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Masters course: Homological algebra

Homological algebra is the algebra that was invented in order to define and study the homology and cohomology of topological spaces, but it has applications all over mathematics.

My aim is to cover the properties of projective, injective and flat modules, complexes of modules and Ext and Tor groups, homological dimensions, homology and cohomology of groups, and more abstractly, abelian and triangulated categories.

Students are expected to already have some familiarity with rings and modules.

Some suggested books:

- C. A. Weibel, An introduction to homological algebra, CUP 1994.
- J. J. Rotman, An introduction to homological algebra, Springer 2009.
- M. S. Osborne, Basic homological algebra, Springer 2000.
- S. I. Gelfand and Yu. I. Manin, Methods of homological algebra, 2nd ed., Springer 2010.
- H. Krause, Homological theory of representations, CUP 2022.
- The Stacks Project, https://stacks.math.columbia.edu/
- T. Leinster, Basic category theory, CUP 2014, now published as arXiv:1612.09375

Contents

More coming...

1 Abelian categories

The basic setting setting for homological algebra, for example used in the book by Henri Cartan and Samuel Eilenberg, 'Homological algebra', 1956, is complexes of additive groups, or more generally complexes of modules for a ring R.

Algebraic geometers also want to work with complexes of sheaves on an algebraic variety, and in his paper 'Sur quelques points d'algèbre homologique', Tohoku Math. J. 9 (1957), 119–221, Alexander Grothendieck showed that you can unify the two settings by working with abelian categories.

Although we won't work with sheaves, it is good to start with abelian categories: modern homological algebra uses triangulated categories and other concepts, and abelian categories are a necessary preparation.

We begin with the language of categories, although students may have seen this already.

1.1 Categories and functors

Definition. A *category* $\mathcal{C}$ consists of

(i) a collection $\mathrm{ob}(\mathcal{C})$ of *objects*

(ii) For any $X, Y \in \mathrm{ob}(\mathcal{C})$, a set $\mathrm{Hom}(X, Y)$ (also denoted $\mathcal{C}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$) of *morphisms* (or *homomorphisms*) $\theta : X \to Y$, and

(iii) For any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(\theta, \phi) \mapsto \theta\phi$.

satisfying

(a) Associativity: $(\theta\phi)\psi = \theta(\phi\psi)$ for $X \xrightarrow{\psi} Y \xrightarrow{\phi} Z \xrightarrow{\theta} W$, and

(b) For each object X there is an identity morphism $\mathrm{Id}_X \in \mathrm{Hom}(X, X)$, with $\mathrm{Id}_Y\theta = \theta = \theta\mathrm{Id}_X$ for all $\theta : X \to Y$.

An *inverse* of a morphism $\theta : X \to Y$ is a morphism $\phi : Y \to X$ with $\theta\phi = \mathrm{Id}_Y$ and $\phi\theta = \mathrm{Id}_X$. If θ has an inverse, it is unique, and denoted θ^{-1}. An *isomorphism* is a morphism with an inverse.

Examples. (1) The categories of Sets, Groups, Rings, etc. The category R-Mod of (left) R-modules for a ring R. These are *concrete categories*: the objects are sets, possibly with extra structure, and the morphisms are maps of sets preserving the extra structure. (Note that I use the words 'morphism' and 'homomorphism' interchangeably, but perhaps homomorphism should only be used for concrete categories.)

(2) If $\mathcal{C}$ is a category, the *opposite category* C^{op} is given by $\mathrm{ob}(\mathcal{C}^{op}) = \mathrm{ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$, with composition derived from that in $\mathcal{C}$.

(3) If $\mathcal{C}$ and $\mathcal{D}$ are categories, the product $\mathcal{C} \times \mathcal{D}$ is the category with $\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$ and $\mathrm{Hom}((X,U),(Y,V)) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{D}}(U,V)$. Note that any morphism (θ, ϕ) can be written as a composition $(\theta, \mathrm{Id}_V)(\mathrm{Id}_X, \phi)$ or $(\mathrm{Id}_Y, \phi)(\theta, \mathrm{Id}_U)$.

(4) Given a group G or a ring R, there is a category with one object $*$, $\mathrm{Hom}(*,*) = G$ or R and composition given by multiplication.

(5) A partially ordered set $(S, \leq)$ gives a category with objects $s \in S$ and

$$\mathrm{Hom}(s,t) = \begin{cases} i_{st} & (s \leq t) \\ \emptyset & (s \not\leq t). \end{cases}$$

The composition must be given by $i_{tu} i_{st} = i_{su}$ for $s \leq t \leq u$, so $\mathrm{Id}_s = i_{ss}$.

(6) A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows and mappings $s, t : Q_1 \to Q_0$ giving the source and target of each arrow, so $s(a) \xrightarrow{a} t(a)$. It is like a category without a composition. The *path category* of a quiver has objects the vertices, and the morphisms $i \to j$ are the *paths* $a_n \dots a_2 a_1$ given by sequences of arrows

$$i = i_0 \xrightarrow{a_1} i_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} i_n = j.$$

There is also a trivial path Id_i for each vertex i. Composition is given by concatenation. For example the category given by the poset $(\mathbb{N}, \leq)$ is isomorphic to the path category of the quiver $0 \to 1 \to 2 \to \dots$.

Definition. Because of Russell's paradox, there is no set of all sets. One solution is to allow normal sets and 'big sets' called classes. There is a class of all sets.

- *Normal category:* $\mathrm{ob}(\mathcal{C})$ is a class, $\mathrm{Hom}(X,Y)$ are sets. For example the concrete categories above.
- *BIG category:* $\mathrm{ob}(\mathcal{C})$ is a class, $\mathrm{Hom}(X,Y)$ are classes. We only rarely need this.
- *Small category:* $\mathrm{ob}(\mathcal{C})$ is a set, $\mathrm{Hom}(X,Y)$ are sets. For example the category given by a partially ordered set.
- *Skeletally small category:* A normal category, such that there is a set S of objects such that every object is isomorphic to one in S.

Definition. A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ such that

- $\mathrm{ob}(\mathcal{D})$ is a subclass of $\mathrm{ob}(\mathcal{C})$.
- $\mathrm{Hom}_{\mathcal{D}}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all $X,Y \in \mathrm{ob}(\mathcal{D})$.
- $\mathrm{Id}_X^{\mathcal{C}} \in \mathrm{Hom}_{\mathcal{D}}(X,X)$ for all $X \in \mathrm{ob}(\mathcal{D})$.
- Composition in $\mathcal{D}$ is the same as composition in $\mathcal{C}$.

It is a *full subcategory* if $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all $X,Y \in \mathrm{ob}(\mathcal{D})$. Thus a full subcategory of $\mathcal{C}$ is determined by a subclass $\mathrm{ob}(\mathcal{D})$ of $\mathrm{ob}(\mathcal{C})$.

Examples. (a) The category Ab of abelian groups is a full subcategory of the category of all groups.

(b) The category R-mod of finitely generated R-modules is a full subcategory of R-Mod. It is skeletally small, with $S = \{R^n/U : n \in \mathbb{N}, U \subseteq R^n\}$.

(c) The category whose objects are sets and with $\mathrm{Hom}(X,Y) =$ the injective functions $X \to Y$ is a subcategory of the category of sets.

Definition. A *monomorphism* in a category is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Z \to X$, if $\theta\alpha = \theta\beta$ then $\alpha = \beta$.

An *epimorphism* is a morphism $\theta : X \to Y$ such that for all pairs of morphisms $\alpha, \beta : Y \to Z$, if $\alpha\theta = \beta\theta$ then $\alpha = \beta$.

Examples. (1) In the categories of sets or of R-modules, monomorphism = injective map, epimorphism = surjective map. For example we show epi = surjection for modules. Say $\theta : X \to Y$ is surjective and $\alpha\theta = \beta\theta$. Since θ is surjective, for all $y \in Y$ there is $x \in X$ with $\theta(x) = y$. Then $\alpha(y) = \alpha(\theta(x)) = \beta(\theta(x)) = \beta(y)$. Thus $\alpha = \beta$. Say $\theta : X \to Y$ is an epimorphism. The natural map $Y \to Y/\operatorname{Im}\theta$ and the zero map have the same composition with θ, so they are equal. Thus $\operatorname{Im}\theta = Y$.

(2) In the category of rings, the inclusion map $\theta : \mathbb{Z} \to \mathbb{Q}$ is not surjective, but it is an epimorphism.

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is given by

- For each object $X \in \mathrm{ob}(\mathcal{C})$, an object $F(X) \in \mathrm{ob}(\mathcal{D})$, and
- For each morphism $\theta : X \to Y$ in $\mathcal{C}$, a morphism $F(\theta) : F(X) \to F(Y)$ in $\mathcal{D}$

such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for all $X \in \mathrm{ob}(\mathcal{C})$ and $F(\theta\phi) = F(\theta)F(\phi)$ for composable morphisms $X \xrightarrow{\phi} Y \xrightarrow{\theta} Z$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the same thing as a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$. Thus it is an assignment of

- For each object $X \in \text{ob}(\mathcal{C})$, an object $F(X) \in \text{ob}(\mathcal{D})$, and
- For each morphism $\theta : X \to Y$ in $\mathcal{C}$ a morphism $F(\theta) : F(Y) \to F(X)$ in $\mathcal{D}$,

such that $F(\text{Id}_X) = \text{Id}_{F(X)}$ and $F(\theta\phi) = F(\phi)F(\theta)$ for composable morphisms $X \xrightarrow{\phi} Y \xrightarrow{\theta} Z$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- *faithful* if the map $F : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective for all $X, Y \in \text{ob}(\mathcal{C})$,
- *full* if the map $F : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective for all $X, Y \in \text{ob}(\mathcal{C})$,
- *fully faithful* if it is full and faithful.
- *dense* or *essentially surjective* if every object in $\mathcal{D}$ is isomorphic to $F(X)$ for some object X in $\mathcal{C}$.
- An *equivalence* if it is fully faithful and dense.
- An *isomorphism* if it has an inverse, or equivalently if it is fully faithful and a bijection on objects. Thus every isomorphism is an equivalence.

Examples. (1) The inclusion functor of a subcategory, for example Ab to *Group*, is always faithful. It is full if and only if the subcategory is full.

(2) A composition of functors is a functor. (Thus there is a category of small categories.)

(3) There are many examples of *forgetful functors* for concrete categories, which forget some structure. For example *Group* $\to$ *Set*, or R-Mod $\to$ Ab. They are faithful.

(4) Given a ring homomorphism $\theta : R \to S$, restriction defines a faithful functor S-Mod $\to$ R-Mod. [It is full if and only if θ is a epimorphism in the category of rings, but that is another story.]

(5) If K is a field, then duality $V \rightsquigarrow V^* = \text{Hom}_K(V, K)$ gives a contravariant functor $* : K\text{-Mod} \to K\text{-Mod}$, sending a morphism $\theta : V \to W$ to $\theta^* : W^* \to V^*$ given by $\theta^*(\xi)(v) = \xi(\theta(v))$.

Thus it gives a covariant functor $(K\text{-Mod})^{op} \to K\text{-Mod}$.

Considering only finite-dimensional vector spaces, this gives a functor $(K\text{-mod})^{op} \to K\text{-mod}$. This is an equivalence since every n-dimensional vector space is isomorphic to V^* with V an n-dimensional vector space, and the map $* : \text{Hom}(V, W) \to \text{Hom}(W^*, V^*)$ is injective, so an isomorphism by dimensions.

(6) Let $\mathcal{C}$ be a category and let $\mathrm{Hom}(X, Y)$ denote the Hom sets for $\mathcal{C}$. Fix an object $X \in \mathrm{ob}(\mathcal{C})$. There is a functor $\mathrm{Hom}(X, -) : \mathcal{C} \to Set$ sending an object Y to $\mathrm{Hom}(X, Y)$, and sending a morphism $\theta \in \mathrm{Hom}(Y, Z)$ to the mapping $\mathrm{Hom}(X, \theta) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ defined by $\mathrm{Hom}(X, \theta)(\phi) = \theta\phi$.

Dually, fixing Y, we get a contravariant functor $\mathrm{Hom}(-, Y) : \mathcal{C} \to Set$.

Varying both X and Y, we get a functor $\mathrm{Hom}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to Set$.

1.2 Natural transformations and functor categories

Definition. Let F, G be functors $\mathcal{C} \to \mathcal{D}$. A *natural transformation* $\alpha : F \to G$ is given by morphisms $\alpha_X : F(X) \to G(X)$ for all $X \in \mathrm{ob}(\mathcal{C})$ such that $G(\theta)\alpha_X = \alpha_Y F(\theta)$ for every morphism $\theta : X \to Y$ in $\mathcal{C}$.

It is a *natural isomorphism* if all α_X are isomorphisms in $\mathcal{D}$.

Example. If K is a field and X is a K-vector space, there is a map $\alpha_X : X \to X^{**}$ given by $\alpha_X(x)(\phi) = \phi(x)$ for $\phi \in X^*$. Together, these give a natural transformation $\alpha : \mathrm{Id} \to (-)^{**}$ of functors from K-Mod to K-Mod. Namely, if $\theta : X \to Y$ is a linear map, then $\theta^* : Y^* \to X^*$ is given by $\theta^*(\psi) = \psi\theta$ for $\psi \in Y^*$, and $\theta^{**} : X^{**} \to Y^{**}$ is given by $\theta^{**}(\xi) = \xi\theta^*$ for $\xi \in X^{**}$. Then

$$\theta^{**}(\alpha_X(x))(\psi) = \alpha_X(x)(\theta^*(\psi)) = \alpha_X(x)(\psi\theta) = (\psi\circ\theta)(x) = \psi(\theta(x)) = \alpha_Y(\theta(x))(\psi),$$

so $\theta^{**}\alpha_X = \alpha_Y\theta$. If we used K-mod, then α would be a natural isomorphism.

Definition. The *functor category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has objects the functors $F : \mathcal{C} \to \mathcal{D}$. The morphisms are the natural transformations. The composition $\beta\alpha$ of natural transformations $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ is given by $(\beta\alpha)_X = \beta_X\alpha_X$. The identity natural transformation $\mathrm{Id}_F : F \to F$ is given by $(\mathrm{Id}_F)_X = \mathrm{Id}_{F(X)}$.

Remarks. (1) In general this is a BIG category. To get a normal category, we can take $\mathcal{C}$ small, or more generally skeletally small. We need to check that the collection of natural transformations $F \to G$ is a set. Every object in $\mathcal{C}$ is isomorphic to an object in a set S. A natural transformation $\alpha : F \to G$ is determined by the morphisms α_X for $X \in S$, for if $\theta : Y \to X$ is an isomorphism, then $\alpha_Y = G(\theta^{-1})\alpha_X F(\theta)$.

(2) The natural isomorphisms $F \to G$ are the isomorphisms in the functor category, e.g. if α is a natural isomorphism, it has inverse α^{-1} defined by $(\alpha^{-1})_X = (\alpha_X)^{-1}$.

