# Functorial Filtrations for Semiperfect generalisations of Gentle Algebras

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

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# Abstract

In this thesis we introduce a class of semiperfect rings which generalise the class of finite-dimensional gentle algebras. We consider complexes of modules over these rings which have finitely generated projective homogeneous components. We then classify them up to homotopy equivalence. The method we use to solve this classification problem is called the functorial filtrations method. The said method was previously only used to classify modules.

To my family, friends, and my partner.

# Contents

# Chapter 1

# Background and Preliminaries.

## Introduction.

In this thesis we present and solve a classification problem. The objects we classify are complexes of projectives up to homotopy, and so our results apply to derived categories. For certain finite-dimensional algebras, indecomposables in their bounded derived categories have been classified. For example if $Q$ is a finite quiver then the indecomposable objects in $\mathcal{D}^b(kQ\text{-}\mathbf{mod})$ are essentially given by the indecomposables in $kQ$ (1.5.11). So, by Gabriel's theorem (1.5.5) $kQ$ is derived-finite (1.5.10) provided $Q$ is a disjoint union of connected subquivers of Dynkin type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$. For another example we can consider a gentle algebra $\Gamma$, as introduced by Assem and Skowroński [4, p.272, Proposition] (1.2.10). By a theorem (1.5.29) of Bekkert and Merklen, the indecomposables in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ are shifts of *string* and *band* complexes. The problem we solve was inspired by and generalises this theorem.

The class of *complete gentle* algebras we work over contains the Assem-Skowroński gentle algebras discussed above. This containment is strict. For example, this class contains infinite-dimensional algebras such as $k[[x,y]]/(xy)$, and algebras where the ground ring is not a field, such as the $\widehat{\mathbb{Z}}_p$-algebra

$$\left\{ \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in \mathbb{M}_2(\widehat{\mathbb{Z}}_p) \mid \gamma_{11} - \gamma_{22}, \gamma_{12} \in p\widehat{\mathbb{Z}}_p \right\}$$

The method used to solve this problem is sometimes called the *functorial filtration* method. Functorial filtrations have been written in MacLane's language of linear relations [49], and were used in the past to classify modules (with certain finiteness conditions) up to isomorphism. Gel'fand and Ponomarev [32] seem to be the first to solve classification problems in this way. Their work was interpreted in the language of functors by Gabriel [30]. Since then the method was adapted to new settings by Ringel [55], Donovan and Freislich [23], Butler and Ringel [15], Crawley-Boevey [18, 19, 21] and Ricke [54].

The first chapter of this thesis comprises a literature review and an introduction to the algebras and representations we work with in the second chapter. The first chapter is organised as follows: generalisations of special biserial algebras are studied in section 1.1; in section 1.2 we restrict our focus to generalisations of gentle algebras; the string and band representations (which consitute complete lists of pairwise non-isomorphic indecomposables) are looked at in section 1.3; in section 1.4 we look at the functorial filtration method; and known classification results about derived categories are looked at in section 1.5.

The second chapter contains the main research presented in this thesis, where we adapt the functorial filtration method to homotopy categories. The second chapter is structured as follows: in section 2.1 we introduce linear relations which we work with for the remainder of the chapter; these relations are used to define functors in section 2.2; in sections 2.3, 2.4 and 2.5 we verify certain compatability conditions between these functors and string and band complexes; these verifications are used to complete the proofs of the main results in section 2.6, and in section 2.7 we apply our results and state some conjectures. The third chapter is an appendix.

**Conventions:** Unless stated otherwise, all categories are assumed to be additive and locally small, and all functors are covariant and additive. All rings are associative, although they need not be unital. By a module we mean a unital left module (see definition 3.1.30).

**General Notation:** In most definitions THIS FONT is used to highlight the terminology or notation being defined. This is to help the reader find the definitions they want with greater ease. We also use *this font* to emphasise the words being defined. THIS FONT is also used in giving names to certain results in this thesis.

The short-hand resp. will be used to mean respectively. The short hand iff will be used to mean if and only if. We write $\mathbb{N}$ for the set of non-negative integers. For any set $X$ we write $\#X$ for the cardinality of $X$.

## 1.1 Some Biserial Rings.

Throughout the thesis we consider algebras defined using quivers and relations. The quivers involved may have infinitely many vertices. For example in section 1.5 we look at the *string algebras* considered by Butler and Ringel [15]. In chapter 2 we study algebras where the ground ring is not necessarily a field (see example 1.1.6). To avoid repetition, in this section we start by describing a class of rings (see definition 1.1.21) which contains all those we want to study.

### 1.1.1 Path Algebras and Relations.

**Assumption:** Throughout the thesis we assume $R$ is a unital, commutative, noetherian and local ground ring with maximal ideal $\mathfrak{m}$.

During the literature review (sections 1.3 and 1.5) and in many examples we often restrict to the case where $R$ is a field (which is sometimes algebraically closed).

**Example 1.1.1.** (Completions) For any field $k$ the ring $k[[t]]$ of formal power series $\sum_{i\geq 0} a_i t^i$ (where $a_i \in k$) is commutative, noetherian and local, whose maximal ideal is the ideal $(t)$ generated by $t$ (see [6, p.11, Exercise 5 (i)]). Note that $k$ is isomorphic to the quotient field $k[[t]]/(t)$, and the exact sequence of abelian groups $0 \to (t) \to k[[t]] \to k \to 0$ splits.

For an example where this sequence does not split, consider the ring of $p$-adic integers $\widehat{\mathbb{Z}}_p$ where $p \in \mathbb{Z}$ is prime. Elements here are formal sums $\sum_{i=0}^{\infty} \alpha_i p^i$ where each $\alpha_i$ is an element of $\{0, \dots, p-1\}$. This defines a local noetherian ring with maximal ideal $p\widehat{\mathbb{Z}}_p = \{\sum_i \alpha_i p^i \in \widehat{\mathbb{Z}}_p \mid \alpha_0 = 0\}$. The exact sequence $0 \to p\widehat{\mathbb{Z}}_p \to \widehat{\mathbb{Z}}_p \to \mathbb{Z}/p\mathbb{Z} \to 0$ does not split, as there are no non-zero additive homomorphisms $\mathbb{Z}/p\mathbb{Z} \to \widehat{\mathbb{Z}}_p$.

**Example 1.1.2.** (Localisations) The ring of fractions $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid b \in \mathbb{Z}_{>0} \text{ and } p \nmid b\}$ for a fixed but arbitrary prime number $p$ defines a commutative local noetherian ring with with maximal ideal $p\mathbb{Z}_{(p)}$. For another example we can take $R$ to be the localisation $k[t]_{(t)} = \{\frac{f(t)}{g(t)} \mid g(0) \neq 0\}$ where $k$ is any field.

**Definition 1.1.3.** (Quivers) In what follows $Q = (Q_0, Q_1, h, t)$ will denote a quiver. The functions $h$ and $t$ map the set of *arrows* $Q_1$ to the set of *vertices* $Q_0$ taking an arrow $a$ to its *head* $h(a)$ and *tail* $t(a)$ respectively.

(Paths) A *non-trivial path* of *length* $n > 0$ is a sequence $p = a_1 \dots a_n$ with $h(a_{i+1}) = t(a_i)$ for $0 < i < n$ (contrary[1] to [5]). A *trivial path* has the form $e_v$ (where $v$ is a vertex) and should be considered a path of length 0. A *path* will mean a trivial or non-trivial path, and we extend the domain of $h$ and $t$ to the set of all paths by stipulating $h(p) = h(a_1)$ and $t(p) = t(a_n)$ for any non-trivial path $p$ (as above), and $h(e_v) = v = t(e_v)$ for any vertex $v$.

(Cycles) A *cycle* is a non-trivial path whose head and tail coincide, a *loop* is a cycle which is also an arrow, and a quiver is called *acyclic* if it has no cycles.

(Path Algebras) $RQ$ will be the *path algebra* of $Q$ (*over* $R$), the $R$-algebra defined by an $R$-basis consisting of the paths, and where the product $ab$ of two paths $a$ and $b$ is given by the concatenation of arrows in case $t(a) = h(b)$, and $ab = 0$ otherwise. Note that $RQ$ is an associative $R$-algebra which is unital when $Q$ is finite (see [5, 1.4 Lemma]).

**Example 1.1.4.** In general our quivers contain cycles. They can also be infinite and disconnected. In all of our examples there are countably many vertices, and each vertex is labelled by a natural number. For example

$$\begin{tikzcd}
0 \arrow[loop left, "\alpha"] \arrow[r, "\gamma"] & 1 \arrow[r, "\eta"] & 3 \arrow[dl, "\lambda"'] \arrow[d, "\sigma"] & & 6 \arrow[d, "\omega"'] & 8 \arrow[l, "\xi"] & 10 & 11 \arrow[l, "\delta"'] & 12 \arrow[l, "\delta'"'] & \cdots \arrow[l] \\
& 2 \arrow[ul, "\beta"] \arrow[u, "\mu"] & 4 \arrow[r, "\theta"] \arrow[r, bend right, "\tau"'] & 5 & 7 \arrow[r, "\nu"] & 9 \arrow[u, "\zeta"']
\end{tikzcd}$$

**Definition 1.1.5.** (Relations) A set $\rho$ of *relations* is a subset of $\bigcup_{u,v} e_v RQ e_u$, where $u$ and $v$ each run through the vertices of $Q$. Let $(\rho)$ denote the two-sided ideal of $RQ$ generated by the elements in $\rho$.

[1] In [5] the function $h$ (resp. $t$) is written using the symbol $t$ (resp. $s$) and called the *target* (resp. *source*). Furthermore, in [5] the unique length 2 path in the quiver $Q = \; 1 \xleftarrow[\alpha]{} 2 \xleftarrow[\beta]{} 3$ is written $\beta\alpha$, where as in this thesis it is written $\alpha\beta$.

($R$-ALGEBRAS SURJECTIVELY GIVEN BY $(Q,\rho,\theta)$) An $R$-algebra $\Lambda$ is said to be *surjectively given by* $(Q,\rho,\theta)$ if there is a quiver $Q$, a set of relations $\rho \subseteq \bigcup_{u,v} e_v RQe_u$ and a surjective $R$-algebra homomorphism $\theta : RQ \to \Lambda$ where: $(\rho) \subseteq \ker(\theta)$; $e_v \notin (\rho)$ for any vertex $v$; and $\theta(p) \neq 0$ for any path $p \notin (\rho)$. In this case for any such $p$ we abuse notation by using the same symbol to denote the coset $p + (\rho)$ and the image of this coset in $\Lambda$. Note that $\theta$ factors through the canonical projection $RQ \to RQ/(\rho)$, however in general we have $(\rho) \neq \ker(\theta)$.

**Example 1.1.6.** Let $Q$ be the quiver given by two loops $\alpha$ and $\beta$ at a single vertex $v$ and let $\rho = \{\alpha^2, \beta^2\}$. Let $R = \widehat{\mathbb{Z}}_p$ , $\mathfrak{m} = p\widehat{\mathbb{Z}}_p$ (as in example 1.1.1) and let $\Lambda$ be the $\widehat{\mathbb{Z}}_p$-subalgebra of $2 \times 2$ matrices $(\gamma_{ij}) \in \mathbb{M}_2(\widehat{\mathbb{Z}}_p)$ with $\gamma_{11} - \gamma_{22}, \gamma_{12} \in p\widehat{\mathbb{Z}}_p$. There is a $\widehat{\mathbb{Z}}_p$-algebra homomorphism $\theta : \widehat{\mathbb{Z}}_p Q \to \Lambda$ defined by extending the assignments

$$\theta(\alpha) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \theta(\beta) = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$$

multiplicatively. Hence $(\rho) \subseteq \ker(\theta)$, and for any integer $n \geq 1$ we have

$$\begin{aligned} \theta((\alpha\beta)^n) &= \begin{pmatrix} 0 & 0 \\ 0 & p^n \end{pmatrix}, \quad \theta((\alpha\beta)^n\alpha) = \begin{pmatrix} 0 & 0 \\ p^n & 0 \end{pmatrix}, \\ \theta((\beta\alpha)^n) &= \begin{pmatrix} p^n & 0 \\ 0 & 0 \end{pmatrix}, \quad \theta((\beta\alpha)^n\beta) = \begin{pmatrix} 0 & p^{n+1} \\ 0 & 0 \end{pmatrix} \end{aligned} \qquad (\Diamond)$$

This shows any path $\sigma \notin (\rho)$ has a non-zero image in $\Lambda$. For any $\lambda \in \Lambda$ we have $\lambda = (\sum_{k=0}^{\infty} r_{ij,k}p^k)_{i,j}$ for elements $r_{ij,k} \in \{0, \dots, p-1\}$ where $r_{11,0} = r_{22,0}$ and $r_{12,0} = 0$. Hence $\lambda = \theta(r + \sum_{t>0} \sum_{\sigma \in \mathbf{P}(t)} r_{p,t}\sigma)$ where $r = r_{22,0}$ and for each $t > 0$ we let $r_{\alpha,t} = r_{21,t}$, $r_{\beta\alpha,t} = r_{11,t+1}$, $r_{\alpha\beta,t} = r_{22,t+1}$ and $r_{\beta,t} = r_{12,t+1}$. Thus $\theta$ is surjective, and so $\Lambda$ is given by $(Q,\rho,\theta)$. Note that the kernel of $\theta$ is the ideal in $\widehat{\mathbb{Z}}_p Q$ generated by $\alpha^2$, $\beta^2$ and $\alpha\beta + \beta\alpha - p$.

This example illustrates a scenario where the ground ring $R$ is not a field, and where $(\rho) \neq \ker(\theta)$. This will be a running example throughout the thesis.

**Definition 1.1.7.** (NOTATION: $\mathbf{P}(t, v \rightarrow)$, $\mathbf{P}(t)$) For each $t \geq 0$ and each vertex $v$ let $\mathbf{P}(t, v \rightarrow)$ (resp. $\mathbf{P}(t, \rightarrow v)$) be the set of paths $\sigma$ of length $t$ with $\sigma \notin (\rho)$ and tail (resp. head) $v$, and let $\mathbf{P}(t)$ denote the set $\bigcup_v \mathbf{P}(t, v \rightarrow) \cup \mathbf{P}(t, \rightarrow v)$ of all paths outside $(\rho)$ of length $t$.

(NOTATION: $\mathbf{P}(v \rightarrow)$, $\mathbf{P}$, $\mathbf{A}(v \rightarrow)$) For each vertex $v$ let $\mathbf{P}(v \rightarrow)$ (resp. $\mathbf{P}(\rightarrow v)$) denote the set $\bigcup_{t>0} \mathbf{P}(t, v \rightarrow)$ (resp. $\bigcup_{t>0} \mathbf{P}(t, \rightarrow v)$) of all *non-trivial* paths outside $(\rho)$ with tail (resp. head) $v$; and $\mathbf{P}$ denote the union $\bigcup_{t>0} \mathbf{P}(t)$. We also set $\mathbf{A}(v \rightarrow) = \mathbf{P}(1, v \rightarrow)$ and $\mathbf{A}(\rightarrow v) = \mathbf{P}(1, \rightarrow v)$.

**Example 1.1.8.** Recall the quiver $Q$ from example 1.1.4. Let

$$\rho = \{\alpha\beta,\ \gamma\alpha,\ \eta\mu,\ \sigma\eta,\ \mu\lambda,\ \theta\sigma,\ \tau\sigma,\ \nu\omega\xi\zeta\nu\omega\xi\zeta,\ \beta\lambda\eta\gamma - \alpha\}$$

The non-trivial paths with tail 0 which lie outside $(\rho)$ are precisely

$$\mathbf{P}(0 \rightarrow) = \{\alpha^{n+1}, \gamma\alpha^n, \eta\gamma\alpha^n, \lambda\eta\gamma\alpha^n \mid n \in \mathbb{N}\}$$

**Definition 1.1.9.** [35, §2] (see also [5, p.50]) (GENERALISED TRIANGULAR MATRIX RINGS) For a fixed integer $n > 1$ we shall define a ring $T_n(R_i, M_{ij}, \varphi_{ij}^t)$ by fixing the following data. For each integer $i$ with $n \geq i \geq 1$ let:

(a) let $R_i$ be a unital ring, and let $M_{ii} = R_i$ considered as an $R_i$-$R_i$ bimodule, and let $I_{ii}$ be the identity map on $R_i$; and for each integer $j$ with $n \geq i > j \geq 1$,

(b) $M_{ij}$ be an $R_i$-$R_j$ bimodule, let $I_{ij}$ be the identity map on $M_{ij}$, and let $\varphi_{ij}^i$ and $\varphi_{ij}^j$ be the canonical $R_i$-$R_j$ bimodule isomorphisms $M_{ij} \otimes_{R_j} R_j \rightarrow M_{ij}$ and $R_i \otimes_{R_i} M_{ij} \rightarrow M_{ij}$ respectively; and for each integer $t$ with $n \geq i > t > j \geq 1$ let

(c) $\varphi_{ij}^t : M_{it} \otimes_{R_t} M_{tj} \rightarrow M_{ij}$ be an $R_i$-$R_j$ -bimodule homomorphism, where the diagram

$$\begin{tikzcd}
M_{ab} \otimes_{R_b} M_{bc} \otimes_{R_c} M_{cd} \arrow[rr, "I_{ab}\otimes\varphi^c_{bd}"] \arrow[d, "\varphi^b_{ac}\otimes I_{ac}"] & & M_{ab} \otimes_{R_b} M_{bd} \arrow[d, "\varphi^b_{ad}"] \\
M_{ac} \otimes_{R_c} M_{cd} \arrow[rr, "\varphi^c_{ad}"] & & M_{ad}
\end{tikzcd}$$

commutes for any integers $a, b, c, d$ with $n > a > b > c > d > 1$.

We can now define the *generalised (lower) triangular matrix ring* $T_n(R_i, M_{ij}, \varphi^t_{ij})$ by the set of $n \times n$ matrices $(m_{ij})$ where $m_{ij} = 0$ for $i < j$ and $m_{ij} \in M_{ij}$ otherwise. Addition is defined component-wise and the product of $(m_{ij})$ and $(m'_{ij})$ is given by $(\sum_{t=1}^{n} \varphi^t_{ij}(m_{it} \otimes m'_{tj}))$.

**Example 1.1.10.** Let $p \in \mathbb{Z}$ be prime. Let $R = \mathbb{Z}/p^3\mathbb{Z}$ and $\mathfrak{m} = p\mathbb{Z}/p^3\mathbb{Z}$. Let $Q$ be the quiver

$$\begin{tikzcd}
 & & & 2 \arrow[dl, "\gamma"'] & \\
5 & 4 \arrow[l, "\lambda"'] \arrow[loop above, "\varepsilon"] & 3 \arrow[l, "\delta"'] & & 1 \arrow[ll, "\alpha"] \arrow[ul, "\beta"']
\end{tikzcd}$$

and let $\rho = \{\lambda\varepsilon, \varepsilon^3, \varepsilon\delta, \delta\alpha, \alpha - \gamma\beta\}$. Our aim is to define a generalised lower triangular matrix ring $\Lambda$ given by $(Q, \rho, \theta)$, where $\theta$ is yet to be defined.

Let $R_1 = R_2 = R_3 = R_5 = R/\mathfrak{m}$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, the field with $p$ elements. Let $R_4 = R = \mathbb{Z}/p^3\mathbb{Z}$. For $5 \geq i > j \geq 1$ define the $R_i$-$R_j$-bimodules $M_{ij}$ by setting $M_{i1} = 0$ for $i = 4, 5$ and $M_{ij} = \mathbb{Z}/p\mathbb{Z}$ otherwise. Hence for $5 \geq i > t > j \geq 1$ there are three possibly non-zero $R_i$-$R_j$-bimodule homomorphisms $\varphi^t_{ij} : M_{it} \otimes_{R_t} M_{tj} \to M_{ij}$, namely $\varphi^2_{31}$, $\varphi^3_{51}$ and $\varphi^4_{53}$.

Let $\varphi^2_{31} = \varphi^3_{51}$ be the $\mathbb{Z}/p\mathbb{Z}$-$\mathbb{Z}/p\mathbb{Z}$ bimodule homomorphism $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ given by $\bar{n} \otimes \bar{m} \mapsto \bar{n}\bar{m}$, and let $\varphi^4_{53}$ be the $\mathbb{Z}/p\mathbb{Z}$-$\mathbb{Z}/p\mathbb{Z}$ bimodule homomorphism $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}/p^3\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ given by the same formula. In this example, to check the appropriate diagrams commute, one need only consider the case $(a, b, c, d) = (5, 4, 3, 2)$ (which is straightforward).

We label elements in the generalised triangular matrix ring $\Lambda = T_5(R_i, M_{ij}, \varphi^t_{ij})$ by

$$[r_{ij}] = \begin{pmatrix} \overline{r_{11}} & 0 & 0 & 0 & 0 \\ \overline{r_{21}} & \overline{r_{22}} & 0 & 0 & 0 \\ \overline{r_{31}} & \overline{r_{32}} & \overline{r_{33}} & 0 & 0 \\ 0 & \overline{r_{42}} & \overline{r_{43}} & r_{44} & 0 \\ 0 & \overline{r_{52}} & \overline{r_{53}} & \overline{r_{54}} & \overline{r_{55}} \end{pmatrix}$$

where for any $r + p^3\mathbb{Z} \in \mathbb{Z}/p^3\mathbb{Z}$ we let $\overline{r} = r + p\mathbb{Z} \in \mathbb{Z}/p\mathbb{Z}$. We can now define a $\mathbb{Z}/p^3\mathbb{Z}$-algebra homomorphism $\theta : (\mathbb{Z}/p^3\mathbb{Z})Q \to \Lambda$ by sending $\sum_{i=1}^5 r_i e_i + \sum_{\sigma\in\mathbf{P}} r_\sigma \sigma$ to

$$\begin{pmatrix} \overline{r_1} & 0 & 0 & 0 & 0 \\ \overline{r_\beta} & \overline{r_2} & 0 & 0 & 0 \\ \overline{r_{\gamma\beta} - r_\alpha} & \overline{r_\gamma} & \overline{r_3} & 0 & 0 \\ 0 & \overline{r_{\delta\gamma}} & \overline{r_\delta} & r_4 + pr_\varepsilon + p^2 r_{\varepsilon^2} & 0 \\ 0 & \overline{r_{\lambda\delta\gamma}} & \overline{r_{\lambda\delta}} & \overline{r_\lambda} & \overline{r_5} \end{pmatrix}$$

It is straightforward, but tedious, to check: $\theta$ is surjective; $\theta$ is a homomorphism of rings; $(\rho) \subseteq \ker(\theta)$; and that $\sigma \in \mathbf{P}$ implies $\theta(\sigma) \neq 0$. Hence $\Lambda$ is a $\mathbb{Z}/p^3\mathbb{Z}$-algebra given by $(Q, \rho, \theta)$.

### 1.1.2 Quasi-Bounded Special Biserial Algebras.

**Assumption:** In section 1.1.2 we assume $\Lambda$ is an $R$-algebra surjectively given by $(Q, \rho, \theta)$.

In definitions 1.1.11 and 1.1.21 we introduce conditions on $\Lambda$ to ensure the (unital) projective indecomposable $\Lambda$-modules are biserial (see definition 1.1.26 and proposition 1.1.28).

**Definition 1.1.11.** [61, §1, (SP)] (Special Conditions) We say $(Q, \rho)$ satisfies *special conditions* if:

SPI) given any vertex $v$ we have $\#\mathbf{A}(v \to) \leq 2$ and $\#\mathbf{A}(\to v) \leq 2$; and

SPII) given any arrow $y$ there is at most one arrow $x$ with $xy \in \mathbf{P}(2)$, and at most one arrow $z$ with $yz \in \mathbf{P}(2)$.

**Example 1.1.12.** The pair $(Q, \rho)$ from example 1.1.8 satisfies special conditions.

**Definition 1.1.13.** (First And Last Arrows, notation: $\mathrm{f}(p)$, $\mathrm{l}(p)$) Any non-trivial path $p$ in $Q$ has a *first arrow* $\mathrm{f}(p)$ and a *last arrow* $\mathrm{l}(p)$ satisfying $\mathrm{l}(p)p' = p = p''\mathrm{f}(p)$ for some (possibly trivial) paths $p'$ and $p''$. That is, $p$ may be depicted by

$$\overbrace{h(p) \longleftarrow}^{\mathrm{l}(p)}\overbrace{\bullet \longleftarrow \cdots \longleftarrow \bullet \longleftarrow t(p)}^{p'}$$

$$\underbrace{h(p) \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet}_{p''}\underbrace{\longleftarrow t(p)}_{\mathrm{f}(p)}$$

(Parallel, Initial And Terminal Subpaths) Two distinct paths in $Q$ are called *parallel* if they have the same head and the same tail. We say a path $p$ is a *subpath* of a path $p'$, and write $p \leq p'$, if $p' = \gamma p \delta$ for some (possibly trivial) paths $\gamma$ and $\delta$. If $\delta$ is trivial we call $p$ an *initial* subpath, and if $\gamma$ is trivial we call $p$ a *terminal* subpath. If $p \leq p'$ and $p \neq p'$ then we say $p$ is a *proper subpath* of $p'$.

(Maximal Paths) A path $p \in \mathbf{P}$ is called *maximal* if it is not a proper subpath of some path $p' \in \mathbf{P}$.

The following was adapted from [61, p.175, Corollary].

**Lemma 1.1.14.** *Let $(Q, \rho)$ satisfy special conditions.*

*For all $p, p' \in \mathbf{P}$ such that $p$ is not longer than $p'$,*

(ia) *if* $\mathrm{f}(p) = \mathrm{f}(p')$ *then $p$ is an initial subpath of $p'$,*

(ib) *if* $\mathrm{l}(p) = \mathrm{l}(p')$ *then $p$ is a terminal subpath of $p'$,*

(ii) *We have $(\rho) = (\rho')$ in $RQ$ where elements in $\rho'$ have the form $\sum_q r_q q$ where $q$ runs through parallel paths in $Q$.*

*Proof.* (i) It suffices to find some path $\gamma$ such that $p' = \gamma p$, by symmetry. Note $p = qa$ and $p' = q'a$ for $a = \mathrm{f}(p)$ and some paths $q$ and $q'$. If $q$ is trivial then $q = e_{t(a)}$ in which case $p = a$ and it suffices to take $\gamma = q'$. Otherwise $q$ has length $n > 0$, and so $q'$ is also non-trivial. If $\mathrm{f}(q) \neq \mathrm{f}(q')$ we have $\mathrm{f}(q)a, \mathrm{f}(q')a \notin (\rho)$ which contradicts SPII). Hence $\mathrm{f}(q) = \mathrm{f}(q')$. If $q$ has length $n > 0$ then iterating this argument on the remaining $n - 1$ arrows in $q$ gives the claim.

(ii) Recall definition 3.1.30. Note that the set consisting of the elements $e_v$ (for each vertex $v$) defines a complete set of orthogonal idempotents for $\Lambda$. By example 3.1.31, for each $z \in \rho$ there is a finite set of vertices $v(1), \dots, v(n)$ for which $z = ez = ze$ where $e = \sum_{i=1}^{n} e_{v(i)}$. Note that the number $n > 0$ and the vertices $v(i)$ all depend on $z$. Since $z \in \rho$ we have $e_{v(i)} z e_{v(j)} \in (\rho)$ for each $i$ and $j$, and we write $e_{v(i)} z e_{v(j)} = \sum_q r_q q$ where $q$ runs through the parallel paths with head $v(i)$ and tail $v(j)$, and $r_q \in R$ is non-zero for all but finitely many $q$. Let $\rho'$ be the set of all such $e_{v(i)} z e_{v(j)}$ where $z$ runs through $\rho$. By definition $\rho' \subseteq (\rho)$ and $\rho \subseteq (\rho')$ since $z = \sum_{i,j} e_{v(i)} z e_{v(j)}$, and hence $(\rho) = (\rho')$. □

**Definition 1.1.15.** (Point-Wise Local Algebras) We say $\Lambda$ is *left* (resp. *right*) *pointwise local* if for each vertex $v$ of $Q$ the left module $\Lambda e_v$ (resp. right module $e_v \Lambda$) is local with maximal submodule $\sum_{a \in \mathbf{A}(v \to)} \Lambda a$ (resp. $\sum_{a \in \mathbf{A}(\to v)} a\Lambda$).

We say $\Lambda$ is *pointwise local* if it is left pointwise local and right pointwise local. In this case $\mathrm{rad}(\Lambda)$ is the ideal of $\Lambda$ generated by the arrows by lemma 3.1.34 (ii).

**Example 1.1.16.** Let $k$ be a field and $\Lambda = k[x,y]/(xy)$. Note that $\Lambda$ is surjectively given by $(Q,\rho,\theta)$ where $Q$ is the quiver with two loops $X$ and $Y$ at one vertex, $\rho = \{XY, YX\}$ and $\theta$ is given by $X \mapsto x$ and $Y \mapsto y$.

In case $k$ is algebraically closed the maximal ideals of $\Lambda$ have the form $(X-\lambda, Y-\mu)+(XY)/(XY)$ for $\lambda,\mu \in k$. This shows $\mathrm{rad}(\Lambda)=0$ and so $\Lambda$ is *not* pointwise local. Consequently algebras arising in this way shall be omitted from focus.

Part (b) in the corollary below motivated definition 1.1.15. Part (c) motivated definition 1.1.19 and SPIII) from definition 1.1.21. The corollary itself was motivated by [5, II.2.10].

**Corollary 1.1.17.** *Suppose:*

(a) $(Q,\rho)$ *satisfies special conditions;*

(b) $\Lambda$ *is left (resp. right) pointwise local; and*

(c) $\bigcap_{n\geq 1}(\mathrm{rad}(\Lambda))^n a = 0$ *(resp.* $\bigcap_{n\geq 1} a(\mathrm{rad}(\Lambda))^n = 0$*) for any arrow* $a$*.*

*Then for any* $p \in \mathbf{P}$*:*

(i) *any non-zero submodule of* $\Lambda p$ *(resp.* $p\Lambda$*) has the form* $\Lambda qp$ *(resp.* $pq\Lambda$*) with* $q \in \mathbf{P}$*;*

(iia) $\Lambda p$ *is local with* $\mathrm{rad}(\Lambda p) = \mathrm{rad}(\Lambda)p$*;*

(iib) *if* $\mathrm{rad}(\Lambda p) \neq 0$ *then* $\mathrm{rad}(\Lambda p) = \Lambda a p$ *for an arrow* $a$ *with* $ap \in \mathbf{P}$*; and*

(iii) *if* $p' \in \mathbf{P}$, $\Lambda p = \Lambda p'$ *and* $\mathrm{f}(p) = \mathrm{f}(p')$ *(resp.* $p\Lambda = p'\Lambda$ *and* $\mathrm{l}(p) = \mathrm{l}(p')$*) then* $p = p'$*.*

*Proof.* (i) The proofs for the respective statements about $p\Lambda$ will follow by symmetry.

Suppose $N$ is a non-zero submodule of $\Lambda p$. Consider the set $T$ consisting of all paths $q$ such that $qp \in \mathbf{P}$ and $N \subseteq \Lambda qp$. Clearly $e_v \in T$ where $v$ is the head of $p$. Suppose $T$ is infinite, and so by lemma 1.1.14 (ia) there is a sequence of consecutive arrows $a_1, a_2 \dots$ (that is, where $h(a_i) = t(a_{i+1})$) such that $a_n \dots a_1 p \in \mathbf{P}$ and $N \subseteq \Lambda a_n \dots a_1 p$ for each $n \geq 1$.

However as $\Lambda$ is left pointwise local this means $N \subseteq \bigcap_{n\geq 1} \mathrm{rad}(\Lambda)^n \mathrm{f}(p) = 0$, and hence $T$ must be finite as $N \neq 0$. So we let $q'$ be the longest path in $T$. For a contradiction assume that $N \neq \Lambda q'p$. Consider the submodule $M$ of $\Lambda e_{h(q')}$ consisting of all $\mu \in \Lambda e_{h(q')}$ with $\mu q'p \in N$. Since $N \neq \Lambda q'p$ we have $M \neq \Lambda e_{h(q')}$, and since $\Lambda$ is left pointwise local this gives $M \subseteq \sum_{a\in \mathbf{A}(h(q')\rightarrow)} \Lambda a$. Since $N \neq 0$ we can choose $n \in N \subseteq \Lambda q'p$ with $n \neq 0$.

Writing $n = \lambda q'p$ gives $\lambda \in M$ by definition. As $\lambda \neq 0$ there must exist an arrow $a'$ with tail $h(q')$ and $a'q'p \notin (\rho)$, and note that $a'$ is unique as $(Q, \rho)$ saitsfies special conditions. But now we have $n = \lambda' a'q'p$ for some $\lambda' \in \Lambda$ which shows $N \subseteq \Lambda a'q'p$ and so $a'q' \in T$ contradicts the maximality of $q'$.

(ii) Consider the $\Lambda$-module homomorphism $\alpha : \Lambda e_{h(p)} \to \Lambda p$ given by $\lambda \mapsto \lambda p$. As $\Lambda$ is surjectively given by $(Q, \rho, \theta)$ we have $\alpha(e_{h(p)}) \neq 0$ and so $\alpha \neq 0$. Let $\pi : \Lambda p \to \Lambda p/\mathrm{rad}(\Lambda)p$ be the canonical projection. By [65, 49.7 (b)] we have $\Lambda p \neq \mathrm{rad}(\Lambda p)$ and so as $\mathrm{rad}(\Lambda)p \subseteq \mathrm{rad}(\Lambda p)$ we have $p \notin \mathrm{rad}(\Lambda)p$ which means $\pi\alpha \neq 0$.

By lemma 3.1.34 (ic) we have $\alpha(\mathrm{rad}(\Lambda e_{h(p)})) \subseteq \mathrm{rad}(\Lambda)p \subseteq \ker(\pi)$ and so $\mathrm{rad}(\Lambda e_{h(p)}) \subseteq \ker(\pi\alpha)$. Since $\Lambda$ is pointwise local this gives $\mathrm{rad}(\Lambda e_{h(p)}) = \ker(\pi\alpha)$ and so $\Lambda p/\mathrm{rad}(\Lambda)p \simeq \Lambda e_{h(p)}/\mathrm{rad}(\Lambda e_{h(p)})$ which is simple and so $\mathrm{rad}(\Lambda)p$ is a maximal submodule, and thus $\mathrm{rad}(\Lambda)p = \mathrm{rad}(\Lambda p)$.

(iii) We will just show that ($\Lambda p = \Lambda p' \neq 0$ and $\mathrm{f}(p) = \mathrm{f}(p')$) implies $p = p'$. The other statement will hold by symmetry. By lemma 1.1.14 (ia) we have $p' = \gamma p$ with for some path $\gamma$. For a contradiction we assume $\gamma$ is non-trivial.

Consider the map $\nu : \Lambda e_{h(p)} \to \Lambda p$ sending $\lambda$ to $\lambda p$. Since $\Lambda p = \Lambda\gamma p$ we have $e_{h(p)} - \mu\gamma \in \ker(\nu)$ for some $\mu \in \Lambda e_{h(p)}$. Since $\Lambda p \neq 0$ we have $\ker(\nu) \neq \Lambda e_{h(p)}$ and so as $\Lambda$ is pointwise local we have that $\ker(\nu) \subseteq \sum_{a\in\mathbf{A}(h(p)\rightarrow)} \Lambda a$. Since $\mu\gamma \in \sum_{a\in\mathbf{A}(h(p)\rightarrow)} \Lambda a$ this gives $e_{h(p)} \in \sum_{a\in\mathbf{A}(h(p)\rightarrow)} \Lambda a$ which contradicts that $\Lambda$ is pointwise local. $\square$

The quiver $Q$ may be infinite (see example 1.1.4), and so $\Lambda$ need not be unital. However $\Lambda$ will always have a *complete set of (orthogonal) idempotents* (see definition 3.1.30) given by the vertices of $Q$ (which follows from the surjectivity of $\theta$).

The equivalence of (iia) and (iib) in the following[2] will be used in section 1.2.1.

**Lemma 1.1.18.** *Suppose that*

(a) $(Q, \rho)$ *satisfies special conditions,*

*and suppose that for each vertex* $v$*,*

(b) $\Lambda \mathfrak{m} e_v \subseteq \sum_{a \in \mathbf{A}(v \to)} \Lambda a$ *(resp.* $e_v \mathfrak{m} \Lambda \subseteq \sum_{a \in \mathbf{A}(\to v)} a\Lambda$*), and*

(c) $\mathbf{P}(t, v \to) = \emptyset$ *(resp.* $\mathbf{P}(t, \to v) = \emptyset$*) for* $t \gg 0$.

*Then the following statements hold.*

(i) $\Lambda$ *is left (resp. right) pointwise local.*

(ii) *If* $a$ *is an arrow then* $\Lambda a$ *(resp.* $a\Lambda$*) is finitely generated over* $R$.

(iii) *For distinct* $p, p' \in \mathbf{P}$ *we have (1) iff (2), where*

(1) $\Lambda p = \Lambda p'$ *(resp.* $p\Lambda = p'\Lambda$*) which is simple.*

(2) $p$ *and* $p'$ *are parallel and* $Rp = Rp'$ *in* $\Lambda$.

(iv) *If* $((1)$ *or* $(2))$ *then* $\Lambda p = \mathrm{soc}(\Lambda e_{t(p)}) = \Lambda p'$ *(resp.* $p\Lambda = \mathrm{soc}(e_{h(p)}\Lambda) = p'\Lambda$*).*

*Proof.* The respective claims hold by symmetry. We begin by showing any proper submodule of $\Lambda e_v$ is contained in $\sum_{a \in \mathbf{A}(v \to)} \Lambda a$. We also set up some notation for later in the proof.

By assumption (c) for all $a \in \mathbf{A}(v \to)$ there is some integer $l$ such that there are no paths $p$ of length $n > l$ with $\mathrm{f}(p) = a$. Hence by lemma 1.1.14 (ia) there are arrows $a_1, \ldots, a_{n(a)}$ such that: $n(a) \leq l$; $a_1 \ldots a_{n(a)} \in \mathbf{P}$; $a_{n(a)} = a$; and if $p \in \mathbf{P}$ and $\mathrm{f}(p) = a$ then $p \in \{p_{a,i} \mid 1 \leq i \leq n(a)\}$ where $p_{a,i} = a_i \ldots a_{n(a)}$.

So any $\mu \in \Lambda e_v$ has the form $\mu = re_v + \sum_{a,i} r_{i,a} p_{a,i}$ where $r, r_{a,i} \in R$. Now fix $a \in \mathbf{A}(v \to)$, $i, j \geq 1$ and $q \in \mathbf{P}(j)$. If $i + j \leq n(a)$ then $qp_{a,i} = p_{a,i+j}$ or $qp_{a,i} = 0$, and if $i + j > n(a)$ then $qp_{a,i} = 0$. Hence for any $i$ and $j$ we have $qp_{a,i} \in Rp_{a,i+j}$.

[2] Compare with corollary 1.1.17 (iii).

We now assume $r \notin \mathfrak{m}$ and show this gives $\Lambda\mu = \Lambda e_v$. Without loss of generality we can assume $a, a' \in \mathbf{A}(v \to)$ and $a \neq a'$. For simplicity write $n(a) = n$ and $n(a') = m$. Since $R$ lies in the centre of $\Lambda$,and there are no paths $p$ of length greater than 1 with $\mathrm{f}(p) = a$, we have $p_{a,n}\mu = rp_{a,n}$ and so $Rp_{a,n} \subseteq \Lambda\mu$ as $r$ is a unit.

By symmetry we can assume $n \geq m$. If $n-1 \geq m$ then $p_{a,n-1}\mu = rp_{a,n-1} + r_{a,1}p_{a,n-1}p_{a,1}$ and so by the above we have similarly that $Rp_{a,n-1} \subseteq \Lambda\mu$. Proceeding this way gives $Rp_{a,i} \subseteq \Lambda\mu$ when $m \leq i \leq n$, and hence $\eta_m \in \Lambda\mu$ where we let $\eta_d = \sum_{a,i \leq d} r_{i,a}p_{a,i}$ for each $d$ with $1 \leq d \leq m$.

We now proceed as above, but we deal with the paths $p_{a,d}$ and $p_{a',d}$ simultaneously. Again $p_{a,m}\eta_m = rp_{a,m}$ and so $Rp_{a,m} \subseteq \Lambda\eta_m$. Writing

$$p_{a',m}\eta_m = rp_{a',m} + r_{a,n-m}p_{a',m}p_{a,n-m} + \cdots + r_{a,1}p_{a',m}p_{a,1}$$

shows that $p_{a',m} \in \Lambda\eta_m$ as each one of the following products of (pairs of) paths $p_{a',m}p_{a,n-m}, \ldots, p_{a',m}p_{a,1}$ lies in $Rp_{a,i}$ for some $i$ with $m+1 \leq i \leq n$. Thus $p_{a,m}, p_{a',m} \in \Lambda\mu$, and proceeding this way we can show $p_{a,i}, p_{a',i} \in \Lambda\mu$ for each $i$ with $1 \leq i \leq m$.

We have already shown that $p_{a,i} \in \Lambda\mu$ when $m \leq i \leq n$, and so $re_v \in \Lambda\mu$ which means $\Lambda\mu = \Lambda e_v$. Since we assumed $r \notin \mathfrak{m}$, we have shown that for any proper submodule $M$ of $\Lambda e_v$, if $\mu \in M$ then $\mu \subseteq \Lambda\mathfrak{m}e_v + \sum_{a\in\mathbf{A}(v\to)} \Lambda a$ which by (b) means $\Lambda\mu \subseteq \sum_{a\in\mathbf{A}(v\to)} \Lambda a$.

(i) To recap, if $M$ is a proper submodule of $\Lambda e_v$ then $M \leqslant \sum_{a\in\mathbf{A}(v\to)} \Lambda a$. If $\sum_{a\in\mathbf{A}(v\to)} \Lambda a = \Lambda e_v$ then we have $e_v \in \sum_{a\in\mathbf{A}(v\to)} \Lambda a$. This means we can write $e_v$ as $\mu$ above where $r = 0$, which gives $p_{a,n} = p_{a,n}e_v = 0$ as above, which contradicts that $p_{a,n} \in \mathbf{P}(v \to)$ and that $\Lambda$ is surjectively given by $(Q, \rho, \theta)$.

We have now shown there are no proper submodules containing $\sum_{a\in\mathbf{A}(v\to)} \Lambda a$, which is a proper submodule. Hence $\sum_{a\in\mathbf{A}(v\to)} \Lambda a$ is the unique maximal submodule of $\Lambda e_v$.

(ii) Note that $\Lambda a$ is generated as an $R$-module by the paths $p$ with $\mathrm{f}(p) = a$, of which there are finitely many, by (c).

(iii) (1) $\Rightarrow$ (2) By assumption there is some $\lambda \in \Lambda e_{h(p')}$ such that $p = \lambda p'$.

Since $\Lambda$ is left pointwise local (by (i)), we can write $\lambda = re_{h(p')} + z$ for some $z \in \mathrm{rad}(\Lambda)e_{h(p')}$. This gives $zp' \in \mathrm{rad}(\Lambda)\Lambda p' = \mathrm{rad}(\Lambda p')$ by corollary 1.1.17 (ii), and as $\Lambda p'$ is simple this means $zp' = 0$ and so $p = rp'$. If $r \in \mathfrak{m}$ then $re_{h(p')} \in \Lambda\mathfrak{m}e_{h(p')}$ which is contained in $\mathrm{rad}(\Lambda)e_{h(p')}$ by assumption (b) (and again as $\Lambda$ is left pointwise local).

As above this means $rp' = 0$ which contradicts that $p \in \mathbf{P}$. Hence $r \notin \mathfrak{m}$ which means $r$ is a unit and $p = rp'$ gives $Rp = Rp'$. Since $p$ and $p'$ are distinct this shows they are parallel.

(iii) (1) $\Leftarrow$ (2) If $b$ is an arrow with $bl(p) \in \rho$ then $bp' \in bRp = Rbp = 0$. Without loss of generality this means $\mathrm{rad}(\Lambda)e_{h(p)} \subseteq \ker(\psi)$ where $\psi$ is the $\Lambda$-module epimorphism $\Lambda e_{h(p)} \to \Lambda p$ given by $\lambda \mapsto \lambda p$. Note that $\psi(e_{h(p)}) \neq 0$ and $\Lambda$ is pointwise local (by part (i)), so we must have $\mathrm{rad}(\Lambda)e_{h(p)} = \ker(\psi)$. Thus $\Lambda p$ is simple because $\Lambda$ is pointwise local, and similarly $\Lambda p'$ is simple.

(iv) Since $\Lambda p$ and $\Lambda p'$ are assumed to be simple we have $\Lambda p', \Lambda p \subseteq \mathrm{soc}(\Lambda e_v)$. It suffices to show $\Lambda p$ and (by symmetry $\Lambda p'$) is an essential submodule of $\Lambda e_v$ so that $\Lambda p \supseteq \mathrm{soc}(\Lambda e_v)$ by lemma 3.1.34 (ib). Hence we assume $M$ is a non-zero submodule of $\Lambda e_{h(p)}$, and we show $M \cap \Lambda p \neq 0$. If $M = \Lambda e_{h(p)}$ then clearly $M \cap \Lambda = \Lambda p \neq 0$ since $p \in \mathbf{P}$ and there is nothing to prove.

Otherwise $M \leqslant \sum_{a \in \mathbf{A}(h(p)\to)} \Lambda a$ as $\Lambda$ is pointwise local, and we choose $\mu \in M$ which is non-zero. It suffices to find $\lambda \in \Lambda$ such that $0 \neq \lambda\mu \in \Lambda p$. Let $\mu = \sum_a \mu_a$ where $\mu_a \in \Lambda a$ and $\mu_{a'} \in \Lambda a'$ for $a, a' \in \mathbf{A}(v \to)$ with $a \neq a'$ (which exist as $p$ and $p'$ are distinct and parallel). In this notation $p_{a,n} = p$ and $p_{a',m} = p'$, and so $\mathrm{f}(p) = a$ and $\mathrm{f}(p') = a'$ up to reordering.

Since $\mu \neq 0$ we can assume $\mu_a \neq 0$ and, without loss of generality, $\mu_{a'} \neq 0$. So there exist $m', n' > 0$ maximal for which $\mu_a \in \Lambda p_{a,n'}$ and $\mu_{a'} \in \Lambda p_{a',m'}$. Hence $\mu_a = \sum_{i=n'}^{n} r_{a,i}p_{a,i}$ and $\mu_{a'} = \sum_{i=m'}^{m} r_{a',i}p_{a',i}$. If $r_{a,n'} \in \mathfrak{m}$ then $\Lambda\mathfrak{m}e_v \subseteq \sum_{a \in \mathbf{A}(v\to)} \Lambda a$ implies $r_{a,n'}p_{a,n'} \in \Lambda p_{a,n'+1}$. Hence by the maximality of $n'$ and $m'$ we have $r_{a,n'}, r_{a',m'} \notin \mathfrak{m}$. By symmetry we can assume $n - n' \geq m - m'$. Let $q = a_n \dots a_{n'+1}$. If $n = n'$ then $m = m'$ and so $\mu_a \in \Lambda p$ and $\mu_{a'} \in \Lambda p' \subseteq \Lambda Rp \subseteq \Lambda p$, in which case it suffices to let $\lambda$ be the local unit of $\mu$.

Otherwise $n > n'$, and so q is non-trivial. Note that

$$q\mu_a = \sum_{i=n'}^{n} r_{a,i} q p_{a,i} = r_{a,n'} q p_{a,n'} = r_{a,n'} p$$

as $qp_{a,i}$ has length greater than $n$ if $i > n'$, and similarly $q\mu_{a'} = 0$ as $qp_{a',j}$ has length $n - n' + j$ which is greater than $m$ if $j > m'$, recalling we assumed (using symmetry) that $n - n' + m' > m$.

Hence setting $\lambda = q$ gives $\lambda\mu = q\mu_a \neq 0$ as $r_{a,n'} \notin \mathfrak{m}$. $\square$

The assumptions (b) and (c) from lemma 1.1.18 motivated the following definition.

**Definition 1.1.19.** (Rad-Nilpotency) For an ideal $I$ in $R$ we say $\Lambda$ is *pointwise rad-nilpotent modulo* $I$ if for each vertex $v$ there is some $n(v) \geq 1$ for which

$$(\mathrm{rad}(\Lambda))^{n(v)} e_v \subseteq \Lambda I e_v \subseteq \mathrm{rad}(\Lambda) e_v.$$

(Admissible Ideals, notation $J$) [5, II.2.1] Let $J$ be the ideal of $RQ$ generated by the arrows. We say the ideal $(\rho)$ in $RQ$ is an *admissible* ideal provided there is some integer $m \geq 2$ for which $J^m \subseteq (\rho) \subseteq J^2$.

**Example 1.1.20.** Recall the algebra $k[x, y]/(xy)$ from example 1.1.16. Recall $\mathrm{rad}(k[x, y]/(xy)) = 0$ and so by definition $k[x, y]/(xy)$ is pointwise rad-nilpotent modulo $I$ iff $I = 0$. Note that $\rho = \{XY, YX\}$ and since $X^m \notin (\rho)$ for all $m \geq 1$ the ideal $(\rho)$ cannot be admissible.

**Definition 1.1.21.** (Quasi-Bounded Special Biserial Algebras) $\Lambda$ is called a *quasi-bounded special biserial algebra over* $R$ if $(Q, \rho)$ satisfies special conditions, and:

SPIII) $\Lambda$ is pointwise local and pointwise rad-nilpotent modulo $\mathfrak{m}$;

and for any arrow $a$

SPIV) the $R$-modules $\Lambda a$ and $a\Lambda$ are finitely generated; and

SPV) the $\Lambda$-modules $\Lambda a \cap \Lambda a'$ and $a\Lambda \cap a'\Lambda$ are simple or trivial for any arrow $a' \neq a$.

**Example 1.1.22.** Recall the $\widehat{\mathbb{Z}}_p$-subalgebra $\Lambda = \{(\gamma_{ij}) \in \mathbb{M}_2(\widehat{\mathbb{Z}}_p) \mid \gamma_{11} - \gamma_{22}, \gamma_{12} \in p\widehat{\mathbb{Z}}_p\}$ surjectively given by $(Q, \rho, \theta)$ from example 1.1.6. The calculations labeled $(\lozenge)$ give

$$\Lambda\alpha = \begin{pmatrix} p\widehat{\mathbb{Z}}_p & 0 \\ \widehat{\mathbb{Z}}_p & 0 \end{pmatrix}, \ \Lambda\beta = \begin{pmatrix} 0 & p\widehat{\mathbb{Z}}_p \\ 0 & p\widehat{\mathbb{Z}}_p \end{pmatrix},$$

$$\alpha\Lambda = \begin{pmatrix} 0 & 0 \\ p\widehat{\mathbb{Z}}_p & p\widehat{\mathbb{Z}}_p \end{pmatrix}, \ \beta\Lambda = \begin{pmatrix} p\widehat{\mathbb{Z}}_p & p\widehat{\mathbb{Z}}_p \\ 0 & 0 \end{pmatrix}$$

So, SPIV) and SPV) both hold. We now check SPIII). If $\Lambda\alpha + I = \Lambda$ for a left ideal $I$ then we have $1 = \lambda + \gamma$ for some $\gamma = (\gamma_{ij}) \in I$ and some $\lambda = (\lambda_{ij}) \in \Lambda\alpha$. This means $\lambda_{12} = \lambda_{22} = 0$ and so $\gamma_{12} = 0$, $\lambda_{11} + \gamma_{11} = \gamma_{22} = 1$ and $\gamma_{21} + \lambda_{21} = 0$. Consequently $\det(\gamma) = 1 - \lambda_{11}$ which is a unit as $\lambda_{11} \in p\widehat{\mathbb{Z}}_p$ and so $I = \Lambda$. Hence $\Lambda\alpha$ is a superfluous left ideal of $\Lambda$. Similarly $\Lambda\beta$ is superfluous and so $\mathrm{rad}(\Lambda e_v) \supseteq \Lambda\alpha \oplus \Lambda\beta$.

Conversely there is an $R$-module isomorphism $\mathbb{Z}/p\mathbb{Z} \to \Lambda/\Lambda\alpha \oplus \Lambda\beta$ sending the coset of $n$ to the coset of $n\mathbb{I}_2$ (where $\mathbb{I}_2$ is the unit in $\mathbb{M}_2(\widehat{\mathbb{Z}}_p)$) and so $\Lambda\alpha \oplus \Lambda\beta$ is a maximal left submodule of $\Lambda$ which proves $\mathrm{rad}(\Lambda e_v) = \Lambda\alpha \oplus \Lambda\beta$. This shows $\Lambda$ is left pointwise local. By symmetry we have $\mathrm{rad}(e_v\Lambda) = \alpha\Lambda \oplus \beta\Lambda$ and hence $\Lambda$ is right pointwise local. Moreover any path of length greater than 2 which lies outside $(\rho)$ ends in $\alpha\beta$ or $\beta\alpha$, and since $\alpha\beta + \beta\alpha = p$ in $\Lambda$ this means $(\mathrm{rad}(\Lambda))^3 \subseteq \Lambda p \subseteq \mathrm{rad}(\Lambda)$. Hence $\Lambda$ is pointwise rad-nilpotent modulo $p\widehat{\mathbb{Z}}_p$ and so $\Lambda$ is a quasi-bounded special biserial algebra over $\widehat{\mathbb{Z}}_p$.

**Example 1.1.23.** Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime $p$. Recall the $\mathbb{Z}/p^3\mathbb{Z}$-algebra $\Lambda$ from example 1.1.10

$$\Lambda = T_5(R_i, M_{ij}, \varphi^t_{ij}) = \begin{pmatrix} \mathbb{F}_p & 0 & 0 & 0 & 0 \\ \mathbb{F}_p & \mathbb{F}_p & 0 & 0 & 0 \\ \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p & 0 & 0 \\ 0 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{Z}/p^3\mathbb{Z} & 0 \\ 0 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \end{pmatrix}$$

Writing left submodules of $\Lambda$ as columns gives

$$\left.\begin{aligned} \Lambda e_1 &= \begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p & 0 & 0 \end{pmatrix}^t \\ \Lambda\alpha &= \begin{pmatrix} 0 & 0 & \mathbb{F}_p & 0 & 0 \end{pmatrix}^t \\ \Lambda\beta &= \begin{pmatrix} 0 & \mathbb{F}_p & \mathbb{F}_p & 0 & 0 \end{pmatrix}^t \end{aligned}\right\} \quad \begin{pmatrix} \text{submodules of} \\ \text{column 1} \end{pmatrix}$$

$$\left.\begin{aligned} \Lambda e_2 &= \begin{pmatrix} 0 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \end{pmatrix}^t \\ \Lambda\gamma &= \begin{pmatrix} 0 & 0 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \end{pmatrix}^t \end{aligned}\right\} \quad \begin{pmatrix} \text{submodules of} \\ \text{column 2} \end{pmatrix}$$

$$\left.\begin{aligned} \Lambda e_3 &= \begin{pmatrix} 0 & 0 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \end{pmatrix}^t \\ \Lambda\delta &= \begin{pmatrix} 0 & 0 & 0 & \mathbb{F}_p & \mathbb{F}_p \end{pmatrix}^t \end{aligned}\right\} \quad \begin{pmatrix} \text{submodules of} \\ \text{column 3} \end{pmatrix}$$

$$\left.\begin{aligned} \Lambda e_4 &= \begin{pmatrix} 0 & 0 & 0 & \mathbb{Z}/p^3\mathbb{Z} & \mathbb{F}_p \end{pmatrix}^t \\ \Lambda\varepsilon &= \begin{pmatrix} 0 & 0 & 0 & p\mathbb{Z}/p^3\mathbb{Z} & 0 \end{pmatrix}^t \\ \Lambda\lambda &= \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbb{F}_p \end{pmatrix}^t \end{aligned}\right\} \quad \begin{pmatrix} \text{submodules of} \\ \text{column 4} \end{pmatrix}$$

$$\left.\Lambda e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbb{F}_p \end{pmatrix}^t\right\} \quad \begin{pmatrix} \text{submodule of} \\ \text{column 5} \end{pmatrix}$$

This shows $\Lambda$ is left pointwise local, since for each vertex $v \in \{1, 2, 3, 4, 5\}$ the quotients $\Lambda e_v / \sum_{a \in \mathbf{A}(v\rightarrow)} \Lambda a$ have $p$ elements, and so they must be simple. Similarly one can show $\Lambda$ is (right and hence) pointwise local.

It is also clear that for each arrow $a$ the $\mathbb{Z}/p^3\mathbb{Z}$-modules $\Lambda a$ and $a\Lambda$ are finitely generated. The intersections $\Lambda\alpha \cap \Lambda\beta = \Lambda\alpha$ and $\alpha\Lambda \cap \gamma\Lambda = \alpha\Lambda$ have $p$ elements, so they must also be simple. Since $\Lambda a \cap \Lambda a' = 0$ and $a\Lambda \cap a'\Lambda = 0$ for all other pairs of distinct arrows $a$ and $a'$, $\Lambda$ is a quasi-bounded special biserial algebra over $\mathbb{Z}/p^3\mathbb{Z}$.

We now explain how the *special algebras* studied by Pogorzały and Skowroński [53] are examples of quasi-bounded special biserial algebras.

**Example 1.1.24.** (Pogorzały-Skowroński Special Algebras) Let $\Lambda' = kQ/(\rho)$ be a *special* algebra in the terminology used by Pogorzały and Skowroński [53, pp.492 - 493]. This means: $k$ is an algebraically closed field; $\Lambda'$ is surjectively given by $(Q, \rho, \theta)$ where $\theta$ is the quotient map $kQ \to \Lambda'$; (R1) and (R2) from [53, p.492] hold; and $(\rho)$ is admissible.

