subseteq \operatorname{rad} N$, for if $\phi : N \to S$ and $x \in \operatorname{rad} M$, then $\phi\theta(x) = 0$. Thus soc, rad and top define additive functors R-Mod $\to$ R-Mod. It follows that $\operatorname{soc}(M \oplus N) = \operatorname{soc} M \oplus \operatorname{soc} N$ and $\operatorname{rad}(M \oplus N) = \operatorname{rad} M \oplus \operatorname{rad} N$ and $\operatorname{top}(M \oplus N) \cong \operatorname{top} M \oplus \operatorname{top} N$.

(iii) $\operatorname{rad}(M/\operatorname{rad} M) = 0$ since the maximal submodules of M all contain $\operatorname{rad} M$, so are in 1:1 correspondence with the maximal submodules of $M/\operatorname{rad} M$.

(iv) If M is semisimple, then $\operatorname{rad} M = 0$. For if $M \cong \bigoplus_{i \in I} S_i$, the projections $M \to S_i$ show that $\operatorname{rad} M = 0$.

(v) In general it is not true that if $M/\operatorname{rad} M$ is semisimple. For example $\operatorname{rad}({}_{\mathbb{Z}}\mathbb{Z}) = \cap_{p \text{ prime}} p\mathbb{Z} = 0$, but ${}_{\mathbb{Z}}\mathbb{Z}$ is not semisimple.

However, if M is artinian (has dcc on submodules), e.g. if M is a finite-dimensional module for an algebra, then $M/\operatorname{rad} M$ is semisimple, and it is the unique largest quotient of M which is semisimple.

Namely, we can write $\operatorname{rad} M$ as a finite intersection of maximal submodules $U_1 \cap \cdots \cap U_n$. Then $M/\operatorname{rad} M$ embeds in $(M/U_1) \oplus \cdots \oplus (M/U_n)$, so it is semisimple. Conversely if M/N is semisimple, the canonical map $M \to M/N$ sends $\operatorname{rad} M$ into $\operatorname{rad}(M/N) = 0$, so $\operatorname{rad} M \subseteq N$.

Recall that the *Jacobson radical* $J(R)$ of a ring R is the intersection of its maximal left ideals, so $J(R) = \operatorname{rad}({}_R R)$. It is an ideal in R, by functoriality or by the following.

Theorem. *If R is a ring and $x \in R$, the following are equivalent*
(i) $xS = 0$ for any simple left module S.
(ii) $x \in I$ for every maximal left ideal I, i.e. $x \in J(R)$.
(iii) $1 - ax$ has a left inverse for all $a \in R$.
(iv) $1 - ax$ is invertible for all $a \in R$.
(i')-(iv') The right-hand analogues of (i)-(iv).

Proof. (i) implies (ii). If I is a maximal left ideal in R, then R/I is a simple left module, so $x(R/I) = 0$, so $x(I + 1) = I + 0$, so $x \in I$.

(ii) implies (iii). If there is no left inverse, then $R(1 - ax)$ is a proper left ideal in R, so contained in a maximal left ideal I by Zorn's Lemma. Now $x \in I$, and $1 - ax \in I$, so $1 \in I$, so $I = R$, a contradiction.

(iii) implies (iv) $1 - ax$ has a left inverse u, and $1 + uax$ has a left inverse v. Then $u(1-ax) = 1$, so $u = 1+uax$, so $vu = 1$. Thus u has a left and right inverse, so it is invertible and these inverses are equal, and are themselves invertible. Thus $1 - ax$ is invertible.

(iv) implies (i'). Suppose T is a simple right R-module with $Tx \neq 0$. Then there is $t \in T$ with $tx \neq 0$. By simplicity, there is $a \in R$ with $txa = t$. Let b be an inverse to $1 - ax$. Then

$$0 = t(1 - xa)(1 + xba) = t(1 - xa + xba - xaxba) = t(1 - xa + x(1 - ax)ba) = t.$$

Contradiction. □

Lemma. *If I is a left ideal in R which is nil, meaning that every element is nilpotent, then $I \subseteq J(R)$.*

Proof. If $x \in I$ and $a \in R$ then $ax \in I$, so $(ax)^n = 0$, so $1 - ax$ is invertible with inverse $1 + ax + (ax)^2 + \dots$. □

Lemma (Nakayama's Lemma)**.** *Suppose M is a finitely generated module for a ring R.*
(i) If $J(R)M = M$, then $M = 0$.
(ii) If $N \subseteq M$ is a submodule with $N + J(R)M = M$, then $N = M$.

Proof. (i) If $M \neq 0$ then by Zorn's lemma (using that M is finitely generated), it has a maximal submodule N. Then M/N is simple, so $J(R)(M/N) = 0$, so $J(R)M \subseteq N$. Contradiction.

(ii) Apply (i) to M/N. □

Examples. (a) If $R = KQ/I$ with I an admissible ideal, then $J(R)$ is equal to the ideal $L = (KQ)_+/I$. Namely, for some n we have $(KQ)_+^n \subseteq I$, so $L^n = 0$, so $L \subseteq J(R)$ by the lemma. On the other hand,

$$R/L \cong KQ/(KQ)_+ \cong K \times \cdots \times K$$

is semisimple as an algebra, so as an R-module. Now the canonical map $R \to R/L$ sends $\operatorname{rad} R$ to $\operatorname{rad}(R/L) = 0$, so $J(R) = \operatorname{rad} R \subseteq L$.

(b) If Q is a finite quiver then $J(KQ)$ is spanned by the paths from i to j such that there is no path from j to i.

The set I spanned by these paths is an ideal, and if Q has n vertices, then any path in this ideal has length less than n, so $I^n = 0$. Thus $I \subseteq J(KQ)$.

Conversely suppose that $a \in J(KQ)$ involves a path p from i to j, and suppose there exists a path q from j to i.

Then $b = qae_i \in e_i KQ e_i$ involves the path qp. Also $b \in J(KQ)$, so if $\lambda \in K$, then $1 - \lambda b$ is invertible, say with inverse c. Then $e_i - \lambda b$ is invertible in $e_i KQ e_i$ with inverse $e_i c e_i$. But $e_i KQ e_i$ is isomorphic to a free algebra $K\langle X \rangle$, so its only invertible elements are the elements of K. Thus $e_i - \lambda b$ is a multiple of e_i. Thus $p = q = e_i$, but then b is a multiple of e_i and then for suitable λ, $e_i - \lambda b$ is not invertible in $e_i KQ e_i$.

Proposition/Definition. *A ring R is called a* local ring *if it satisfies the following equivalent conditions.*

(i) $R/J(R)$ is a division ring.

(ii) The non-invertible elements of R form an ideal.

(iii) There is a unique maximal left ideal in R.

If so, then the ideal in (ii) and the left ideal in (iii) are equal to $J(R)$.

Proof. (i) implies (ii). The elements of $J(R)$ are not invertible, so it suffices to show that any $x \notin J(R)$ is invertible. Now $J(R) + x$ is an invertible element in $R/J(R)$, say with inverse $J(R) + a$. Then $1 - ax, 1 - xa \in J(R)$. But this implies ax and xa are invertible, hence so is x.

(ii) implies (iii). Clear.

(iii) implies (i). Since $J(R)$ is the intersection of the maximal left ideals, it is the unique maximal left ideal. Thus $\overline{R} = R/J(R)$ is a simple R-module, and so a simple $\overline{R}$-module. Then $\overline{R} \cong \operatorname{End}_{\overline{R}}(\overline{R})^{op}$, which is a division ring by Schur's Lemma. □

Examples. (i) A ring of power series $K[[x]]$. The elements of the ideal (x) are non-invertible, and all other elements are invertible.

(ii) If I is an admissible ideal in KQ, then KQ/I is local if and only if Q has exactly one vertex. For example $R = K[x]/(x^n)$ is local.

(iii) The set of upper triangular matrices with equal diagonal entries is a subalgebra of $M_n(K)$, e.g.

$$\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in K\}$$

It is local since if $a = 0$ the matrix is nilpotent, and if $a \neq 0$ the matrix is invertible, and the inverse is still in the subalgebra.

(iv) The exterior algebra

$$R = \Lambda(V) \cong K\langle x_1, \dots, x_n\rangle/(x_i^2, x_i x_j + x_j x_i).$$

The ideal I generated by the x_i is nil and $R/I \cong K$.

Remark. Let Q be a finite quiver. Sometimes it is useful to consider the power series path algebra $K\langle\langle Q\rangle\rangle$, consisting of sums

$$\sum_{p \text{ path}} a_p p$$

with $a_p \in K$, but with no requirement that only finitely many are non-zero. Multiplication makes sense because any path p can be obtained as a product qq' in only finitely many ways. In the special case of a loop one gets the power series algebra $K[[x]]$. Alternatively

$$K\langle\langle Q\rangle\rangle \cong \varprojlim_n KQ/(KQ)_+^n,$$

the $(KQ)_+$-adic completion of KQ. Some properties:

(i) An element of $K\langle\langle Q\rangle\rangle$ is invertible if and only if the coefficient of each trivial path e_i is nonzero.

(ii) $J(K\langle\langle Q\rangle\rangle)$ consists of the elements in which the trivial paths all have coefficient zero, so it is the ideal generated by the arrows.

(iii) f.d. $K\langle\langle Q\rangle\rangle$-modules correspond exactly to f.d. modules M for KQ which are *nilpotent*, meaning that $(KQ)_+^d M = 0$ for some d.

1.6 Finite length indecomposable modules

Definition. A *composition series* for an R-module M is a chain of submodules

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M$$

such that the quotients M_i/M_{i-1} are simple. If so the *length* of the composition series is n and the *composition factors* are the quotients $M_1/M_0, M_2/M_1, \dots, M_n/M_{n-1}$.

It is easy to see that M has a composition series if and only if it has the acc and the dcc on submodules, that is, it is noetherian and artinian.