Lemma (Yoneda's Lemma). *For a functor $F : \mathcal{C} \to Set$ and $X \in \mathrm{ob}(\mathcal{C})$, there is a 1-1 correspondence between natural transformations $\alpha : \mathrm{Hom}(X, -) \to F$ and elements $f \in F(X)$.*

Proof. A natural transformation α gives a mapping $\alpha_X : \mathrm{Hom}(X,X) \to F(X)$, and hence an element $f = \alpha_X(\mathrm{Id}_X) \in F(X)$. Conversely, given $f \in F(X)$ and $Y \in \mathrm{ob}(\mathcal{C})$ we get a morphism $\alpha_Y : \mathrm{Hom}(X,Y) \to F(Y)$, $\theta \mapsto F(\theta)(f)$. This defines a natural transformation α. These constructions are inverses. $\square$

Remark. In particular, if $F = \mathrm{Hom}(Y,-)$, this shows that we get a bijection

$$\mathrm{Hom}(Y,X) \rightsquigarrow \{\text{natural transformations } \mathrm{Hom}(X,-) \to \mathrm{Hom}(Y,-)\}$$

Thus we get a fully faithful functor $\mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathit{Set})$.

Also $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(Y,-)$ are naturally isomorphic if and only if X and Y are isomorphic.

Definition. A functor $\mathcal{C} \to \mathit{Set}$ is *representable* if it is naturally isomorphic to a functor of the form $\mathrm{Hom}(X,-)$ for some object X. Thus it is equivalent that there is an object X and $f \in F(X)$ such that for all objects Y, the map $\mathrm{Hom}(X,Y) \to F(Y)$, $\theta \mapsto F(\theta)(f)$ is a bijection.

Dually, a contravariant functor from $\mathcal{C}$ to Set is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}(-,Y)$.

Definition. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say that (F,G) is an *adjoint pair*, or that F is *left adjoint* to G or G is *right adjoint* to F if there is a natural isomorphism $\alpha : \mathrm{Hom}(F(-),-) \to \mathrm{Hom}(-,G(-))$ of functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathit{Sets}$.

Thus one needs bijections

$$\alpha_{X,Y} : \mathrm{Hom}(F(X),Y) \to \mathrm{Hom}(X,G(Y))$$

for all $X \in \mathrm{ob}(\mathcal{C})$ and $Y \in \mathrm{ob}(\mathcal{D})$, such that

$$\begin{array}{ccc} \mathrm{Hom}(F(X'),Y) & \xrightarrow{\alpha_{X',Y}} & \mathrm{Hom}(X',G(Y)) \\ {\scriptstyle \cdot F(\theta)}\downarrow & & {\scriptstyle \cdot\theta}\downarrow \\ \mathrm{Hom}(F(X),Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}(X,G(Y)) \end{array}$$

commutes for all $\theta : X \to X'$, and

$$\begin{array}{ccc} \mathrm{Hom}(F(X),Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}(X,G(Y)) \\ {\scriptstyle \phi\cdot}\downarrow & & {\scriptstyle G(\phi)\cdot}\downarrow \\ \mathrm{Hom}(F(X),Y') & \xrightarrow{\alpha_{X,Y'}} & \mathrm{Hom}(X,G(Y')) \end{array}$$

commutes for all $\phi : Y \to Y'$. (Here we have used that every morphism in a product of categories is a composition of morphisms in which one of the components is the identity.)

Examples. (1) Let R be a ring. We have a forgetful functor $Forget_R : R\text{-Mod} \to Sets$. Given a set X, let $Free_R(X)$ be the free left R-module with basis X. Thus

$$Free_R(X) = \{\sum_{x \in X} r_x x : r_x \in R \text{ for } x \in X, \text{ all but finitely many zero}\}.$$

Any mapping $\phi : X \to Y$ gives a module homomorphism $Free_R(X) \to Free_R(Y)$. This gives a functor $Free_R : Sets \to R\text{-Mod}$. For M a left R-module, we have a bijection

$$\alpha_{X,M} : \mathrm{Hom}_R(Free_R(X), M) \to \mathrm{Hom}_{Sets}(X, Forget_R(M))$$

This is natural in both X and M, so it turns $(Free_R, Forget_R)$ into an adjoint pair of functors.

(2) By defining things with morphisms in the natural way, we get adjoint functors $(Path, Forget)$ where $Forget$ is the functor from small categories to quivers which forgets the composition and $Path$ sends a quiver to its path category.

Proposition. *A functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint G if and only if the contravariant functors $\mathrm{Hom}_{\mathcal{D}}(F(-), Y)$ are representable for all $Y \in \mathrm{ob}(\mathcal{D})$. Moreover, if a right adjoint exists, it is unique up to natural isomorphism.*

Dually, $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if $\mathrm{Hom}_{\mathcal{C}}(X, G(-))$ is representable for all $X \in \mathrm{ob}(\mathcal{C})$, and if a left adjoint exists, it is unique up to natural isomorphism.

Proof. If there is a right adjoint, then $\mathrm{Hom}(F(-), Y) \cong \mathrm{Hom}(-, G(Y))$, so it is representable. Conversely, we show how to define G. For each $Y \in \mathrm{ob}(\mathcal{D})$, we choose an object $G(Y) \in \mathrm{ob}(\mathcal{C})$ and a natural isomorphism $\mathrm{Hom}(F(-), Y) \to \mathrm{Hom}(-, G(Y))$. For each $X \in \mathrm{ob}(\mathcal{C})$ this gives bijections $\alpha_{XY} : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, G(Y))$. Now if $\phi : Y \to Y'$ is a morphism in $\mathcal{D}$, we get maps

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\alpha_{XY}^{-1}} \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\phi \cdot} \mathrm{Hom}_{\mathcal{D}}(F(X), Y') \xrightarrow{\alpha_{X,Y'}} \mathrm{Hom}_{\mathcal{C}}(X, G(Y')).$$

In particular, taking $X = G(Y)$, this composition sends $\mathrm{Id}_{G(Y)}$ to an element of $\mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y'))$. We define $G(\phi)$ to be this element.

If G and G' are right adjoints, then we must have a natural isomorphism

$$\beta : \mathrm{Hom}_{\mathcal{C}}(-, G(-)) \to \mathrm{Hom}_{\mathcal{C}}(-, G'(-)).$$

For any $Y \in \mathrm{ob}(\mathcal{D})$, this gives a bijection

$$\beta_{G(Y),Y} : \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(G(Y), G'(Y)).$$

Let γ_Y be the image of $\mathrm{Id}_{G(Y)}$. It is straightforward to check that this defines a natural isomorphism $\gamma : G \to G'$. □

Remark. If (F, G) is an adjoint pair as in the definition, then there are natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ called the *unit* and *counit*, defined by $\eta_X = \alpha_{X,F(X)}(\mathrm{Id}_{F(X)})$ and $\alpha_{G(Y),Y}(\epsilon_Y) = \mathrm{Id}_{G(Y)}$. Moreover α can be recovered from them. For example, if $\phi \in \mathrm{Hom}(F(X), Y)$, then the commutative square

$$\begin{array}{ccc} \mathrm{Hom}(F(X), F(X)) & \xrightarrow{\alpha_{X,F(X)}} & \mathrm{Hom}(X, GF(X)) \\ \phi\cdot \downarrow & & G(\phi)\cdot \downarrow \\ \mathrm{Hom}(F(X), Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}(X, G(Y)) \end{array}$$

gives $\alpha_{XY}(\phi) = G(\phi)\eta_X$.

Theorem. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if there is a functor $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\alpha : FG \cong \mathrm{Id}_{\mathcal{D}}$ and $\beta : GF \cong \mathrm{Id}_{\mathcal{C}}$. In this case (F, G) and (G, F) are both adjoint pairs.*

Proof. If G exists, then the existence of α implies that F is dense. If $\theta : X \to X'$ is a morphism in $\mathcal{C}$, there is a commutative square

$$\begin{array}{ccc} GF(X) & \xrightarrow{\beta_X} & X \\ GF(\theta) \downarrow & & \theta \downarrow \\ GF(X') & \xrightarrow{\beta_{X'}} & X' \end{array}$$

so $\theta = \beta_{X'} GF(\theta) \beta_X^{-1}$. It follows that F is faithful. By symmetry G is also faithful. Now if $\phi : F(X) \to F(X')$ is a morphism, let $\theta = \beta_{X'} G(\phi) \beta_X^{-1}$. Then also $\theta = \beta_{X'} GF(\theta) \beta_X^{-1}$. It follows that $G(\phi) = GF(\theta)$, so $\phi = F(\theta)$ since G is faithful. Thus F is full.

Conversely, suppose that F is an equivalence, so fully faithful and dense. If $Y \in \mathrm{ob}(\mathcal{D})$, then $Y \cong F(X')$ for some X', so

$$\mathrm{Hom}(F(-), Y) \cong \mathrm{Hom}(F(-), F(X')) \cong \mathrm{Hom}(-, X').$$

It follows that F has a right adjoint G. Then

$$\mathrm{Hom}(-, GF(-)) \cong \mathrm{Hom}(F(-), F(-)) \cong \mathrm{Hom}(-, -),$$

which gives a natural isomorphism $GF \cong \mathrm{Id}_{\mathcal{C}}$. Now if $Y, Y' \in \mathrm{ob}(\mathcal{D})$ we can choose X, X' and isomorphisms $\psi_Y : Y \to F(X)$ and $\psi_{Y'} : Y' \to F(X')$. Then

$$\mathrm{Hom}(Y, Y') \cong \mathrm{Hom}(F(X), F(X')) \cong \mathrm{Hom}(X, X')$$

$$\cong \mathrm{Hom}(GF(X), GF(X')) \cong \mathrm{Hom}(G(Y), G(Y'))$$

and the composition is the map given by G, for if $\phi \in \mathrm{Hom}(Y, Y')$ corresponds to $\theta \in \mathrm{Hom}(X, X')$, then $\phi = \psi_{Y'}^{-1} F(\theta)\psi_Y$, and ϕ is sent to

$$G(\psi_{Y'})^{-1} GF(\theta) G(\psi_Y) = G(\phi).$$

Thus G is fully faithful. Then

$$\mathrm{Hom}(FG(-), -) \cong \mathrm{Hom}(G(-), G(-)) \cong \mathrm{Hom}(-, -),$$

so $FG \cong \mathrm{Id}_{\mathcal{D}}$, and

$$\mathrm{Hom}(G(-), -) \cong \mathrm{Hom}(FG(-), F(-)) \cong \mathrm{Hom}(-, F(-))$$

so (G, F) is an adjoint pair. □

1.3 Limits and colimits

Definition. Let $\mathcal{C}$ be a category. Let $\mathcal{I}$ be a small category. An *$\mathcal{I}$-diagram* in $\mathcal{C}$ is a functor $M : \mathcal{I} \to \mathcal{C}$. For $i \in \mathrm{ob}(\mathcal{I})$, we write M_i instead of $M(i)$ and for a morphism $a : i \to j$ in $\mathcal{I}$, we write M_a for the morphism $M_i \to M_j$.

For example if $\mathcal{I}$ is the category

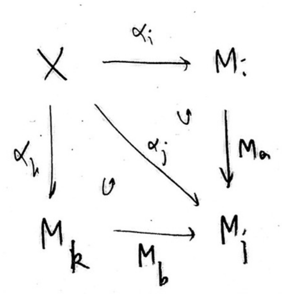
$$\begin{array}{ccc} & & i \\ & & \downarrow{\scriptstyle a} \\ k & \xrightarrow{b} & j \end{array}$$

with three objects i, j, k and two non-identity morphisms a, b, then an $\mathcal{I}$-diagram is given by objects and morphisms

$$\begin{array}{ccc} & & M_i \\ & & \downarrow{\scriptstyle M_a} \\ M_k & \xrightarrow{M_b} & M_j. \end{array}$$

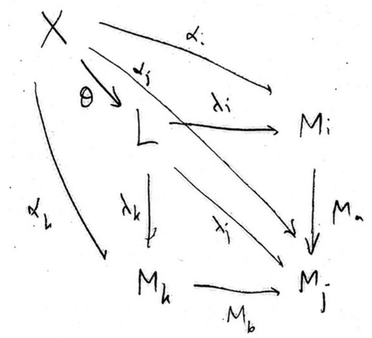
in $\mathcal{C}$ By a *cone over M* we mean a collection (X, α_i) consisting of an object $X \in \mathcal{C}$ and morphisms $\alpha_i : X \to M_i$ for all $i \in \mathrm{ob}(\mathcal{I})$ satisfying $M_a \alpha_i = \alpha_j$ for all morphisms $a : i \to j$ in $\mathcal{I}$.

In the example:



A *limit* for M is a cone (L, λ_i) over M, which is universal in the sense that if (X, α_i) is any cone over M, then there is a unique morphism $\theta : X \to L$ such that $\alpha_i = \lambda_i \theta$ for all $i \in \mathrm{ob}(\mathcal{I})$.

In the example:



Remarks. (i) If M has a limit, then it is unique up to a unique isomorphism, so we can talk about *the limit*, and denote it

$$L = \lim_{i \in \mathcal{I}} M_i \in \mathrm{ob}(\mathcal{C}).$$

(ii) Suppose $\phi : M \to N$ is a natural transformation between $\mathcal{I}$-diagrams, and suppose that $\lim_{i \in \mathcal{I}} M$ and $\lim_{i \in \mathcal{I}} N_i$ both exist. Then for each i we get a morphism

$$\lim_{i \in \mathcal{I}} M_i \xrightarrow{\lambda_i^M} M_i \xrightarrow{\phi_i} N_i.$$

These turn $\lim_{i \in \mathcal{I}} M_i$ into a cone over N. Thus we get a unique morphism

$$\lim_{i \in \mathcal{I}} \phi_i : \lim_{i \in \mathcal{I}} M_i \to \lim_{i \in \mathcal{I}} N_i$$

such that for any i the diagram

$$\begin{array}{ccc} \lim_{i \in \mathcal{I}} M_i & \xrightarrow{\lambda_i^M} & M_i \\ {\scriptstyle \lim_{i \in \mathcal{I}} \phi_i} \downarrow & & {\scriptstyle \phi_i} \downarrow \\ \lim_{i \in \mathcal{I}} N_i & \xrightarrow{\lambda_i^N} & N_i \end{array}$$

commutes. Thus if every $\mathcal{I}$-diagram has a limit, we get a functor

$$\lim_{i\in\mathcal{I}} : \mathrm{Fun}(\mathcal{I},\mathcal{C}) \to \mathcal{C}.$$

(iii) Given $X \in \mathrm{ob}(\mathcal{C})$, we define a *constant functor* $c(X) : \mathcal{I} \to \mathcal{C}$ sending each object of $\mathcal{I}$ to X and each morphism in $\mathcal{I}$ to Id_X. Given a morphism $\theta : X \to Y$ we get a natural transformation $c(X) \to c(Y)$ whose components are all θ. This defines a functor $c : \mathcal{C} \to \mathrm{Fun}(\mathcal{I},\mathcal{C})$. A cone over M is the same thing as a natural transformation $\alpha : c(X) \to M$. To ask that there is a limit is to ask that the contravariant functor

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(c(-), M) : \mathcal{C} \to \mathit{Set}$$

is representable, so naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, L)$ for some L. If all $\mathcal{I}$-diagrams have a limit, then we get adjoint functors $(c(-), \lim_{i\in\mathcal{I}})$.