Conditions (R1) and (R2) are the same as SP1) and SPII) from definition 1.1.7. Since $(\rho)$ is admissible we then have that $\Lambda'$ is pointwise local by lemma 1.1.18, and so $J/(\rho) = \mathrm{rad}(\Lambda)$ which together gives $(\rho) \subseteq (\mathrm{rad}(\Lambda'))^2 \subseteq \mathrm{rad}(\Lambda')$ and $(\mathrm{rad}(\Lambda))^m = (J/(\rho))^m = 0$ for some $m$.

Hence $\Lambda'$ is pointwise rad-nilpotent modulo 0 and SPIII) holds. By lemma 1.1.18 (ii) it is clear that SPIV) holds. To see that $\Lambda'$ is a quasi-bounded special biserial algebra over $k$, one observes that the statement and proof of [61, Lemma 1] is precisely the verification of SPV). Note that the definition introduced by Skowroński and Waschbüsch [61, §1 (SP)] requires $Q$ to be finite, a restriction we are omitting.

We now give some immediate consequences of definition 1.1.21.

**Corollary 1.1.25.** *If $\Lambda$ is a quasi-bounded special biserial algebra over $R$ then for any vertex $v$,*

(ia) $\Lambda e_v / \sum_{a\in\mathbf{A}(v\rightarrow)} \Lambda a \simeq k \simeq e_v\Lambda / \sum_{a\in\mathbf{A}(\rightarrow v)} a\Lambda$ *as $R$-modules, and*

(ib) $\Lambda e_v$ *and* $e_v\Lambda$ *are finitely generated as $R$-modules.*

*Consequently*

(ii) *if $Q$ is finite then the ring $\Lambda$ is unital, noetherian and semilocal, and*

(iii) $\bigcap_{n>0}(\mathrm{rad}(\Lambda))^n M = 0$ *for any $\Lambda$-module $M$ which is finitely generated as an $R$-module.*

*Proof.* (ia) Consider the map $\tau_v : R \to \Lambda e_v / \sum_{a\in\mathbf{A}(v\rightarrow)} \Lambda a$ sending $r$ to $re_v + \sum_{a\in\mathbf{A}(v\rightarrow)} \Lambda a$. For any $\lambda + \sum_{a\in\mathbf{A}(v\rightarrow)} \Lambda a \in \Lambda e_v / \sum_{a\in\mathbf{A}(v\rightarrow)} \Lambda a$ there is some $\mu \in RQ$ with $\theta(\mu) = \lambda$ since $\theta$ is onto. Let $A$ denote the ideal of $RQ/(\rho)$ generated by the arrows. Since $\lambda = \lambda e_v$ we have that $\mu = \mu e_v$ and so writing $\mu e_v = re_v + ae_v$ for some $r \in R$ and $a \in A$ gives $re_v - \lambda e_v \in \sum_{a\in\mathbf{A}(v\rightarrow)} \Lambda a$. Hence $\tau_v$ is surjective.

Suppose for a contradiction that $\mathfrak{m}e_v = \Lambda e_v$. Then there is some $x \in \mathfrak{m}$ for which $(1-x)e_v = 0$ in $\Lambda$. As $1-x$ is a unit in $R$ this contradicts that any path $p \notin (\rho)$ has a non-zero image in $\Lambda$. Thus $\mathfrak{m}e_v \subseteq \sum_{a\in\mathbf{A}(v\rightarrow)} \Lambda a$ and so $\mathfrak{m} \subseteq \ker(\tau_v)$.

Since $\theta$ is onto $\ker(\tau_v)$ is a left $R$-submodule of $R$, and as $\ker(\tau_v) \neq R$ this gives $\mathfrak{m} = \ker(\tau_v)$. This gives an isomorphism between $\Lambda e_v / \sum_{a \in \mathbf{A}(v\to)} \Lambda a$ and $R/\mathfrak{m} = k$. By symmetry we also have an $R$-module isomorphism $k \simeq e_v\Lambda / \sum_{a \in \mathbf{A}(v\to)} \Lambda a$.

(ib) By SPIV) the $R$-modules $\sum_{a\in\mathbf{A}(v\to)} \Lambda a$ and $\sum_{a\in\mathbf{A}(\to v)} a\Lambda$ are finitely generated over $R$. By corollary 1.1.25 (i) the quotients $\Lambda e_u / \sum_{a\in\mathbf{A}(u\to)} \Lambda a$ and $e_u\Lambda / \sum_{a\in\mathbf{A}(\to u)} a\Lambda$ are isomorphic to $k$, and hence by applying the horseshoe lemma (see lemma 3.2.8) $\Lambda e_u$ and $e_u\Lambda$ are also finitely generated over $R$.

(ii) In general we have $\Lambda = \bigoplus_v \Lambda e_v$ which is a finite direct sum if $Q$ is finite, and each summand is finitely generated by (ib). Hence $\Lambda$ is finitely generated over $R$ and has a 1 defined by $\sum_v e_v$. Since $\Lambda$ is finitely generated over $R$, $\Lambda$ is semilocal by [48, (20.6) Proposition].

Since $R$ is a noetherian ring, $\Lambda$ is noetherian as an $R$-module as it is finitely generated over $R$. Any ascending chain of left (or right) ideals in the ring $\Lambda$ defines an ascending chain of submodules in the $R$-module $\Lambda$. This means $\Lambda$ is a noetherian ring.

(iii) Note that $\bigcap_{n>0}(\mathrm{rad}(\Lambda))^n M \subseteq \bigcap_{n>0} \mathfrak{m}^n M$ by SPIII). The claim now follows from Krull's intersection theorem (see for example [48, Ex.4.23]). $\square$

We now motivate the terminology introduced in definition 1.1.21.

**Definition 1.1.26.** [28, p.62] (Uniserial And Biserial Modules) Let $\Gamma$ be a (possibly non-unital) ring. We say a $\Gamma$-module $M$ is *uniserial* if $N \subseteq N'$ or $N' \subseteq N$ for any submodules $N$ and $N'$ of $M$. Thus if $M'$ is a maximal (resp. simple) submodule of a uniserial module $M$ then $M$ is local (resp. colocal) and $M' = \mathrm{rad}(M)$ (resp. $M' = \mathrm{soc}(M)$).

If $M$ is indecomposable we say $M$ is *biserial* if there are uniserial submodules $L$ and $L'$ of $M$ such that

(a) $L + L' = M$ or $M$ is local with $L + L' = \mathrm{rad}(M)$, and

(b) $L \cap L' = 0$ or $M$ is colocal with $\mathrm{soc}(M) = L \cap L'$.

In [28, p.65] Fuller calls a unital artinian ring $\Gamma$ *biserial* provided every indecomposable projective left or right $\Gamma$-module is biserial. To motivate the study of algebras with this property we note the following theorem due to Crawley-Boevey.

**Theorem 1.1.27.** [20, Theorem A] *Let $k$ be an algebraically closed field and let $\Gamma$ be a finite-dimensional $k$-algebra with a multiplicative unit. If $\Gamma$ is biserial then $\Gamma$ is representation-tame* (see definition 1.5.4)*.*

The following result generalises [61, Lemma 1]. The reader is advised to recall section 3.1.5 in which some language and theory is developed about rings with enough idempotents.

**Proposition 1.1.28.** *Let $\Lambda$ be a quasi-bounded special biserial algebra over $R$. Then:*

(i) *any unital projective indecomposable left (resp. right) module is isomorphic to $\Lambda e_v$ (resp. $e_v\Lambda$) for some vertex $v$; and*

(ii) *for each vertex $v$ the left (resp. right) module $\Lambda e_v$ (resp. $e_v\Lambda$) is unital, indecomposable, projective and biserial.*

*Proof.* The respective claims will follow by symmetry.

(i) By SPIII) $\Lambda$ has a complete set of orthogonal local idempotents $E$ defined by the elements $e_v$ (where $v$ runs through the vertices). By lemma 3.1.37 this means is any unital indecomposable projective $\Lambda$-module $P$ is isomorphic to $\Lambda e_v$ for some vertex $v$.

(ii) By SPIII) we have $\mathrm{rad}(\Lambda e_v) = \sum_{a\in\mathbf{A}(v\to)} \Lambda a$ and by SPI) the set $\mathbf{A}(v \to)$ has at most two elements. Without loss of generality we can assume there are two distinct arrows $a$ and $a'$ with tail $v$. Hence $\mathrm{rad}(\Lambda e_v) = \Lambda a + \Lambda a'$.

By corollary 1.1.25 (iii) and SPIV), part (c) of corollary 1.1.17 holds. Parts (a) and (b) of corollary 1.1.17 hold by assumption, and so by corollary 1.1.17 (i) the modules $\Lambda a$ and $\Lambda a'$ are uniserial. SPV) says the intersections $\Lambda a \cap \Lambda a'$ are simple or trivial. $\square$

There exists an algebra which is not special (see example 1.1.24) but whose projective indecomposable modules are biserial.

**Example 1.1.29.** Recall the special algebras introduced in example 1.1.24.

Suppose $R$ is a field $k$ and $\rho = \{\beta\alpha - \delta\gamma\alpha, \xi\beta\}$ where $Q$ is the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \arrow[rr, "\beta"] \arrow[dr, "\gamma"'] & & 4 \arrow[r, "\xi"] & 5 \\
& & 3 \arrow[ur, "\delta"'] & &
\end{tikzcd}$$

Pogorzały and Skowroński note in [53, Example 2, p.503] that $kQ/(\rho)$ is not special biserial. In a remark above [61, Lemma 2] Skowroński and Waschbuch have shown $kQ/(\rho)$ is biserial.

## 1.2 Gentle Algebras.

### 1.2.1 Quasi-Bounded String Algebras.

**Assumption:** In section 1.2.1 we assume $\Lambda$ is a quasi-bounded special biserial algebra over $R$ surjectively given by $(Q, \rho, \theta)$.

**Definition 1.2.1.** (QUASI-BOUNDED STRING ALGEBRAS) We call $\Lambda$ a *quasi-bounded string algebra over* $R$ if $\Lambda a \cap \Lambda a' = a\Lambda \cap a'\Lambda = 0$ for any distinct arrows $a$ and $a'$.

(BUTLER-RINGEL STRING ALGEBRAS) [15, p.157] By a *(Butler-Ringel) string algebra* we mean an algebra of the form $kQ/(\rho)$ where: $k$ is a field; $Q$ is a (possibly infinite) quiver; the elements of $\rho$ are paths of length at least 2; and conditions (1), (1*), (2), (2*), (3) and (3*) from [15, p.157] hold.

Conditions (1) and (1*) are the same as SPI) from definition 1.1.7. Similarly conditions (2) and (2*) are the same as SPII). Condition (3) (resp. (3*)) says that for each vertex $v$ we have $\mathbf{P}(t, v \rightarrow) = \emptyset$ (resp. $\mathbf{P}(t, \rightarrow v) = \emptyset$) for some $t \gg 0$.

**Example 1.2.2.** Recall the quasi-bounded special biserial algebra $\Lambda$ over $\mathbb{Z}/p^3\mathbb{Z}$ from example 1.1.23. As presented, $\Lambda$ is not a string algebra since $\Lambda\alpha \cap \Lambda\beta = \Lambda\alpha$ and $\alpha\Lambda \cap \gamma\Lambda = \alpha\Lambda$ have $p > 0$ elements.

We now explain how the two notions from definition 1.2.1 relate to one-another.

**Lemma 1.2.3.** *If $k$ is a field then $\Lambda$ is a quasi-bounded string algebra over $k$ iff it is a Butler-Ringel string algebra.*

*Proof.* Let $\Lambda$ be a Butler-Ringel string algebra. As conditions (3) and (3*) hold $\Lambda$ is pointwise rad-nilpotent modulo 0. As in example 1.1.24, lemma 1.1.18 applies, and so SPIII) and SPIV) both hold. Now suppose $a$ and $a'$ are distinct arrows with the same tail. Since $\rho$ consists of paths, the set of $p$ such that $\mathrm{f}(p) = a$ defines a $k$-basis for $\Lambda a$, and so $\Lambda a \cap \Lambda a' = 0$. Hence SPV) holds and so $\Lambda$ is a quasi-bounded string algebra over $k$.

Conversely, suppose $\Lambda$ is a quasi-bounded string algebra over $k$.

Hence $\mathfrak{m} = 0$, and so $\Lambda$ is surjectively given by $(Q, \rho, \theta)$ where $\theta : kQ \to \Lambda$ is some $k$-algebra surjection where $(\rho) \subseteq \ker(\theta)$ and $\theta(p) \neq 0$ for any path $p \notin (\rho)$. As above, since SPI) and SPII) hold, conditions (1), (1*), (2) and (2*) from [15, p.157] hold. $\Lambda$ is pointwise local and pointwise rad-nilpotent modulo 0, and so conditions (3) and (3*) also hold. Choose a subset $\varrho$ of $\bigcup_{u,v} e_v kQ e_u$ such that $\ker(\theta) = (\varrho)$, and so $\Lambda \simeq kQ/(\varrho)$.

We now claim $(\varrho) = (\widehat{\varrho})$ where $\widehat{\varrho}$ consists only of paths. By lemma 1.1.14 (ii) we have $(\varrho) = (\varrho')$ as ideals in $kQ$ where any element in $\rho'$ has the form $\sum_q r_q q$ where $q$ runs through parallel paths in $Q$, say from $u$ to $v$. Let $\sum_q r_q q \in \rho'$, and so $-r_{e_u} e_u = \sum_{q \neq e_u} r_q q$ as elements in $\Lambda$.

If $r_{e_u} \neq 0$ then $\Lambda e_u \subseteq \sum_{a \in \mathbf{A}(u \to)} (\Lambda a)$ which contradicts corollary 1.1.25 (ia). Hence $r_{e_u} = 0$. By lemma 1.1.14 (i) there are at most two non-trivial paths $p$ and $p'$ with head $u$ and tail $v$ such that $p, p' \notin (\varrho) = (\widehat{\varrho})$. Without loss of generality assume $p$ and $p'$ exist.

This means $r_p + r_{p'} = \sum_{q \neq p, p'} r_q q = 0$ as elements in $\Lambda$. Without loss of generality we can assume $p$ is not longer than $p'$. Let $r = r_p$ and $r' = r_{p'}$. By definition $p = r^{-1} r' p'$ in $\Lambda$ and hence $kp = kp'$, and by lemma 1.1.18 this means $\Lambda p = \Lambda p'$. Since $\Lambda$ is a quasi-bounded string algebra and $\Lambda \mathrm{f}(p) \cap \Lambda \mathrm{f}(p') \neq 0$, we must have $\mathrm{f}(p) = \mathrm{f}(p')$.

By SPIV) and corollary 1.1.25 (iii), part (c) of corollary 1.1.17 holds. Parts (a) and (b) of corollary 1.1.17 hold by assumption. Hence by corollary 1.1.17 (iii), we must have $p = p'$ which contradicts that $p$ and $p'$ are distinct (recall parallel paths are distinct by definition). Hence $(\varrho) = (\widehat{\varrho})$ where $\widehat{\varrho}$ consists only of paths.

An issue still is that $\widehat{\varrho}$ may contain arrows. Define a new quiver $\widetilde{Q}$ as follows. Define a vertex $\widetilde{v}$ in $\widetilde{Q}$ for each vertex $v$ in $Q$. Define an arrow $\widetilde{a}$ with head $\widetilde{v}$ and tail $\widetilde{u}$ for each arrow $a \notin (\varrho)$ with head $v$ and tail $u$. Let $\widetilde{\varrho}$ be the set of non-trivial paths $p$ in $Q$ such that $a$ is not a subpath of $p$. Define $\omega : k\widetilde{Q} \to kQ$ by extending the assingments $e_{\widetilde{v}} \mapsto e_v$ and $\widetilde{a} \mapsto a$ linearly over $k$. The composition with the canonical surjection $kQ \to kQ/(\varrho)$ defines a $k$-algebra surjection $\pi : k\widetilde{Q} \to kQ/(\varrho)$ such that $(\widetilde{\varrho}) \subseteq \ker(\pi)$.

Futhermore any $\sum_p r_p p \in \ker(\pi)$ (where $p$ runs through the paths in $\widetilde{Q}$) satisfies $\sum_p s_p p = 0$ where $s_p = r_p$ for any path $p \notin (\varrho)$, and $s_p = 0$ otherwise.

Since paths define a linearly independent subset of $kQ$ this means $s_p = 0$ for any $p$, and so $\sum_p r_p p = \sum_q r_q q$ where $q$ runs through the paths in $\widetilde{Q}$ with $q \in (\varrho)$. This gives $kQ/(\rho) \simeq kQ'/(\rho')$ as $k$-algebras, so we can assume $\rho$ consists of paths of length at least 2. Thus $\Lambda$ is a Butler-Ringel string algebra, as required. □

**Example 1.2.4.** The algebra $k[x,y]/(xy)$ from example 1.1.16 is an example of a *string algebra* in the sense of Crawley-Boevey [21]. However $k[x,y]/(xy)$ is not an example of a quasi-bounded string algebra over $k$, because it is not a Butler-Ringel string algebra, since $x^m \notin (xy)$ for all $m > 0$ (and so condition (3) fails to hold).

By definition every quasi-bounded string algebra over $R$ is a quasi-bounded special biserial algebra over $R$. Next we prove the following analogue of [25, II.1.3].

**Lemma 1.2.5.** *Let $e_v\Lambda$ and $\Lambda e_v$ be artinian $R$-modules for each $v$. Let $X$ be the set of vertices $x$ such that $Rp_x = Rp'_x$ (in $\Lambda$) for some distinct parallel $p_x, p'_x \in \mathbf{P}(x \to)$. Then:*

(i) *if $x \in X$ then $\Lambda p_x = \mathrm{soc}(\Lambda e_x)$ and $p_x\Lambda = \mathrm{soc}(e_{h(p_x)}\Lambda)$, which are simple;*

(ii) *$I = \bigoplus_{x\in X} \mathrm{soc}(\Lambda e_x)$ defines a two sided ideal of $\Lambda$;*

(iii) *$\Lambda/I$ is a quasi-bounded string algebra over $R$; and*

(iv) *if $M$ is an indecomposable $\Lambda$-module, then $IM = 0$ or $M \simeq \Lambda e_x$ for some $x \in X$.*

To achieve this goal we need some preliminary results.

**Lemma 1.2.6.** *Let $v$ and $w$ be vertices such that $\Lambda e_v$ and $e_w\Lambda$ are artinian as $R$-modules. Consider the map $\mathrm{mult} : e_w\Lambda \times \Lambda e_v \to e_w\Lambda e_v$ defined by $\mathrm{mult}(\lambda, \mu) = \lambda\mu$. Then:*

(i) *as $R$ modules $\Lambda e_v$ and $e_w\Lambda$ are of finite length;*

(ii) *the rings $e_v\Lambda e_v$ and $e_w\Lambda e_w$ are local;*

(iii) $\mathrm{rad}(e_v\Lambda e_v) = e_v\mathrm{rad}(\Lambda)e_v$ *and* $\mathrm{rad}(e_w\Lambda e_w) = e_w\mathrm{rad}(\Lambda)e_w$*;*

(iv) $\mathrm{soc}(\Lambda e_v) \leqslant_{\mathrm{e}} \Lambda e_v$ *and* $\mathrm{soc}(e_w\Lambda) \leqslant_{\mathrm{e}} e_w\Lambda$*;*

(v) *the map* mult *is $e_w\Lambda e_w$-$e_v\Lambda e_v$ bilinear* (see [2, p.280])*;*

(vi) $\mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(T, e_w\Lambda e_v) \simeq \mathrm{soc}(e_w\Lambda)e_v$ *in* $\mathbf{Mod}\text{-}e_v\Lambda e_v$ *if* ${}_{e_w\Lambda e_w}T$ *is simple;*

(vii) $\mathrm{Hom}_{\mathbf{Mod}\text{-}e_v\Lambda e_v}(T', e_w\Lambda e_v) \simeq e_w\mathrm{soc}(\Lambda e_v)$ *in* $e_w\Lambda e_w\text{-}\mathbf{Mod}$ *if* $T'_{e_v\Lambda e_v}$ *is simple; and*

(viii) *if (*$\Lambda p = \mathrm{soc}(\Lambda e_v)$*,* $p\Lambda = \mathrm{soc}(e_w\Lambda)$*, and these modules are simple) for some* $p \in \mathbf{P}$*, then* mult *is non-degenerate.*

*Proof.* For parts (i), (ii) and (iii) we only show the left $R$ module $\Lambda e_v$ has finite length, and that this means the ring $e_v\Lambda e_v$ is local with $\mathrm{rad}(e_v\Lambda e_v) = e_v\mathrm{rad}(\Lambda)e_v$. The other proofs are similar.

(i) By assumption the left $R$-module $\sum_{a\in\mathbf{A}(v\to)}\Lambda a$ is noetherian since it is finitely generated over $R$ by SPIV), and $R$ is a noetherian ring. By corollary 1.1.25 (i) the left $R$-module $\Lambda e_u/\sum_{a\in\mathbf{A}(v\to)}\Lambda a$ is isomorphic to $k$, and hence noetherian. Together this shows $\Lambda e_v$ is noetherian as a left $R$-module. By assumption $\Lambda e_v$ is artinian as a left $R$-module, and so it has finite length as a left $R$-module.

(ii) By part (i) $\Lambda e_v$ is artinian and noetherian as a left $\Lambda$ module. By [65, 32.4, (1)] this means $\Lambda e_v$ is a finite length left $\Lambda$-module.

Since $\Lambda$ is pointwise local, by lemma 3.1.37 the $\Lambda$-module $\Lambda e_w$ is indecomposable. It is straightforward to check the map $\mathrm{End}_\Lambda(\Lambda e_v) \to e_v\Lambda e_v$ given by $f \mapsto f(e_v)$ is a ring isomorphism. By [65, 32.4, (3iii)] this means the (unital) ring $e_v\Lambda e_v$ is local, since the left $\Lambda$ module $\Lambda e_v$ has finite length.

(iii) Suppose $I$ is any proper left ideal of $e_v\Lambda e_v$. Note that $I = e_v\Lambda I$ and the $\Lambda$-module $\Lambda I$ is contained in $\Lambda e_v$. If $\Lambda I = \Lambda e_v$ then we can write $e_v$ as a finite sum of elements of the form $e_u\lambda e_v x$ for $\lambda \in \Lambda$ and $x \in I$ and where $u$ runs through (finitely many of) the vertices. Multiplication on the left by $e_v$ shows that $e_v \in I$ which is a contradiction. Hence $\Lambda I$ is strictly contained in $\mathrm{rad}(\Lambda e_v)$.

Since $\Lambda$ is pointwise local $\mathrm{rad}(\Lambda e_v) = \mathrm{rad}(\Lambda)e_v$, and so $I = e_v\Lambda I \subseteq e_v\mathrm{rad}(\Lambda)e_v$. This shows $e_v\mathrm{rad}(\Lambda)e_v$ is a maximal ideal of $e_v\Lambda e_v$. Together with part (ii), this shows $\mathrm{rad}(e_v\Lambda e_v) = e_v\mathrm{rad}(\Lambda)e_v$.

(iv) Since $\Lambda e_v$ is artinian as a left $R$-module it is artinian as a left $\Lambda$-module. Consequently $\mathrm{soc}(\Lambda e_v)$ can be written as the intersection of finitely many essential submodules. By lemma 3.1.34 (ib) this shows $\mathrm{soc}(\Lambda e_v)$ is an essential $\Lambda$-submodule of $\Lambda e_v$. Similarly $e_w\Lambda$ is artinian as a right $\Lambda$-module and again we can show $\mathrm{soc}(e_w\Lambda)$ is an essential $\Lambda$-submodule of $e_w\Lambda$.

(v) This follows from the properties of ring multiplication.

(vi) If $S$ is any simple left $e_w\Lambda e_w$-module and $s \in S$ is any non-zero element then the $e_w\Lambda e_w$-module homomorphism $e_w\Lambda e_w \to S$ given by $\lambda \mapsto \lambda s$ is onto. By parts (ii) and (iii) the kernel of this map is $e_w\mathrm{rad}(\Lambda)e_w$ because $S$ is simple. Hence $S$ is isomorphic to the quotient $e_w\Lambda e_w/e_w\mathrm{rad}(\Lambda)e_w$. Consider the following assignment $\theta$ of hom-sets

$$\mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda e_w, \mathrm{soc}(e_w\Lambda)e_v) \to \mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda e_w/e_w\mathrm{rad}(\Lambda)e_w, e_w\Lambda e_v)$$

sending $f : e_w\Lambda e_w \to \mathrm{soc}(e_w\Lambda)e_v$ to a map $g^f : e_w\Lambda e_w/e_w\mathrm{rad}(\Lambda)e_w \to e_w\Lambda e_v$ given by $g^f(e_w\lambda e_w + e_w\mathrm{rad}(\Lambda)e_w) = f(e_w\lambda e_w)$. If $e_w\lambda e_w \in e_w\mathrm{rad}(\Lambda)e_w$ then by lemma 3.1.34 (ic) we have $f(e_w\lambda e_w) \in \mathrm{rad}(\mathrm{soc}(e_w\Lambda)e_v)) = 0$, so $\theta$ is well defined. Since the right-action of $e_v\Lambda e_v$ on $\mathrm{soc}(e_w\Lambda)e_v$ is the restriction of $e_v\Lambda e_v$ acting on $e_w\Lambda e_v$, we must have that $\theta$ is a homomorphism of right $e_v\Lambda e_v$-modules.

Clearly $g^f = 0$ implies ($f(e_w) = 0$ and so) $f = 0$, and so $\theta$ is a monomorphism. Finally, any non-zero homomorphism $g : e_w\Lambda e_w/e_w\mathrm{rad}(\Lambda)e_w \to e_w\Lambda e_v$ of $e_w\Lambda e_w$-modules must be injective since $e_w\Lambda e_w/e_w\mathrm{rad}(\Lambda)e_w$ is simple, and so $\mathrm{im}(g) \subseteq \mathrm{soc}(e_w\Lambda)e_v$. Hence we have $g = g^f$ where $f$ is given by $f(e_w\lambda e_w) = g^f(e_w\lambda e_w + e_w\mathrm{rad}(\Lambda)e_w)$. Thus $\theta$ is an isomorphism.

The proof of (vii) is similar and omitted.

(viii) See [2, p.327]. Suppose the submodule $X = \{\lambda \in \Lambda e_v \mid \lambda\mu = 0\ \ \forall \mu \in e_w\Lambda\}$ of $\Lambda e_v$ is non-zero. Since $\Lambda p = \mathrm{soc}(\Lambda e_v)$ which is essential by (iv), we must have $X \cap \Lambda p \neq 0$, and since $\Lambda p$ is simple this means $\Lambda p \subseteq X$ and so we have $0 \neq p = e_w p e_v = 0$. Altogether this shows $X = 0$, and similarly we can show the submodule $Y = \{\mu \in e_w\Lambda \mid \lambda\mu = 0\ \ \forall \lambda \in \Lambda e_v\}$ of $e_w\Lambda$ is zero since $p\Lambda = \mathrm{soc}(e_w\Lambda)$ is simple. $\square$

**Lemma 1.2.7.** *Let $v$ and $w$ be vertices. If*

(a) $\Lambda e_v$ *and* $e_w\Lambda$ *are artinian* $R$*-modules, and*

(b) $\Lambda p = \mathrm{soc}(\Lambda e_v)$ *and* $p\Lambda = \mathrm{soc}(e_w\Lambda)$ *are simple for some* $p \in \mathbf{P}$,

*then* $\Lambda e_v$ *is injective.*

For the proof of this lemma we use the following.

**Theorem 1.2.8.** [2, 30.1. Theorem. (2), (4)] *Let* $\Gamma$ *and* $\Gamma'$ *be unital rings. Let* ${}_\Gamma M$ *be a left* $\Gamma$ *module,* $N_{\Gamma'}$ *be a right* $\Gamma'$ *module,* ${}_\Gamma U_{\Gamma'}$ *be a* $\Gamma$*-*$\Gamma'$*-bimodule, and* $\beta : {}_\Gamma M \times N_{\Gamma'} \to {}_\Gamma U_{\Gamma'}$ *be a* $\Gamma$*-*$\Gamma'$ *bilinear map. Suppose:*

(a) *either* ${}_\Gamma M$ *or* $N_{\Gamma'}$ *has finite length;*

(b) *the map* $\beta$ *is non-degenerate;*

(c) $\mathrm{Hom}_{\Gamma\text{-}\mathbf{Mod}}(T, U)$ *is simple in* $\mathbf{Mod}\text{-}\Gamma'$ *whenever* ${}_\Gamma T$ *is simple; and*

(d) $\mathrm{Hom}_{\mathbf{Mod}\text{-}\Gamma'}(T', U)$ *is simple in* $\Gamma\text{-}\mathbf{Mod}$ *whenever* $T'_{\Gamma'}$ *is simple.*

*Then:*

(i) *the map* $\alpha : M \to \mathrm{Hom}_{\mathbf{Mod}\text{-}\Gamma'}(N, U)$ *given by* $\alpha(m) : n \mapsto \beta(m, n)$ *is a bijection;*

(ii) *the map* $\gamma : N \to \mathrm{Hom}_{\Gamma\text{-}\mathbf{Mod}}(M, U)$ *given by* $\gamma(n) : m \mapsto \beta(m, n)$ *is a bijection; and*

(iii) ${}_\Gamma U$ *is* ${}_\Gamma M$ *injective, and* $U_{\Gamma'}$ *is* $N_{\Gamma'}$ *injective.*

Conclusions (1) and (3) from [2, 30.1. Theorem.] are missing from the above, because these will not be needed.

*Proof of lemma 1.2.7.* We follow the proof of the implication $\Leftarrow$ in [2, 31.3]. Note that $e_w\Lambda e_w$ and $e_v\Lambda e_v$ are unital rings with multiplicative identities $e_w$ and $e_v$ respectively.

By lemma 1.2.6 (v) and (viii) the map $\mathrm{mult} : e_w\Lambda \times \Lambda e_v \to e_w\Lambda e_v$ defined by $\mathrm{mult}(\lambda, \mu) = \lambda\mu$ is a non-degenerate bilinear form. By lemma 1.2.6 (vi) $\mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(T, e_w\Lambda e_v) \simeq \mathrm{soc}(e_w\Lambda)e_v$. By assumption $\mathrm{soc}(e_w\Lambda)e_v$ is a simple right $e_v\Lambda e_v$-module, and so together we have that $\mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(T, e_w\Lambda e_v)$ is simple in $\mathbf{Mod}\text{-}e_v\Lambda e_v$ if $T$ is simple in $e_w\Lambda e_w\text{-}\mathbf{Mod}$.

Similarly $\mathrm{Hom}_{\mathbf{Mod}\text{-}e_v\Lambda e_v}(T', e_w\Lambda e_v)$ is simple in $e_w\Lambda e_w$-**Mod** if $T'$ is simple in **Mod**-$e_v\Lambda e_v$ by lemma 1.2.6 (vii). Note that $e_w\Lambda e_v$ is a submodule of a finite length $R$-module by lemma 1.2.6 (i). Hence $e_w\Lambda e_v$ has finite length as a left $e_w\Lambda e_w$-module.

We have now verified parts (a), (b), (c) and (d) from theorem 1.2.8 hold, where $\Gamma = e_w\Lambda e_w$, $\Gamma' = e_v\Lambda e_v$, $M = e_w\Lambda$, $N = \Lambda e_v$, $U = e_w\Lambda e_v$, and $\beta = \mathrm{mult}$.

By theorem 1.2.8 (i) the map $\alpha : \Lambda e_v \to \mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda, e_w\Lambda e_v)$ given by $\alpha(\mu) : \lambda \mapsto \mathrm{mult}(\lambda, \mu) = \lambda\mu$ (for each $\mu \in \Lambda e_v$) is a bijection. Note $\mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda, e_w\Lambda e_v)$ is a left $\Lambda$-module, where (for each $f \in \mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda, e_w\Lambda e_v)$, $\kappa \in \Lambda$ and $\lambda \in e_w\Lambda$) one has $(\kappa f)(\lambda) = f(\lambda\kappa)$. In particular,

$$\begin{aligned}(\kappa(\alpha(\mu)))(\lambda) &= (\alpha(\mu))(\lambda\kappa) = \mathrm{mult}(\lambda\kappa, \mu) = (\lambda\kappa)\mu \\ &= \lambda(\kappa\mu) = \mathrm{mult}(\lambda, \kappa\mu) = (\alpha(\kappa\mu))(\lambda).\end{aligned}$$

This means $\alpha$ is ($\Lambda$-linear, and hence) an isomorphism $\Lambda e_v \simeq \mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda, e_w\Lambda e_v)$ of left $\Lambda$-modules. Since $e_w\Lambda$ contains $e_w\Lambda e_w$ as a direct summand $e_w\Lambda e_w$ is subgenerated by $e_w\Lambda$ (see [65, §15]). So, by [65, 15.3 (a) iff (b)] and [65, 45.8 (1)], the $e_w\Lambda e_w$-$\Lambda$ bimodule $e_w\Lambda$ gives a pair of adjoint functors[3]

$$e_w\Lambda \otimes_\Lambda - : \Lambda\text{-}\mathbf{Mod} \to e_w\Lambda e_w\text{-}\mathbf{Mod} \text{ and } \mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda, -) : e_w\Lambda e_w\text{-}\mathbf{Mod} \to \Lambda\text{-}\mathbf{Mod}$$

Note this is well-known as the tensor-hom adjunction. The existence of this adjunction implies there is a natural isomorphism

$$\mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}((e_w\Lambda \otimes_\Lambda -), e_w\Lambda e_v) \simeq \mathrm{Hom}_{\Lambda\text{-}\mathbf{Mod}}(-, \mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda, e_w\Lambda e_v))$$

of functors $(\Lambda\text{-}\mathbf{Mod})^{\mathrm{op}} \to \mathbf{Ab}$ (where **Ab** is the category of abelian groups). Note that $e_w\Lambda$ is a projective object in **Mod**-$\Lambda$ by the appropriate analogue of lemma 3.1.37, and so $e_w\Lambda \otimes_\Lambda - : \Lambda\text{-}\mathbf{Mod} \to e_w\Lambda e_w\text{-}\mathbf{Mod}$ is exact (see [65, §18]).

[3] We use $-$ to denote the argument for the various functors we consider.

$e_w\Lambda$ contains $e_w\Lambda e_w$ as a direct summand, and so $e_w\Lambda$ is a generator for $e_w\Lambda e_w$-**Mod** (see [2, §8, Exercise 4]). By theorem 1.2.8 (iii), $e_w\Lambda e_v$ is $e_w\Lambda$ injective as a left $e_w\Lambda e_w$-module. Hence the left $e_w\Lambda e_w$-module $e_w\Lambda e_v$ is injective (see [2, 16.14. Corollary. (1)]).

From the above we have that $\mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}((e_w\Lambda \otimes_\Lambda -), e_w\Lambda e_v)$ is exact. Since $\Lambda e_v \simeq \mathrm{Hom}_{e_w\Lambda e_w\text{-}\mathbf{Mod}}(e_w\Lambda, e_w\Lambda e_v)$ as left $\Lambda$-modules, $\mathrm{Hom}_{\Lambda\text{-}\mathbf{Mod}}(-, \Lambda e_v)$ is exact, and so $\Lambda e_v$ is injective. □

*Proof of lemma 1.2.5.* By SPIV) and corollary 1.1.25 (iii) part (c) of corollary 1.1.17 holds. Parts (a) and (b) of corollary 1.1.17 hold by assumption, and so (for the remainder of the proof) we can apply corollary 1.1.17. Let $v$ be a vertex. We start by showing $\mathbf{P}(t, v \rightarrow) = \emptyset$ and $\mathbf{P}(t, \rightarrow v) = \emptyset$ for all $t \gg 0$. For a contradiction we assume otherwise, say that ($\mathbf{P}(t, v \rightarrow) = \emptyset$ for all $t \gg 0$) is false.

At most two arrows have tail $v$, and so this assumption means there is an arrow $a$ with tail $v$ and a strictly increasing sequence of integers $t_1 < t_2 < t_3 < \dots$ such that for each integer $i > 0$ there is a path $p_i \in \mathbf{P}(t_i, v \rightarrow)$ with $\mathrm{f}(p_i) = a$. By lemma 1.1.14 (ia) $p_i$ is an initial subpath of $p_{i+1}$ for each $i$, and so there is a strict chain of $R$-modules

$$\Lambda e_v \supset \Lambda p_1 \supset \Lambda p_2 \supset \Lambda p_3 \supset \dots$$

which contradicts the assumption that $\Lambda e_v$ is an artinian $R$-module. This means part (c) of lemma 1.1.18 holds. Parts (a) and (b) of lemma 1.1.18 hold by assumption, and so (for the remainder of the proof) we can apply lemma 1.1.18.

(i) If $x \in X$ then $Rp_x = Rp'_x$ for some distinct parallel $p_x, p'_x \in \mathbf{P}(x \rightarrow)$. By lemma 1.1.18 (iv) this means $\Lambda p_x = \mathrm{soc}(\Lambda e_x) = \Lambda p'_x$ and $p_x\Lambda = \mathrm{soc}(e_{h(p_x)}\Lambda) = p'_x\Lambda$. By lemma 1.1.18 (iii) these modules are simple.

(ii) Let $x \in X$. By lemma 1.2.6 (i) $\Lambda e_x$ is the injective hull of $\mathrm{soc}(\Lambda e_x)$. By lemma 3.1.34 (iiic) we can write $\mathrm{soc}(\Lambda e_x)$ as the direct sum $S_1 \oplus \dots \oplus S_n$ where each $S_i$ is simple. By lemma 3.1.34 (iia) and (iiib) $\Lambda e_x \simeq E_1 \oplus \dots \oplus E_n$ where each $E_i$ is the injective hull of $S_i$.

Now $\Lambda e_x$ is indecomposable by lemma 3.1.37, and so $n = 1$ and thus $\mathrm{soc}(\Lambda e_x)$ is simple, and hence there is a unique simple $\Lambda$-submodule of $\Lambda e_x$.

Since $I$ is a left ideal it suffices to show $z\mu \in I$ for any $z \in I$ and $\mu \in \Lambda$. Defining the $\Lambda$-module endomorphism $g : \Lambda \to \Lambda$ by $\lambda \mapsto \lambda\mu$ it suffices to show the image of $g$ restricted to $I$ lies in $I$.

For a contradiction suppose there is some vertex $t \in X$ such that $\mathrm{im}(gs_t) \nsubseteq I$ where $s_t : \mathrm{soc}(\Lambda e_t) \to I$ is the canonical inclusion. If $\mathrm{im}(gs_t) \subseteq \bigoplus_{y\in X} \Lambda e_y$ then as $\mathrm{soc}(\Lambda e_y)$ is simple for each $y$, so too is $\mathrm{im}(gs_t)$, and so by lemma 3.1.34 (iib) $\mathrm{im}(gs_t)$ is contained in $\mathrm{soc}(\bigoplus_{y\in X} \Lambda e_y) = I$.

This is a contradiction, and so there is some vertex $v \notin X$ for which $\pi_v g s_t \neq 0$ where $\pi_v : \Lambda \to \Lambda e_v$ is the canonical projection. This means $\pi_v g \iota_t \neq 0$ where $\iota_t : \Lambda e_t \to \Lambda$ is the canonical inclusion. If $\ker(\pi_v g \iota_t) \neq 0$ then as $\mathrm{soc}(\Lambda e_t)$ is essential we have $\ker(\pi_v g \iota_t) \cap \mathrm{soc}(\Lambda e_t) \neq 0$, and since $\mathrm{soc}(\Lambda e_t)$ is simple this means $\mathrm{soc}(\Lambda e_t) \subseteq \ker(\pi_v g \iota_t)$ which contradicts that $\pi_v g s_t \neq 0$.

Hence $\ker(\pi_v g \iota_t) = 0$ and so as $\Lambda e_t$ is injective and $\pi_v g \iota_t$ is an inclusion we have that $\Lambda e_t$ is a summand of $\Lambda e_v$. By lemma 3.1.37 this means $\Lambda e_t \simeq \Lambda e_v$ and so $\Lambda e_t = \Lambda e_t e_v = 0$ since $v \notin X \ni t$. Hence assuming (there is some vertex $t \in X$ such that $\mathrm{im}(gs_t) \nsubseteq I$) gives the contradiction $\Lambda e_t = 0$.

(iii) We claim firstly that $\Lambda/I$ is surjectively given by $(Q, \rho_X, \pi\theta)$ where

$$\rho_X = \rho \cup \{p_x \mid x \in X\} \text{ and } \pi : \Lambda \to \Lambda/I \text{ is the canonical projection.}$$

Clearly any element of $\rho_X$ is sent to 0 under $\pi\theta$ and so $(\rho_X) \subseteq \ker(\pi\theta)$. Now suppose $p$ is a path with $p \notin (\rho_X)$. For a contradiction suppose $\pi(\theta(p)) = 0$ and so $\theta(p) \in I$.

Note there is a finite subset $V$ of $X$ where $\theta(p) \in \bigoplus_{u\in V} \mathrm{soc}(\Lambda e_u)$, and so $\theta(p) = \sum_{u\in V} \lambda_u$ where $\lambda_u \in \mathrm{soc}(\Lambda e_u)$ for each $u \in V$. Since $\Lambda$ is surjectively given by $(Q, \rho, \theta)$ and $p \notin (\rho)$ we must have $\theta(p) \neq 0$, and so $\theta(p)\theta(e_w) \neq 0$ for some $w \in V$ (otherwise $\lambda_u = 0$ for each $u \in V$ which would mean $\theta(p) = 0$).

Since $\theta$ is an algebra homomorphism this shows $\theta(pe_w) \neq 0$, and so $w = t(p)$, and so $\theta(p) = \theta(p)\theta(e_w)$. Altogether this gives $\theta(p) = \lambda_w \in \mathrm{soc}(\Lambda e_w) = \Lambda p_w$. We now have that $\Lambda p \subseteq \Lambda p_w = \Lambda p'_w$. Without loss of generality we have $\mathrm{f}(p_w) = \mathrm{f}(p)$. By corollary 1.1.17 (i) $\Lambda p \subseteq \Lambda p_w$ implies $\Lambda p = \Lambda q p_w$ for some path $q$, and so $p = qp_w$, by corollary 1.1.17 (iii), which contradicts that $p \notin (\rho_X)$.

This contradiction tells us that $\Lambda/I$ is surjectively given by $(Q, \rho_X, \pi\theta)$. Since $\rho_X \supseteq \rho$ and $(Q, \rho)$ satisfies special conditions the pair $(Q, \rho_X)$ satisfies special conditions. Hence SPI) holds and SPII) holds. Furthermore any path $p \in \rho_X$ must be non-trivial, since otherwise $e_v \in \rho$ for some vertex $v$ which contradicts that $\Lambda$ is surjectively given by $(Q, \rho, \theta)$.

Since $I \subseteq \sum_{x\in X, a\in \mathbf{A}(x\to)} \Lambda a \subseteq \mathrm{rad}(\Lambda)$ and $\Lambda$ is pointwise rad-nilpotent modulo $\mathfrak{m}$, for each vertex $v$ there is an integer $n(v) \geq 1$ such that

$$(\mathrm{rad}(\Lambda/I))^{n(v)}e_v = (((\mathrm{rad}(\Lambda))^{n(v)} + I)/I)e_v = (\mathrm{rad}(\Lambda))^{n(v)}e_v + I/I$$

which lies in $(\Lambda\mathfrak{m}e_v + I)/I = (\Lambda/I)\mathfrak{m}e_v$, and

$$(\Lambda/I)\mathfrak{m}e_v = (\Lambda\mathfrak{m}e_v + I)/I \subseteq (\mathrm{rad}(\Lambda)e_v + I)/I = (\mathrm{rad}(\Lambda)/I)e_v = \mathrm{rad}(\Lambda/I)e_v$$

and so $\Lambda/I$ is pointwise rad-nilpotent modulo $\mathfrak{m}$. For any vertex $v$ and any ideal $J$ in $\Lambda$ we have $(J/I)e_v \simeq Je_v/Ie_v$ as $\Lambda$-modules. As above we have

$$\mathrm{rad}(\Lambda/I)e_v = (\mathrm{rad}(\Lambda)/I)e_v = \mathrm{rad}(\Lambda)e_v/Ie_v = (\textstyle\sum_{a\in\mathbf{A}(v\to)} \Lambda a)/Ie_v = \sum_{a\in\mathbf{A}(v\to)}(\Lambda/I)a,$$
$$(\Lambda/I)e_v/\mathrm{rad}(\Lambda/I)e_v \simeq (\Lambda e_v/Ie_v)/(\mathrm{rad}(\Lambda)e_v/Ie_v) \simeq \Lambda e_v/\mathrm{rad}(\Lambda)e_v$$

and so $\Lambda/I$ is left pointwise local. By symmetry $\Lambda$ is also right pointwise local and hence SPIII) holds. If $a$ is an arrow then $(\Lambda/I)a \simeq \Lambda a/Ia$ as $\Lambda$-modules and hence as $R$-modules, which means $(\Lambda/I)a$ is finitely generated as an $R$-module because ($\Lambda a$ is finitely generated as an $R$-module and the quotient map $\Lambda a \to \Lambda a/Ia$ is an $R$-module epimorphism). So, SPIV) holds.

To prove $\Lambda/I$ is a quasi-bounded string algebra over $R$ surjectively given by $(Q, \rho_X, \pi\theta)$, by symmetry it suffices to prove $(\Lambda/I)a \cap (\Lambda/I)a' = 0$ whenever $a$ and $a'$ are distinct arrows.

It suffices to assume $a$ and $a'$ have the same tail, say $v$. Choosing $\lambda + I \in (\Lambda/I)a \cap (\Lambda/I)a'$ with $\lambda \in \Lambda e_v$ it suffices to prove $\lambda \in Ie_v$. By corollary 1.1.17 we have $\Lambda a \cap \Lambda a' = \Lambda p$ and $\Lambda a \cap \Lambda a' = \Lambda p'$ for some $p, p' \in \mathbf{P}$ where $\mathrm{f}(p) = a$ and $\mathrm{f}(p') = a'$. ¿Since $\Lambda$ is a quasi-bounded special biserial algebra, $\Lambda p = \Lambda p'$ is simple, and so by lemma 1.1.18 (iii) the paths $p, p' \in \mathbf{P}$ are parallel and $Rp = Rp'$. Since $t(p) = t(p') = v$ this means $v \in X$, and so after reordering we have $p = p_v$ and $p' = p'_v$. Hence if $\lambda \in \Lambda a \cap \Lambda a'$ then $\lambda \in \Lambda p_v \subseteq Ie_v$ and so we can assume $\lambda \notin \Lambda a \cap \Lambda a'$.

By definition there are some $\mu \in \Lambda e_{h(a)}$ and $\mu' \in \Lambda e_{h(a')}$ such that $\lambda - \mu a \in I$ and $\lambda - \mu' a' \in I$, and so $\mu a - \mu' a' \in I$. Since $v \in X$ this means means $\lambda - \mu a, \lambda - \mu' a' \in \Lambda p_v$ and so $\mu a - \mu' a' \in \Lambda p_v$. As above we can apply corollary 1.1.17 (i) to show $\Lambda \mu a = \Lambda q a$ for some path $q$, and since $\lambda \notin \Lambda a \cap \Lambda a'$ we can assume $\mu a \neq 0$, and so by corollary 1.1.17 (iii) we can assume $q$ is of maximal length such that $\Lambda \mu a = \Lambda q a$. Let $\mu a = \eta q a$ for some $\eta \in \Lambda e_{h(q)}$. Writing $\eta = r e_{h(a)} + z$ where $r \in R$ and $z \in \mathrm{rad}(\Lambda) e_{h(a)}$ gives $\mu a = rqa + zqa$, and so by the maximality of $q$ we have $r \notin \mathfrak{m}$ since $\Lambda \mathfrak{m} e_{h(a)} \subseteq \mathrm{rad}(\Lambda) e_{h(a)}$. Similarly there is some path $q'$, some $r' \in R \setminus \mathfrak{m}$ and some $z' \in \mathrm{rad}(\Lambda) e_{h(a')}$ such that $\mu' a' = r' q' a' + z' q' a'$.

Without loss of generality $\mathrm{f}(p) = a$ which means $\Lambda p_v \subseteq \Lambda q a$ and so $p_a q a = p_v$ for some path $p_a$ (by another application of corollary 1.1.17). By symmetry we can also assume $p_{a'} q' a' = p'_v$ for some paths $q', p'_a$. For a contradiction we now assume $p_a$ and $p_{a'}$ are both nontrivial. Without loss of generality we can assume $p_a$ is not shorter than $p_{a'}$.

Suppose $\mathrm{l}(p_a) = \mathrm{l}(p_{a'})$, in which case $\mathrm{l}(p_v) = \mathrm{l}(p_a q a) = \mathrm{l}(p_{a'} q' a') = \mathrm{l}(p_{v'})$. Since $\Lambda p_v = \Lambda p_{v'}$, by corollary 1.1.17) (iii) we have $p_v = p_{v'}$ which contradicts that these paths are parallel. So we must have $\mathrm{l}(p_a) \neq \mathrm{l}(p_{a'})$. We now claim this means $p_a q' a' = 0$.

Note that $qa$ and $q'a'$ are non-trivial paths. If $\mathrm{f}(p_a) \neq \mathrm{f}(p_{a'})$ then $p_a q' = 0$ by SPI) and the claim holds. Otherwise $\mathrm{f}(p_a) = \mathrm{f}(p_{a'})$, and so $p_{a'}$ is an initial subpath of $p_a$ by lemma 1.1.14 (ia), and so $p_a$ is longer than $p_{a'}$ since $\mathrm{l}(p_a) \neq \mathrm{l}(p_{a'})$.

So, $p_a q' a' \in \mathrm{rad}(\Lambda) p_{a'} q' a' = \mathrm{rad}(\Lambda p'_v)$. The claim now follows, since $\Lambda p_v$ is simple by part (i) and so $\mathrm{rad}(\Lambda p'_v) = 0$. Altogether this gives $p_a(\mu a - \mu' a') = r p_a q a = r p_v$ which is non-zero as $r$ is a unit. Since $\mu a - \mu' a' \in \Lambda p_v$ and $p_a$ is non-trivial we have the contradiction $\mathrm{rad}(\Lambda p_v) \neq 0$. Hence $p_a$ must be trivial, and so $qa = p_v$ which means $\mu a \in I$ as required.

(iv) It suffices to assume $p_x m \neq 0$ for some $m \in M$ and some $x \in X$, and then prove $M \simeq \Lambda e_x$. Consider the $\Lambda$-module homomorphism $\varphi : \Lambda e_x \to M$ sending $\lambda e_x$ to $\lambda m$. If $\ker(\varphi) \neq 0$ then $\ker(\varphi) \cap \mathrm{soc}(\Lambda e_x) \neq 0$ as $\mathrm{soc}(\Lambda e_x)$ is essential and therefore $\mathrm{soc}(\Lambda e_x) \subseteq \ker(\varphi)$ as $\mathrm{soc}(\Lambda e_x)$ is simple. However this is impossible as $p_x \notin \ker(\varphi)$ as $p_x m \neq 0$. Hence $\ker(\varphi) = 0$. Since $\Lambda e_x$ is injective by lemma 1.2.7 this means $\Lambda e_x$ is a summand of $M$, and so $M \simeq \Lambda e_x$ as $M$ is indecomposable and $\Lambda e_x \neq 0$. $\square$

**Example 1.2.9.** Recall the quasi-bounded special biserial algebra $\Lambda$ over $\mathbb{Z}/p^3\mathbb{Z}$ from example 1.1.23. Note $\Lambda\alpha \cap \Lambda\beta = \Lambda\alpha \neq 0$, and $\Lambda a \cap \Lambda a' = 0$ for all other pairs of distinct arrows $a, a'$. In the notation of the proof of lemma 1.2.5 we have $X = \{1\}$ and

$$I = \mathrm{soc}(\Lambda e_1) = \Lambda\alpha = \Lambda\gamma\beta = \begin{pmatrix} 0 & 0 & \mathbb{F}_p & 0 & 0 \end{pmatrix}^t$$

as a submodule of column 1, which gives (as rings)

$$\Lambda/I \simeq \begin{pmatrix} \mathbb{F}_p & 0 & 0 & 0 & 0 \\ \mathbb{F}_p & \mathbb{F}_p & 0 & 0 & 0 \\ 0 & \mathbb{F}_p & \mathbb{F}_p & 0 & 0 \\ 0 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{Z}/p^3\mathbb{Z} & 0 \\ 0 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \end{pmatrix}$$

which is the quasi-bounded string algebra over $\mathbb{Z}/p^3\mathbb{Z}$ surjectively given by $(Q, \rho_X, \pi\theta)$ where $\rho_X = \rho \cup \{\alpha\}$.

### 1.2.2 Complete Gentle Algebras.

Recall definitions 1.1.11 and 1.2.1.

**Assumption:** In section 1.2.2 we assume $\Lambda$ is a quasi-bounded string algebra over $R$ surjectively given by $(Q, \rho, \theta)$.

**Definition 1.2.10.** (Quasi-Bounded Gentle Algebras) By the assumption above the pair $(Q, \rho)$ satisfies special conditions SPI) and SPII).
We say the pair $(Q, \rho)$ satisfies *gentle conditions* if in addition:

GI) any path $p \notin \mathbf{P}$ has a subpath $q \notin \mathbf{P}$ of length 2; and

GII) given any arrow $y$ there is at most one arrow $x$ with $xy \notin \mathbf{P}$ and at most one arrow $z$ with $yz \notin \mathbf{P}$.

We call $\Lambda$ a *quasi-bounded gentle algebra over $R$* if $(Q, \rho)$ satisfies gentle conditions.

(Assem-Skowroński Gentle Algebras) [4, p.272, Proposition] By an *Assem-Skowroński gentle algebra* we mean a Butler-Ringel string algebra $kQ/(\rho)$ where $Q$ is finite and $(Q, \rho)$ satisfies gentle conditions[4].

By lemma 1.2.3 $\Lambda$ is a quasi-bounded string algebra over $k$ iff it is a Butler-Ringel string algebra, which gives the following.

**Corollary 1.2.11.** *Let $k$ be a field and suppose $Q$ is a finite quiver. Then $\Lambda$ is a quasi-bounded gentle algebra over $k$ iff $\Lambda$ is an Assem-Skowroński gentle algebra.*

**Example 1.2.12.** [3, p.4] Let $k$ be field and $\Lambda = kQ/(\rho)$ where $\rho = \{ba, cb, ac, sr, ts, rt\}$ and $Q$ is the quiver

$$\begin{tikzcd}
1 \arrow[dd, "b"'] & & 3 \arrow[dl, "t"'] \\
& 0 \arrow[ul, "a"'] \arrow[dr, "r"] & \\
2 \arrow[ur, "c"] & & 4 \arrow[uu, "s"']
\end{tikzcd}$$

[4]Note GI) and GII) here correspond to (R3) and (R4) in [4]

Hence $\Lambda$ is a quasi-bounded gentle algebra over $k$ given by $(Q, \rho, \theta)$ by corollary 1.2.11.

**Example 1.2.13.** The calculations from example 1.1.6 show the $\widehat{\mathbb{Z}}_p$-subalgebra $\Lambda = \{(\gamma_{ij}) \in \mathbb{M}_2(\widehat{\mathbb{Z}}_p) \mid \gamma_{11} - \gamma_{22}, \gamma_{12} \in p\widehat{\mathbb{Z}}_p\}$ is also a quasi-bounded gentle algebra.

**Corollary 1.2.14.** *If $v$ is a vertex and $t > 0$ is an integer, then:*

(i) $\mathrm{rad}(\bigoplus_{p\in\mathbf{P}(t-1,v\to)} \Lambda p) = \bigoplus_{p\in\mathbf{P}(t,v\to)} \Lambda p = (\mathrm{rad}(\Lambda))^t e_v$*; and*

(ii) $\mathrm{rad}(\bigoplus_{p\in\mathbf{P}(t-1,\to v)} p\Lambda) = \bigoplus_{p\in\mathbf{P}(t,\to v)} p\Lambda = e_v(\mathrm{rad}(\Lambda))^t$.

*Also, for any $q \in \mathbf{P}$:*

(iii) *if $\lambda \in \Lambda e_v$ and $\lambda q = 0$ where $v = h(q)$ then $\lambda e_v \in \bigoplus_{a\in\mathbf{A}(v\to)} \Lambda a$;*

(iv) *if $\lambda \in e_v\Lambda$ and $q\lambda = 0$ where $v = t(q)$ then $e_v\lambda \in \bigoplus_{a\in\mathbf{A}(\to v)} a\Lambda$; and*

(v) $\mathrm{rad}(\Lambda)q \cap \mathrm{l}(q)\Lambda \subseteq q\mathrm{rad}(\Lambda)$ *and* $q\mathrm{rad}(\Lambda) \cap \Lambda\mathrm{f}(q) \subseteq \mathrm{rad}(\Lambda)q$.

