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Masters course: Homological algebra

Homological algebra is the algebra that was invented in order to define and study the homology and cohomology of topological spaces, but it has applications all over mathematics.

My aim is to cover the properties of projective, injective and flat modules, complexes of modules and Ext and Tor groups, homological dimensions, homology and cohomology of groups, and more abstractly, abelian and triangulated categories.

Students are expected to already have some familiarity with rings and modules.

Some suggested books:

- C. A. Weibel, An introduction to homological algebra, CUP 1994.
- J. J. Rotman, An introduction to homological algebra, Springer 2009.
- M. S. Osborne, Basic homological algebra, Springer 2000.
- S. I. Gelfand and Yu. I. Manin, Methods of homological algebra, 2nd ed., Springer 2010.
- H. Krause, Homological theory of representations, CUP 2022.
- The Stacks Project, https://stacks.math.columbia.edu/

Contents

More coming...

1 Abelian categories

The basic setting setting for homological algebra, for example used in the book by Henri Cartan and Samuel Eilenberg, 'Homological algebra', 1956, is complexes of additive groups, or more generally complexes of modules for a ring R.

Algebraic geometers also want to work with complexes of sheaves on an algebraic variety, and in his paper 'Sur quelques points d'algèbre homologique', Tohoku Math. J. 9 (1957), 119–221, Alexander Grothendieck showed that you can unify the two settings by working with abelian categories.

Although we won't work with sheaves, it is good to start with abelian categories: modern homological algebra uses triangulated categories and other concepts, and abelian categories are a necessary preparation.

We begin with the language of categories, although students may have seen this already.

1.1 Categories and functors

Definition. A *category* $\mathcal{C}$ consists of

(i) a collection $\mathrm{ob}(\mathcal{C})$ of *objects*

(ii) For any $X, Y \in \mathrm{ob}(\mathcal{C})$, a set $\mathrm{Hom}(X, Y)$ (also denoted $\mathcal{C}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$) of *morphisms* (or *homomorphisms*) $\theta : X \to Y$, and

(iii) For any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(\theta, \phi) \mapsto \theta\phi$.

satisfying

(a) Associativity: $(\theta\phi)\psi = \theta(\phi\psi)$ for $X \xrightarrow{\psi} Y \xrightarrow{\phi} Z \xrightarrow{\theta} W$, and

(b) For each object X there is an identity morphism $\mathrm{Id}_X \in \mathrm{Hom}(X, X)$, with $\mathrm{Id}_Y\theta = \theta = \theta\mathrm{Id}_X$ for all $\theta : X \to Y$.

An *inverse* of a morphism $\theta : X \to Y$ is a morphism $\phi : Y \to X$ with $\theta\phi = \mathrm{Id}_Y$ and $\phi\theta = \mathrm{Id}_X$. If θ has an inverse, it is unique, and denoted θ^{-1}. An *isomorphism* is a morphism with an inverse.

Examples. (1) The categories of Sets, Groups, Rings, etc. The category R-Mod of (left) R-modules for a ring R. These are *concrete categories*: the objects are sets, possibly with extra structure, and the morphisms are maps of sets preserving the extra structure. (Note that I use the words 'morphism' and 'homomorphism' interchangeably, but perhaps homomorphism should only be used for concrete categories.)

(2) If $\mathcal{C}$ is a category, the *opposite category* C^{op} is given by $\mathrm{ob}(\mathcal{C}^{op}) = \mathrm{ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$, with composition derived from that in $\mathcal{C}$.

(3) If $\mathcal{C}$ and $\mathcal{D}$ are categories, the product $\mathcal{C} \times \mathcal{D}$ is the category with $\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$ and $\mathrm{Hom}((X,U),(Y,V)) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{D}}(U,V)$. Note that any morphism (θ, ϕ) can be written as a composition $(\theta, \mathrm{Id}_V)(\mathrm{Id}_X, \phi)$ or $(\mathrm{Id}_Y, \phi)(\theta, \mathrm{Id}_U)$.

(4) Given a group G or a ring R, there is a category with one object $*$, $\mathrm{Hom}(*,*) = G$ or R and composition given by multiplication.

(5) A partially ordered set $(S, \leq)$ gives a category with objects $s \in S$ and

$$\mathrm{Hom}(s,t) = \begin{cases} i_{st} & (s \leq t) \\ \emptyset & (s \not\leq t). \end{cases}$$

The composition must be given by $i_{tu} i_{st} = i_{su}$ for $s \leq t \leq u$, so $\mathrm{Id}_s = i_{ss}$.