Examples. (a) A *pullback* of a diagram

$$\begin{array}{ccc} & & U \\ & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & W \end{array}$$

of objects and morphisms in $\mathcal{C}$ consists of an object X and morphisms p, q giving a commutative square

$$\begin{array}{ccc} X & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & W \end{array}$$

and which is universal for such commutative squares, that is for any X', $p' : X' \to U$, $q' : X' \to V$ with $fp' = gq'$ there is a unique $\theta : X' \to X$ with $p' = p\theta$ and $q' = q\theta$.
This is a limit for the category $\mathcal{I}$ already given.

(b) An *equalizer* of a pair of morphisms $f, g : U \to W$ consists of an object E and a morphism $p : E \to U$ with $fp = gp$ and with the universal property, that for all $q : X \to U$ with $fq = gq$ there is a unique $\theta : X \to E$ with $q = p\theta$. Here $\mathcal{I}$ is the category

$$\circ \rightrightarrows \circ$$

with two objects and two non-identity morphisms.

(c) Let I be a set. A *product* of a family of objects $M_i \in \mathrm{ob}(\mathcal{C})$ $(i \in I)$ is an object $P = \prod_{i\in I} M_i \in \mathrm{ob}(\mathcal{C})$ equipped with morphisms $p_i : P \to M_i$ such that for any object X and morphisms $q_i : X \to M_i$ there is a unique morphism $\theta : X \to P$ with $q_i = p_i\theta$ for all $i \in I$, that is, the map

$$\mathrm{Hom}(X,P) \to \prod_i \mathrm{Hom}(X,M_i), \quad \theta \mapsto (p_i\theta)$$

is a bijection, where on the right hand side we are taking the cartesian product of sets. Here we take the category $\mathcal{I}$ with object set I and only identity morphisms.

(d) A *terminal object* in a category $\mathcal{C}$ is an object T such that for every object X there is a unique morphism $X \to T$. This is the same thing as a product of objects indexed by the empty set, or a limit over an empty category.

Theorem. *A category $\mathcal{C}$ is* (finitely) complete*, meaning that for all (finite) small categories $\mathcal{I}$ and $\mathcal{I}$-diagrams M the limit exists in $\mathcal{C}$, if and only if $\mathcal{C}$ has products indexed by any (finite) set and equalizers.*

Proof. We will need the explicit construction of limits. Suppose M is an $\mathcal{I}$-diagram in $\mathcal{C}$. Consider the products and associated morphisms

$$\prod_{i\in\mathrm{ob}(\mathcal{I})} M_i \xrightarrow{p_i} M_i, \quad \prod_a M_{t(a)} \xrightarrow{p_a} M_{t(a)}$$

where the second product is indexed by the morphisms a in $\mathcal{I}$ and $s(a), t(a)$ are the source and target of a. By the universal property of the second product, there are unique morphisms

$$\prod_{i\in\mathrm{ob}(\mathcal{I})} M_i \begin{array}{c}\phi \\ \longrightarrow \\ \longrightarrow \\ \psi\end{array} \prod_a M_{t(a)}$$

with $p_{t(a)} = p_a\phi$ and $M_a p_{s(a)} = p_a\psi$. Let E equipped with a morphism

$$p : E \xrightarrow{p} \prod_{i\in\mathrm{ob}(\mathcal{I})} M_i$$

be an equalizer for this diagram. It is straightforward to show that E equipped with the morphisms $\lambda_i = p_i p : E \to M_i$ for $i \in \mathrm{ob}(\mathcal{I})$ is a limit for M. (If you want the details, here they are. To show that this data forms a cone over M, we need that if a is a morphism in $\mathcal{I}$, then $\lambda_{t(a)} = M_a\lambda_{s(a)}$. Now $\lambda_{t(a)} = p_{t(a)}p = p_a\phi p$ and $M_a\lambda_{s(a)} = M_a p_{s(a)}p = p_a\psi p$ and these are equal since $\phi p = \psi p$. Now suppose that X equipped with morphisms $\alpha_i : X \to M_i$ for $i \in \mathrm{ob}(\mathcal{I})$ is a cone over M. Then $\alpha_{t(a)} = M_a\alpha_{s(a)}$ for all morphisms a in $\mathcal{I}$. By the property of the product there is a unique morphism

$$\gamma : X \to \prod_{i\in\mathrm{ob}(\mathcal{I})} M_i$$

with $p_i\gamma = \alpha_i$ for all $i \in \mathrm{ob}(\mathcal{I})$. Now if a is a morphism in $\mathcal{I}$, then $p_a\phi\gamma = p_{t(a)}\gamma = \alpha_{t(a)} = M_a\alpha_{s(a)} = M_a p_{s(a)}\gamma = p_a\psi\gamma$. Thus by the uniqueness property in the definition of the product $\prod_a M_{t(a)}$, we must have $\phi\gamma = \psi\gamma$. Thus by the equalizer property there is a unique morphism $\theta : X \to E$ with $\gamma = p\theta$. Now by the uniqueness property in the definition of the product $\prod_{i\in\mathrm{ob}(\mathcal{I})} M_i$ the condition $\gamma = p\theta$ is equivalent to the condition $p_i\gamma = p_i p\theta$ for all $i \in \mathrm{ob}(\mathcal{I})$, which can be rewritten as $\alpha_i = \lambda_i\theta$, as appearing in the definition of a limit.) □

Examples. The categories *Set* and R-Mod are complete. The product is the usual one. The terminal object is a one-point set or the zero module. The equalizer of $f, g : U \to W$ is the inclusion

$$\{u \in U : f(u) = g(u)\} \to U.$$

For R-modules this is the same as $\mathrm{Ker}(f - g)$. The pullback is $\{(u, v) \in U \times V : f(u) = g(v)\}$, etc.

Lemma. *In an equalizer, p is mono. A pullback of a mono is a mono, that is, in a pullback diagram, if f is mono, so is the parallel morphism q.*

Proof. For the equalizer, suppose $\alpha, \beta : X \to E$ and $p\alpha = p\beta = p'$. Since $fp' = gp'$, there is a unique $\theta : X \to E$ with $p' = p\theta$. But both $\theta = \alpha$ and $\theta = \beta$ satisfy this, so $\alpha = \beta$.

For the pullback. Suppose $\alpha, \beta : X' \to X$ with $q\alpha = q\beta$. Then $gq\alpha = gq\beta$, so $fp\alpha = fp\beta$. Since f is mono, $p\alpha = p\beta$. Thus by the uniqueness part of the universal property for a pullback, $\alpha = \beta$. □

Now we do the dual notion.

Definition. A *colimit* of a diagram $M : \mathcal{I} \to \mathcal{C}$ is the same thing as a limit of M considered as a functor $\mathcal{I}^{op} \to \mathcal{C}^{op}$. Thus it is an object

$$C = \operatorname*{colim}_{i\in\mathcal{I}} M_i \in \mathrm{ob}(\mathcal{C})$$

equipped with morphisms $\alpha_i : M_i \to C$ for each $i \in \mathrm{ob}(\mathcal{I})$ such that $\alpha_j M_a = \alpha_i$ for any $a : i \to j$ and such that if $X \in \mathrm{ob}(\mathcal{C})$ and $\beta_i : M_i \to X$ satisfy $\beta_j M_a = \beta_i$ for any $a : i \to j$, then there is a unique $\theta : C \to X$ such that $\beta_i = \theta\alpha_i$ for all i.

Examples. (a) A *pushout* of a pair of morphisms $f : W \to U$ and $g : W \to V$, consists of an object X and morphisms $p : U \to X$ and $q : V \to X$ giving a commutative square $pf = qg$, and which is universal for such commutative squares, that is for any X', $p' : U \to X'$, $q' : V \to X'$ with $p'f = q'g$ there is a unique $\theta : X \to X'$ with $p' = \theta p$ and $q' = \theta q$.

(b) A *coequalizer* of a pair of morphisms $f, g : U \to W$ consists of an object X and a morphism $p : W \to X$ with $pf = pg$ and the universal property.

(c) A *coproduct* of a family of objects M_i ($i \in I$) is an object $C = \coprod_{i \in I} M_i$ equipped with morphisms $i_i : M_i \to C$ such that for any object X and morphisms $j_i : M_i \to X$ there is a unique morphism $\theta : C \to X$ with $j_i = \theta i_i$. That is, the map

$$\mathrm{Hom}(C, X) \to \prod_i \mathrm{Hom}(M_i, X), \quad \theta \mapsto (\theta i_i)$$

is a bijection.

(b) An *initial* object is an object X with a unique morphism to any other object. It is a coproduct over the empty set or colimit over the empty category.

Definition. A category $\mathcal{C}$ is *(finitely) cocomplete* if all (finite) colimits exist. It is equivalent that $\mathcal{C}$ has all (finite) coproducts and coequalizers.

Examples. (i) The categories *Sets* and R-Mod are cocomplete.

For *Sets* the coproduct is the disjoint union

$$\dot{\bigcup} M_i.$$

The initial object is the empty set. The coequalizer of morphisms $f, g : U \to W$ is $W \to W/\sim$ where $\sim$ is the smallest equivalence relation with $f(u) \sim g(u)$ for all $u \in U$. The pushout of morphisms $f : W \to U$ and $g : W \to V$ is $U \dot{\cup} V/\sim$ where $\sim$ is the equivalence relation generated by $f(w) \sim g(w)$ for $w \in W$.

For R-Mod coproducts are direct sums

$$\bigoplus_{i \in I} M_i = \{(m_i) \in \prod_{i \in I} M_i : \text{all but finitely many } m_i = 0\}.$$

The initial object is the the zero module 0. The coequalizer of morphisms $f, g : U \to W$ in R-Mod is the map $W \to W/\mathrm{Im}(f - g)$. The pushout of morphisms $f : W \to U$ and $g : W \to V$ is $(U \oplus V)/\mathrm{Im}\,\theta$, where $\theta : W \to U \oplus V$ is $\theta(w) = (f(w), -g(w))$.

Lemma. *A pushout of an epi is an epi, that is, in a pushout diagram, if f is epi, so is the parallel morphism q.*

Proposition. *If (F, G) is a pair of adjoint functors, $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, then F preserves colimits and G preserves limits, if they exist.*

Proof. If M is an $\mathcal{I}$-diagram in $\mathcal{D}$ with a limit in $\mathcal{D}$, then GM is a functor $\mathcal{I} \to \mathcal{C}$ and

$$\begin{aligned} \mathrm{Hom}(c(-), GM) &\cong \mathrm{Hom}(F(c(-)), M) \cong \mathrm{Hom}(c(F(-)), M) \\ &\cong \mathrm{Hom}(F(-), \lim_{i \in \mathcal{I}} M_i) \cong \mathrm{Hom}(-, G(\lim_{i \in cI} M_i)), \end{aligned}$$

so $G(\lim_{i \in \mathcal{I}} M_i)$ is a limit for GM, that is, $G(\lim_{i \in \mathcal{I}} M_i) \cong \lim_{i \in \mathcal{I}} G(M_i)$. □

1.4 Additive categories

Definition. Let K be a commutative ring. A *K-category* is a category $\mathcal{C}$ with the extra structure that the sets $\mathrm{Hom}(X,Y)$ are K-modules for all $X,Y \in \mathrm{ob}(\mathcal{C})$ and the composition maps

$$\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z), \quad (\theta,\phi) \mapsto \theta\phi$$

are K-bilinear. In particular, for any objects $X,Y \in \mathrm{ob}(\mathcal{C})$, there is a zero morphism $0 \in \mathrm{Hom}(X,Y)$.

Recall that a $\mathbb{Z}$-module is the same thing as an additive group. A $\mathbb{Z}$-category is also called a *preadditive* category, so any K-category is preadditive.

Examples. The category Ab of abelian groups is preadditive. So is R-Mod for a ring R. If R is a K-algebra, then R-Mod is a K-category.

Definition. If $\mathcal{C}$ and $\mathcal{D}$ are K-categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *K-linear* if the mapping

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

is a homomorphism of K-modules for all $X,Y \in \mathrm{ob}(\mathcal{C})$. A $\mathbb{Z}$-linear functor is also called an *additive* functor.

If $\mathcal{C}$ and $\mathcal{D}$ are K-categories, we denote by $\mathrm{Fun}_K(\mathcal{C},\mathcal{D})$ the category whose objects are the K-linear functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations. It is naturally a K-category: if $\alpha,\alpha' : F \to G$ are natural transformations and $\lambda,\lambda' \in K$, we define $(\lambda\alpha + \lambda'\alpha')_X = \lambda\alpha_X + \lambda'\alpha'_X \in \mathrm{Hom}_{\mathcal{D}}(F(X),G(X))$.

Example. Let R be a ring, and consider it as a category with one object. It is preadditive, and

$$\mathrm{Fun}_{\mathbb{Z}}(R,\mathrm{Ab}) \cong R\text{-Mod}.$$

Remark. If $\mathcal{C}$ is a preadditive category and $X \in \mathrm{ob}(\mathcal{C})$, then the representable functor $\mathrm{Hom}(X,-)$ gives an additive functor $\mathcal{C} \to \mathrm{Ab}$, so an object in $\mathrm{Fun}_{\mathbb{Z}}(\mathcal{C},\mathrm{Ab})$. An appropriate version of Yoneda's Lemma gives that if $F : \mathcal{C} \to \mathrm{Ab}$ is an additive functor, then there is a 1-1 correspondence between natural transformations $\mathrm{Hom}(X,-) \to F$ and elements $f \in F(X)$.

Definition. The *kernel* of a morphism $f : U \to W$ in a preadditive category is the equalizer of f and 0. Thus it is an object X and a morphism $p : X \to U$ with $fp = 0$, such that for any morphism $p' : X' \to U$ with $fp' = 0$ there is a unique morphism $\theta : X' \to X$ with $p' = p\theta$. Conversely the equalizer of f,g = kernel of $f - g$.

The cokernel of a morphism $f : U \to W$ in a preadditive category is the coequalizer of f and 0. Thus it is an object X and a morphism $p : W \to X$ with $pf = 0$, such that for any morphism $p' : W \to X'$ with $p'f = 0$ there is a unique morphism $\theta : X \to X'$ with $p' = \theta p$.

For example the cokernel of a morphism $f : U \to W$ in R-Mod is $W \to W/\operatorname{Im} f$.