We define $\operatorname{length} M$ to be the length of a composition series, or ∞ if there is none. The Jordan-Hölder Theorem (proof omitted) says that any two composition series have the same length, and the composition factors are the same, up to reordering. Clearly if $0 \to X \to Y \to Z \to 0$ is exact, then

$$\operatorname{length} Y = \operatorname{length} X + \operatorname{length} Z.$$

Clearly a finite-dimensional module for an algebra has finite length.

Definition. A module M for a ring R is *indecomposable* if $M \neq 0$ and there is no direct sum decomposition $M = X \oplus Y$ with X and Y non-zero submodules of M. It is equivalent that $\mathrm{End}_R(M)$ contains no idempotents except 0,1.

Examples. (i) A semisimple module is indecomposable if and only if it is simple.

(ii) For a quiver Q, the projective KQ-modules $P[i] = KQe_i$ are indecomposable. If not, identifying

$$\mathrm{End}(P[i]) = e_i KQ e_i$$

we get an idempotent $e \in e_i KQ e_i$ with $e \neq 0, e_i$. Then $0 \neq e \in KQe_i$ and $0 \neq f = e_i - e \in e_i KQ$ and $ef = 0$. Contradiction.

Proposition. *For a nonzero ring R we have*

Every element of R is nilpotent or invertible $\Rightarrow$ *R is local* $\Rightarrow$ *R has no idempotents except 0,1*

Thus if M is a nonzero module, we have

Every endomorphism of M is nilpotent or invertible $\Rightarrow$ $\mathrm{End}(M)$ *is local* $\Rightarrow$ *M is indecomposable*

Proof. Suppose every element of R is nilpotent or invertible. We claim that the non-invertible elements form an ideal I. Say $x \in I$ and $ax \notin I$. Then $x^n = 0$, so $0 = [(ax)^{-1}a]^n x^n = 1$. Now if $x, y \in I$ and $x + y$ is invertible, then letting $a = (x+y)^{-1}$ we have $ax = 1 - ay$, so ax is invertible. Contradiction.

Now suppose R is local. If e is a non-trivial idempotent, then e and $1 - e$ are non-invertible (else $e = e1 = eee^{-1} = ee^{-1} = 1$). Thus both are in $J(R)$, so $1 \in J(R)$. Contradiction. □

The next result shows that for a finite length module, the three conditions are equivalent.

Lemma (Fitting's Lemma)**.** *If M is a finite length module and $\theta \in \mathrm{End}(M)$, then there is a decomposition as a direct sum of submodules*

$$M = M_0 \oplus M_1$$

such that $\theta|_{M_0}$ is a nilpotent endomorphism of M_0 and $\theta|_{M_1}$ is an invertible endomorphism of M_1.

In particular, if M is indecomposable, then any endomorphism is nilpotent or invertible, so $\mathrm{End}(M)$ *is local.*

Proof. There are chains of submodules

$$\mathrm{Im}(\theta) \supseteq \mathrm{Im}(\theta^2) \supseteq \mathrm{Im}(\theta^3) \supseteq \dots$$

$$\mathrm{Ker}(\theta) \subseteq \mathrm{Ker}(\theta^2) \subseteq \mathrm{Ker}(\theta^3) \subseteq \dots$$

which must stabilize since M has finite length. Thus there is some n with $\mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$ and $\mathrm{Ker}(\theta^n) = \mathrm{Ker}(\theta^{2n})$. We show that

$$M = \mathrm{Ker}(\theta^n) \oplus \mathrm{Im}(\theta^n).$$

If $m \in \mathrm{Ker}(\theta^n) \cap \mathrm{Im}(\theta^n)$ then $m = \theta^n(m')$ and $\theta^{2n}(m') = \theta^n(m) = 0$, so $m' \in \mathrm{Ker}(\theta^{2n}) = \mathrm{Ker}(\theta^n)$, so $m = \theta^n(m') = 0$. If $m \in M$ then $\theta^n(m) \in \mathrm{Im}(\theta^n) = \mathrm{Im}(\theta^{2n})$, so $\theta^n(m) = \theta^{2n}(m'')$ for some m''. Then $m = (m - \theta^n(m'')) + \theta^n(m'') \in \mathrm{Ker}(\theta^n) + \mathrm{Im}(\theta^n)$.

Now it is easy to see that the restriction of θ to $\mathrm{Ker}(\theta^n)$ is nilpotent, and its restriction to $\mathrm{Im}(\theta^n)$ is invertible. □

We now apply the idea of the Jacobson radical to the module category.

Proposition/Definition. *If X and Y are R-modules, we define* $\mathrm{rad}(X, Y)$ *to be the set of all* $\theta \in \mathrm{Hom}(X, Y)$ *satisfying the following equivalent conditions.*
(i) $1_X - \phi\theta$ *is invertible for all* $\phi \in \mathrm{Hom}(Y, X)$.
(ii) $1_Y - \theta\phi$ *is invertible for all* $\phi \in \mathrm{Hom}(Y, X)$.
Thus by definition $\mathrm{rad}(X, X) = J(\mathrm{End}(X))$.

Proof. (i) implies (ii). If u is an inverse for $1_X - \phi\theta$ then $1_Y + \theta u\phi$ is an inverse for $1_Y - \theta\phi$. □

Lemma. *(a)* rad *defines an ideal in the module category, that is* $\mathrm{rad}(X, Y)$ *is an additive subgroup of* $\mathrm{Hom}(X, Y)$*, and given maps $X \to Y \to Z$, if one is in the radical, so is the composition.*

(b) $\mathrm{rad}(X \oplus X', Y) = \mathrm{rad}(X, Y) \oplus \mathrm{rad}(X', Y)$ *and* $\mathrm{rad}(X, Y \oplus Y') = \mathrm{rad}(X, Y) \oplus \mathrm{rad}(X, Y')$.

Proof. (a) For a sum $\theta + \theta'$, let f be an inverse for $1 - \phi\theta$. Then $1 - \phi(\theta + \theta') = (1 - \phi\theta)(1 - f\phi\theta')$, a product of invertible maps.

(b) Straightforward. □

Definition. A module map $\theta : X \to Y$ is a *split mono* if it has a retraction, that is, there is a map $\phi : Y \to X$ with $\phi\theta = 1_X$. Equivalently if θ is an isomorphism of X with a direct summand of Y.

A module map $\theta : X \to Y$ is a *split epi* if it has a section, that is, there is a map $\psi : Y \to X$ with $\theta\psi = 1_Y$. Equivalently if θ identifies Y with a direct summand of X.

Lemma. *(i) If X has local endomorphism ring, then* $\operatorname{rad}(X,Y)$ *is the set of maps which are not split monos.*

(ii) If Y has local endomorphism ring, then $\operatorname{rad}(X,Y)$ *is the set of maps which are not split epis.*

(iii) If X and Y have local endomorphism ring, then $\operatorname{rad}(X,Y)$ *is the set of non-isomorphisms.*

Proof. (i) Suppose $\theta \in \operatorname{Hom}(X,Y)$. If θ is a split mono there is $\phi \in \operatorname{Hom}(Y,X)$ with $\phi\theta = 1_X$, so $1_X - \phi\theta$ is not invertible. Conversely if there is some ϕ with $f = 1_X - \phi\theta$ not invertible, then $\phi\theta = 1_X - f$ is invertible. Then $(\phi\theta)^{-1}\phi\theta = 1_X$, so θ is split mono.

(ii) is dual and (iii) follows. □

Theorem (Krull-Remak-Schmidt Theorem)**.** *Every finite length module M is isomorphic to a direct sum of indecomposable modules,*

$$M \cong X_1 \oplus \cdots \oplus X_n.$$

Moreover if $M \cong Y_1 \oplus \cdots \oplus Y_m$ is another decomposition into indecomposables, then $m = n$ and the X_i and Y_j can be paired off so that corresponding modules are isomorphic.

Proof. The existence of a decomposition holds by induction on the length. Given any two modules X and Y, we set

$$\operatorname{top}(X,Y) = \operatorname{Hom}(X,Y)/\operatorname{rad}(X,Y).$$

It is naturally an $\operatorname{End}(Y)$-$\operatorname{End}(X)$-bimodule, and in fact an $\operatorname{End}(Y)/J(\operatorname{End}(Y))$-$\operatorname{End}(X)/J(\operatorname{End}(X))$-bimodule. We apply this to an indecomposable X of finite length and the module M. Then $D = \operatorname{End}(X)/J(\operatorname{End}(X))$ is a division ring and $\operatorname{top}(X,M)$ is a right D-module. Moreover as a right D-module,

$$\operatorname{top}(X,M) = \operatorname{top}(X, X_1 \oplus \cdots \oplus X_n) \cong \operatorname{top}(X,X_1) \oplus \cdots \oplus \operatorname{top}(X,X_n)$$

and

$$\operatorname{top}(X,X_i) \cong \begin{cases} D & (X_i \cong X) \\ 0 & (X_i \not\cong X) \end{cases}$$

so the number of X_i isomorphic to X is equal to the length of $\operatorname{top}(X,M)$ as a right D-module, so it is the same in any decomposition of M. □

Definition. Clearly any finite length module M is isomorphic to a direct sum

$$\underbrace{M_1 \oplus \cdots \oplus M_1}_{r_1} \oplus \cdots \oplus \underbrace{M_n \oplus \cdots \oplus M_n}_{r_n}$$

with the M_i indecomposable and $M_i \not\cong M_j$ for $i \neq j$.

We define $\#M = n$, the number of non-isomorphic indecomposable summands in a decomposition of M.

We say M is *basic* if all $r_i = 1$, that is, M can be written as a direct sum of pairwise non-isomorphic indecomposable modules.

Given any R-module M, we write $\operatorname{add} M$ for the full subcategory of R-Mod consisting of all modules isomorphic to a direct summand of a finite direct sum of copies of M.

For example $\operatorname{add} R$ is the category of f.g. projective R-modules.