(6) A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows and mappings $s, t : Q_1 \to Q_0$ giving the source and target of each arrow, so $s(a) \xrightarrow{a} t(a)$. It is like a category without a composition. The *path category* of a quiver has objects the vertices, and the morphisms $i \to j$ are the *paths* $a_n \dots a_2 a_1$ given by sequences of arrows

$$i = i_0 \xrightarrow{a_1} i_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} i_n = j.$$

There is also a trivial path Id_i for each vertex i. Composition is given by concatenation. For example the category given by the poset $(\mathbb{N}, \leq)$ is isomorphic to the path category of the quiver $0 \to 1 \to 2 \to \dots$.

Definition. Because of Russell's paradox, there is no set of all sets. One solution is to allow normal sets and 'big sets' called classes. There is a class of all sets.

- *Normal category:* $\mathrm{ob}(\mathcal{C})$ is a class, $\mathrm{Hom}(X,Y)$ are sets. For example the concrete categories above.
- *BIG category:* $\mathrm{ob}(\mathcal{C})$ is a class, $\mathrm{Hom}(X,Y)$ are classes. We only rarely need this.
- *Small category:* $\mathrm{ob}(\mathcal{C})$ is a set, $\mathrm{Hom}(X,Y)$ are sets. For example the category given by a partially ordered set.
- *Skeletally small category:* A normal category, such that there is a set S of objects such that every object is isomorphic to one in S.

Definition. A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ such that

- $\mathrm{ob}(\mathcal{D})$ is a subclass of $\mathrm{ob}(\mathcal{C})$.
- $\mathrm{Hom}_{\mathcal{D}}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \mathrm{ob}(\mathcal{D})$.
- $\mathrm{Id}_X^{\mathcal{C}} \in \mathrm{Hom}_{\mathcal{D}}(X,X)$ for all $X \in \mathrm{ob}(\mathcal{D})$.
- Composition in $\mathcal{D}$ is the same as composition in $\mathcal{C}$.

It is a *full subcategory* if $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \mathrm{ob}(\mathcal{D})$. Thus a full subcategory of $\mathcal{C}$ is determined by a subclass $\mathrm{ob}(\mathcal{D})$ of $\mathrm{ob}(\mathcal{C})$.

Examples. (a) The category Ab of abelian groups is a full subcategory of the category of all groups.

(b) The category R-mod of finitely generated R-modules is a full subcategory of R-Mod. It is skeletally small, with $S = \{R^n/U : n \in \mathbb{N}, U \subseteq R^n\}$.

(c) The category whose objects are sets and with $\mathrm{Hom}(X,Y) =$ the injective functions $X \to Y$ is a subcategory of the category of sets.

Definition. A *monomorphism* in a category is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Z \to X$, if $\theta\alpha = \theta\beta$ then $\alpha = \beta$.

An *epimorphism* is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Y \to Z$, if $\alpha\theta = \beta\theta$ then $\alpha = \beta$.

Examples. (1) In the categories of sets or of R-modules, monomorphism = injective map, epimorphism = surjective map. For example we show epi = surjection for modules. Say $\theta : X \to Y$ is surjective and $\alpha\theta = \beta\theta$. Since θ is surjective, for all $y \in Y$ there is $x \in X$ with $\theta(x) = y$. Then $\alpha(y) = \alpha(\theta(x)) = \beta(\theta(x)) = \beta(y)$. Thus $\alpha = \beta$. Say $\theta : X \to Y$ is an epimorphism. The natural map $Y \to Y/\operatorname{Im}\theta$ and the zero map have the same composition with θ, so they are equal. Thus $\operatorname{Im}\theta = Y$.

(2) In the category of rings, the inclusion map $\theta : \mathbb{Z} \to \mathbb{Q}$ is not surjective, but it is an epimorphism.