**Example 1.2.15.** RecaLl the $\widehat{\mathbb{Z}}_p$-subalgebra $\Lambda = \{(\gamma_{ij}) \in \mathbb{M}_2(\widehat{\mathbb{Z}}_p) \mid \gamma_{11} - \gamma_{22}, \gamma_{12} \in p\widehat{\mathbb{Z}}_p\}$ given by $(Q, \rho, \theta)$ from examples 1.1.6 and 1.1.22. The calculations labeled $(\Diamond)$ gave

$$\mathrm{rad}(\Lambda) = \Lambda\alpha \oplus \Lambda\beta = \begin{pmatrix} p\widehat{\mathbb{Z}}_p & 0 \\ \widehat{\mathbb{Z}}_p & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & p\widehat{\mathbb{Z}}_p \\ 0 & p\widehat{\mathbb{Z}}_p \end{pmatrix} = \begin{pmatrix} p\widehat{\mathbb{Z}}_p & p\widehat{\mathbb{Z}}_p \\ \widehat{\mathbb{Z}}_p & p\widehat{\mathbb{Z}}_p \end{pmatrix}$$

Which is the case $t = 1$ in the above. More generally corollary 1.2.14 and the equations in $(\Diamond)$ tell us that

$$\mathrm{rad}(\Lambda)^t = \left\{ \begin{array}{lr} \Lambda \begin{pmatrix} 0 & 0 \\ 0 & p^t \end{pmatrix} \oplus \Lambda \begin{pmatrix} p^t & 0 \\ 0 & 0 \end{pmatrix} & (\text{if } t \text{ is even}) \\ \\ \Lambda \begin{pmatrix} 0 & 0 \\ p^t & 0 \end{pmatrix} \oplus \Lambda \begin{pmatrix} 0 & p^{t+1} \\ 0 & 0 \end{pmatrix} & (\text{if } t \text{ is odd}) \end{array} \right\} = \begin{pmatrix} p^t\widehat{\mathbb{Z}}_p & p^t\widehat{\mathbb{Z}}_p \\ p^{t-1}\widehat{\mathbb{Z}}_p & p^t\widehat{\mathbb{Z}}_p \end{pmatrix}$$

*Proof of corollary 1.2.14.* We only prove (i), (iii) and the statement $\mathrm{rad}(\Lambda)q \cap \mathrm{l}(q)\Lambda \subseteq q\mathrm{rad}(\Lambda)$ for (v). The remaining claims will follow by symmetry, as they did in the proof of lemma 1.1.17 (ii).

(i) By lemma 3.1.34 (iib) and corollary 1.1.17 (ii) we have $\mathrm{rad}(\bigoplus_{p\in\mathbf{P}(t-1,v\to)}\Lambda p) = \bigoplus_{p\in\mathbf{P}(t,v\to)}\Lambda p$. We now show $\bigoplus_{p\in\mathbf{P}(s,v\to)}\Lambda p = (\mathrm{rad}(\Lambda))^{s}e_v$ by induction on $s\geq 0$. The case $s=0$ is immediate because $\Lambda$ is pointwise local. Now suppose we have $\bigoplus_{p\in\mathbf{P}(s,v\to)}\Lambda p = (\mathrm{rad}(\Lambda))^{s}e_v$ for each vertex $v$. This shows $(\mathrm{rad}(\Lambda))^{s+1}e_v$ is equal to $\mathrm{rad}(\Lambda)(\bigoplus_{p\in\mathbf{P}(s,v\to)}\Lambda p)$.

Let $M = \mathrm{rad}(\bigoplus_{p\in\mathbf{P}(s,v\to)}\Lambda p)$ which is $\bigoplus_{p\in\mathbf{P}(s+1,v\to)}\Lambda p$ by the above in case $t=s+1$. Hence it suffices to show $\mathrm{rad}(M)/\mathrm{rad}(\Lambda)M=0$. We now follow the proof of [48, (24.4)]. Clearly $\overline{M}=M/\mathrm{rad}(\Lambda)M$ is a $\overline{\Lambda}=\Lambda/\mathrm{rad}(\Lambda)$-module, and so (as in the proof of lemma 3.1.37) there is a set $T$ and an epimorphism of $\overline{\Lambda}$-modules $\varepsilon:\bigoplus_{(v,t)\in Q_0\times T}\overline{\Lambda}e_v\to\overline{M}$.

Again as $\Lambda$ is pointwise local each $\overline{\Lambda}e_v$ is simple and so by lemma 3.1.34 (iiic) there is a subset $S\subseteq Q_0\times T$ with $\ker(\varepsilon)=\bigoplus_{(v,t)\in S}\overline{\Lambda}e_v$ and so $\overline{M}\simeq\bigoplus_{(v,t)\notin S}\overline{\Lambda}e_v$ by lemma 3.1.34 (iiia). This shows $\overline{M}$ is simple and so $\mathrm{rad}(\overline{M})=0$. Hence $\mathrm{rad}(\overline{M})=\mathrm{rad}(M)/\mathrm{rad}(\Lambda)M=0$ as required.

(iii) Suppose $\lambda q=0$, and assume $\lambda e_v\notin\bigoplus_{a\in\mathbf{A}(v\to)}\Lambda a$ for a contradiction. So by corollary 1.1.25 (i) there must be an element $r\in R\setminus\mathfrak{m}$ with $re_v-\lambda e_v\in\bigoplus_{a\in\mathbf{A}(h(q)\to)}\Lambda a$. There must be an arrow $a$ such that $aq\in\mathbf{P}$ as otherwise $rq=rq-\lambda q\in(\bigoplus_{a\in\mathbf{A}(h(q)\to)}\Lambda a)q=0$ which contradicts that $\Lambda$ was surjectively given by $(Q,\rho,\theta)$, since $r$ is a unit. This means $rq-\lambda q=rq\in\Lambda aq$ which gives $q\in\Lambda aq$. This implies $\Lambda\gamma=\Lambda a\gamma$, which contradicts lemma 1.1.17 (ii).

(v) Suppose $\lambda q\in \mathrm{l}(q)\Lambda$ for some $\lambda\in\mathrm{rad}(\Lambda)$. Write $q=\mathrm{l}(q)q'$ for some path $q'$, and write $\lambda=\sum_p r_p p$ for some finite support subset $\{r_p\mid p\in\mathbf{P}\}\subseteq R$. Since $\lambda q\in e_{h(q)}\Lambda$ we can assume $r_p=0$ whenever $h(p)\neq h(q)$. Since $\Lambda$ is a quasi-bounded string algebra over $R$ the sum $\sum_{a\in\mathbf{A}(\to h(q))}a\Lambda$ is direct, so we can assume $r_p=0$ whenever $\mathrm{l}(p)\neq\mathrm{l}(q)$.

If $p$ is a path from $\mathbf{P}$ with $\mathrm{l}(p)=\mathrm{l}(q)$ and $pq\in\mathbf{P}$ then there is a path $p'\in\mathbf{P}$ with $pq=\mathrm{l}(q)p'$. As $p$ and $q$ are non-trivial paths, $p'$ is non-trivial. If $q'$ is non-trivial then $\mathrm{l}(p')=\mathrm{l}(q')$ by SPII) since $\mathrm{l}(q)q',\mathrm{l}(q)p'\in\mathbf{P}$ in which case $p'=q'q''$ for some path $q''$ by lemma 1.1.14 (ib). If $q'$ is trivial then $\mathrm{l}(q)=q$. In any case we have $pq=qp''$ for some path $p''$.

If $q'$ is non-trivial then $p'' = q''$ which is trivial: otherwise $p' = q'$ and so $pq = \mathrm{l}(q)q' = q$ (which contradicts that $p \in \mathbf{P}$). If $q'$ is trivial then $p'' = p'$ which is non-trivial. In any case, if $p$ is a path from $\mathbf{P}$ with $\mathrm{l}(p) = \mathrm{l}(q)$ and $pq \in \mathbf{P}$ then $pq = qp''$ for some non-trivial path $p''$. Since $\lambda q = \sum_p r_p pq$ where the sum runs through $p \in \mathbf{P}$ with $pq \in \mathbf{P}$ and $\mathrm{l}(p) = \mathrm{l}(q)$, we have altogether $\lambda q \in qp''\Lambda \subseteq q\mathrm{rad}(\Lambda)$. □

**Definition 1.2.16.** (NOTATION: $\mathbf{P}a^{-1}$, $a^{-1}\mathbf{P}$) For a quiver $Q$, a set of relations $\rho$ and an arrow $a$ we let $\mathbf{P}a^{-1}$ (resp. $a^{-1}\mathbf{P}$) denote the set of paths $p \in \mathbf{P}$ such that $pa \in \mathbf{P}$ (resp. $ap \in \mathbf{P}$).

**Example 1.2.17.** Consider the quasi-bounded gentle algebra over $\widehat{\mathbb{Z}}_p$ from example 1.2.13. Here $\mathbf{P}\alpha^{-1}$ (resp. $\alpha^{-1}\mathbf{P}$) is the set of alternating sequences in $\alpha$ and $\beta$ which end (resp. start) with $\beta$. For example, $\mathbf{P}\alpha^{-1} \cap \alpha^{-1}\mathbf{P} \ni \beta\alpha\beta \notin \mathbf{P}\beta^{-1} \cup \beta^{-1}\mathbf{P}$.

**Corollary 1.2.18.** *Let $\Lambda$ be a quasi-bounded gentle algebra over $R$, and fix $q \in \mathbf{P}$. Suppose $\lambda \in \Lambda$ is a finite sum $\sum_{p\in\mathbf{P}(\mathrm{l}(q))^{-1}} r_p p$ (resp. $\sum_{p\in(\mathrm{f}(q))^{-1}\mathbf{P}} r_p p$) with $r_p \in R$. If $\lambda \neq 0$ then $\lambda q \neq 0$ (resp. $q\lambda \neq 0$).*

*Proof.* By symmetry it is enough to find a contradiction assuming $\lambda\gamma = 0$ and $0 \neq \lambda = \sum_{p\in\mathbf{P}(\mathrm{l}(q))^{-1}} r_p p$. So there is an arrow $a \in \mathbf{P}\mathrm{l}(q)$ and we let $T$ be the set of $p \in \mathbf{P}(\mathrm{l}(q))^{-1}$ with $\lambda \in \Lambda p$. If $T$ is infinite then $\lambda \in \bigcap_{n\geq 1}(\mathrm{rad}(\Lambda))^n\Lambda a$ which means $\lambda = 0$ by corollary 1.1.25 (iii). Hence $T$ is finite, and by lemma 1.1.14 (ia) any path $q' \in \mathbf{P}\mathrm{l}(q)$ must satisfy $\mathrm{f}(q') = a$. Altogether this means there is a path $p' \in T$ of maximal length $l > 0$.

Since $\lambda \in \Lambda p'$ and $\sum_{t\geq l}\sum_{p\in\mathbf{P}(\mathrm{l}(q))^{-1}\cap\mathbf{P}(t)} r_p p \in \Lambda p'$ we have $\sum_{t<l}\sum_{p\in\mathbf{P}(\mathrm{l}(q))^{-1}\cap\mathbf{P}(t)} r_p p \in \Lambda p'$. After rewriting $\sum_{t<l}\sum_{p\in\mathbf{P}(\mathrm{l}(q))^{-1}\cap\mathbf{P}(t)} r_p p$ we can assume $r_p = 0$ for all paths $p \in T$ of length at most $l-1$. Since $\Lambda$ is a quasi-bounded gentle algebra and $a\mathrm{l}(q) \in \mathbf{P}$ we have $p'q \in \mathbf{P}$ and so $\Lambda p'q \neq 0$ (otherwise $p'q$ is a path outside $(\rho)$ whose image is 0 in $\Lambda$, contradicting definition 1.1.5).

Since $\lambda q = 0$ we have $r_{p'}p'q = -\sum_{p'\neq p\in\mathbf{P}(\mathrm{l}(q))^{-1}} r_p pq$ which lies in $\mathrm{rad}(\Lambda p'q) = \mathrm{rad}(\Lambda)p'q$ by corollary 1.1.17 (ii). If $r_{p'} \notin \mathfrak{m}$ then $r_{p'}$ is a unit in which case $r_{p'}p'q \in \mathrm{rad}(\Lambda p'q)$. By corollary 1.2.14 there is some $a \in \mathbf{A}(h(p') \rightarrow)$ with $ap' \in \mathbf{P}$ and so $\mathrm{rad}(\Lambda)p'q = \Lambda ap'q$.

Since $\Lambda r_{p'}p'q = \Lambda p'q$, altogether we have $\Lambda p'q = \Lambda ap'q$ which contradicts lemma 1.1.17 (ii). Hence $r_{p'} \in \Lambda\mathfrak{m} \subseteq \mathrm{rad}(\Lambda)$ since $\Lambda$ is pointwise rad-nilpotent modulo $\mathfrak{m}$, which again means $r_{p'}p' \in \Lambda p''$ where $p'$ is an initial subpath of $p''$. But now $\lambda \in \Lambda p''$ which contradicts the maximality of $l$. $\square$

We now consider a particular family of rings which will be the focus of chapter 2. Recall definitions 1.1.5, 1.1.21 and 1.2.10.

**Definition 1.2.19.** (Complete Gentle Algebras) Let $\Lambda$ be a quasi-bounded gentle algebra over $R$ surjectively given by $(Q, \rho, \theta)$. We say $\Lambda$ is a *complete gentle algebra over* $R$ if $Q$ is finite and the ground ring $R$ is $\mathfrak{m}$-adically complete.

**Example 1.2.20.** Since $\widehat{\mathbb{Z}}_p$ is the completion of $\mathbb{Z}$ in its $p$-adic topology, the $\widehat{\mathbb{Z}}_p$-algebra $\Lambda$ surjectively given by $(Q, \rho, \theta)$ from example 1.2.15 is a complete gentle algebra, since in this case $Q$ is (given by two loops at one vertex, and hence) finite.

The next corollary follows immediately from corollary 1.1.25 (ii) and [48, (23.3)].

**Corollary 1.2.21.** *Any complete gentle algebra is a semiperfect ring.*

This corollary motivated the title of the thesis. It is not used until section 2.4.

### 1.2.3 Path-Complete Gentle Algebras.

We now discuss a way of constructing many examples of complete gentle algebras. This work was motivated by the thesis of Ricke [54].

**Assumption:** In section 1.2.3 we assume: $k$ is a field, $Q$ is a quiver, $\rho$ is a set of paths in $Q$ of length at least 2, and $(Q, \rho)$ satisfies special conditions (recall definition 1.1.11).

Note that under the assumptions above the ring $kQ/(\rho)$ is a *string algebra* in the terminology used by Crawley-Boevey [21].

**Definition 1.2.22.** (Completed Path Algebra The *completed path algebra* $\overline{kQ}$ of $Q$ consists of possibly infinite sums $\sum \lambda_p p$ where $p$ runs through the set of all paths and the elements $\lambda_p$ are scalars from $k$. If $Q$ has infinitely many paths it is possible to have $\lambda_p \neq 0$ for infinitely many $p$. For elements $a = \sum \lambda_p p$ and $a' = \sum \lambda'_p p$ of $\overline{kQ}$ we let $a + a' = \sum(\lambda_p + \lambda'_p)p$ and $aa' = \sum_q(\sum_{(p,p')} \lambda_p \lambda'_{p'})q$ where the last sum runs over all paths $q$ and all pairs of paths $(p, p')$ such that $pp' = q$. Hence $\overline{kQ}$ is an associative $k$-algebra. Note that $\overline{kQ}$ has a 1 given by $\sum_v e_v$, even if $Q$ has infinitely many vertices.

(Notation: $p_{\leq n}$, $p_{<n}$, $p_{\geq n}$, $p_{>n}$) For $r \geq 0$ and $p$ a path of length $r$ we let $p_{\leq r} = p_{>0} = p$, $p_{\leq 0} = e_{t(p)}$ and $p_{>r} = e_{h(p)}$. In case $r > 0$ for any $s$ with $0 < s < r$ let $p_{\leq s} = p_1 \dots p_s$ and $p_{>s} = p_{s+1} \dots p_r$ where $p = p_1 \dots p_r$ for arrows $p_i$. Thus for any $s$ with $0 \leq s \leq r$ we have $p = p_{\leq s} p_{>s}$.

(Notation: $\overline{I}$) If $I$ is an ideal of $kQ$ we let $\overline{I}$ denote the ideal of $\overline{kQ}$ generated by the elements of $I$.

**Proposition 1.2.23.** *If $J$ is the ideal of $kQ$ generated by the arrows then* $\mathrm{rad}(\overline{kQ}) = \overline{J}$.

*Proof.* For each vertex $v$ let $\overline{J}_v$ be the subset of $\overline{kQ}$ consisting of all $\sum \lambda_p p$ for which $\lambda_{e_v} = 0$. Note $\overline{J}_v$ is a left ideal of $\overline{kQ}$, and since $\overline{kQ}/\overline{J}_v$ is one dimensional, each $\overline{J}_v$ is a maximal left ideal and therefore $\mathrm{rad}(\overline{kQ}) \subseteq \bigcap_v \overline{J}_v = \overline{J}$ so it suffices to choose $y \in \overline{J}$ and show $y \in \mathrm{rad}(\overline{kQ})$. By [48, p.51, Lemma 4.3] it is enough to prove $1 - xyz$ is a unit for any $x, z \in \overline{kQ}$.

If $xyz$ is nilpotent, say $(xyz)^d = 0$ for some minimal $d \geq 1$, then (setting $(xyz)^0 = 1$) $\sum_{i=0}^{d-1}(xyz)^i$ is the inverse of $1 - xyz$. Suppose instead $xyz$ is not nilpotent. The idea of the proof from here is to show the symbol $\sum_{n\geq 0}(xyz)^n$ defines an element of $\overline{kQ}$.

For all $n \geq 1$ we have $xyz \in \overline{J}$ so we may write $(xyz)^n = \sum_p \lambda_{p,n}\sigma$ for some $\lambda_{p,n} \in k$ where $p$ runs through all non-trivial paths. Since $(xyz)^n \in \overline{J^n}$ there is a minimal integer $m(n)$ such that $\lambda_{p,n} \neq 0$ for some path $p$ of length $m(n)$, and we have $m(1) < m(2) < \dots$ By construction we have $\lambda_{q,n} = 0$ for each path $q$ of length at most $m(n) - 1$, and so after writing $(xyz)^{n+1} = (xyz)(xyz)^n$ we have $\lambda_{p,n+1} = \sum_{i=0}^{r} \lambda_{p_{\leq i},1}\lambda_{p_{>i},n}$ for each $n \geq 1$, $r \geq 0$ and each path $p$ of length $r$.

For each $d \geq 1$ and each path $p$ of length $d$ there is a unique integer $l_p \geq 1$ such that $m(l_p) \leq d \leq m(l_p + 1) - 1$. In this case we have $\lambda_{p,n} = 0$ whenever $n > l_p$, and hence

$$\begin{aligned}\textstyle\sum_{n=1}^{l_p-1} \lambda_{p,n+1} &= \textstyle\sum_{n=1}^{l_p} \sum_{i=0}^{r} \lambda_{p_{\leq i},1}\lambda_{p_{>i},n} \\ = \textstyle\sum_{i=0}^{r} \lambda_{p_{\leq i},1}(\sum_{n=1}^{l_p} \lambda_{p_{>i},n}) &= \textstyle\sum_{i=0}^{r} \lambda_{p_{\leq i},1}(\sum_{n=1}^{l_{p_{>i}}} \lambda_{p_{>i},n})\end{aligned}$$

Let $\alpha_p = \beta_p = 0$ if $p$ is a trivial path, and let $\alpha_p = \lambda_{p,1}$ and $\beta_p = \sum_{n=1}^{l_p} \lambda_{p,n}$ otherwise. Let $a = \sum_{s\geq 0}\sum_{q'} \alpha_{q'}q'$ and $b = \sum_{t\geq 0}\sum_{q''} \beta_{q''}q''$ where $q'$ (resp. $q''$) runs through the paths of length $s$ (resp. $t$). Altogether we have

$$ab = \textstyle\sum_{r\geq 1}\sum_p(\sum_{i=0}^{r} \lambda_{p_{\leq i},1}(\sum_{n=1}^{l_{p_{>i}}} \lambda_{p_{>i},n}))p = \sum_{r\geq 1}\sum_p\sum_{n=2}^{l_p} \lambda_{p,n} = b - a$$

and so $(1 - a)(1 + b) = 1$. Similarly one can show $ba = b - a$ and hence $(1 + b)(1 - a) = 1$. As $xyz = a$ this shows $1 + b$ is the inverse of $1 - xyz$ in $\overline{kQ}$. □

**Definition 1.2.24.** Let $\rho$ be a set of relations in $Q$. Recall (from definition 1.1.7) that for $t \geq 0$ we use $\mathbf{P}(t)$ to denote the set $\bigcup_v \mathbf{P}(t, v \rightarrow) \cup \mathbf{P}(t, \rightarrow v)$ of all paths outside $(\rho)$ of length $t$.

(Notation: $\mathbf{P}(\geq s)$, $\mathbf{P}(> s)$, $\mathbf{P}(\leq s)$ and $\mathbf{P}(< s)$) For $s \geq 0$ let $\mathbf{P}(\geq s)$, $\mathbf{P}(> s)$, $\mathbf{P}(\leq s)$ and $\mathbf{P}(< s)$ be the unions $\bigcup_{t\geq s}\mathbf{P}(t)$, $\bigcup_{t>s}\mathbf{P}(t)$, $\bigcup_{t\leq s}\mathbf{P}(t)$ and $\bigcup_{t<s}\mathbf{P}(t)$ respectively.

(Notation: $A$) Let $A$ be the ideal of $kQ/(\rho)$ generated by the arrows in $Q$.

(PATH-COMPLETE ALGEBRAS) By a *path-complete algebra* we mean an algebra of the form $\overline{kQ}/\overline{(\rho)}$ where $\overline{(\rho)}$ is the ideal in $\overline{kQ}$ generated by the elements of $\rho$.

(NOTATION: $\overline{I}$) For an ideal $I$ of $kQ/(\rho)$ let $\overline{I}$ denote the ideal in $\overline{kQ}/\overline{(\rho)}$ generated by the elements of $I$. Note that this is consistent with the notation from defintion 1.2.22 in case $\rho = \emptyset$.

**Proposition 1.2.25.** [54, Theorem 3.2.7, Corollary 3.2.8] *Let $Q$ be finite. Then every path-complete algebra of the form $\Lambda = \overline{kQ}/\overline{(\rho)}$ is isomorphic to the completion $\widehat{kQ/(\rho)}_A$ of $kQ/(\rho)$ with respect to the $A$-adic topology. Furthermore,* $\mathrm{rad}(\Lambda) = \overline{A}$ *and hence $\Lambda$ is complete with respect to its* $\mathrm{rad}(\Lambda)$*-adic topology.*

*Proof.* Let $A^0 = kQ$ and $\Gamma = \widehat{kQ/(\rho)}_A$. For any $h \geq 0$ an arbitrary element of $(kQ/(\rho))/A^h$ may be written as $[\sum_{p\in\mathbf{P}(<h)} \mu_p p] + A^h$ for scalars $\mu_\sigma \in k$. There is also an inverse system of $kQ/(\rho)$-module epimorphisms

$$\cdots \to (kQ/(\rho))/A^h \to (kQ/(\rho))/A^{h-1} \to \cdots \to (kQ/(\rho))/A^2 \to (kQ/(\rho))/A^1$$

where the map $\theta_h : (kQ/(\rho))/A^h \to (kQ/(\rho))/A^{h-1}$ is defined by sending $[\sum_{p\in\mathbf{P}(<h)} \mu_p p]+A^h$ to $[\sum_{p\in\mathbf{P}(<h-1)} \mu_p p]+A^{h-1}$. By the discussion in [6, p.103] it is sufficient to show there is a $k$-algebra isomorphism $\Lambda \to \Gamma'$ where $\Gamma'$ is the inverse limit $\mathrm{Lim}_{\leftarrow}((kQ/(\rho))/A^h)$ with respect to the above system $\{\theta_h\}_{h>0}$. Consider the assignment $\Theta : \overline{kQ} \to \Gamma'$ sending $\sum \lambda_p p$ to $([\sum_{p\in\mathbf{P}(<h)} \lambda_p p] + A^h)_h$. This is clearly a well-defined surjective $k$-linear map. For $\sum_p \lambda_p p, \sum_q \lambda'_q q \in \overline{kQ}$ we have

$$([\textstyle\sum_{p\in\mathbf{P}(<h)} \lambda_p p] + A^h)_h([\sum_{q\in\mathbf{P}(<l)} \lambda'_q q] + A^l)_l = ([\sum_{l+h<t,p\in\mathbf{P}(<h),q\in\mathbf{P}(<l)} \lambda_p \lambda'_q pq] + A^t)_t$$

and so $\Theta(\sum_p \lambda_p p)\Theta(\sum_p \lambda'_p p) = \Theta(\sum_p \lambda_p p \sum_{p'} \lambda'_{p'} p')$ which means $\Theta$ is a $k$-algebra epimorphism. It is clear that $\overline{(\rho)} \subseteq \ker(\Theta)$ so it remains to prove $\overline{(\rho)} \supseteq \ker(\Theta)$.

Let $m = \sum_p \lambda_p p \in \ker(\Theta)$. By assumption $(\sum_{p\in\mathbf{P}(<h)} \lambda_p p + A^h)_h = (0)_h$ so for each integer $l > 0$ we have $[\sum_{p\in\mathbf{P}(<l)} \lambda_p p] \in A^l$ which is only possible if $\lambda_p = 0$ whenever $p \in \mathbf{P}$. By definition this means $\sum_p \lambda_p p \in \overline{(\rho)}$ and so $\overline{(\rho)} = \ker(\Theta)$.

Hence $\Lambda$ and $\Gamma$ are isomorphic $k$-algebras. Since $(\rho) \subseteq J$ we have $\overline{(\rho)} \subseteq \overline{J}$ which is $\mathrm{rad}(\overline{kQ})$ by proposition 1.2.23 and so $\overline{A} = \overline{J}/\overline{(\rho)} = \mathrm{rad}(\overline{kQ}/\overline{(\rho)})$ by [48, (4.6) Proposition, p.51]. Thus $\Lambda$ is $\mathrm{rad}(\Lambda)$-adically complete. $\square$

We now define an equivalent topology on $\Lambda$. By a cycle we mean a non-trivial path with the same head and tail. By lemma 1.1.14 (i), if $P$ and $P'$ are cycles outside $(\rho)$ that begin (or end) with the same arrow, then one must be a power of the other. The following definition follows the terminology of [21, §3].

**Definition 1.2.26.** (Primitive Cycles, notation: $\mathbf{P}(v \circlearrowleft)$) We call a cycle *primitive* if it lies outside $(\rho)$ and is not the power of another cycle. Let $\mathbf{P}(v \circlearrowleft)$ denote the set of primitive cycles at a vertex $v$. Hence when $\bigcup_v \mathbf{P}(v \circlearrowleft)$ is finite we may consider the sum $z = \sum_v z_v$ where $z_v = \sum_{p \in \mathbf{P}(v\circlearrowleft)} p$ is the sum of all primitive cycles at a vertex $v$. Write $(z)$ for the corresponding ideal generated by $z$ in $kQ/(\rho)$.

**Lemma 1.2.27.** *If $Q$ is finite there exists some $t > 0$ for which $A^t \subseteq (z)$ as ideals in $kQ/(\rho)$.*

*Proof.* Consider the ideal $I$ in $kQ/(\rho)$ generated by $\bigcup_v \mathbf{P}(v \circlearrowleft)$, and the set S of all paths $p$ which lie outside $I$. By [21, p.9, Lemma 4.1] the algebra $kQ/(\rho)$ is a finitely generated $k[z]$ module as we assume there are finitely many vertices. This shows S must be finite, and as $\bigcup_v \mathbf{P}(v \circlearrowleft)$ is also finite we can consider the length $l > 0$ of (any of) the longest path(s) in $\mathrm{S} \cup \bigcup_v \mathbf{P}(v \circlearrowleft)$. Now let $t = l + 1$.

By the construction of $t$ for any path $\beta \in \mathbf{P}(\geq t)$ we have a factorisation $\beta = \alpha p \gamma$ for some primitive cycle $p$ at vertex $v$ and some paths $\alpha$ and $\gamma$, one of which must be non-trivial, and where $h(\gamma) = v = t(\alpha)$. We now claim that $\beta - \alpha z \gamma \in (\rho)$. Proving this claim is sufficient to show $A^t \subseteq (z)$. We prove this claim below.

Without loss of generality $\alpha$ is non-trivial. So either $z_v = p$ in which case $\alpha p = \alpha z_v$, or there is another primitive cycle $p'$ in which case $z_v = p + p'$ and so $\alpha p = \alpha z_v - \alpha p'$ since $(Q, \rho)$ satisfies special conditions. Similarly we have that $\alpha z_v = \alpha z$ and $\alpha p' \in (\rho)$ as $\alpha p \notin (\rho)$. In either case this shows $\beta - \alpha z \gamma \in (\rho)$, giving the claim above. $\square$

**Corollary 1.2.28.** *Let $Q$ be finite and suppose $(Q,\rho)$ satisfies special conditions. Then $\overline{kQ}/\overline{(\rho)}$ is isomorphic to the completion $\widehat{kQ/(\rho)}_{(z)}$ of $kQ/(\rho)$ with respect to the $(z)$-adic topology, and $\overline{kQ}/\overline{(\rho)}$ is a finitely generated $k[[z]]$-module.*

*Proof.* Assuming a sequence of elements $\underline{x} = (x_n + A^n)_{n\in\mathbb{N}}$ is Cauchy in the $A$-adic topology, given any $s > 0$ there is some $N_s \geq 0$ such that for any integers $n, m \geq N_s$, $x_n - x_m \in A^s$. Hence also there is some $M_s = N_{ts} \in \mathbb{N}$ such that for any integers $n, m \geq M_s$, $x_n - x_m \in A^{ts} \subseteq (z)^s$. Similarly if $\underline{x}$ is Cauchy with respect to the $(z)$-adic topology then given any $s > 0$ there is some $L_s \in \mathbb{N}$ such that for any integers $n, m \geq L_s$, $x_n - x_m \in (z)^s \subseteq A^s$. Setting $\underline{y} = (x_n + (z)^n)_{n\in\mathbb{N}}$ this proves $\underline{x} = 0$ in $\widehat{kQ/(\rho)}_A$ iff $\underline{y} = 0$ in $\widehat{kQ/(\rho)}_{(z)}$ and so the assignment $\widehat{kQ/(\rho)}_A \to \widehat{kQ/(\rho)}_{(z)}$ sending $\underline{x}$ to $\underline{y}$ is a well defined $k$-algebra isomorphism.

Since $kQ/(\rho)$ is a finitely generated $k[z]$-module by [21, p.9, Lemma 4.1] there is a free $k[z]$-module $F = \bigoplus_{i=1}^{n} k[z]$ and an epimorphism of $k[z]$-modules $f : F \to \Lambda$. Consider the completions $\widehat{F}_{(z)}$ of $F$ in the $(z)$-adic topology. By [6, p.108, Proposition 10.12] there is an epimorphism $\hat{f} : \widehat{F}_{(z)} \to \widehat{\Lambda}_{(z)}$ of $\widehat{k[z]}_{(z)} \simeq k[[z]]$-modules and by [6, p.108, Proposition 10.13] $\widehat{F}_{(z)} \simeq \widehat{k[z]}_{(z)} \otimes_{k[z]} F$ which is $k[[z]] \otimes_{k[z]} F \simeq \bigoplus_{i=1}^{n} k[[z]]$. This shows $\widehat{kQ/(\rho)}_{(z)}$ is a finitely generated $k[[z]]$-module as required. □

**Corollary 1.2.29.** *If $Q$ is finite and $(Q,\rho)$ satisfies gentle conditions then $\Lambda$ is a complete gentle algebra over $k[[t]]$ surjectively given by $(Q,\rho,\theta)$ for some $\theta$.*

*Proof.* By definition SPI) and SPII) hold. Let $R = k[[t]]$, $\mathfrak{m} = (t)$ and $\Lambda = \overline{kQ}/\overline{(\rho)}$. Let $\theta$ be the map $k[[t]]Q \to \Lambda$ sending $\sum_p \lambda_p(t)p$ to $\sum_p \lambda_p(z)p$. By the above this map is surjective, and so $\Lambda$ is surjectively given by $(Q,\rho,\theta)$. By lemma 1.2.27 the ring $\Lambda$ is pointwise rad-nilpotent modulo $(t)$.

By proposition 1.2.25 we have $\mathrm{rad}(\Lambda) = \overline{A}$ and so for any vertex $v$ we have $\mathrm{rad}(\Lambda)e_v = \sum_{a\in\mathbf{A}(v\to)} \Lambda a$ and $e_v\mathrm{rad}(\Lambda) = \sum_{a\in\mathbf{A}(\to v)} a\Lambda$. Hence SPIII) holds. Furthermore as $\Lambda$ is finitely generated as a $k[[t]]$-module, so too are the $k[[t]]$-modules $\Lambda a$ and $a\Lambda$ for any arrow $a$, since $k[[t]]$ is noetherian. Hence SPIV) holds.

Since there are no commutativity relations in $\rho$ these sums must be direct and so (SPV) holds, and) $\Lambda$ is a quasi-bounded string algebra over $k[[t]]$. Furthermore GI) and GII) hold by assumption, and $Q$ is finite, so $\Lambda$ is a complete gentle algebra over $k[[t]]$ since $k[[t]]$ is $(t)$-adically complete. □

**Example 1.2.30.** Let $Q$ be the quiver with two loops $X$ and $Y$ at one vertex, and let $\rho = \{XY, YX\}$. Clearly $\overline{kQ}/(\rho) \simeq k[[x, y]]/(xy)$ and so by corollary 1.2.29 $\Lambda = k[[x, y]]/(xy)$ is a complete gentle algebra over $k[[t]]$ given by $(Q, \rho, \theta)$ where $\theta(X) = x$, $\theta(Y) = y$ and $\theta(t) = x + y$.

## 1.3 String and Band Representations.

We now introduce and study so-called *string* and *band* representations. Such objects are defined by quivers of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$, and constitute a complete list of indecomposable modules (with finiteness conditions) over some of the algebras we have introduced so far. In section 1.4 we introduce what is sometimes called the *functorial filtration* method as a way to verify the completeness of the above list. We will then sketch the application of this method to Butler-Ringel string algebras.

### 1.3.1 Modules Given By Words.

**Assumption:** In section 1.3.1 we assume $\Lambda$ is a quasi-bounded string algebra over $R$ surjectively given by $(Q, \rho, \theta)$ (see definitions 1.1.5 and 1.1.21).

We now define some combinatorial data from $Q$ and $\rho$. Recall definition 1.1.7.

**Definition 1.3.1.** (Quiver Homomorphisms) For quivers $U = (U_0, U_1, h_u, t_u)$ and $V = (V_0, V_1, h_v, t_v)$ a quiver homomorphism $\mathrm{r} : U \to V$ is given by functions $\mathrm{r}_0 : U_0 \to V_0$ and $\mathrm{r}_1 : U_1 \to V_1$ such that $\mathrm{r}_0 h_u = h_v \mathrm{r}_1$ and $\mathrm{r}_0 t_u = t_v \mathrm{r}_1$.

[63, p.481, Basic concepts] (Walks and Tours) A *walk* in $Q$ is a quiver homomorphism $\mathrm{w} : L \to Q$ where the underlying graph of $L$ is a *connected* subgraph of

$$_{\infty}\mathbb{A}_{\infty} : \qquad \cdots \text{———} \mathsf{v}_{-1} \overset{\mathtt{f}_0}{\text{———}} \mathsf{v}_0 \overset{\mathtt{f}_1}{\text{———}} \mathsf{v}_1 \overset{\mathtt{f}_2}{\text{———}} \mathsf{v}_2 \text{———} \cdots$$

If it exists, we shall assume that the left-most edge of $L$ is $\mathtt{f}_1$. A *tour* in $Q$ is a quiver homomorphism $\mathrm{t} : Z \to Q$ where the underlying graph of $Z$ is $\tilde{\mathbb{A}}_m$ for some $m \geq 0$, where

$$\begin{tikzcd}[column sep=small] \tilde{\mathbb{A}}_m : & \mathsf{u}_0 \arrow[r, dash] \arrow[dr, dash, "\mathsf{g}_m"'] & \cdots \arrow[r, dash] & \mathsf{u}_{m-1} \arrow[dl, dash, "\mathsf{g}_{m-1}"] \\ & & \mathsf{u}_m & \end{tikzcd}$$

for $m \geq 1$ and $\tilde{\mathbb{A}}_0$ is the graph with one edge $\mathsf{g}_0$ (which is a loop) at one vertex $\mathsf{u}_0$.

For example, $\tilde{\mathbb{A}}_1$ is the underlying graph of the Kronecker quiver.

(Linear Walks And Tours, Finite And Closed Walks) We call a walk $\mathrm{w} : L \to Q$ (resp. tour $\mathrm{t} : Z \to Q$) *linear*[5] if $\#\mathbf{A}(\to \mathsf{v}_i) \leq 1$ and $\#\mathbf{A}(\mathsf{v}_i \to) \leq 1$ for each $i$ (resp. $\#\mathbf{A}(\to \mathsf{u}_j) = 1$ and $\#\mathbf{A}(\mathsf{u}_j \to) = 1$ for each $j$). We say a walk $\mathrm{w} : L \to Q$ is *finite* if $L$ is finite, in which case (if $L_0 = \{\mathsf{v}_0, \dots, \mathsf{v}_{r-1}\}$ for $r \geq 1$) we say w is *closed* if $\mathrm{w}(\mathsf{v}_0) = \mathrm{w}(\mathsf{v}_{r-1})$.

**Example 1.3.2.** Recall example 1.2.20, where we considered the quiver $Q$ consisting of two loops $\alpha$ and $\beta$ at single vertex $v$. Let $L$ be the quiver

$$\mathsf{v}_0 \xrightarrow{\mathsf{a}_1} \mathsf{v}_1 \xrightarrow{\mathsf{b}_2} \mathsf{v}_2 \xleftarrow{\mathsf{a}_3} \mathsf{v}_3 \xleftarrow{\mathsf{b}_4} \mathsf{v}_4 \xleftarrow{\mathsf{a}_5} \mathsf{v}_5 \xrightarrow{\mathsf{b}_6} \mathsf{v}_6 \xrightarrow{\mathsf{a}_7} \mathsf{v}_7 \xrightarrow{\mathsf{b}_8} \mathsf{v}_8 \xrightarrow{\mathsf{a}_9} \mathsf{v}_9$$

We can define a finite closed non-linear walk w by the assignments $\mathsf{a}_n \mapsto \alpha$, $\mathsf{b}_m \mapsto \beta$ and $\mathsf{v}_i \mapsto v$ for each appropriate $n$, $m$ and $i$. Define a tour $\mathrm{t} : Z \to Q$ again given by $\mathsf{a}_n \mapsto \alpha$, $\mathsf{b}_m \mapsto \beta$ and $\mathsf{v}_i \mapsto v$; where $Z$ is the quiver

$$\begin{tikzcd}
\mathsf{u}_0 \arrow[r, "\mathsf{a}_1"] & \mathsf{u}_1 \arrow[r, "\mathsf{b}_2"] & \mathsf{u}_2 & \mathsf{u}_3 \arrow[l, "\mathsf{a}_3"'] & \mathsf{u}_4 \arrow[l, "\mathsf{b}_4"'] & \mathsf{u}_5 \arrow[l, "\mathsf{a}_5"'] \arrow[lllll, bend left, "\mathsf{b}_6"]
\end{tikzcd}$$

For an example of a linear tour $\mathrm{t}' : Z' \to Q$ consider the same assignments where $Z'$ is

$$\begin{tikzcd}
\mathsf{u}_0 \arrow[rr, "\mathsf{a}_1"] & & \mathsf{u}_1 \arrow[ll, bend left, "\mathsf{b}_2"]
\end{tikzcd}$$

Similarly for an example of a right-infinite walk, consider

$$\mathsf{v}_0 \xleftarrow{\mathsf{a}_1} \mathsf{v}_1 \xleftarrow{\mathsf{b}_2} \mathsf{v}_2 \xleftarrow{\mathsf{a}_3} \mathsf{v}_3 \xrightarrow{\mathsf{b}_4} \mathsf{v}_4 \xrightarrow{\mathsf{a}_5} \mathsf{v}_5 \xrightarrow{\mathsf{b}_6} \mathsf{v}_6 \xrightarrow{\mathsf{a}_7} \cdots$$

**Definition 1.3.3.** [63, p.481] (Subwalks) If r is a walk or a tour, then a *subwalk* of r is the restriction of r to any connected proper subquiver of its domain.

(Runs Through) Let $\mathrm{r} : W \to Q$ be a walk or tour. If $V$ is a collection of vertices in $Q$ we say r *runs through* (the elements of) $V$ if for each $v \in V$ there is a vertex $\mathsf{v}$ in $W$ such that $\mathrm{r}(\mathsf{v}) = v$.

[5] Wald and Waschbüsch refer to finite linear walks (resp. linear tours) as *paths* (resp. *circuits*).

For any non-trivial path $p$ in the quiver $Q$, say $p = \alpha_1 \dots \alpha_n$ (for $n > 0$ and arrows $\alpha_i$), there is an associated linear walk $\mathrm{p} : P \to Q$ where $P$ is the quiver

$$\mathsf{v}_0 \xleftarrow{\mathsf{a}_1} \cdots \xleftarrow{\mathsf{a}_n} \mathsf{v}_n$$

defined by setting $\mathrm{p}(\mathsf{v}_i) = v_i$ for $0 \leq i \leq n$ and $\mathrm{p}(\mathsf{a}_i) = \alpha_i$ when $i > 0$.

(Occurs In) We say $p$ *occurs in* r if $P$ is a subquiver of $W$ and the linear walk p is the subwalk of r defined by the restriction to $P$.

**Example 1.3.4.** The arrows $a$ and $b$ occur in each walk and tour from example 1.3.2. Note that the path $\alpha\beta\alpha$ occurs in the walk w and the tour t but not the tour t′.

**Definition 1.3.5.** [21, p.2, Words] (Letters, Heads, Tails) By a *letter* we mean an arrow $x$ or the formal inverse $y^{-1}$ of an arrow $y$. The *head* and *tail* of a letter $l$ are, respectively: ($h(l) = h(x)$ and $t(l) = t(x)$) if $l = x$ for some arrow $x$; or ($h(l) = t(y)$ and $t(l) = h(y)$) if $l = y^{-1}$ for some arrow $y$.

($\{0\} \neq I$-words) Let $I$ be one of the sets $\{0, \dots, n\}$ (for some $n \in \mathbb{N}$), $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$. For $I \neq \{0\}$, an $I$-*word* will mean a sequence of letters

$$w = \begin{cases} w_1 \dots w_n & (\text{if } I = \{0, \dots, n\}) \\ w_1 w_2 \dots & (\text{if } I = \mathbb{N}) \\ \dots w_{-1} w_0 & (\text{if } I = -\mathbb{N}) \\ \dots w_{-1} w_0 \mid w_1 w_2 \dots & (\text{if } I = \mathbb{Z}) \end{cases}$$

(a bar $\mid$ shows the position of $w_0$ and $w_1$ when $I = \mathbb{Z}$) satisfying:

(a) if $w_i$ and $w_{i+1}$ are consecutive letters, then the tail of $t(w_i) = h(w_{i+1})$,

(b) if $w_i$ and $w_{i+1}$ are consecutive letters, then $w_i^{-1} \neq w_{i+1}$,

(c) if $p$ is any non-trivial path in $Q$ such that $p = \alpha_1 \dots \alpha_m$ or its formal inverse $p^{-1} = \alpha_m^{-1} \dots \alpha_1^{-1}$ occurs as a sequence of consecutive letters in $w$, then $p \in \mathbf{P}$.

(TRIVIAL WORDS) In case $I = \{0\}$ there are *trivial words* $1_{v,\epsilon}$ for each vertex $v$ of $Q$ and each $\epsilon = \pm 1$. By a *word* we mean an $I$-word for some $I$, and such a word is a *finite word of length* $n$ if $I = \{0, \dots, n\}$.

(NOTATION: $v_i(w)$) For any $I$-word $w$ and any $i \in I$ there is an associated vertex $v_i(w)$ given by: the head of $w_{i+1}$ in case $i+1 \in I$; the tail of $w_i$ in case $i-1 \in I$; and otherwise $v_0(1_{v,\epsilon}) = v$. Note that if $i-1, i+1 \in I$ then these vertices coincide (by condition (a)).

**Example 1.3.6.** For the quasi-bounded string algebra over $\widehat{\mathbb{Z}}_p$ from example 1.1.6, $\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ is a word defined by the walk $\mathrm{w} : L \to Q$ from example 1.3.2.

We now give a brief comparison with some different terminology.

**Definition 1.3.7.** [63, (2.1) Definition] (V-SEQUENCES) A walk $\mathrm{w} : L \to Q$ is called a *V-sequence* if:

(a) if $p$ is a path in $Q$ and $p$ occurs in w, then $p \in \mathbf{P}$; and

(b) $\mathrm{w}(\mathtt{a}) \neq \mathrm{w}(\mathtt{a}')$ for all arrows $\mathtt{a}$ and $\mathtt{a}'$ in $L$ such that $h(\mathtt{a}) = h(\mathtt{a}')$ or $t(\mathtt{a}) = t(\mathtt{a}')$.

**Corollary 1.3.8.** *There is a one-to-one correspondence between words and V-sequences.*

*Proof.* Let $\mathrm{w} : L \to Q$ be a V-sequence. As in definition 1.3.7, the underlying graph of $L$ is a connected subgraph of ${}_\infty\mathbb{A}_\infty$. Hence we can (and shall) assume the set $L_1$ of arrows of $L$ is a subset of $\{\mathtt{f}_i \mid i \in \mathbb{Z}\}$. So we can consider a set $I_\mathrm{w}$ of integers $i$ such that $\mathtt{f}_i \in L_1$. Now we can define an $I_\mathrm{w}$-word $w[\mathrm{w}]$ by $w[\mathrm{w}]_i = \mathrm{s}(\mathtt{f}_i)$ whenever $i, i+1 \in I_\mathrm{w}$.

Conversely each $I$-word $w$ defines a quiver $L_w$ as follows. The vertices of $L_w$ are given by $\{\mathsf{v}_i \mid i \in I\}$. Whenever $i, i+1 \in I$ there is an arrow $\mathtt{f}_i$ with head (resp. tail) $\mathsf{v}_i$ and tail (resp. head) $\mathsf{v}_{i+1}$ provided $w_{i+1} = x$ (resp. $w_{i+1} = x^{-1}$) for some arrow $x$. There is now an associated V-sequence $\mathrm{w}[w] : L_w \to Q$ defined by setting $\mathrm{w}[w](\mathsf{v}_i) = v_i(w)$ whenever $i \in I$ and $\mathrm{w}[w](\mathtt{f}_i) = w_i$ whenever $i, i+1 \in I$.

This is a one-to-one correspondence because $w[\mathrm{w}[u]] = u$ for any word $u$, and $\mathrm{w}[w[\mathrm{u}]] = \mathrm{u}$ for any V-sequence u. $\square$

**Lemma 1.3.9.** *Let $\mathcal{L}$ be the set of letters. There is a function $s : \mathcal{L} \to \{1, -1\} = \{\pm 1\}$ such that:*

(i) *for each vertex $v$ and each $\epsilon \in \{\pm 1\}$ we have $\#\{q \in \mathcal{L} \mid s(q) = \epsilon, h(q) = v\} \leq 2$; and*

(ii) *for distinct $q, q' \in \mathcal{L}$ with $s(q) = s(q')$ and $h(q) = h(q')$ we have $\{q, q'\} = \{\alpha^{-1}, \beta\}$ with $\alpha\beta \in (\rho)$ (that is, $\alpha\beta \notin \mathbf{P}$).*

*Proof.* Fix an arbitrary vertex $v$. Let $\mathcal{L}_v$ be the set of letters with head $v$. By SPI), the arrows in $Q$ which are incident at $v$ (that is, have head or tail $v$) define a connected subquiver $Q_v$ of $\widehat{Q}_v$, where $\widehat{Q}_v$ is the quiver

$$\begin{tikzcd}
t(\beta_v) \arrow[dr, "\beta_v"] & & t(\delta_v) \arrow[dl, "\delta_v"] \\
& v \arrow[dl, "\alpha_v"'] \arrow[dr, "\gamma_v"'] & \\
h(\alpha_v) & & h(\gamma_v)
\end{tikzcd}$$

Furthermore, if we let $\widehat{\mathcal{L}}_v = \{\alpha_v^{-1}, \beta_v, \gamma_v^{-1}, \delta_v\}$, after relabelling we have that: $\mathcal{L}_v$ is the set of $x \in \widehat{\mathcal{L}}_v$ such that $x$ is an arrow in $Q$; if ($\alpha_v$ and $\beta_v$ are arrows in $Q$) then $\alpha_v\beta_v \in (\rho)$; and if ($\gamma_v$ and $\delta_v$ are arrows in $Q$) then $\gamma_v\delta_v \in (\rho)$.

Consider the function $\widehat{s} : \bigcup_v \widehat{\mathcal{L}}_v \to \{\pm 1\}$ defined by setting $\widehat{s}(\alpha_v^{-1}) = \widehat{s}(\beta_v) = 1$ and $\widehat{s}(\gamma_v^{-1}) = \widehat{s}(\delta_v) = -1$ for each $v$. Note $\widehat{s}$ is well defined because $\widehat{\mathcal{L}}_v \cap \widehat{\mathcal{L}}_{v'} = \emptyset$ for distinct vertices $v$ and $v'$.

Let $s$ be the restriction of $\widehat{s}$ to the subset $\bigcup_v \mathcal{L}_v$ of $\bigcup_v \widehat{\mathcal{L}}_v$.

(i) For $\epsilon = 1$ we have $\{q \in \mathcal{L} \mid s(q) = 1, h(q) = v\} \subseteq \{\alpha_v^{-1}, \beta_v$. For $\epsilon = -1$ we have $\{q \in \mathcal{L} \mid s(q) = -1, h(q) = v\} \subseteq \{\gamma_v^{-1}, \delta_v\}$.

(ii) Let $s(q) = s(q') = \epsilon$ and $h(q) = h(q') = v$. If $\epsilon = 1$ then as $q = q'$ we have $\{q \in \mathcal{L} \mid s(q) = 1, h(q) = v\} = \alpha_v^{-1}, \beta_v$ and, from our relabelling, $\alpha_v\beta_v \in (\rho)$. As above, the proof in case $\epsilon = -1$ is ommited.

**Definition 1.3.10.** [18, §2] (SIGN, NOTATION: $s(w)$) If $w$ is an $I$-word where $\{0\} \neq I \subseteq \mathbb{N}$ we define the *sign* of $w$ to be $s(w_1)$. We define the sign of a trivial word to be $s(1_{v\pm 1}) = \pm 1$.

(COMPOSING WORDS) Suppose $w$ and $w'$ are non-trivial words where $I_w \subseteq -\mathbb{N}$ and $I_{w'} \subseteq \mathbb{N}$. If $u = h(w^{-1})$ and $\epsilon = -s(w^{-1})$ let $w1_{u,\epsilon} = w$. If $v = h(w')$ and $\delta = s(w')$ we let $1_{v,\delta}w' = w'$. Write $ww'$ for the concatenation of the letters in $w$ with the letters in $w'$. In case $D = \dots w_{-1}w_0$ is a $-\mathbb{N}$-word and $E = w_1w_2\dots$ is an $\mathbb{N}$-word, write $ww' = \dots w_0 \mid w_1 \dots$

(NOTATION: $w_{<i}$, $w_{\leq i}$, $w_{>i}$, $w_{\geq i}$) If $w$ is an $I$-word and $i \in I$ is arbitrary we let $w_{\leq i} = \dots w_i$ given $i - 1 \in I$, and otherwise $w_i = w_{\leq i} = 1_{h(w),s(w)}$. Similarly we let $w_{>i} = w_{i+1}\dots$ given $i + 1 \in I$ and otherwise $w_{>i} = 1_{h(w),s(w)}$. Hence (for appropriate $i$) there are unique words $w_{<i}$ and $w_{\geq i}$ satisfying $w_{\leq i} = w_{<i}w_i$ and $w_iw_{>i} = w_{\geq i}$.

**Example 1.3.11.** Recall the word $\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ from example 1.3.6. Setting $s(\alpha) = 1$ gives $s(\alpha^{-1}) = 1$, $s(\beta) = -1$ and $s(\beta^{-1}) = -1$: which gives the table

| $i$ | $w_i$ | $s(w_i)$ | $(w_{\leq i})^{-1}$ | $w_{>i}$ |
|---|---|---|---|---|
| 0 | - | - | $1_{v,-1}$ | $\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ |
| 1 | $\alpha^{-1}$ | 1 | $\alpha$ | $\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ |
| 2 | $\beta^{-1}$ | $-1$ | $\beta\alpha$ | $\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ |
| 3 | $\alpha$ | 1 | $\alpha^{-1}\beta\alpha$ | $\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ |
| 4 | $\beta$ | $-1$ | $\beta^{-1}\alpha^{-1}\beta\alpha$ | $\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ |
| 5 | $\alpha$ | 1 | $\alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha$ | $\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ |
| 6 | $\beta^{-1}$ | $-1$ | $\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha$ | $\alpha^{-1}\beta^{-1}\alpha^{-1}$ |
| 7 | $\alpha^{-1}$ | 1 | $\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha$ | $\beta^{-1}\alpha^{-1}$ |
| 8 | $\beta^{-1}$ | $-1$ | $\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha$ | $\alpha^{-1}$ |
| 9 | $\alpha^{-1}$ | 1 | $\alpha\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha$ | $1_{v,-1}$ |

**Definition 1.3.12.** (SHIFTING WORDS, NOTATION: $w[t]$) For $t \in \mathbb{Z}$ and a $\mathbb{Z}$-word $w = \dots w_0 \mid w_1 \dots$ the *shift* $w[t]$ *of* $w$ *by* $t$ will be the word $\dots w_t \mid w_{t+1}\dots$ We extend this definition to all $I$-words $w$ where $I \neq \mathbb{Z}$ by setting $w = w[t]$ for all $t \in \mathbb{Z}$.

**Corollary 1.3.13.** (Equivalence Of Words, notation: $\sim$) *There is an equivalence relation $\sim$ on the set words, given by $w \sim w'$ iff $w' = w[t]$ or $w' = w^{-1}[t]$ for some $t$.*

*Proof.* Since $w = w[0]$ the relation is reflexive. For symmetry, one has $w = w[-t]$ if $w' = w^{-1}[t]$ and $w = (w')^{-1}[-t]$ if $w' = w^{-1}[t]$. For transitivity (letting $w^1 = w$): if $w' = w^{\pm 1}[t]$ and $w'' = w'[t']$ then $w'' = w^{\pm 1}[t + t']$; and if $w' = w^{\pm 1}[t]$ and $w'' = (w')^{-1}[t']$ then $w'' = w^{\mp 1}[t + t']$. $\square$

**Definition 1.3.14.** (Cyclic and Primitive Words) For $t > 0$ we say a $\{0, \ldots, t\}$-word is *cyclic* if $w^n = w \ldots w$ ($w$ composed with itself $n$-times) is a word for any $n > 0$. In this case we say $w$ is primitive if $w \neq (w')^n$ for all cyclic words $w'$ and all $n > 0$.

(Periodic Words) We say a $\mathbb{Z}$-word $w$ is *periodic* if $w = w\,[p]$ for some $p > 0$ and the *period* of $w$ describes the minimal such $p$. In this case we write $w = {}^{\infty}(w')^{\infty}$, $w_{\geq 0} = (w')^{\infty}$ and $w_{\leq 0} = {}^{\infty}(w')$ where $w' = w_1 \ldots w_p$.

**Example 1.3.15.** Consider again the quasi-bounded string algebra over $\widehat{\mathbb{Z}}_p$ from example 1.1.6. The $\{0, \ldots, 6\}$-word $\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}$ and the $\{0, 1, 2\}$-word $\alpha\beta$ are both cyclic. To say it another way,

$$\ldots \alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\alpha \mid \beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1} \ldots$$

and $\ldots \alpha\beta\alpha\beta \mid \alpha\beta\alpha\beta \ldots$ define periodic $\mathbb{Z}$-words of periods 6 and 2 respectively.

**Definition 1.3.16.** [63, (2.2) Definition] (Primitive V-Sequences) A tour $\mathrm{t} : Z \to Q$ is called a *primitive V-sequence* if

(a) if $p$ is a path in $Q$ and $p$ occurs in $\mathrm{t}$, then $p \in \mathbf{P}$;

(b) $\mathrm{t}(\mathtt{a}) \neq \mathrm{t}(\mathtt{a}')$ for all arrows $\mathtt{a}$ and $\mathtt{a}'$ in $W$ such that $h(\mathtt{a}) = h(\mathtt{a}')$ or $t(\mathtt{a}) = t(\mathtt{a}')$; and

(c) there is no automorphism $\sigma \neq \mathrm{id}$ of $Z$ such that $\mathrm{t}\sigma = \mathrm{t}$.

The definition in [63] requires that a V-sequence must not be a linear tour, however we omit this restriction (for example, consider corollary 1.3.17 together with the periodic $\mathbb{Z}$-word of period 2 from example 1.3.15).

**Corollary 1.3.17.** *There is a one-to-one correspondence between periodic $\mathbb{Z}$-words and primitive V-sequences.*

*Proof.* There is a one-to-one correspondence (for each $p > 0$) between perodic $\mathbb{Z}$-words $w$ of period $p$ and the primitive cyclic $\{0, \dots, p\}$-words $w'$ given by $w \leftrightarrow w'$ iff $w = {}^{\infty}(w')^{\infty}$. Hence it is enough to define a one-to-one correspondence between primitive cyclic words and primitive V-sequences.