Clearly if M has finite length, then $\operatorname{add} M$ consists of the modules isomorphic to a finite direct sum of copies of the M_i. The module

$$M' = M_1 \oplus \cdots \oplus M_n$$

is the unique basic module, up to isomorphism, with $\operatorname{add} M = \operatorname{add} M'$.

Definition. Let $\theta : X \to Y$ be a map of R-modules.

(i) We say that θ is *left minimal* if for $\alpha \in \operatorname{End}(Y)$, if $\alpha\theta = \theta$, then α is invertible.

(ii) We say that θ is *right minimal* if for $\beta \in \operatorname{End}(X)$, if $\theta\beta = \theta$, then β is invertible.

Lemma. *Given a map $\theta : X \to Y$ of finite length modules.*

(i) There is a decomposition $Y = Y_0 \oplus Y_1$ such that $\operatorname{Im}(\theta) \subseteq Y_1$ and $X \to Y_1$ is left minimal.

(ii) There is a decomposition $X = X_0 \oplus X_1$ such that $\theta(X_0) = 0$ and $X_1 \to Y$ is right minimal.

Proof. (i) Of all decompositions $Y = Y_0 \oplus Y_1$ with $\operatorname{Im}(\theta) \subseteq Y_1$ choose one with Y_1 of minimal length. Let θ_1 be the map $X \to Y_1$. Let $\alpha \in \operatorname{End}(Y_1)$ with $\alpha\theta_1 = \theta_1$. By the Fitting decomposition, $Y_1 = \operatorname{Im}(\alpha^n) \oplus \operatorname{Ker}(\alpha^n)$ for $n \gg 0$. Now $\alpha^n\theta_1 = \theta_1$, so $\operatorname{Im}(\theta_1) \subseteq \operatorname{Im}(\alpha^n)$, and we have another decomposition $Y = [Y_0 \oplus \operatorname{Ker}(\alpha^n)] \oplus \operatorname{Im}(\alpha^n)$. By minimality, $\operatorname{Ker}(\alpha^n) = 0$, so α is injective, and hence an isomorphism.

(ii) is dual. □

Lemma. *Let $\theta_i : X_i \to Y_i$ be finitely many maps between finite length modules. If the θ_i are left (respectively right) minimal, then so is the map $\bigoplus_i X_i \to \bigoplus_i Y_i$.*

Proof. We prove it for right minimal (left minimal is similar). If not, then by the lemma, there is a non-zero summand X' of $\bigoplus_i X_i$ on which the map is zero. We may assume that X' is indecomposable, so has local endomorphism ring. Let $f_i : X' \to X_i$ be the projections. Since $\theta(X') = 0$ we have $\theta_i f_i = 0$ for all i. Since X' is a summand there are $g_i : X_i \to X'$ with $1_{X'} = \sum_i g_i f_i$. Thus some $g_i f_i$ is invertible, so f_i is a split mono, with retraction $r = (g_i f_i)^{-1} g_i$. Then $\beta = 1_{X_i} - f_i r$ satisfies $\theta_i \beta = \theta_i$, so by minimality β is invertible, but $\beta f_i = 0$, so $f_i = 0$, a contradiction. □

1.7 Left artinian rings

We're interested in f.d. algebras over a field K, but some things we can do more generally for left artinian rings.

Lemma. *Let R be a left artinian ring and M an R-module. Then*

(i) $J = J(R)$ is a nilpotent ideal.

(ii) R/J is a semisimple ring.

(iii) R is left noetherian, so has finite length as a left R-module. Thus finite length modules are the same as finitely generated modules.

(iv) There are only finitely many simple R-modules

(v) If M is an R-module, then $\operatorname{rad} M = JM$ *and* $\operatorname{soc} M = \{m \in M : Jm = 0\}$.

(vi) If $M =$ $\operatorname{rad} M$ *or* $\operatorname{soc} M = 0$ *then $M = 0$.*

Proof. (i) By the dcc we have $J^n = J^{2n}$ for some n. Suppose this is nonzero. Then there is a nonzero left ideal I with $J^n I = I$. Thus there is a minimal one. Let $L = \{x \in I : J^n x = 0\}$. Clearly it is a left ideal and a proper subset of I. If $x \in I \setminus L$, then $J^n x \subseteq I$ and $J^n(J^n x) = J^n x \neq 0$, so by minimality $J^n x = I$. Thus $Rx = I$. Thus I/L is simple. Thus $J^n(I/L) = 0$, so $I = J^n I \subseteq L$. Contradiction.

(ii) Now R/J is semisimple as an R-module, so as an R/J-module, so it is a semisimple ring.

(iii) Each J^i/J^{i+1} is an R/J-module, so semisimple. Since they are also artinian, they are finite direct sums of simples, so they are also noetherian. Thus R is noetherian.

(iv) Any simple module is a composition factor of the finite-length module R/J.

(v) If N is a maximal submodule of M, then M/N is simple, and so $J(M/N) = 0$, so $JM \subseteq N$. Thus $JM \subseteq \operatorname{rad} M$. On the other hand M/JM is an R/J-module, so semisimple. Then by functoriality, the map $M \to M/JM$ sends $\operatorname{rad} M$ to $\operatorname{rad}(M/JM) = 0$, so $\operatorname{rad} M \subseteq JM$.

Any simple submodule S of M satisfies $JS = 0$, so $Jm = 0$ for all $m \in \operatorname{soc} M$, so $\operatorname{soc} M$ is contained in the RHS. Now the RHS is an R/J-module, so semisimple, so contained in $\operatorname{soc} M$.

(vi) If $M = JM$ then $M = J^n M = 0$. Any non-zero module has a non-zero f.g. submodule, and that has a simple submodule by the dcc. □

Notation. Let R be left artinian. We decompose ${}_R R$ into indecomposables, and collect isomorphic terms, so

$$R \cong P[1]^{r_1} \oplus \cdots \oplus P[n]^{r_n}$$

with the $P[i]$ non-isomorphic modules. The modules $P[1], \dots, P[n]$ are are called the *principal indecomposable modules* (pims).

Let $D_i = (\mathrm{End}(P[i])/J(\mathrm{End}(P[i])))^{op}$. Since $P[i]$ is indecomposable of finite length, it is a division algebra.

Let $S[i] = P[i]/\operatorname{rad} P[i]$.

Lemma. *(i) The $P[i]$ are a complete set of non-isomorphic indecomposable f.g. projective R-modules.*

(ii) The $S[i]$ are a complete set of non-isomorphic simple R-modules, and $D_i \cong \mathrm{End}(S[i])^{op}$.

(iii) $R/J(R) \cong M_{r_1}(D_1) \times \cdots \times M_{r_n}(D_n)$, and under this isomorphism, the simple module $S[i]$ corresponds to the module $D_i^{r_i}$.

Note that in case $J(R) = 0$, part (iii) recovers the Artin-Wedderburn decomposition.

Proof. (i) They are projective and nonisomorphic. Any f.g. projective module is a direct summand of a f.g. free module, so by the Krull-Remak-Schmidt Theorem isomorphic to one of the $P[i]$.

(ii) Since the construction of $S[i] = P[i]/\operatorname{rad} P[i]$ is functorial there is a natural map

$$\mathrm{End}(P[i]) \to \mathrm{End}(S[i])$$

and since $P[i]$ is projective, it is surjective. Now $\mathrm{End}(P[i])$ is a local ring, hence so also is $\mathrm{End}(S[i])$, so $S[i]$ is indecomposable. Since it is semisimple, it is simple, so $\mathrm{End}(S[i])$ is a division ring. Thus we must have an isomorphism

$$\mathrm{End}(P[i])/J(\mathrm{End}(P[i])) \to \mathrm{End}(S[i]).$$

Now the $S[i]$ are non-isomorphic, for inverse isomorphisms between $S[i]$ and $S[j]$ would lift to maps $P[i] \to P[j] \to P[i]$ whose composition can't be nilpotent, so must be invertible, so $P[i] \cong P[j]$, so $i = j$.

Any simple module S has a non-zero map from some $P[i]$, but then the map $P[i] \to S$ must give a non-zero map $S[i] \to S$, and this must be an isomorphism.

(iii) As an R-module, we have

$$R/J = R/\operatorname{rad} R \cong \bigoplus_i (P[i]/\operatorname{rad} P[i])^{r_i} = \bigoplus S[i]^{r_i}.$$

Since $\mathrm{Hom}(S[i], S[j]) = 0$ for $i \neq j$ we get

$$\mathrm{End}_R(R/J) \cong M_{r_1}(\mathrm{End}(S[1])) \times \cdots \times M_{r_n}(\mathrm{End}(S[n])).$$

Now use that

$$R/J \cong \mathrm{End}_{R/J}(R/J)^{op} = \mathrm{End}_R(R/J)^{op}$$

Then $S[i] \cong \mathrm{Hom}(R, S[i]) \cong \bigoplus_j \mathrm{Hom}(P[j], S[i])^{r_j} \cong \mathrm{End}(S[i])^{r_i}$. □

Example. Let R be the set of matrices of shape

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \subseteq M_3(K).$$

It is a subalgebra, so an algebra. We can write it as $R = S \oplus I$ for a subalgebra S and ideal I with

$$S = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Now I is a nil ideal, so $I \subseteq J(R)$. Also

$$R/I \cong S \cong M_2(K) \times K$$

which is semisimple, so $J(R) \subseteq I$. Thus $J(R) = I$. Now we get the decomposition $R = Re^{11} \oplus Re^{22} \oplus Re^{33}$ where

$$P[1] = Re^{11} = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \cong Re^{22}, \quad P[2] = Re^{33} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

$$\operatorname{rad} P[1] = J(R)P[1] = 0 \quad \operatorname{rad} P[2] = J(R)P[2] = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \cong P[1].$$

Then $D_i = K$, $S[1] = P[1]$ is 2-dimensional and $S[2] = P[2]/\operatorname{rad} P[2]$ is 1-dimensional.