Theorem. *For objects X, $X_1, \dots, X_n$ $(n \geq 0)$ in a preadditive category the following are equivalent*

(i) X is the product of $X_1, \dots, X_n$ for some morphisms $p_i : X \to X_i$

(ii) X is the coproduct of $X_1, \dots, X_n$ for some morphisms $i_i : X_i \to X$,

(iii) X is a biproduct *of $X_1, \dots, X_n$, meaning that there are morphisms $p_i : X \to X_i$ and $i_i : X_i \to X$ with $p_i i_i = \mathrm{Id}_{X_i}$, $p_i i_j = 0$ for $i \neq j$ and $\sum_{i=1}^n i_i p_i = \mathrm{Id}_X$.*

In this case we write $X = \bigoplus_{i=1}^n X_i$ and call it a direct sum.

Proof. (i)⇒(iii) For any object X' we have a bijection

$$\mathrm{Hom}(X', X) \to \prod_{i=1}^n \mathrm{Hom}(X', X_i), \quad \phi \mapsto (p_i \phi).$$

In particular, taking $X' = X_j$, there is a morphism $i_j : X_j \to X$ such that

$$p_i i_j = \begin{cases} \mathrm{Id}_{X_i} & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Now if $\phi = \sum_{i=1}^n i_i p_i$ then $p_j \phi = \sum_{i=1}^n p_j i_i p_i = p_j$, so $\phi = \mathrm{Id}_X$ by the uniqueness part of the definition of a product.

(iii)⇒(i) For any X' one has inverse bijections

$$\mathrm{Hom}(X', X) \underset{\phi \mapsto (p_i \phi)}{\overset{(\alpha_i) \mapsto \sum i_i \alpha_i}{\rightleftarrows}} \prod_{i=1}^n \mathrm{Hom}(X', X_i)$$

so the p_i turn X into a product.

(ii)⇔(iii) Dual. □

Remark. The case $n = 0$ gives the following. In a preadditive category, an object X is terminal if and only if it is initial if and only if $\mathrm{Id}_X = 0$. This is called a *zero* object, and denoted 0.

Definition. A category is *additive* if it is preadditive, it has a zero object and every pair of objects has a direct sum (equivalently it has all finite direct sums).

Examples. (1) Ab, R-Mod, R-mod.

(2) If $\mathcal{C}$ is a preadditive category and $\mathcal{D}$ is additive, then $\mathrm{Fun}_{\mathbb{Z}}(\mathcal{C}, \mathcal{D})$ is additive. The direct sum of functors $F_1, \dots, F_n$ is the functor F with

$$F(X) = F_1(X) \oplus \cdots \oplus F_n(X)$$

for $X \in \mathrm{ob}(\mathcal{C})$.

Corollary. *If F is an additive functor between additive categories, then F preserves finite direct sums, so $F(0) = 0$ and $F(X \oplus Y) \cong F(X) \oplus F(Y)$.*

Proof. If X is a biproduct of $X_1, \dots, X_n$, then clearly $F(X)$ is a biproduct of $F(X_1), \dots, F(X_n)$. □

1.5 Abelian categories

Definition. A category is *abelian* if

(i) it is additive,

(ii) every morphism has a kernel and a cokernel,

(iii) every epi is a cokernel and every mono is a kernel.

Remarks. (1) The opposite of an abelian category is abelian. This saves work in proofs.

(2) An abelian category has all finite limits and colimits.

(3) Every mono is the kernel of its cokernel and every epi is the cokernel of its kernel. For example, suppose $f : X \to Y$ is mono, say a kernel of $g : Y \to W$, and suppose f has cokernel $c : Y \to Z$. Then $g = kc$ for some $k : Z \to W$. Now if $s : U \to Y$ is a morphism with $cs = 0$, then $gs = kcs = 0$, so s factors through f. It follows that f is a kernel of c.

Lemma. *In an abelian category a pullback of an epi is an epi and a pushout of a mono is a mono.*

Proof. Say

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow{d} & W \end{array}$$

is a pullback with d epi. We want to show that a is epi. We have morphisms

$$X \xrightarrow{\binom{a}{-b}} Y \oplus Z \xrightarrow{(c\ d)} W$$

where $(c\ d)$ comes from considering $Y \oplus Z$ as the coproduct of Y and Z and $\binom{a}{-b}$ comes from considering $Y \oplus Z$ as the product of Y and Z. Since the square is a pullback, $\binom{a}{-b}$ is the kernel of $(c\ d)$. Since d is an epi, so is $(c\ d)$. Thus by the remark above, $(c\ d)$ is the cokernel of $(a\ \ -b)$. Thus the square is a pushout. Suppose $f : Y \to U$ is a morphism with $fa = 0$. Since $fa = 0 = 0b$, by the pushout property there is a unique morphism $h : W \to U$ with $hc = f$ and $hd = 0$. Since d is epi, $h = 0$. Thus $f = 0$. □

Lemma. *Every morphism $f : X \to Y$ in an abelian category factors as a product $f = gh$ where h is an epi and g is a mono, and this decomposition is unique up to isomorphism, in fact h is a cokernel of the kernel of f and g is a kernel of the cokernel of f.*

Proof. Let $k : U \to X$ be a kernel of f and let $h : X \to Z$ be a cokernel of k. Let $h : X \to Z$ be a cokernel of k. Then f factors as gh for some $g : Z \to Y$. We show that g is mono, so suppose that $s : W \to Z$ is a morphism with $gs = 0$. Take the pullback

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ W & \xrightarrow{s} & Z \end{array}$$

By the previous result, q is an epi. Now $gsq = 0$, so $ghp = 0$, so $fp = 0$, so $p = kr$ for some $r : P \to U$. Then $sq = hp = hkr = 0$, so $s = 0$.

For uniqueness suppose that f factors as $X \xrightarrow{h} Z \xrightarrow{g} Y$ with h an epi and g a mono. Since g is mono, a kernel of f is also a kernel of h, so h is a cokernel of this. Similarly for g. □

Lemma. *A morphism in an abelian category is an isomorphism if and only if it is mono and epi.*

Proof. If $f : X \to Y$ is mono, then its kernel is $0 \to X$, and the cokernel of this is $X \to X$. □

Examples. (1) Ab is abelian and R-Mod is abelian. If R is a left noetherian ring, the category R-mod of finitely generated left modules is abelian. (The noetherian hypothesis ensures that the kernel of a morphism between f.g. modules is f.g.)

(2) If $\mathcal{C}$ is a preadditive category then $\mathrm{Fun}_{\mathbb{Z}}(\mathcal{C}, \mathrm{Ab})$ is abelian. Kernels and cokernels are computed objectwise: if $\alpha : F \to G$ is a natural transformation, then

$$(\operatorname{Ker} \alpha)(X) = \operatorname{Ker}(F(X) \to G(X)), \quad (\operatorname{Coker} \alpha)(X) = \operatorname{Coker}(F(X) \to G(X)).$$

Remark. A *subobject* of an object X in an abelian category is an equivalence class of monos to X, where $\alpha : U \to X$ is equivalent to $\alpha' : U' \to X \Leftrightarrow \alpha = \alpha'\theta$ for some isomorphism $\phi : U \to U'$. [There is possibly a set-theoretic problem here, which we ignore.]

Given a subobject $U \to X$ we denote its cokernel by $X \to X/U$.

Given a morphism $\theta : X \to Y$, the kernel of θ gives a subobject $\operatorname{Ker}\theta$ of X. The image $\operatorname{Im}\theta$ is the subobject of Y given by the morphism g in a factorization $\theta = gh$ with h epi and g mono.

We get analogues of the isomorphism theorems - details omitted.

1.6 Exact sequences

We work in an abelian category.

Definition. A sequence of objects and morphisms

$$\cdots \to L \xrightarrow{f} M \xrightarrow{g} N \to \cdots$$

is said to be *exact* at M if $\operatorname{Im} f = \operatorname{Ker} g$. The sequence is *exact* if it is exact at every place where morphisms come in and out. A *short exact sequence* is an exact sequence of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

Remarks. (1) Write f and g as compositions ba and dc with

$$L \xrightarrow{a} \operatorname{Im} f \xhookrightarrow{b} M \xrightarrow{c} \operatorname{Im} g \xhookrightarrow{d} N$$

Then we have: exact at M
$\Leftrightarrow$ b is a kernel for g (this is the definition)
$\Leftrightarrow$ b is a kernel for c (d is mono, so g and c have the same kernel)
$\Leftrightarrow$ c is a cokernel for b (since any epi is a cokernel for it kernel and any mono is a kernel for its cokernel)
$\Leftrightarrow$ c is a cokernel for f (since a is epi)

(2) $0 \to M \xrightarrow{g} N$ is exact at M if and only if g is a mono and $L \xrightarrow{f} M \to 0$ is exact at M if and only if f is an epi.

(3) A sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ is exact if and only if f is a kernel for g. A sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact if and only if g is a cokernel for f.

(4) $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence if and only if f is a kernel for g and g is a cokernel for f.

(5) Any subobject $U \to M$ gives a short exact sequence $0 \to U \to M \to M/U \to 0$.

(6) Any morphism $f : M \to N$ gives an exact sequence

$$0 \to \operatorname{Ker} f \to M \xrightarrow{f} N \to \operatorname{Coker} f \to 0$$

with $\operatorname{Coker} f = N/\operatorname{Im} f$ and short exact sequences

$$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0 \quad \text{and} \quad 0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0.$$

(7) If L and N are objects, their direct sum has morphisms

$$L \underset{p_L}{\overset{i_L}{\rightleftarrows}} L \oplus N \underset{p_N}{\overset{i_N}{\leftrightarrows}} N$$

and the sequence

$$0 \to L \xrightarrow{i_L} L \oplus N \xrightarrow{p_N} N \to 0,$$

is exact. For example, if $\theta : L \oplus N \to X$ is a morphism with $\theta i_L = 0$, then

$$\theta = \theta \operatorname{Id}_{L\oplus N} = \theta(i_L p_L + i_N p_N) = \theta i_N p_N$$

so θ factors through p_N.

Lemma. *For a short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

in an abelian category, the following conditions are equivalent, in which case the sequence is said to be split.

(i) f *is a* split monomorphism, *meaning that it has a* retraction, *a morphism* $r : M \to L$ *with* $rf = \operatorname{Id}_L$.

(ii) g *is a* split epimorphism, *meaning that it has a* section, *a morphism* $s : N \to M$ *with* $gs = \operatorname{Id}_N$.

(iii) There are morphisms

$$L \underset{f}{\overset{r}{\leftrightarrows}} M \underset{s}{\overset{g}{\rightleftarrows}} N$$

turning M *into a biproduct of* L *and* N.

(iv) There is an isomorphism $\theta : M \to L \oplus N$ giving a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{i_L} & L \oplus N & \xrightarrow{p_N} & N & \longrightarrow & 0. \end{array}$$

Proof. (i)⇒(iii). We have $(\mathrm{Id}_M - fr)f = f - frf = f - f = 0$. Thus since g is a cokernel for f we have $\mathrm{Id}_M - fr = sg$ for some $s : N \to M$. Now $gsg = g(\mathrm{Id}_M - fr) = g = \mathrm{Id}_N g$, so $gs = \mathrm{Id}_N$ since g is epi. Also $rsg = r(\mathrm{Id}_M - fr) = r - r = 0$, so $rs = 0$ since g is epi.

(ii)⇒(iii) is dual.

(iii)⇒(iv) is clear, since M is identified with $L \oplus N$.

(iv)⇒(i) and (ii) taking $r = p_L\theta$ and $s = \theta^{-1}i_N$. □

Lemma (Snake Lemma)**.** *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} (0 & \longrightarrow &)L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N'(& \longrightarrow & 0) \end{array}$$

there is a morphism $c : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ giving an exact sequence

$$(0 \to) \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{c} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma (\to 0).$$

Lemma (Five Lemma)**.** *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, so is γ.

Proof. For the category R-Mod, these are most easily proved by diagram chasing. For proofs in general, see §1 of B. Iversen, Cohomology of sheaves, Springer 1986. Alternatively, in the exercises starting on page 118 of Gelfand and Manin, Methods of Homological Algebra, Springer 2002, the results are proved by a generalized type of diagram chasing. □

Lemma. *Given a short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

The pullback of g along a morphism $\theta : N' \to N$ fits in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0 \\ & & \| & & \theta' \downarrow & & \theta \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

and the pushout of f along a morphism $\phi : L \to L''$ fits in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \phi \downarrow & & \phi' \downarrow & & \| & & \\ 0 & \longrightarrow & L'' & \xrightarrow{f''} & M'' & \xrightarrow{g''} & N & \longrightarrow & 0. \end{array}$$

Proof. Given θ there is a pullback given by g' and θ' and we have already seen that g' is epi. By the pullback property there is f' such that $\theta' f' = f$ and $g' f' = 0$. Now f' is clearly mono. It is a kernel for g', for if $h : X \to M'$ and $g'h = 0$ then $g\theta' h = \theta g' h = 0$, so $\theta' h = fk$ for some $k : X \to L$. Thus $\theta' f' k = \theta' h$. Now $f'k = h$ by the uniqueness property of the pullback. □

1.7 Exact functors

Definition. If F is an additive functor between abelian categories, we say that F is *exact* (respectively *left exact*, respectively *right exact*) if given any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$

is exact (respectively $0 \to F(X) \to F(Y) \to F(Z)$ is exact, respectively $F(X) \to F(Y) \to F(Z) \to 0$ is exact).

Similarly, if F is a contravariant functor, we want the sequence

$$0 \to F(Z) \to F(Y) \to F(X) \to 0$$

to be exact (respectively $0 \to F(Z) \to F(Y) \to F(X)$ exact, respectively $F(Z) \to F(Y) \to F(X) \to 0$ exact).

Remarks. (i) Any additive functor between abelian categories sends split exact sequences to split exact sequences (since it preserves direct sums).

(ii) An exact functor sends any exact sequence (not just a short exact sequence) to an exact sequence.

(iii) The following are equivalent:
(a) F is left exact
(b) F sends any exact sequence $0 \to X \to Y \to Z$ to an exact sequence $0 \to F(X) \to F(Y) \to F(Z)$
(c) F preserves kernels
(d) F preserves finite limits.
For (a)⇒(b), the exact sequence gives two short exact sequences. For (c)⇒(d) note that one can pass between kernels and equalizers, and construct finite limits using an equalizer and finite products.

Dually, the following are equivalent:
(a) F is right exact
(b) F sends an exact sequence $X \to Y \to Z \to 0$ to an exact sequence $F(X) \to F(Y) \to F(Z) \to 0$
(c) F preserves cokernels
(d) F preserves finite colimits.