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is given by

- For each object $X \in \mathrm{ob}(\mathcal{C})$, an object $F(X) \in \mathrm{ob}(\mathcal{D})$, and
- For each morphism $\theta : X \to Y$ in $\mathcal{C}$, a morphism $F(\theta) : F(X) \to F(Y)$ in $\mathcal{D}$

such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for all $X \in \mathrm{ob}(\mathcal{C})$ and $F(\theta\phi) = F(\theta)F(\phi)$ for composable morphisms $X \xrightarrow{\phi} Y \xrightarrow{\theta} Z$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the same thing as a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$. Thus it is an assignment of

- For each object $X \in \mathrm{ob}(\mathcal{C})$, an object $F(X) \in \mathrm{ob}(\mathcal{D})$, and
- For each morphism $\theta : X \to Y$ in $\mathcal{C}$ a morphism $F(\theta) : F(Y) \to F(X)$ in $\mathcal{D}$,

such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ and $F(\theta\phi) = F(\phi)F(\theta)$ for composable morphisms $X \xrightarrow{\phi} Y \xrightarrow{\theta} Z$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- *faithful* if the map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective for all $X, Y \in \mathrm{ob}(\mathcal{C})$,
- *full* if the map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective for all $X, Y \in \mathrm{ob}(\mathcal{C})$,
- *fully faithful* if it is full and faithful.
- *dense* or *essentially surjective* if every object in $\mathcal{D}$ is isomorphic to $F(X)$ for some object X in $\mathcal{C}$.
- An *equivalence* if it is fully faithful and dense.
- An *isomorphism* if it has an inverse, or equivalently if it is fully faithful and a bijection on objects. Thus every isomorphism is an equivalence.

Examples. (1) The inclusion functor of a subcategory, for example Ab to *Group*, is always faithful. It is full if and only if the subcategory is full.

(2) A composition of functors is a functor. (Thus there is a category of small categories.)

(3) There are many examples of *forgetful functors* for concrete categories, which forget some structure. For example *Group* $\to$ *Set*, or R-Mod $\to$ Ab. They are faithful.

(4) Given a ring homomorphism $\theta : R \to S$, restriction defines a faithful functor S-Mod $\to$ R-Mod. [It is full if and only if θ is a epimorphism in the category of rings, but that is another story.]

(5) If K is a field, then duality $V \rightsquigarrow V^* = \mathrm{Hom}_K(V, K)$ gives a contravariant functor $* : K\text{-Mod} \to K\text{-Mod}$, sending a morphism $\theta : V \to W$ to $\theta^* : W^* \to V^*$ given by $\theta^*(\xi)(v) = \xi(\theta(v))$.

Thus it gives a covariant functor $(K\text{-Mod})^{op} \to K\text{-Mod}$.

Considering only finite-dimensional vector spaces, this gives a functor $(K\text{-mod})^{op} \to K\text{-mod}$. This is an equivalence since every n-dimensional vector space is isomorphic to V^* with V an n-dimensional vector space, and the map $* : \mathrm{Hom}(V, W) \to \mathrm{Hom}(W^*, V^*)$ is injective, so an isomorphism by dimensions.

(6) Let $\mathcal{C}$ be a category and let $\mathrm{Hom}(X,Y)$ denote the Hom sets for $\mathcal{C}$. Fix an object $X \in \mathrm{ob}(\mathcal{C})$. There is a functor $\mathrm{Hom}(X,-) : \mathcal{C} \to Set$ sending an object Y to $\mathrm{Hom}(X,Y)$, and sending a morphism $\theta \in \mathrm{Hom}(Y,Z)$ to the mapping $\mathrm{Hom}(X,\theta) : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$ defined by $\mathrm{Hom}(X,\theta)(\phi) = \theta\phi$.

Dually, fixing Y, we get a contravariant functor $\mathrm{Hom}(-,Y) : \mathcal{C} \to Set$.

Varying both X and Y, we get a functor $\mathrm{Hom}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to Set$.

1.2 Natural transformations and functor categories

Definition. Let F, G be functors $\mathcal{C} \to \mathcal{D}$. A *natural transformation* $\alpha : F \to G$ is given by morphisms $\alpha_X : F(X) \to G(X)$ for all $X \in \mathrm{ob}(\mathcal{C})$ such that $G(\theta)\alpha_X = \alpha_Y F(\theta)$ for every morphism $\theta : X \to Y$ in $\mathcal{C}$.

It is a *natural isomorphism* if all α_X are isomorphisms in $\mathcal{D}$.