Definition. Let R be a K-algebra. We say that a finite-dimensional R-module M is *split* if in its decomposition into indecomposables, for each summand, the top of the endomorphism ring is K.

We say that a finite-dimensional algebra R is *basic* or *split* if ${}_RR$ has this property. It is equivalent that all $r_i = 1$, respectively that all $D_i = K$.

Proposition. *(i) Any f.d. algebra is Morita equivalent to a basic one.*

(ii) If K is algebraically closed, any f.d. module or algebra is split.

(iii) If I is an admissible ideal in KQ, then KQ/I is basic and split.

Proof. (i) Let $P = P[1] \oplus \cdots \oplus P[n]$. It is a basic module. Since it involves all of the indecomposable projective R-modules, it is a finitely generated projective generator for R-Mod, so R is Morita equivalent to $A = \operatorname{End}_R(P)^{op}$. Now

$$\operatorname{End}_R(P/\operatorname{rad} P)^{op} \cong \operatorname{End}_R(S[1] \oplus \cdots \oplus S[n])^{op} \cong D_1 \times \cdots \times D_n$$

and since the construction of $P/\operatorname{rad} P$ is functorial, there is a natural map

$$\operatorname{End}_R(P) \to \operatorname{End}_R(P/\operatorname{rad} P),$$

and it is surjective since P is projective. The kernel is a nil ideal since if θ is in the kernel, then $\theta(P) \subseteq \operatorname{rad} P = JP$, so $\theta^n(P) \subseteq J^n P = 0$ for $n \gg 0$. Thus the kernel is the radical of $\operatorname{End}_R(P)$, and so A is basic. □

Theorem (Gabriel's less famous theorem about quivers)**.** *If R is a f.d. K-algebra which is basic and split, then $R \cong KQ/I$ for some quiver Q and admissible ideal I.*

Proof. We have a decomposition $R = P[1] \oplus \cdots \oplus P[n]$ without multiplicities. Using the isomorphism $R \cong \operatorname{End}(R)^{op}$, the projections onto the $P[i]$ give a complete family of orthogonal idempotents $e_1, \dots, e_n$ with $P[i] = Re_i$.

Let $J = J(R)$. By assumption $e_1, \dots, e_n$ induce a basis of R/J. We have

$$J = \bigoplus_{i,j} e_j J e_i.$$

and

$$J^2 = \bigoplus_{i,j} e_j J^2 e_i$$

so

$$J/J^2 \cong \bigoplus_{i,j} (e_j J e_i)/(e_j J^2 e_i).$$

Let Q be the quiver with $Q_0 = \{1, \dots, n\}$ and with

$$\dim(e_j J e_i)/(e_j J^2 e_i)$$

arrows from i to j, for all i, j. Define an algebra homomorphism

$$\theta : KQ \to R$$

sending e_i to e_i, and sending the arrows from i to j to elements in $e_j J e_i$ inducing a basis of the quotient. Let $U = \theta(KQ_+)$. We have $U \subseteq J$ and $U + J^2 = J$. Thus by Nakayama's Lemma, $U = J$. It follows that θ is surjective.

Let $I = \operatorname{Ker} \theta$. If m is sufficiently large that $J^m = 0$, then $\theta(KQ_+^m) \subseteq U^m = 0$, so $KQ_+^m \subseteq I$. Suppose $x \in I$. Write it as $x = u + v + w$ where u is a linear combination of trivial paths, v is a linear combination of arrows, and w is in KQ_+^2. Since $\theta(e_i) = e_i$ and $\theta(v), \theta(w) \in J$, we must have $u = 0$. Now $\theta(v) = -\theta(w) \in J^2$, so that $\theta(v)$ induces the zero element of J/J^2. Thus $v = 0$. Thus $x = w \in KQ_+^2$. □

1.8 Injective modules and duality

Definition. Recall that an R-module E is *injective* if it satisfies the following equivalent conditions.

(i) $\mathrm{Hom}(-, E)$ is an exact (contravariant) functor.

(ii) Any short exact sequence $0 \to E \to Y \to Z \to 0$ is split.

(iii) Given an injective map $\theta : X \hookrightarrow Y$, any map $X \to E$ factors through θ.

(iv) (Baer's criterion) Given any left ideal I in R, any map $I \to E$ lifts to a map $R \to E$.

Definition. Let R be a K-algebra, as usual with our assumption that K is a field. If M is a left (respectively right) R-module, then

$$DM = \mathrm{Hom}_K(M, K)$$

is a right (respectively left) R-module.

Properties. (i) If P is a projective R-module, then DP is injective. Namely DR is injective since

$$\mathrm{Hom}_R(-, DR) \cong \mathrm{Hom}_K(- \otimes_R R, K) \cong \mathrm{Hom}_K(-, K) = D(-)$$

which is exact, and any P is a direct summand of a free module $R^{(I)}$, and so $D(P)$ is a direct summand of $D(R^{(I)}) \cong D(R)^I$, a product of copies of $D(R)$, which is injective. Alternatively,

$$\mathrm{Hom}_R(-, DP) \cong \mathrm{Hom}_K(- \otimes_R P, K).$$

Since P is projective, it is flat. Thus this functor is exact.

(ii) If M is finite dimensional, then $\dim DM = M$ and we have a natural isomorphism $M \to D(DM)$. Thus D gives antiequivalences

$$R\text{-mod} \rightleftarrows \text{mod-}R.$$

(iii) If R is a finite-dimensional K-algebra, and E is a f.d. injective R-module, then DE is projective.

Namely, choose a f.d. free left R-module F with a surjective map $F \to DE$. Then E embeds in DF, but E is injective, so E is a direct summand of DF. Then DE is a direct summand of F, so DE is projective.

Thus D induces an antiequivalence between the category of f.d. projective modules on one side and the category of f.d. injective modules on the other side.

Remark. Many results about finite-dimensional K-algebras generalize to *artin algebras*, that is, algebras over a commutative artinian ring K which are finitely generated as a K-module. One needs to replace D by $\mathrm{Hom}_K(-, E)$ where E is the injective envelope of the direct sum of the simple K-modules. (Injective envelopes will be discussed later.)

Definition. For R a f.d. algebra, the *Nakayama functor* is the functor

$$\nu(-) = DR \otimes_R - : R\text{-mod} \to R\text{-mod}.$$

Properties. (i) ν has right adjoint $\nu^-(-) = \mathrm{Hom}_R(DR, -)$.

(ii) We have $\nu(X) \cong D\,\mathrm{Hom}_R(X, R)$. Namely,

$$D\nu(X) = \mathrm{Hom}_K(DR \otimes_R X, K) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_K(DR, K))$$

by Hom-tensor adjointness, and then

$$\mathrm{Hom}_K(DR, K) = D^2 R \cong R,$$

so $D\nu(X) \cong \mathrm{Hom}_R(X, R)$. Now apply D.

(iii) $\mathrm{Hom}_R(X, \nu P) \cong D\,\mathrm{Hom}_R(P, X)$ for X, P left R-modules with P projective.

Namely there is a map $\mathrm{Hom}_R(P, R) \otimes_R X \to \mathrm{Hom}_R(P, X)$ sending $\theta \otimes x$ to the map sending p to $\theta(p)x$. This is a natural transformation between functors of P. Now for $P = R$ it is easy to see that it is an isomorphism, so by functoriality it is an isomorphism for any direct sum of copies of R, so for any f.g. free module F, and also it is an is an isomorphism for any direct summand of F, so for any f.g. projective module P. Now applying D we get an isomorphism

$$D\,\mathrm{Hom}_R(P, X) \cong D(\mathrm{Hom}_R(P, R) \otimes_R X) \cong \mathrm{Hom}_R(X, D\,\mathrm{Hom}_R(P, R)) \cong \mathrm{Hom}_R(X, \nu P).$$

(iv) ν restricts to an equivalence from the category of f.d. projective left modules to the category of f.d. injective left modules.

We know that ν sends f.d. projective modules to f.d. injective modules. Moreover if P, P' are f.d. projective modules, then using (iii) twice we get

$$\mathrm{Hom}_R(\nu P, \nu P') \cong D\,\mathrm{Hom}_R(P', \nu P) \cong \mathrm{Hom}_R(P, P')$$

so ν is fully faithful on the category of f.d. projective modules. Now if I is a f.d. injective module, then there is a f.g. free module with a surjective map onto DI, say $R^n \to DI$. Then the map $I \to DR^n$ is injective, so a split mono. Thus I is isomorphic to the image of an idempotent endomorphism of $DR^n \cong \nu(R^n)$. This comes from an idempotent endomorphism of R^n, and if this has image P, then $I \cong \nu(P)$. Thus the functor is dense.

Notation. Let R be a finite-dimensional algebra, and let $P[i]$ and $S[i]$ be the indecomposable projective and simple modules. We define $I[i] = \nu(P[i])$. They are a complete set of non-isomorphic indecomposable f.d. injective modules. Note also that $\operatorname{soc} I[i] \cong S[i]$ since

$$\dim \operatorname{Hom}(S[j], I[i]) = \dim \operatorname{Hom}(P[i], S[j]) = \begin{cases} \dim D_i & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Note that if $R = KQ/I$ with I admissible, then $P[i] = Re_i$ and

$$I[i] = \nu(Re_i) = D \operatorname{Hom}_R(Re_i, R) = D(e_i R).$$

Thus considering $I[i]$ as a representation of Q, the vector space at vertex j is

$$I[i]_j = e_j D(e_i R) = D(e_i R e_j),$$

which has as basis the dual basis associated a basis of $e_i R e_j$ given by the paths from j to i modulo the relations.