Lemma. *For an object M in an abelian category $\mathcal{C}$, the functor $\mathrm{Hom}(M, -)$ is a left exact functor $\mathcal{C} \to Ab$ and the functor $\mathrm{Hom}(-, M)$ is a left exact contravariant functor from $\mathcal{C}$ to Ab. That is, if $0 \to X \to Y \to Z \to 0$ is exact, then so are*

$$0 \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y) \to \mathrm{Hom}(M, Z)$$

and

$$0 \to \mathrm{Hom}(Z, M) \to \mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M).$$

Proof. The first sequence is exact at $\mathrm{Hom}(M, Y)$ since $X \to Y$ is a kernel for $Y \to Z$, and it is exact at $\mathrm{Hom}(M, X)$ since $X \to Y$ is a mono. □

Remark. Note that if the sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is not split, then $\mathrm{Hom}(Z.-)$ and $\mathrm{Hom}(-, X)$ are not exact functors, for example if $\mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, Z)$ is surjective, then Id_Z lifts to an element $s \in \mathrm{Hom}(Z, Y)$ which is a section for g.

Lemma. *If (L, R) are a pair of adjoint additive functors between abelian categories, $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$, then L is right exact and R is left exact.*

Proof. We already did this for limits and colimits. Let's see it again. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact sequence in $\mathcal{C}$. For any object U in $\mathcal{D}$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Z, R(U)) \to \mathrm{Hom}_{\mathcal{C}}(Y, R(U)) \to \mathrm{Hom}_{\mathcal{C}}(X, R(U))$$

is exact. Hence so is

$$0 \to \mathrm{Hom}_{\mathcal{C}}(L(Z), U) \to \mathrm{Hom}_{\mathcal{C}}(L(Y), U) \to \mathrm{Hom}_{\mathcal{C}}(L(X), U).$$

Thus $L(g)$ is a cokernel of $L(f)$, so $L(X) \to L(Y) \to L(Z) \to 0$ is exact. Thus L is right exact. □

1.8 Filtered colimits

Remark. A poset $(I, \leq)$ is *directed* if it is non-empty and for all $x, y \in I$ there exists $z \in I$ with $x \leq z$ and $y \leq z$. For example the poset $\mathbb{N}$ is directed.

An *inverse limit* is a limit over the opposite of a directed poset. For example the ring of p-adic integers is

$$\hat{\mathbb{Z}}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/\mathbb{Z}p^n \quad \text{where} \quad \cdots \to \mathbb{Z}/\mathbb{Z}p^3 \to \mathbb{Z}/\mathbb{Z}p^2 \to \mathbb{Z}/\mathbb{Z}p \to \mathbb{Z}/\mathbb{Z}1.$$

On the other hand, a *direct limit* is a colimit over a directed poset. For example

$$\underrightarrow{\mathrm{colim}}_{n \in \mathbb{N}} \mathbb{Z}/\mathbb{Z}p^n \quad \text{where} \quad \mathbb{Z}/\mathbb{Z}1 \xrightarrow{p} \mathbb{Z}/\mathbb{Z}p \xrightarrow{p} \mathbb{Z}/\mathbb{Z}p^2 \xrightarrow{p} \mathbb{Z}/\mathbb{Z}p^3 \to \dots$$

is the union of the groups, the Prüfer group

$$\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \{z \in \mathbb{C} : z^{p^n} = 1 \text{ for some } n \geq 0\}.$$

More generally we shall consider colimits over small filtered categories. (In fact any filtered colimit can be turned into a direct limit, see Proposition 8.1.6 in Exposé I of SGA 4.)

Definition. A category $\mathcal{I}$ is *filtered* if

- it is non-empty
- for any objects i, j there is an object k and morphisms $i \to k$ and $j \to k$, and
- for any morphisms $a, b : i \to j$ there is a morphism $c : j \to k$ with $ca = cb$.

Lemma. *Let $\mathcal{I}$ be a small filtered category and M an $\mathcal{I}$-diagram in R-*Mod*. On the disjoint union*

$$\dot{\bigcup_{i\in \mathrm{ob}(\mathcal{I})}} M_i$$

consider the equivalence relation $\sim$ generated by the condition that $M_a(m) \sim m$ whenever $a : i \to j$ is a morphism in $\mathcal{I}$ and $m \in M_i$. Then:

(i) $m \in M_i \sim m' \in M_j \Leftrightarrow$ there exist $i \xrightarrow{a} k \xleftarrow{b} j$ in $\mathcal{I}$ with $M_a(m) = M_b(m')$. In particular, if $m \in M_i$, then $[m] = 0$ if and only if there is a morphism $a : i \to k$ in $\mathcal{I}$ such that $M_a(m) = 0$.

(ii) The set of equivalence classes $C := (\dot{\bigcup}_{i\in \mathrm{ob}(\mathcal{I})} M_i)/\sim$ is naturally an R-module in such a way that the mappings $\alpha_i : M_i \to C$, $m \mapsto [m]$ are module homomorphisms.

(iii) The mappings α_i turn C into a colimit for M, that is, $C = \mathrm{colim}_{i\in\mathcal{I}} M$.

The same thing works for filtered colimits in the category of sets.

Proof. (i) Consider the relation $\approx$ defined by this condition. It is clearly reflexive and symmetric. It suffices to show that it is transitive. Suppose $m \approx m'$ and $m' \approx m''$ with $m \in M_i$, $m' \in M_j$, $m'' \in M_k$. By filteredness there are

$$i \xrightarrow{a} p \xleftarrow{b} j \xrightarrow{c} q \xleftarrow{d} k$$

with $M_a(m) = M_b(m')$ and $M_c(m') = M_d(m'')$. By filteredness there are morphisms $p \xrightarrow{a'} r \xleftarrow{d'} q$. And then $a'b$ and $d'c$ are morphisms $j \to r$, so there is a morphism $f : r \to s$ with $fa'b = fd'c$. Then $M_{fa'a}(m) = M_{fa'b}(m') = M_{fd'c}(m') = M_{fd'd}(m'')$, so $m \approx m''$.

(ii) We turn C into an R-module as follows:
- If $m \in M_i$ and $r \in R$, then $r[m] := [rm]$.
- If $m \in M_i$ and $m' \in M_j$ then

$$[m] + [n] := [M_a(m) + M_b(m')]$$

for morphisms $i \xrightarrow{a} k \xleftarrow{b} j$ in $\mathcal{I}$.

Using filteredness one can show that this is well-defined. For example if $c : i \to i'$ we have $[m] = [M_c(m)]$, and we want

$$[M_a(m) + M_b(m')] = [M_{a'}(M_c(m)) + M_{b'}(m')]$$

where $i' \xrightarrow{a'} k' \xleftarrow{b'} j$. By filteredness there is are $k \xrightarrow{d} s \xleftarrow{d'} k'$ and then $f : s \to t$ such that $fda = fd'a'c$ and $fdb = fd'b'$. Then

$$[M_{a'}(M_c(m)) + M_{b'}(m')] = [M_{fd'}(M_{a'c}(m) + M_{b'}(m'))] = [M_{fd'a'c}(m) + M_{fd'b'}(m')]$$

$$= [M_{fda}(m) + M_{fdb}(m')] = [M_{fd}(M_a(m) + M_b(m'))] = [M_a(m) + M_b(m')].$$

Clearly this turns C into an R-module and the α_i are homomorphisms.

(iii) Clearly, if $a : i \to j$ then $\alpha_j M_a = \alpha_i$. Given a module X and homomorphisms $\beta_i : M \to X$ satisfying $\beta_j M_a = \beta_i$ for all $a : i \to j$, the β_i give a mapping

$$\dot{\bigcup}_{i \in \mathrm{ob}(\mathcal{I})} M_i \to X$$

and it is constant on equivalence classes, so it defines a homomorphism $\theta : C \to X$ satisfying $\theta\alpha_i = \beta_i$ for all i. Clearly θ is uniquely determined. Thus we have the universal property. □

Theorem. *The category* R-Mod *has exact filtered colimits. That is, suppose $\mathcal{I}$ is a small filtered category. Let L, M, N be $\mathcal{I}$-diagrams in* R-Mod *and let $\alpha : L \to M$ and $\beta : M \to N$ be natural transformations. If for all i, the sequences of R-modules*

$$L_i \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} N_i$$

are exact, then so is the induced sequence

$$\operatorname*{colim}_{i \in \mathcal{I}} L \to \operatorname*{colim}_{i \in \mathcal{I}} M \to \operatorname*{colim}_{i \in \mathcal{I}} N$$

Proof. Note that for any index set $\mathcal{I}$, $\lim_{i \in \mathcal{I}} : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ is right adjoint to the constant functor $c : \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$, so is left exact. In the same way, $\mathrm{colim}_{i \in \mathcal{I}} : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ is left adjoint to the constant functor, so is right exact. We get exactness using the lemma. Take an element $x \in \mathrm{colim}_{i \in \mathcal{I}} M$ sent to zero in $\mathrm{colim}_{i \in \mathcal{I}} N$. Now x is represented by an element $m \in M_i$ for some i. Then $\beta_i(m)$ represents the zero element, so there is some $a : i \to j$ such that $N_a(\beta_i(m)) = 0$. Thus $\beta_i(M_a(m)) = 0$. Thus $M_a(m) = \alpha_j(\ell)$ for some $\ell \in L_j$. But then x is the image of the element in $\mathrm{colim}_{i \in \mathcal{I}} L$ represented by ℓ. □

Definition. A *Grothendieck category* is an abelian category with the following additional properties:

- It is cocomplete. (Since it is abelian, it is equivalent that it has arbitrary coproducts, which is (AB3) in Grothendieck's terminology.)
- It has a *generator* that is, an object G such that for any object X there is an epimorphism from a coproduct of copies of G to X.

- It has exact filtered colimits (or equivalently, in Grothendieck's terminology, (AB5)).

Examples. Module categories are Grothendieck categories. As are functor categories with values in a Grothendieck category, such as Ab. Also categories of graded modules. Also the category of quasicoherent sheaves on a noetherian scheme.

Remark. Given exact sequences $0 \to X_i \to Y_i \to Z_i \to 0$ $(i \in I)$ in a cocomplete abelian category, we get a natural sequence

$$0 \to \coprod_i X_i \to \coprod_i Y_i \to \coprod_i Z_i \to 0$$

which is in general only right exact. It is exact on the left in the following cases:
(a) For finite index sets I, since then the coproducts are also finite products, so the sequence is naturally left exact.
(b) For module categories - easy to prove using that the coproduct is the direct sum of modules.
(c) In case filtered colimits are exact, since

$$\coprod_{i \in I} X_i \cong \operatorname*{colim}_{\text{finite } F \subseteq I} \coprod_{i \in F} X_i$$

where the colimit is over the directed poset of finite subsets F of I.

Definition. An R-module M is *finitely presented* (f.p.) if it is a quotient of a finitely generated free module by a finitely generated submodule. Equivalently if there is an exact sequence $R^m \to R^n \to M \to 0$.

Any quotient of a f.p. module by a f.g. submodule is f.p. If R is left noetherian, any f.g. left R-module is f.p.

Theorem. *Every R-module is a filtered colimit of f.p. modules. More generally, if M is a module and $\mathcal{C}$ is a full subcategory of the category of f.p. R-modules such that every map from a f.p. module to M factors through a module in $\mathcal{C}$, then M is a filtered colimit of modules in $\mathcal{C}$.*

Proof. We may assume that $\mathcal{C}$ is small. Let $\mathcal{I}$ be the category with:

- Objects are pairs (X, f) with $X \in \operatorname{ob}(\mathcal{C})$ and $f \in \operatorname{Hom}(X, M)$.

- Morphisms $(X, f) \to (X', f')$ are morphisms $\theta : X \to X'$ with $f'\theta = f$.

This category is usually denoted $\mathcal{C}/M$. It is filtered:
- It is nonempty since the zero map $0 \to M$ must factor.
- Given objects (X, f) and (X', f'), the morphism

$$(f\ f') : X \oplus X' \to M$$

factors through an object in $\mathcal{C}$, say as

$$X \oplus X' \xrightarrow{(g\ g')} X'' \xrightarrow{f''} M.$$

Then we have morphisms $g : (X, f) \to (X'', f'')$ and $g' : (X', f') \to (X'', f'')$.
- Given morphisms $\alpha, \beta : (X, f) \to (X', f')$, we have $f'(\alpha - \beta) = 0$, so taking the cokernel

$$X \xrightarrow{\alpha-\beta} X' \xrightarrow{\gamma} \mathrm{Coker}(\alpha - \beta) \to 0,$$

we get $f' = h\gamma$ for some $h : \mathrm{Coker}(\alpha - \beta) \to M$. But then h factors through an object X'' in $\mathcal{C}$

$$\mathrm{Coker}(\alpha - \beta) \xrightarrow{\phi} X'' \xrightarrow{f''} M.$$

Then $\phi\gamma : (X', f) \to (X'', f'')$ and $\phi\gamma\alpha = \phi\gamma\beta$.

Let $F : \mathcal{I} \to R\text{-Mod}$ be the $\mathcal{I}$-diagram sending an object (X, f) to X and a morphism θ to θ. Let

$$L = \operatorname*{colim}_{(X,f)\in\mathcal{I}} F(X, f) = \operatorname*{colim}_{(X,f)\in\mathcal{I}} X.$$

It is equipped with morphisms $\alpha_{(X,f)} : X \to L$ for each (X, f). For each object (X, f) in $\mathcal{I}$, we have the morphism $f : X \to M$. Thus by the universal property, there is a unique morphism $\beta : L \to M$ such that $\beta\alpha_{(X,f)} = f$ for each (X, f). We want to show β is an isomorphism.

For an element x in a module X we write $\hat{x}$ for the map $R \to X$, $r \mapsto rx$. For any $m \in M$, the map $\hat{m} : R \to M$ factors through an object X in $\mathcal{C}$, say as

$$R \xrightarrow{\hat{x}} X \xrightarrow{f} M.$$

Then (X, f) is an object in $\mathcal{I}$ and

$$m = f(x) = \beta(\alpha_{(X,f)}(x)) \in \mathrm{Im}(\beta)$$

so β is surjective. Suppose $\ell \in L$ and $\beta(\ell) = 0$. Then ℓ is represented by an element $x \in X$ for an object $(X, f) \in \mathrm{ob}(\mathcal{I})$. Since it is sent to 0 in M, we have $f(x) = 0$. Taking the cokernel

$$R \xrightarrow{\hat{x}} X \xrightarrow{\phi} \mathrm{Coker}(\hat{x}) \to 0$$

we have $f = h\phi$ for some $h : \operatorname{Coker}(\hat{x}) \to M$. Then h factors as

$$\operatorname{Coker}(\hat{x}) \xrightarrow{g} X' \xrightarrow{f'} M$$

with X' in $\mathcal{C}$. Then $g\phi : (X, f) \to (X', f')$ and $g\phi(x) = 0$, so x represents the zero element in the colimit, that is, $\ell = 0$. □

Proposition. *A module X is finitely presented if and only if* $\operatorname{Hom}(X, -)$ *commutes with filtered colimits, that is, for any filtered category $\mathcal{I}$ and $\mathcal{I}$-diagram M, the canonical map*

$$\operatorname*{colim}_{i\in\mathcal{I}} \operatorname{Hom}(X, M_i) \to \operatorname{Hom}(X, \operatorname*{colim}_{i\in\mathcal{I}} M_i)$$

is bijective.