Example. If K is a field and X is a K-vector space, there is a map $\alpha_X : X \to X^{**}$ given by $\alpha_X(x)(\phi) = \phi(x)$ for $\phi \in X^*$. Together, these give a natural transformation $\alpha : \mathrm{Id} \to (-)^{**}$ of functors from K-Mod to K-Mod. Namely, if $\theta : X \to Y$ is a linear map, then $\theta^* : Y^* \to X^*$ is given by $\theta^*(\psi) = \psi\theta$ for $\psi \in Y^*$, and $\theta^{**} : X^{**} \to Y^{**}$ is given by $\theta^{**}(\xi) = \xi\theta^*$ for $\xi \in X^{**}$. Then

$$\theta^{**}(\alpha_X(x))(\psi) = \alpha_X(x)(\theta^*(\psi)) = \alpha_X(x)(\psi\theta) = (\psi\circ\theta)(x) = \psi(\theta(x)) = \alpha_Y(\theta(x))(\psi),$$

so $\theta^{**}\alpha_X = \alpha_Y\theta$. If we used K-mod, then α would be a natural isomorphism.

Definition. The *functor category* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ has objects the functors $F : \mathcal{C} \to \mathcal{D}$. The morphisms are the natural transformations. The composition $\beta\alpha$ of natural transformations $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ is given by $(\beta\alpha)_X = \beta_X\alpha_X$. The identity natural transformation $\mathrm{Id}_F : F \to F$ is given by $(\mathrm{Id}_F)_X = \mathrm{Id}_{F(X)}$.

Remarks. (1) In general this is a BIG category. To get a normal category, we can take $\mathcal{C}$ small, or more generally skeletally small. We need to check that the collection of natural transformations $F \to G$ is a set. Every object in $\mathcal{C}$ is isomorphic to an object in a set S. A natural transformation $\alpha : F \to G$ is determined by the morphisms α_X for $X \in S$, for if $\theta : Y \to X$ is an isomorphism, then $\alpha_Y = G(\theta^{-1})\alpha_X F(\theta)$.

(2) The natural isomorphisms $F \to G$ are the isomorphisms in the functor category, e.g. if α is a natural isomorphism, it has inverse α^{-1} defined by $(\alpha^{-1})_X = (\alpha_X)^{-1}$.

Lemma (Yoneda's Lemma). *For a functor $F : \mathcal{C} \to Set$ and $X \in \mathrm{ob}(\mathcal{C})$, there is a 1-1 correspondence between natural transformations $\alpha : \mathrm{Hom}(X,-) \to F$ and elements $f \in F(X)$.*

Proof. A natural transformation α gives a mapping $\alpha_X : \mathrm{Hom}(X, X) \to F(X)$, and hence an element $f = \alpha_X(\mathrm{Id}_X) \in F(X)$. Conversely, given $f \in F(X)$ and $Y \in \mathrm{ob}(\mathcal{C})$ we get a morphism $\alpha_Y : \mathrm{Hom}(X, Y) \to F(Y)$, $\theta \mapsto F(\theta)(f)$. This defines a natural transformation α. These constructions are inverses. $\square$

Remark. In particular, if $F = \mathrm{Hom}(Y, -)$, this shows that we get a bijection

$$\mathrm{Hom}(Y, X) \rightsquigarrow \{\text{natural transformations } \mathrm{Hom}(X, -) \to \mathrm{Hom}(Y, -)\}$$

Thus we get a fully faithful functor $\mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, Set)$.

Also $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(Y, -)$ are naturally isomorphic if and only if X and Y are isomorphic.

Definition. A functor $\mathcal{C} \to Set$ is *representable* if it is naturally isomorphic to a functor of the form $\mathrm{Hom}(X, -)$ for some object X. Thus it is equivalent that there is an object X and $f \in F(X)$ such that for all objects Y, the map $\mathrm{Hom}(X, Y) \to F(Y)$, $\theta \mapsto F(\theta)(f)$ is a bijection.

Dually, a contravariant functor from $\mathcal{C}$ to Set is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}(-, Y)$.

Definition. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say that (F, G) is an *adjoint pair*, or that F is *left adjoint* to G or G is *right adjoint* to F if there is a natural isomorphism $\alpha : \mathrm{Hom}(F(-), -) \to \mathrm{Hom}(-, G(-))$ of functors $\mathcal{C}^{op} \times \mathcal{D} \to Sets$.