Examples. (1) For the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4$$

with relations $a_{i+1} a_i = 0$, the injective are

$$I[1] = K \to 0 \to 0 \to 0 \cong S[1],$$

$$I[2] = K \to K \to 0 \to 0 \cong P[1],$$
$$I[3] = 0 \to K \to K \to 0 \cong P[2],$$
$$I[4] = 0 \to 0 \to K \to K \cong P[3].$$

(2) For the commutative square

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ 3 & \xrightarrow{d} & 4 \end{array}$$

the injective $I[4]$ is

$$\begin{array}{ccc} K & \xrightarrow{1} & K \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ K & \xrightarrow{1} & K \end{array}$$

so $I[4] \cong P[1]$.

1.9 Module classes, envelopes and covers

Definition. We shall call a subcategory $\mathcal{C}$ of R-Mod a *module class* provided
(i) It is a full subcategory,
(ii) It is closed under isomorphisms, that is, if $X \cong Y$ and $X \in \mathcal{C} \Rightarrow Y \in \mathcal{C}$,
(iii) It is closed under finite direct sums and summands, that is, $X \oplus Y \in \mathcal{C}$ iff $X, Y \in \mathcal{C}$.

If a module class consists of finite length modules, it is determined by the indecomposables it contains.

Examples. (i) All modules, finite length modules, f.d. modules for an algebra, the zero module, the projective modules, the injective modules, the semisimple modules.

(2) Any intersection of module classes.

(3) If $\mathcal{M}$ is any collection of modules, then $\operatorname{add}\mathcal{M}$, is the smallest module class containing $\mathcal{M}$. It consists of all modules isomorphic to a direct summand of a finite direct sum of modules in $\mathcal{M}$.

Definition. Let $\mathcal{C}$ be a module class and X a module, not necessarily in $\mathcal{C}$.

(i) A *left $\mathcal{C}$-approximation* (or preenvelope) of X is a morphism $\theta : X \to C$ with $C \in \mathcal{C}$, such that the induced map

$$\operatorname{Hom}(C, C') \to \operatorname{Hom}(X, C')$$

is surjective for all $C' \in \mathcal{C}$. That is, for any $\theta' : X \to C'$ there is $f : C \to C'$ with $\theta' = f\theta$.

A *$\mathcal{C}$-envelope* (or *hull*) of X is a left minimal left $\mathcal{C}$-approximation of X.

(ii) A *right $\mathcal{C}$-approximation* (or precover) of X is a morphism $\theta : C \to X$ with $C \in \mathcal{C}$, such that the induced map

$$\operatorname{Hom}(C', C) \to \operatorname{Hom}(C', X)$$

is surjective for all $C' \in \mathcal{C}$. That is, for any $\theta' : C' \to X$ with C' in $\mathcal{C}$, there is $f : C' \to C$ with $\theta' = \theta f$.

A *$\mathcal{C}$-cover* of X is a right minimal right $\mathcal{C}$-approximation.

Lemma. *If X has a $\mathcal{C}$-envelope (resp. cover), then it is unique up to isomorphism, and it is a direct summand of any left (resp. right) $\mathcal{C}$-approximation.*

Proof. Straightforward. □

Lemma. *(a) A morphism $\theta : X \to I$ is an injective envelope of X if and only if I is injective, θ is a monomorphism, and $\operatorname{Im}\theta$ is an essential submodule of I, meaning that if U is a nonzero submodule of I, then $U \cap \operatorname{Im}\theta \neq 0$.*

(b) A morphism $\phi : P \to X$ is a projective cover of X if and only if P is projective. ϕ is an epimorphism, and $\operatorname{Ker}\phi$ is a superfluous submodule of P, meaning that if U is a submodule of P with $U + \operatorname{Ker}\phi = P$, then $U = P$.

Proof. (a) By the injective property, and the fact that every module can be embedded in some injective module, θ is a left injective approximation if and only if I is injective and θ is a monomorphism.

Suppose that θ is left minimal and U is a submodule of I with $U \cap \operatorname{Im}\theta = 0$. Then $U \oplus \operatorname{Im}\theta$ is a submodule of I, and by the injective property there is a morphism α such that the diagram

$$\begin{array}{ccccc} X & \longrightarrow & U \oplus \operatorname{Im}\theta & \longrightarrow & I \\ \Vert & & {\scriptstyle p}\downarrow & & {\scriptstyle \alpha}\downarrow \\ X & \longrightarrow & U \oplus \operatorname{Im}\theta & \longrightarrow & I \end{array}$$

commutes, where p is the projection onto $\operatorname{Im}\theta$. Then α is an isomorphism, but $U \subseteq \operatorname{Ker}\alpha$, so $U = 0$.

Suppose that $\operatorname{Im}\theta$ is essential and $\alpha\theta = \theta$. Then $\operatorname{Im}\theta \cap \operatorname{Ker}\alpha = 0$ so $\operatorname{Ker}\alpha = 0$, so α is mono. Since I is injective, α must be a split mono, so $I = \operatorname{Im}\alpha \oplus Y$. But then $Y \cap \operatorname{Im}\theta = 0$, so $Y = 0$, so α is an epi.

(b) Dual. □

Remark. For an arbitrary ring, injective envelopes always exist.

Projective covers do not always exist: observe that the canonical map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not a projective cover of $\mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}$-module, since $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$. Now if $P \to \mathbb{Z}/2\mathbb{Z}$ were a projective cover, it would be a summand of this map. But $\mathbb{Z}$ is indecomposable, so it would be isomorphic to this map.

Injective envelopes and projective covers (when they exist) are denoted $X \to E(X)$ and $P(X) \to X$. They exist for f.d. algebras. We show how to construct them.

Lemma. *Suppose R is left artinian and X is an R-module.*

(a) A homomorphism to an injective module $\theta : X \to I$ is an injective envelope if and only if the induced map $\operatorname{soc} X \to \operatorname{soc} I$ is an isomorphism.

(b) A homomorphism from a projective module $\phi : P \to X$ is a projective cover if and only if the induced map $\operatorname{top} P \to \operatorname{top} X$ is an isomorphism.

Proof. (a) Since $\operatorname{soc} I$ is semisimple, we have $\operatorname{soc} I = \theta(\operatorname{soc} X) \oplus U$ for some U. If θ is an injective envelope, then $U = 0$, so the map on socles is an isomorphism.

Conversely if the map on socles is an isomorphism, then $\operatorname{soc}\operatorname{Ker}\theta = 0$, so θ is injective, and if U is a non-zero submodule of I with $U \cap \operatorname{Im}\theta = 0$, then $U \cap \operatorname{soc} I = 0$, so $\operatorname{soc} U = 0$ so $U = 0$.

(b) Similar, using part (vi) of the first lemma about left artinian rings. □

Remark. Suppose R is a finite-dimensional algebra and X is an R-module.

(a) One gets an injective envelope of X as follows. Write $\operatorname{soc} X$ as a direct sum of copies of the simple modules $S[i]$. Let I be the corresponding direct sum of the injective modules $I[i]$. Since R is noetherian, an arbitrary direct sum of injective modules is injective, so I is injective. Let $\theta_0 : \operatorname{soc} X \to I$ be the map given by the inclusions $S[i] \cong \operatorname{soc} I[i] \hookrightarrow I[i]$. By the injective property, it extends to a map $\theta : X \to I$, which is an injective envelope by the lemma.

(b) One gets an projective cover of X as follows. Write $\operatorname{top} X$ as a direct sum of copies of the simple modules $S[i]$. Let P be the corresponding direct sum of the projective modules $P[i]$. Let $\phi_0 : P \to \operatorname{top} X$ be the map given by the canonical maps $P[i] \to S[i]$. Then by the projective property, it lifts to a map $\phi : P \to X$, which is a projective cover by the lemma.

Remark. To use this explicitly, it is useful to be able to compute the socle and top of an R-module X. This is very easy when $R = KQ/I$ with I an admissible ideal. Then R-modules are identified with representations of Q satisfying the relations defining the ideal I, and recall that a representation X is given by a vector space X_i for each vertex i and a linear map $X_a : X_i \to X_j$ for each arrow $a : i \to j$. Now the simple R-modules are the simples $S[i]$, so a semisimple R-module is exactly a representation X in which all the linear maps X_a are zero. Now the socle of a representation X is the unique largest semisimple subrepresentation, so given by the subspaces

$$(\operatorname{soc} X)_i = \bigcap_{a \text{ an arrow with tail at } i} \operatorname{Ker} X_a.$$

Now $J(R) = KQ_+/I$ and

$$KQ_+ = \sum_{\text{a an arrow}} aKQ,$$

so

$$J(R) = \sum_{\text{a an arrow}} aR$$

where if a is an arrow in Q then a also denotes its image in R. By the lemma at the start of the section on left artinian rings, we have $\operatorname{rad} X = J(R)X$, so

$$\operatorname{rad} X = \sum_{\text{a an arrow}} aRX = \sum_{\text{a an arrow}} aX.$$

This means that if X is considered as a representation of Q, then $\operatorname{rad} X$ is the subrepresentation given by the subspaces

$$(\operatorname{rad} X)_i = \sum_{a \text{ an arrow with head at } i} \operatorname{Im} X_a.$$

Let's explore left and right approximations a little more, for use later on. Henceforth work inside the category R-mod of finite-dimensional modules for an algebra R.

Definition. Let $\mathcal{C}$ be a module class in R-mod.

We say that $\mathcal{C}$ is *covariantly finite* if every f.d. module X has a left $\mathcal{C}$-approximation. If so, it has a $\mathcal{C}$-envelope. Namely, if $\theta : X \to C$ is a left $\mathcal{C}$-approximation, then by the lemma in section 1.6 there is a decomposition $C = C_0 \oplus C_1$ such that $\operatorname{Im} \theta \subseteq C_1$ and the map $X \to C_1$ is left minimal. Now clearly this map is also a left $\mathcal{C}$-aproximation, so it is a $\mathcal{C}$-envelope.

We say that $\mathcal{C}$ is *contravariantly finite* if every f.d. module X has a right $\mathcal{C}$-approximation. If so, it has a $\mathcal{C}$-cover.