Proof. Given a presentation $R^m \to R^n \to X \to 0$, since filtered colimits preserve exact sequences we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{colim}\operatorname{Hom}(X, M_i) & \longrightarrow & \operatorname{colim}\operatorname{Hom}(R^n, M_i) & \longrightarrow & \operatorname{colim}\operatorname{Hom}(R^m, M_i) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \operatorname{Hom}(X, \operatorname{colim} M_i) & \longrightarrow & \operatorname{Hom}(R^n, \operatorname{colim} M_i) & \longrightarrow & \operatorname{Hom}(R^m, \operatorname{colim} M_i)
\end{array}$$

Now the right hand vertical maps are isomorphisms (for example this follows easily from our construction of filtered colimits for modules), hence so is the left hand vertical map by the Five Lemma.

Conversely, suppose that $\operatorname{Hom}(X, -)$ commutes with filtered colimits. Write $X = \operatorname{colim}_{i\in\mathcal{I}} M_i$, a filtered colimit of f.p. modules. Then

$$\operatorname{Id}_X \in \operatorname{Hom}(X, X) = \operatorname{Hom}(X, \operatorname*{colim}_{i\in\mathcal{I}} M_i) = \operatorname*{colim}_{i\in\mathcal{I}} \operatorname{Hom}(X, M_i).$$

This is a colimit of $\mathbb{Z}$-modules, so Id_X is represented by some element of $\operatorname{Hom}(X, M_i)$. This means that Id_X can be factored as a composition $X \to M_i \to \operatorname{colim}_{i\in\mathcal{I}} M_i = X$. This means that X is a direct summand of M_i. Now M_i is f.p. and hence so is X. □

2 Projective, injective and flat modules

2.1 Projective modules

Proposition/Definition. *An object P in an abelian category is* projective *if it satisfies the following equivalent conditions.*
(i) $\mathrm{Hom}(P,-)$ *is an exact functor.*
(ii) Any short exact sequence $0 \to X \to Y \to P \to 0$ *is split.*
(iii) Given any epimorphism $\theta : Y \twoheadrightarrow Z$*, any morphism* $P \to Z$ *factors through* θ*.*

Proof. (i)⇒(ii) $\mathrm{Hom}(P,Y) \to \mathrm{Hom}(P,P)$ is onto. A lift of Id_P is a section.
(ii)⇒(iii) Take the pullback along the map $P \to Z$. The resulting exact sequence has P as third term, so is split. This gives a map from P to the pullback. Composing with the map to Y gives the map $P \to Y$.
(iii)⇒(i) Clear. □

Proposition. *A coproduct* $\coprod_i M_i$ *is projective* ⇔ *all* M_i *are projective.*

Proof. $\coprod_i M_i$ is projective
⇔ the functor $\mathrm{Hom}(\coprod_i M_i,-) = \prod_i \mathrm{Hom}(M_i,-)$ is exact
⇔ $0 \to \mathrm{Hom}(\coprod_i M_i,X) \to \mathrm{Hom}(\coprod_i M_i,Y) \to \mathrm{Hom}(\coprod_i M_i,Z) \to 0$ exact for all exact sequences $0 \to X \to Y \to Z \to 0$
⇔ $0 \to \prod_i \mathrm{Hom}(M_i,X) \to \prod_i \mathrm{Hom}(M_i,Y) \to \prod_i \mathrm{Hom}(M_i,Z) \to 0$ exact
⇔ $0 \to \mathrm{Hom}(M_i,X) \to \mathrm{Hom}(M_i,Y) \to \mathrm{Hom}(M_i,Z) \to 0$ are exact. (To see this, use that products of abelian groups are just the cartesian products.)
⇔ all M_i are projective. □

Theorem. *Let R be a ring. An R-module is projective if and only if it is a direct summand of a free module. A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module R^n, for some n. Any f.g. projective module is f.p.*

Proof. $\mathrm{Hom}_R(R,X) \cong X$, so R is a projective module, hence so is any direct sum of copies of R. If $F \to P$ is onto with F free and P projective, then P is isomorphic to a summand of F. □

We write R-proj for the category of finitely generated projective left R-modules.

Lemma. *The functor* $\mathrm{Hom}_R(-,R)$ *defines an antiequivalence between* R-proj *and* R^{op}-proj.

Proof. Observe that if M is a left R-module, then $\mathrm{Hom}_R(M,R)$ is naturally a right module, and if M is free of rank n, then

$$\mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_R(R^n,R) \cong \mathrm{Hom}_R(R,R)^n \cong (R_R)^n$$

is a free right R-module of rank n. If P is f.g. projective, then there is Q with $P \oplus Q \cong R^n$. Then

$$\mathrm{Hom}_R(P,R) \oplus \mathrm{Hom}_R(Q,R) \cong (R_R)^n$$

so $\mathrm{Hom}_R(P,R)$ is f.g. projective. The inverse equivalence is given by the same construction, but for right R-modules. There is a natural transformation

$$X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X,R),R), \quad x \mapsto (\theta \mapsto \theta(x)).$$

It is an isomorphism for $X = R$, so for finite direct sums of copies of R, so for f.g. projective modules. □

Examples. (i) Every R-module is projective ⇔ Every short exact sequence is split ⇔ every submodule of a module has a complement ⇔ Every module is semisimple ⇔ R is a semisimple (artinian) ring R ⇔ (the Artin-Wedderburn Theorem) R is a finite direct sum of matrix rings over division rings.

(ii) Sometimes all f.g. projective modules are free. For example:
(a) If R is a principal ideal domain. Namely, a standard theorem says that any f.g. module is a finite direct sum of cyclic modules. Now if $0 \neq a \in R$, then

$$\mathrm{Hom}_R(R/Ra,R) = \{r \in R : ar = 0\} = 0$$

so R/Ra cannot be projective (unless it is 0). Thus every f.g. projective module is a direct sum of copies of R.
(b) If R is a commutative local ring. This is an exercise using Nakayama's lemma.
(c) If R is a polynomial ring $K[X_1,\dots,X_n]$ with K a field. For $n > 1$ this is the (hard) Quillen-Suslin Theorem (1976).

(iii) Idempotents (elements with $e^2 = e$) give projective modules. Recall that for an R-module M, there is a 1-1 correspondence

$$\text{Decompositions } M = X \oplus Y \quad \underset{e \mapsto M = \mathrm{Im}\, e \oplus \mathrm{Im}(1-e)}{\overset{X\oplus Y \mapsto \text{proj onto } X}{\rightleftarrows}} \quad \text{Idempotents } e \in \mathrm{End}_R(M).$$

Taking $M = R$ and $\mathrm{End}(M) = R^{op}$, idempotents $e \in R$ give decompositions $R = Re \oplus R(1-e)$. For example, in $R = M_n(K)$ the idempotents E^{ii} give the decomposition

$$R = RE^{11} \oplus \cdots \oplus RE^{nn} = C_1 \oplus \cdots \oplus C_n$$

where C_i is the matrices living in the ith column. Since $E^{ij}E^{ji} = E^{ii}$ and $E^{ji}E^{ij} = E^{jj}$, right multiplication by E^{ij} gives an isomorphism $C_i \to C_j$. On the other hand, let R be the ring of 2×2 upper triangular matrices with entries in K. Then

$$R = RE^{11} \oplus RE^{22} = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}.$$

Now the two summands are not isomorphic.

(iv) An example of a ring R with a module $P \ncong R$ such that $P^2 \cong R^2$, so P is projective. Let

$$R = \{\text{continuous } f : [0, \pi] \to \mathbb{R} : f(0) = f(\pi)\}.$$

If $f \in R$ is idempotent, then $f(x)^2 = f(x)$ for all x, so $f(x) \in \{0, 1\}$. So by continuity $f = 0$ or 1. Let

$$P = \{\text{continuous } f : [0, \pi] \to \mathbb{R} : f(0) = -f(\pi)\}.$$

It is naturally an R-module. Now $R \ncong P$ since if there is an isomorphism sending $1 \in R$ to $g \in P$, then it sends any f to fg. By the Intermediate Value Theorem $g(x) = 0$ for some $0 < x < \pi$. But then every element of P vanishes at x, which is nonsense. On the other hand, there are inverse isomorphisms between R^2 and P^2 given by

$$\begin{pmatrix} f & g \end{pmatrix} \mapsto \begin{pmatrix} f & g \end{pmatrix} \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}$$

(See page 28 of T.-Y. Lam, Lectures on Modules and Rings, Springer 1999.)

(v) Connection with vector bundles. The example above is the following theorem with X the circle and the vector bundle given by the Möbius band.

Swan's Theorem (1962). The global section functor gives an equivalence between the category of topological vector bundles on a compact Hausdorff topological space X and the category of f.g. projective modules for its ring of continuous functions $C(X)$.

Earlier was:

Serre's Theorem (1955). The global section functor gives an equivalence between the category of algebraic vector bundles on an affine variety X and the category of f.g. projective modules for its coordinate ring $K[X]$.

Thus by the Quillen-Suslin Theorem, every algebraic vector bundle on affine n-space $\mathbb{A}^n$ is trivial.

Definition. (The beginnings of K-theory) The *Grothendieck group* $K_0(R)$ of a ring R is the $\mathbb{Z}$-module generated by the isomorphism classes $[P]$ of f.g. projective R-modules P, subject to the relations $[P \oplus Q] = [P] + [Q]$ for all P, Q.

This definition would not be interesting without the restriction that the modules P are finitely generated, since for every projective module P, there is a free module F (not finitely generated), with $P \oplus F \cong P$. Namely, there is some free module

F_0 with $P \oplus Q \cong F_0$. Let F be the direct sum of a countable number of copies of F_0. Then

$$\begin{aligned} F &\cong F_0 \oplus F_0 \oplus \ldots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \ldots \\ &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \ldots \cong P \oplus F_0 \oplus F_0 \oplus \ldots \cong P \oplus F. \end{aligned}$$

This is 'Eilenberg's swindle'.

Definition. Suppose R is an integral domain (commutative) with field of fractions K. A *fractional ideal* is a nonzero R-submodule I of K such that $I \subseteq d^{-1}R$ for some nonzero $d \in R$. For example any nonzero ideal in R is a fractional ideal. If I and J are fractional ideals, then

$$IJ := \{\sum_{i=1}^{n} x_i y_i : n \geq 0, x_i \in I, y_i \in J\}$$

is another fractional ideal and so is

$$I^{-1} := \{y \in K : Iy \subseteq R\}.$$

Clearly $II^{-1} \subseteq R$ always.

Lemma. *If R is an integral domain and I is a fractional ideal, the following are equivalent:*
(a) $II^{-1} = R$.
(b) I is invertible*, meaning that there is a fractional ideal J with $IJ = R$.*
(c) I is f.g. projective
(d) I is projective

Proof. (a)⇒(b) is trivial.

(b)⇒(c). Write $1 = \sum_{k=1}^{n} i_k j_k$ with $i_k \in I$ and $j_k \in J$. Consider $f : I \to R^n$ and $g : R^n \to I$ given by $f(a) = (aj_k)$ and $g((r_k)) = \sum_k r_k i_k$. The former makes sense since $aj_k \in R$ for all $a \in I$. Then $gf = \mathrm{Id}_I$, so I is a direct summand of R^n.

(c)⇒(d) is trivial.

(d)⇒(a). For some indexing set Λ there are maps $I \xrightarrow{f} R^{(\Lambda)} \xrightarrow{g} I$ with $gf = \mathrm{Id}_I$. Now f is given by maps $f_\lambda : I \to R$ such that for any $a \in I$, only finitely many $f_\lambda(a)$ are nonzero, and g is of the form $(r_\lambda) \mapsto \sum_\lambda r_\lambda i_\lambda$ for some elements $i_\lambda \in I$. By assumption there is nonzero $d \in R$ with $dI \subseteq R$. Choose $0 \neq a \in I$. Then $a^{-1} f_\lambda(a) \in I^{-1}$, for if $b \in I$, then $da, db \in R$, so

$$ba^{-1} f_\lambda(a) = (ad)^{-1} db f_\lambda(a) = (ad)^{-1} f_\lambda(dba) = (ad)^{-1} ad f_\lambda(b) = f_\lambda(b) \in R.$$

Then

$$II^{-1} \ni \sum_{\lambda \in \Lambda} i_\lambda a^{-1} f_\lambda(a) = a^{-1} g(f(a)) = a^{-1} a = 1.$$

so $II^{-1} = R$. □

Example. In $R = \mathbb{Z}[\sqrt{-5}]$, consider the ideal

$$I = (2, 1 + \sqrt{-5}) = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}, a \equiv b(\mathrm{mod} 2)\}.$$

We have

$$\begin{aligned} I^{-1} &= \{x = a + b\sqrt{-5} : a, b \in \mathbb{Q}, 2x, (1 + \sqrt{-5})x \in R\} \\ &= \{x = a + b\sqrt{-5} : 2a, 2b, a - 5b, a + b \in \mathbb{Z}\} \\ &= \frac{1}{2}I = R1 + R\frac{1 + \sqrt{-5}}{2} \end{aligned}$$

and

$$1 = (1 + \sqrt{-5}) \cdot (1 - \frac{1 + \sqrt{-5}}{2}) - 2 \cdot 1 \in II^{-1}.$$

So $II^{-1} = R$. Thus I is a projective module. But it is not free:
- It is not a principal ideal, so not isomorphic to R.
- It is not a free module on more than one generator, since for a nonzero commutative ring, there can be no embedding of the module $R \oplus R$ into R (exercise: consider the product of the images of $(1, 0)$ and $(0, 1)$.).

Remark. For an integral domain R, one can show that the following are equivalent, in which case R is a *Dedekind domain*.
- All ideals in R are projective
- All fractional ideals are invertible.
- R is noetherian of Krull dimension ≤ 1 (that is, all non-zero prime ideals are maximal) and integrally closed in its field of fractions.

For example the ring of integers of a number field. In this case one can show that $K_0(R) \cong \mathbb{Z} \oplus Cl(R)$, where $Cl(R)$ is the *ideal class group*, the group of fractional ideals modulo the subgroup of principal fractional ideals.

2.2 Tensor products

If X is a right R-module and Y is a left R-module, the tensor product $X \otimes_R Y$ is a $\mathbb{Z}$-module $X \otimes_R Y$ equipped with a mapping

$$X \times Y \to X \otimes_R Y, \quad (x, y) \mapsto x \otimes y$$

such that the mapping is a homomorphism of additive groups in each argument, and R-balanced, meaning that

$$xr \otimes y = x \otimes ry$$

for all $x \in X$, $y \in Y$ and $r \in R$, and such that it is universal for this property, that is, if

$$f : X \times Y \to M$$

is additive in each argument and R-balanced, then there is a unique $\mathbb{Z}$-module homomorphism $\alpha : X \otimes_R Y \to M$ such that $f(x, y) = \alpha(x \otimes y)$.