If $w' = w_1 \dots w_p$ is a primitive cyclic word, define a quiver $Q_{w'}$ by giving any orientation to $\tilde{\mathbb{A}}_0$ if $p = 1$, for $p > 1$ orientating the edges $\mathrm{g}_i$ in

$$\begin{tikzcd} \tilde{\mathbb{A}}_p : & \mathsf{u}_0 \arrow[r, dash] \arrow[dr, dash, "\mathsf{g}_{p-1}"'] & \cdots \arrow[r, dash] & \mathsf{u}_{p-2} \arrow[dl, dash, "\mathsf{g}_{p-2}"] \\ & & \mathsf{u}_{p-1} & \end{tikzcd}$$

by

$$\left.\begin{array}{ll} \mathsf{u}_0 \longleftarrow \mathsf{u}_{p-1} & (\text{if } w_p = x) \\ \mathsf{u}_0 \longrightarrow \mathsf{u}_{p-1} & (\text{if } w_p = x^{-1}) \\ \mathsf{u}_i \longleftarrow \mathsf{u}_{i+1} & (\text{if } w_{i+1} = x \text{ and } i+1 < p) \\ \mathsf{u}_i \longrightarrow \mathsf{u}_{i+1} & (\text{if } w_{i+1} = x^{-1} \text{ and } i+1 < p) \end{array}\right. \quad (\text{where } x \text{ is an arrow.})$$

where $i$ runs through all integers with $i, i{+}1 \in I$. We can now define a tour $\mathrm{t}[w'] : Q_{w'} \to Q$ by $(\mathrm{t}[w'](\mathsf{u}_i) = v_i(w')$ and $\mathrm{t}[w'](\mathsf{g}_i) = x$ if $i + 1 < p$ and $w_{i+1}$ is $x$ or its inverse), and $(\mathrm{t}[w'](\mathsf{u}_{p-1}) = v_{p-1}(w')$ and $\mathrm{t}[w'](\mathsf{g}_i) = x$ if $i = p - 1$ and $w_p$ is $x$ or its inverse). Since $w'$ is cyclic $\mathrm{t}[w']$ is closed, and since $w'$ is primitive there is no automorphism $\sigma \neq \mathrm{id}$ of $Q_{w'}$ such that $\mathrm{t}[w']\sigma = \mathrm{t}[w']$.

Given any primitive V-sequence $\mathrm{t} : Z \to Q$ we let $I_{\mathrm{t}} = \{0, \dots, p-1\}$ where ($p = 1$ if the underlying graph of $Z$ is $\tilde{\mathbb{A}}_0$) and $p > 1$ otherwise. We can now define an $I_{\mathrm{t}}$-word $w[\mathrm{t}]$ by $w[\mathrm{t}]_i = \mathrm{t}_1(\mathsf{g}_i)$ whenever $i, i+1 \in I_{\mathrm{t}}$. Since t is a tour satisfying conditions (a) and (b) from definition 1.3.16, $w[\mathrm{t}]$ is a cyclic word. Since t is a tour satisfying condition (c) from definition 1.3.16, $w[\mathrm{t}]$ is primitive. $\square$

From now on we shall choose to use the language of words, instead of the language of V-sequences, knowing these languages may be translated into one-another. We now define string and band modules for quasi-bounded string algebras over $R$. Recall definition 1.3.5.

**Definition 1.3.18.** [21, Modules given by words, p.3] Given an $I$-word $w$ let $M(w)$ be the $R$-module generated by elements $b_i$ (as $i$ runs through $I$) subject to the relations

$$e_v b_i = \begin{cases} b_i & (\text{if } v_w(i) = v) \\ 0 & (\text{otherwise}) \end{cases}$$

for any vertex $v$ in $Q$ and

$$x b_i = \begin{cases} b_{i-1} & (\text{if } i-1 \in I \text{ and } w_i = x) \\ b_{i+1} & (\text{if } i+1 \in I \text{ and } w_{i+1} = x^{-1}) \\ 0 & (\text{otherwise}) \end{cases}$$

for any arrow $x$ in $Q$. Since $\Lambda$ is surjectively given by $(Q, \rho, \theta)$ the assignments above define an action of $\Lambda$ on $M(w) = \sum_{i \in I} R b_i$. From now on $M(w)$ will be considered as a $\Lambda$-module, unless specified otherwise.

**Example 1.3.19.** Consider the complete gentle algebra $\Lambda$ over $\widehat{\mathbb{Z}}_p$ from example 1.1.6. Recall the word $w = \alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}$ from example 1.3.6. Here $M(w) = \sum_{i=0}^{9} \widehat{\mathbb{Z}}_p b_i$ where the action of $\Lambda$ is described as follows.

In the following schema the action of $\alpha$ and $\beta$ are given by the solid arrows, and the action of $p$ is described by the dashed arrows (recall $p = \alpha\beta + \beta\alpha$ in $\Lambda$) as follows. If two dashed arrows leave $b_i$ (when $i = 5$ here) $pb_i$ is given by the sum of their targets (so $pb_5 = b_3 + b_7$).

$$
\begin{tikzcd}
 & & & & & b_5 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[ddll, dashed, bend left, "p"] \arrow[ddrr, dashed, bend left, "p"] & & & & \\
b_0 \arrow[dr, "\alpha"'] \arrow[ddrr, dashed, bend left, "p"] & & & & b_4 \arrow[dl, "\beta"'] \arrow[ddll, dashed, bend left, "p"] & & b_6 \arrow[dr, "\alpha"'] \arrow[ddrr, dashed, bend left, "p"] & & & \\
 & b_1 \arrow[dr, "\beta"'] & & b_3 \arrow[dl, "\alpha"'] & & & & b_7 \arrow[dr, "\beta"'] \arrow[ddrr, dashed, bend left, "p"] & & \\
 & & b_2 & & & & & & b_8 \arrow[dr, "\alpha"'] & \\
 & & & & & & & & & b_9
\end{tikzcd}
$$

In this example we have $p^4 M(w) = 0$. As the next example shows, $\mathfrak{m}$ need not act nilpotently on $M(w)$.

**Example 1.3.20.** Let $\Lambda$ be the complete gentle algebra from example 1.1.6. In example 1.3.19 we described the $\Lambda$-module $M(\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1})$ using a finite schema. The module $M(\alpha\beta\alpha(\beta^{-1}\alpha^{-1})^{\infty})$ is depicted in a similar way by the right-infinite schema

$$
\begin{tikzcd}
 & & & b_3 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[ddll, dashed, bend left, "p"] \arrow[ddrr, dashed, bend left, "p"] & & & & & \\
 & & b_2 \arrow[dl, "\beta"'] \arrow[ddll, dashed, bend left, "p"] & & b_4 \arrow[dr, "\alpha"'] \arrow[ddrr, dashed, bend left, "p"] & & & & \\
 & b_1 \arrow[dl, "\alpha"'] & & & & b_5 \arrow[dr, "\beta"'] \arrow[ddrr, dashed, bend left, "p"] & & & \\
b_0 & & & & & & b_6 \arrow[dr, "\alpha"'] \arrow[ddrr, dashed, bend left, "p"] & & \\
 & & & & & & & b_7 \arrow[dr] & \\
 & & & & & & & & \ddots
\end{tikzcd}
$$

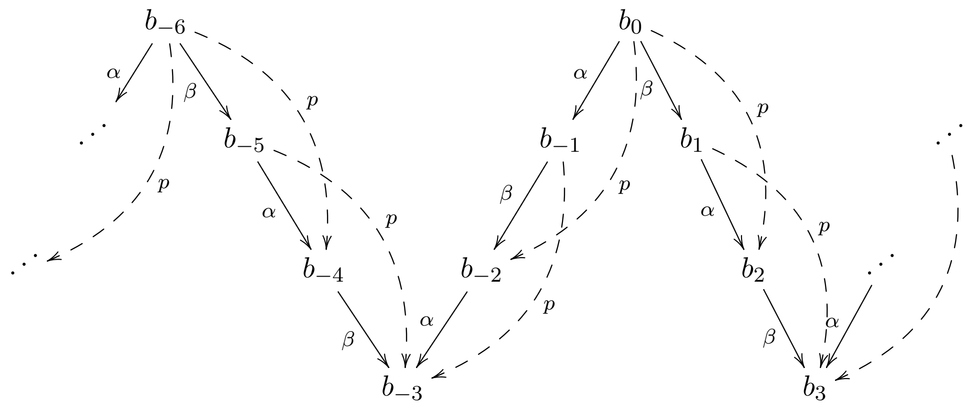
Similarly $-\mathbb{N}$-words define left-infinite schemas, and $\mathbb{Z}$ define left-right-infinite schemas.

**Definition 1.3.21.** (String Modules) Any module of the form $M(w)$ will be called a *string module.*

**Example 1.3.22.** Let $\Lambda$ be the complete gentle algebra from example 1.3.20. For the periodic $\mathbb{Z}$-word

$$w = \ldots\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\alpha \mid \beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\ldots$$

the schema



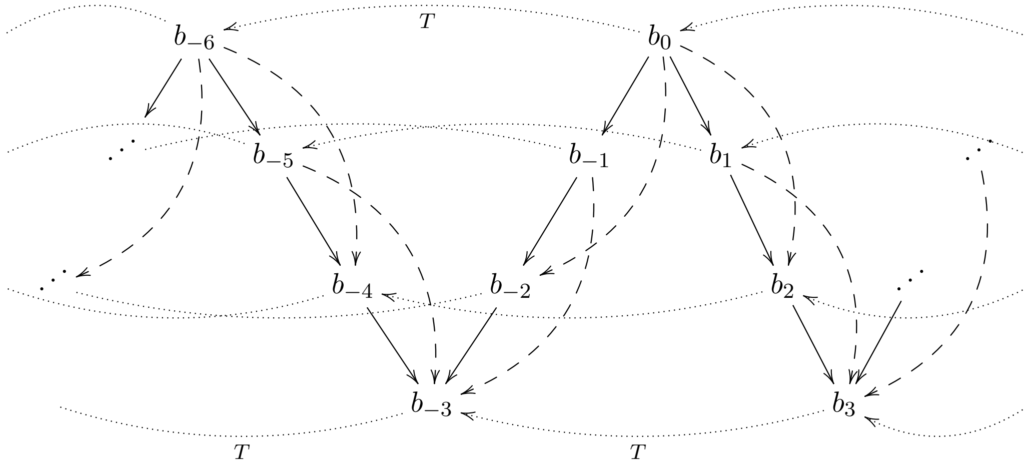

defining $M(w)$ has translational symmetry.

**Definition 1.3.23.** (Action of $T$ on $M(w)$ for periodic $w$) Suppose now $w$ is a periodic word of period $p$, and consider the module $M(w)$. Consider the $R$-linear map $T : M(w) \to M(w)$ defined on generators by $b_i \mapsto b_{i-p}$. Since $w$ is a $\mathbb{Z}$-word and periodic, $T$ is an automorphism (describing translational symmetry in the walk s$[w]$ given by $w$) and hence $M(w)$ has the structure of a right $R[T, T^{-1}]$-module.

(Notation: $M(w, V)$) For any left $R[T, T^{-1}]$-module $V$ we let $M(w, V)$ be the tensor product $M(w) \otimes_{R[T,T^{-1}]} V$ (considered as a left $\Lambda$-module).

**Example 1.3.24.** Let $\Lambda$ be the complete gentle algebra from example 1.3.22. Note that $p-1 \notin p\widehat{\mathbb{Z}}_p$ and so it is a unit, say with inverse $q$. Let $V$ be the free $\widehat{\mathbb{Z}}_p$-module $\widehat{\mathbb{Z}}_p^2 = \widehat{\mathbb{Z}}_p \oplus \widehat{\mathbb{Z}}_p$ and consider the $\widehat{\mathbb{Z}}_p$-linear automorphism $T : \widehat{\mathbb{Z}}_p^2 \to \widehat{\mathbb{Z}}_p^2$ given by $(\gamma_1, \gamma_2) \mapsto (\gamma_1 + (p - 1)\gamma_2, \gamma_2)$. Let $w$ be the word ${}^{\infty}(\beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\alpha)^{\infty}$ from example 1.3.22.

Adding dotted arrows for the right-action of $T$ on $M(w)$ in the schema from example 1.3.22 gives




For any $v \in V$ we have $\alpha b_0 \otimes v = b_5 \otimes Tv$, and so $M(w, V)$ may be interpreted by the schema

$$\begin{tikzcd}
 & & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & b_0 \otimes \widehat{\mathbb{Z}}_p^2 \arrow[dlll, "\beta"'] \arrow[drrr, "\alpha"] & \begin{pmatrix}1 & p-1\\ 0 & 1\end{pmatrix} & & \\
b_1 \otimes \widehat{\mathbb{Z}}_p^2 \arrow[dr, "\alpha"] & & & & & & b_5 \otimes \widehat{\mathbb{Z}}_p^2 \arrow[dl, "\beta"'] \\
\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & b_2 \otimes \widehat{\mathbb{Z}}_p^2 \arrow[drr, "\beta"] & & & & b_4 \otimes \widehat{\mathbb{Z}}_p^2 \arrow[dll, "\alpha"'] & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\
 & & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & b_3 \otimes \widehat{\mathbb{Z}}_p^2 & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & &
\end{tikzcd}$$

**Definition 1.3.25.** (Band Modules) A module of the form $M(w, V)$ will be called a *band* module provided the left $R[T, T^{-1}]$-module $V$ is indecomposable.

### 1.3.2 Complexes Given By Homotopy Words.

Recall definitions 1.1.5, 1.1.21, 1.2.10 and 1.2.19.

**Assumption:** In section 1.3.2 we assume $\Lambda$ is a complete gentle algebra over $R$ surjectively given by $(Q, \rho, \theta)$, and that $I$ is one of the sets $\{0, \dots, m\}$ (for some $m \geq 0$), $\mathbb{N}$, $-\mathbb{N} = \{-n \mid n \in \mathbb{N}\}$, or $\mathbb{Z}$.

**Definition 1.3.26.** (Homotopy Letters) A *homotopy letter* $q$ is one of $\gamma$, $\gamma^{-1}$, $d_\alpha$, or $d_\alpha^{-1}$ for $\gamma \in \mathbf{P}$ and an arrow $\alpha$. Those of the form $\gamma$ or $d_\alpha$ will be called *direct*, and those of the form $\gamma^{-1}$ or $d_\alpha^{-1}$ will be called *inverse*. The *inverse* $q^{-1}$ of a homotopy letter $q$ is defined by setting $(\gamma)^{-1} = \gamma^{-1}$, $(\gamma^{-1})^{-1} = \gamma$, $(d_\alpha)^{-1} = d_\alpha^{-1}$ and $(d_\alpha^{-1})^{-1} = d_\alpha$.

(Homotopy Words) For $I \neq \{0\}$ a *homotopy $I$-word* is a sequence of homotopy letters

$$C = \begin{cases} l_1^{-1} r_1 \dots l_m^{-1} r_m & (\text{if } I = \{0, \dots, m\}) \\ l_1^{-1} r_1 l_2^{-1} r_2 \dots & (\text{if } I = \mathbb{N}) \\ \dots l_{-1}^{-1} r_{-1} l_0^{-1} r_0 & (\text{if } I = -\mathbb{N}) \\ \dots l_{-1}^{-1} r_{-1} l_0^{-1} r_0 \mid l_1^{-1} r_1 l_2^{-1} r_2 \dots & (\text{if } I = \mathbb{Z}) \end{cases}$$

(which will be written as $C = \dots l_i^{-1} r_i \dots$ to save space) such that:

(a) any homotopy letter occuring in $C$ of the form $l_i^{-1}$ (resp. $r_i$) is inverse (resp. direct);

(b) any list of 2 consecutive homotopy letters, which occurs in $C$ and has the form $l_i^{-1} r_i$, is one of $\gamma^{-1} d_{\mathrm{l}(\gamma)}$ or $d_{\mathrm{l}(\gamma)}^{-1} \gamma$ for some $\gamma \in \mathbf{P}$; and

(c) any list of 4 consecutive homotopy letters, which occurs in $C$ and has the form $l_i^{-1} r_i l_{i+1}^{-1} r_{i+1}$, is one of

$$\begin{array}{cc} \gamma^{-1} d_{\mathrm{l}(\gamma)} d_{\mathrm{l}(\lambda)}^{-1} \lambda & (\text{where } h(\gamma) = h(\lambda) \text{ and } \mathrm{l}(\gamma) \neq \mathrm{l}(\lambda)) \\ d_{\mathrm{l}(\gamma)}^{-1} \gamma d_{\mathrm{l}(\lambda)}^{-1} \lambda & (\text{where } \mathrm{f}(\gamma)\mathrm{l}(\lambda) \in (\rho)) \\ d_{\mathrm{l}(\gamma)}^{-1} \gamma \lambda^{-1} d_{\mathrm{l}(\lambda)} & (\text{where } t(\gamma) = t(\lambda) \text{ and } \mathrm{f}(\gamma) \neq \mathrm{f}(\lambda)) \\ \text{or } \gamma^{-1} d_{\mathrm{l}(\gamma)} \lambda^{-1} d_{\mathrm{l}(\lambda)} & (\text{where } \mathrm{f}(\lambda)\mathrm{l}(\gamma) \in (\rho)) \end{array}$$

For $I = \{0\}$ there are *trivial homotopy words* $\underline{1}_{v,1}$ and $\underline{1}_{v,-1}$ for each vertex $v$.

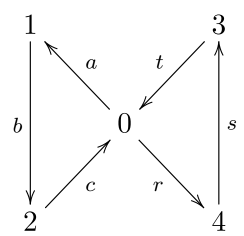
(Inverting Homotopy Words) The *inverse* $C^{-1}$ of $C$ is defined by $(\underline{1}_{v,\delta})^{-1} = \underline{1}_{v,-\delta}$ if $I = \{0\}$, and otherwise inverting the homotopy letters and reversing their order.

So: the inverse of a homotopy $\mathbb{N}$-word is a homotopy $-\mathbb{N}$ word; the inverse of a homotopy $-\mathbb{N}$-word is a homotopy $\mathbb{N}$-word; and if $I$ is (finite or $\mathbb{Z}$) then the inverse of a homotopy $I$-word is a homotopy $I$-word. Note the homotopy $\mathbb{Z}$-words are indexed so that

$$\left(\dots l_{-1}^{-1} r_{-1} l_0^{-1} r_0 \mid l_1^{-1} r_1 l_2^{-1} r_2 \dots\right)^{-1} = \dots r_2^{-1} l_2 r_1^{-1} l_1 \mid r_0^{-1} l_0 r_{-1}^{-1} l_{-1} \dots$$

Our aim in this section (1.3.2) is to give an analogue to section 1.3.1, by replacing modules with complexes of projectives. The definition above (1.3.26) appears to be new.

**Example 1.3.27.** Recall (example 1.2.12) the Assem-Skowroński gentle algebra $\Lambda = kQ/(\rho)$ where $\rho = \{ba, cb, ac, sr, ts, rt\}$ and $Q$ is the quiver




Then $C = s^{-1}d_s t^{-1} d_t d_c^{-1} c$ is a homotopy $\{0,1,2,3\}$-word.

**Example 1.3.28.** Recall the complete gentle algebra $\Lambda = k[[x,y]]/(xy)$ from example 1.2.30. Write $x^{-m}$ and $y^{-m}$ for $(x^m)^{-1}$ and $y^{-m}$ for $(y^m)^{-1}$ for each $n, m > 0$. Then

$$C = x^{-2}d_x y^{-1} d_y x^{-2} d_x d_y^{-1} y^3 d_x^{-1} x y^{-1} d_y x^{-2} d_x y^{-1} d_y x^{-2} d_x \dots$$

is a homotopy $\mathbb{N}$-word.

The pairs $\gamma^{-1} d_{\mathrm{l}(\gamma)}$ and $d_{\mathrm{l}(\lambda)}^{-1}\lambda$ (for $\gamma \in \mathbf{P}$) are in bijective correspondence with the alphabet used by Bekkert and Merklen [7] to define *generalised words*.

**Definition 1.3.29.** Let $C = \dots l_i^{-1} r_i \dots$ be a homotopy $I$-word.

(GENERALISED WORDS) Define a sequence $[C] = \dots [C]_i \dots$ by setting $[C]_i = [\gamma^{-1}]$ when $l_i^{-1} r_i = d_{\mathrm{l}(\gamma)}^{-1} \gamma$ and $[C]_i = [\gamma]$ when $l_i^{-1} r_i = \gamma^{-1} d_{\mathrm{l}(\gamma)}$. We call $[C]$ the *generalised word associated to $C$*.

(HEADS AND TAILS) The head and tail of any path $\gamma \in \mathbf{P}$ are already defined and we extend this notion to all homotopy letters by setting $h(d_a^{\pm 1}) = h(a)$ and $h(q^{-1}) = t(q)$.

(VERTICES, NOTATION: $v_C(i)$) For each $i \in I$ there is an associated vertex $v_C(i)$ defined by $v_C(i) = t(l_{i+1})$ for $i \leq 0$ and $v_C(i) = t(r_i)$ for $i > 0$ provided $C = \dots l_i^{-1} r_i \dots$ is non-trivial, and $v_{1_{v,\pm 1}}(0) = v$ otherwise.

(NOTATION: $H((\gamma^{-1} d_{\mathrm{l}(\gamma)}))^{-1})$, $\mu_C$) Let $H(\gamma^{-1} d_{\mathrm{l}(\gamma)}) = -1$ and $H(d_{\mathrm{l}(\gamma)}^{-1} \gamma) = 1$ for any $\gamma \in \mathbf{P}$. Define a function $\mu_C : I \to \mathbb{Z}$ by

$$\mu_C(i) = \begin{cases} \sum_{t=1}^{i} H(l_t^{-1} r_t) & (\text{if } i > 0) \\ 0 & (\text{if } i = 0) \\ -\sum_{t=i+1}^{0} H(l_t^{-1} r_t) & (\text{if } i < 0) \end{cases}$$

(CONTROLLED HOMOGENY) We say $C$ has *controlled homogeny* if the preimage $\mu_C^{-1}(t)$ is finite for each $t \in \mathbb{Z}$.

**Example 1.3.30.** Recall the complete gentle algebra and the homotopy $\{0,1,2,3\}$-word $C = s^{-1} d_s t^{-1} d_t d_c^{-1} c$ from example 1.3.27. Here we have the table

| $i \in \mathbb{N}$ | $l_i^{-1} r_i$ | $[C]_i$ | $v_C(i)$ | $\mu_C(i)$ |
|---|---|---|---|---|
| 0 | - | - | 4 | 0 |
| 1 | $s^{-1} d_s$ | $[s]$ | 3 | $-1$ |
| 2 | $t^{-1} d_t$ | $[t]$ | 0 | $-2$ |
| 3 | $d_c^{-1} c$ | $[c]^{-1}$ | 2 | $-1$ |

and so $[C] = [s][t][c]^{-1}$.

For infinite homotopy words we can construct the beginning of such a table.

**Example 1.3.31.** Recall the complete gentle algebra $k[[x,y]]/(xy)$ and the $\mathbb{N}$-word $C$ from example 1.3.28. Here we have

| $i \in \mathbb{N}$ | $l_i^{-1}r_i$ | $[C]_i$ | $v_C(i)$ | $\mu_C(i)$ |
|---|---|---|---|---|
| 0 | - | - | $v$ | 0 |
| 1 | $x^{-2}d_x$ | $[x^2]$ | $v$ | $-1$ |
| 2 | $y^{-1}d_y$ | $[y]$ | $v$ | $-2$ |
| 3 | $x^{-2}d_x$ | $[x^2]$ | $v$ | $-3$ |
| 4 | $d_y^{-1}y^3$ | $[y^3]^{-1}$ | $v$ | $-2$ |
| 5 | $d_x^{-1}x$ | $[x]^{-1}$ | $v$ | $-1$ |
| 6 | $y^{-1}d_y$ | $[y]$ | $v$ | $-2$ |
| 7 | $x^{-2}d_x$ | $[x^2]$ | $v$ | $-3$ |
| 8 | $y^{-1}d_y$ | $[y]$ | $v$ | $-4$ |
| | | $\vdots$ | | |

**Definition 1.3.32.** (SHIFTING HOMOTOPY WORDS, NOTATION: $C[t]$) If $t \in \mathbb{Z}$ and $C = \dots l_0^{-1}r_0 \mid l_1^{-1}r_1 \dots$ is a homotopy $\mathbb{Z}$-word, we define the *shift* $C[t]$ *of* $C$ *by* $t$ to be the homotopy $\mathbb{Z}$-word $\dots l_t^{-1}r_t \mid l_{t+1}^{-1}r_{t+1} \dots$

If $C$ is a homotopy $I$-word and $I \neq \mathbb{Z}$ let $C = C[t]$ for all $t \in \mathbb{Z}$.

**Lemma 1.3.33.** *Let $C$ be a homotopy $I$-word and $i \in I$. Then:*

(i) $v_{C^{-1}}(i) = v_C(m-i)$ *and* $\mu_{C^{-1}}(i) = \mu_C(m-i) - \mu_C(m)$ *when* $I = \{0, \dots, m\}$*;*

(ii) $v_{C^{-1}}(i) = v_C(-i)$ *and* $\mu_{C^{-1}}(i) = \mu_C(-i)$ *when $I$ is infinite; and*

(iii) $v_C(i+t) = v_{C[t]}(i)$ *and* $\mu_C(i+t) = \mu_{C[t]}(i) + \mu_C(t)$ *for any integer $t$ when $I = \mathbb{Z}$.*

*Proof.* If $C$ is trivial then there is nothing to prove so we assume otherwise.

(i) Suppose $C$ is finite, so $C = l_1^{-1}r_1 \dots l_m^{-1}r_m$ and hence $C^{-1} = r_m^{-1}l_m \dots r_1^{-1}l_1$. This means $v_{C^{-1}}(0) = t(r_m) = v_C(m)$ and for $i > 0$ we have $t(l_{m+1-i}) = t(r_{m-i})$ and so $v_{C^{-1}}(i) = v_C(m-i)$.

Similarly $\mu_C(0) = 0 = \mu_{C^{-1}}(0)$ and for $i > 0$ we have $\sum_{t=1}^{i} H(l_{m+1-t}^{-1} r_{m+1-t}) = \sum_{t=1}^{m} H(l_t^{-1} r_t) - \sum_{t=1}^{m-i} H(l_t^{-1} r_t)$ which gives $\sum_{t=1}^{i} H(r_{m+1-t}^{-1} l_{m+1-t}) = \sum_{t=1}^{m-i} H(l_t^{-1} r_t) - \sum_{t=1}^{m} H(l_t^{-1} r_t)$ because $H(l_s^{-1} r_s) = -H(r_s^{-1} l_s)$ for any $s$ with $1 \leq s \leq m$. This shows $\mu_{C^{-1}}(i) = \mu_C(m-i) - \mu_C(m)$.

(ii) Suppose now $C$ is infinite. Without loss of generality $C$ is a $\mathbb{Z}$-word. Writing $C = \dots l_0^{-1} r_0 \mid l_1^{-1} r_1 \dots$ gives $C^{-1} = \dots l_0'^{-1} r_0' \mid l_1'^{-1} r_1' \dots$ where $l_{i+1}' = r_{-i}$ and $r_i' = l_{1-i}$ for each $i \in \mathbb{Z}$. As above one can show $v_C(-i) = v_{C^{-1}}(i)$ and $\mu_{C^{-1}}(i) = \mu_C(-i)$ by considering the cases $i < 0$ and $i > 0$.

(iii) There is nothing to prove when $t = 0$. Suppose firstly $t > 0$. Clearly the formula holds for $i = 0$. Writing $C = \dots l_0^{-1} r_0 \mid l_1^{-1} r_1 \dots$ gives $C[t] = \dots l_t^{-1} r_t \mid l_{t+1}^{-1} r_{t+1} \dots$ and since $v_C(t) = t(r_t) = t(l_{t+1}) = v_{C[t]}(0)$ it is clear $v_C(i+t) = v_{C[t]}(i)$ for each $i \in \mathbb{Z}$.

In case $-t \leq i \leq -1$ writing $\mu_{C[t]}(i) = -\sum_{s=i+1}^{0} H(l_{s+t}^{-1} r_{s+t})$ as the sum of $-\sum_{s=1}^{t} H(l_s^{-1} r_s) = -\mu_C(t)$ and $\mu_C(i+t) = \sum_{s=1}^{t+i} H(l_s^{-1} r_s)$ shows $\mu_C(i+t) = \mu_C(t) + \mu_{C[t]}(i)$. The case where $i < -t$ is similar, writing $\mu_{C[t]}(i) =$ as the sum of $\mu_C(i+t)$ and $-\mu_C(t)$. In case $i + t > 0$ one has $\mu_C(i+t) - \mu_C(t) = \sum_{s=1}^{i+t} H(l_s^{-1} r_s) = \mu_{C[t]}(i)$.

Now suppose instead $t < 0$. Then $-t > 0$ and by the above $v_{C[t]}(i) = v_{C[t]}(i + t - t) = v_{C[t][-t]}(i+t) = v_C(i+t)$ and (as $\mu_{C[t]}(-t) = -\mu_C(t)$) for each $i \in \mathbb{Z}$ we have $\mu_{C[t]}(i) + \mu_C(t) = \mu_C(i+t)$ as $\mu_{C[t]}(i+t-t) = \mu_{C[t][-t]}(i+t) + \mu_{C[t]}(-t)$. $\square$

**Definition 1.3.34.** (NOTATION: $P(C)$) Let $C$ be a homotopy $I$-word. Let $P(C) = \bigoplus_{n\in\mathbb{Z}} P^n(C)$ where for $n \in \mathbb{Z}$ we let $P^n(C) = \bigoplus_{i\in\mu_C^{-1}(n)} \Lambda e_{v_C(i)}$. For each $i \in I$ let $\underline{b}_i$ denote the coset of $e_{v_C(i)}$ in the summand $\Lambda e_{v_C(i)}$ of $P^{\mu_C(i)}(C)$. We define the complex $P(C)$ by extending the assignment $d_{P(C)}(\underline{b}_i) = \underline{b}_i^- + \underline{b}_i^+$ linearly over $\Lambda$ for each $i \in I$, where

$$\underline{b}_i^+ = \left\{ \begin{array}{ll} \alpha \underline{b}_{i+1} & (\text{if } i+1 \in I,\ l_{i+1}^{-1} r_{i+1} = d_{1(\alpha)}^{-1} \alpha) \\ 0 & (\text{otherwise}) \end{array} \right\}$$

$$\underline{b}_i^- = \left\{ \begin{array}{ll} \beta \underline{b}_{i-1} & (\text{if } i-1 \in I,\ l_i^{-1} r_i = \beta^{-1} d_{1(\beta)}) \\ 0 & (\text{otherwise}) \end{array} \right\}$$

**Example 1.3.35.** Recall (example 1.2.12) the finite-dimensional gentle algebra $\Lambda = kQ/(\rho)$ where $\rho = \{ba,\, cb,\, ac,\, sr,\, ts,\, rt\}$ and $Q$ is the quiver

$$\begin{tikzcd}
1 \arrow[dd, "b"'] & & 3 \arrow[dl, "t"'] \\
 & 0 \arrow[ul, "a"'] \arrow[dr, "r"] & \\
2 \arrow[ur, "c"'] & & 4 \arrow[uu, "s"']
\end{tikzcd}$$

For $C = s^{-1}d_s t^{-1}d_t d_c^{-1}c$ we have $\underline{b}_0^- = \underline{b}_0^+ = \underline{b}_1^+ = \underline{b}_3^- = \underline{b}_3^+ = 0$, $\underline{b}_1^- = s\underline{b}_0$, $\underline{b}_2^- = t\underline{b}_1$ and $\underline{b}_2^+ = c\underline{b}_3$. The generalised word $[C] = [s][t][c^{-1}]$ associated to $C$ helps us draw the schema below

$$\begin{tikzcd}
 & & \Lambda e_0 \arrow[dl, "t"'] \arrow[dr, "c"] \arrow[rr, dotted, no head] & & P^{-2}(C) \arrow[d, "d^{-2}_{P(C)}"] \\
 & \Lambda e_3 \arrow[dl, "s"'] \arrow[rr, dotted, no head] & & \Lambda e_2 \arrow[r, dotted, no head] & P^{-1}(C) \arrow[d, "d^{-1}_{P(C)}"] \\
\Lambda e_4 \arrow[rrrr, dotted, no head] & & & & P^0(C)
\end{tikzcd}$$

which depicts $P(C)$, where $P^n(C) = 0$ for $n \neq -2, -1, 0$.

**Remark 1.3.36.** For each of the following possibilities of $l_i^{-1} r_i l_{i+1}^{-1} r_{i+1}$ (from part (c) of definition 1.3.26), consider the corresponding schema (with the same label).

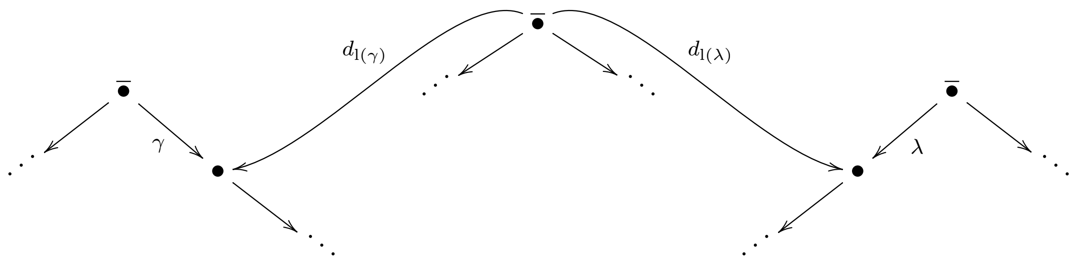
(1) $\gamma^{-1} d_{\mathrm{l}(\gamma)} d_{\mathrm{l}(\lambda)}^{-1} \lambda$

(2) $d_{\mathrm{l}(\gamma)}^{-1} \gamma \lambda^{-1} d_{\mathrm{l}(\lambda)}$

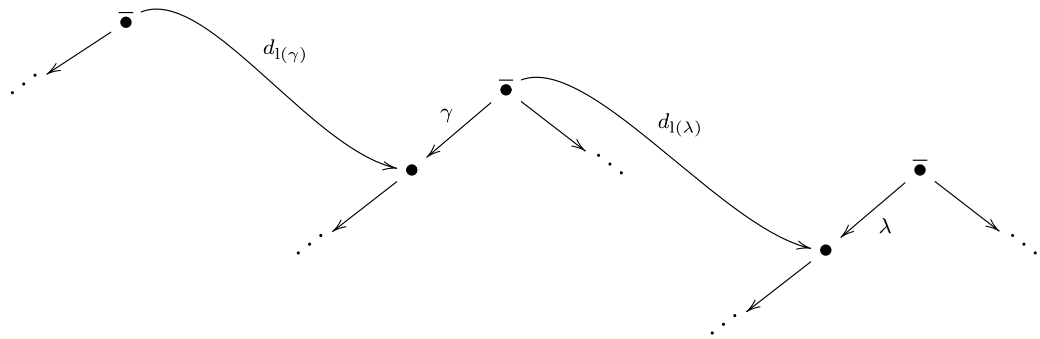
(3) $d_{\mathrm{l}(\gamma)}^{-1} \gamma d_{\mathrm{l}(\lambda)}^{-1} \lambda$

(4) $\gamma^{-1} d_{\mathrm{l}(\gamma)} \lambda^{-1} d_{\mathrm{l}(\lambda)}$

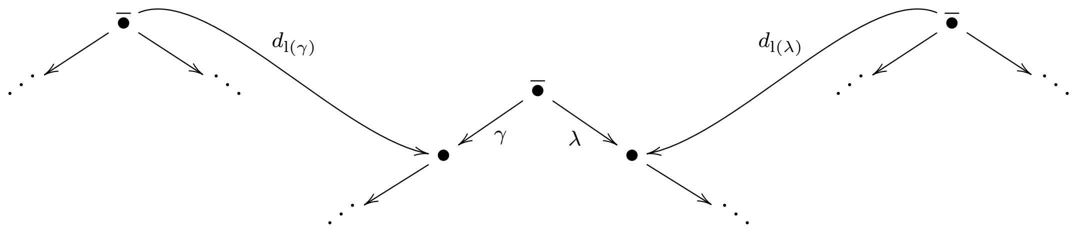
(1)



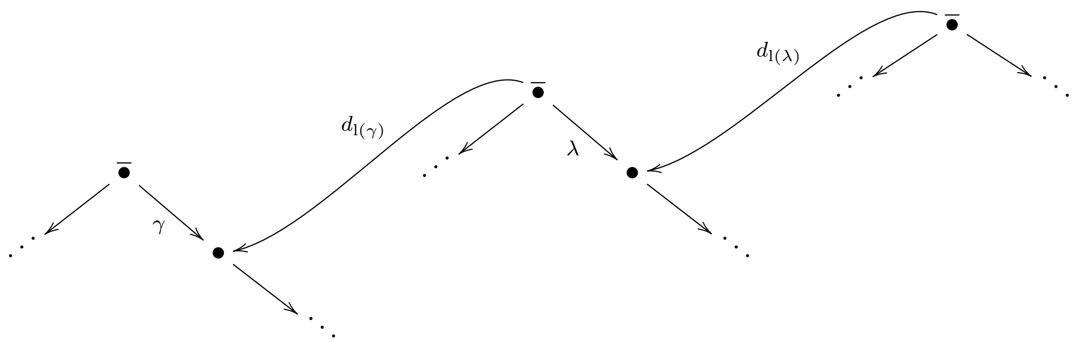

(2)




(3)




(4)




This may be useful later when considering the constructive and refined functors as defined in section 2.2. Each $\bar{\bullet}$ corresponds to an element of $I$, and symbolizes the head of an indecomposable projective $\Lambda$-module.

**Example 1.3.37.** The corresponding schema for example 1.3.35 is

$$\begin{tikzcd}
 & & & & & & \underline{b}_2 \arrow[dl, "a"'] \arrow[dr, "r"] \arrow[dddlll, bend right, "d_t"'] \arrow[dddrrr, bend left, "d_c"] & & & & \\
 & & & & & a\underline{b}_2 & & r\underline{b}_2 & & & \\
 & & \underline{b}_1 \arrow[dr, "t"] \arrow[dddl, bend right, "d_s"'] & & & & & & & & \underline{b}_3 \arrow[dl, "c"'] \\
 & & & t\underline{b}_1 \arrow[dr, "a"'] & & & & & & c\underline{b}_3 \arrow[dl, "r"'] & \\
\underline{b}_0 \arrow[dr, "s"'] & & & & at\underline{b}_1 & & & & rc\underline{b}_3 & & \\
 & s\underline{b}_0 & & & & & & & & &
\end{tikzcd}$$

**Remark 1.3.38.** The above definition of complexes $P(C)$ is equivalent to the following construction. For each $n \in \mathbb{Z}$ and $i \in \mu_C^{-1}(n)$ let

$$(d^n_{P(C)})_{i+1,i} = \begin{Bmatrix} {}_\times\alpha & (\text{if } l_{i+1}^{-1} r_{i+1} = d_{\mathrm{l}(\alpha)}^{-1}\alpha) \\ 0 & (\text{otherwise}) \end{Bmatrix}$$

$$(d^n_{P(C)})_{i-1,i} = \begin{Bmatrix} {}_\times\beta & (\text{if } l_i^{-1} r_i = \beta^{-1} d_{\mathrm{l}(\beta)}) \\ 0 & (\text{otherwise}) \end{Bmatrix}$$

where ${}_\times\lambda$ denotes the left $\Lambda$-module map $\Lambda e_{h(\lambda)} \to \Lambda e_{t(\lambda)}$ sending $\mu$ to $\mu\lambda$ (for $\lambda \in \mathbf{P}$). Set $(d^n_{P(C)})_{j,i} = 0$ for $j \in \mu_C^{-1}(n+1)$ where $j \neq i \pm 1$ so that $(d^n_{P(C)})_{j,i}$ defines an element from $\mathrm{Hom}_\Lambda(\Lambda e_{v_C(i)}, \Lambda e_{v_C(j)})$ for each $i \in \mu_C^{-1}(n)$ and $j \in \mu_C^{-1}(n+1)$. We then have

$$d^n_{P(C)}(\textstyle\sum_{i \in \mu_C^{-1}(n)} m_i) = \sum_{j \in \mu_C^{-1}(n+1)} (\sum_{i \in \mu_C^{-1}(n)} (d^n_{P(C)})_{j,i}(m_i))$$

Now fix $l \in \mu_C^{-1}(n+2)$, $j \in \mu_C^{-1}(n+1)$, and $i \in \mu_C^{-1}(n)$.

Using the rules that define homotopy words, given $(d^{n+1}_{P(C)})_{l,j} = {}_{\times}\lambda$ and $(d^{n}_{P(C)})_{j,i} = {}_{\times}\gamma$ we must have either

$$\begin{Bmatrix} j = i+1 = l-1 \text{ and } l^{-1}_{i+1} r_{i+1} l^{-1}_{i+2} r_{i+2} = d^{-1}_{\mathrm{l}(\gamma)} \gamma d^{-1}_{\mathrm{l}(\lambda)} \lambda, \text{ or} \\ j = i-1 = l+1 \text{ and } l^{-1}_{i-1} r_{i-1} l^{-1}_{i} r_{i} = \lambda^{-1} d_{\mathrm{l}(\lambda)} \gamma^{-1} d_{\mathrm{l}(\gamma)} \end{Bmatrix}$$

In either case we have $\mathrm{f}(\gamma)\mathrm{l}(\lambda) \in \rho$ which shows $(d^{n+1}_{P(C)})_{l,j}\,(d^{n}_{P(C)})_{j,i} = {}_{\times}\gamma\lambda = 0$. By the above $d^{n}_{P(C)}$ defines a $\Lambda$-module map whose image is contained in $\mathrm{rad}(P^{n+1}(C))$ and $P(C)$ defines a complex of projectives.

**Example 1.3.39.** Recall the complete gentle algebra $k[[x,y]]/(xy)$ and the homotopy $\mathbb{N}$-word $C$ from example 1.3.28. Here the generalised word associated to $C$ is $[C] = [x^2][y][x^2][y^3]^{-1}[x]^{-1}[y][x^2][y][x^2]\dots$ and so $P(C)$ may be depicted by

$$\begin{tikzcd}[column sep=small]
 & & & & & & & & & \iddots \arrow[dl, "x^2"] & \vdots \arrow[d] \\
 & & & & & & & & \Lambda \arrow[rr, dotted, no head] \arrow[dl, "y"] & & P^{-4}(C) \arrow[d, "d^{-4}_{P(C)}"] \\
 & & & \Lambda \arrow[rrrr, dotted, no head] \arrow[dl, "x^2"'] \arrow[dr, "y^3"] & & & & \Lambda \arrow[rrr, dotted, no head] \arrow[dl, "x^2"] & & & P^{-3}(C) \arrow[d, "d^{-3}_{P(C)}"] \\
 & & \Lambda \arrow[rr, dotted, no head] \arrow[dl, "y"] & & \Lambda \arrow[rr, dotted, no head] \arrow[dr, "x"] & & \Lambda \arrow[rrrr, dotted, no head] \arrow[dl, "y"] & & & & P^{-2}(C) \arrow[d, "d^{-2}_{P(C)}"] \\
 & \Lambda \arrow[rrrr, dotted, no head] \arrow[dl, "x^2"] & & & & \Lambda \arrow[rrrrr, dotted, no head] & & & & & P^{-1}(C) \arrow[d, "d^{-1}_{P(C)}"] \\
\Lambda \arrow[rrrrrrrrrr, dotted, no head] & & & & & & & & & & P^{0}(C)
\end{tikzcd}$$

As above we can interpret $P(C)$ using a different diagram, such as

$$\cdots \xrightarrow{x^2} \Lambda \xrightarrow{\begin{pmatrix} 0 \\ y \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} x^2 & 0 \\ y^3 & 0 \\ 0 & x^2 \end{pmatrix}} \Lambda \oplus \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} y & 0 & 0 \\ 0 & x & y \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} x^2 & 0 \end{pmatrix}} \Lambda \longrightarrow \cdots$$

**Remark 1.3.40.** Not all complexes of the form $P(C)$ have finitely generated homogeneous components. This is because not all homotopy words have controlled homogeny.

**Example 1.3.41.** For $\Lambda = k[[x, y]]/(xy)$ as above, the homotopy word

$$C = \dots x^{-1}d_x d_y^{-1} y x^{-1} d_x \mid d_y^{-1} y x^{-1} d_x d_y^{-1} y \dots = {}^{\infty}(d_y^{-1} y x^{-1} d_x)(d_y^{-1} y x^{-1} d_x)^{\infty}$$

defines the complex depicted by

$$\begin{tikzcd}
\ddots \arrow[dr, "y"] & & \Lambda \arrow[dl, "x"'] \arrow[dr, "y"] & & \Lambda \arrow[dl, "x"'] \arrow[dr, "y"] & & \Lambda \arrow[dl, "x"'] \arrow[dr, "y"] & & & P^0(C) \arrow[d, "d^0_{P(C)}"] \\
& \Lambda & & \Lambda & & \Lambda & & \ddots & & P^1(C)
\end{tikzcd}$$

whose homogeneous component of degree 1 is isomorphic to $\bigoplus_{\mathbb{Z}} \Lambda$. The homotopy word $C$ is an example of what we call a *periodic* homotopy $\mathbb{Z}$-word with *period* 2.

Let us recall and adapt some terminology from [21].

**Definition 1.3.42.** Let $C$ be a homotopy word. (NOTATION: $I_C$) Write $I_C$ for the subset of $\mathbb{Z}$ where $C$ is a homotopy $I_C$-word.

(PERIODICITY) We say a homotopy word $C$ is *periodic* if $C = C\,[p]$ and $\mu_C(p) = 0$ for some $p > 0$. In this case the *period* of $C$ describes the minimal such $p$.

Recall definition 3.2.1. Before we introduce band complexes it is necessary to note some isomorphisms (induced by certain symmetries between homotopy words) between the complexes introduced above (and certain degree shifts of such complexes). We now make use of the book keeping from lemma 1.3.33.

**Corollary 1.3.43.** *Let $C$ be a homotopy $I$-word. Then:*

(i) *if $I = \{0, \dots, m\}$ there is an isomorphism of complexes $P(C^{-1}) \to P(C)[\mu_C(m)]$;*

(ii) *if $I$ is infinite there is an isomorphism of complexes $P(C^{-1}) \to P(C)$; and*

(iii) *if $I = \mathbb{Z}$ and $t \in \mathbb{Z}$ there is an isomorphism of complexes $P(C[t]) \to P(C)[\mu_C(t)]$.*

*Proof.* Let $D = C^{-1}$. Fix $n \in \mathbb{Z}$. For each $i \in \mu_D^{-1}(n)$ let $\underline{b}_{i,D}$ denote the coset of $e_{v_D(i)}$ in the summand $\Lambda e_{v_D(i)}$ of $P^n(D)$. For each $j \in \mu_C^{-1}(n + \mu_C(m))$ let $\underline{b}'_{j,C}$ denote the coset of $e_{v_C(j)}$ in the summand $\Lambda e_{v_C(j)}$ of $P^{n+\mu_C(m)}(C)$.

We now extend the assignment $\theta^n(\underline{b}_{i,D}) = \underline{b}'_{m-i,C}$ linearly over $\Lambda$. By lemma 1.3.33 (i) we have $v_D(i) = v_C(m-i)$ for each $i \in I_{C^{-1}}$, and hence $\theta^n$ defines $\Lambda$-module homomorphism. Part (iii) of lemma 1.3.33 shows that $i \in \mu_D^{-1}(n)$ iff $m - i \in \mu_C^{-1}(n + \mu_C(m))$; and so $\theta^n$ is an isomorphism $P^n(D) \to P^n(C)[\mu_C(m)]$.

Writing $C = l_1^{-1} r_1 \dots l_m^{-1} r_m$ and $C^{-1} = l_1'^{-1} r_1' \dots l_m'^{-1} r_m'$ gives $l_{m-i+1} = r_i'$ and $r_{m-i+1} = l_i'$ for $1 \leq i \leq m$. By definition $d_{P(C)}(\underline{b}'_{m-i,C}) = \underline{b}'^{+}_{m-i,C} + \underline{b}'^{-}_{m-i,C}$ and $d_{P(D)}(\underline{b}_{i,D}) = \underline{b}^{+}_{i,C^{-1}} + \underline{b}^{-}_{i,C^{-1}}$ where we have the following. If ($i+1 \in I_D$ and $l'^{-1}_{i+1} r'_{i+1} = d^{-1}_{\mathrm{l}(\alpha)} \alpha$) then $\underline{b}'^{+}_{m-i,C} = \alpha \underline{b}'_{m-(i-1),C}$ and $\underline{b}^{+}_{i,D} = \alpha \underline{b}_{i+1,D}$, and otherwise $\underline{b}'^{+}_{m-i,C} = \underline{b}^{+}_{i,D} = 0$. Similarly if ($i - 1 \in I_D$ and $l_i'^{-1} r_i' = \beta^{-1} d_{\mathrm{l}(\beta)}$) then $\underline{b}'^{-}_{m-i,C} = \beta \underline{b}'_{m-(i+1),C}$ and $\underline{b}^{-}_{i,D} = \beta \underline{b}_{i-1,D}$, and otherwise $\underline{b}'^{-}_{m-i,C} = \underline{b}^{-}_{i,D} = 0$.

Together this shows $\theta^{n+1}(\underline{b}^{+}_{i,D}) = \underline{b}'^{+}_{m-i,C}$ and $\theta^{n+1}(\underline{b}^{-}_{i,D}) = \underline{b}'^{-}_{m-i,C}$ and so $\theta^{n+1}(d^n_{P(C^{-1})}(\underline{b}_{i,D})) = d^{n+\mu_C(m)}_{P(C)}(\theta^n(\underline{b}_{i,D}))$ which means $\theta$ defines an morphism of complexes. This gives (i). The proofs for parts (ii) and (iii) are similar to the above. For (ii) we apply lemma 1.3.33 (ii), and for (iii) we apply lemma 1.3.33 (iii). $\square$

**Example 1.3.44.** Recall example 1.3.35. Here $C = s^{-1} d_s t^{-1} d_t d_c^{-1} c$ and so $C^{-1} = c^{-1} d_c d_t^{-1} t d_s^{-1} s$. Since $\mu_C(3) = -1 + -1 + 1 = -1$ there is an isomorphism $\theta : P(C^{-1}) \to P(C)[-1]$ which may be depicted by

$$\begin{tikzcd}
P^{-1}(C^{-1}) \arrow[d, "d^{-1}_{P(C^{-1})}"] & & \Lambda e_0 \arrow[dl, "c"'] \arrow[dr, "t"] \arrow[rrrrr, dashed, "\theta^{-1}"] & & & & & \Lambda e_0 \arrow[dl, "t"'] \arrow[dr, "c"] & & P^{-2}(C) \arrow[d, "d^{-2}_{P(C)}"] \\
P^0(C^{-1}) \arrow[d, "d^0_{P(C^{-1})}"] & \Lambda e_2 \arrow[rrrrrrr, dashed, "\theta^0"'] & & \Lambda e_3 \arrow[dr, "s"'] \arrow[rrr, dashed, "\theta^0"] & & & \Lambda e_3 \arrow[dl, "s"] & & \Lambda e_2 & P^{-1}(C) \arrow[d, "d^{-1}_{P(C)}"] \\
P^1(C^{-1}) & & & & \Lambda e_4 \arrow[r, dashed, "\theta^1"] & \Lambda e_4 & & & & P^0(C)
\end{tikzcd}$$

The next definition highlights the importance of part (iii) of corollary 1.3.43.

**Definition 1.3.45.** (Notation: $P(C,V)$) Let $V$ be an $R[T,T^{-1}]$-module. By corollary 1.3.43, $P^n(C)$ is a left $\Lambda \otimes_R R[T,T^{-1}]$-module where $T$ acts by $\underline{b}_i \mapsto \underline{b}_{i-p}$. By translational symmetry the map $d^n_{P(C)} : P^n(C) \to P^{n+1}(C)$ is $\Lambda \otimes_R R[T,T^{-1}]$-linear. We define the complex $P(C,V)$ by letting $P^n(C,V) = P^n(C) \otimes_{R[T,T^{-1}]} V$ and $d^n_{P(C,V)} = d^n_{P(C)} \otimes 1_V$ for each $n \in \mathbb{Z}$.

**Example 1.3.46.** Recall example 1.3.41 where $\Lambda = k[[x,y]]/(xy)$. The automorphisms $\theta^0$ of $P^0(C)$ and $\theta^1$ of $P^1(C)$ may be depicted as

$$\begin{tikzcd}
\cdots \arrow[dr] & & \Lambda \arrow[ll, dashed, "\theta^0"'] \arrow[dl, "x"] \arrow[dr, "y"] & & \Lambda \arrow[ll, dashed, "\theta^0"'] \arrow[dl, "x"] \arrow[dr, "y"] & & \Lambda \arrow[ll, dashed, "\theta^0"'] \arrow[dl, "x"] \arrow[dr, "y"] & & \cdots \arrow[ll, dashed, "\theta^0"'] & P^0(C) \arrow[d, "d^0_{P(C)}"] \\
\cdots & \Lambda \arrow[l, dashed, "\theta^1"] & & \Lambda \arrow[ll, dashed, "\theta^1"] & & \Lambda \arrow[ll, dashed, "\theta^1"] & & \cdots \arrow[ll, dashed, "\theta^1"] & & P^1(C)
\end{tikzcd}$$

Let $V$ be the $(k[[t]])[T,T^{-1}]$ module $k[[t]]$ where $T$ acts as multiplication by a unit $u \in k[[t]] \setminus (t)$. For any $f(t) \in k[[t]]$ we have $y\underline{b}_1 \otimes f(t) = y\underline{b}_{-1} \otimes u^{-1}f(t)$ in $P^1(C,V)$ and so $d^0_{P(C)}(\underline{b}_0) \otimes f(t) = (x + yu^{-1})\underline{b}_{-1} \otimes 1(t)$. This together with the isomorphism $\Lambda \otimes_{k[[t]]} k[[t]] \simeq \Lambda$ shows $P(C,V)$ may be interpreted by the schema

$$\begin{tikzcd}
\Lambda \arrow[d, bend right=60, "x"'] \arrow[d, bend left=60, "y"', "u^{-1}"] & P^0(C,V) \arrow[d, "d^0_{P(C,V)}"] \\
\Lambda & P^1(C,V)
\end{tikzcd}$$

**Lemma 1.3.47.** *Let $n \in \mathbb{Z}$ and $C$ be a periodic homotopy $\mathbb{Z}$-word of period $p$. Let $V$ be an $R[T,T^{-1}]$-module which is free as an $R$-module with $R$-basis $\{v_\lambda \mid \lambda \in \Omega\}$. Write $\langle n,p\rangle$ for the set $\mu_C^{-1}(n) \cap [0,p-1]$. Then the map*

$$\kappa_n : P^n(C,V) = (\textstyle\bigoplus_{i \in \mu_C^{-1}(n)} \Lambda e_{v_C(i)}) \otimes V \to \bigoplus_{i \in \langle n,p\rangle} \Lambda e_{v_C(i)} \otimes_R V,$$
$$(\textstyle\sum_{i \in \mu_C^{-1}(n)} m_i) \otimes v \mapsto \sum_{i \in \langle n,p\rangle} (\sum_{s \in \mathbb{Z}} m_{i+ps} T^s \otimes T^{-s} v)$$

*is a $\Lambda$-module isomorphism. Hence $P(C,V)$ is a projective $\Lambda$-module generated by $\{\underline{b}_i \otimes v_\lambda \mid 0 \leq i \leq p-1, \lambda \in \Omega\}$.*

*Proof.* Let $\Gamma = \Lambda \otimes_R R[T, T^{-1}]$. Fix $n \in \mathbb{Z}$. For each $l \in \{0, \ldots, p-1\}$ the $\Lambda$-submodule $N_l = \sum_{s \in \mathbb{Z}} \Lambda e_{v_C(l+ps)}$ of $P^n(C)$ is a $\Gamma$-submodule since it is closed under the right action of $T$. Any element of $N_l \otimes_{R[T,T^{-1}]} V$ has the form $\sum_s \sum_\lambda \sum_\sigma \mu_{\sigma,l+ps,\lambda} \sigma \underline{b}_{l+ps} \otimes v_\lambda$ for scalars $\mu_{\sigma,l+ps,\lambda} \in R$ for each $s \in \mathbb{Z}$, $\lambda \in \Omega$ and $\sigma \in \mathbf{P}(\geq 0)$ with tail $v_C(l+ps) = v_C(l)$. Since $\underline{b}_{l+ps} T^s = \underline{b}_l$ for each $s$, we have $\sum_\sigma \mu_{\sigma,l+ps,\lambda} \sigma \underline{b}_{l+ps} \otimes v_\lambda = \sum_\sigma \mu_{\sigma,l+ps,\lambda} \sigma \underline{b}_l \otimes T^{-s} v_\lambda$

Consider the $\Lambda$-module homomorphism $\chi_l : N_l \otimes_{R[T,T^{-1}]} V \to \Lambda e_{v_C(l)} \otimes_R V$ for each $l \in \mu_P^{-1}(n)$ sending $\underline{b}_{l+ps} \otimes v_\lambda$ to $\underline{b}_l \otimes T^{-s} v_\lambda$. By the above $\chi_{l,n}$ is clearly bijective. Since $C$ is periodic of period $p$ we have $\mu_C^{-1}(n) = \{l + ps \mid l \in \langle n, p\rangle, s \in \mathbb{Z}\}$ which gives a $\Gamma$-module isomorphism $P^n(C) \to \bigoplus_{l \in \langle n,p\rangle} N_l$ defined by sending $\sum_{i \in \mu_C^{-1}(n)} m_i$ (for $m_i \in \Lambda \underline{b}_i$) to $\sum_{l \in (n,p)} \sum_{s \in \mathbb{Z}} m_{l+ps}$. This defines a $\Lambda$-module isomorphism $\tau_n$ from $P^n(C, V)$ to $\bigoplus_{l \in \langle n,p\rangle} N_l \otimes_{R[T,T^{-1}]} V$ defined by sending $(\sum_{i \in \mu_C^{-1}(n)} m_i) \otimes v$ to $\sum_{l \in \langle n,p\rangle} (\sum_{s \in \mathbb{Z}} m_{l+ps}) \otimes v$.