Thus one needs bijections

$$\alpha_{X,Y} : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, G(Y))$$

for all $X \in \mathrm{ob}(\mathcal{C})$ and $Y \in \mathrm{ob}(\mathcal{D})$, such that

$$\begin{array}{ccc} \mathrm{Hom}(F(X'), Y) & \xrightarrow{\alpha_{X',Y}} & \mathrm{Hom}(X', G(Y)) \\ {\scriptstyle \cdot F(\theta)}\downarrow & & {\scriptstyle \cdot\theta}\downarrow \\ \mathrm{Hom}(F(X), Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}(X, G(Y)) \end{array}$$

commutes for all $\theta : X \to X'$, and

$$\begin{array}{ccc} \mathrm{Hom}(F(X), Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}(X, G(Y)) \\ {\scriptstyle \phi\cdot}\downarrow & & {\scriptstyle G(\phi)\cdot}\downarrow \\ \mathrm{Hom}(F(X), Y') & \xrightarrow{\alpha_{X,Y'}} & \mathrm{Hom}(X, G(Y')) \end{array}$$

commutes for all $\phi : Y \to Y'$. (Here we have used that every morphism in a product of categories is a composition of morphisms in which one of the components is the identity.)

Examples. (1) Let R be a ring. We have a forgetful functor $Forget_R : R\text{-Mod} \to Sets$. Given a set X, let $Free_R(X)$ be the free left R-module with basis X. Thus

$$Free_R(X) = \{\sum_{x\in X} r_x x : r_x \in R \text{ for } x \in X, \text{ all but finitely many zero}\}.$$

Any mapping $\phi : X \to Y$ gives a module homomorphism $Free_R(X) \to Free_R(Y)$. This gives a functor $Free_R : Sets \to R\text{-Mod}$. For M a left R-module, we have a bijection

$$\alpha_{X,M} : \mathrm{Hom}_R(Free_R(X), M) \to \mathrm{Hom}_{Sets}(X, Forget_R(M))$$

This is natural in both X and M, so it turns $(Free_R, Forget_R)$ into an adjoint pair of functors.

(2) By defining things with morphisms in the natural way, we get adjoint functors $(Path, Forget)$ where $Forget$ is the functor from small categories to quivers which forgets the composition and $Path$ sends a quiver to its path category.

Proposition. *A functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint G if and only if the contravariant functors $\mathrm{Hom}_{\mathcal{D}}(F(-), Y)$ are representable for all $Y \in \mathrm{ob}(\mathcal{D})$. Moreover, if a right adjoint exists, it is unique up to natural isomorphism.*

Dually, $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if $\mathrm{Hom}_{\mathcal{C}}(X, G(-))$ is representable for all $X \in \mathrm{ob}(\mathcal{C})$, and if a left adjoint exists, it is unique up to natural isomorphism.

End of lecture on 2026-04-15. Provisional script for the next lecture follows (subject to change).

Proof. If there is a right adjoint, then $\mathrm{Hom}(F(-), Y) \cong \mathrm{Hom}(-, G(Y))$, so it is representable. Conversely, we show how to define G. For each $Y \in \mathrm{ob}(\mathcal{D})$, we choose an object $G(Y) \in \mathrm{ob}(\mathcal{C})$ and a natural isomorphism $\mathrm{Hom}(F(-), Y) \to \mathrm{Hom}(-, G(Y))$. For each $X \in \mathrm{ob}(\mathcal{C})$ this gives bijections $\alpha_{XY} : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, G(Y))$. Now if $\phi : Y \to Y'$ is a morphism in $\mathcal{D}$, we get maps

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\alpha_{XY}^{-1}} \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\phi\cdot} \mathrm{Hom}_{\mathcal{D}}(F(X), Y') \xrightarrow{\alpha_{X,Y'}} \mathrm{Hom}_{\mathcal{C}}(X, G(Y')).$$

In particular, taking $X = G(Y)$, this composition sends $\mathrm{Id}_{G(Y)}$ to an element of $\mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y'))$. We define $G(\phi)$ to be this element.

If G and G' are right adjoints, then we must have a natural isomorphism

$$\beta : \mathrm{Hom}_{\mathcal{C}}(-, G(-)) \to \mathrm{Hom}_{\mathcal{C}}(-, G'(-)).$$

For any $Y \in \mathrm{ob}(\mathcal{D})$, this gives a bijection

$$\beta_{G(Y),Y} : \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(G(Y), G'(Y)).$$

Let γ_Y be the image of $\mathrm{Id}_{G(Y)}$. It is straightforward to check that this defines a natural isomorphism $\gamma : G \to G'$. □

Remark. If (F, G) is an adjoint pair as in the definition, then there are natural transformations $u : \mathrm{Id}_{\mathcal{C}} \to GF$ and $c : FG \to \mathrm{Id}_{\mathcal{D}}$ called the *unit* and *counit*, defined by $u_X = \alpha_{X,F(X)}(\mathrm{Id}_{F(X)})$ and $\alpha_{G(Y),Y}(c_Y) = \mathrm{Id}_{G(Y)}$.