We say that $\mathcal{C}$ is *functorially finite* if it is covariantly and contravariantly finite.

Example. If the inclusion $i : \mathcal{C} \to R$-mod has a left adjoint L, then any module $\mathcal{C}$ is covariantly finite. Namely, by assumption for any module X and module $C \in \mathcal{C}$, there is a bijection

$$\operatorname{Hom}(X, C) \to \operatorname{Hom}(LX, C), \quad \theta \mapsto \theta'$$

and this is a natural transformation in X, meaning that

$$(\theta f)' = \theta' L(f) \quad \text{for all } f : X' \to X$$

and a natural transformation in C, meaning that

$$(g\theta)' = g\theta' \quad \text{for all } g : C \to C' \text{ in } \mathcal{C}. \quad (*)$$

Now given a module X, the identity map $1_{LX} : LX \to LX$ is θ' for some $\theta : X \to LX$. Then θ is a left $\mathcal{C}$-approximation of X, since if $\phi : X \to C$ with $C \in \mathcal{C}$, then $(\phi'\theta)' = \phi'\theta' = \phi'$ by (*) and using that $\theta' = 1_{LX}$. Since the map $\theta \mapsto \theta'$ is a bijection, we deduce $\phi'\theta = \phi$, so ϕ factors through θ. Also θ is left minimal in the strong sense, for if $g : LX \to LX$ and $g\theta = \theta$, then

$$1_{LX} = \theta' = (g\theta)' = g\theta' = g1_{LX} = g$$

and for left minimality we only need to know that g is an isomorphism.

Similarly if i has a right adjoint R, then the morphism $i(RM) \to M$ is a $\mathcal{C}$-cover, so $\mathcal{C}$ is contravariantly finite.

Lemma. *If M is a f.d. R-module, then* $\operatorname{add} M$ *is functorially finite in R-mod.*

Proof. For any f.d. module X we take a basis of $\operatorname{Hom}_R(X, M)$, say with n elements. This gives a map $X \to M^n$ which is a left $\operatorname{add} M$-approximation. Similarly for a right $\operatorname{add} M$-approximation use a basis of $\operatorname{Hom}_R(M, X)$ to get a map $M^n \to X$. □

For injective envelopes and projective covers of finite-dimensional modules for a finite-dimensional algebra R we could have used $\operatorname{add} R$ and $\operatorname{add} DR$. For use much later, we record the following.

Definition. If $\mathcal{M}$ is a collection of f.d. modules, the modules *generated* by $\mathcal{M}$ are the module class

$$\operatorname{gen} \mathcal{M} = \{N : \exists \text{ epimorphism } M' \twoheadrightarrow N \text{ with } M' \in \operatorname{add} \mathcal{M}\}.$$

The modules *cogenerated* by $\mathcal{M}$ are the module class

$$\operatorname{cogen} \mathcal{M} = \{N : \exists \text{ monomorphism } N \hookrightarrow M' \text{ with } M' \in \operatorname{add} \mathcal{M}\}.$$

Proposition. *If R is f.d. and M is a f.d. R-module, then* $\operatorname{gen} M$ *is covariantly finite, and dually* $\operatorname{cogen} M$ *is contravariantly finite.*

Proof. Given X, take a projective cover $P \to X$. Take a left $\operatorname{add} M$-approximation $P \to M'$. Take the pushout

$$\begin{array}{ccc} P & \longrightarrow & M' \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & G \end{array}$$

Since $P \to X$ is onto, so is $M' \to G$, so $G \in \operatorname{gen} M$. If $f : X \to G'$ with $G' \in \operatorname{gen} M$, then there is a map from M'' onto G' with $M'' \in \operatorname{add} M$. Since P is projective, the composition $P \to X \to G'$ lifts to a map $P \to M''$. Since the map $P \to M'$ is an approximation, the map $P \to M''$ factors as $P \to M' \to M''$. Now the two maps $X \to G'$ and $M' \to M'' \to G'$ agree on P, so there is an induced map of the pushout $G \to G'$. Thus the map $X \to G'$ factors as $X \to G \to G'$. Thus the map $X \to G$ is a left $\operatorname{gen} M$-approximation. □

1.10 Homological algebra for finite-dimensional algebras

We consider modules for a f.d. algebra R.

Definition. Recall that a projective resolution of a module M is an exact sequence

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

with the P_i projective. Letting $\Omega_0 M = M$ and $\Omega_i M = \operatorname{Im} d_i$ for $i > 0$, it breaks into short exact sequences

$$0 \to \Omega_{i+1} M \to P_i \to \Omega_i M \to 0$$

for all $i \geq 0$. It is a *minimal projective resolution* if the maps $P_i \to \Omega_i M$ are projective covers for all $i \geq 0$. Dually for an injective resolution

$$0 \to M \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d_1} I^2 \to \dots,$$

setting $\Omega^0 M = M$ and $\Omega^i M = \operatorname{Im} d^{i-1}$ we get exact sequences

$$0 \to \Omega^i M \to I^i \to \Omega^{i+1} M \to 0$$

for all $i \geq 0$ and it is a *minimal injective resolution* if the maps $\Omega^i M \to I^i$ are injective envelopes for all i.

The minimal projective and injective resolutions of M exist and are unique up to (non-unique) isomorphism. For example one constructs the minimal projective resolution of M be taking a projective cover of M. This has kernel $\Omega_1 M$. Then take a projective cover of this, and so on.

Example. Recall that the cyclically oriented square

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle d}\uparrow & & \downarrow{\scriptstyle b} \\ 4 & \xleftarrow[c]{} & 3 \end{array}$$

with admissible relations cba and dc, has

$$P[1] = \begin{array}{ccc} K & \xrightarrow{1} & K \\ \uparrow & & \downarrow{\scriptstyle 1} \\ 0 & \longleftarrow & K \end{array} \quad P[2] = \begin{array}{ccc} 0 & \longrightarrow & K \\ \uparrow & & \downarrow{\scriptstyle 1} \\ K & \xleftarrow[1]{} & K \end{array} \quad P[3] = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \downarrow \\ K & \xleftarrow[1]{} & K \end{array} \quad P[4] = \begin{array}{ccc} K & \xrightarrow{1} & K \\ {\scriptstyle 1}\uparrow & & \downarrow{\scriptstyle 1} \\ K & \xleftarrow[0]{} & K \end{array}$$

The simple modules have minimal projective resolutions

$$\begin{aligned} 0 \to P[1] \to P[4] \to P[2] \to P[1] \to & S[1] \to 0, \\ 0 \to P[3] \to P[2] \to & S[2] \to 0, \\ 0 \to P[1] \to P[4] \to P[3] \to & S[3] \to 0, \\ 0 \to P[1] \to P[4] \to & S[4] \to 0. \end{aligned}$$

For example the projective cover of $S[1]$ is $P[1]$, giving an exact sequence

$$0 \to \Omega^1 S[1] \to P[1] \to S[1] \to 0$$

which is

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & K & \xrightarrow{1} & K & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow 0 \\
 & \uparrow & & \downarrow{\scriptstyle 1} & & \uparrow & & \downarrow{\scriptstyle 1} & & \uparrow & & \downarrow & \\
 & 0 & \longleftarrow & K & & 0 & \longleftarrow & K & & 0 & \longleftarrow & 0 &
\end{array}$$

and the projective cover of $\Omega^1 S[1]$ is $P[2]$, giving an exact sequence

$$0 \to \Omega^2 S[1] \to P[2] \to \Omega^1 S[1] \to 0$$

which is

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow 0 \\
 & \uparrow & & \downarrow & & \uparrow & & \downarrow{\scriptstyle 1} & & \uparrow & & \downarrow{\scriptstyle 1} & \\
 & K & \longleftarrow & 0 & & K & \xleftarrow[1]{} & K & & 0 & \longleftarrow & K &
\end{array}$$

so $\Omega^2 S[1] \cong S[3]$, etc.

Lemma (1)**.** *$\dim \operatorname{Ext}^k(S[i], M)$ is equal to $\dim D_i$ times the multiplicity of $I[i]$ as a summand of I^k in the minimal injective resolution of M.*

$\dim \operatorname{Ext}^k(M, S[j])$ is equal to $\dim D_j$ times the multiplicity of $P[j]$ as a summand of P_k in the minimal projective resolution of M.

Proof. Let $0 \to M \to I^0 \to I^1 \to \dots$ be the minimal injective resolution of M. Recall that $\operatorname{Ext}^k(S[i], M)$ is the kth cohomology of the complex

$$0 \to \operatorname{Hom}(S[i], I^0) \to \operatorname{Hom}(S[i], I^1) \to \dots$$

Now the differential in this complex is zero, for a homomorphism $S[i] \to I^n$ has image contained in $\operatorname{soc} I^n$. The map $\Omega^n M \to I^n$ is an injective envelope, so $\operatorname{soc} I^n$ is contained in the image of this map, so it is killed by the map $I^n \to I^{n+1}$, and hence the composition $S[i] \to I^n \to I^{n+1}$ is zero.