Letting $\kappa_n = (\bigoplus_{l \in \mu_P^{-1}(n)} \chi_l) \tau_n$ gives the first part of the lemma. Setting $M = \bigoplus_{i \in \langle n,p\rangle} \Lambda e_{v_C(i)}$ gives $M \otimes_R V \simeq \bigoplus_{\lambda \in \Omega} M$ as $\Lambda$-modules as $V$ is a free $R$-module, giving the second part of the lemma. □

**Definition 1.3.48.** (String Complexes) If $C$ is a homotopy word we call $P(C)$ a *string complex* provided $C$ is not a periodic homotopy $\mathbb{Z}$-word.

(Band Complexes) If $V$ is an $R[T, T^{-1}]$-module we call $P(C, V)$ a *band complex* provided: $C$ is a periodic homotopy $\mathbb{Z}$-word; $V$ is an indecomposable $R[T, T^{-1}]$-module; and $V$ is free as an $R$-module. In this case we often choose an $R$-basis $\{v_\lambda \mid \lambda \in \Omega\}$ for $V$.

## 1.4 Functorial Filtrations.

This section will include a literature review for the *functorial filtration method*, which (as it has been presented in the current literature) is written in the the language of relations. In chapter 2 we adapt the method to a new setting, which requires the notion of an *$R$-linear relation*. For convenience we study this notion before explaining how the functorial filtration method works and where it has been applied.

### 1.4.1 Linear Relations.

**Assumption:** In section 1.4.1 we assume $\Lambda$ is an $R$-algebra surjectively given by $(Q, \rho, \theta)$.

We follow the work of Maclane, however we use notation from work of Gel'fand and Ponomarev [32], Ringel [55] and Crawley-Boevey [21].

**Definition 1.4.1.** [49, §2] (see also [32, §1, Definition]) Let $M$, $N$ and $L$ be $R$-modules. (Relations) An *$R$-linear relation from $M$ to $N$* is an $R$-submodule $V$ of $M \oplus N$. If the context is clear, $V$ will be called a relation. A *relation on $M$* is a relation from $M$ to $M$.

(Converse And Composition) The *converse*[6] $V^{-1} = \{(n, m) \mid (m, n) \in V\}$ of a relation $V$ from $M$ to $N$ defines a relation from $N$ to $M$. If $W$ is a relation from $L$ to $M$ the *composition* $VW$ is the relation from $L$ to $N$ consisting of all pairs $(l, n) \in L \oplus N$ such that $(m, n) \in V$ and $(l, m) \in W$ for some $m \in M$.

(Image, Kernel, Domain Of Definition, Indeterminacy) For any $m \in M$ the *image of $V$ at $m$* is $Vm = \{n \in N \mid (m, n) \in V\}$ and for a subset $S \subseteq M$ we let $VS = \bigcup_{m\in S} Vm$. The *kernel* $\ker(V) = V^{-1}0$ and the *domain of definition* $\mathrm{dom}(V) = V^{-1}M$ define $R$-submodules of $M$. The *image* $\mathrm{im}(V) = VM$ and the *indeterminacy* $\mathrm{ind}(V) = V0$ define $R$-submodules of $N$. Note that $\mathrm{ind}(V) = \ker(V^{-1})$ and $\mathrm{im}(V) = \mathrm{dom}(V^{-1})$.

As suggested in [32, p.28] we consider the case where $V$ is the graph of an $R$-linear map.

[6] In [32] this is called the inverse relation, and denoted $V^{\sharp}$. We avoid this notation as it conflicts with notation we use following Crawley-Boevey [21] (see definition 1.4.29).

**Example 1.4.2.** (Relations Given By Graphs) If $f : M \to N$ is any $R$-linear map then $\mathrm{graph}(f) = \{(m, f(m)) \mid m \in M\}$ defines a relation from $M$ to $N$. Since $f$ is well-defined and $R$-linear we have $\mathrm{dom}(\mathrm{graph}(f)) = M$ and $\mathrm{ind}(\mathrm{graph}(f)) = 0$. Note that

$$\ker(f) = \{m \in M \mid f(m) = 0\} = \{m \in M \mid (m, 0) \in \mathrm{graph}(f)\} = (\mathrm{graph}(f))^{-1}0$$

and

$$\mathrm{im}(f) = \{f(m) \mid m \in M\} = \textstyle\bigcup_{m \in M}\{n \in N \mid (m, n) \in \mathrm{graph}(f)\} = \mathrm{graph}(f)M$$

If $g : L \to M$ is an $R$-linear map we similarly have $\mathrm{graph}(f)\mathrm{graph}(g) = \mathrm{graph}(fg)$.

We now introduce some language to explain the word *functorial* in functorial filtrations.

**Definition 1.4.3.** (notation: $R$-**Rel**) The *category of (R-linear) relations* $R$-**Rel** is defined as follows. The objects are pairs $(V, M)$ where $V$ is an $R$-linear relation on an $R$-module $M$. A morphism $\langle f\rangle : (V, M) \to (W, N)$ is given by an $R$-linear map $f : M \to N$ such that $(f(m), f(m')) \in W$ for any $(m, m') \in V$.

(Notation: ${}_R(-)$, $(\ ,-)$, $((-)^{-1},\ )$, im, ind) There are some canonical functors defined as follows.

$$
\begin{gathered}
{}_R(-) : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod},\ \ {}_\Lambda M \mapsto {}_R M,\ \ [f : {}_\Lambda M \to {}_\Lambda N] \mapsto [{}_R M \ni m \mapsto f(m) \in {}_R N] \\
(\ ,-) : R\text{-}\mathbf{Rel} \to R\text{-}\mathbf{Mod},\ \ (V, M) \mapsto M,\ \ \langle f\rangle \mapsto f \\
((-)^{-1},\ ) : R\text{-}\mathbf{Rel} \to R\text{-}\mathbf{Rel},\ \ (V, M) \mapsto (V^{-1}, M),\ \ \langle f\rangle \mapsto \langle f\rangle \\
\mathrm{im} : R\text{-}\mathbf{Rel} \to R\text{-}\mathbf{Mod},\ \ (V, M) \mapsto VM,\ \ \langle f\rangle \mapsto [VM \ni m \mapsto f(m) \in VN] \\
\mathrm{ind} : R\text{-}\mathbf{Rel} \to R\text{-}\mathbf{Mod},\ \ (V, M) \mapsto V0,\ \ \langle f\rangle \mapsto [V0 \ni m \mapsto f(m) \in V0]
\end{gathered}
$$

(Subfunctors of ${}_R(-)$, Intervals) A *subfunctor* $\mathsf{S}$ *of* ${}_R(-)$ is given by an $R$-submodule $\mathsf{S}({}_\Lambda M) \subseteq {}_R M$ for each $\Lambda$-module ${}_\Lambda M$, such that $f(m) \in \mathsf{S}({}_\Lambda N)$ for any $f \in \mathrm{Hom}_{\Lambda\text{-}\mathbf{Mod}}({}_\Lambda M, {}_\Lambda N)$ and $m \in \mathsf{S}({}_\Lambda M)$. For subfunctors $\mathsf{S}$ and $\mathsf{S}'$ of ${}_R(-)$ we write $\mathsf{S} \leqslant \mathsf{S}'$ if we have $\mathsf{S}(M) \subseteq \mathsf{S}'(M)$ for each $\Lambda$-module $M$. In this case $[\mathsf{S}, \mathsf{S}']$ is called an *interval*, and we say intervals $[\mathsf{S}, \mathsf{S}']$ and $[\mathsf{T}, \mathsf{T}']$ *avoid each other* if $\mathsf{S}' \leqslant \mathsf{T}$ or $\mathsf{T}' \leqslant \mathsf{S}$.

**Example 1.4.4.** The functor $\mathsf{0} : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ taking every module and homomorphism to 0 is a subfunctor of ${}_R(-)$ satisfying $\mathsf{0} \leqslant {}_R(-)$.

**Definition 1.4.5.** (Functorial Relations On $\Lambda$-**Mod**) An assignment $G : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Rel}$ of objects and arrows will be called a *functorial relation* (*on* $\Lambda$-**Mod**) if:

(a) ($G({}_\Lambda M) = (V, {}_R M)$ is an object in $R$-**Rel**) for each object ${}_\Lambda M$ in $\Lambda$-**Mod**; and

(b) ($G(f) = \langle f \rangle$ is an arrow in $R$-**Rel**) for each arrow $f : {}_\Lambda M \to {}_\Lambda N$ in $\Lambda$-**Mod**.

Note that if $G$ is a functorial relation then $G$ is a functor such that $(\ ,-)G = {}_R(-)$.

(Point-wise Composition) Given any functorial relations $G$ and $G'$ we define their *pointwise composition* as the assignment $G$-$G' : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Rel}$ defined on objects by $(G\text{-}G')({}_\Lambda M) = (VW, {}_R M)$ (where $G({}_\Lambda M) = (V, {}_R M)$ and $G'({}_\Lambda M) = (W, {}_R M)$) and on arrows $f$ by $(G\text{-}G')(f) = \langle f \rangle$.

**Corollary 1.4.6.** *If* $G : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Rel}$ *is a functorial relation then:*

(i) $[\operatorname{ind} G, \operatorname{im} G]$ *is an interval of subfunctors of* ${}_R(-)$*;*

(ii) *setting* $G^{-1} = ((-)^{-1},\ )G$ *defines another functorial relation (on* $\Lambda$-**Mod***); and*

(iii) *if* $G'$ *is another functorial relation then* $G$-$G'$ *is a functorial relation.*

It is straightforward to check that the composition of relations is associative: and hence so too is the pointwise composition of functorial relations.

**Definition 1.4.7.** (Notation: $\mathrm{im}G_\infty(M)$, $\mathrm{ind}G_\infty(M)$) Suppose $G_i : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Rel}$ is a functorial relation for each $i \in \mathbb{N}$. For each $\Lambda$-module $M$ let

$$\mathrm{im}G_\infty(M) = \{m \in M \mid \exists (m_n) \in M^{\mathbb{N}} : m = m_0 \text{ and } (m_i, m_{i-1}) \in V_i \ \ \forall i > 0\}, \text{ and}$$
$$\mathrm{ind}G_\infty(M) = \{m \in M \mid \exists (m_n) \in M^{(\mathbb{N})} : m = m_0 \text{ and } (m_i, m_{i-1}) \in V_i \ \ \forall i > 0\}$$

where $G_i({}_\Lambda M) = (V_i, {}_R M)$ for each $i$, $M^{\mathbb{N}} = \prod_{\mathbb{N}} M$ and $M^{(\mathbb{N})} = \bigoplus_{\mathbb{N}} M$.

**Corollary 1.4.8.** *If* $G_i : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Rel}$ *is a functorial relation for each* $i \in \mathbb{N}$ *then* $\mathrm{ind}G_\infty$ *and* $\mathrm{im}G_\infty$ *define an interval* $[\mathrm{ind}G_\infty, \mathrm{im}G_\infty]$ *of subfunctors of* ${}_R(-)$.

We now use our assumption that $\Lambda$ is an $R$-algebra surjectively given by $(Q, \rho, \theta)$. Let $M$ be a $\Lambda$-module.

**Definition 1.4.9.** (NOTATION: $\mathrm{rel}^{\lambda}(M)$, $\lambda^{-1}0$, $\lambda M$) For any vertices $u$ and $v$ and any $\lambda \in e_v \Lambda e_u$ there is an $R$-linear relation from $e_u M$ to $e_v M$ (and hence an $R$-linear relation on $M$) denoted and defined by $\mathrm{rel}^{\lambda}(M) = \{(m, \lambda m) \mid m \in e_u M\}$. We shorthand notation by setting $\ker(\mathrm{rel}^{\lambda}(M)) = \lambda^{-1}0$ and $\mathrm{im}(\mathrm{rel}^{\lambda}(M)) = \lambda M$.

**Corollary 1.4.10.** (NOTATION: $\mathcal{R}^{\lambda}$) *For any vertices $u$ and $v$ and any $\lambda \in e_v \Lambda e_u$ the assignment ${}_{\Lambda}M \mapsto (\mathrm{rel}^{\lambda}(M), {}_{R}M)$ defines a functorial relation $\mathcal{R}^{\lambda}$.*

*Proof.* Note $\mathcal{R}^{\lambda}({}_{\Lambda}M) = (\mathrm{rel}^{\lambda}(M), {}_{R}M)$ gives a well-defined assignment of objects $\Lambda$-**Mod** $\to$ $R$-**Rel**. This means (a) from definition 1.4.5 holds. If $f : M \to N$ is a homomorphism of $\Lambda$-modules and $(m, \lambda m) \in \mathrm{rel}^{\lambda}(M)$ then $(f(m), f(\lambda m)) \in \mathrm{rel}^{\lambda}(N)$ because $f(\lambda m) = \lambda f(m)$, and so $\mathcal{R}^{\lambda}(f) = \langle f \rangle$ defines an arrow in $R$-**Rel**. This means (b) from definition 1.4.5 holds. $\square$

### 1.4.2 Words and Relations.

**Assumption:** In section 1.4.2 $\Lambda$ will denote a quasi-bounded string algebra over $R$ surjectively given by $(Q, \rho, \theta)$, and $M$ will be any (unital, left) $\Lambda$-module.

We now apply corollary 1.4.10 to the case where $\lambda$ is the coset of an arrow. Recall the notation from definition 1.4.9.

**Remark 1.4.11.** (Relations Given By Arrows) Recall that if $x$ is an arrow with head $v$ and tail $u$, then by corollaries 1.4.6 and 1.4.10 the assignment ${}_{\Lambda}M \mapsto (\mathrm{rel}^x(M), {}_R M)$ (resp. ${}_{\Lambda}M \mapsto ((\mathrm{rel}^x(M))^{-1}, {}_R M))$ defines a functorial relation, where $\mathrm{rel}^x(M)$ is the set of $(m, xm)$ as $m$ runs through $e_u M$.

(Notation: $x^{-1}0$, $xM$) The notation introduced in definition 1.4.9 gives $xM = \{xm \mid m \in e_u M\}$, $x^{-1}0 = \{m \in e_u M \mid xm = 0\}$, $x^{-1}M = e_u M$ and $x0 = 0$.

**Example 1.4.12.** Let $Q$ be the quiver with two loops $a$ and $b$ at a single vertex $v$, and let $\rho = \{a^n, b^m, ab, ba\}$ for some $m, n > 1$. If $k$ is a field then $\Lambda = kQ/(\rho)$ is a Butler-Ringel string algebra, and so it is a quasi-bounded string algebra over $k$ by lemma 1.2.3. Note that $\Lambda \simeq k[a, b]/(ab, a^n, b^m)$.

Since $ab = 0 = ba$ in $\Lambda$ we have $aM \subseteq b^{-1}0$ and $bM \subseteq a^{-1}0$ (this is [32, p.29, Proposition 2.1]). Later (in example 1.4.23) we see more inclusions of this sort. In the meantime some more notation and theory will be introduced.

**Definition 1.4.13.** (Notation: $\mathrm{rel}^{v,\pm}(M)$) If $w = 1_{v,\pm}$ let $\mathrm{rel}^w(M) = \mathrm{rel}^{v,\pm}(M)$ be the $R$-linear relation $\{(m, m) \mid m \in e_v M\}$ on $e_v M$.

(Relations Given By Finite Words, Notation: $\mathrm{rel}^w(M)$) Let $w = w_1 \dots w_n$ be a non-trivial finite word. If $x$ is an arrow let

$$\mathrm{rel}_i^w(M) = \begin{cases} \mathrm{rel}^x(M) & (\text{if } w_i = x) \\ (\mathrm{rel}^x(M))^{-1} & (\text{if } w_i = x^{-1}) \end{cases}$$

and let $\mathrm{rel}^w(M) = \mathrm{rel}_1^w(M) \dots \mathrm{rel}_n^w(M)$, the $n$-fold composition of these relations.

**Example 1.4.14.** Consider the Butler-Ringel string algebra $\Lambda \simeq k[a,b]/(ab, a^n, b^m)$ from example 1.4.12. Let $m > 2$ and $w$ be the $\{0,1,2,3\}$-word $a^{-1}bb$. By definition, for any $z, z' \in M$ we have $z \in \mathrm{rel}_w(M)z'$ iff

$$\begin{aligned} (z', z) \in \mathrm{rel}^w(M) &= \mathrm{rel}^w_1(M)\,\mathrm{rel}^w_2(M)\,\mathrm{rel}^w_3(M) \\ = \{(y', y) : (y', u) \in \mathrm{rel}^w_3(M),\ &(u, v) \in \mathrm{rel}^w_2(M) \text{ and } (v, y) \in \mathrm{rel}^w_1(M) \text{ for some } u, v\} \\ = \{(y', y) : (y', u),\ (u, v) &\in \mathrm{rel}^b(M) \text{ and } (v, y) \in (\mathrm{rel}^a(M))^{-1} \text{ for some } u, v\} \\ = \{(y', y) : u &= by',\ bu = v \text{ and } v = ay \text{ for some } u, v\} \end{aligned}$$

iff there are elements $u, v \in M$ for which $u = bz'$, $bu = v$ and $v = az$.

**Definition 1.4.15.** (Notation: $wm'$, $wS$) Let $w$ be a finite word. If $m' \in e_{t(w)}M$ let $wm' = \{m \in e_{h(w)}M : (m', m) \in \mathrm{rel}^w(M)\}$. For any $S \subseteq e_{t(w)}M$ let $wS = \bigcup_{m' \in S} wm'$.

**Example 1.4.16.** Let $\Lambda$ be the complete gentle algebra from example 1.3.24, which is surjectively given by $(Q, \rho, \theta)$, where $Q$ consists of two loops $\alpha$ and $\beta$ at one vertex, and $\rho = \{\alpha^2, \beta^2\}$. Let $w = \beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}$. Here we have

$$\mathrm{rel}^w(M) = \left\{ (m', m) \;\middle|\; \begin{array}{c} \exists m_0, \dots, m_7 \in M \;:\; m = m_0,\ \beta m_0 = m_1, \\ m_1 = \alpha m_2,\ m_2 = \beta m_3,\ m_3 = \alpha m_4, \\ \beta m_4 = m_5,\ \alpha m_5 = m_6,\ \beta m_6 = m_7,\ m_7 = m' \end{array} \right\}$$

It is helpful to depict the relations above by

$$\begin{tikzcd}
 & & & & m_4 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & & \\
 & & & m_3 \arrow[dl, "\beta"'] & & m_5 \arrow[dr, "\alpha"] & & \\
m \arrow[dr, "\beta"'] & & m_2 \arrow[dl, "\alpha"'] & & & & m_6 \arrow[dr, "\beta"'] & \\
 & m_1 & & & & & & m'
\end{tikzcd}$$

For any $S \subseteq M$ we have $m \in wS$ iff there is a sequence of $m_i$'s as above with $m' \in S$.

Recall that if $x$ is an arrow with head $v$ and tail $u$, by corollary 1.4.10 there is a functorial relation $\mathcal{R}^x : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Rel}$ where $\mathcal{R}^x({}_\Lambda M) = (\mathrm{rel}^x(M), {}_R M)$.

**Definition 1.4.17.** Let $w$ be an $I$-word. If $\{0\} \neq I$ is finite the calculations from examples 1.4.14 and 1.4.16 generalise to

$$\mathrm{rel}^w(M) = \left\{ (m', m) \;\middle|\; \begin{array}{c} \exists m_0, \dots, m_n \\ \text{where } m_i \in e_{v_i(w)} M \end{array} \;:\; \begin{array}{c} m_0 = m,\ m_n = m';\ \text{and for each } i > 0, \\ (m_{i-1} = x m_i \text{ if } w_i = x) \text{ and} \\ (x m_{i-1} = m_i \text{ if } w_i = x^{-1}). \end{array} \right\}$$

(Notation: $\mathcal{R}^w_i$) Suppose $I \neq \{0\}$. Let ($\mathcal{R}^w_i = \mathcal{R}^x$ if $w_i = x$) and ($\mathcal{R}^w_i = (\mathcal{R}^x)^{-1}$ if $w_i = x^{-1}$) for each $i > 0$ in $I$ and each arrow $x$. Recall (corollary 1.4.6 (ii)) that the pointwise composition of functorial relations is a functorial relation.

(Notation: $\mathcal{R}^w$ For Non-Trivial Finite $w$) If $I = \{0, \dots, n\}$ and $n > 0$ we let $\mathcal{R}^w = \mathcal{R}^w_1\text{-}\dots\text{-}\mathcal{R}^w_n$, the functorial relation given by the $n$-fold pointwise composition.

(Notation: $\mathcal{R}^{v,\pm 1}$) If $w = 1_{v,\pm 1}$ (where $v = h(w)$) we let $\mathcal{R}^w(M) = \mathcal{R}^{v,\pm 1}(M) = (\mathrm{rel}^{v,\pm}(M), {}_R M)$, recalling $\mathrm{rel}^{v,\pm}(M)$ is the set of pairs $(m, m)$ with $m \in e_v M$.

**Corollary 1.4.18.** *If $w$ is a finite word then $\mathcal{R}^w : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Rel}$ is a functorial relation such that $\mathcal{R}^w(M) = (\mathrm{rel}^w(M), {}_R M)$ for any $\Lambda$-module $M$.*

**Definition 1.4.19.** (notation: $\mathcal{W}_{v,\delta}$) For each vertex $v$ and $\delta \in \{\pm 1\}$ let $\mathcal{W}_{v,\delta}$ be the set of all $I$-words $w$ with $I \subseteq \mathbb{N}$, $h(w) = v$ and $s(w) = \delta$. Let $w$ be an $I$-word from $\mathcal{W}_{v,\delta}$ we define $R$-submodules $w^-(M) \subseteq w^+(M) \subseteq e_v M$ as follows.

(Notation: $w^\pm(M)$) First suppose $I$ is finite. Then we let $w^+(M) = wx^{-1}0$ if there is an arrow $x$ such that $wx^{-1}$ is a word, and $w^+(M) = wM$ otherwise. We let $w^-(M) = wyM$ if there is an arrow $y$ such that $wy$ is a word, and otherwise we let $w^-(M) = w0$.

Now suppose $I = \mathbb{N}$. Let $w^+(M)$ be the set of $m \in e_{h(w)} M$ such that there is a sequence $m_n$ ($n \geq 0$) with $m_0 = m$ and $m_{n-1} \in w_n m_n$ for all $n$. One defines $w^-(M)$ to be the subset of $m \in w^+(M)$ where the sequence above is eventually zero. Equivalently $w^-(M) = \bigcup_{n \geq 0} w_{\leq n} 0$.

We now see how the functorial relations $\mathcal{R}^w$ and the $R$-submodules $w^{\pm}(M)$ are related.

**Remark 1.4.20.** Let $w$ be an $I$-word where $I \subseteq \mathbb{N}$, and let $M$ be a $\Lambda$-module.

Recall the functors im and ind from definition 1.4.3 taking $R$-**Rel** to $R$-**Mod**. Note that $\mathrm{im}(U, N) = UN$ and $\mathrm{ind}(U, N) = U0$ whenever $U$ is an $R$-linear relation on $N$.

By corollary 1.4.6, for any functorial relation $G$ the compositions $\mathrm{im}\, G$ and $\mathrm{ind}\, G$ (taking $\Lambda$-**Mod** to $R$-**Rel** to $R$-**Mod**) are subfunctors of the forgetful functor ${}_R(-)$. In definition 1.4.7 we defined functors $\mathrm{im}G_\infty$ and $\mathrm{ind}G_\infty$ for a collection of functorial relations $\{G_i \mid i \in \mathbb{N}\}$. By corollary 1.4.8, $\mathrm{im}G_\infty$ and $\mathrm{ind}G_\infty$ are also subfunctors of ${}_R(-)$.

If $I$ is finite then

$$wM = \left\{ m \in M \;\middle|\; \begin{array}{c} \exists m_0, \ldots, m_n \\ \text{where } m_i \in e_{v_i(w)}M \end{array} : \begin{array}{c} m_0 = m \text{ and for each } i > 0, \\ (m_{i-1} = x m_i \text{ if } w_i = x) \text{ and} \\ (x m_{i-1} = m_i \text{ if } w_i = x^{-1}). \end{array} \right\}$$
$$= \{ m \in e_{h(w)}M \mid m_0, \ldots, m_n : m = m_0 \text{ and } (m_i, m_{i-1}) \in \mathrm{rel}_i^w(M) \text{ whenever } n \geq i \geq 1 \}$$

since $m = xm'$ iff $m \in xm'$ iff $(m', m) \in \mathrm{rel}^x(M)$ for any arrow $x$ and any $m, m' \in M$.

The above shows $wM = \mathrm{im}(\mathcal{R}^w(M))$, and similarly we have $w0 = (\mathrm{ind}\, \mathcal{R}^w)(M)$. If instead $I = \mathbb{N}$ then $\{\mathcal{R}_i^w \mid i \in \mathbb{N}\}$ is a collection of functorial relations (defined in definition 1.4.17), and (as above, we can show) $w^+(M) = (\mathrm{im}\mathcal{R}_\infty^w)(M)$ and $w^-(M) = (\mathrm{ind}\mathcal{R}_\infty^w)(M)$.

**Definition 1.4.21.** (Notation: $w^{\pm}$) Let $w \in \mathcal{W}_{v,\pm 1}$ be an $I$-word.

If $I$ is finite let: $w^+ = \mathrm{ind}\, \mathcal{R}^{wx^{-1}}$ if $wx^{-1}$ is a word for some arrow $x$ and $w^+ = \mathrm{im}\, \mathcal{R}^w$ otherwise); and ($w^- = \mathrm{im}\, \mathcal{R}^{wy}$ if $wy$ is a word for some arrow $y$ and $w^- = \mathrm{ind}\, \mathcal{R}^w$ otherwise). If $I = \mathbb{N}$ let $w^+ = \mathrm{im}\mathcal{R}_\infty^w$ and $w^- = \mathrm{ind}\mathcal{R}_\infty^w$.

**Remark 1.4.22.** If $w$ is a finite word then $\mathrm{im}(\mathcal{R}^{wz^{-1}}(M)) = \mathrm{im}(\mathcal{R}^w(M))$ (resp. $\mathrm{ind}(\mathcal{R}^{wz}(M)) = \mathrm{ind}(\mathcal{R}^w(M))$) if $wz^{-1}$ (resp. $wz$) is a word for some arrow $z$.

**Example 1.4.23.** Let $\Lambda$ be the complete gentle algebra from example 1.4.16. Recall we had $s(\alpha) = 1$, $s(\alpha^{-1}) = 1$, $s(\beta) = -1$ and $s(\beta^{-1}) = -1$ from example 1.3.11. Hence if $w = w_1 \ldots$ is non-trivial, then ($w$ lies in $\mathcal{W}_{v,1}$ (resp. $\mathcal{W}_{v,-1}$) iff $w_1 = \alpha^{\pm 1}$ (resp. $w_1 = \beta^{\pm}$)).

By definition (1.4.19) we have $w^-(M) = w\alpha M$ and $w^+(M) = w\alpha^{-1}0$ (resp. $w^-(M) = w\beta M$ and $w^+(M) = w\beta^{-1}0$) if $w$ is finite and $s(w^{-1}) = -1$ (resp. $s(w^{-1}) = 1$). Since $\alpha^2 = \beta^2 = 0$ in $\Lambda$ we have $\alpha M \subseteq \alpha^{-1}0$ and $\beta M \subseteq \beta^{-1}0$. In the notation from definition 1.4.3): for any finite word $w$ there is an interval $[w^-, w^+]$ of subfunctors of the forgetful functor ${}_{\widehat{\mathbb{Z}}_p}(-) : \Lambda\text{-}\mathbf{Mod} \to \widehat{\mathbb{Z}}_p\text{-}\mathbf{Mod}$.

If $w \in \mathcal{W}_{v,\pm 1}$ is non-trivial and finite such that $w_i$ is an inverse (resp. direct) letter for each $i \neq 0$, then (by remark 1.4.22) we have ${}_{\widehat{\mathbb{Z}}_p}(-) = \mathrm{im}\,\mathcal{R}^w$ (resp. $\mathsf{0} = \mathrm{ind}\,\mathcal{R}^w$) as functors $\Lambda\text{-}\mathbf{Mod} \to \widehat{\mathbb{Z}}_p\text{-}\mathbf{Mod}$. One may find it useful to arrange the intervals $[w^-, w^+]$ (as $w$ runs through the $I = \{0, \ldots, n\}$-words in $\mathcal{W}_{v,1}$ for (say) $n \leq 2$) as follows.

$$
\begin{tikzcd}
{}_{\widehat{\mathbb{Z}}_p}(-) \arrow[r, equal] \arrow[ddddddddddddddd, dash] & \mathrm{im}\mathcal{R}^{\alpha^{-1}} \arrow[r, equal] \arrow[ddddddd, dash] & \mathrm{im}\mathcal{R}^{\alpha^{-1}\beta^{-1}} \arrow[r, equal] \arrow[ddd, dash] & \mathrm{im}\mathcal{R}^{\alpha^{-1}\beta^{-1}\alpha^{-1}} \arrow[d, dash] \\
 & & & (\alpha^{-1}\beta^{-1})^+ = \mathrm{ind}\mathcal{R}^{\alpha^{-1}\beta^{-1}\alpha^{-1}} \\
 & & & (\alpha^{-1}\beta^{-1})^- = \mathrm{im}\mathcal{R}^{\alpha^{-1}\beta^{-1}\alpha} \arrow[d, dash] \\
 & & (\alpha^{-1})^+ = \mathrm{ind}\mathcal{R}^{\alpha^{-1}\beta^{-1}} \arrow[r, equal] & \mathrm{ind}\mathcal{R}^{\alpha^{-1}\beta^{-1}\alpha} \\
 & & (\alpha^{-1})^- = \mathrm{im}\mathcal{R}^{\alpha^{-1}\beta} \arrow[r, equal] \arrow[ddd, dash] & \mathrm{im}\mathcal{R}^{\alpha^{-1}\beta\alpha^{-1}} \arrow[d, dash] \\
 & & & (\alpha^{-1}\beta)^+ = \mathrm{ind}\mathcal{R}^{\alpha^{-1}\beta\alpha^{-1}} \\
 & & & (\alpha^{-1}\beta)^- = \mathrm{im}\mathcal{R}^{\alpha^{-1}\beta\alpha} \arrow[d, dash] \\
 & (1_{v,1})^+ = \mathrm{ind}\mathcal{R}^{\alpha^{-1}} \arrow[r, equal] & \mathrm{ind}\mathcal{R}^{\alpha^{-1}\beta} \arrow[r, equal] & \mathrm{ind}\mathcal{R}^{\alpha^{-1}\beta\alpha} \\
 & (1_{v,1})^- = \mathrm{im}\mathcal{R}^{\alpha} \arrow[r, equal] \arrow[ddddddd, dash] & \mathrm{im}\mathcal{R}^{\alpha\beta^{-1}} \arrow[r, equal] \arrow[ddd, dash] & \mathrm{im}\mathcal{R}^{\alpha\beta^{-1}\alpha^{-1}} \arrow[d, dash] \\
 & & & (\alpha\beta^{-1})^+ = \mathrm{ind}\mathcal{R}^{\alpha\beta^{-1}\alpha^{-1}} \\
 & & & (\alpha\beta^{-1})^- = \mathrm{im}\mathcal{R}^{\alpha\beta^{-1}\alpha} \arrow[d, dash] \\
 & & (\alpha)^+ = \mathrm{ind}\mathcal{R}^{\alpha\beta^{-1}} \arrow[r, equal] & \mathrm{ind}\mathcal{R}^{\alpha\beta^{-1}\alpha} \\
 & & (\alpha)^- = \mathrm{im}\mathcal{R}^{\alpha\beta} \arrow[r, equal] \arrow[ddd, dash] & \mathrm{im}\mathcal{R}^{\alpha\beta\alpha^{-1}} \arrow[d, dash] \\
 & & & (\alpha\beta)^+ = \mathrm{ind}\mathcal{R}^{\alpha\beta\alpha^{-1}} \\
 & & & (\alpha\beta)^- = \mathrm{im}\mathcal{R}^{\alpha\beta\alpha} \arrow[d, dash] \\
\mathsf{0} \arrow[r, equal] & \mathrm{ind}\mathcal{R}^{\alpha} \arrow[r, equal] & \mathrm{ind}\mathcal{R}^{\alpha\beta} \arrow[r, equal] & \mathrm{ind}\mathcal{R}^{\alpha\beta\alpha}
\end{tikzcd}
$$

Let $\mathbf{C}_0 = [0,1]$ and iteratively define the sets $\mathbf{C}_n$ by setting $\mathbf{C}_n = \mathbf{C}_n^+ \cup \mathbf{C}_n^-$ for each integer $n > 0$ where

$$\mathbf{C}_n^- = \{\frac{x}{3} \mid x \in \mathbf{C}_{n-1}\} \text{ and } \mathbf{C}_n^+ = \{\frac{2}{3} + \frac{x}{3} \mid x \in \mathbf{C}_{n-1}\}$$

Using this notation the *Cantor set* is $\mathbf{C} = \bigcap_{n>0} \mathbf{C}_n$. Recall $\mathbf{C}$ together with the distance metric on $\mathbb{R}$ is a metric space. Any element $x \in \mathbf{C}$ defines a sequence $\mathtt{x} = (\mathtt{x}_i)_{i\geq 0} \in \{-1,+1\}^{\mathbb{N}}$ by setting $\mathtt{x}_i = \pm 1$ if $x \in \mathbf{C}_{i+1}^{\pm}$, and if $x$ lies in the boundary of $\mathbf{C}$ then $\mathtt{x}_i = \mathtt{x}_{i+1} = \dots$ for large enough $i$.

Consider a subfunctor of $\widehat{\mathbb{Z}_p}(-)$ of the form $\mathsf{S} = w^{\pm}$ where $w \in \mathcal{W}_{v,1}$ is a $\{0,\dots,n\}$-word. We can associate a sequence $\mathtt{x} = (\mathtt{x}_i)_{i\geq 0} \in \{-1,+1\}^{\mathbb{N}}$ as follows. If $n > 0$ we define $\{\mathtt{x}_i \mid 0 \leq i < n\}$ by

$$w = \begin{cases} \alpha^{\mathtt{x}_0}\beta^{\mathtt{x}_1}\dots\beta^{\mathtt{x}_{n-1}} & \text{(if } n \text{ is even)} \\ \alpha^{\mathtt{x}_0}\dots\alpha^{\mathtt{x}_{n-1}} & \text{(if } n \text{ is odd)} \end{cases}$$

and for any $n \geq 0$ we let ($\mathtt{x}_n = \pm 1$ and $\mathtt{x}_i = \mp 1$ for all $i > n$) if $\mathsf{S} = w^{\pm}$. We have now defined an element of the boundary of $\mathbf{C}$ for any functor of the form $w^{\pm}$ where $w$ is finite. For another example of this relationship see [30, §9, Proposition].

**Definition 1.4.24.** (Ordering Words) [55, p.24, proof] (see also [21, §6]) For distinct words $w, w'$ from $\mathcal{W}_{v,\delta}$ we say $w < w'$ if one of the following hold:

(a) there are letters $l$ and $l'$ and words $u, u', u''$ for which $w = ulu'$, $w' = u(l')^{-1}u''$,

(b) there is some arrow $x$ for which $w' = wx^{-1}u$ for some word $u$,

(c) there is some arrow $y$ for which $w = w'yu'$ for some word $u'$.

**Remark 1.4.25.** To see that $<$ defines a total order on $\mathcal{W}_{v,\delta}$ one may follow the proof of the analogous statement for homotopy words, which is given in lemma 2.1.27.

**Example 1.4.26.** Let $\Lambda$ be the complete gentle algebra from example 1.4.23. Here we have

$$\alpha^{-1}\beta^{-1}\alpha^{-1} > \alpha^{-1}\beta^{-1} > \alpha^{-1}\beta^{-1}\alpha > \alpha^{-1}\beta\alpha^{-1} > \alpha^{-1}\beta > \alpha^{-1}\beta\alpha^{-1} > \alpha^{-1}\beta$$

Note how an interval (say $[(\alpha\beta^{-1})^{-}, (\alpha\beta^{-1})^{+}]$) from example 1.4.23 corresponds to an inequality of words (say $\alpha\beta^{-1}\alpha < \alpha\beta^{-1}\alpha^{-1}$) above. Let us try to make this more precise using the terminology introduced in definition 1.4.3.

**Lemma 1.4.27.** [32, Chapter II, Lemma 1.1, Proposition 1.7] (Interval Avoidance) *Let $\delta \in \{\pm 1\}$ and $v$ be a vertex. For distinct $u, w \in \mathcal{W}_{v,\delta}$ the intervals $[u^{-}, u^{+}]$ and $[w^{-}, w^{+}]$ avoid each other.*

In Gabriel's exposition this result is [30, p. Proposition]. The result in Ringel's paper [55, p.23 Lemma], [55, p.24, Proposition] is slightly different. In Crawley-Boevey's paper it is [21, Lemma 6.2]. A proof of the analogue of this statement for homotopy words is proposition 2.1.30. Hence, we omit a proof. For convenience we now introduce what shall be referred to as *refined functors.*

**Definition 1.4.28.** Let $\mathsf{S}$ and $\mathsf{S}'$ be subfunctors of the forgetful functor ${}_{R}(-)$.

(Notation: $\mathsf{S} \cap \mathsf{S}'$, $\mathsf{S} + \mathsf{S}'$) The *meet* $\mathsf{S} \cap \mathsf{S}'$ and *join* $\mathsf{S} + \mathsf{S}'$ of $\mathsf{S}$ and $\mathsf{S}'$ are subfunctors of ${}_{R}(-)$ defined by setting $(\mathsf{S} \cap \mathsf{S}')(M) = \mathsf{S}(M) \cap \mathsf{S}'(M)$ and $(\mathsf{S} + \mathsf{S}')(M) = \mathsf{S}(M) + \mathsf{S}'(M)$ for each $\Lambda$-module $M$.

(Notation: $\mathsf{S}/\mathsf{S}'$) If $\mathsf{S} \leqslant \mathsf{S}'$ then the *quotient* $\mathsf{S}'/\mathsf{S}$ is a functor $\Lambda$-**Mod** $\to$ $R$-**Mod** defined by $(\mathsf{S}'/\mathsf{S})(M) = \mathsf{S}'(M)/\mathsf{S}(M)$ for each $\Lambda$-module $M$, and $(\mathsf{S}'/\mathsf{S})(f)(m + \mathsf{S}(M)) = f(m) + \mathsf{S}(N)$ for any $f \in \mathrm{Hom}_{\Lambda\text{-}\mathbf{Mod}}({}_{\Lambda}M, {}_{\Lambda}N)$ and $m \in \mathsf{S}'({}_{\Lambda}M)$. This map is well defined since $f(m) \in \mathsf{S}({}_{\Lambda}N)$ whenever $m \in \mathsf{S}({}_{\Lambda}M)$.

(Refined Functors for Modules) [15, p.162] (see also [21, §7] and [55, pp.24-25]) Let $w$ and $w'$ be (finite or $\mathbb{N}$)-words. The *refined functor* $F_{w,w'}$ given by the pair $(w, w')$ is defined as the quotient $F_{w,w'} = F^{+}_{w,w'}/F^{-}_{w,w'}$ : $\Lambda$-**Mod** $\to$ $R$-**Mod** where

$$F^{+}_{w,w'} = w^{+} \cap w'^{+} \text{ and } F^{-}_{w,w'} = (w^{+} \cap w'^{-}) + (w^{-} \cap w'^{+})$$

### 1.4.3 Reduction Lemma.

We now need some more theory about relations in general.

**Assumption:** In section 1.4.3 we assume $M$ is an $R$-module.

**Definition 1.4.29.** [21, Definitions 4.1 and 4.3] Let $V$ be a relation on $M$. For any integer $n > 0$ let $V^n$ be the $n$-fold composition of $V$ with itself (so $V^1 = V$ and $V^2 = VV$).

(Notation: $V^n$, $V'$, $V''$, $V^\flat$, $V^\sharp$) The $R$-submodules $V^\sharp = V'' \cap (V^{-1})''$ and $V^\flat = V'' \cap (V^{-1})' + V' \cap (V^{-1})''$ of $M$ are defined by setting

$$U' = \bigcup_{n>0} U^n 0, \quad U'' = \{m \in M \mid \exists m_0, m_1, m_2, \dots \in M \,:\, m_0 = m,\, m_i \in U m_{i+1} \;\; \forall i\}$$

for any relation $U$ on $M$.

**Lemma 1.4.30.** [21, Lemma 4.4] *For any relation $V$ on $M$ we have*

(i) $V^\sharp \subseteq VV^\sharp$, (ii) $V^\flat = V^\sharp \cap VV^\flat$, (iii) $V^\sharp \subseteq V^{-1}V^\sharp$, *and* (iv) $V^\flat = V^\sharp \cap V^{-1}V^\flat$.

*Proof.* (i) and (iii) For $z \in V^\sharp$ there are some elements $z_i \in M$ where $i$ runs through the integers and $z_0 = z$ for which $z_{i-1} \in Vz_i$ for each $i$. Let $z_1 = y_0$ and more generally $z_{i+1} = y_i$ for each $i$. By construction there are elements $y_i \in M$ for which $y_{i-1} \in Vy_i$ for each $i$, and $z_0 \in Vy_0$. (iii) now follows as a corollary since $(V^{-1})^\sharp = V^\sharp$.

(ii) and (iv) If $z \in V^\flat = (V'' \cap (V^{-1})') + (V' \cap (V^{-1})'')$ then there exists $z^+ \in V'' \cap (V^{-1})'$ and $z^- \in V' \cap (V^{-1})''$ for which $z = z^+ + z^-$. This means there exist $z_i^+, z_i^- \in M$ for each integer $i$ where $z_{i-1}^\pm \in Vz_i^\pm$, $z_0^\pm = z^\pm$ and there exist some $n_-, n_+ \in \mathbb{Z}$ for which $z_i^\pm = 0$ whenever $\pm i > \pm n_\pm$. Letting $y_i^\pm = z_{i+1}^\pm$ shows $y_{i-1}^\pm \in Vy_i^\pm$ for each $i$ and $y_i^\pm = 0$ whenever $\pm i > \pm(n_\pm - 1)$, so $y_0^+ \in V'' \cap (V^{-1})'$ and $y_0^- \in V' \cap (V^{-1})''$.

So $z^+ = z_0^+$ is an element of $Vy_0^+ \subseteq V(V'' \cap (V^{-1})')$ and similarly $z^- \in V((V^{-1})'' \cap V')$. Since $V' \cap (V^{-1})''$ and $V'' \cap (V^{-1})'$ define $R$-modules we have that $z \in VV^\flat$. This shows $V^\flat \subseteq VV^\flat \cap V^\sharp$. Now suppose $x \in VV^\flat \cap V^\sharp$ and $x \in Vz$ with $z \in V^\flat$ as above. We then have $x - z_{-1}^+ - z_{-1}^- \in V0 \cap V^\sharp$ and $z_{-1}^+, z_{-1}^- \in V^\flat$ so $x \in V^\flat$ as required. Again this gives (iv) as $(V^{-1})^\flat = V^\flat$. $\square$

As above, the proof in [21] generalises to a proof of the following lemma with no further complications. For this reason, the proof is omitted.

**Lemma 1.4.31.** [21, Lemma 4.5] (AUTOMORPHISM LEMMA) *If $V$ is a linear relation on $M$ then there is an $R$-module automorphism $\theta$ on $V^\sharp/V^\flat$ defined by $\theta(m+V^\flat)=m'+V^\flat$ iff $m'\in V^\sharp\cap(V^\flat+Vm)$.*

**Definition 1.4.32.** Let $V$ be a relation on $M$.

(REDUCTIONS) A *reduction of* $V$ is a pair $(U,f)$ such that $U$ is an $R[T,T^{-1}]$-module which is free over $R$, and $f:U\to M$ is an $R$-module map for which $V^\sharp=\mathrm{im}(f)+V^\flat$ and $f(Tu)\in Vf(u)$ for each $u\in U$.

(MEETS IN $\mathfrak{m}$) We say a reduction $(U,f)$ of $V$ *meets in* $\mathfrak{m}$ if the pre-image $f^{-1}(V^\flat)$ is contained in $\mathfrak{m}U$.

($R$-RANK) If $F$ is a free $R$-module with an $R$-basis consisting of $d\geq 0$ elements we will say $F$ *has free $R$-rank $d$*.

(SPLIT RELATIONS) [21, p.9] If $R$ is a field we say $V$ is *split* if there is an $R$-linear subspace $W$ of $M$ such that $V^\sharp=W\oplus V^\flat$ and $\#Vm\cap W=1$ for each $m\in W$.

**Corollary 1.4.33.** *If $R$ is a field and $(U,f)$ is a reduction of a relation $V$ on $M$ which meets in $\mathfrak{m}=0$, then $V$ is split.*

*Proof.* By definition $U$ is an $R[T,T^{-1}]$-module and $f:U\to M$ is an $R$-linear map for which $V^\sharp=\mathrm{im}(f)+V^\flat$ and $f(Tu)\in Vf(u)$ for each $u\in U$. Let $W=\mathrm{im}(f)$. For any $m\in W$ we have $m=f(u)$ for some $u\in U$. Since $U$ is an $R[T,T^{-1}]$ we have $Tu\in U$ which means $f(Tu)\in Vm\cap W$, and so $\#Vm\cap W\geq 1$. It remains to show $Vm\cap W\subseteq\{f(Tu)\}$.

Let $m'\in Vm\cap W$ be arbitrary. By definition $m'=f(u')\in W$ for some $u'\in U$, and so $(m,f(u')),(m,f(Tu))\in V$. This means $f(u'-Tu)\in V0\subseteq VV^\flat=(V^{-1})^{-1}(V^{-1})^\flat$, and since $f(u'-Tu)\in\mathrm{im}(f)\subseteq V^\sharp=(V^{-1})^\sharp$ we have $f(u'-Tu)\in(V^{-1})^\sharp\cap(V^{-1})^{-1}(V^{-1})^\flat$ and so $f(u'-Tu)\in(V^{-1})^\flat=V^\flat$ by lemma 1.4.30. This means $u'=Tu$ since $(U,f)$ is a reduction of $V$ which meets in $\mathfrak{m}=0$ (and $u'-Tu\in f^{-1}(V^\flat)$). This gives $m'=f(Tu)$ as required. $\square$

The lemma (1.4.34) below is a generalisation of [21, Lemma 4.6] (by corollary 1.4.33).

**Lemma 1.4.34.** (REDUCTION LEMMA) *Let $M$ be an $R$-module and $V$ a relation on $M$ such that $V^\sharp/V^\flat$ is a finite-dimensional $R/\mathfrak{m} = k$-vector space. Then there is a reduction $(U, f)$ of $V$ which meets in $\mathfrak{m}$ where $U$ has free $R$-rank $\dim_k(V^\sharp/V^\flat)$.*

In our setting $V^\flat$ need not have an complement as an $R$-submodule of $V^\sharp$.

**Example 1.4.35.** Let $p$ be an odd prime and let $R$ be the quotient ring $\mathbb{Z}/p^3\mathbb{Z}$, so $\mathfrak{m} = p\mathbb{Z}/p^3\mathbb{Z}$. Let $M$ be the quotient $F/K$ of the free module $F = Rz_0 \oplus Rz_1$ by the submodule $K = Rpz_1$. Let $g : M \to M$ be the $r$-module homomorphism defined by $g(r\overline{z}_0 + s\overline{z}_1) = r\overline{z}_1$ where $\overline{z}_i = z_i + K$ for each $i \in \{0, 1\}$.

Let $V = (\text{graph}(g))^{-1}\text{graph}(g)$ considered as an $R$-linear relation on $M$. By definition, for elements $m = r\overline{z}_0 + s\overline{z}_1$ and $m' = r'\overline{z}_0 + s'\overline{z}_1$ in $M$, $(m, m') \in V$ iff $g(m) = g(m')$ iff $(r - r')\overline{z}_1 = 0$ iff $r - r' \in \mathfrak{m}$. This means $V = V^{-1}$ and ($m \in Vm$ for any $m \in M$, and so) $V'' = M$, and so $V^\sharp = M$. For any $n > 0$ we have $r\overline{z}_0 + s\overline{z}_1 \in V^n 0$ iff $r \in \mathfrak{m}$, and so $V^\flat = Rp\overline{z}_0 + R\overline{z}_1$.

The automorphism $\theta$ of $V^\sharp/V^\flat$ from lemma 1.4.31 is defined by $\theta(r\overline{z}_0 + s\overline{z}_1 + V^\flat) = r'\overline{z}_0 + s'\overline{z}_1 + V^\flat$ iff $r - r' \in \mathfrak{m}$. Let $U$ be the submodule of $M$ generated by $\overline{z}_0$, and let $f$ be the inclusion of $U$ into $M$. Make $U$ a right $R[T, T^{-1}]$-module by setting $(\overline{z}_0)T = \overline{z}_0$. Note that $U = R\overline{z}_0 = Rz_0 + K/K \simeq Rz_0/Rz_0 \cap K \simeq Rz_0$ which is free over $R$ of rank 1.

Altogether this means $(U, f)$ is a reduction of $V$ which meets in $\mathfrak{m}$. Note that $V^\sharp/V^\flat \simeq \mathbb{F}_p = k$ and so $\dim_k(V^\sharp/V^\flat) = 1$, which verifies lemma 1.4.34 in this example. Note also that the exact sequence $0 \to V^\flat \to V^\sharp \to V^\sharp/V^\flat \to 0$ does not split.

We now check the proof [21, Lemma 4.6] works in our more general setting.

*Proof of lemma 1.4.34.* Let $\theta$ denote the induced $R$-module automorphism of $V^\sharp/V^\flat$ from lemma 1.4.31. Let $\overline{A} = (\overline{a_{ij}})$ be the matrix of $\theta$ (with entries from $k$) with respect to a $k$-basis $\overline{v_1}, \ldots, \overline{v_d}$ of $V^\sharp/V^\flat$. For each $i$ choose $v_i \in V^\sharp$ such that $\overline{v_i} = v_i + V^\flat$ and for each $j$ choose $a_{ij} \in R$ such that $\overline{a_{ij}} = a_{ij} + \mathfrak{m}$.

As $\overline{A} \in \mathrm{GL}_d(k)$, $\det(\overline{A}) \neq 0$ and so if we let $A$ be the matrix $(a_{ij}) \in \mathrm{M}_d(R)$ we have $\det(A) \notin \mathfrak{m}$. This means $\det(A)$ is a unit as $R$ is local, and hence $A \in \mathrm{GL}_d(R)$. Now fix $j \in \{1, \dots, d\}$.

We have $\theta(\overline{v_j}) = \sum_{i=1}^{d} a_{ij}v_i + V^{\flat}$ as $\mathfrak{m}V^{\sharp} \subseteq V^{\flat}$, and so by definition there is some $w_j \in Vv_j$ for which $\sum_{i=1}^{d} a_{ij}v_i - w_j \in V^{\flat}$. Let $z_j = w_j - \sum_{i=1}^{d} a_{ij}v_i$. Write $z_j = z_j^{+} + z_j^{-}$ for elements $z_j^{+} \in (V^{-1})' \cap V''$ and $z_j^{-} \in (V^{-1})'' \cap V'$. Hence there are some integers $n_{-}$ and $n_{+}$, and a collection $\{z_{j,n}^{\pm} \mid n \in \mathbb{Z}\} \subseteq M$ for which: $z_{j,n}^{\pm} \in Vz_{j,n+1}^{\pm}$ for each $n \in \mathbb{Z}$; $z_{j,n}^{-} = 0$ for each $n > n_{-}$; and $z_{j,n}^{+} = 0$ for each $n < n_{+}$. Now for each $n \in \mathbb{Z}$ define the matrices $L^{\pm,n}$ by

$$L^{+,n} = \begin{cases} 0 & \text{(if } n > 0) \\ (A^{-1})^{1-n} & \text{(otherwise)} \end{cases} \quad L^{-,n} = \begin{cases} -A^{n-1} & \text{(if } n > 0) \\ 0 & \text{(otheriwse)} \end{cases}$$

Write $L^{\pm,n} = (m_{ij}^{\pm,n})_{i,j}$ for elements $m_{ij}^{\pm,n} \in R$. Note that (if $m_{ij}^{+,n}z_{i,n}^{+} \neq 0$ then $n_{+} \leq n \leq 0$) and (if $m_{ij}^{-,n}z_{i,n}^{-} \neq 0$ then $n_{-} \geq n \leq 1$). This means the sum $\sum_{n\in\mathbb{Z}} \sum_{i=1}^{d} m_{ij}^{+,n}z_{i,n}^{+} + \sum_{n\in\mathbb{Z}} \sum_{i=1}^{d} m_{ij}^{-,n}z_{i,n}^{-}$ is finite. Since $V$ is an $R$-linear relation and $(z_{i,n}^{\pm}, z_{i,n-1}^{\pm}) \in V$ we have $(m_{ij}^{\pm,n}z_{i,n}^{\pm}, m_{ij}^{\pm,n}z_{i,n-1}^{\pm}) \in V$, and hence

$$\textstyle\sum_{n\in Z} \sum_{i=1}^{d} m_{ij}^{+,n}z_{i,n-1}^{+} + m_{ij}^{-,n}z_{i,n-1}^{-} \in V(\sum_{n\in\mathbb{Z}} \sum_{i=1}^{d} m_{ij}^{+,n}z_{i,n}^{+} + m_{ij}^{-,n}z_{i,n}^{-}) \quad (\star)$$

Letting $u_i = v_i + \sum_{n\in\mathbb{Z}}(\sum_{k=1}^{d} m_{ki}^{+,n}z_{k,n}^{+})) + \sum_{n\in\mathbb{Z}}(\sum_{k=1}^{d} m_{ki}^{-,n}z_{k,n}^{-})$ gives

$$\begin{aligned} \textstyle\sum_{i=1}^{d} a_{ij}u_i &= \textstyle\sum_{i=1}^{d} a_{i,j}(v_i + \sum_{n\in\mathbb{Z}}(\sum_{k=1}^{d} m_{ki}^{+,n}z_{k,n}^{+}) + \sum_{n\in\mathbb{Z}}(\sum_{k=1}^{d} m_{ki}^{-,n}z_{k,n}^{-})) \\ &= \textstyle\sum_{i=1}^{d} a_{ij}v_i + \sum_{n\in\mathbb{Z}} \sum_{k=1}^{d}(\sum_{i=1}^{d} a_{ij}m_{ki}^{+,n})z_{k,n}^{+} + \sum_{n\in\mathbb{Z}} \sum_{k=1}^{d}(\sum_{i=1}^{d} a_{ij}m_{ki}^{-,n})z_{k,n}^{-}. \end{aligned}$$

For $n \leq 0$ we have

$$\begin{aligned} &\textstyle(\sum_{i=1}^{d} a_{ij}m_{ki}^{+,n})z_{k,n}^{+} = (\sum_{i=1}^{d} m_{ki}^{+,n}a_{ij})z_{k,n}^{+} = (L^{+,n}A)_{k,j}z_{k,n}^{+} = ((A^{-1})^{1-n}A)_{k,j}z_{k,n}^{+} \\ &= ((A^{-1})^{-n})_{k,j}z_{k,n}^{+} = \begin{cases} ((L^{+,n+1})_{k,j})z_{k,n}^{+} = m_{k,j}^{+,n+1}z_{k,n}^{+} & \text{if } n < 0, \\ \delta_{k,j}z_{k,0}^{+} & \text{if } n = 0. \end{cases} \qquad (\star\star) \end{aligned}$$

For $n > 0$ we have

$$(\sum_{i=1}^d a_{ij} m_{ki}^{-,n}) z_{k,n}^- = (\sum_{i=1}^d m_{ki}^{-,n} a_{ij}) z_{k,n}^- = (L^{-,n}A)_{k,j} z_{k,n}^- = (-A^{n-1}A)_{k,j} z_{k,n}^-$$
$$= (-A^n)_{k,j} z_{k,n}^- = ((L^{-,n+1})_{k,j}) z_{k,n}^- = m_{k,j}^{-,n+1} z_{k,n}^- \quad (\star\star\star)$$

Combining $(\star\star)$ and $(\star\star\star)$ together with the definition of $L^{\pm,n}$ gives

$$\sum_{i=1}^d a_{ij} u_i = \sum_{i=1}^d a_{ij} v_i + \sum_{n\in\mathbb{Z}} \sum_{k=1}^d (\sum_{i=1}^d a_{ij} m_{ki}^{+,n}) z_{k,n}^+ + \sum_{n\in\mathbb{Z}} \sum_{k=1}^d (\sum_{i=1}^d a_{ij} m_{ki}^{-,n}) z_{k,n}^-$$
$$= \sum_{i=1}^d a_{ij} v_i + z_{j,0}^+ + \sum_{n\in\mathbb{Z}\,:\,n<0} \sum_{k=1}^d m_{k,j}^{+,n+1} z_{k,n}^+ + \sum_{n\in\mathbb{Z}\,:\,n>0} \sum_{k=1}^d m_{k,j}^{-,n+1} z_{k,n}^-$$

Note that

$$\sum_{n\in\mathbb{Z}\,:\,n>0} \sum_{k=1}^d m_{k,j}^{-,n+1} z_{k,n}^- = \sum_{n\in\mathbb{Z}\,:\,n>0} \sum_{k=1}^d m_{k,j}^{-,n+1} z_{k,n}^- - z_{j,0}^- + z_{j,0}^-$$
$$= z_{j,0}^- + \sum_{n\in\mathbb{Z}\,:\,n>0} (\sum_{k=1}^d m_{k,j}^{-,n+1} z_{k,n}^- - z_{j,0}^-)$$
$$= z_{j,0}^- + \sum_{n\in\mathbb{Z}\,:\,n>0} (\sum_{k=1}^d m_{k,j}^{-,n+1} z_{k,n}^- + \sum_{i=1}^d m_{k,j}^{-,0} z_{j,0}^-)$$
$$= z_{j,0}^- + \sum_{n\in\mathbb{Z}\,:\,n>0} (\sum_{k=1}^d m_{k,j}^{-,n} z_{k,n-1}^-)$$

as $L^{-,0} = -I_d$, and so altogether

$$\sum_{i=1}^d a_{ij} u_i = \sum_{i=1}^d a_{ij} v_i + z_j + \sum_{n\in\mathbb{Z}} \sum_{k=1}^d m_{k,j}^{+,n} z_{k,n-1}^+ + \sum_{n\in\mathbb{Z}} \sum_{k=1}^d m_{k,j}^{-,n} z_{k,n-1}^-$$
$$\in V v_j + V(\sum_{n\in\mathbb{Z}} \sum_{i=1}^d m_{i,j}^{+,n} z_{i,n}^+ + \sum_{n\in\mathbb{Z}} \sum_{i=1}^d m_{i,j}^{-,n} z_{i,n}^-) \text{ by } (\star)$$
$$\subseteq V(v_j + \sum_{n\in\mathbb{Z}} \sum_{i=1}^d m_{i,j}^{+,n} z_{i,n}^+ + \sum_{n\in\mathbb{Z}} \sum_{i=1}^d m_{i,j}^{-,n} z_{i,n}^-)$$

So $\sum_{j=1}^n a_{ij} u_i \in V u_j$. Let $U = \bigoplus_{i=1}^d R$ and define the action on U by $T(r_i)_i = (\sum_{j=1}^d a_{ij} r_j)_i$, multiplication by the matrix $A = (a_{ij})$. Define $f$ by $f((r_i)) = \sum_{t=1}^d r_t u_t$.