Theorem. *(i) The tensor product exists and it is unique up to isomorphism.*
(ii) Any element can be written (non-uniquely) as a finite sum

$$x_1 \otimes y_1 + \cdots + x_n \otimes y_n.$$

(iii) $X \otimes_R R \cong X$ *and* $R \otimes_R Y \cong Y$ *via the maps* $x \otimes r \mapsto xr$ *and* $r \otimes y \mapsto ry$.
(iv) If $\theta : X \to X'$ *and* $\phi : Y \to Y'$ *are module homomorphisms, then there is a unique* $\mathbb{Z}$*-module homomorphism*

$$\theta \otimes \phi : X \otimes_R Y \to X' \otimes_R Y'$$

with $(\theta \otimes \phi)(x \otimes y) = \theta(x) \otimes \phi(y)$.
(v) We can identify $X \otimes_R Y$ *with* $Y \otimes_{R^{op}} X$.

For a proof see my Algebra II notes.

Definition. If S and R are rings, an S-R-*bimodule* X is given by a left S-module and a right R-module with the same underlying additive group, and such that the actions commute: $(sx)r = s(xr)$.

Theorem. *Let* X *be an* S-R*-bimodule. If* Y *is a left* R*-module, then* $X \otimes_R Y$ *becomes an* S*-module with* $s(x \otimes y) = (sx) \otimes y$*. This gives a tensor product functor*

$$X \otimes_R - : R\text{-Mod} \to S\text{-Mod}.$$

If Z *is an* S*-module, then* $\mathrm{Hom}_S(X, Z)$ *becomes an* R*-module via* $(r\theta)(x) = \theta(xr)$*. This gives a functor*

$$\mathrm{Hom}_S(X, -) : S\text{-Mod} \to R\text{-Mod}.$$

Moreover there is an isomorphism of additive groups

$$\mathrm{Hom}_S(X \otimes_R Y, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z))$$

which is natural in Y *and* Z*. Thus* $(X \otimes_R -, \mathrm{Hom}_S(X, -))$ *is an adjoint pair of functors.*

Proof. The first parts are straightforward. Given $\theta \in \mathrm{Hom}_S(X \otimes_R Y, Z)$ we get $\phi \in \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z))$ by $\phi(y)(x) = \theta(x \otimes y)$, and given ϕ we get θ by the same formula. □

After the results about adjoint functors, we get.

Corollary. *If X is an S-R-bimodule, then the tensor product functor $X \otimes_R - : R\text{-Mod} \to S\text{-Mod}$ preserves colimits, so it is right exact and commutes with direct sums (coproducts):*

$$X \otimes_R \left(\bigoplus_{i \in I} Y_i \right) \cong \bigoplus_{i \in I} (X \otimes_R Y_i) .$$

(Similarly for the functor $- \otimes_R Y : \text{Mod-}R \to \text{Mod-}T$ for an R-T-bimodule Y.)

Theorem (Eilenberg, Watts)**.** *Any functor $F : R\text{-Mod} \to S\text{-Mod}$ which preserves colimits, that is, is right exact and commutes with direct sums, is naturally isomorphic to a tensor product functor $X \otimes_R -$ for some S-R-bimodule X.*

Proof. $F(R)$ is an S-module, and it becomes an S-R-bimodule via the map

$$R^{op} \xrightarrow{\cong} \text{End}_R(R) \xrightarrow{F} \text{End}_S(F(R)).$$

Now for any R-module Y there is a R-module map

$$Y \xrightarrow{\cong} \text{Hom}_R(R, Y) \xrightarrow{F} \text{Hom}_S(F(R), F(Y)).$$

By Hom-Tensor adjointness this corresponds to an S-module map $F(R) \otimes_R Y \to F(Y)$. This is natural in Y, so it is Φ_Y for some natural transformation $\Phi : F(R) \otimes_R - \to F$. Clearly Φ_R is an isomorphism. Then for any free module $R^{(I)}$ we have $F(R^{(I)}) = F(R)^{(I)} \cong F(R) \otimes R^{(I)}$, so $\Phi_{R^{(I)}}$ is an isomorphism. Then for any module Y there is a presentation $R^{(I)} \to R^{(J)} \to Y \to 0$ and the first two vertical maps in the diagram

$$\begin{array}{ccccccc}
F(R) \otimes R^{(I)} & \longrightarrow & F(R) \otimes R^{(J)} & \longrightarrow & F(R) \otimes Y & \longrightarrow & 0 \\
\Phi_{R^{(I)}} \downarrow & & \Phi_{R^{(J)}} \downarrow & & \Phi_Y \downarrow & & \\
F(R^{(I)}) & \longrightarrow & F(R^{(J)}) & \longrightarrow & F(Y) & \longrightarrow & 0
\end{array}$$

are isomorphisms. Also the rows are exact. Hence the third vertical map is an isomorphism. Thus Φ is a natural isomorphism. □

Lemma. *If X is an S-R-bimodule, then there is a homomorphism of additive groups*

$$\phi_{U,Y} : \text{Hom}_S(U, X) \otimes_R Y \to \text{Hom}_S(U, X \otimes_R Y), \quad \theta \otimes y \mapsto (u \mapsto \theta(u) \otimes y)$$

for U an S-module and Y an R-module, which is a natural transformation in U and Y. It is an isomorphism if U is f.g. projective. Conversely, taking $X = R = S$, if Id_U is in the image of the map

$$\phi_{U,U} : \text{Hom}_S(U, S) \otimes_S U \to \text{Hom}_S(U, S \otimes_S U) \cong \text{End}_S(U),$$

then U is finitely generated projective.

Proof. The first part is clear. The map $\phi_{S,Y}$ is an isomorphism since it is identified with the identity map since $\mathrm{Hom}_S(S,X)\otimes_R Y$ and $\mathrm{Hom}(S, X\otimes_R Y)$ can both be identified with $X\otimes_R Y$. Now given a direct sum $U = U_1 \oplus \cdots \oplus U_n$ we get

$$\mathrm{Hom}_S(U,X)\otimes_R Y \cong \bigoplus_{i=1}^{n}(\mathrm{Hom}_S(U_i,X)\otimes_R Y)$$

and

$$\mathrm{Hom}_S(U, X\otimes_R Y) \cong \bigoplus_{i=1}^{n}\mathrm{Hom}_S(U_i, X\otimes_R Y)$$

so $\phi_{U,Y}$ corresponds to the mapping whose components are $\phi_{U_i,Y}$, so $\phi_{U,Y}$ is a bijection if and only if all $\phi_{U_i,Y}$ are bijections. Thus $\mathrm{Hom}(S^n, Y)$ is an isomorphism, and hence so is $\phi_{U,Y}$ for any direct summand U of a f.g. free module S^n.

Say Id_U comes from $\sum_i \theta_i \otimes u_i$, then the composition of the maps

$$U \xrightarrow{(\theta_i)} S^n \xrightarrow{(u_i)} U$$

is the identity, so U is a direct summand of S^n, so f.g. projective. □

2.3 Morita equivalence

Recall that an R-module P is a generator if for any module M there is an epi from a direct sum of copies of P to M, $P^{(I)} \to M$.

Theorem (Morita equivalence)**.** *Let R and S be rings. The following are equivalent.*
(i) The categories R-Mod and S-Mod are equivalent
(ii) $R \cong \mathrm{End}_S(X)^{op}$ for some f.g. projective generator X in S-Mod.
(iii) There is an S-R-bimodule X giving an equivalence $X\otimes_R - : R\text{-Mod} \to S\text{-Mod}$.

Proof. (i)⇒(ii) Let $F : R\text{-Mod} \to S\text{-Mod}$ be an equivalence and let $X = F(R)$. Since F is fully faithful we have $R \cong \mathrm{End}_S(X)^{op}$. Since F is an equivalence, X is projective. Clearly $\mathrm{Hom}_R(R,-)$ commutes with filtered colimits, so $\mathrm{Hom}_S(X,-)$ commutes with filtered colimits, so X is finitely presented, so it is a f.g. projective. Also, for any R-module M there is an epimorphism from a free module $R^{(I)} \to M$. Since F is an equivalence, applying F we get an epimorphism $X^{(I)} \to F(M)$. Since any S module is isomorphic to one of the form $F(M)$, this shows that X is a generator.

(ii)⇒(iii) For any S-module T we have a mapping

$$X\otimes_R \mathrm{Hom}_S(X,T) \to T, \quad x\otimes\theta \mapsto \theta(x)$$

This is natural in T, and it is an isomorphism for $T = X$. Thus it is an isomorphism for $T = X^n$. Thus it is an isomorphism for T any summand of X^n. Now X is a generator as an S-module, and ${}_SS$ is finitely generated, so there is an epimorphism $X^n \to S$ for some n. Then since ${}_SS$ is projective, S is isomorphic to a summand of X^n. Thus we get an isomorphism

$$X \otimes_R \mathrm{Hom}_S(X, S) \to S, \quad x \otimes \theta \mapsto \theta(x)$$

This is an isomorphism of S-S-bimodules. Also, by the lemma above applied to the S-S-bimodule S, we have an isomorphism

$$\mathrm{Hom}_S(X, S) \otimes_S X \mapsto \mathrm{Hom}_S(X, S \otimes_S X) \cong R$$

and this is an isomorphism of R-R-bimodules. Thus the functors $X \otimes_R - :$ R-Mod $\to$ S-Mod and $\mathrm{Hom}_S(X, S) \otimes_S - :$ S-Mod $\to$ R-Mod are inverses (up to natural isomorphisms) so they are equivalences.

(iii)⇒(i) is trivial □

Examples. (a) Any ring S is Morita equivalent to $M_n(S)$ for $n \geq 1$. Namely the module S^n is a finitely generated projective generator for S-Mod, with $\mathrm{End}_S(S^n)^{op} \cong M_n(S)$.

(b) If $e \in S$ is idempotent, and $SeS = S$, then S is Morita equivalent to eSe (which is a ring under multiplication with identity element e). Namely, the condition ensures that the multiplication map $Se \otimes_{eSe} eS \to S$ is onto. Taking a map from a free eSe-module onto eS, say $eSe^{(I)} \to eS$, we get a map $Se^{(I)} \to S$, so Se is a generator. Then $\mathrm{End}_S(Se)^{op} \cong eSe$.

(c) If S is a finite-dimensional algebra over a field (or more generally a left artinian ring), we can write ${}_SS = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ with the P_i indecomposable modules. We say that S is *basic* if the P_i are pairwise non-isomorphic. (By the Krull-Remak-Schmidt Theorem this doesn't depend on the choice of decomposition.)

Now any finite-dimensional algebra S is Morita equivalent to a basic one. Namely, take X to be one a direct sum of representatives of the isomorphism classes of the P_i, and let $R = \mathrm{End}_S(X)^{op}$. Then X is a f.g. projective generator for S, so by (ii)⇒(iii) in the theorem we get an equivalence $X \otimes_R - :$ R-Mod $\to$ S-Mod. This sends ${}_RR$ to X. Now X is a direct sum of pairwise nonisomorphic indecomposable summands, hence so is ${}_RR$.

For example if S is the subalgebra of $M_3(K)$ given by matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

then the decomposition of ${}_SS = P_1 \oplus \cdots \oplus P_n$ has the form

$$ {}_SS = \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}. $$

Now $P_1 \cong P_2$, so we take $X = P_1 \oplus P_3 = Se$ where

$$ e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

Then $R = \mathrm{End}_S(P_1 \oplus P_3)^{op} \cong eSe$. This is isomorphic to the subalgebra of $M_2(K)$ given by matrices of shape

$$ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}. $$

2.4 Injective modules

Proposition/Definition. *An object E in an abelian category is* injective *if it satisfies the following equivalent conditions.*
(i) $\mathrm{Hom}(-, E)$ *is an exact (contravariant) functor.*
(ii) Any short exact sequence $0 \to E \to Y \to Z \to 0$ is split.
(iii) For any monomorphism $\theta : X \hookrightarrow Y$, any morphism $X \to E$ factors through θ.

Proposition. *A product $\prod_{i \in I} M_i$ is injective $\Leftrightarrow$ all M_i are injective. Thus a finite direct sum is injective if and only if each term is injective.*

Proof. This is the opposite category version of the result for projectives. Then a finite direct sum is the same as a finite product. □

Definition. An inclusion of R-modules $M \subseteq N$ is an *essential extension* of M if every non-zero submodule S of N has $S \cap M \neq 0$.

Theorem. *For an R-module M, following conditions are equivalent.*
(a) M is injective.
(b) (Baer's criterion) Every homomorphism $f : I \to M$ from a left ideal I of R can be extended to a homomorphism $R \to M$.
(c) M has no non-trivial essential extensions

Proof. (a)$\Rightarrow$(b) is trivial.

(b)⇒(c) Let $M \subseteq N$ be a non-trivial essential extension and fix $x \in N \setminus M$. We consider the pullback

$$\begin{array}{ccc} I & \longrightarrow & R \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

where $R \to N$ is the map $r \mapsto rx$. Then $I \to R$ is injective, so I is identified with a left ideal in R. Explicitly

$$I = \{r \in R : rx \in M\}$$

and then $f : I \to M$ is given by $f(r) = rx \in M$ for $r \in I$. By (b), the map f extends to a map $g : R \to M$. This must be of the form $g(r) = rm$ where $m = g(1) \in M$.

Suppose $r \in R$ satisfies $r(x - m) \in M$. Then $rx \in M$, and it follows that $r \in I$. Then $rx = rm$, so $r(x - m) = 0$. Thus $M \cap R(x - m) = 0$ and $R(x - m) \neq 0$, contradicting that $M \subseteq N$ is an essential extension.

(c)⇒(a). Given an inclusion $M \subseteq N$, we need to show that M is a direct summand of N. By Zorn's Lemma, the set of submodules in N with zero intersection with M has a maximal element C. If $M + C = N$, then C is a complement. Otherwise, $M \cong (M + C)/C \subseteq N/C$ is a non-trivial extension. By (c) it cannot be an essential extension, so there is a non-zero submodule U/C with zero intersection with $(M+C)/C$. Then $U \cap (M+C) = C$, so $U \cap M \subseteq C \cap M = 0$. This contradicts the maximality of C. □

Definition. If R is an integral domain and M is an R-module, then

- M is *divisible* if for all $m \in M$ and $0 \neq a \in R$, there is $m' \in M$ with $m = am'$. For example the field of fractions of R is divisible.

- M is *torsion-free* if $am \neq 0$ for all nonzero $a \in R$ and $m \in M$. For example R and its field of fractions are torsion-free.

Lemma. *If R is an integral domain, then any injective module is divisible. If R is a principal ideal domain, then any divisible module is injective.*

Proof. Divisibility says that any map $Ra \to M$ lifts to a map $R \to M$. If R is a pid these are all ideals in R. □

Note that if R is an integral domain, but not a pid, then divisible modules are not necessarily injective. But still torsion-free divisible modules are injective. Thus the field of fractions is injective. We leave this as an exercise.