Theorem. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if there is a functor $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\alpha : FG \cong \mathrm{Id}_{\mathcal{D}}$ and $\beta : GF \cong \mathrm{Id}_{\mathcal{C}}$. In this case (F, G) and (G, F) are both adjoint pairs.*

Proof. If G exists, then the existence of α implies that F is dense. If $\theta : X \to X'$ is a morphism in $\mathcal{C}$, there is a commutative square

$$\begin{array}{ccc} GF(X) & \xrightarrow{\beta_X} & X \\ {\scriptstyle GF(\theta)}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\ GF(X') & \xrightarrow{\beta_{X'}} & X' \end{array}$$

so $\theta = \beta_{X'} GF(\theta) \beta_X^{-1}$. It follows that F is faithful. By symmetry G is also faithful. Now if $\phi : F(X) \to F(X')$ is a morphism, let $\theta = \beta_{X'} G(\phi) \beta_X^{-1}$. Then also $\theta = \beta_{X'} GF(\theta) \beta_X^{-1}$. It follows that $G(\phi) = GF(\theta)$, so $\phi = F(\theta)$ since G is faithful. Thus F is full.

Conversely, suppose that F is an equivalence, so fully faithful and dense. If $Y \in \mathrm{ob}(\mathcal{D})$, then $Y \cong F(X')$ for some X', so

$$\mathrm{Hom}(F(-), Y) \cong \mathrm{Hom}(F(-), F(X')) \cong \mathrm{Hom}(-, X').$$

It follows that F has a right adjoint G. Then

$$\mathrm{Hom}(-, GF(-)) \cong \mathrm{Hom}(F(-), F(-)) \cong \mathrm{Hom}(-, -),$$

which gives a natural isomorphism $GF \cong \mathrm{Id}_{\mathcal{C}}$. Now if $Y, Y' \in \mathrm{ob}(\mathcal{D})$ we can choose X, X' with $Y \cong F(X)$ and $Y' \cong F(X')$. Then

$$\mathrm{Hom}(Y, Y') \cong \mathrm{Hom}(F(X), F(X')) \cong \mathrm{Hom}(X, X')$$

$$\cong \mathrm{Hom}(GF(X), GF(X')) \cong \mathrm{Hom}(G(Y), G(Y'))$$

and one can check that the composition is the map given by G. Thus G is fully faithful. Then

$$\mathrm{Hom}(FG(-),-) \cong \mathrm{Hom}(G(-),G(-)) \cong \mathrm{Hom}(-,-),$$

so $FG \cong \mathrm{Id}_{\mathcal{D}}$, and

$$\mathrm{Hom}(G(-),-) \cong \mathrm{Hom}(FG(-),F(-)) \cong \mathrm{Hom}(-,F(-))$$

so (G,F) is an adjoint pair. □

1.3 Limits and colimits

Definition. Let $\mathcal{C}$ be a category. Let $\mathcal{I}$ be a small category. An *$\mathcal{I}$-diagram* in $\mathcal{C}$ is a functor $M : \mathcal{I} \to \mathcal{C}$. For $i \in \mathrm{ob}(\mathcal{I})$, we write M_i instead of $M(i)$ and for a morphism $a : i \to j$ in $\mathcal{I}$, we write M_a for the morphism $M_i \to M_j$.

By a *cone over* M we mean a collection (X, α_i) consisting of an object $X \in \mathcal{C}$ and morphisms $\alpha_i : X \to M_i$ for all $i \in \mathrm{ob}(\mathcal{I})$ satisfying $M_a \alpha_i = \alpha_j$ for all morphisms $a : i \to j$ in $\mathcal{I}$.

A *limit* for M is a cone (L, λ_i) over M, which is universal in the sense that if (X, α_i) is any cone over M, then there is a unique morphism $\theta : X \to L$ such that $\alpha_i = \lambda_i \theta$ for all $i \in \mathrm{ob}(\mathcal{I})$.