Since $V^\sharp / V^\flat$ has $k$-basis $\overline{v_1}, \ldots, \overline{v_d}$ for any $m \in V^\sharp$ there are elements $s_1, \ldots, s_n \in R$ such that writing $\overline{s_i} = s_i + \mathfrak{m}$ for each $i$ gives $m + V^\flat = \sum_{i=1}^d \overline{s_i}(v_i + V^\flat)$ which equals $\sum_{i=1}^d s_i u_i + V^\flat$.

There is an element $x = \sum_{i=1}^d s_i u_j = f((s_j)) \in \mathrm{im}(f)$ and an element $c \in V^\flat$ with $m - t = c$ and thus $m = t + c$. This shows $V^\sharp \subseteq \mathrm{im}(f) + V^\flat$ and as $u_i \in V^\sharp$ for each $i$ this inclusion is an equality.

Since $f$ maps into $V^\sharp$ and $\mathfrak{m}V^\sharp \subseteq V^\flat$ we have $\mathfrak{m}U \subseteq \{u \in U : f(u) \in V^\flat\}$. Conversely if $f(u) \in V^\flat$ where $f(u) = \sum_{t=1}^d r_t u_t$ then $0 = \sum_{t=1}^d \overline{r_t u_t} = \sum_{t=1}^d \overline{r_t v_t}$ and as $\overline{v_1}, \ldots, \overline{v_d}$ was an $R/\mathfrak{m} = k$-basis for $V^\sharp/V^\flat$, which means $r_i + \mathfrak{m} = 0$ in $k$ (and hence $r_i \in \mathfrak{m}$) for each $i$. Hence $\mathfrak{m}U \supseteq \{u \in U : f(u) \in V^\flat\}$. Now fix $u = (r_i) \in U$. By definition we have $Tu = (\sum_{j=1}^d a_{ij} r_j)_i$ and so $f(Tu) = \sum_{j=1}^d r_j \sum_{t=1}^d a_{tj} u_t$ which is an element of $\sum_{j=1}^d r_j V u_j \subseteq V f(u)$, as required. $\square$

### 1.4.4 The Structure Theorem.

So far we have developed the language of linear relations, and explained how such relations may be defined using words. We are now ready to explain what the functorial filtration method is, how it works and how it has been used in the past. Since section 1.4.4 is essentially a literature review, we now restrict our focus.

**Assumption:** In section 1.4.4 we let $R$ be a field $k$, and $\Lambda$ be a quasi-bounded string algebra over $k$ (equivalently, $\Lambda$ is a Butler-Ringel string algebra, by lemma 1.2.3).

We now highlight the significance of example 1.4.12 in the context of the functorial filtration method. In [32] Gel'fand and Ponomarev pose and solve the following.

**Problem 1.4.36.** [32, p.26] Given a field $k$, $P_1$ and $P_2$ two finite dimensional $k$-vector spaces, and $d_- : P_1 \to P_2$, $d_+ : P_2 \to P_1$ and $\delta : P_2 \to P_2$ three $k$-linear maps such that $d_-\delta = 0$, $\delta d_+ = 0$, and the endomorphisms $d_+d_-$ and $\delta$ of $P_1$ and $P_2$ are nilpotent: Classify the canonical form of the matrices of $d_-$, $d_+$ and $\delta$ with respect given bases of $P_1$ and $P_2$. Hence the problem is to classify the finite dimensional representations of the quiver

$$\begin{tikzcd} 2 \arrow[r, shift left, "s_+"] & 1 \arrow[l, shift left, "s_-"] \arrow[loop right, "\sigma"] \end{tikzcd}$$

subject to relations $s_-\sigma = 0$, $\sigma s_+ = 0$, $(s_+s_-)^n = 0$ and $\sigma^m = 0$ for some $n, m > 0$. Writing $P = P_1 \oplus P_2$ we may introduce the $k$-vector space endomorphisms

$$a = \begin{pmatrix} 0 & s_+ \\ s_- & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}$$

and the above conditions on $d_\pm$ and $\delta$ correspond to $ab = ba = 0$ and $a^n = b^m = 0$. Correspondingly it suffices to classify the indecomposable objects in $\Lambda$-**mod** where $\Lambda$ is the $k$-algebra from example 1.4.12.

Our aim in section 1.4.4 is to explain the following theorem (which is due to Butler and Ringel) using example 1.4.12. Consider the functor $\mathrm{res}_\iota : k[T, T^{-1}]$-**Mod** $\to$ $k[T, T^{-1}]$-**Mod** which swaps the action of $T$ and $T^{-1}$ (see definition 2.0.3 for details).

Note that a string module $M(w)$ (resp. a band module $M(w, V)$) is finite-dimensional iff $w$ is finite (resp. $V$ is finite-dimensional).

**Theorem 1.4.37.** [15, p.161, Theorem] *The following statements hold.*

(i) *Any object in* $\Lambda$-**mod** *is a direct sum of string and band modules.*

(ii) *Any indecomposable object in* $\Lambda$-**mod** *is isomorphic to a string or band module.*

(iii) *Every finite-dimensional string or band module is indecomposable.*

(iva) *There cannot exist an isomorphism between a finite-dimensional string module $M(w)$ and a finite-dimensional band module $M(w, V)$.*

(ivb) *Finite dimensional string modules $M(w)$ and $M(w')$ are isomorphic iff $w' = w^{\pm 1}$.*

(ivc) *Finite dimensional band modules $M(w, V)$ and $M(w', V')$ are isomorphic iff*

$$(w' = w[t] \text{ and } V \simeq V') \text{ or } (w' = w^{-1}[t] \text{ and } V \simeq \mathrm{res}_\iota(V'))$$

For the proof of this theorem, Butler and Ringel use [15, p.163, Proposition]. This result was based on the following lemma due to Gabriel. Our presentation follows Ringel's interpretation [55, §3, Lemma].

**Lemma 1.4.38.** [30, §4, Structure theorem] *Let $\mathfrak{N}$ be an abelian category and let $J$ be an index set. For each $j \in J$ let: $\mathfrak{C}_j$ be an abelian category; and let $T_j : \mathfrak{C}_j \to \mathfrak{N}$ and $G_j : \mathfrak{N} \to \mathfrak{C}_j$ be additive functors.*

*Suppose that:*

(a) $G_j T_j \simeq \mathrm{id}$ *for each $j \in J$ and $G_l T_j = 0$ for each $j, l \in J$ with $l \neq j$;*

(b) *for every object $M$ in $\mathfrak{N}$ we have $G_j(M) = 0$ for all but finitely many $j$;*

(c) *if $\alpha : \bigoplus_j T_j(G_j(V_j)) \to M$ is an arrow in $\mathfrak{N}$ (for some $V_j$ and $M$) such that $G_j(\alpha)$ is an isomorphism for each $j$, then $\alpha$ is an isomorphism; and*

(d) *for every object $M$ in $\mathfrak{N}$ and every $j \in J$ there is an arrow $\kappa_{j,M} : T_j(G_j(M)) \to M$ such that $G_j(\kappa_{j,M})$ is an isomorphism.*

*Then the following statements hold.*

(i) *Any object in* $\mathfrak{N}$ *is isomorphic to* $\bigoplus_j T_j(G_j(M))$ (see [15, p.163, Proposition (iv)]).

(ii) *Any indecomposables in* $\mathfrak{N}$ *has the form* $T_j(C)$ *with* $C$ *indecomposable in* $\mathfrak{C}_j$.

(iii) *If* $C$ *is indecomposable in* $\mathfrak{C}_j$ *then* $T_j(C)$ *is indecomposable in* $\mathfrak{N}$.

(iv) $T_j(C)$ *and* $T_l(D)$ *are isomorphic iff* $j = l$ *and* $C \simeq D$ *in* $\mathfrak{C}_j$.

We have omitted the proof of lemma 1.4.38. This is because later in the thesis (see definition 2.6.4 and lemma 2.6.5) we have adapted this result for our own purposes, and in our proof (of lemma 2.6.5) we closely follow Ringel's proof of [55, §3, Lemma].

In the remainder of section 1.4.4 we sketch a proof of theorem 1.4.37. We start by defining: a category $\mathfrak{N}$; an index set $J$; categories $\mathfrak{C}_j$ for each $j \in J$; and functors $G_j : \mathfrak{N} \to \mathfrak{C}_j$ and $T_j : \mathfrak{C}_j \to \mathfrak{N}$ for each $j \in J$. We then explain why parts (a), (b), (c) and (d) from lemma 1.4.38 hold in this notation. We start by defining $\mathfrak{N}$, $J$ and $\mathfrak{C}_j$.

**Definition 1.4.39.** (NOTATION: $\mathfrak{N}$) Let $\mathfrak{N} = \Lambda$-**mod**, the full subcategory of $\Lambda$-**Mod** consisting of finitely generated modules. By lemma 3.1.34 (id) any finitely generated $\Lambda$-modules is finite-dimensional over $k$ (note $\Lambda e_v$ is finite-dimensional for any vertex $v$ because $\Lambda$ is a Butler-Ringel string algebra). Hence $\mathfrak{N}$ is the abelian subcategory of $\Lambda$-**Mod** consisting of all finite-dimensional modules.

(NOTATION: $J$) Recall corollary 1.3.13. There is an equivalence relation $\sim$ on the subset $\Delta$ of pairs $(w, w') \in \bigsqcup_v \mathcal{W}_{v,1} \times \mathcal{W}_{v,-1}$ such that $w^{-1}w'$ is a word, given by $(w, w') \sim (u, u')$ iff $u^{-1}u' = w^{-1}w'[t]$ or $u^{-1}u' = (w')^{-1}w[t]$ for some integer $t$. Define $J \subseteq \Delta$ by choosing a representative $(w, w')$, one for each equivalence class $\overline{(w, w')} \in \Delta/\sim$ such that $w^{-1}w'$ is finite or a periodic $\mathbb{Z}$-word.

(NOTATION: $\mathfrak{C}_j$) Fix $j \in J$, say $j = (w, w')$. If $w^{-1}w'$ is finite let $\mathfrak{C}_j = k$-**mod** (the full subcategory of $k$-**Mod** consisting of finite-dimensional vector spaces). If $w^{-1}w'$ is a periodic $\mathbb{Z}$-word let $\mathfrak{C}_j = k[T, T^{-1}]$-$\mathbf{Mod}_{k\text{-}\mathbf{mod}}$, the full subcategory of $k[T, T^{-1}]$-**Mod** consisting of modules which are finite-dimensional over $k$.

**Remark 1.4.40.** ($F_{w,w'}:\Lambda$-**Mod** $\to k[T,T^{-1}]$-**Mod** FOR $w^{-1}w'$ PERIODIC) Let $w$ and $w'$ be words such that $w^{-1}w'$ is a periodic $\mathbb{Z}$-word. This means $w={}^{\infty}u^{\infty}$ where $u$ is a primitive cyclic $\{0,\dots,p\}$-word for some $p>0$.

Consider the relation $\mathrm{rel}^u(M)$ on $e_{h(u)}M$ given by the word $u$, and hence recall the notation $(\mathrm{rel}^u(M))''$ and $(\mathrm{rel}^u(M))'$ from definition 1.4.29. As in [21, §7] we have

$$w^+(M)=\mathrm{im}\mathcal{R}^w_\infty(M)=((\mathrm{rel}^u(M))^{-1})'',\ w^-(M)=\mathrm{ind}\mathcal{R}^w_\infty(M)=((\mathrm{rel}^u(M))^{-1})',$$
$$w'^+(M)=\mathrm{im}\mathcal{R}^{w'}_\infty(M)=(\mathrm{rel}^u(M))'' \text{ and } w'^-(M)=\mathrm{ind}\mathcal{R}^{w'}_\infty(M)=(\mathrm{rel}^u(M))'.$$

Recall the refined functor $F_{w,w'}:\Lambda$-**Mod** $\to k$-**Mod** from definition 1.4.28. The above, together with lemma 1.4.31, shows that $F_{w,w'}(M)$ is a $k[T,T^{-1}]$-module. So in this case we can (and from now on, we shall) consider $F_{w,w'}$ as a functor taking $\Lambda$-**Mod** to $k[T,T^{-1}]$-**Mod**.

In lemma 2.2.11 and corollary 2.2.12 we prove statements analogous to those in remark 1.4.40, for the case of homotopy words (instead of words).

**Definition 1.4.41.** (CONSTRUCTIVE FUNCTORS FOR MODULES) Let $w$ and $w'$ be words such that $w^{-1}w'$ is an $I$-word for some subset $I\subset\mathbb{Z}$.

If $w^{-1}w'$ is not a periodic $\mathbb{Z}$-word define the functor $S_{w,w'}:k$-**Mod** $\to\Lambda$-**Mod** on objects by $S_{w,w'}(V)=M(w^{-1}w')\otimes_k V$. For a $k$-linear map $f:V\to V'$ and bases $\{v_\lambda\mid\lambda\in\Omega\}$ for $V$ and $\{v'_{\lambda'}\mid\lambda'\in\Omega'\}$ for $V'$, write $f(v_\lambda)=\sum a_{\lambda',\lambda}v_{\lambda'}$ for scalars $a_{\lambda',\lambda}\in k$ for each $\lambda,\lambda'\in\Omega$. Define $S_{w,w'}$ on morphisms by extending the assignment $S_{w,w'}(f)(b_i\otimes v_\lambda)=\sum_{\lambda'}a_{\lambda',\lambda}b_i\otimes v'_{\lambda'}$ (for each $i\in I$ and $\lambda\in\Omega$) linearly over $\Lambda$.

If $w^{-1}w'$ is a periodic $\mathbb{Z}$-word $V$ and $V'$ have the structure of left $k[T,T^{-1}]$-modules, and the $k$-linear map $f:V\to V'$ above is also $k[T,T^{-1}]$-linear. Hence $T$ defines automorphisms $\varphi_V:V\to V$ and $\varphi_{V'}:V'\to V'$ satisfying $f\varphi_V=\varphi_{V'}f$. Define the functor $S_{w,w'}$ : $k[T,T^{-1}]$-**Mod** $\to\Lambda$-**Mod** on objects by $S_{w,w'}(V)=M(u,V)$.

The formula $S_{w,w'}(f)(b_i\otimes v_\lambda)=\sum_{\lambda'\in\Omega'}a_{\lambda',\lambda}b_i\otimes v'_{\lambda'}$ gives $S_{w,w'}(f)(b_{i-p}\otimes v_\lambda)=T(\sum_{\lambda'}a_{\lambda',\lambda}b_i\otimes v'_{\lambda'})$ and consequently $S_{w,w'}(f)(Tb_i\otimes v_\lambda)=T(S_{w,w'}(f)(b_i\otimes v_\lambda))$.

Hence if $w^{-1}w'$ is a periodic $\mathbb{Z}$-word then $S_{w,w'}(f)$ defines a $\Lambda$-module homomorphism $M(w) \otimes_{k[T,T^{-1}]} V \to M(w) \otimes_{k[T,T^{-1}]} V'$.

**Remark 1.4.42.** Let $j \in J$, and so $j = (w, w')$ where $w$ and $w'$ are words such that $w^{-1}w'$ is a $\{0, \dots, n\}$-word (for some $n \geq 0$) or a periodic $\mathbb{Z}$-word.

If $V$ is an object in $\mathfrak{C}_j$ with (finite) basis $\{v_\lambda \mid \lambda \in \Omega\}$ then the $\Lambda$-module $S_{w,w'}(V)$ is generated by the elements $b_i \otimes v_\lambda$ where $\lambda$ runs through $\Omega$ and $i$ runs through the integers with ($0 \leq i \leq n$ if $w^{-1}w'$ is finite) and ($0 \leq i \leq p-1$ if $w^{-1}w'$ is periodic of period $p$). Hence if $(w, w') \in J$ then $S_{w,w'}(V)$ is an object in $\Lambda$-**mod** if $V$ is an object in $\mathfrak{C}_j$.

Since $F^{+}_{w,w'}(M) \subseteq e_v M$ where $v$ is the head of $w$ and $w'$, when $M$ is finite-dimensional so is $F_{w,w'}(M)$. Hence if $(w, w') \in J$ then $F_{w,w'}(M)$ is an object in $\mathfrak{C}_j$ if $M$ is an object in $\Lambda$-**mod**.

In corollary 2.2.12 we state and prove a result analogous to remark 1.4.42 for the case of homotopy words. We can now defined the functors $G_j$ and $T_j$.

**Definition 1.4.43.** (NOTATION: $G_j$, $T_j$) Fix $j \in J$, say $j = (w, w')$.

If $w^{-1}w'$ is finite let $G_j$ be the restriction of $F_{w,w'} : \Lambda\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ to $\mathfrak{N}$, which (by the above) defines a functor $\mathfrak{N} \to \mathfrak{C}_j$. Let $T_j$ be the restriction of $S_{w,w'} : k\text{-}\mathbf{Mod} \to \Lambda\text{-}\mathbf{Mod}$ to $\mathfrak{C}_j$, which again defines a functor $\mathfrak{C}_j \to \mathfrak{N}$.

If $w^{-1}w'$ is a periodic $\mathbb{Z}$-word let $G_j$ be the restriction of $F_{w,w'} : \Lambda\text{-}\mathbf{Mod} \to k[T, T^{-1}]\text{-}\mathbf{Mod}$ to $\mathfrak{N}$, and let $T_j$ be the restriction of $S_{w,w'} : k[T, T^{-1}]\text{-}\mathbf{Mod} \to \Lambda\text{-}\mathbf{Mod}$ to $\mathfrak{C}_j$. Again $G_j$ and $T_j$ define functors $\mathfrak{N} \to \mathfrak{C}_j$ and $\mathfrak{C}_j \to \mathfrak{N}$ respectively.

**Lemma 1.4.44.** [21, Lemma 7.1] *If $w \in \mathcal{W}_{v,1}$ and $w' \in \mathcal{W}_{v,-1}$ then*

(i) *if $w^{-1}w'$ is not a word then $F_{w,w'} \simeq 0$,*

(ii) *if $w^{-1}w'$ is a word which is not a periodic $\mathbb{Z}$-word then $F_{w,w'} \simeq F_{w',w}$,*

(iii) *if $w^{-1}w'$ is a word which is a periodic $\mathbb{Z}$-word then $F_{w,w'} \simeq \mathrm{res}_\iota F_{w',w}$, and*

(iv) *if $(u, u') \in \mathcal{W}_{v,1} \times \mathcal{W}_{v,-1}$ with $u^{-1}u' = w^{-1}w'[t]$ for some $t$ then $F_{w,w'} \simeq F_{u,u'}$.*

There is a corresponding result for the functors $S_{w,w'}$. Note however that the functors $S_{w,w'}$ have only been defined for $(w, w') \in \Delta$, the set of such pairs where $w^{-1}w$ a word.

**Lemma 1.4.45.** [21, p.3] *If $(w, w') \in \Delta$, then*

(i) *if $w^{-1}w'$ is not a periodic $\mathbb{Z}$-word then $S_{w,w'} \simeq S_{w',w}$,*

(ii) *if $w^{-1}w'$ is a periodic $\mathbb{Z}$-word then $S_{w,w'} \simeq S_{w',w}\mathrm{res}_\iota$, and*

(iii) *if $(u, u') \in \Delta$ with $u^{-1}u' = w^{-1}w'[t]$ for some $t$ then $S_{w,w'} \simeq S_{u,u'}$.*

**Remark 1.4.46.** In definitions 1.4.39 and 1.4.43 we have defined the index set $J$, the categories $\mathfrak{N}$ and $\mathfrak{C}_j$ and the functors $G_j$ and $T_j$. In this notation: parts (i), (ii), (iii) and (iv) of lemma 1.4.38 are precisely parts (i), (ii), (iii) and ((iva), (ivb) and (ivc)) of theorem 1.4.37.

In the remainder of section 1.4.4 we sketch the proof of theorem 1.4.37. To do this we explain why parts (a), (b), (c) and (d) from lemma 1.4.38 hold. Before we begin we justify why our verification (of parts (a), (b), (c) and (d) from lemma 1.4.38) does not depend on the choice of $J$.

Choose another index set $J' \subset \Delta$ of representatives $(w, w')$, one for each equivalence class $\overline{(w, w')} \in \Delta/\sim$ such that $w^{-1}w'$ is finite or a periodic $\mathbb{Z}$-word. For each $j' = (u, u') \in J'$ if $u^{-1}u'$ is finite (resp. periodic) let $\mathfrak{C}'_{j'}$ be $k$-**mod** (resp. $k[T, T^{-1}]$-$\mathbf{Mod}_{k\text{-}\mathbf{mod}}$) (as in definition 1.4.39), let $G'_{j'}$ be the restriction of $F_{u,u'}$ to the category $\mathfrak{N}$, and let $T'_{j'}$ be the restriction of $S_{u,u'}$ to the category $\mathfrak{C}'_{j'}$ (as in definition 1.4.43).

By lemmas 1.4.44 and 1.4.45 for any $j' \in J'$ with $j \sim j'$ for some $j \in J$ we have ($G_j \simeq G'_{j'}$ and $T_j \simeq T'_{j'}$) or ($G_j \simeq \mathrm{res}_\iota G'_{j'}$ and $T_j \simeq T'_{j'}\mathrm{res}_\iota$). Since $(\mathrm{res}_\iota)^{-1} = \mathrm{res}_\iota$, if the choice of $J$ and $\{\mathfrak{C}_j, G_j, T_j \mid j \in J\}$ verify parts (a), (b), (c) and (d) from lemma 1.4.38, then so do the choice of $J'$ and $\{\mathfrak{C}'_{j'}, G'_{j'}, T'_{j'} \mid j' \in J'\}$.

**Definition 1.4.47.** Let $v$ be a vertex, $\delta \in \{\pm 1\}$, $u \in \mathcal{W}_{v,\delta}$ and $w$ be an $I$-word. Recall the notation from definition 1.3.10: for $i \in I$ we have $w_{\leq i} = \dots w_i$ (resp. $w_{>i} = w_{i+1} \dots$) given $i - 1 \in I$ (resp. $i + 1 \in I$), and otherwise $w_{\leq i} = 1_{h(w),s(w)}$ (resp. $w_{>i} = 1_{h(w^{-1}),s(w^{-1})}$).

The following notation for the *truncated subwords* $w(i, \delta)$ of $w$ came from [21, §8].

Recall the ordering on words from definition 1.4.24, and the corresponding avoidance of intervals (of subfunctors of the forgetful functor) from lemma 1.4.27.

(Notation: $w(i,\delta)$) For any $i \in I$ the words $w_{>i}$ and $(w_{\leq i})^{-1}$ have head $v_i(w)$ and opposite signs. Let $w(i,\delta)$ be the one with sign $\delta$.

(Notation: $(I,u,\pm)$) If $w$ is not a periodic $\mathbb{Z}$-word let $(I,u,+)$ (resp. $(I,u,-)$) be the set of $i \in I$ such that $v_i(w) = v$ and $w(i,\delta) \leq u$ (resp. $w(i,\delta) < u$). If $w$ is a periodic $\mathbb{Z}$-word of period $p > 0$ let $(I,u,+)$ (resp. $(I,u,-)$) be the set of integers $i$ with $0 \leq i \leq p-1$ such that $v_i(w) = v$ and $w(i,\delta) \leq u$ (resp. $w(i,\delta) < u$).

We now collect together some results from [21]. Recall definition 1.4.32.

**Lemma 1.4.48.** *Let $w$ be an $I$-word and let $u = w(i,1)$ and $u' = w(i,-1)$ for some $i \in I$. Let $V$ be a $k$-vector space with $k$-basis $\{v_\lambda \mid \lambda \in \Omega\}$, and suppose that $V$ is a $k[T,T^{-1}]$-module if $w$ is periodic.*

(i) [21, Lemmas 8.1 and 8.4] *If $v$ is a vertex, $\delta \in \{\pm 1\}$ and $w' \in \mathcal{W}_{v,\delta}$ then $\{b_i \otimes v_\lambda \mid i \in (I,u,\pm), \lambda \in \Omega\}$ is a $k$-basis of ${w'}^{\pm}(S_{u,u'}(V))$.*

(ii) [21, Lemmas 8.2 and 8.5] $F^{+}_{u,u'}(S_{u,u'}(V)) = F^{-}_{u,u'}(S_{u,u'}(V)) \oplus \bigoplus_{\lambda\in\Omega} k(b_i \otimes v_\lambda)$, *and* $F^{+}_{v,v'}(S_{u,u'}(V)) = F^{-}_{v,v'}(S_{u,u'}(V))$ *for all $(v,v') \in \mathcal{W}_{v,1} \times \mathcal{W}_{v,-1}$ with $(u,u') \nsim (v,v')$.*

(iii) [21, Lemmas 8.3 and 8.6] *Let $M$ be a $\Lambda$-module. If ($w$ is not periodic) or ($u = {w'}^{\infty}$, $u' = ({w'}^{-1})^{\infty}$ and $\mathrm{rel}^{w'}(F_{u,u'}(M))$ is split for a primitive cyclic word $w'$): then there is a homomorphism $\kappa_{u,u',M} : S_{u,u'}(F_{u,u'}(M)) \to M$ such that $F_{u,u'}(\kappa_{u,u',M})$ is an isomoprhism.*

In chapter 2 we adapt the statement and proofs of lemma 1.4.48 (ii) (see part (ii) in lemmas 2.3.20 and 2.3.21).

**Lemma 1.4.49.** [21, Lemma 10.5], see also [55, §6] *Let $M$ be a finite-dimensional $\Lambda$-module, and let $v$ be a vertex. Suppose $U$ is a subspace of $e_vM$ where $m \notin U$ for some $m \in e_vM$. Then there is a pair $(u,u')$ of words such that: $u^{-1}u'$ is a word;*

$$(U+m) \cap (u^{-}(M) + {u'}^{+}(M) \cap u^{+}(M)) \neq \emptyset; \text{ and}$$
$$(U+m) \cap (u^{-}(M) + {u'}^{-}(M) \cap u^{+}(M)) = \emptyset.$$

**Lemma 1.4.50.** *Let $\theta : N \to M$ be a $\Lambda$-module homomorphism such that $F_{w,w'}(\theta)$ is an isomorphism for each $j = (w, w') \in J$.*

(i) [21, Lemma 9.4] *If $N$ is a direct sum of string and band modules then $\theta$ is injective.*

(ii) [21, Lemma 10.6 (i)] *If $M$ is finite-dimensional then $\theta$ is surjective.*

The proof of lemma 1.4.48 (ii) uses lemma 1.4.48 (i), and the proof of lemma 1.4.50 (ii) uses lemma 1.4.49. Using these results, we now verify parts (a), (b), (c) and (d) from lemma 1.4.38.

*Verification of lemma 1.4.38 (a).* Let $j \in J$, say $j = (u, u')$ with $u \in \mathcal{W}_{v,1}$ and $u' \in \mathcal{W}_{v,-1}$ for some vertex $v$. Consider the $k$-linear maps $\theta_V : F_{u,u'}(S_{u,u'}(V)) \to V$ defined by

$$\theta_V(\textstyle\sum_{i\in I,\, \lambda\in\Omega} r_{i,\lambda}(b_i \otimes v_\lambda) + F^-_{u,u'}(S_{u,u'}(V))) = \textstyle\sum_{\lambda\in\Omega} r_{t,\lambda} v_\lambda$$

where $r_{i,\lambda} \in k$, $\{v_\lambda \mid \lambda \in \Omega\}$ is a $k$-basis for $V$ and $t \in I$ satisfies $u = w(t, 1)$ and $u' = w(t, -1)$ where $w = u^{-1}u'$. Note that when $u = w(s, 1)$ and $u' = w(s, -1)$ for some $s \in I$, if ($w$ isn't periodic, or $w$ is periodic of period $p > |s - t|$) then $s = t$. By the first part of lemma 1.4.48 (ii) the map $\theta_V$ is an isomorphism of vector spaces. It is straightforward to show $\theta_V$ is $k[T, T^{-1}]$-linear if $w$ is periodic. We omit the proof that $\theta : G_j S_j \to \mathrm{id}$ defines a natural isomorphism. For details see the proof of [55, p.26, second Lemma] (which was adapted to lemma 2.3.19 (i) in chapter 2). □

*Verification of lemma 1.4.38 (b).* Suppose $M$ is a finite-dimensional $\Lambda$-module, say with basis $m_1, \dots, m_n$. For each integer $t$ with $1 \le t$ there is a vertex $v(t)$ for which $e_{v(t)} m_t \neq 0$, and so by lemma 1.4.49 for each such $t$ there is a pair $(u_t, u'_t) \in \mathcal{W}_{v(t),1} \times \mathcal{W}_{v(t),-1}$ of words such that: $u_t^{-1} u'_t$ is a word; $m_t \in (u_t^-(M) + {u'_t}^+(M) \cap u_t^+(M))$, and $m_t \notin (u_t^-(M) + {u'_t}^-(M) \cap u_t^+(M))$. Order pairs $(u, u')$ of words from $\mathcal{W}_{v,1} \times \mathcal{W}_{v,-1}$ lexicographically, by setting $(u, u') < (w, w')$ if $u < w$ or ($u = w$ and $u' < w'$). Let $(u, u')$ be a pair from $J$ with $(u, u') \neq (u_t, u'_t)$ for all $t$. Using lemma 1.4.27 one can show $F_{u,u'}(M) = 0$, and so $F_{w,w'}(M) \neq 0$ implies $(w, w') = (u_t, u'_t)$ for some $t$ (that is, $F_{w,w'}(M) = 0$ for all but finitely many $j = (w, w') \in J$). □

*Verification of lemma 1.4.38 (c).* Suppose $\theta : N \to M$ is a homomorphism of finite-dimensional $\Lambda$-modules where $N$ is a direct sum of string and band modules. If $F_{w,w'}(\theta)$ is an isomorphism for each pair $(w, w') \in J$, then $\theta$ is injective and surjective by lemma 1.4.50. $\square$

*Verification of lemma 1.4.38 (d).* Let $M$ be a finite dimensional module and let $j \in J$, say $j = (u, u')$ for words $u$ and $u'$. If $u = w'^{\infty}$ and $u' = (w'^{-1})^{\infty}$ are words for some primitive cyclic word $w'$ then $F_{u,u'}(M) = (\mathrm{rel}^{w'}(F_{u,u'}(M)))^{\sharp}/(\mathrm{rel}^{w'}(F_{u,u'}(M)))^{\flat}$ is finite dimensional, and so by lemma 1.4.34 and corollary 1.4.33 the relation $\mathrm{rel}^{w'}(F_{u,u'}(M))$ on $F_{u,u'}(M)$ is split. By lemma 1.4.48 (iii) this shows there is a homomorphism $\kappa_{u,u',M} : T_j(G_j(M)) \to M$ such that $G_j(\kappa_{u,u',M})$ is an isomoprhism. $\square$

At this point we conjecture that the results and set-up described above may be used to adapt Butler and Ringel's theorem.

**Conjecture 1.4.51.** *The functorial filtration method, as presented above, may be used to classify the finitely generated modules over any quasi-bounded string algebra over $R$ (that is, theorem 1.4.37 may be generalised from the case where $R$ is the field $k$).*

In the proofs of the main results of this thesis (theorem 2.0.1, theorem 2.0.4 and theorem 2.0.5) we show that the said method may be used to classify objects in the derived catgeory of any complete gentle algebra.

## 1.5 Some Derived Categories.

### 1.5.1 Representation Type.

In section 1.5.1 we recall what one means by the *representation type* of a finite-dimensional algebra (over an algebraically closed field). While doing so we recall some results that aid and motivate the notion of derived representation type (which we look at in section 1.5.2). We start with a remark.

**Remark 1.5.1.** Let $\Gamma$ be a finite-dimensional algebra over a field $k$. Iteratively decomposing the idempotent 1 of $\Gamma$ yields a complete set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$. We can choose a subset $\{e_{b(1)}, \dots, e_{b(t)}\}$ of $\{e_1, \dots, e_n\}$ so that, where $e_b = \sum_i e_{b(i)}$, the subalgebra $\Gamma^b := e_b\Gamma e_b$ of $\Gamma$ is *basic*, that is[7] for all $m$ with $1 \leq m \leq n$ there is a unique $i$ with $1 \leq i \leq t$ and $\Gamma e_m \simeq \Gamma e_{b(i)}$ (see [5, I.6, 6.3 Definition]). By [5, I.6, 6.10 Corollary] the categories $\Gamma$-**mod** and $\Gamma^b$-**mod** are equivalent, so to study the objects in $\Gamma$-**mod** we can assume $\Gamma$ is basic. Furthemore we can assume $\Gamma$ is not the direct product of two subalgebras[8], that is, we can assume $\Gamma$ is *connected* [5, I.4, p.18].

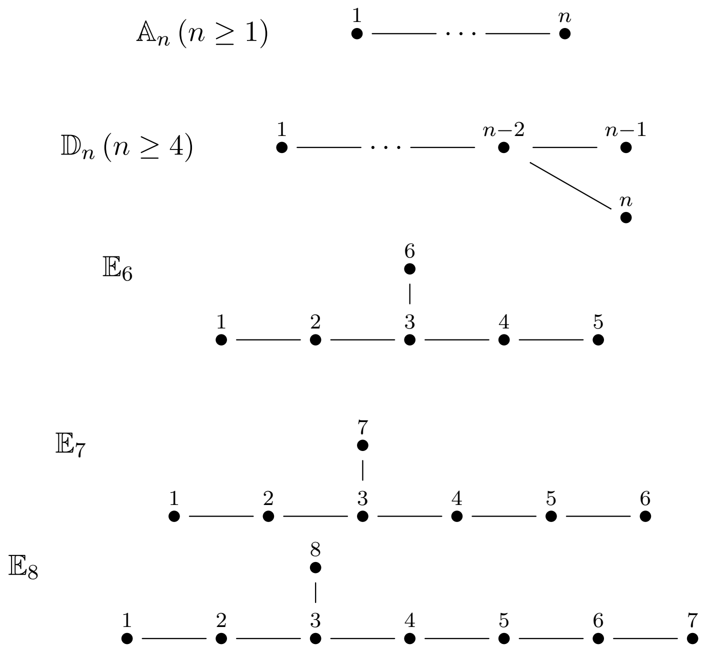
**Assumption:** In section 1.5.2 we let $k$ be an algebraically closed field and $\Gamma$ be a (unital) $k$-algebra. Unless stated otherwise we assume $\Gamma$ is finite-dimensional, connected and basic. We also assume $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents for $\Gamma$.

**Definition 1.5.2.** [5, II.3, 3.1 Definition] The *ordinary quiver* $Q_\Gamma = (Q_{\Gamma,0}, Q_{\Gamma,1}, h_\Gamma, t_\Gamma)$ *of* $\Gamma$ is defined by setting $Q_{\Gamma,0} = \{1, \dots, n\}$ and drawing $\dim_k(e_j(\text{rad}(\Gamma)/\text{rad}^2(\Gamma))e_i)$ distinct arrows with tail $i$ and head $j$. Recall definition 1.1.19. By [5, II.3, 3.7 Theorem] there is an admissible ideal $I_\Gamma$ of $kQ_\Gamma$ for which $\Gamma \simeq kQ_\Gamma/I_\Gamma$.

We consider momentarilly the case where $I_\Gamma = 0$ in the above. That is, we recall well-known results about the representation type of quivers. This should serve as motivation for the notion of representation type, and hence derived representation type.

[7] By [5, I.6, 6.2 Proposition (a)] this is the same as requiring $\Lambda^b/\text{rad}(\Lambda^b)$ has the form $k \times \cdots \times k$ .
[8] Equivalently, $\Lambda$ has no non-trivial central idempotents.

**Definition 1.5.3.** The *Dynkin diagrams* are:




(Dynkin Type Quivers) A quiver $Q$ is said to be *of Dynkin type* $X_n$ where $X \in \{\mathbb{A}, \mathbb{D}, \mathbb{E}\}$ if there is some integer $n \geq 1$ for which $X_n$ is a Dynkin diagram with $n$ vertices and $Q$ is the quiver defined by choosing an orientation of the unorientated edges of $X_n$. Sometimes the subscript $n$ (resp. $m$) is omitted if the number of vertices in the quiver has little importance.

The next definition together with the remaining theorems in section 1.5.2 highlight the importance of quivers of Dynkin or Euclidean type.

**Definition 1.5.4.** The *representation type* of $\Gamma$ is defined as follows. Let $k\langle x, y\rangle$-$\mathbf{Mod}_{k\text{-}\mathbf{mod}}$ denote the full subcategory of $k\langle x, y\rangle$-$\mathbf{Mod}$ consisting of finite-dimensional modules.

(Representation-Finite, Representation-Infinite) [5, §I.4, 4.11 Definition] We say $\Gamma$ is *representation-finite* if there are finitely many isomorphism classes of finite-dimensional indecomposable $\Gamma$-modules. Otherwise we say $\Gamma$ is *representation-infinite*.

(Representation-Tame) [17, 6.5, Definition] We say $\Gamma$ is *representation-tame* if for each $d > 0$ there are a finite number of $\Gamma$-$k[x]$-bimodules $M_i$ which are free as right $k[x]$-modules, such that every indecomposable $\Gamma$-module of dimension $d$ is isomorphic to $M_i \otimes_{k[x]} N$ for some $i$ and some simple $k[x]$-module $N$.

(Representation-Wild) [17, 6.4, Definition] We say $\Gamma$ is *representation-wild* if there is a finitely generated $\Gamma$-$k\langle x, y\rangle$-bimodule $M$, which is free as a right $k\langle x, y\rangle$-module, and such that the functor $M \otimes_{k\langle x,y\rangle} - : k\langle x, y\rangle\text{-}\mathbf{Mod}_{k\text{-}\mathbf{mod}} \to \Gamma\text{-}\mathbf{mod}$ preserves indecomposable objects and reflects isomorphisms.

If $Q$ is a finite acyclic quiver then the zero ideal of the path algebra $kQ$ is admissible by [5, §II.2, 2.2 (b)], and $kQ$ is a finite-dimensional, basic and conneted associative $k$-algebra by [5, §II.2, 2.12]. If $Q$ is a finite quiver then we say $Q$ is representation-finite (resp. infinite, tame, wild) if $kQ$ is representation-finite (resp. infinite, tame, wild).

**Theorem 1.5.5.** [29, p.3] *A quiver $Q$ is representation-finite iff it is a disjoint union of finitely many Dynkin quivers of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$.*

To motivate definition 1.5.4 it helps to recall Drozd's so-called *tame-wild dichotomy.*

**Theorem 1.5.6.** [24] (see also [17, Corollary C]) *The following statements hold.*

(i) *Either $\Gamma$ is representation-tame or $\Gamma$ is representation-wild.*

(ii) *$\Gamma$ is representation-tame iff $\Gamma$ is not representation-wild.*

We now motivate the study of *special algebras* by Pogorzały and Skowroński [53]. Recall that if $kQ/(\rho)$ is a *special* algebra then: $k$ is an algebraically closed field; $kQ/(\rho)$ is surjectively given by $(Q, \rho, \theta)$ where $\theta$ is the quotient map $kQ \to kQ/(\rho)$; SP1) and SPII) hold; and $(\rho)$ is admissible.

**Lemma 1.5.7.** [61, Lemma 2] *If $\Gamma e_v$ and $e_v\Gamma$ are biserial for each vertex $v$ and $\Gamma$ is representation-finite then $\Gamma \simeq kQ/(\rho)$ is a special algebra (where $Q$ is finite).*

In fact, any special algebra $kQ/(\rho)$ is representation-tame. For this see [63, (2.4) Corollary], the proof of which uses theorem 1.4.37 and the following.

**Proposition 1.5.8.** [63, (2.3) Proposition] *Let $\Gamma$ be a special algebra (not necessarily finite-dimensional). If $M$ is an indecomposable in $\Gamma$-**mod** then one (and only one) of the following statements hold.*

(i) *There is a finite word $w$ such that $M \simeq M(w)$.*

(ii) *There is a periodic $\mathbb{Z}$-word $w$ and a finite-dimensional $k[T, T^{-1}]$-module $V$ such that $M \simeq M(w, V)$.*

(iii) *There is a vertex $v$ such that $M \simeq \Gamma e_v$ is a non-uniserial projective-injective $\Gamma$-module.*

**Remark 1.5.9.** It should be noted that Wald and Waschbüch proved proposition 1.5.8 only in case $\Gamma$ is finite-dimensional. Their proof involves reducing the task to *symmetric*[9] special algebras using [63, (1.4) Theorem]. Symmetric special algebras are the same as *Brauer graph algebras* (for example, see [59, Theorem 1.1]). They then use the classification of finite-dimensional indecomposable modules over Brauer graph algebras, which is due to Donovan and Freislich [23, Theorem 1] (who use the functorial filtration method). Instead we prove proposition 1.5.8 using theorem 1.4.37 and lemma 1.2.5.

*Proof of proposition 1.5.8.* By lemma 1.2.5 letting $X$ be the set of vertices $x$ such that $kp_x = kp'_x$ (in $\Gamma$) for some distinct parallel $p_x, p'_x \in \mathbf{P}(x \to)$, we have that: setting $I = \bigoplus_{x \in X} \mathrm{soc}(\Gamma e_x)$ defines a two sided ideal of $\Gamma$, $\Gamma/I$ is a quasi-bounded string algebra over $k$; and $IM = 0$ or $M \simeq \Gamma e_x$ for some $x \in X$.

By lemma 1.2.3, since $\Gamma/I$ is a quasi-bounded string algebra over $k$ it is a string algebra in the sense of Butler and Ringel (what we call a Butler-Ringel string algebra). Hence if $M$ is a finite-dimensional indecomposable $\Gamma$-module with $IM = 0$, then $M \simeq M(w)$ where $w$ is finite, or $M \simeq M(w, V)$ where $V$ is finite-dimensional by theorem 1.4.37. □

[9] The ring $\Gamma$ is called *symmetric* provided $\Gamma' \simeq \mathrm{Hom}_k(\Gamma', k)$ as $\Gamma'$-$\Gamma'$ bimodules.

### 1.5.2 Derived Representation Type.

We shall freely use (without reference) definitions and certain results from homological algebra which are given in appendix (see sections 3.2 and 3.3). For an abelian category $\mathcal{A}$ (which will be $\Gamma$-**Mod** or $\Gamma$-**mod**) recall $\mathcal{C}(\mathcal{A})$ is the category of cochain complexes, $\mathcal{K}(\mathcal{A})$ is the homotopy category and $\mathcal{D}(\mathcal{A})$ is the derived catgeory.

For full additive subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{A}$ recall $\mathcal{C}_{\mathcal{Y}}^{\delta,\epsilon}(\mathcal{X})$, $\mathcal{K}_{\mathcal{Y}}^{\delta,\epsilon}(\mathcal{X})$ and $\mathcal{D}_{\mathcal{Y}}^{\delta,\epsilon}(\mathcal{X})$ denote the full subcategories of $\mathcal{C}(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ (resp.) consisting of complexes $M$ in $\mathcal{C}^{\delta}(\mathcal{X})$ whose homology complex $H(M)$ lies in $\mathcal{C}^{\epsilon}(\mathcal{Y})$, where: $\delta, \epsilon \in \{+,-\}$; $\mathcal{C}^{\pm}(\mathcal{X})$ is the full subcategory of $\mathcal{C}(\mathcal{X})$ consisting of complexes $X$ such that $X^i = 0$ for $\pm i \ll 0$; $\mathcal{C}^{\emptyset}(\mathcal{X}) = \mathcal{C}(\mathcal{X})$; and $\mathcal{C}^{b}(\mathcal{X})$ is the full subcategory of $\mathcal{C}(\mathcal{X})$ consisting of objects $X$ which lie in both $\mathcal{C}^{+}(\mathcal{X})$ and $\mathcal{C}^{-}(\mathcal{X})$. Let $\mathbf{0}$ be the subcategory of $\mathcal{A}$ consisting of the zero complex. We simplify notation in certain cases by setting

$$\begin{array}{ccc}
\mathcal{C}_{\mathcal{A}}^{\delta,\epsilon}(\mathcal{X}) = \mathcal{C}^{\delta,\epsilon}(\mathcal{X}), & \mathcal{K}_{\mathcal{A}}^{\delta,\epsilon}(\mathcal{X}) = \mathcal{K}^{\delta,\epsilon}(\mathcal{X}), & \mathcal{D}_{\mathcal{A}}^{\delta,\epsilon}(\mathcal{X}) = \mathcal{D}^{\delta,\epsilon}(\mathcal{X}), \\
\mathcal{C}_{\mathcal{Y}}^{\delta,\emptyset}(\mathcal{X}) = \mathcal{C}_{\mathcal{Y}}^{\delta}(\mathcal{X}), & \mathcal{K}_{\mathcal{Y}}^{\delta,\emptyset}(\mathcal{X}) = \mathcal{K}_{\mathcal{Y}}^{\delta}(\mathcal{X}), & \mathcal{D}_{\mathcal{Y}}^{\delta,\emptyset}(\mathcal{X}) = \mathcal{D}_{\mathcal{Y}}^{\delta}(\mathcal{X}), \\
\mathcal{C}_{\mathcal{Y}}^{\emptyset}(\mathcal{X}) = \mathcal{C}_{\mathcal{Y}}(\mathcal{X}), & \mathcal{K}_{\mathcal{Y}}^{\emptyset}(\mathcal{X}) = \mathcal{K}_{\mathcal{Y}}(\mathcal{X}), & \mathcal{D}_{\mathcal{Y}}^{\emptyset}(\mathcal{X}) = \mathcal{D}_{\mathcal{Y}}(\mathcal{X}), \\
\mathcal{C}_{\mathbf{0}}^{\delta,\epsilon}(\mathcal{X}) = \mathcal{C}_{\mathrm{acyc}}^{\delta,\epsilon}(\mathcal{X}), & \mathcal{K}_{\mathbf{0}}^{\delta,\epsilon}(\mathcal{X}) = \mathcal{K}_{\mathrm{acyc}}^{\delta,\epsilon}(\mathcal{X}), & \mathcal{D}_{\mathbf{0}}^{\delta,\epsilon}(\mathcal{X}) = \mathcal{D}_{\mathrm{acyc}}^{\delta,\epsilon}(\mathcal{X}).
\end{array}$$

Recall $\mathcal{P}$ (resp. $\mathcal{I}$) denotes the full subcategory of $\mathcal{A}$ consisting of projective (resp. injective) objects. This means the objects of $\mathcal{P}$ (resp. $\mathcal{I}$) are the objects $X$ in $\mathcal{A}$ such that the functor $\mathrm{Hom}_{\mathcal{A}}(P,-) : \mathcal{A} \to \mathbf{Ab}$ (resp. $\mathrm{Hom}_{\mathcal{A}}(-,I) : \mathcal{A}^{op} \to \mathbf{Ab}$) is exact. By applying [64, p.378, Corollary 10.2.5] in the context above, since $\mathcal{P}$ and $\mathcal{I}$ are additive, $\mathcal{K}^{\delta,\epsilon}(\mathcal{P})$ and $\mathcal{K}^{\delta,\epsilon}(\mathcal{I})$ are triangulated subcategories of $\mathcal{K}(\mathcal{A})$. Recall the following (triangle) equivalences from corollary 3.3.28 (given by horizontal arrows)

$$\begin{tikzcd}
\mathcal{K}_{\mathrm{p}}(\mathcal{A}) \arrow[r, "\sim"] & \mathcal{D}(\mathcal{A}) \\
\mathcal{K}^{-}(\mathcal{P}) \arrow[r, "\sim"] \arrow[u, phantom, "\cup\uparrow"] & \mathcal{D}^{-}(\mathcal{A}) \arrow[u, phantom, "\cup\uparrow"] \\
\mathcal{K}^{-,b}(\mathcal{P}) \arrow[r, "\sim"] \arrow[u, phantom, "\cup\uparrow"] & \mathcal{D}^{b}(\mathcal{A}) \arrow[u, phantom, "\cup\uparrow"]
\end{tikzcd}$$

where $\mathcal{K}_{\mathrm{p}}(\mathcal{A})$ is the full subcategory of $\mathcal{K}(\mathcal{A})$ consisting of K-projective complexes (see defintion 3.3.17). In what follows we use this notation without reference.

**Definition 1.5.10.** Let $[t]$ denote the automorphism of $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ defined by $t$-copies of the shift $[1]$ (so that $M[t]^i = M^{i+t}$ for any object $M$ in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$).

(Derived-Finite Algebras) We say $\Gamma$ is *derived-finite* if there are finitely many indecomposable objects $M_1, \dots, M_n$ in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ such that, if $M$ is an indecomposable object in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$, then $M \simeq M_i[t]$ for some $i$ and $t$.

**Example 1.5.11.** Let $Q$ be a finite quiver. Consider complexes of the form

$$M[t] = \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where $M$ in $kQ$-**mod** is indecomposable, and concentrated in homogeneous degree $t$. A well-known result of Happel [34, p.49, Lemma] tells us that any indecomposable in $\mathcal{D}^b(kQ\text{-}\mathbf{mod})$ has this form. Clearly they are pair-wise non-isomorphic objects in $\mathcal{D}^b(kQ\text{-}\mathbf{mod})$. By (Gabriel's) theorem 1.5.5 this means $kQ$ is derived-finite provided $Q$ is a disjoint union of finitely many Dynkin quivers of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$.

**Definition 1.5.12.** (Cohomological Dimension Vectors) For each object $M$ in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ the *cohomological dimension vector* is denoted $\underline{\mathrm{Dim}}(M)$ and defined as the $\mathbb{Z}$-sequence $(\dim_k(H^i(M)))_{i\in\mathbb{Z}}$ where $H^i(M) = \ker(d^i_M) \,/\, \mathrm{im}(d^{i-1}_M)$ for each $i \in \mathbb{Z}$.

(Derived-Discrete Algebras) We say $\Gamma$ is *derived-discrete* if for each $(t_j)_{j\in\mathbb{Z}} \in \prod_{\mathbb{Z}} \mathbb{N}$ there are finitely many isoclasses $[M]$ of indecomposable objects $M$ in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ such that $\underline{\mathrm{Dim}}(M) = (t_j)$.

**Example 1.5.13.** Every finite-dimensional algebra which is derived-finite is derived-discrete. Hence by example 1.5.11 if $Q$ is a disjoint union of finitely many Dynkin quivers of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$, then $kQ$ is derived-discrete.

**Example 1.5.14.** [12, p.20] Let $n \geq r \geq 1$ and $m \geq 0$ be integers.

Let $Q(r, n, m)$ be the quiver

$$\begin{tikzcd}
 & & & & 1 \arrow[r, "\alpha_1"] & \cdots \arrow[r] & n-r-1 \arrow[dr, "\alpha_{n-r-1}"] & \\
(-m) \arrow[r, "\alpha_{-m}"] & \cdots \arrow[r, "\alpha_{-2}"] & (-1) \arrow[r, "\alpha_{-1}"] & 0 \arrow[ur, "\alpha_0"] & & & & n-r \arrow[dl, "\alpha_{n-r}"] \\
 & & & & n-1 \arrow[ul, "\alpha_{n-1}"] & \cdots \arrow[l] & n-r+1 \arrow[l] &
\end{tikzcd}$$

Let $I(r, n, m)$ be the ideal in $kQ(r, n, m)$ generated by $\alpha_0\alpha_{n-1}, \alpha_{n-1}\alpha_{n-2}, \ldots, \alpha_{n-r+1}\alpha_{n-r}$. By [62, p.171, Theorem: (ii) iff (iv)] $kQ(r, n, m)/I(r, n, m)$ is derived-discrete.

**Definition 1.5.15.** We say $\Gamma$ is of *derived* $\mathbb{A}$-$\mathbb{D}$-$\mathbb{E}$ *Dynkin type* if $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod}) \simeq \mathcal{D}^b(kQ\text{-}\mathbf{mod})$ where $Q$ is a disjoint union of finitely many Dynkin quivers of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$.

The next theorem describes all the other derived-discrete finite-dimensional algebras, up to *derived-equivalence*.

**Theorem 1.5.16.** [12, Theorem A: (i) iff (ii)] *If* $\Gamma$ *is not of derived* $\mathbb{A}$-$\mathbb{D}$-$\mathbb{E}$ *Dynkin type, then* $\Gamma$ *is derived-discrete iff* $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod}) \simeq \mathcal{D}^b(kQ(r, n, m)/I(r, n, m)\text{-}\mathbf{mod})$ *for some* $n \geq r \geq 1$ *and* $m \geq 0$.

**Definition 1.5.17.** For each integer $j$ with $1 \leq j \leq l$ suppose $C_j$ is a bounded complex of $\Gamma$-$k[T]$-bimodules (that is, $C_j$ is an object in $\mathcal{D}^b(\Gamma \otimes_k k[T]\text{-}\mathbf{mod})$) where $C_j^n$ is finitely generated and free as a right $k[T]$-module for each $j$ and $n \in \mathbb{Z}$.

[9, Definition 1.1] (Rational Parameterising Families) We say $C_1, \ldots, C_l$ is a *rational parameterising family for* $(t_j)_{j\in\mathbb{Z}} \in \prod_{\mathbb{Z}} \mathbb{N}$ if every indecomposable object $M$ in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ with $\underline{\mathrm{Dim}}(M) = (t_j)$ is isomorphic (in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$) to a complex of the form $C_j \otimes_{k[T]} S$ where $S$ is a simple left $k[T]$-module[10].

(Derived-Tame Algebras) [7, p.289, Definition 1] $\Gamma$ is called *derived-tame* if, for each $(t_j) \in \prod_{\mathbb{Z}} \mathbb{N}$ there exists a rational parameterising family for $(t_j)$.

[10] $C_i \otimes_{k[T]} S$ is defined by setting $(C_i \otimes_{k[T]} S)^n = C_i^n \otimes_{k[T]} S$ and $d^n_{C_i \otimes_{k[T]} S} = d^n_{C_i} \otimes \mathrm{id}$ for each $n \in \mathbb{Z}$.

**Example 1.5.18.** Suppose $\Gamma$ is derived-discrete. Let $(t_j) \in \prod_{\mathbb{Z}} \mathbb{N}$. There are finitely many indecomposable objects in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$, say $M_1, \dots, M_l$, such that any indecomposable $M$ in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ such that $\underline{\mathrm{Dim}}(M) = (t_j)$ satisfies $M \simeq M_i$ for some $i$. Setting $C_i = M_i \otimes_k k[T]$ defines a bounded complex of $\Gamma$-$k[T]$-bimodules. Furthermore as each $M_i^n$ is a finitely generated $\Gamma$-module (which is a finite-dimensional algebra) we have that $C_i^n \simeq k[T] \oplus \cdots \oplus k[T]$ (with $\dim_k(M_i^n)$ summands). For any $M$ as above we have $M \simeq M_i$, and as $k \simeq k[T] \otimes_{k[T]} S$ where $S$ is the simple (one-dimensional) $k[T]$-module $k[T]/(T)$, this gives $M_i \simeq C_i \otimes_{k[T]} S$.

Consequently $\Gamma$ is derived-tame. We have therefore already seen many examples of derived-tame algebras (see example 1.5.14 and theorem 1.5.16). By example 1.5.13 we also know that derived-finite algebras are also derived-tame.

**Definition 1.5.19.** [8, Definition 1.2] (Derived-Wild Algebras) We say $\Gamma$ is *derived-wild* if there is a bounded complex of projective $\Gamma \otimes_k k\langle x, y\rangle$-modules $M$, such that for any finite-dimensional $k\langle x, y\rangle$-module $X$:

(a) if $X'$ is another finite-dimensional $k\langle x, y\rangle$-module, then $M \otimes_{k\langle x,y\rangle} X \simeq M \otimes_{k\langle x,y\rangle} X'$ in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ iff $X \simeq X'$ in $k\langle x, y\rangle\text{-}\mathbf{mod}$; and

(b) the object $M \otimes_{k\langle x,y\rangle} X$ is indecomposable in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ iff $X$ is indecomposable in $k\langle x, y\rangle\text{-}\mathbf{mod}$.