Definition. For the rest of this section we assume that R is a K-algebra, where K is a field or a principal ideal domain. In particular, we can consider any ring R as a K-algebra with $K = \mathbb{Z}$.

We define $(-)^* = \mathrm{Hom}_K(-, E_K)$, where

$$E_K = \begin{cases} K & \text{(if } K \text{ is a field)} \\ F/K & \text{(if } K \text{ is a pid with fraction field } F \neq K) \end{cases}$$

For example $E_{\mathbb{Z}} = \mathbb{Q}/\mathbb{Z}$.

Lemma. *(i) E_K is an injective K-module, and $(-)^*$ defines an exact functor from R-modules on one side to R-modules on the other side.*

(ii) If M is an R-module, the map $M \to M^{**}$, $m \mapsto (\theta \mapsto \theta(m))$ is an injective map of R-modules. (It is an isomorphism if K is a field and M is a finite-dimensional K-vector space).

Proof. (i) Any R-module M also gets an action of K via $\lambda m = (\lambda 1_R)m$, and these two actions commute, so $M^* = \mathrm{Hom}_K(M, E_K)$ becomes an R-module on the other side.

Now any quotient of a divisible module is clearly divisible, so E_K is a divisible K-module, so an injective K-module, so $(-)^*$ is an exact functor.

(ii) Given $0 \neq m \in M$, let Km be the cyclic K-submodule of M generated by m. It suffices to find a K-module map $f : Km \to E_K$ with $f(m) \neq 0$, for then since E_K is injective, f lifts to a map $\theta : M \to E_K$.

If K is a field there is an isomorphism $Km \to E_K$.

If K is a principal ideal domain and not a field, choose a maximal ideal Ka containing $ann(m) = \{x \in K : xm = 0\}$. Since K is not a field, $a \neq 0$. Then there is a map $Km \to E_K$ sending xm to $K + a^{-1}x \in F/K$. This is well-defined since if $xm = x'm$, then $x - x' \in ann(m)$, so $x - x' = ba$ for some $b \in K$, and then $a^{-1}x - a^{-1}x' = b \in K$. It is clearly a K-module homomorphism. Now it sends m to $K + a^{-1}$. If this is zero, then $a^{-1} \in K$, so a is invertible in K, so $Ka = K$, contradicting that Ka is a maximal ideal.

If K is a field, and M is K-vector space of dimension d, then so is M^*, and so also M^{**}, so the map $M \to M^{**}$ must be an isomorphism. $\square$

Theorem. *Any R-module embeds in a product of copies of R^*, and such a product is an injective R-module. An R-module is injective if and only if it is isomorphic to a direct summand of such a product.*

Proof. We have natural isomorphisms of functors R-Mod $\to$ Ab,

$$\operatorname{Hom}_R(-, R^*) = \operatorname{Hom}_R(-, \operatorname{Hom}_K(R, E_K)) \cong \operatorname{Hom}_K(R \otimes_R -, E_K)$$

$$\cong \operatorname{Hom}_K(-, E_K)^* = (-)^*,$$

which is exact, so R^* is injective. Thus any product of copies of R^* is injective.

If M is any R-module, then M embeds in M^{**}. Now M^* is a right R-module, so can be written as a quotient of a free right R-module, say $R^{(X)}$. Then

$$M \hookrightarrow M^{**} \hookrightarrow (R^{(X)})^* = \operatorname{Hom}_K(R^{(X)}, E_K) \cong \operatorname{Hom}_K(R, E_K)^X = (R^*)^X.$$

The last part is clear. □

Corollary. *Any module over any ring embeds in an injective module.*

Remark. More generally one can show that any object in a Grothendieck category has a monomorphism to an injective object.

Theorem (Bass, Papp)**.** *For a ring R the following are equivalent*
(i) R is left noetherian
(ii) Any filtered colimit of injective left R-modules is injective
(iii) Any direct sum of injective left R-modules is injective.

Proof. (i)⇒(ii). Let $M = \operatorname{colim}_{i\in\mathcal{I}} M_i$ be a filtered colimit of injective modules. Suppose I is a left ideal in R. It gives an exact sequence $0 \to I \to R \to R/I \to 0$. Since the M_i are injective, we get exact sequences

$$0 \to \operatorname{Hom}(R/I, M_i) \to \operatorname{Hom}(R, M_i) \to \operatorname{Hom}(I, M_i) \to 0.$$

A colimit of exact sequences is exact, so

$$0 \to \operatorname*{colim}_{i\in\mathcal{I}} \operatorname{Hom}(R/I, M_i) \to \operatorname*{colim}_{i\in\mathcal{I}} \operatorname{Hom}(R, M_i) \to \operatorname*{colim}_{i\in\mathcal{I}} \operatorname{Hom}(I, M_i) \to 0$$

is exact. Since the modules R/I, R and I are finitely presented, this is isomorphic to

$$0 \to \operatorname{Hom}(R/I, M) \to \operatorname{Hom}(R, M) \to \operatorname{Hom}(I, M) \to 0.$$

Thus by Baer's criterion M is injective.

(ii)⇒(iii). We have

$$\bigoplus_{i\in I} M_i \cong \operatorname*{colim}_{J\subseteq I} \bigoplus_{j\in J} M_j \cong \operatorname*{colim}_{J\subseteq I} \prod_{j\in J} M_j$$

where J runs over the finite subsets of I, a filtered colimit of injective modules.

(iii)⇒(i). Consider an ascending chain of left ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq R.$$

Let I be their union. For each $n \geq 1$, choose an embedding $\phi_n : R/I_n \to E_n$ with E_n injective. We have a well-defined map

$$\theta : I \to E := \bigoplus_{n=1}^{\infty} E_n, \quad \theta(a)_n = \phi_n(I_n + a)$$

Since E is injective, θ extends to a map $R \to E$. Let that map send 1 to $e \in E$. Then $\theta(a) = ae$ for $a \in R$. But e only has finitely many non-zero components, so there is some n such that $e_n = 0$. Then $\theta(a)_n = 0$ for all $a \in I$, so $a \in I_n$. Thus $I = I_n$, so the chain of ideals stabilizes. Thus R is left noetherian. □

2.5 Flat modules

In this section R is a K-algebra and K is a field or pid, for example R is a ring and $K = \mathbb{Z}$.

Definition. A right R-module X is *flat* if $X \otimes_R -$ is an exact functor, either considered as a functor R-Mod $\to$ $\mathbb{Z}$-Mod or equivalently as a functor R-Mod $\to$ K-Mod.

Remark. (i) A direct sum of modules is flat if and only if each summand is flat, since if X_i are right R-modules and $0 \to L \to M \to N \to 0$ is an exact sequence of left R-modules, then

$$0 \to \bigoplus_{i\in I} X_i \otimes_R L \to \bigoplus_{i\in I} X_i \otimes_R M \to \bigoplus_{i\in I} X_i \otimes_R N \to 0$$

is exact if and only if it is exact for each sequence

$$0 \to X_i \otimes_R L \to X_i \otimes_R M \to X_i \otimes_R N \to 0$$

is exact.

(ii) Any projective module is flat, for $R \otimes_R X \cong X$, so R is flat.

(iii) Any filtered colimit of flat modules is flat. If $\mathcal{I}$ is a small filtered category, X is an $\mathcal{I}$-diagram of flat right R-modules and $0 \to L \to M \to N \to 0$ is an exact sequence of left R-modules, then since the X_i are flat, we get exact sequences

$$0 \to X_i \otimes_R L \to X_i \otimes_R M \to X_i \otimes_R N \to 0.$$

Since R-Mod has exact filtered colimits, the sequence

$$0 \to \operatorname*{colim}_{i\in\mathcal{I}}(X_i \otimes_R L) \to \operatorname*{colim}_{i\in\mathcal{I}}(X_i \otimes_R M) \to \operatorname*{colim}_{i\in\mathcal{I}}(X_i \otimes_R N) \to 0$$

is exact. Since tensor products commute with colimits, this is

$$0 \to (\operatorname*{colim}_{i\in\mathcal{I}} X_i) \otimes_R L \to (\operatorname*{colim}_{i\in\mathcal{I}} X_i) \otimes_R M \to (\operatorname*{colim}_{i\in\mathcal{I}} X_i) \otimes_R N \to 0$$

so $\operatorname{colim}_{i\in\mathcal{I}} X_i$ is flat.

End of lecture on 2026-06-01. Provisional script for the next lecture follows (subject to change).

Proposition. *A right R-module X is flat if and only if X^* is injective.*

Proof. If Y is a left R-module, then $\operatorname{Hom}_R(Y, X^*) \cong (X \otimes_R Y)^*$. If X is flat, then this is exact as a functor of Y, so X^* is injective. Conversely, if X^* is injective then again this is exact as a functor of Y. Suppose X is not flat. Given an exact sequence of left R-modules

$$0 \to L \to M \to N \to 0$$

we get

$$0 \to H \to X \otimes_R L \to X \otimes_R M \to X \otimes_R N \to 0.$$

Then we get

$$(X \otimes_R M)^* \to (X \otimes_R L)^* \to H^* \to 0$$

Thus $H^* = 0$. But H embeds in H^{**}, so $H = 0$. □

Proposition. *A module X_R is flat if and only if the multiplication map $X \otimes_R I \to X$ is injective for every left ideal I in R.*

Proof. If X is flat, tensoring it with the exact sequence $0 \to I \to R \to R/I \to 0$ shows that the map is injective.

If the map is injective, then the map $X^* \to (X \otimes_R I)^*$ is surjective. We can write this as $\operatorname{Hom}_R(R, X^*) \to \operatorname{Hom}_R(I, X^*)$. By Baer's criterion X^* is injective. Thus X is flat. □

Example. If R is an integral domain, then any flat right R-module X is torsion-free, that is, if $x \in X$ and $a \in R$ and $xa = 0$, then $x = 0$ or $a = 0$. Namely, if $I = Ra$ with $0 \neq a \in R$, then the map $X \otimes_R I \to X$ is the identified with the map $X \to X$ of multiplication by a.

If R is a pid, then a right module X is flat if and only if it is torsion-free, since any non-zero ideal in R is of this form.

Thus $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. This also follows from the next construction.

Example. Let R be a commutative ring. A subset $S \subseteq R$ is *multiplicative* if $1 \in S$ and $st \in S$ for all $s, t \in S$. The *localization* of an R-module M with respect to S is

$$S^{-1}M = S \times M/\sim$$

where $\sim$ is the equivalence relation given by

$$(s, m) \sim (s', m') \Leftrightarrow t(sm' - s'm) = 0 \text{ for some } t \in S$$

It is equivalent that $um = u'm'$ for some $u, u' \in S$ with $us = u's'$. To see this, take $t = us$ or $u = ts'$ and $u' = ts$.

The equivalence class containing (s, m) is denoted $s^{-1}m$. Now $S^{-1}M$ has an addition given by the usual formula for adding fractions

$$s^{-1}m + t^{-1}n = (st)^{-1}(tm + sn).$$

Moreover $S^{-1}R$ becomes a ring and $S^{-1}M$ becomes an $S^{-1}R$-module with the usual formula for multiplication

$$(s^{-1}a)(t^{-1}b) = (st)^{-1}(ab).$$

This was all on an exercise sheet for Algebra II. It was also shown on the exercise sheet that $S^{-1}M \cong S^{-1}R \otimes_R M$. We can deduce this here from Eilenberg-Watts. The construction gives a localization functor

$$R\text{-Mod} \to S^{-1}R\text{-Mod}, \quad M \mapsto S^{-1}M.$$

and it is easy to see that this is an exact functor. It is easy to see that an exact sequence $0 \to L \to M \to N \to 0$ of R-modules gives an exact sequence

$$0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$$

so this functor is exact. It also commutes with arbitrary direct sums. Thus by the Eilenberg-Watts theorem,

$$S^{-1}M \cong X \otimes_R M$$

for all M, for some bimodule X. Then $X \cong S^{-1}R$ considered as a left $S^{-1}R$ module in the usual way, and as a right R-module via $(s^{-1}r)r' = s^{-1}(rr')$. Thus

$$S^{-1}M \cong S^{-1}R \otimes_R M.$$

Since the localization functor is exact, $S^{-1}R$ is a flat R-module. Here is another way to see this. Consider S as a the set of objects in a category, with

$$\text{Hom}(s, t) = \{u \in S : us = t\}$$

It is filtered since it has object 1, if $s, s' \in S$ then they both have morphisms to ss', and if $u, u' : s \to t$, then $t = us = u's$. Thus considering s as a morphism $t \to st$, the compositions with u and u' are equal.

Consider the functor $S \to R$-Mod sending all $s \in S$ to $M_s = M$ and $u \in \mathrm{Hom}(s,t)$ to multiplication by u. Then our description of the colimit gives

$$\operatorname*{colim}_{s\in S} M_s = \dot{\bigcup}_{s\in S} M/\sim \; = \; (S \times M)/\sim$$

where $(s,m) \sim (s',m') \Leftrightarrow$ there are morphisms $u : s \to v$ and $u' : s' \to v$ with $um = u'm'$. Thus

$$S^{-1}M = \operatorname*{colim}_{s\in S} M_s$$

Thus if M is a flat R-module, so is $S^{-1}M$. In particular $S^{-1}R$ is flat.

Lemma. *Let X be an S-R-bimodule. If U is a f.p. left S-module and Y is a flat left R-module, then the natural map*

$$\mathrm{Hom}_S(U,X) \otimes_R Y \to \mathrm{Hom}_S(U, X \otimes_R Y)$$

is an isomorphism

Proof. It is clear for $U = S$. Then it follows for $U = S^n$. In general there is an exact sequence $S^m \to S^n \to U \to 0$, and in the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_S(U,X)\otimes_R Y & \longrightarrow & \mathrm{Hom}_S(S^n,X)\otimes_R Y & \longrightarrow & \mathrm{Hom}_S(S^m,X)\otimes_R Y \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_S(U,X\otimes_R Y) & \longrightarrow & \mathrm{Hom}_S(S^n,X\otimes_R Y) & \longrightarrow & \mathrm{Hom}_S(S^m,X\otimes_R Y) \end{array}$$

the rows are exact and the right two vertical maps are isomorphisms, hence so is the first. □

Recall that any f.g. projective module is finitely presented.

Theorem. *(i) A finitely presented flat module is projective.*
(ii) (Lazard, Govorov) Any flat module is a filtered colimit of finitely generated projective (even free) modules.

Proof. (i) If Y is a f.p. flat left R-module, then the natural map $\mathrm{Hom}_R(Y,R) \otimes_R Y \to \mathrm{End}_R(Y)$ is an isomorphism by the last lemma. Thus by the last lemma in the section on projective modules, Y is f.g. projective.

(ii) If M is a flat left R-module and X is f.p., then the map $\mathrm{Hom}(X,R) \otimes_R M \to \mathrm{Hom}(X,M)$ is an isomorphism. It follows that any map $f : X \to M$ can be factored as

$$X \xrightarrow{\theta} R^n \xrightarrow{g} M.$$

Now use the result in the section on filtered colimits. □