Remarks. (i) If M has a limit, then it is unique up to a unique isomorphism, so we can talk about *the limit*, and denote it

$$L = \lim_{i \in \mathcal{I}} M_i \in \mathrm{ob}(\mathcal{C}).$$

(ii) Suppose $\phi : M \to N$ is a natural transformation between $\mathcal{I}$-diagrams, and suppose that $\lim_{i\in\mathcal{I}} M$ and $\lim_{i\in\mathcal{I}} N_i$ both exist. Then for each i we get a morphism

$$\lim_{i\in\mathcal{I}} M_i \xrightarrow{\lambda_i^M} M_i \xrightarrow{\phi_i} N_i.$$

These turn $\lim_{i\in\mathcal{I}} M_i$ into a cone over N. Thus we get a unique morphism

$$\lim_{i\in\mathcal{I}} \phi_i : \lim_{i\in\mathcal{I}} M_i \to \lim_{i\in\mathcal{I}} N_i$$

such that for any i the diagram

$$\begin{array}{ccc} \lim_{i\in\mathcal{I}} M_i & \xrightarrow{\lambda_i^M} & M_i \\ {\scriptstyle \lim_{i\in\mathcal{I}} \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_i} \\ \lim_{i\in\mathcal{I}} N_i & \xrightarrow{\lambda_i^N} & N_i \end{array}$$

commutes. Thus if every $\mathcal{I}$-diagram has a limit, we get a functor

$$\lim_{i\in\mathcal{I}} : \mathrm{Fun}(\mathcal{I},\mathcal{C}) \to \mathcal{C}.$$

(iii) Given $X \in \mathrm{ob}(\mathcal{C})$, we define a *constant functor* $c(X) : \mathcal{I} \to \mathcal{C}$ sending each object of $\mathcal{I}$ to X and each morphism in $\mathcal{I}$ to Id_X. Given a morphism $\theta : X \to Y$ we get a natural transformation $c(X) \to c(Y)$ whose components are all θ. This defines a functor $c : \mathcal{C} \to \mathrm{Fun}(\mathcal{I},\mathcal{C})$. A cone over M is the same thing as a natural transformation $\alpha : c(X) \to M$. To ask that there is a limit is to ask that the contravariant functor

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(c(-), M) : \mathcal{C} \to \mathit{Set}$$

is representable, so naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, L)$ for some L. If all $\mathcal{I}$-diagrams have a limit, then we get adjoint functors $(c(-), \lim_{i\in\mathcal{I}})$.

Examples. (a) Let I be a set. A *product* of a family of objects $M_i \in \mathrm{ob}(\mathcal{C})$ $(i \in I)$ is an object $P = \prod_{i\in I} M_i \in \mathrm{ob}(\mathcal{C})$ equipped with morphisms $p_i : P \to M_i$ such that for any object X and morphisms $q_i : X \to M_i$ there is a unique morphism $\theta : X \to P$ with $q_i = p\theta$, that is, the map

$$\mathrm{Hom}(X, P) \to \prod_i \mathrm{Hom}(X, M_i), \quad \theta \mapsto (p_i\theta)$$

is a bijection. Here we take the category $\mathcal{I}$ with object set I and only identity morphisms.

(b) A *terminal object* in a category $\mathcal{C}$ is an object T such that for every object X there is a unique morphism $X \to T$. This is the same thing as a product of objects indexed by the empty set or a limit over an empty category.

(c) An *equalizer* of a pair of morphisms $f, g : U \to W$ consists of an object E and a morphism $p : E \to U$ with $fp = gp$ and with the universal property, that for all $q : X \to U$ with $fq = gq$ there is a unique $\theta : X \to E$ with $q = p\theta$. Here $\mathcal{I}$ is the category

$$\circ \rightrightarrows \circ$$

with two objects and two non-identity morphisms.

(d) A *pullback* of a diagram

$$\begin{array}{ccc} & & U \\ & & \big\downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & W \end{array}$$

of objects and morphisms in $\mathcal{C}$ consists of an object X and morphisms p, q giving a commutative square

$$\begin{array}{ccc} X & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & {\scriptstyle f}\downarrow \\ V & \xrightarrow{g} & W \end{array}$$

and which is universal for such commutative squares, that is for any X', $p' : X' \to U$, $q' : X' \to V$ with $fp' = gq'$ there is a unique $\theta : X' \to X$ with $p' = p\theta$ and $q' = q\theta$.