We may now state an analogue to theorem 1.5.6.

**Theorem 1.5.20.** [8, Theorem 1.3] (Derived Tame-Wild Dichotomy) *The following statements hold.*

(i) *Either* $\Gamma$ *is derived-tame or* $\Gamma$ *is derived-wild.*

(ii) $\Gamma$ *is derived-tame iff* $\Gamma$ *is not derived-wild.*

### 1.5.3 Gentle Algebras.

**Assumption:** In section 1.5.3 let $k$ be a field, $Q$ be a finite connected quiver, and $\rho$ a set of zero relations and commutativity relations such that $(\rho)$ is an admissible ideal in $kQ$. We also let $\Gamma = kQ/(\rho)$.

**Definition 1.5.21.** (DUALITY) Let $D(\Gamma) = \mathrm{Hom}_k(\Gamma, k)$, considered as a $\Gamma$-$\Gamma$-bimodule whose action on the left is given by $(\lambda\varphi) : \mu \mapsto \varphi(\mu\lambda)$ and whose action on the right is given by $(\varphi\lambda)\mu \mapsto \varphi(\lambda\mu)$ for each $\mu \in \Gamma$.

[37, 2.2, p.351] (REPETITIVE ALGEBRA) We define the *repetitive algebra* $\widehat{\Gamma}$ as follows. As a vector space we let $\widehat{\Gamma} = \bigoplus_{z\in\mathbb{Z}} \Lambda[z] \oplus \bigoplus_{z\in\mathbb{Z}} D(\Lambda)[z]$, and we define mutliplication by the formula

$$(a_z,\, \phi_z) \times (b_z,\, \psi_z) = (ab_z,\, (a\psi)_z + (\phi b)_z)$$

With a view towards highlighting known classifications of derived categories, we now motivate the introduction of the repetitive algebra.

**Theorem 1.5.22.** [34, §4] *There is a (triangulated) full and faithful functor* $\mathrm{H}$ : $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod}) \to \underline{\widehat{\Gamma}\text{-}\mathbf{mod}}$*, which is dense if* $\Gamma$ *has finite global dimension.*

**Definition 1.5.23.** (THE EXPANSION $\widehat{Q}$ OF $Q$) [53, p.497][11] Let $\mathbf{M}$ denote the set of maximal paths in $\mathbf{P}$. The quiver $\widehat{Q} = (\widehat{Q_0}, \widehat{Q_1}, \widehat{h}, \widehat{t})$ is defined as follows. Let $\widehat{Q_0} = \{v_i \mid v \in Q_0, i \in \mathbb{Z}\}$ and $\widehat{Q_1} = \{a_i, p[i] \mid a \in Q_1, p \in \mathbf{M}, i \in \mathbb{Z}\}$. The functions $\widehat{h}$ and $\widehat{t}$ are defined by setting $\widehat{h}(a_i) = h(a)_i$, $\widehat{t}(a_i) = t(a)_i$, $\widehat{h}(p[i]) = t(p)_i$ and $\widehat{t}(p[i]) = h(p)_{i+1}$. Any $q \in \mathbf{P}$ can be written as $\mathrm{l}(q)\dots\mathrm{f}(q)$ and we let $q_i$ denote the path $\mathrm{l}(q)_i \dots \mathrm{f}(q)_i$ in $\widehat{Q}$ . The arrows $p[i]$ are called the *connecting arrows*.

(FULL PATHS AND THE EXPANSION $\widehat{\rho}$ OF $\rho$) [58, §2] A *full path* in $\widehat{Q}$ is a path of the form $p''_i p[i] p'_{i+1}$ where $p \in \mathbf{M}$ such that $p = p'p''$ for some paths $p', p''$ (one of which may be trivial). We shall write $\widehat{\mathbf{F}}$ for the set of full paths in $\widehat{Q}$. Let $\widehat{\rho} = \overline{\rho} \cup \breve{\rho} \cup \widetilde{\rho}$ where $\overline{\rho}$, $\breve{\rho}$ and $\widetilde{\rho}$ are defined as follows.

[11] See also work by Ringel [56, §4].

(a) $\overline{\rho}$ consists of all paths $p_i$ where $p \in \rho$.

(b) $\breve{\rho}$ consists of all paths $\sigma$ in $\widehat{Q}$ such that a connecting arrow $p[i]$ is a subpath of $\sigma$, and $\sigma$ is not a subpath of a full path.

(c) $\widetilde{\rho}$ consists of all $p''_i p[i] p'_{i+1} - q''_i q[i] q'_{i+1}$ for distinct $p, q \in \mathbf{M}$ such that $p = p'p''$ and $q = q'q''$ with $t(p') = t(q')$.

**Remark 1.5.24.** Conditions (a) and (b) above are precicely (R1) and (R2) from [58, §2]. Condition (c) above is not the same as (R3) from [58, §2], however they are equivalent provided $\rho$ consists of paths and $(Q, \rho)$ satisfies special and gentle conditions (an assumption omitted by Schröer: see [58, §5, Examples]).

**Example 1.5.25.** Let $\rho = \{\delta\beta, \beta\alpha\}$ where $Q$ is the quiver

$$\begin{tikzcd}
& 1 \arrow[d, "\alpha"'] & \\
& 2 \arrow[d, "\beta"'] & \\
& 3 \arrow[dl, "\gamma"] \arrow[dr, "\delta"] & \\
4 \arrow[rr, "\varepsilon"'] & & 5
\end{tikzcd}$$

The set of maximal paths is $\mathbf{M} = \{\alpha, \varepsilon\gamma\beta, \delta\}$. Hence $\widehat{Q}$ is the quiver

$$\begin{tikzcd}
& & 1_1 \arrow[d, "\alpha_1"'] & & & 1_0 \arrow[d, "\alpha_0"'] & & & 1_{-1} \arrow[d, "\alpha_{-1}"'] & & \\
\cdots & & 2_1 \arrow[d, "\beta_1"'] \arrow[urrr, "\alpha[0]"'] & & & 2_0 \arrow[d, "\beta_0"'] \arrow[urrr, "\alpha[-1]"'] & & & 2_{-1} \arrow[d, "\beta_{-1}"'] & & \cdots \\
\cdots & & 3[1] \arrow[dl, "\gamma_1"] \arrow[dr, "\delta_1"] & & & 3[0] \arrow[dl, "\gamma_0"] \arrow[dr, "\delta_0"] & & & 3[-1] \arrow[dl, "\gamma_{-1}"] \arrow[dr, "\delta_{-1}"] & & \cdots \\
& 4_1 \arrow[rr, "\varepsilon_1"'] & & 5_1 \arrow[uurr, "\varepsilon\gamma\beta[0]"] \arrow[urr, "\delta[0]"'] & 4_0 \arrow[rr, "\varepsilon_0"'] & & 5_0 \arrow[uurr, "\varepsilon\gamma\beta[-1]"] \arrow[urr, "\delta[-1]"'] & 4_{-1} \arrow[rr, "\varepsilon_{-1}"'] & & 5_{-1} &
\end{tikzcd}$$

and the set of full paths is

$$\widehat{\mathbf{F}} = \{\alpha_i\alpha[i],\ \alpha[i]\alpha_{i+1},\ \varepsilon_i\gamma_i\beta_i\varepsilon\gamma\beta[i],\ \gamma_i\beta_i\varepsilon\gamma\beta[i]\varepsilon_{i+1},\ \beta_i\varepsilon\gamma\beta[i]\varepsilon_{i+1}\gamma_{i+1},\ \varepsilon\gamma\beta[i]\varepsilon_{i+1}\gamma_{i+1}\beta_{i+1},\ \delta_i\delta[i],\ \delta[i]\delta_{i+1} \mid i \in \mathbb{Z}\}$$

The expansion of the pair $(Q, \rho)$ is the pair $(\widehat{Q}, \widehat{\rho})$ for $\widehat{Q}$ as above and $\widehat{\rho} = \overline{\rho} \cup \breve{\rho} \cup \widetilde{\rho}$ where $\overline{\rho} = \{\delta_i\beta_i, \beta_i\alpha_i \mid i \in \mathbb{Z}\}$,

$$\breve{\rho} = \{\alpha_i\alpha[i]\alpha_{i+1},\ \varepsilon_i\gamma_i\beta_i\varepsilon\gamma\beta[i]\varepsilon_{i+1},\ \gamma_i\beta_i\varepsilon\gamma\beta[i]\varepsilon_{i+1}\gamma_{i+1},\ \beta_i\varepsilon\gamma\beta[i]\varepsilon_{i+1}\gamma_{i+1}\beta_{i+1},\ \delta_i\delta[i]\delta_{i+1} \mid i \in \mathbb{Z}\}$$

and

$$\widetilde{\rho} = \{\alpha_i\alpha[i] - \varepsilon\gamma\beta[i]\varepsilon_{i+1}\gamma_{i+1}\beta_{i+1},\ \delta[i]\delta_{i+1} - \beta_i\varepsilon\gamma\beta[i]\varepsilon_{i+1}\gamma_{i+1},\ \delta_i\delta[i] - \varepsilon_i\gamma_i\beta_i\varepsilon\gamma\beta[i] \mid i \in \mathbb{Z}\}$$

**Theorem 1.5.26.** *The following statements hold.*

(i) [58, p.428, Theorem] *There is a $k$-algebra isomorphism $\widehat{\Gamma} \simeq k\widehat{Q}/(\widehat{\rho})$.*

(ii) [58, p.429, Proposition] (see also [4, 1.3 Proposition] and [53, Lemma 8]) *The pair $(Q, \rho)$ satisfies (special, and) gentle conditions iff $(\widehat{Q}, \widehat{\rho})$ satisfies special conditions.*

**Remark 1.5.27.** Let $\Gamma$ be an Assem-Skowroński gentle algebra. Recall our aim was to describe the indecomposable objects in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$. Suppose $\Gamma$ has finite global dimension. By theorem 1.5.22 it is enough to describe the objects in $\underline{\widehat{\Gamma}\text{-}\mathbf{mod}}$.

By theorem 1.5.26 (i) there is an isomorphism $\widehat{\Gamma} \simeq k\widehat{Q}/(\widehat{\rho})$ and so it is enough to classify the (non-projective) objects in $k\widehat{Q}/(\widehat{\rho})$-**mod**. By theorem 1.5.26 (ii) $k\widehat{Q}/(\widehat{\rho})$ is a Pogorzały-Skowroński special algebra. Hence by proposition 1.5.8 the indecomposable modules over $\widehat{\Gamma}$ are the non-uniserial projective-injective indecomposables, together with the indecomposable modules over the associated Butler-Ringel string algebra.

By theorem 1.4.37 the indecomposable modules over a Butler-Ringel string algebra are classified into string modules and band modules. Correspondences between string and band modules over $\widehat{\Gamma}$ and objects in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ (via the functor $\mathrm{H} : \mathcal{D}^b(\Gamma\text{-}\mathbf{mod}) \to \underline{\widehat{\Gamma}\text{-}\mathbf{mod}}$ from theorem 1.5.22) have been studied by Bobinski [11].

Instead of following this approach, we state an explicit classification of the objects in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$, due to Bekkert and Merklen.

**Definition 1.5.28.** (Full Cycles of Zero-Relations) A *full cycle of zero-relations* $\alpha_1 \dots \alpha_n$ will refer to a cycle in $Q$ which is not the product of shorter cycles in $Q$, and for which $\alpha_n\alpha_1 \in (\rho)$ and $\alpha_i\alpha_{i+1} \in (\rho)$ for each $i \in \{1, \dots, n\}$ such that $i + 1 \leq n$.

**Theorem 1.5.29.** [7, Theorem 3] *Let* $\mathsf{S}$ *be the set of homotopy* $I$*-words* $A$ *which are not periodic homotopy* $\mathbb{Z}$*-words, and such that:*

(a) *if* $I \supseteq -\mathbb{N}$ *then* $[A_{<t}] = {}^{\infty}([\alpha_1]^{-1} \dots [\alpha_n]^{-1})$ *for some* $t \gg 0$ *and some full cycle of zero-relations* $\alpha_1 \dots \alpha_n$*; and*

(b) *if* $I \supseteq \mathbb{N}$ *then* $[A_{>t}] = ([\beta_m] \dots [\beta_1])^{\infty}$ *for some* $t \gg 0$ *and some full cycle of zero-relations* $\beta_1 \dots \beta_m$*.*

*Let* $\mathsf{B}$ *be the set of periodic homotopy* $\mathbb{Z}$*-words* $E$*. The following statements hold.*

(i) *For any* $A \in \mathsf{S}$ *and any integer* $t$ *the complex* $P(A)[t]$ *is an indecomposable object in* $\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj})$ *(which is equivalent to* $\mathcal{D}^b(\Lambda\text{-}\mathbf{mod})$*).*

(ii) *For any* $E \in \mathsf{B}$*, any integer* $t$ *and any finite-dimensional* $k[T, T^{-1}]$*-module* $V$ *the complex* $P(E, V)[t]$ *is an indecomposable object in* $\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj})$*.*

(iii) *If* $M$ *is an indecomposable object in* $\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj})$ *then (*$M \simeq P(A)[t]$ *for some* $A$ *and* $t$ *as in* (i)*) or (*$M \simeq P(E, V)[t]$ *for some* $E$*,* $V$ *and* $t$ *as in* (ii)*).*

**Remark 1.5.30.** For each isomorphism class of the indecomposable objects in $\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj})$ fix a representative, and then write $\mathtt{Ind}(\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj}))$ for the full subcategory of $\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj})$ given by these chosen representatives.

The claims in the theorem [7, Theorem 3] by Bekkert and Merklen stricly contain parts (i), (ii) and (iii) of theorem 1.5.29. It is also shown that $\mathtt{Ind}(\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj}))$ can be described as a subset of the string and band complexes described in (parts (i) and (ii) of) the above. In corollary 2.7.6 we generalise theorem 1.5.29 to all complete gentle algebras. In theorem 2.0.4 we discuss these isomorphism classes, and in theorem 2.0.5 we give a decomposition property.

# Chapter 2

# Classification of Complexes for Complete Gentle Algebras.

**Assumption:** Unless specified otherwise, throughout chapter 2 we assume $\Lambda$ is a complete gentle algebra over $R$ given by $(Q, \rho, \theta)$.

In this thesis our main result is as follows.

**Theorem 2.0.1.** (Description of Objects) *The following statements hold.*

(i) *Every object in $\mathcal{K}(\Lambda\text{-}\mathbf{proj})$ is isomorphic to a (possibly infinite) direct sum of shifts of string and band complexes.*

(ii) *Each (shift of a) string or band complex is an indecomposable object in $\mathcal{K}(\Lambda\text{-}\mathbf{Proj})$.*

**Remark 2.0.2.** Note that in part (ii) of theorem 2.0.1 we have not restricted to string and band complexes which have finitely generated homogeneous components. Consider the string complex $P({}^{\infty}(x^{-1}d_x d_y^{-1}y)^{\infty}))$ from example 1.3.41 depicted by

$$\begin{tikzcd}
\ddots \arrow[dr, "y"] & & \Lambda \arrow[dl, "x"'] \arrow[dr, "y"] & & \Lambda \arrow[dl, "x"'] \arrow[dr, "y"] & & \Lambda \arrow[dl, "x"'] \arrow[dr, "y"] & \\
& \Lambda & & \Lambda & & \Lambda & & \ddots
\end{tikzcd}$$

By theorem 2.0.1 (ii) $P({}^{\infty}(x^{-1}d_x d_y^{-1}y)^{\infty}))$ defines an indecomposable object in $\mathcal{K}(k[[x,y]]/(xy)\text{-}\mathbf{Proj})$. In proposition 2.7.1 we describe exactly when (direct sums of shifts of) string and band complexes define objects in $\mathcal{K}(\Lambda\text{-}\mathbf{proj})$.

**Definition 2.0.3.** (Swapping $T$ and $T^{-1}$) Let $\iota$ define the $R$-algebra automorphism of $R[T,T^{-1}]$ which exchanges $T$ and $T^{-1}$. Define a functor $\mathrm{res}_\iota : R[T,T^{-1}]\text{-}\mathbf{Mod} \to R[T,T^{-1}]\text{-}\mathbf{Mod}$ by setting $\mathrm{res}_\iota(V)$ to have underlying $R$-module structure $V$ (for any $R[T,T^{-1}]$-module $V$) but where the action of $T$ on $v \in \mathrm{res}_\iota(V)$ is defined by $T.v = T^{-1}v$. Hence $\iota$ is an involution (that is $\mathrm{res}_\iota \circ \mathrm{res}_\iota \simeq 1_{R[T,T^{-1}]-\mathbf{Mod}}$). Clearly $\mathrm{res}_\iota$ restricts to an involution of the full subcategory $R[T,T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$ of $R[T,T^{-1}]\text{-}\mathbf{Mod}$ consisting of $R[T,T^{-1}]$-modules which are free over $R$.

In section 2.2.2 we show $P(C,V)$ and $P(C^{-1},\mathrm{res}_\iota(V))$ are isomorphic complexes. The next theorem is an analogue of [55, p.21, Theorem], and informs the reader how to construct isomorphism classes of indecomposables.

**Theorem 2.0.4.** (Description of Isoclasses) *Let $C$ be a homotopy $I_C$-word, and let $D$ be a homotopy $I_D$-word.*

(i) *If $C$ and $D$ are not periodic homotopy $\mathbb{Z}$-words then $P(C) \simeq P(D)[m]$ iff:*

$$I_C = I_D = \{0,\dots,t\} \text{ and } ((D = C \text{ and } m = 0) \text{ or } (D = C^{-1} \text{ and } m = \mu_C(t))); \text{ or}$$
$$((I_C = I_D = \pm\mathbb{N} \text{ and } C = D) \text{ or } (I_C = \pm\mathbb{N}, I_D = \mp\mathbb{N} \text{ and } D = C^{-1})) \text{ and } m = 0; \text{ or}$$
$$I_C = I_D = \mathbb{Z},\ D = C^{\pm 1}[t] \text{ and } m = \mu_C(\pm t).$$

(ii) *If $C$ and $D$ are periodic homotopy $\mathbb{Z}$-words, and if $V,W$ lie in $R[T,T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$ then $P(C,V) \simeq P(D,W)[m]$ iff*

$$D = C[t],\ V \simeq W \text{ and } m = \mu_C(t), \text{ or}$$
$$D = C^{-1}[t],\ V \simeq \mathrm{res}_\iota W \text{ and } m = \mu_C(-t).$$

(iii) *There is no isomorphism between any shift of a string complex and any shift of a band complex.*

It is known that the Krull-Remak-Schmidt-Azumaya property holds for the homotopy category of a complete local noetherian ring ([14, p.85, Proposition A.2]). The next theorem verifies this property in our setting.

**Theorem 2.0.5.** (Krull-Remak-Schmidt-Azumaya Decomposition Property) *If an object of* $\mathcal{K}(\Lambda\text{-}\mathbf{proj})$ *is written as a direct sum of string and band complexes in two different ways, there is an isomorphism preserving bijection between the summands.*

## 2.1 Homotopy Words and Relations.

**Assumption:** (Notation: $M^\bullet$) In section 2.1 we assume

$$M^\bullet \quad = \qquad \cdots \longrightarrow M^{-1} \xrightarrow{d_M^{-1}} M^0 \xrightarrow{d_M^0} M^1 \longrightarrow \cdots$$

is a fixed complex in the category $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$. Hence $M^i$ is a projective $\Lambda$-module and $\mathrm{im}(d_M^i) \subseteq \mathrm{rad}(M^{i+1})$ for each integer $i$.

(Notation: $M$, $d_M$, $e_vM$, $d_M|_v$). Let $M$ be the projective $\Lambda$-module $\bigoplus_{i\in\mathbb{Z}} M^i$. Let $d_M$ be the $\Lambda$-module endomorphism $\bigoplus_{i\in\mathbb{Z}} d_M^i$ of $M$ sending $\sum_i m_i$ to $\sum_i d_M^i(m_i)$. For any vertex $v$ we we write $d_M|_v$ for the $R$-module endomorphism of $e_vM = \bigoplus_{i\in\mathbb{Z}} e_vM^i$ defined by the restriction of $d_M$. Hence, as $R$-module endomorphisms of $M = \bigoplus_{i,v} e_vM^i$, we have $d_M = \sum_v d_M|_v$ where the sum runs through all vertices $v$ of the quiver $Q$.

This section shall be the analogue of section 1.4.2 for homotopy words (insetad of words). In case $\Lambda$ is an Assem-Skowroński gentle algebra, part (ii) of the following lemma is [7, p. 299, Lemma 5].

**Lemma 2.1.1.** *For any arrows $a$ and $b$,*

(i) *if $v = h(b) = t(a)$ and $ab \in \mathbf{P}$, $abm = 0$ implies $bm = 0$ for all $m \in M$,*

(ii) *if $v = t(a)$ then $\{m' \in e_vM \mid am' = 0\} = \sum_{b'\in\mathbf{A}(\to v)\,:\,ab'\notin\mathbf{P}} b'M$, and*

(iii) *if $v = h(b) = h(a)$ and $a \neq b$ the sum $aM + bM$ is direct.*

*Proof.* Since $M$ is a projective $\Lambda$-module there is a split embedding $\varphi : M \to \bigoplus_x \Lambda$ of $\Lambda$-modules where $x$ runs through some index set $X$.

(i) Write $\varphi(m) = \sum_x \lambda_x$ for some $\lambda_x \in \Lambda$. For a contradiction assume $bm \neq 0$. Hence for some $x \in X$ we have $b\lambda_x \neq 0$. Since $abm = 0$ we have $ab\lambda_x = 0$ and so $b\lambda_x$ contradicts corollary 1.2.18.

(ii) Suppose $m' \in e_{t(a)}M$ with $am' = 0$. Write $\varphi(m') = \sum_x \lambda'_x$ for some $\lambda'_x \in \Lambda$. Without loss of generality it suffices to assume there are arrows $b, b'$ with head $v$ satisfying $ab \in \mathbf{P}$ and $ab' \notin \mathbf{P}$. Since $am' = 0$ we have $a\lambda'_x = 0$ for each $x \in X$. By corollary 1.2.14 (iv) this means $\lambda'_x \in b\Lambda \oplus b'\Lambda$. For each $x \in X$ write $\lambda'_x = bm^x_- + b'm^x_+$ for elements $m^x_-, m^x_+ \in \Lambda$.

Since $ab'm^x_+ = 0$ and $am' = 0$ we have $abm^x_- = 0$ and so by part (i) $bm^x_- = 0$. As this holds for each $x$ we have $\varphi(m') = b' \sum_x m^x_+$. As $\varphi$ splits there is a $\Lambda$-module map $\tau$ for which $\tau\varphi = 1_M$ and so $m' = b'\tau(\sum_x m^x_+) \in b'M$. This shows $\{m' \in e_{t(a)}M \mid am' = 0\} \subseteq \sum b'M$ and the other inclusion is trivial.

(iii) For $n \in aM \cap bM$ write $n = am = bm'$ for some $m, m' \in M$. As above write $\varphi(m) = \sum_x \lambda_x$ and $\varphi(m') = \sum_x \lambda'_x$ for $\lambda_x, \lambda'_x \in \Lambda$. Applying $\varphi$ to $n$ gives $a\lambda_x = b\lambda'_x$ for each $x \in X$, and as $a\Lambda \cap b\Lambda = 0$ this means $a\lambda_x = 0$ for each $x \in X$. Hence $\varphi(n) = 0$ and as $\varphi$ is an embedding we have $n = 0$ as required. $\square$

In the next lemma we begin to give some meaning to the letters $d_\alpha^{\pm 1}$ for arrows $\alpha$. This lemma should also motivate why we assume $\operatorname{im}(d^i_M) \subseteq \operatorname{rad}(M^{i+1})$ for each integer $i$.

**Lemma 2.1.2.** *For each arrow $\alpha$ there is an $R$-module endomorphism $d_{\alpha,M}$ of $e_{h(\alpha)}M$ such that $d_M|_v = \sum_\beta d_{\beta,M}$ running over all arrows $\beta$ with head $v$.*
*Furthermore for any $\tau \in \mathbf{P}$ and any $x \in e_{t(\tau)}M$,*

(i) $d_{\mathrm{l}(\tau),M}(\tau x) = \begin{cases} \tau d_{\sigma,M}(x) & (\textit{if } \tau\sigma \in \mathbf{P} \textit{ for some arrow } \sigma) \\ 0 & (\textit{otherwise}) \end{cases}$

(ii) *if $h(\theta) = h(\tau)$ for some arrow $\theta \neq \mathrm{l}(\tau)$ then $d_{\theta,M}(\tau x) = 0$,*

(iii) *if $h(\phi) = h(\tau)$ for some arrow $\phi$ then $d_{\phi,M}d_{\mathrm{l}(\tau),M} = 0$, and*

(iv) *if $\tau x \in \operatorname{im}(d_{\mathrm{l}(\tau),M})$ then $d_{\varsigma,M}(x) = 0$ for any arrow $\varsigma$ such that $\tau\varsigma \in \mathbf{P}$.*

*Proof.* Any complete gentle algebra is semilocal by corollary 1.1.25 (ii). Hence $\operatorname{rad}(M) = \operatorname{rad}(\Lambda)M$ by [48, p.348, (24.4) Proposition]. By assumption $\operatorname{im}(d_M) \subseteq \operatorname{rad}(\Lambda)M$ and so the image of $d_M$ upon restriction to $e_vM$ is contained in $e_v\operatorname{rad}(\Lambda)M$.

Recall $e_v\mathrm{rad}(\Lambda) = (\bigoplus_{\beta\in\mathbf{A}(\to v)}\beta\Lambda)$ by corollary 1.2.14. By definition $(\bigoplus_{\beta\in\mathbf{A}(\to v)}\beta\Lambda)M = \sum_{\beta\in\mathbf{A}(\to v)}\beta M$, and this sum is direct by lemma 2.1.1. For any arrow $\gamma$ with head $v$ let $\pi_\gamma : \bigoplus_{\beta\in\mathbf{A}(\to v)}\beta M \to \gamma M$ and $\iota_\gamma : \gamma M \to \bigoplus_{\beta\in\mathbf{A}(\to v)}\beta M$ be the natural projections and inclusions in the category of $R$-modules. Define $d_{\alpha,M} : e_vM \to e_vM$ to be the map sending $m$ to $\iota_\alpha(\pi_\alpha(d_M|_v(m)))$ which lies in $\bigoplus_{\beta\in\mathbf{A}(\to v)}\beta M$. Then we have

$$\textstyle\sum_\beta d_{\beta,M}(m) = \sum_\beta \iota_\beta(\pi_\beta(d_M|_v(m))) = (\sum_\beta \iota_\beta\pi_\beta)(d_M|_v(m)) = d_M|_v(m) \quad (\star_v).$$

Let $v = h(\tau)$ and $u = t(\tau)$. By definition $d_{\mathrm{l}(\tau),M}(\tau x) = \iota_{\mathrm{l}(\tau)}(\pi_{\mathrm{l}(\tau)}(d_M(\tau x))) = \tau\, d_M|_u(x)$.

(i), (ii) For distinct arrows $\sigma$ and $\sigma'$ with head $u$ we have ($\tau\sigma\in\mathbf{P}$ iff $\tau\sigma'\notin\mathbf{P}$) by SPII) and GII). The equations $\tau\, d_M|_u(x) = d_{\mathrm{l}(\tau),M}(\tau x)$ and $(\star_u)$ together show

$$d_{\mathrm{l}(\tau),M}(\tau x) = \begin{cases} \tau(d_{\sigma,M}(x) + d_{\sigma',M}(x)) & (\text{if } \mathbf{A}(\to u) = \{\sigma,\sigma'\}, \tau\sigma\in\mathbf{P} \text{ and } \tau\sigma'\notin\mathbf{P}) \\ \tau(d_{\sigma,M}(x)) & (\text{if } \mathbf{A}(\to u) = \{\sigma\} \text{ and } \tau\sigma\in\mathbf{P}) \\ \tau(d_{\sigma',M}(x)) & (\text{if } \mathbf{A}(\to u) = \{\sigma'\} \text{ and } \tau\sigma'\notin\mathbf{P}) \\ 0 & (\text{if } \mathbf{A}(\to u) = \emptyset) \end{cases}$$

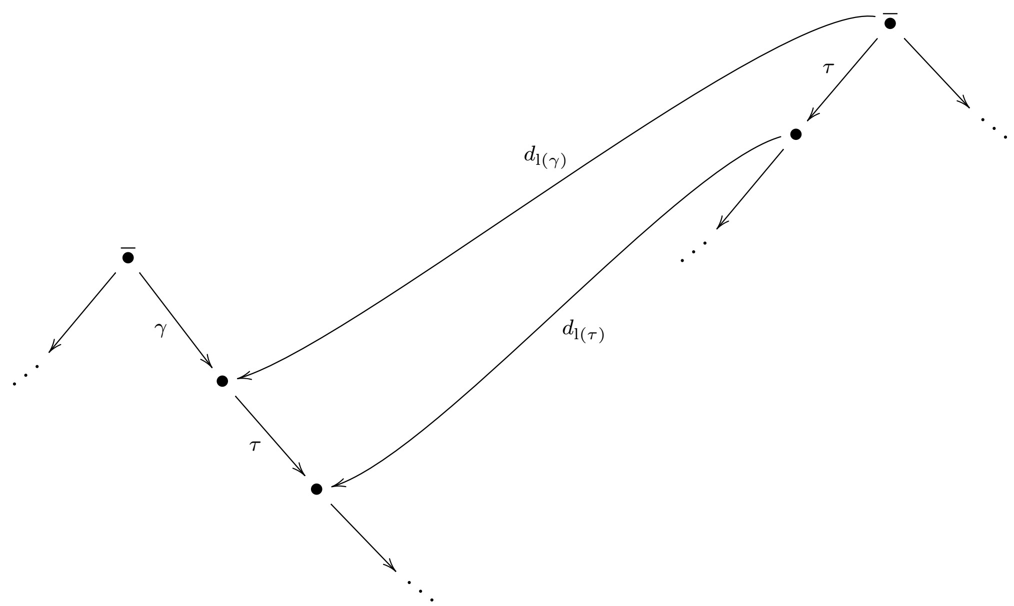
giving $d_{\mathrm{l}(\tau),M}(\tau x) = \tau(d_{\mathrm{l}(\sigma),M}(x))$ if $\sigma$ exists, and $d_{\mathrm{l}(\tau),M}(\tau x) = 0$ otherwise, and so (i) holds. For part (ii) note that $\pi_\theta(\tau x) = 0$ by definition.

(iii) We fix $x \in e_vM$, let $v' = t(\mathrm{l}(\tau))$ and choose $x' \in e_{v'}M$ for which $\mathrm{l}(\tau)x' = d_{\mathrm{l}(\tau),M}(x)$. By (ii) we can assume $\phi = \mathrm{l}(\tau)$ and by (i) we can assume there is some arrow $\sigma$ for which $\phi\sigma\in\mathbf{P}$. By (i) this gives $d_{\phi,M}(d_{\phi,M}(x)) = \phi d_{\sigma,M}(x')$. Given any arrow $\sigma'$ with head $v'$ where $\sigma\neq\sigma'$ we have $\phi\sigma'\notin\mathbf{P}$ by SPII), and so $\phi(d_{\sigma,M}(x')) = \phi(\sum_{\gamma\in\mathbf{A}(\to v')} d_{\gamma,M}(x'))$. By $(\star_{v'})$ we have $\sum_{\gamma\in\mathbf{A}(\to v')} d_{\gamma,M}(x') = d_M|_{v'}(x')$. So far this gives $d_{\phi,M}(d_{\phi,M}(x)) = \phi d_M(x')$.

By $(\star_v)$ we have $d_{\phi,M}(x) = d_M|_v(x) - \sum_\vartheta d_{\vartheta,M}(x)$ where the sum runs through all arrows $\vartheta$ with head $v$ and where $\vartheta\neq\phi$. At most one such $\vartheta$ exists by SPI). Without loss of generality we may assume $\vartheta$ exists. Since $d_M(d_M|_v(x) - d_{\vartheta,M}(x)) = -d_M(d_{\vartheta,M}(x))$ we have $d_{\phi,M}(d_{\phi,M}(x)) \in \vartheta M$. We already have that $d_{\phi,M}(d_{\phi,M}(x)) = d_{\phi,M}(d_{\mathrm{l}(\tau),M}(x)) \in \phi M$, and so $d_{\phi,M}(d_{\mathrm{l}(\tau),M}(x')) = 0$ by lemma 2.1.1.

(iv) By definition $x \in \tau^{-1}\mathrm{im}(d_{l(\tau),M})$. Hence there is some $x' \in e_{h(\tau)}M$ for which $\tau x = d_{l(\tau),M}(x')$. We know that $d_{\varsigma,M}(x) = \varsigma x''$ for some $x'' \in e_{t(\varsigma)}M$. By hypothesis we have that $\tau\varsigma \in \mathbf{P}$. Also by (i) and (ii) above we have $\tau d_{\varsigma,M}(x) = d_{l(\tau),M}(\tau x) = d_{l(\tau),M}(d_{l(\tau),M}(x'))$ and so $(\tau\varsigma)x'' = 0$. By lemma 2.1.1 we have $\varsigma x'' = 0$ as required. □

**Remark 2.1.3.** Part (i) of lemma 2.1.2 may be interpreted as the commutativity in the square drawn from the following schema (where $\tau, \gamma \in \mathbf{P}$ such that $\tau\gamma \in \mathbf{P}$).




The reader is advised to look out for such commuting squares when drawing the diagrams from remark 1.3.36.

Recall during sections 1.4.1 and 1.4.2 we defined functorial relations on $\Lambda$-**Mod**.

**Definition 2.1.4.** If $f^\bullet : M^\bullet \to N^\bullet$ is an arrow in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ let $f = \bigoplus_{i\in\mathbb{Z}} f^i$, the underlying $\Lambda$-module homomorphism. Sometimes we consider $f$ as an $R$-module homomorphism.

(Notation: ${}_R(-)^\bullet$) There is a functor ${}_R(-)^\bullet : \mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to R\text{-}\mathbf{Mod}$ defined on objects by sending $M^\bullet$ to the underlying $R$-module ${}_RM$ of $M$, and defined on arrows by sending $f^\bullet$ to the underlying $R$-module homomorphism $f : {}_RM \to {}_RN$.

(SUBFUNCTORS OF ${}_R(-)^\bullet$, INTERVALS) A *subfunctor* $\mathsf{S}$ *of* ${}_R(-)^\bullet$ is given by an $R$-submodule $\mathsf{S}(M^\bullet) \subseteq {}_RM$ for each object $M^\bullet$ in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$, such that $f(m) \in \mathsf{S}(N^\bullet)$ for any $f \in \mathrm{Hom}_{\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})}(M^\bullet, N^\bullet)$ and $m \in \mathsf{S}(M^\bullet)$. For subfunctors $\mathsf{S}$ and $\mathsf{S}'$ of ${}_R(-)^\bullet$ we write $\mathsf{S} \leqslant \mathsf{S}'$ if we have $\mathsf{S}(M^\bullet) \subseteq \mathsf{S}'(M^\bullet)$ for each $M^\bullet$. In this case $[\mathsf{S}, \mathsf{S}']$ is called an *interval*, and we say intervals $[\mathsf{S}, \mathsf{S}']$ and $[\mathsf{T}, \mathsf{T}']$ *avoid each other* if $\mathsf{S}' \leqslant \mathsf{T}$ or $\mathsf{T}' \leqslant \mathsf{S}$.

Just as words $w$ define subfunctors $w^\pm$ of the forgetful functor ${}_R(-) : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$, we aim to show homotopy words $C$ define subfunctors $C^\pm : \mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to R\text{-}\mathbf{Mod}$ of the forgetful functor ${}_R(-)^\bullet$. Let us start to make this precise. Recall the notation from definition 1.4.9 and example 1.4.2. For any arrow $a$ and any path $p \in \mathbf{P}$ we have $\mathrm{rel}^p(M) = \{(m, pm) \mid m \in e_{t(p)}M\}$ and $\mathrm{graph}(d_{a,M}) = \{(m, d_{a,M}(m)) \mid m \in e_{h(a)}M\}$.

**Definition 2.1.5.** (NOTATION: $\underline{\mathrm{rel}}^{v,\pm}(M^\bullet)$) If $v$ is a vertex and $C = \underline{1}_{v,\pm}$ let $\underline{\mathrm{rel}}^C(M^\bullet) = \underline{\mathrm{rel}}^{v,\pm}(M^\bullet) = \mathrm{rel}^{v,\pm}(M)$, the relation $\{(m, m) \mid m \in e_vM\}$ on $e_vM$.

(RELATIONS GIVEN BY HOMOTOPY WORDS, NOTATION: $\underline{\mathrm{rel}}^C(M^\bullet)$) Let $C = l_1^{-1}r_1 \ldots l_n^{-1}r_n$ be a homotopy $\{0, \ldots, n\}$-word where $n > 0$. For each $i$ with $0 < i \leq n$ let

$$\underline{\mathrm{rel}}_i^C(M^\bullet) = \begin{cases} (\mathrm{rel}^\gamma(M))^{-1}\mathrm{graph}(d_{\mathrm{l}(\gamma),M}) & (\text{if } l_i^{-1}r_i = \gamma^{-1}d_{\mathrm{l}(\gamma)}) \\ (\mathrm{graph}(d_{\mathrm{l}(\gamma),M}))^{-1}\mathrm{rel}^\gamma(M) & (\text{if } l_i^{-1}r_i = d_{\mathrm{l}(\gamma)}^{-1}\gamma) \end{cases}$$
$$= \begin{cases} \{(m', m) \in e_{h(\gamma)}M \oplus e_{t(\gamma)}M \mid \gamma m = d_{\mathrm{l}(\gamma)}(m')\} & (\text{if } l_i^{-1}r_i = \gamma^{-1}d_{\mathrm{l}(\gamma)}) \\ \{(m', m) \in e_{t(\gamma)}M \oplus e_{h(\gamma)}M \mid d_{\mathrm{l}(\gamma)}(m) = \gamma m'\} & (\text{if } l_i^{-1}r_i = d_{\mathrm{l}(\gamma)}^{-1}\gamma) \end{cases}$$

and let $\underline{\mathrm{rel}}^C(M) = \underline{\mathrm{rel}}_1^C(M^\bullet) \ldots \underline{\mathrm{rel}}_n^C(M^\bullet)$, the $n$-fold composition of these relations.

**Example 2.1.6.** Consider the (finite-dimensional) Assem-Skowroński gentle algebra $\Lambda = kQ/(\rho)$ where $\rho = \{ba, cb, ac, sr, ts, rt\}$ and $Q$ is the quiver

$$\begin{tikzcd}
1 \arrow[dd, "b"'] & & 3 \arrow[dl, "t"'] \\
& 0 \arrow[ul, "a"] \arrow[dr, "r"] & \\
2 \arrow[ur, "c"'] & & 4 \arrow[uu, "s"']
\end{tikzcd}$$

Consider the homotopy $\{0,1,2,3\}$-word $C = s^{-1}d_s t^{-1} d_t d_c^{-1} c$. By definition

$$\begin{aligned}
\underline{\mathrm{rel}}^C(M^\bullet) &= \underline{\mathrm{rel}}_1^C(M^\bullet)\,\underline{\mathrm{rel}}_2^C(M^\bullet)\,\underline{\mathrm{rel}}_3^C(M^\bullet)\\
&= \{(w,z) \mid \exists x,y : (w,x) \in \underline{\mathrm{rel}}_3^C(M^\bullet),\ (x,y) \in \underline{\mathrm{rel}}_2^C(M^\bullet),\ (y,z) \in \underline{\mathrm{rel}}_1^C(M^\bullet)\}\\
&= \left\{ (w,z) \mid \begin{array}{c} (w,x) \in (\mathrm{graph}(d_{c,M}))^{-1}\mathrm{rel}^c(M) \\ (x,y) \in (\mathrm{rel}^t(M))^{-1}\mathrm{graph}(d_{t,M}) \\ \text{and } (y,z) \in (\mathrm{rel}^s(M))^{-1}\mathrm{graph}(d_{s,M}) \text{ for some } x,y \end{array} \right\}\\
&= \{(w,z) \mid d_{c,M}(x) = cw,\ ty = d_{t,M}(x),\ sz = d_{s,M}(y) \text{ for some } x,y\}
\end{aligned}$$

**Definition 2.1.7.** (NOTATION: $\gamma^{\pm 1}U$, $d_\alpha^{\pm 1}U$) Let $q$ be a homotopy letter (that is, let $q$ be one of $\gamma$, $\gamma^{-1}$,$\mathrm{d}_\alpha$ or $d_\alpha^{-1}$ for some path $\gamma \in \mathbf{P}$ or some arrow $\alpha$. If $U$ is a subset of $e_{t(q)}M$ then define the subset $qU$ of $e_{h(q)}M$ by

$$\begin{aligned}
\gamma U &= \{\gamma m \in e_{h(\gamma)}M \mid m \in U\}, & \gamma^{-1}U &= \{m \in e_{t(\gamma)}M \mid \gamma m \in U\},\\
d_\alpha U &= \{d_{\alpha,M}(m) \in e_{h(\alpha)}M \mid m \in U\}, & d_\alpha^{-1}U &= \{m \in e_{h(\alpha)}M \mid d_{\alpha,M}(m) \in U\}.
\end{aligned}$$

(NOTATION: $\underline{1}_{v,\pm 1}U$) For any vertex $v$ and any subset $U$ of $e_vM$ let $\underline{1}_{v,\pm 1}U = U$.

Recall the category $R$-**Rel** whose objects are pairs $(V,N)$ (where $N$ is an $R$-module and $V$ is an $R$-linear relation on $N$). Recall the functor im (resp. ind) from $R$-**Rel** to $R$-**Mod** sending $(V,N)$ to $VN = \{m \in N \mid (m',m) \in V \text{ for some } m' \in M\}$ (resp. $V0 = \{m \in N \mid (0,m) \in V\}$. By definition if $C$ is any finite homotopy word then $CM = \mathrm{im}(\underline{\mathrm{rel}}^C(M^\bullet))$ and $C0 = \mathrm{ind}(\underline{\mathrm{rel}}^C(M^\bullet))$.

**Example 2.1.8.** Consider the Assem-Skowroński gentle algebra $\Lambda = kQ/(\rho)$, the homotopy $\{0,1,2,3\}$-word $C = s^{-1}d_s t^{-1} d_t d_c^{-1} c$ and the calculation of $\underline{\mathrm{rel}}^C(M^\bullet)$ from example 2.1.6. For any subset $U \subseteq e_2M$ we have

$$CU = \{z \in e_4M \mid \begin{array}{c} d_{c,M}(x) = cw,\ ty = d_{t,M}(x),\ sz = d_{s,M}(y) \\ \text{for some } x \in e_0M,\ y \in e_3M,\ w \in U\} \end{array}$$

It is helpful to depict the relations above by

$$
\begin{tikzcd}
 & & & & & & x \arrow[dl, "a"'] \arrow[dr, "r"] \arrow[dddlll, bend right, "d_{t,M}"'] \arrow[dddrrr, bend left, "d_{c,M}"] & & & & \\
 & & & & & ax \arrow[dddl, bend right, "d_{a,M}"] & & rx \arrow[dddr, bend left, "d_{r,M}"'] & & & \\
 & & y \arrow[dr, "t"'] \arrow[dddl, bend right, "d_{s,M}"'] & & & & & & & & w \arrow[dl, "c"'] \\
 & & & ty \arrow[dr, "a"'] & & & & & & cw \arrow[dl, "r"'] & \\
z \arrow[dr, "s"'] & & & & aty & & & & rcw & & \\
 & sz & & & & & & & & &
\end{tikzcd}
$$

Notice the similarities between the schema above and the picture from example 1.3.37.

Before defining the functors $C^{\pm} : \mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to R\text{-}\mathbf{Mod}$ we collect some results about the sets introduced in definition 2.1.7.

**Corollary 2.1.9.** *If $a$ is an arrow then $a^{-1}d_a\mathrm{rad}(M) \subseteq e_{t(a)}\mathrm{rad}(M)$. Furthermore, given an arrow $b$ with $ab \in \mathbf{P}$ we have $(ab)^{-1}ad_bM = b^{-1}d_bM$.*

*Proof.* By lemma 2.1.2 (ii) $d_a\mathrm{rad}(M) = d_aaM$ and so it is enough to show that $a^{-1}d_aaM \subseteq e_{t(a)}\mathrm{rad}(M)$. By lemma 2.1.2 (i) $a^{-1}d_aaM = a^{-1}ad_bM$ if there is an arrow $b$ for which $ab \in \mathbf{P}$ and otherwise $a^{-1}d_aaM = a^{-1}d_a0$.

If $b$ exists then any $m \in a^{-1}d_aaM$ satisfies $am = ad_{b,M}(m')$ for some $m' \in e_{h(b)}M$ and so by lemma 2.1.1 we have that $m - d_{b,M}(m')$ lies in the subspace $a^{-1}0 = \sum b'M$ of $e_{t(\alpha)}\mathrm{rad}(M)$ where the sum ranges over all arrows $b'$ with $ab' \notin \mathbf{P}$. As $\mathrm{im}(d_{b,M}) \subseteq \mathrm{rad}(M)$ this shows $a^{-1}d_aaM \subseteq e_{t(\alpha)}\mathrm{rad}(M)$. If $b$ does not exist we have $a^{-1}d_aaM = a^{-1}d_a0 = a^{-1}0 = \sum b'M$ which is a subset of $e_{t(a)}\mathrm{rad}(M)$. In any case we have shown $a^{-1}d_a\mathrm{rad}(M) \subseteq e_{t(a)}\mathrm{rad}(M)$.

Now assume $b$ exists for the second part of the corollary. For any $m \in b^{-1}a^{-1}ad_b.M$ we have some $m' \in e_{h(b)}M$ for which $bm - d_{b,M}(m')$ lies in $a^{-1}0$. As $bm - d_{b,M}(m') = bm''$ for some $m'' \in M$ we have $abm'' = 0$ which means $bm'' = 0$ by lemma 2.1.1. This gives $b^{-1}a^{-1}ad_bM \subseteq b^{-1}d_bM$ and the reverse inclusion is obvious. □

We now gather some consequences of lemma 2.1.2 in the language of linear relations. The next corollary follows from lemma 2.1.2.

**Corollary 2.1.10.** *Let $\alpha$, $\beta$, $\gamma$ and $\sigma$ be paths in $\mathbf{P}$ with $\alpha\beta \in \mathbf{P}$, $h(\gamma) = h(\sigma)$ and $\mathrm{l}(\gamma) \neq \mathrm{l}(\sigma)$. Then we have*

$$\beta^{-1}d_{\mathrm{l}(\beta)}M \subseteq (\alpha\beta)^{-1}d_{\mathrm{l}(\alpha)}M, \qquad d^{-1}_{\mathrm{l}(\alpha)}\alpha\beta M \subseteq d^{-1}_{\mathrm{l}(\alpha)}\alpha M,$$
$$\alpha^{-1}d_{\mathrm{l}(\alpha)}M \subseteq d^{-1}_{\mathrm{l}(\beta)}\beta 0, \qquad \gamma M \subseteq d^{-1}_{\mathrm{l}(\sigma)}\sigma 0, \qquad d_{\mathrm{l}(\sigma)}M \subseteq d^{-1}_{\mathrm{l}(\sigma)}\sigma 0.$$

### 2.1.1 One-Sided Functors.

We now look toward defining functors $C^{\pm}$ for each homotopy word $C$. Our definition will adapt notions used by Ringel [55, p.23] and Crawley-Boevey [21, p.11] (see definition 1.4.19). At this point it is necessary to describe how homotopy words are composed, adapted from [21] for our purposes.

**Definition 2.1.11.** (Sign for Homotopy Letters) Let $\mathbf{A}^{\pm}$ be the set of homotopy letters of the form $\alpha$ or $\alpha^{-1}$ where $\alpha$ is an arrow. Note that any such homotopy letter defines a letter (in the sense of definition 1.3.5). Hence (from definition 1.3.10) we have chosen a *sign* $s(q) \in \{\pm 1\}$ for each homotopy letter $q$ in $\mathbf{A}^{\pm}$, such that if distinct letters $q$ and $q'$ from $\mathbf{A}^{\pm}$ have the same head, they have the same sign only if $\{q, q'\} = \{\alpha^{-1}, \beta\}$ with $\alpha\beta \notin \mathbf{P}$.

We extend this notion to all letters by letting $s(\gamma) = s(\mathrm{l}(\gamma))$, $s(\gamma^{-1}) = s(\mathrm{f}(\gamma)^{-1})$, and $s(d_{\alpha}^{\pm 1}) = -s(\alpha)$ for each $\gamma \in \mathbf{P}$ and each arrow $\alpha$.

(Sign for Homotopy Words) For a (non-trivial finite or $\mathbb{N}$)-word $C$ we let $h(C)$ and $s(C)$ be the head and sign of the first letter of $C$. For the trivial words $1_{v,\pm 1}$ we let $s(1_{v,\pm 1}) = \pm 1$ and $h(1_{v,\pm 1}) = v$.

(Composing Homotopy Words) Suppose $D$ and $E$ are non-trivial homotopy words where $I_D \subseteq -\mathbb{N}$ and $I_E \subseteq \mathbb{N}$. If $u = h(D^{-1})$ and $\epsilon = -s(D^{-1})$ let $D\underline{1}_{u,\epsilon} = D$. If $v = h(E)$ and $\delta = s(E)$ we let $\underline{1}_{v,\delta}E = E$. The *composition* $DE$ is the sequence of homotopy letters given by concatenating the letters in $D$ with the letters in $E$. In case $D = \dots l_{-1}^{-1} r_{-1} l_0^{-1} r_0$ and is a $-\mathbb{N}$-word and $E = l_1^{-1} r_1 l_2^{-1} r_2 \dots$ is an $\mathbb{N}$-word, write $DE = \dots l_0^{-1} r_0 \mid l_1^{-1} r_1 \dots$

**Example 2.1.12.** Recall $\Lambda = k[[x, y]]/(xy)$ from example 1.3.28. Let $s(x) = 1$. Hence $s(x^{-1}) = -1$, $s(y^{-1}) = 1$ and $s(y) = -1$ because $xy, yx \notin \mathbf{P}$. So we have $s(d_x^{\pm 1}) = -1$ and $s(d_y^{\pm 1}) = 1$. Hence for the homotopy $\mathbb{N}$-word

$$C = x^{-2} d_x y^{-1} d_y x^{-2} d_x d_y^{-1} y^3 d_x^{-1} x y^{-1} d_y x^{-2} d_x y^{-1} d_y x^{-2} d_x \dots$$

we have the table

| $i \in \mathbb{N}$ | $s(l_i)$ | $s(r_i)$ | $i \in \mathbb{N}$ | $s(l_i)$ | $s(r_i)$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | $-1$ | 5 | $-1$ | 1 | |
| 2 | $-1$ | 1 | 6 | 1 | 1 | $\cdots$ |
| 3 | 1 | $-1$ | 7 | $-1$ | $-1$ | |
| 4 | 1 | $-1$ | 8 | 1 | 1 | |

We now characterise when the two words may be composed.

**Proposition 2.1.13.** *If $D$ and $E$ are non-trivial homotopy words where $I_D \subseteq -\mathbb{N}$ and $I_E \subseteq \mathbb{N}$ then ($DE$ is a homotopy word iff $h(D^{-1}) = h(E)$ and $s(D^{-1}) = -s(E)$).*

*Proof.* There is nothing to prove if $D$ is a trivial homotopy word or $E$ is a trivial homotopy word. So we can assume otherwise, and write $D = \dots l_0^{-1} r_0$ and $E = l_1^{-1} r_1 \dots$ Suppose firstly that $DE$ is a homotopy word. We know that there are some paths $\lambda, \gamma \in \mathbf{P}$ where $l_0^{-1} r_0 l_1^{-1} r_1$ is one of $\gamma^{-1} d_{\mathrm{l}(\gamma)} d_{\mathrm{l}(\lambda)}^{-1} \lambda$, $d_{\mathrm{l}(\gamma)}^{-1} \gamma d_{\mathrm{l}(\lambda)}^{-1} \lambda$, $d_{\mathrm{l}(\gamma)}^{-1} \gamma \lambda^{-1} d_{\mathrm{l}(\lambda)}$ or $\gamma^{-1} d_{\mathrm{l}(\gamma)} \lambda^{-1} d_{\mathrm{l}(\lambda)}$.

Suppose $l_0^{-1} r_0 l_1^{-1} r_1 = \gamma^{-1} d_{\mathrm{l}(\gamma)} d_{\mathrm{l}(\lambda)}^{-1} \lambda$. Here $h(D^{-1}) = h(d_{\mathrm{l}(\gamma)}^{-1}) = h(\gamma)$, $h(E) = h(d_{\mathrm{l}(\lambda)}^{-1}) = h(\lambda)$, $s(D^{-1}) = s(d_{\mathrm{l}(\gamma)}^{-1}) = -s(\mathrm{l}(\gamma))$ and $-s(E) = -s(d_{\mathrm{l}(\lambda)}^{-1}) = s(\mathrm{l}(\lambda))$. Hence ($h(D^{-1}) = h(E)$ and $s(D^{-1}) = -s(E)$) iff ($h(\gamma) = h(\lambda)$ and $\mathrm{l}(\gamma) \neq \mathrm{l}(\lambda)$) which are precisely the conditions for $DE$ to be a word.

Similarly, we have that: if $l_0^{-1} r_0 l_1^{-1} r_1 = d_{\mathrm{l}(\gamma)}^{-1} \gamma \lambda^{-1} d_{\mathrm{l}(\lambda)}$ then (($h(D^{-1}) = h(E)$ and $s(D^{-1}) = -s(E)$) iff ($t(\gamma) = t(\lambda)$ and $\mathrm{f}(\gamma) \neq \mathrm{f}(\lambda)$)); if $l_0^{-1} r_0 l_1^{-1} r_1 = d_{\mathrm{l}(\gamma)}^{-1} \gamma d_{\mathrm{l}(\lambda)}^{-1} \lambda$ then (($h(D^{-1}) = h(E)$ and $s(D^{-1}) = -s(E)$) iff ($t(\gamma) = h(\lambda)$ and $s(\gamma^{-1}) = -s(d_{\mathrm{l}(\lambda)}^{-1})$) iff $\mathrm{f}(\gamma)\mathrm{l}(\lambda) \notin \mathbf{P}$); and if $l_0^{-1} r_0 l_1^{-1} r_1 = \gamma^{-1} d_{\mathrm{l}(\gamma)} \lambda^{-1} d_{\mathrm{l}(\lambda)}$ then (($h(D^{-1}) = h(E)$ and $s(D^{-1}) = -s(E)$) iff $\mathrm{f}(\lambda)\mathrm{l}(\gamma) \notin \mathbf{P}$). □

**Lemma 2.1.14.** *Let $C$ be a homotopy $I$-word where $I = -\mathbb{N}$ or $I$ is finite.*

(i) *If $C\gamma^{-1} d_{\mathrm{l}(\gamma)}$ and $C\beta^{-1} d_{\mathrm{l}(\beta)}$ are homotopy words and $\beta$ isn't longer than $\gamma$ then $\beta$ is an initial subpath of $\gamma$.*

(ii) *If $Cd_{\mathrm{l}(\gamma)}^{-1}\gamma$ and $Cd_{\mathrm{l}(\alpha)}^{-1}\alpha$ are homotopy words and $\alpha$ isn't longer than $\gamma$ then $\alpha$ is a terminal subpath of $\gamma$.*

*Proof.* We only prove (i) as (ii) is similar. By proposition 2.1.13 $\gamma^{-1}$ and $\beta^{-1}$ both have head $h(C^{-1}$ and sign $-s(C^{-1})$. This means $t(\beta) = t(\gamma)$ and $s(\mathrm{f}(\beta)) = s(\mathrm{f}(\gamma))$ and so $\mathrm{f}(\beta) = \mathrm{f}(\gamma)$ by definition. Hence $\beta$ is an initial subpath of $\gamma$ by lemma 1.1.14. □

**Corollary 2.1.15.** *Let $C$ be a finite homotopy word and suppose $Ca^{-1}d_a$ and $Cd_b^{-1}b$ are homotopy words for arrows $a$ and $b$. Then*

(i) $C\gamma^{-1}d_{\mathrm{l}(\gamma)}$ *is a homotopy word iff* $\mathrm{f}(\gamma) = a$*,*

(ii) $Cd_{\mathrm{l}(\tau)}^{-1}\tau$ *is a homotopy word iff* $\mathrm{l}(\tau) = b$*,*

(iii) *if* $C\gamma^{-1}d_{\mathrm{l}(\gamma)}$ *and* $C\gamma'^{-1}d_{\mathrm{l}(\gamma')}$ *are homotopy words and* $\gamma'$ *is longer than* $\gamma$ *then* $C\gamma^{-1}d_{\mathrm{l}(\gamma)}M \subseteq C\gamma'^{-1}d_{\mathrm{l}(\gamma')}M$*, and*

(iv) *if* $Cd_{\mathrm{l}(\tau)}^{-1}\tau$ *and* $Cd_{\mathrm{l}(\tau')}^{-1}\tau'$ *are homotopy words and* $\tau'$ *is longer than* $\tau$ *then* $Cd_{\mathrm{l}(\tau')}^{-1}\tau'M \subseteq Cd_{\mathrm{l}(\tau)}^{-1}\tau M$*.*

*Proof.* Follows by corollary 2.1.10 and corollary 2.1.14. □

**Example 2.1.16.** For the complete gentle algebra $k[[x,y]]/(xy)$ we have

$$M = d_x^{-1}xM \supseteq d_x^{-1}x^2M \supseteq d_x^{-1}x^3M \supseteq \cdots \supseteq x^{-3}d_xM \supseteq x^{-2}d_xM \supseteq x^{-1}d_xM$$

and

$$M = d_y^{-1}yM \supseteq d_y^{-1}y^2M \supseteq d_y^{-1}y^3M \supseteq \cdots \supseteq y^{-3}d_yM \supseteq y^{-2}d_yM \supseteq y^{-1}d_yM$$

We can now define certain $R$-submodules of $M$ which will be the building blocks of our refined functors.