Thus $\operatorname{Ext}^k(S[i], M) \cong \operatorname{Hom}(S[i], I^k)$, and the dimension of this is $\dim D_i$ times the multiplicity of $I[i]$ as a summand of I^k. □

Lemma (2)**.** *If $R = KQ/I$ with I admissible, then the number of arrows from i to j is $\dim \operatorname{Ext}^1(S[i], S[j])$.*

Proof. Since I is admissible, $I \subseteq (KQ)_+^2$. Now $P[i] = (KQ/I)e_i$, so $\operatorname{rad} P[i] = ((KQ)_+/I)e_i$, and $\operatorname{rad}\operatorname{rad} P[i] = ((KQ)_+^2/I)e_i$. Thus

$$\operatorname{top}\operatorname{rad} P[i] = \frac{\operatorname{rad} P[i]}{\operatorname{rad}\operatorname{rad} P[i]} \cong ((KQ)_+/(KQ)_+^2))e_i \cong \bigoplus_j S[j]^{n_{ij}}$$

where n_{ij} is the number of arrows from i to j. Then in the minimal projective resolution of $S[i]$,

$$\cdots \to P_1 \to P[i] \to S[i] \to 0$$

P_1 is the projective cover of $\operatorname{rad} P[i]$, so also of the top of $\operatorname{rad} P[i]$, so the multiplicity of $P[j]$ is n_{ij}. Thus $\dim \operatorname{Ext}^1(S[i], S[j]) = n_{ij}$. □

Recall that a module X has *projective dimension* $\leq n$ if it has a terminating projective resolution of the form

$$0 \to P_n \to \cdots \to P_0 \to X \to 0.$$

Since this resolution can be used to compute Ext groups, it implies that $\operatorname{Ext}^j(X, Y) = 0$ for all $j > n$ and all modules Y, and it is in fact equivalent to this, for if you take any projective resolution of X and break it into short exact sequences

$$0 \to \Omega_{i+1}X \to P_i \to \Omega_i X \to 0$$

with $\Omega_0 X = X$, then applying $\operatorname{Hom}(-, Y)$, the long exact sequence gives for $j > 0$ an exact sequence

$$\operatorname{Ext}^j(P_i, Y) \to \operatorname{Ext}^j(\Omega_{i+1}X, Y) \to \operatorname{Ext}^{j+1}(\Omega_i X, Y) \to \operatorname{Ext}^{j+1}(P_i, Y)$$

and the outer terms here are zero since P_i is projective and $j > 0$, so $\operatorname{Ext}^j(\Omega_{i+1}X, Y) \cong \operatorname{Ext}^{j+1}(\Omega_i X, Y)$ (*dimension shifting*). Thus we get

$$\operatorname{Ext}^1(\Omega_n X, Y) \cong \operatorname{Ext}^2(\Omega_{n-1}X, Y) \cong \ldots \cong \operatorname{Ext}^{n+1}(\Omega_0 X, Y) = \operatorname{Ext}^{n+1}(X, Y) = 0$$

Thus $\Omega_n X$ is projective, and so X has a terminating projective resolution

$$0 \to \Omega_n X \to P_{n-1} \cdots \to P_0 \to X \to 0.$$

Dually, a module Y has *injective dimension* $\leq n$ if it has a terminating injective resolution

$$0 \to Y \to I^0 \to \cdots \to I^n \to 0.$$

Lemma (3)**.** *The following are equivalent for a module M and $n \geq 0$.*
(i) $\operatorname{proj.dim} M \leq n$.
(ii) $\operatorname{Ext}^{n+1}(M, S) = 0$ *for all simples S.*
(iii) the minimal projective resolution of M has $P_{n+1} = 0$.
Similarly for the injective dimension.

Proof. (iii) implies (i) implies (ii) are clear.
(ii) implies (iii). Use Lemma (1) above. □

Recall that the (left) *global dimension* of a ring R is the supremum of the projective dimensions of its (left) modules.

Proposition. *The global dimension of a f.d. algebra is the maximum of the projective dimensions of its simple modules.*

Proof. If every simple S has a projective resolution of length $\leq n$, then every semisimple module has a projective resolution of length $\leq n$, so every semisimple module has projective dimension $\leq n$.

Now if $0 \to X' \to X \to X'' \to 0$ is exact, then

$$\operatorname{proj.dim} X \leq \max\{\operatorname{proj.dim} X', \operatorname{proj.dim} X''\}$$

since applying $\operatorname{Hom}(-, Y)$ to this short exact sequence gives a long exact sequence

$$\dots \to \operatorname{Ext}^{n+1}(X'', Y) \to \operatorname{Ext}^{n+1}(X, Y) \to \operatorname{Ext}^{n+1}(X', Y) \to \dots$$

so if the outer terms vanish for all Y, so does the middle term.

Now since R is finite-dimensional, every module X has a filtration

$$X \supseteq J(R)X \supseteq J(R)^2 X \supseteq \dots \supseteq J(R)^N X = 0$$

in which the successive quotients are semisimple. By induction on the length of a filtration with semisimple quotients, we deduce that $\operatorname{proj.dim} X \leq n$. Thus $\operatorname{gl.dim} R \leq n$. □

Corollary. *For a f.d. algebra, the left and right global dimensions are the same.*

Proof. Suppose the right global dimension is $\leq n$. Take a simple left module S and its minimal projective resolution

$$\dots \to P_1 \to P_0 \to S \to 0$$

Dualizing it gives an injective resolution of the simple right module DS

$$0 \to DS \to DP_0 \to DP_1 \to \dots$$

Now this is a minimal injective resolution, and $\operatorname{inj.dim} DS \leq n$, so by Lemma (3), $DP_{n+1} = 0$. Thus $P_{n+1} = 0$, so $\operatorname{proj.dim} S \leq n$. Thus the left global dimension is $\leq n$. Now we get the reverse inequality by considering R^{op}. □

A *hereditary* algebra is one with global dimension ≤ 1. Any path algebra over a field is hereditary - see §4.5 of my lecture notes on homological algebra. Here we do it for quivers without oriented cycles.

Theorem. *If Q is a quiver without oriented cycles, then KQ is hereditary. Any f.d. hereditary algebra which is split and basic arises this way.*

Proof. If i is a vertex, $\operatorname{rad} P[i]$ has basis the nontrivial paths with tail at i. Each such path is of the form pa for some arrow a with tail at i and some path p with tail at $h(a)$. These paths give a basis of $P[h(a)]$. This gives an isomorphism

$$\operatorname{rad} P[i] \cong \bigoplus_{\substack{a \in Q_1 \\ t(a)=i}} P[h(a)]$$

so $S[i]$ has projective dimension ≤ 1. For the converse, the algebra can be given as $R = KQ/I$ with I admissible. Consider the exact sequence of KQ-modules

$$0 \to I/(I.KQ_+) \to KQ_+/(I.KQ_+) \to KQ_+/I \to 0.$$

The middle module is annihilated by I, so this is a sequence of R-modules. The RH module is a submodule of $R = KQ/I$, so it is projective as an R-module. Thus the sequence splits. Letting

$$M = KQ_+/(I.KQ_+), \quad N = I/(I.KQ_+) \oplus KQ_+/I.$$

we deduce that $M \cong N$. Thus $M/(KQ_+)M \cong N/(KQ_+)N$, which gives

$$KQ_+/KQ_+^2 \cong (I/(KQ_+.I + I.KQ_+)) \oplus (KQ_+/KQ_+^2).$$

Thus by dimensions, $I = KQ_+.I + I.KQ_+$. Now by admissibility $I \subseteq KQ_+^2$. Then assuming that $I \subseteq KQ_+^k$ we get

$$I = KQ_+.I + I.KQ_+ \subseteq KQ_+^{k+1}.$$

Thus $I \subseteq KQ_+^k$ for all k. But if $I \neq 0$, then it contains a nonzero element x, and this involves a path of some length d, and then $x \notin KQ_+^{d+1}$. □

1.11 Projective-injective modules and uniserial modules

Modules which are both projective and injective can be useful. Any indecomposable projective-injective has simple top and simple socle.

Lemma (1). *Let R be a f.d. algebra and let P be a left ideal which is a direct summand of R, hence projective, and suppose that P is also injective. Let $S = \operatorname{soc} P$ and let $I = SR$ be the ideal generated by S. If M is an indecomposable R-module, then either M is isomorphic to a direct summand of P or $IM = 0$, so that M is an R/I-module.*

Proof. Suppose $IM \neq 0$. Then $SM \neq 0$. Thus there is some $m \in M$ with $Sm \neq 0$. Thus the homomorphism $\theta : R \to M$ given by $\theta(r) = rm$ has $\theta(S) \neq 0$. Now P is a direct sum of some modules $I[i]$, so S is the corresponding direct sum of the $S[i]$. Thus some $\theta(S[i]) \neq 0$ for some i. Thus the restriction of θ to $I[i]$ is injective. Thus $I[i]$ embeds in M. But by injectivity its image must be a direct summand of M. Thus $M \cong I[i]$ by indecomposability. □

Example. The commutative square algebra R with source 1 and sink 4 has $P[1] \cong I[4]$. But the other indecomposable projectives are not injective. By the lemma, any indecomposable R-module is either isomorphic to $P[1]$, or is a module for the algebra given by the square with two zero relations.

End of lecture on 2025-05-19. Provisional script for the next lecture follows (subject to change).

Definition. We define the following classes of f.d. algebras with the obvious implications. They are all left-right symmetric.

$$R \text{ symmetric} \Rightarrow R \text{ Frobenius} \Rightarrow R \text{ self-injective} \Rightarrow R \text{ QF-3}$$

(i) R is *symmetric* if ${}_RR_R \cong {}_RDR_R$. Equivalently if there is a bilinear form $(-,-) : R \times R \to K$ which is
- non-degenerate: $(a,b) = 0 \forall b \Rightarrow a = 0$, $(a,b) = 0 \forall a \Rightarrow b = 0$,
- associative: $(ab,c) = (a,bc)$, and
- symmetric: $(a,b) = (b,a)$.

The corresponding map $R \to DR$ is $a \mapsto (a,-)$. It follows that $I[i] = \nu(P[i]) = DR \otimes_R P[i] \cong R \otimes_R P[i] \cong P[i]$.

(ii) R is *Frobenius* if ${}_RR \cong {}_RDR$. Equivalently if there is a bilinear form which is non-degenerate and associative.

(iii) R is *self-injective* (or *quasi-Frobenius*) if ${}_RR$ is an injective module. Equivalently the modules $P[i]$ and $I[j]$ are the same, up to a permutation. It is equivalent that a module is projective if and only if it is injective.

(iv) R is *QF-3* (in the sense of Thrall) if R has a faithful f.d. projective-injective module.

Recall that a module M is *faithful* if $r \in R$ and $rm = 0$ for all $m \in M$, then $r = 0$, that is, if the map $R \to \mathrm{End}_K(M)$ is injective.

Examples. (1) The group algebra KG of a finite group is symmetric with $(a,b) = \lambda_1$ where $ab = \sum_{g \in G} \lambda_g g$.

(2) If Q is the cyclic quiver with n vertices, then $KQ/(KQ_+)^{k+1}$ is Frobenius, and it is symmetric $\Leftrightarrow n|k$. The bilinear form (a,b) is the sum of the coefficients of the paths of length k in ab.

(3) The commutative square algebra with source 1 and sink 4 is QF-3 because any indecomposable projective has socle $S[4]$, so embeds in $I[4] \cong P[1]$.

(4) For a commutative algebra the concepts are the same [Namely (ii)⇒(i) since $(a, b) = (1a, b) = (1, ab) = (1, ba) = (b, a)$, (iii)⇒(ii) since the algebra is basic, and (iv)⇒(iii) since if there is a faithful projective-injective module, there is one of the form Re for an idempotent e. But then commutativity gives $(1 - e)Re = 0$, contradicting faithfulness unless $e = 1$.] Commutative Frobenius algebras appear in topological quantum field theory.

Lemma (2)**.** *(i) A f.d. R-module M is faithful if and only if there is an embedding $R \to M^n$ for some n.*

(ii) If M is a f.d. faithful projective-injective module, then every indecomposable projective-injective module I must be a summand of M.

(iii) If R is QF-3, then the injective envelope $E(R)$ of R is a faithful projective-injective module.

Proof. (i) If $R \hookrightarrow M^n$, $r \in R$ and $rm = 0$ for all $m \in M$, then $rx = 0$ for all $x \in M^n$, so $r1 = 0$ for $1 \in R$. Thus $r = 0$.

Conversely, if M is faithful, choose a basis $m_1, \dots, m_n$ of M. This gives a map $R \to M^n$, $r \mapsto (rm_1, \dots, rm_n)$. If $r \mapsto 0$, then $rm_i = 0$ for all i, so $rm = 0$ for all $m \in M$.

(ii) I embeds in M^n, so by injectivity it is a direct summand.

(iii) There is a faithful projective-injective module M, and R embeds in M^n. By the injective property, we get a morphism $\theta : E(R) \to M^n$ such that the composition $R \to E(R) \to M^n$ is injective. But then $R \cap \operatorname{Ker}\theta = 0$, so $\operatorname{Ker}\theta = 0$, so $E(R)$ embeds in M^n, so it is a direct summand, so $E(R)$ is projective. □

Definition. A module M is *uniserial* if its submodules are totally ordered by inclusion, that is, if $N, N' \subseteq M$, then either $N \subseteq N'$ or $N' \subseteq N$. Since we are only considering f.d. modules, it is equivalent that M has a unique composition series.

Example. If S and T are simple modules and $0 \to S \to M \to T \to 0$ is non-split, then M is uniserial. (If L is a submodule with $L \neq 0, S, M$, then $L + S = M$, and $L \cap S = 0$, so the sequence splits.)

Lemma (3)**.** *Let M be a f.d. R-module.*

(i) If M is uniserial, it is indecomposable, has simple top and socle, and only finitely many submodules. Moreover any submodule or quotient of M is uniserial.

(ii) M is a uniserial R-module if and only if $D(M)$ is a uniserial R^{op}-module.

(iii) M is uniserial if and only if the chain

$$M \supseteq \operatorname{rad} M \supseteq \operatorname{rad}^2 M \supseteq \cdots \supseteq \operatorname{rad}^{n-1} M \supseteq \operatorname{rad}^n M = 0$$

is a composition series for some n.

Proof. (i) and (ii) are trivial. For (iii) It suffices to show that if the chain is a composition series, then every submodule L of M is equal to $\operatorname{rad}^i M$, some i. Let i be maximal with $L \subseteq \operatorname{rad}^i M$ If $i = n$ then $L = 0$, otherwise $\operatorname{rad}^i M / \operatorname{rad}^{i+1} M$ is simple, so $\operatorname{rad}^{i+1} M$ is the unique maximal submodule of $\operatorname{rad}^i M$. Since L is not contained in $\operatorname{rad}^{i+1} M$, we must have $L = \operatorname{rad}^i M$. □

Definition. A f.d. algebra R is a *Nakayama algebra* if the indecomposable projective left and right R-modules are uniserial. It is equivalent that the indecomposable projective left modules and the indecomposable injective left modules are all uniserial.

Proposition (1). *If $R = KQ/I$ with Q connected and I admissible, then R is Nakayama if and only if Q is a linear or cyclic quiver.*

Proof. If the quiver is linear or cyclic, then for each vertex i there is a unique maximal path $a_n \dots a_1$ with tail at i and not in I. Then $\operatorname{rad}^j P[i]$ is spanned by the paths $a_k \dots a_1$ with $k \geq j$. Thus the radical series is a composition series. Thus $P[i]$ is uniserial. Similarly for the indecomposable projective right modules.

Conversely, if two arrows a, b have tail at i then the submodules Ra and Rb of $Re_i = P[i]$ are incomparable, for if $Ra \subseteq Rb$, then there is $x \in KQ$ with $a - xb \in I \subseteq (KQ_+)^2$, which is impossible. Similarly for right modules if two arrows have tail at i. □

Proposition (2). *For a f.d. algebra R we have the following.*

(i) If R is Nakayama, then R/I is Nakayama for any ideal I.

(ii) If R is Nakayama, then R is QF-3.

(iii) If $R/J(R)^2$ is QF-3, then R Nakayama.

Thus, for example, R is Nakayama if and only if R/I is QF-3 for all I.

Proof. (i) Write $R = \bigoplus P_i$ with P_i indecomposable projective. Then $R/I = \bigoplus P_i/IP_i$, a direct sum of uniserial modules, so the indecomposable projective left R/I-modules are uniserial. Similarly for right modules.

(ii) It suffices to show that if P is indecomposable projective, then so is its injective envelope $E(P)$. Since P has simple socle, so does $E(P)$. Thus it is indecomposable. Thus it is uniserial, so it has simple top. If $\theta : P' \to E(P)$ is its projective cover, then P' is indecomposable. This gives an exact sequence $0 \to \operatorname{Ker}\theta \to \theta^{-1}(P) \to P \to 0$. Now $\theta^{-1}(P)$ is uniserial, so indecomposable, but this sequence splits, so we must have $\operatorname{Ker}\theta = 0$.

(iii) Let $J = J(R)$.

First suppose $J^2 = 0$. We show that any indecomposable projective left R-module P is uniserial. Now JP is semisimple, so we need to show it is zero or simple. By the QF-3 property, $E(P)$ is projective. If $P \subseteq JE(P)$, then $JP = 0$. Thus suppose $P \not\subseteq JE(P)$. We decompose $E(P)$ into indecomposables, $E(P) =$

$\bigoplus P_i$. Then one of the maps $\operatorname{top} P \to \operatorname{top} P_i$ is an isomorphism, so $P \to P_i$ is an isomorphism, so P is injective, so $E(P) = P$. Then JP is semisimple, but P has simple socle, so JP is simple or zero.

Now we show by induction that any indecomposable projective P for R/J^n is uniserial for $n \geq 2$. For $n = 2$ this is done. Suppose $n > 2$. Then P/J^2P is projective for R/J^2, and it has simple top, so it is indecomposable, so JP/J^2P is zero or simple. Thus JP is a module for R/J^{n-1} which is zero or has simple top, so by induction it is uniserial. Thus P is uniserial.

Thus indecomposable projective left R-modules are uniserial. Similarly we have it for right modules. Thus R is Nakayama. □

Theorem. *Any indecomposable module for a Nakayama algebra is uniserial. Thus any indecomposable module is a quotient of an indecomposable projective, so there are only finitely many indecomposable modules - Nakayama algebras have finite representation type.*

Proof. We prove this for Nakayama algebras R by induction on $\dim R$. Now R has an indecomposable projective-injective module P. We can embed it as an ideal in R. Let $I = SR$, the ideal generated by $S = \operatorname{soc} P$. Then any indecomposable module for R is either isomorphic to P, so uniserial, or an indecomposable module for R/I, so uniserial by induction. □

A f.d. representation of a quiver is *nilpotent* if there is some m such that any path of length $\geq m$ is zero in the representation. For a quiver without oriented cycles all representations are nilpotent. If I is an admissible ideal then any KQ/I-module corresponds to a nilpotent representation of Q.

Corollary. *(i) Any f.d. indecomposable nilpotent representation M of a linear or cyclic quiver Q is isomorphic to $(KQ/KQ_+^m)e_i$ for some vertex i and some m.*

(ii) Any f.d. indecomposable representation of a cyclic quiver is either nilpotent or isomorphic to one of the form

$$V \xrightarrow{1} V \xrightarrow{1} \dots \xrightarrow{1} V \xrightarrow{x} V \quad \textit{(the two ends identified)}$$

where $V = K[x]/(f(x)^n)$ with $f(x)$ a monic irreducible polynomial $\neq x$ in $K[x]$. In particular if K is algebraically closed, $f(x) = x - \lambda$, then $V \cong K^n$ and x corresponds to the Jordan block $J_n(\lambda)$.

Proof. (i) M is a module for $KQ/(KQ_+)^k$ for some k, which is Nakayama.

(ii) Let Q be cyclic with N vertices. Let $T \in KQ$ be the sum of all paths of length N. Then T is a central element of KQ, so it induces an element of $\operatorname{End}_{KQ}(M)$. By Fitting's Lemma, this element must be nilpotent or invertible. If nilpotent, then M is nilpotent. If invertible, then all paths of length N in M must be invertible. Thus all arrows in M must be invertible. Thus M is of the indicated form for some for some $K[x]$-module V on which x acts invertibly. Now V must be indecomposable, so it has the stated form. $\square$