**Definition 2.1.17.** (NOTATION: $C_{<i}$, $C_{\leq i}$, $C_{>i}$, $C_{\geq i}$, $\underline{\mathcal{W}}_{v,\delta}$)

If $C = \dots l_i^{-1}r_i \dots$ is a homotopy word and $i \in I_C$ is arbitrary, we let $C_i = l_i^{-1}r_i$ and $C_{\leq i} = \dots l_i^{-1}r_i$ given $i-1 \in I_C$, and otherwise $C_i = C_{\leq i} = \underline{1}_{h(C),s(C)}$. Similarly we let $C_{>i} = l_{i+1}^{-1}r_{i+1}\dots$ given $i+1 \in I_C$ and otherwise $C_{>i} = \underline{1}_{h(C^{-1}),s(C^{-1})}$.

Hence there are unique homotopy words $C_{<i}$ and $C_{\geq i}$ satisfying $C_{\leq i} = C_{<i}C_i$ and $C_iC_{>i} = C_{\geq i}$. For each vertex $v$ and sign $\delta \in \{\pm 1\}$ let $\underline{\mathcal{W}}_{v,\delta}$ be the set of all homotopy $I$-words with $I \subseteq \mathbb{N}$, head $v$ and sign $\delta$.

(NOTATION: $C^{\pm}(M)$) Suppose $C \in \underline{\mathcal{W}}_{v,\delta}$ is finite.

If there is an arrow $a$ for which $Cd_a^{-1}a$ is a homotopy word let $C^+(M) = \bigcap Cd_a^{-1}\alpha\mathrm{rad}(M)$, where the intersection is taken over all $\alpha \in \mathbf{P}$ with $\mathrm{l}(\alpha) = a$. Otherwise let $C^+(M) = CM$.

If there is an arrow $b$ for which $Cb^{-1}d_b$ is a homotopy word let $C^-(M) = \bigcup C\beta^{-1}d_{\mathrm{l}(\beta)}M$, where the union is taken over all $\beta \in \mathbf{P}$ for which $\mathrm{f}(\beta) = b$. Otherwise let $C^-(M) = C(\sum d_{a_-}M + \sum a_+M)$ where $a_\pm$ runs through all arrows with head $h(C^{-1})$ and sign $\pm s(C^{-1})$.

Suppose instead $C \in \underline{\mathcal{W}}_{v,\delta}$ is a homotopy $\mathbb{N}$-word. In this case let $C^+(M)$ be the set of all $m \in e_vM$ with a sequence of elements $(m_i) \in \prod_{i \in \mathbb{N}} e_{v_C(i)}M$ satisfying $m_0 = m$ and $m_i \in l_{i+1}^{-1}r_{i+1}m_{i+1}$ for each $i \geq 0$, and let $C^-(M)$ be the subset of $C^+(M)$ where each sequence $(m_i)$ is eventually zero. Equivalently $C^-(M) = \bigcup_{n \in \mathbb{N}} C_{\leq n}0$.

**Example 2.1.18.** Recall example 1.3.28. For the homotopy $\mathbb{N}$-word

$$C = x^{-2}d_xy^{-1}d_yx^{-2}d_xd_y^{-1}y^3d_x^{-1}xy^{-1}d_yx^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$$

we have the table

| $i$ | $C_{<i}$ | $C_{>i}$ |
|---|---|---|
| 1 | $\underline{1}_{v,-1}$ | $y^{-1}d_yx^{-2}d_xd_y^{-1}y^3d_x^{-1}xy^{-1}d_yx^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$ |
| 2 | $x^{-2}d_x$ | $x^{-2}d_xd_y^{-1}y^3d_x^{-1}xy^{-1}d_yx^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$ |
| 3 | $x^{-2}d_xy^{-1}d_y$ | $d_y^{-1}y^3d_x^{-1}xy^{-1}d_yx^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$ |
| 4 | $x^{-2}d_xy^{-1}d_yx^{-2}d_x$ | $d_x^{-1}xy^{-1}d_yx^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$ |
| 5 | $x^{-2}d_xy^{-1}d_yx^{-2}d_xd_y^{-1}y^3$ | $y^{-1}d_yx^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$ |
| 6 | $x^{-2}d_xy^{-1}d_yx^{-2}d_xd_y^{-1}y^3d_x^{-1}x$ | $x^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$ |

$\vdots$

**Lemma 2.1.19.** *Let $C$ be a finite homotopy word such that the set $S \subseteq \mathbf{P}$ of all $\gamma$ for which $Cd^{-1}_{\mathrm{l}(\gamma)}\gamma$ is a word is non-empty. Then*

(i) *if $S$ is infinite then $C^{+}(M) = \bigcap Cd^{-1}_{\mathrm{l}(\gamma)}\gamma M$ where $\gamma$ runs through $S$, and*

(ii) *if $S$ is finite then $C^{+}(M) = Cd^{-1}_{\mathrm{l}(\gamma)}0$.*

*Proof.* (i) Let $m \in \bigcap Cd^{-1}_{\mathrm{l}(\gamma)}\gamma M$. For any $\beta \in \mathbf{P}$ for which $\mathrm{l}(\beta) = \mathrm{l}(\gamma)$ (we have $Cd^{-1}_{\mathrm{l}(\beta)}\beta$ is a homotopy word and so) there are some $m'_{\beta} \in e_{h(C^{-1})}M$ and some $m''_{\beta} \in e_{t(\beta)}M$ for which $m \in Cm'_{\beta}$ and $d_{\mathrm{l}(\beta),M}(m'_{\beta}) = \beta m''_{\beta}$. We now fix an arbitrary $\mu \in \mathbf{P}$ with $\mathrm{l}(\mu) = \mathrm{l}(\gamma)$ and show that $m \in Cd^{-1}_{\mathrm{l}(\mu)}\mu\mathrm{rad}(M)$. This will show $\bigcap Cd^{-1}_{\mathrm{l}(\gamma)}\gamma M \subseteq \bigcap Cd^{-1}_{\mathrm{l}(\gamma)}\gamma\mathrm{rad}(M)$ and the other inclusion is obvious. By assumption there is some arrow $\alpha$ with $\mu\alpha \in \mathbf{P}$. Since $\mathrm{l}(\mu\alpha) = \mathrm{l}(\mu)$ there are some $m'_{\mu\alpha} \in e_{h(C^{-1})}M$ and some $m''_{\mu\alpha} \in e_{t(\alpha)}M$ for which $m \in Cm'_{\mu\alpha}$ and $d_{\mathrm{l}(\mu),M}(m'_{\mu\alpha}) = \beta\alpha m''_{\beta\alpha}$ by the above. As $d_{\mathrm{l}(\mu),M} = d_{\mathrm{l}(\gamma),M}$ and $\alpha m''_{\beta\alpha} \in e_{t(\beta)}\mathrm{rad}(M)$ we have that $m \in Cd^{-1}_{\mathrm{l}(\mu)}\mu\alpha M \subseteq Cd^{-1}_{\mathrm{l}(\mu)}\mu\mathrm{rad}(M)$.

(ii) By assumption $S$ has $n \geq 1$ elements, and by corollary 2.1.15 (iv) we can write $S = \{\gamma_1 \dots \gamma_i \mid 1 \leq i \leq n\}$ for arrows $\gamma_i$ where $\gamma_1 = \mathrm{l}(\gamma)$. By corollary 2.1.10 we have a chain $Cd^{-1}_{\gamma_1}\gamma_1\mathrm{rad}(M) \supseteq \cdots \supseteq Cd^{-1}_{\gamma_1}\gamma_1 \dots \gamma_n\mathrm{rad}(M)$ and so $C^{+}(M) = Cd^{-1}_{\gamma_1}\gamma_1 \dots \gamma_n\mathrm{rad}(M)$. Since $\gamma_1 \dots \gamma_n$ is a maximal length path with first arrow $\gamma_1$ there can be no arrow $\gamma_{n+1}$ satisfying $\gamma_1 \dots \gamma_{n+1} \in \mathbf{P}$. Since $\Lambda$ is a complete gentle algebra this gives $\gamma_n\mathrm{rad}(M) = 0$ by corollary 1.2.18, and so $Cd^{-1}_{\gamma_1}\gamma_1 \dots \gamma_n\mathrm{rad}(M) \subseteq Cd^{-1}_{\gamma_1}0$ as required. □

**Corollary 2.1.20.** *If $C \in \underline{\mathcal{W}}_{v,\delta}$ then*

(i) *if $g : M^{\bullet} \to N^{\bullet}$ is a morphism in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ then $\mathrm{im}(g|_{C^{\pm}(M^{\bullet})}) \subseteq C^{\pm}(N^{\bullet})$, and*

(ii) *if $M^{\bullet}$ is an object in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ then $C^{-}(M^{\bullet}) \subseteq C^{+}(M^{\bullet})$.*

*Proof.* (i) It is enough to show that if $(m, m')$ lies in $U$ then $(g(m), g(m'))$ lies in $V$ where ($U = \mathrm{rel}^{\lambda}(M)$ and $V = \mathrm{rel}^{\lambda}(N)$ for some path $\lambda \in \mathbf{P}$) or ($U = \mathrm{graph}(d_{a,M})$ and $V = \mathrm{graph}(d_{a,N})$ for some arrow $a$). Since $g$ is a chain map between $\Lambda$-modules $g$ is linear over $\Lambda$, and so we have $g(m') = g(\lambda m) = \lambda g(m)$ if $(m, m') \in \mathrm{rel}^{\lambda}(M)$. If instead $(m, m') \in \mathrm{graph}(d_{a,M})$ then for $v = h(a)$ we have $gd_M\big|_v = d_N\big|_v g$ by lemma 2.1.2 and so $\sum_{\beta \in \mathbf{A} \to v} g(d_{\beta,M}(m)) - d_{\beta,N}(g(m)) = 0$, which is a direct sum by lemma 2.1.1.

(ii) Assume firstly that $C$ is finite. Clearly there is nothing to prove given $C^{+}(M^{\bullet}) = CM$. So we may assume $C^{+}(M^{\bullet}) = \bigcap Cd_{\mathrm{l}(\gamma)}^{-1}\gamma\mathrm{rad}(M)$ where the intersection is taken over all $\gamma \in \mathbf{P}$ for which $Cd_{\mathrm{l}(\gamma)}^{-1}\gamma$ is a word, where such a $\gamma$ exists.

Suppose $C^{-}(M) = \bigcup C\beta^{-1}d_{\mathrm{l}(\beta)}M$ in the above. Then for $m \in C^{-}(M)$ there exists some $\beta \in \mathbf{P}$ for which $C\beta^{-1}d_{\mathrm{l}(\beta)}$ is a word and $m \in C\beta^{-1}d_{\mathrm{l}(\beta)}M$. Hence by corollary 2.1.10 (iv) we have $m \in Cd_{\mathrm{l}(\gamma)}^{-1}0 \subseteq \bigcap Cd_{\mathrm{l}(\gamma)}^{-1}\gamma\mathrm{rad}(M)$. So instead suppose $C^{-}(M^{\bullet}) = C(\sum \mathrm{im}(d_{\alpha_{-},M}) + \sum \alpha_{+}M)$ where the first (resp. second) sum is taken over all arrows $\alpha_{-}$ (resp. $\alpha_{+}$) with head $h(C^{-1})$ and sign $-s(C^{-1})$ (resp. $s(C^{-1})$). So $C\sum\alpha_{+}M \subseteq Cd_{\mathrm{l}(\gamma)}^{-1}\gamma 0 \subseteq C^{+}(M^{\bullet})$ by corollary 2.1.10 (ii) and $C\sum \mathrm{im}(d_{\mathrm{l}(\alpha_{-}),M}) \subseteq Cd_{\mathrm{l}(\gamma)}^{-1}\gamma 0 \subseteq C^{+}(M)$ by corollary 2.1.10 (iii), as required.

Now we can assume $C$ is an $\mathbb{N}$-word. Clearly if $m \in \bigcup_{n\in\mathbb{N}} C_{\leq n}0$ then $m \in C_{\leq n}0$ and so there is a sequence $(x_i \mid i \in \mathbb{N}, 0 < i \leq n) \in \prod e_{v_C(i)}M$ for which $(x_{i-1}, x_i) \in (l_i^{-1}r_i, M)$ for $i > 0$. Hence there is a sequence $(x_i \mid i \in \mathbb{N}) \in \prod e_{v_C(i)}M$ defined by letting $x_j = 0$ for $j > n$ and again we have $(x_{i-1}, x_i) \in (l_i^{-1}r_i, M)$ for $i > 0$, and so $m \in C^{+}(M^{\bullet})$. This shows $C^{-}(M^{\bullet}) \subseteq C^{+}(M^{\bullet})$. □

We have shown that any homotopy word $C$ from $\underline{\mathcal{W}}_{v,\delta}$ defines a pair of subfunctors $C^{-} \leq C^{+}$ of the forgetful functor $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to R\text{-}\mathbf{Mod}$ (which takes a complex $M^{\bullet}$ to the underlying $R$-module $M$).

**Lemma 2.1.21.** *Let $l^{-1}r, C \in \underline{\mathcal{W}}_{v,\delta}$, $\mathcal{Z}$ be an index set, $\{X_z \mid z \in \mathcal{Z}\}$ a set of objects in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ and $X = \bigoplus_{z\in\mathcal{Z}} X_z$. Then,*

(i) *$\gamma X = \bigoplus_{z\in\mathcal{Z}} \gamma X_z$ for each $\gamma \in \mathbf{P}$ and $d_{\alpha,X} = \bigoplus_{z\in\mathcal{Z}} d_{\alpha,X_z}$ for each arrow $\alpha$,*

(ii) *for $x = \sum_{z\in\mathcal{Z}} x_z$ and $x' = \sum_{z\in\mathcal{Z}} x'_z$ in $X$, one has $x \in l^{-1}rx'$ iff $x_z \in l^{-1}rx'_z$ for each $z$,*

(iii) *if $C$ is finite then $C(X) = \bigoplus_{z\in\mathcal{Z}} C(X_z)$,*

(iv) *for any $C$ we have $C^{+}(X) = \bigoplus_{z\in\mathcal{Z}} C^{+}(X_z)$, and*

(v) *for any $C$ we have $C^{-}(X) = \bigoplus_{z\in\mathcal{Z}} C^{-}(X_z)$.*

*Proof.* For any $z \in \mathcal{Z}$ let $p_z$ denote the natural projection $X \to X_z$ and $i_z$ the natural embedding $X_z \to X$. Note that we are now considering relations on the $R$-module $X = \bigoplus_{z\in\mathcal{Z}} X_z$. For any object $M$ in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ and any arrow $a$ with head $v$ recall $\pi_{a,M} : \bigoplus_{c:\to v} cM \to aM$ and $\iota_{a,M} : aM \to \bigoplus_{c:\to v} cM$ denote the natural projections and inclusions in the category of $R$-modules. Recall $d_{a,M}$ is the map sending $m \in e_vM$ to $\iota_{a,M}(\pi_{a,M}(d_M|_v(m)))$.

(i) Since $X_z$ is a complex of $\Lambda$-modules the equality of $R$-modules $\gamma X = \bigoplus_z \gamma X_z$ is obvious. By construction, for any $x = \sum x_z \in \bigoplus_z e_vX_z$ we have $d_X|_{e_vX}(\sum x_z) = \sum_z d_X|_{e_vX}(x_z)$ and $\iota_{\alpha,X}\pi_{\alpha,X}(d_{X_z}|_{e_vX_z}(x_z)) = \iota_{\alpha,X_z}\pi_{\alpha,X_z}(d_{X_z}|_{e_vX_z}(x_z))$ for each $z$, and so $d_{\alpha,X}(x) = \sum_z d_{\alpha,X_z}(x_z)$.

(ii) If $x \in l_n^{-1}r_nx'$ then for each $z$ we have that $x_z = p_z(\sum_z x_z)$ lies in $l_n^{-1}r_n(p_z(\sum_z x'_z)) = l_n^{-1}r_nx'_z$ by corollary 2.1.20. Conversely if $x_z \in l_n^{-1}r_nx'_z$ for each $z$ then we have $x \in l^{-1}rx'$ since $x_z = x'_z = 0$ for all but finitely many $z$, and since $(l^{-1}r)$ defines a linear relation.

(iii) There is nothing to prove when $C$ is trivial so we may assume otherwise. Let $C = l_1^{-1}r_1 \dots l_t^{-1}r_t$. For $x \in C(X)$ there are elements $x_{z,n} \in X_i$ for each $z \in \mathcal{Z}$ and each integer $n$ with $0 \leq n \leq t$ such that; each sum $\sum_z x_{z,n}$ has finite support, $x = \sum_z x_{z,0}$, and $\sum x_{z,n-1} \in l_n^{-1}r_n \sum x_{z,n}$ when $n > 1$. So by (ii) $x_{z,n-1} \in l_n^{-1}r_nx_{z,n}$ for each $z$ when $n > 1$, which shows $x \in \bigoplus_z C(X_z)$. Now suppose $x' \in \bigoplus_z C(X_z)$. Here $x' = \sum_z x_{z,0}$ with finite support where for each $z$ we have $x_{z,0} \in Cx_{z,t}$ for some $x_{z,t} \in e_{h(C^{-1})}X_z$. We can write $x' = x_{z(1),0} + \cdots + x_{z(q),0}$ for some $z(1), \dots, z(q) \in \mathcal{Z}$. By definition there are elements $x_{z(j),n} \in X_{z(j)}$ for each $j$ and $n$ with $1 \leq j \leq q$ and $0 \leq n \leq t$ where $x_{z(j),n-1} \in l_n^{-1}r_nx_{z(j),n}$ for $n > 0$. Again by (ii) this shows $\sum_{j=1}^q x_{z(j),n-1} \in l_n^{-1}r_n \sum_{j=1}^q x_{z(j),n}$ and hence $x = \sum_{j=1}^q x_{z(j),0}$ which lies in $C\sum_{j=1}^q x_{z(j),t} \subseteq C(X)$.

(iv) Suppose $C$ is finite homotopy word. If there is no arrow $a$ for which $Cd_a^{-1}a$ is a homotopy word then $C^+(M) = CM$ and the result follows by (i). Hence we may assume $a$ exists. If only finitely many such $\gamma \in \mathbf{P}$ with $\mathrm{l}(\gamma) = a$ exist then $C^+(M) = Cd_a^{-1}\alpha 0$ (where $\alpha$ is the longest of all such $\gamma$) by lemma 2.1.19 (ii), and by (i) the result holds. Assuming infinitely many such $\gamma$ exist, so $C^+(M) = \bigcap Cd_a^{-1}\gamma M$ by lemma 2.1.19 (ii).

Since arbitrary intersections and arbitrary direct sums commute we have $\bigcap_\gamma \bigoplus_z Cd_a^{-1}\gamma X_z = \bigoplus_z \bigcap_\gamma Cd_a^{-1}\gamma X_z$, and also $\bigcap_\gamma Cd_a^{-1}\gamma \bigoplus_z X_z = \bigcap_\gamma \bigoplus_z Cd_a^{-1}\gamma X_z$ by part (i). Altogether this shows $C^+(X) = \bigoplus_z C^+(X_z)$ which concludes the case where $C$ is finite. Now suppose $C$ is a homotopy $\mathbb{N}$-word, say $C = l_1^{-1}r_1 \dots l_t^{-1}r_t \dots$ in which case

$$C^+(X) = \left\{x_0 \in e_v \bigoplus_z X_z \mid \exists(x_n \mid n \geq 1) \in \prod e_{v_C(n)}X : x_n \in l_{n+1}^{-1}r_{n+1}x_{n+1} \ \forall n \geq 0\right\}$$

For $x \in C^+(X)$ there is some collection $\{x_{z,n} \in X_z \mid z \in \mathcal{Z}, n \in \mathbb{N}\}$ where each sum $\sum_z x_{z,n}$ has finite support, $x = \sum x_{z,0}$, and $(\sum_z x_{z,n-1}) \in l_n^{-1}r_n(\sum x_{z,n})$ when $n > 1$. By (ii) again we have $x \in \bigoplus_z C^+(X_z)$.

Now suppose $x' \in \bigoplus_z C(X_z)$, say $x' = \sum_z x_{z,0}$ where for each $z \in \mathcal{Z}$ we have $x_{z,0} \in C^+(M)$ for some sequence $(x_{z,n} \mid n \geq 1) \in \prod e_{v_C(n)}X_z$ such that $x_{z,n} \in l_{n+1}^{-1}r_{n+1}x_{z,n+1}$ for all $n \geq 1$. As above we can write $x' = x_{z(1),0} + \dots + x_{z(q),0}$ for some $z(1), \dots, z(q) \in \mathcal{Z}$ and defining the sequence $(\sum_{j=1}^q x_{z(j),n} \mid n \geq 1)$ in $\prod e_{v_C(n)}X$ gives $x' \in C(\bigoplus_{z\in\mathcal{Z}} X_z)$ after applying (ii).

(v) Again start by assuming $C$ is finite. Suppose there exists some arrow $b$ for which $Cb^{-1}d_b$ is a homotopy word. In this case we have $C^-(X) = \bigcup \bigoplus_z C\beta^{-1}d_{\mathrm{l}(\beta)}X_z$ by part (iii), where the union is taken over all $\beta$ with $\mathrm{f}(\beta) = b$. Now for $x \in C^-(X)$ the above shows that there is some $\alpha \in \mathbf{P}$ for which $C\alpha^{-1}d_{\mathrm{l}(\alpha)}$ is a word and $x \in \bigoplus_z C\alpha^{-1}d_{\mathrm{l}(\alpha)}X_z$. Hence $x$ is an element from $\bigoplus_z \bigcup C\beta^{-1}d_{\mathrm{l}(\beta)}X_z = \bigoplus_z C^-(X_z)$.

Now suppose $x' \in \bigoplus_z C^-(X_z)$, again say $x' = \sum_{j=1}^q x_{z(j),0}$ where for each $j$ there is some $\alpha_j \in \mathbf{P}$ for which $C\alpha_j^{-1}d_{\mathrm{l}(\alpha_j)}$ is a homotopy word and $x_{z(j),0} \in C\alpha_j^{-1}d_{\mathrm{l}(\alpha_j)}X_{z(j)}$. Without loss of generality we may assume $\alpha_q$ is the longest of the paths $\alpha_1, \dots, \alpha_q \in \mathbf{P}$.

By corollary 2.1.10 (i) this gives $m_{z(j),0} \in C\alpha_q^{-1}d_{\mathrm{l}(\alpha_q)}X_{z(j)}$ for each $j$ and so $x' \in \bigoplus_z C\alpha_q^{-1}d_{\mathrm{l}(\alpha_q)}X_z = C\alpha_q^{-1}d_{\mathrm{l}(\alpha_q)}X$ by part (iii), which gives $m \in \bigcup C\beta^{-1}d_{\mathrm{l}(\beta)} \bigoplus_z X_z$ as required.

We may now assume there is no $\alpha \in \mathbf{P}$ for which $C\alpha^{-1}d_{\mathrm{l}(\alpha)}$ is a homotopy word. Here by parts (i) and (ii) we have $\sum_{\alpha_-} \mathrm{im}(d_{\alpha_-,X}) = \bigoplus_z(\sum_{\alpha_-} \mathrm{im}(d_{\alpha_-,X_z}))$ and $\sum_{\alpha_+} \alpha_+X = \bigoplus_z(\sum_{\alpha_+} \alpha_+X_z))$ and so $C^-(X) = \bigoplus_z C(\sum_{\alpha_-} \mathrm{im}(d_{\alpha_-,X_z}) + \sum_{\alpha_+} \alpha_+X_z)$.

Now assume $C$ is infinite. In this case, by part (iii) we have $C_{\leq n}0_X = \bigcup_{n\geq 0}\bigoplus_z C_{\leq n}0_{X_z}$ which is contained in $\bigoplus_z \bigcup_n C_{\leq n}0_{X_z} = \bigoplus_z C^-(X_z)$. Now suppose $x \in \bigoplus_z C^-(X_z)$ say $x = \sum_{j=1}^q x_{z(j),0}$ where for each $j$ there is some $n_j \geq 0$ for which $x_{z(j),0} \in C_{\leq n_j}0_{X_z}$. Without loss of generality we may assume $n_q$ is the largest of $\{n_1, \dots, n_q\}$ and so $x_{z(j),0} \in C_{\leq n_q}0_{X_z}$ for each $j$. Now $\bigoplus_z C_{\leq n}0_{X_z} = C_{\leq n}0_X$ by part (iii), which together gives $x \in \bigcup_n \bigoplus_z C_{\leq n}0_{X_z} = C^-(X)$ as required. $\square$

### 2.1.2 Ordering Homotopy Words.

Fix some vertex $v$ and some $\delta \in \{\pm 1\}$. We now introduce an ordering on the set $\underline{\mathcal{W}}_{v,\delta}$. To do so an ordering is introduced on the set of pairs $(l, r)$ of letters for which a homotopy word $C$ may be extended to a homotopy word $Cl^{-1}r$.

**Lemma 2.1.22.** *Suppose $l$, $l'$, $r$, and $r'$ are homotopy letters for which $l^{-1}r$ and $l'^{-1}r'$ are distinct homotopy words in $\underline{\mathcal{W}}_{v,\delta}$. Then there exists $\alpha, \alpha' \in \mathbf{P}$ such that $\alpha \neq \alpha'$ and one of the following hold*

(i) $l^{-1}r = d^{-1}_{\mathrm{l}(\alpha)}\alpha$ *and* $l'^{-1}r' = d^{-1}_{\mathrm{l}(\alpha')}\alpha'$ *where* $\mathrm{l}(\alpha) = \mathrm{l}(\alpha')$,

(ii) $l^{-1}r = d^{-1}_{\mathrm{l}(\alpha)}\alpha$ *and* $l'^{-1}r' = \alpha'^{-1}d_{\mathrm{l}(\alpha')}$ *where* $\alpha'\alpha \in \mathbf{P}$,

(iii) $l^{-1}r = \alpha^{-1}d_{\mathrm{l}(\alpha)}$ *and* $l'^{-1}r' = \alpha'^{-1}d_{\mathrm{l}(\alpha')}$ *where* $\mathrm{f}(\alpha) = \mathrm{f}(\alpha')$, *or*

(iv) $l^{-1}r = \alpha^{-1}d_{\mathrm{l}(\alpha)}$ *and* $l'^{-1}r' = d^{-1}_{\mathrm{l}(\alpha')}\alpha'$ *where* $\alpha\alpha' \in \mathbf{P}$.

*Proof.* If $l^{-1}r = d^{-1}_{\mathrm{l}(\alpha)}\alpha$ and $l'^{-1}r' = d^{-1}_{\mathrm{l}(\alpha')}\alpha'$ for some $\alpha, \alpha' \in \mathbf{P}$ then by corollary 2.1.14 (ii) we have $\mathrm{l}(\alpha) = \mathrm{l}(\alpha')$. Since these words are distinct we have $\alpha \neq \alpha'$. In this case we are in situation (i). If instead $l^{-1}r = \alpha^{-1}d_{\mathrm{l}(\alpha)}$ and $l'^{-1}r' = \alpha'^{-1}d_{\mathrm{l}(\alpha')}$ for some $\alpha, \alpha' \in \mathbf{P}$ then by corollary 2.1.14 (i) we have $\mathrm{f}(\alpha) = \mathrm{f}(\alpha')$. In this case we are in situation (iv) because $\alpha \neq \alpha'$ as above. If neither (i) nor (iv) hold then either ($l^{-1}r = \alpha^{-1}d_{\mathrm{l}(\alpha)}$ and $l'^{-1}r' = d^{-1}_{\mathrm{l}(\alpha')}\alpha'$) or ($l^{-1}r = d^{-1}_{\mathrm{l}(\alpha)}\alpha$ and $l'^{-1}r' = \alpha'^{-1}d_{\mathrm{l}(\alpha')}$) for some $\alpha, \alpha' \in \mathbf{P}$. In general, when $\beta^{-1}d_{\mathrm{l}(\beta)}$ and $d^{-1}_{\mathrm{l}(\beta')}\beta'$ are words with the same head and sign (for $\beta, \beta' \in \mathbf{P}$ some) we have $\beta\beta' \in \mathbf{P}$ as $h(\beta^{-1}) = h(d^{-1}_{\mathrm{l}(\beta')})$ and $s(\beta^{-1}) = s(d^{-1}_{\mathrm{l}(\beta')})$ and so $t(\beta) = h(\beta')$ and $s(\mathrm{f}(\beta)^{-1}) = -s(\mathrm{l}(\beta'))$ respectively. $\square$

**Definition 2.1.23.** (Ordering Pairs of Homotopy Letters) If $l^{-1}r$ and $l'^{-1}r'$ are distinct homotopy words in $\underline{\mathcal{W}}_{v,\delta}$ we write $(l, r) < (l', r')$ if

(I) $l^{-1}r = d^{-1}_{\mathrm{l}(\gamma)}\gamma$ and $l'^{-1}r' = d^{-1}_{\mathrm{l}(\gamma)}\gamma\upsilon$ for some $\gamma, \upsilon \in \mathbf{P}$ such that $\gamma\upsilon \in \mathbf{P}$, or

(II) $l^{-1}r = \mu^{-1}d_{\mathrm{l}(\mu)}$ and $l'^{-1}r' = d^{-1}_{\mathrm{l}(\eta)}\eta$ for some $\mu, \eta \in \mathbf{P}$ such that $\mathrm{f}(\mu)\mathrm{l}(\eta) \in \mathbf{P}$, or

(III) $l^{-1}r = \lambda^{-1}d_{\mathrm{l}(\lambda)}$ and $l'^{-1}r' = \lambda^{-1}\kappa^{-1}d_{\mathrm{l}(\kappa)}$ for some $\kappa, \lambda \in \mathbf{P}$ such that $\kappa\lambda \in \mathbf{P}$.

**Lemma 2.1.24.** *For a fixed vertex $v$ and $\delta \in \{\pm 1\}$, the relation $<$ from definition 2.1.23 gives a total order on the set of pairs $(l, r)$ for which $l^{-1}r \in \underline{\mathcal{W}}_{v,\delta}$.*

*Proof.* Since $(Q, \rho)$ satisfies gentle conditions $\mu\eta \in \mathbf{P}$ implies $\mathrm{f}(\mu)\mathrm{l}(\eta) \in \mathbf{P}$ for any $\mu, \eta \in \mathbf{P}$. Together with lemma 2.1.22 this shows distinct pairs $l^{-1}r$ and $l'^{-1}r'$ from $\underline{\mathcal{W}}v, \delta$ are comparable. It is enough to prove transitivity on non-equal pairs. We suppose $(l, r) < (l', r') < (l'', r'')$ and proceed to show $(l, r) < (l'', r'')$ with case analysis.

Suppose $l = l' = d_{\mathrm{l}(\gamma)}$ and $(r, r') = (\gamma\upsilon, \gamma)$ for some $\gamma, \upsilon \in \mathbf{P}$ such that $\gamma\upsilon \in \mathbf{P}$. The only possibility is that $l' = l'' = d_{\mathrm{l}(\lambda)}$ and $(r', r'') = (\lambda\eta, \lambda)$ for some $\lambda, \eta \in \mathbf{P}$ such that $\lambda\eta \in \mathbf{P}$ and $\mathrm{l}(\gamma) = \mathrm{l}(\lambda)$. So $\gamma = \lambda\eta$ and $(r, r'') = (\lambda\eta\upsilon, \lambda)$ which shows $(l, r) < (l'', r'')$.

Suppose instead $(l, l') = (\mu, d_{\mathrm{l}(\eta)})$ for some $\mu, \eta \in \mathbf{P}$ such that $\mathrm{f}(\mu)\mathrm{l}(\eta) \in \mathbf{P}$. The only possibility is that $l' = l'' = d_{\mathrm{l}(\gamma)}$ and $(r', r'') = (\gamma\upsilon, \gamma)$ (for some $\gamma, \upsilon \in \mathbf{P}$ such that $\gamma\upsilon \in \mathbf{P}$) we have that $\mathrm{l}(\gamma) = \mathrm{l}(\eta)$ and so $(l, l'') = (\mu, d_{\mathrm{l}(\gamma)})$ where $\mathrm{f}(\mu)\mathrm{l}(\gamma) \in \mathbf{P}$. Hence $(l, r) < (l'', r'')$.

Finally, suppose $(l, l') = (\lambda, \kappa\lambda)$ for some $\kappa, \lambda \in \mathbf{P}$ such that $\kappa\lambda \in \mathbf{P}$. There are two cases here. Suppose $(l', l'') = (\gamma, \eta\gamma)$ for some $\eta, \gamma \in \mathbf{P}$ such that $\eta\gamma \in \mathbf{P}$. Here $\gamma = \kappa\lambda$ and as $\eta\gamma \in \mathbf{P}$ we have that $\eta\kappa \in \mathbf{P}$ and $(l, l'') = (\lambda, \eta\kappa\lambda)$ which means $(l, r) < (l'', r'')$. Alternatively $(l', l'') = (\mu, d_{\mathrm{l}(\eta)})$ for some $\mu, \eta \in \mathbf{P}$ such that $\mathrm{f}(\mu)\mathrm{l}(\eta) \in \mathbf{P}$. Here $\mu = \kappa\lambda$ and so $\mathrm{f}(\mu) = \mathrm{f}(\lambda)$, and hence $\mathrm{f}(\lambda)\mathrm{l}(\eta) \in \mathbf{P}$. This shows $(l, r) < (l'', r'')$ since $(l, l'') = (\lambda, d_{\mathrm{l}(\eta)})$ where $\mathrm{f}(\lambda)\mathrm{l}(\eta) \in \mathbf{P}$. If $(l, r) < (l, r)$ then $l = d_{\mathrm{l}(\gamma)}$ and $(r, r) = (\gamma\upsilon, \gamma)$ for some $\gamma, \upsilon \in \mathbf{P}$, which means $\upsilon$ is trivial which is impossible. □

**Example 2.1.25.** Let $\Lambda$ be the complete gentle $k[[t]]$-algebra $\Lambda = k[[x, y]]/(xy)$ from example 1.2.30. Then there is a chain of the form

$$(d_x, x) > (d_x, x^2) > (d_x, x^3) > \cdots > (x^3, d_x) > (x^2, d_x) > (x, d_x)$$

Our construction ensures we can extend this lexicographically to a total order on $\underline{\mathcal{W}}_{v,\delta}$ as follows.

**Definition 2.1.26.** (Ordering Homotopy Words) For distinct homotopy words $C, C'$ from $\underline{\mathcal{W}}_{v,\delta}$ we say $C < C'$ if one of the following hold:

(I) there are homotopy letters $l$, $l'$, $r$ and $r'$ and homotopy words $B, D, D'$ for which $C = Bl^{-1}rD$, $C' = Bl'^{-1}r'D'$ and $(l, r) < (l', r')$,

(II) there is some $\beta \in \mathbf{P}$ for which $C' = Cd_{\mathrm{l}(\beta)}^{-1}\beta E$ for some homotopy word $E$,

(III) there is some $\alpha \in \mathbf{P}$ for which $C = C'\alpha^{-1}d_{\mathrm{l}(\alpha)}E'$ for some homotopy word $E'$.

**Lemma 2.1.27.** *The relation $<$ from definition 2.1.26 gives a total order on the set $\underline{\mathcal{W}}_{v,\delta}$.*

*Proof.* Fix distinct words $C$ and $C'$ from $\underline{\mathcal{W}}_{v,\delta}$. Firstly, suppose $C' = CC''$ for some homotopy word $C''$. Since $C$ and $C'$ are distinct homotopy words there is some $\beta \in \mathbf{P}$ for which $C' = Cd_{\mathrm{l}(\beta)}^{-1}\beta E$ for some homotopy word $E$, or there is some $\alpha \in \mathbf{P}$ for which $C' = C\alpha^{-1}d_{\mathrm{l}(\alpha)}E'$ for some homotopy word $E'$. In the former, $C < C'$, and in the latter, $C' < C$. We can now assume $C$ is not the prefix of $C'$, and $C'$ is not the prefix of $C$. So, neither homotopy word is trivial, and we let $B$ denote their longest common prefix. This means there exist letters $l$, $l'$, $r$ and $r'$ and words $B, D, D'$ for which $l^{-1}r \neq l'^{-1}r'$ and $C = Blr^{-1}D$, $C' = Bl'^{-1}r'D'$. We must then have $C < C'$ or $C' < C$ by lemma 2.1.22. By lemma 2.1.24 this shows any two distinct words are comparable. Again it is sufficient to show transitivity. Now suppose for some words $C$, $C'$ that $C < C'$ so:

(I) there are homotopy letters $l_1$, $l'_1$, $r_1$ and $r'_1$ and homotopy words $B_1, D_1, D'_1$ for which $C = B_1l_1^{-1}r_1D_1$, $C'_1 = B_1l_1'^{-1}r'_1D'_1$ and $(l_1, r_1) < (l'_1, r'_1)$;

(II) there is some $\beta_1 \in \mathbf{P}$ for which $C' = Cd_{\mathrm{l}(\beta_1)}^{-1}\beta_1E_1$ for some homotopy word $E_1$; or

(III) there is some $\alpha_1 \in \mathbf{P}$ for which $C = C'\alpha_1^{-1}d_{\mathrm{l}(\alpha_1)}E'_1$ for some homotopy word $E'_1$.

For another homotopy word $C''$ suppose also $C' < C''$ so that:

(I') there are letters $l_2$, $l'_2$, $r_2$ and $r'_2$ and words $B_2, D_2, D'_2$ for which $C' = B_2l_2^{-1}r_2D_2$, $C'' = B_2l_2'^{-1}r'_2D'_2$ and $(l_2, r_2) < (l'_2, r'_2)$;

(II') there is some $\beta_2 \in \mathbf{P}$ for which $C'' = C'd_{\mathrm{l}(\beta_2)}^{-1}\beta_2E_2$ for some homotopy word $E_2$; or

(III') there is some $\alpha_2 \in \mathbf{P}$ for which $C' = C''\alpha_2^{-1}d_{\mathrm{l}(\alpha_2)}E'_2$ for some homotopy word $E'_2$.

There are 9 possible cases, each consisting of one of I, II, or III; and one of I', II', III'. We go through cases to show $C < C''$ and hence prove transitivity.

(I,I') From these assumptions $B_1 l_1'^{-1} r_1' D_1' = B_2 l_2^{-1} r_2 D_2$. If $B_1 = B_2$, then $l_1' = l_2$ and $r_1' = r_2$ in which case we have $C < C''$ since $(l_1, r_1) < (l_1', r_1') < (l_2', r_2')$ and so $(l_1, r_1) < (l_2', r_2')$ from lemma 2.1.22. Otherwise $B_1 l_1'^{-1} r_1'$ is a homotopy subword of $B_2$, or $B_2 l_2^{-1} r_2$ is a homotopy subword of $B_1$. In the former $C'' = B_1 l_1'^{-1} r_1' E_3$ for some homotopy word $E_3$ and $(l_1, r_1) < (l_1', r_1')$, so $C < C''$. In the latter $C = B_2 l_2^{-1} r_2 E_4$ for some homotopy word $E_4$ and $(l_2, r_2) < (l_2', r_2')$, so $C < C''$.

(I,II') $C'' = B_1 l_1'^{-1} r_1' D_1' d_{\mathrm{l}(\beta_2)}^{-1} \beta_2 E_2$ and $(l_1, r_1) < (l_1', r_1')$ so $C < C''$. The cases (III,I') and (III,II') are similar and omitted.

(I,III') Here $C'' \alpha_2^{-1} d_{\mathrm{l}(\alpha_2)} E_2' = B_1 l_1'^{-1} r_1' D_1'$. If $C'' = B_1$ then $l_1' = \alpha_2$ and $r_1' = d_{\mathrm{l}(\alpha_2)}$ and as $(l_1, r_1) < (l_1', r_1')$ we must have $l_1 = \lambda$ and $l_1' = \kappa\lambda$ for some $\kappa \in \mathbf{P}$ with $\kappa\lambda = \alpha_2 \in \mathbf{P}$. Hence $r_1 = d_{\mathrm{l}(\lambda)}$ since $C$ is a homotopy word and so $C = B_1 l_1^{-1} r_1 D_1 = C'' \lambda^{-1} d_{\mathrm{l}(\lambda)} D_1$ which means $C < C''$. Otherwise $C'' \neq B_1$ and so either $C'' \alpha^{-1} d_{\mathrm{l}(\alpha)}$ is a homotopy subword of $B_1$ or $B_1 l_1^{-1} r_1$ is a homotopy subword of $C''$. In the former we have $C = C'' \alpha_2^{-1} d_{\mathrm{l}(\alpha_2)} E_5$ for some homotopy word $E_5$ in which case $C < C''$. Otherwise $C'' = B_1 l_1'^{-1} r_1' E_6$ for some homotopy word $E_6$ and so as $C = B_1 l_1^{-1} r_1 D_1$ with $(l_1, r_1) < (l_1', r_1')$ we have $C < C''$.

(II,I') Here $C d_{\mathrm{l}(\beta_1)}^{-1} \beta_1 E_1 = B_2 l_2^{-1} r_2 D_2$. If $C = B_2$ then $l_2 = d_{\mathrm{l}(\beta_1)}$ and $r_2 = \beta_1$ which is a contradiction as $(d_{\mathrm{l}(\beta_1)}, \beta_1) < (l_2', r_2')$ is impossible for any homotopy letters $l_2'$ and $r_2'$. If $C = B_2 l_2^{-1} r_2 E_7$ for some homotopy word $E_7$ then as $(l_2, r_2) < (l_2', r_2')$ we have $C < C''$. Otherwise $B_2 = C d_{\mathrm{l}(\beta_1)}^{-1} \beta_1 E_8$ which means $C'' = C d_{\mathrm{l}(\beta_1)}^{-1} \beta_1 E_8 l_2^{-1} r_2 D_2$ so again $C < C''$.

(II,II') $C'' = C d_{\mathrm{l}(\beta_1)}^{-1} \beta_1 E_1 d_{\mathrm{l}(\beta_2)}^{-1} \beta_2 E_2'$ and so $C < C''$. The case (III,III') is similar.

(II,III') $C = C''$ gives the contradiction $d_{\mathrm{l}(\beta_1)} = \alpha_2$. If $C$ is longer than $C''$ then $C = C'' \alpha_2^{-1} d_{\mathrm{l}(\alpha_2)} B$ for some word $B$ and if $C''$ is longer than $C$ then $C'' = C d_{\mathrm{l}(\beta_1)}^{-1} \beta_1 B'$ for some homotopy word $B'$. In either case $C < C''$. To complete the proof we need to show $C < C$ is impossible. Otherwise the only possibility is that there are homotopy letters $l$, $l'$, $r$ and $r'$ and words $B, D, D'$ for which $C = B l^{-1} r D$, and $C = B l'^{-1} r' D'$ and $(l, r) < (l', r')$. But as $l = l'$ and $r = r'$, this is impossible. □

An order on the set $\mathcal{W}_{v,\delta}$ of words (with head $v$ and sign $\delta$) was given in definition 1.4.24. Recall that any word $w \in \mathcal{W}_{v,\delta}$ defined subfunctors $w^{\pm}$ of the forgetful functor ${}_R(-) : \Lambda\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$. Furthermore, if $w, w' \in \mathcal{W}_{v,\delta}$ and $w < w'$ then $w^+ \leqslant w'^-$ by lemma 1.4.27 (see [55, p.23, Lemma]). In proposition 2.1.30 we adapt lemma 1.4.27 for homotopy words.

**Lemma 2.1.28.** *Suppose $l^{-1}rD, l'^{-1}r'D' \in \underline{\mathcal{W}}_{v,\delta}$ for homotopy words $D$ and $D'$ and homotopy letters $l$, $l'$, $r$ and $r'$. If $(l,r) < (l',r')$ then $(l^{-1}rD)^+(M) \subseteq (l'^{-1}r'D')^-(M)$.*

*Proof.* The cases (I), (II) and (III) in what follows correspond to those in definition 2.1.23. In case (I) we have $s(\mathrm{l}(v)) = -s(r'^{-1})$, $(l^{-1}rD)^+(M) \subseteq d^{-1}_{\mathrm{l}(\gamma)}\gamma\mathrm{l}(v)M$ and $d^{-1}_{\mathrm{l}(\gamma)}\gamma(D')^-(M) = (l'^{-1}r'D')^-(M)$. This means $v$ has sign $s(D')$ which gives $\mathrm{l}(v)M \subseteq (D')^-(M)$ by lemma 2.1.29.

In case (II), lemma 2.1.2 shows that $\mu^{-1}d_{\mathrm{l}(\mu)}.M \subseteq d^{-1}_{\mathrm{l}(\eta)}0$ and so $(l^{-1}rD)^+(M) = \mu^{-1}d_{\mathrm{l}(\mu)}D^+(M)$ is contained in $(d^{-1}_{\mathrm{l}(\eta)}\eta D')^-(M) = (l'^{-1}r'D')^-(M)$. In case (III), we have $s(D') = s(\mathrm{l}(\kappa))$ which means $\mathrm{l}(\kappa)M \subseteq (D')^-(M)$ as above (by lemma 2.1.29). Furthermore by lemma 2.1.2 we have $\kappa d_{\mathrm{l}(\lambda)}M \subseteq d_{\mathrm{l}(\kappa)}\kappa M$ as $\kappa\lambda \in \mathbf{P}$ and so $d_{\mathrm{l}(\lambda)}M \subseteq \kappa^{-1}d_{\mathrm{l}(\kappa)}\kappa M$. Since $\lambda^{-1}d_{\mathrm{l}(\lambda)}D^+(M)$ is contained in $(\kappa\lambda)^{-1}d_{\mathrm{l}(\kappa)}\kappa M$ the result follows from lemmas 2.1.2 and 2.1.29. $\square$

**Lemma 2.1.29.** (Realisation Lemma) *If $\alpha \in \mathbf{P}$ and $C \in \underline{\mathcal{W}}_{h(\alpha),s(\alpha)}$ then $\mathrm{l}(\alpha)M \subseteq C^-(M^\bullet)$.*

We will use lemma 2.1.29 to prove proposition 2.1.30. Lemma 2.1.29 is also used to prove lemma 2.2.4. Lemma 2.2.4 will be key in proving results about the *refined functors* for homotopy words (introduced and studied in section 2.2.1).

*Proof of lemma 2.1.29.* By definition $s(C) = s(\alpha)$ and $h(C) = h(\alpha)$. Suppose firstly that $C$ is trivial so that $C = \underline{1}_{h(\alpha),s(\alpha)}$. If there is some $\beta \in \mathbf{P}$ for which $C\beta^{-1}d_{\mathrm{l}(\beta)}$ is a homotopy word then $h(\beta^{-1}) = h(C^{-1}) = h(\underline{1}_{h(\alpha),s(\alpha)})$ and so $t(\beta) = h(\alpha)$. Furthermore $s(\beta^{-1}) = -s(C^{-1}) = -s(\underline{1}_{h(\alpha),-s(\alpha)})$ and so $s(\mathrm{f}(\beta)^{-1}) = s(\mathrm{l}(\alpha))$ which gives $\mathrm{f}(\beta)\mathrm{l}(\alpha) \notin \mathbf{P}$. Consequently $\mathrm{l}(\alpha)M \subseteq \beta^{-1}0$ which is contained in $C^-(M)$.

Otherwise there is no $\beta \in \mathbf{P}$ for which $C\beta^{-1}d_{\mathrm{l}(\beta)}$ is a homotopy word and as $s(\mathrm{l}(\alpha)) = s(\underline{1}_{h(\alpha),s(\alpha)}) = s(C^{-1})$ we have $\mathrm{l}(\alpha)M \subseteq \sum_{\gamma} \gamma M$ where $\gamma$ runs through all arrows with head $h(C^{-1})$ and sign $s(C^{-1})$. By definition this means $\mathrm{l}(\gamma)M \subseteq C^{-}(M^{\bullet})$.

Now assume $C$ is non-trivial. If $C = \beta^{-1}d_{\mathrm{l}(\beta)}D$ for some homotopy word $D$ and some $\beta \in \mathbf{P}$ then $t(\beta) = h(C) = h(\alpha)$ and $s(\mathrm{f}(\beta)^{-1}) = s(\alpha) = s(\mathrm{l}(\alpha))$. So as before $\mathrm{f}(\beta)\mathrm{l}(\alpha) \notin \mathbf{P}$ and again $\mathrm{l}(\alpha)M \subseteq \beta^{-1}0 \subseteq C^{-}(M^{\bullet})$. The last possibility is that $C = d^{-1}_{\mathrm{l}(\gamma)}\gamma E$ for some homotopy word $E$ and some $\gamma \in \mathbf{P}$. Here $h(\gamma) = h(\alpha)$ and $s(\mathrm{l}(\alpha)) = s(d^{-1}_{\mathrm{l}(\gamma)}) = -s(\mathrm{l}(\gamma))$. So $\mathrm{l}(\alpha) \neq \mathrm{l}(\gamma)$ and so $\mathrm{l}(\alpha)M \subseteq d^{-1}_{\mathrm{l}(\gamma)}0$ by lemma 2.1.2 (i) which means $\mathrm{l}(\alpha)M \subseteq d^{-1}_{\mathrm{l}(\gamma)}\gamma E^{-}(M) = C^{-}(M^{\bullet})$ as required. □

The next proposition is a key result required to use the functorial filtration method, as we saw in lemma 1.4.27.

**Proposition 2.1.30.** (Homotopy Interval Avoidance) *For any $C, C' \in \underline{\mathcal{W}}_{v,\delta}$ with $C < C'$ we have $C^{+}(M^{\bullet}) \subseteq C'^{-}(M^{\bullet})$.*

*Proof.* We consider the three cases (I), (II), and (III) from definition 2.1.23 in order.

Suppose firstly there are homotopy letters $l$, $l'$, $r$ and $r'$ such that $C = Bl^{-1}rD$ and $C' = Bl'^{-1}r'D'$ and $(l, r) < (l', r')$. Then, by lemma 2.1.28 $C^{+}(M^{\bullet}) = B(l^{-1}rD)^{+}(M^{\bullet})$ is contained in $B(l'^{-1}r'D')^{-}(M^{\bullet}) = (C')^{-}(M^{\bullet})$. Now suppose there is some $\beta \in \mathbf{P}$ for which $C' = Cd^{-1}_{\mathrm{l}(\beta)}\beta E$ for a homotopy word $E$. If $\beta$ is maximal for which $Cd^{-1}_{\mathrm{l}(\beta)}\beta$ is a homotopy word then by lemma 2.1.19 $C^{+}(M^{\bullet}) = Cd^{-1}_{\mathrm{l}(\beta)}0$ which is contained in $Cd^{-1}_{\mathrm{l}(\beta)}\beta(E)^{-}(M^{\bullet})$ .

Otherwise there is some (unique) arrow $\alpha$ for which $\beta\alpha \in \mathbf{P}$ and so $s(\alpha) = -s(\beta^{-1}) = s(E)$ and $h(\alpha) = h(\beta^{-1}) = h(E)$ as $Cd^{-1}_{\mathrm{l}(\beta)}\beta E$ is a homotopy word, which means $C^{+}(M^{\bullet}) \subseteq Cd^{-1}_{\mathrm{l}(\beta)}\beta\mathrm{rad}(M)$ which is contained in $Cd^{-1}_{\mathrm{l}(\beta)}\beta(E)^{-}(M^{\bullet})$ by lemma 2.1.29. Whether $\beta$ is maximal or not we have $C^{+}(M^{\bullet}) \subseteq (Cd^{-1}_{\mathrm{l}(\beta)}\beta E)^{-}(M^{\bullet}) = (C')^{-}(M^{\bullet})$. Finally suppose there is some $\alpha \in \mathbf{P}$ for which $C = C'\alpha^{-1}d_{\mathrm{l}(\alpha)}E'$ for some word $E'$. Here $C'\alpha^{-1}d_{\mathrm{l}(\alpha)}(E')^{+}(M^{\bullet}) \subseteq C'\alpha^{-1}d_{\mathrm{l}(\alpha)}M$ and so $C^{+}(M^{\bullet}) \subseteq (C')^{-}(M^{\bullet})$ □

**Example 2.1.31.** Let $\Lambda$ be the complete gentle $k[[t]]$-algebra $\Lambda = k[[x, y]]/(xy)$ from example 2.1.25. Recall the inclusions given by corollaries 2.1.10 and 2.1.15.

We can arrange these inclusions of sets in a similar way to the arrangement of functors in example 1.4.23.

$$
\begin{tikzcd}
M \arrow[r, equal] \arrow[dddd, dash] & d_y^{-1}yM \arrow[d, dash] & d_y^{-1}y^2M \arrow[r, equal] \arrow[dddd, dash] & d_y^{-1}y^2d_x^{-1}xM \arrow[d, dash] \\
 & d_y^{-1}y^2M \arrow[d, dash] \arrow[ur, dashed, dash, bend right] & & d_y^{-1}y^2d_x^{-1}x^2M \arrow[d, dash] \\
 & d_y^{-1}y^3M \arrow[d, dash] \arrow[rddddddd, dashed, dash] & & d_y^{-1}y^2d_x^{-1}x^3M \arrow[d, dash] \\
 & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
(\underline{1}_{1,1})^+(M^\bullet) \arrow[r, equal] & \bigcap_{n>0} d_y^{-1}y^nM & (d_y^{-1}y^2)^+(M^\bullet) \arrow[r, equal] & \bigcap_{n>0} d_y^{-1}y^2d_x^{-1}x^nM \\
(\underline{1}_{1,1})^-(M^\bullet) \arrow[r, equal] \arrow[dddd, dash] & \bigcup_{n>0} y^{-n}d_yM \arrow[d, dash] & (d_y^{-1}y^2)^-(M^\bullet) \arrow[r, equal] \arrow[dddd, dash] & \bigcup_{n>0} d_y^{-1}y^2x^{-n}d_xM \arrow[d, dash] \\
 & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
 & y^{-2}d_yM \arrow[d, dash] & & d_y^{-1}y^2x^{-1}x^{-1}d_xM \arrow[d, dash] \\
 & y^{-1}d_yM \arrow[d, dash] & & d_y^{-1}y^2x^{-1}d_xM \arrow[d, dash] \\
xM + d_yM \arrow[r, equal] & xM + d_yM & d_y^{-1}y^3M \arrow[r, equal] & d_y^{-1}y^2(yM + d_xM)
\end{tikzcd}
$$

This diagram depicts the construction of the interval $[(d_y^{-1}y^2)^-, (d_y^{-1}y^2)^+]$ of subfunctors of the forgetful functor $\mathcal{C}_{\min}(k[[x,y]]/(xy)\text{-}\mathbf{Proj}) \to k[[t]]\text{-}\mathbf{Mod}$.

## 2.2 Functors.

Recall the functors $S_{w,w'}$ and $F_{w,w'}$ (for words $w, w'$) from definitions 1.4.28, 1.4.39, 1.4.43 and 1.4.41. In section 2.2 we introduce some analogous functors using homotopy words.

**Remark 2.2.1.** In section 2.1 the underlying module structure of an object $M^{\bullet}$ of $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ would be denoted $M$. For the remainder of the thesis we abuse notation by writing $M$ for both the complex and the module.

### 2.2.1 Refined Functors for Complexes.

**Assumption:** In section 2.2.1 $M$ and $N$ will be objects in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$.

Recall that at the end of section 2.1 we gave a way of defining $R$-submodules $C^{-}(M) \subseteq C^{+}(M)$ of $e_v M$ for any given homotopy word $C$ with head $v$.

**Definition 2.2.2.** (NOTATION: $F_{B,D,n}(M)$, $F^{\pm}_{B,D,n}(M)$) For each $n \in \mathbb{Z}$ and each pair $(B, D)$ of homotopy words with head $v$ such that $B^{-1}D$ is a word, define $R$-submodules $F^{-}_{B,D,n}(M) \subseteq F^{+}_{B,D,n}(M) \subseteq e_v M^n$ and $G^{-}_{B,D,n}(M) \subseteq G^{+}_{B,D,n}(M) \subseteq e_v M^n$ by

$$\begin{aligned} F^{+}_{B,D,n}(M) &= M^n \cap \left(B^{+}(M) \cap D^{+}(M)\right), \\ F^{-}_{B,D,n}(M) &= M^n \cap \left(B^{+}(M) \cap D^{-}(M) + B^{-}(M) \cap D^{+}(M)\right), \\ G^{\pm}_{B,D,n}(M) &= M^n \cap \left(B^{-}(M) + D^{\pm}(M) \cap B^{+}(M)\right). \end{aligned}$$

Define the quotients $F_{B,D,n}(M)$ and $G_{B,D,n}(M)$ by

$$F_{B,D,n}(M) = F^{+}_{B,D,n}(M)/F^{-}_{B,D,n}(M), \quad G_{B,D,n}(M) = G^{+}_{B,D,n}(M)/G^{-}_{B,D,n}(M)$$

In corollaries 2.2.8 and 2.2.12 we show $F_{B,D,n}$ and $G_{B,D,n}$ are naturally isomorphic.

**Corollary 2.2.3.** *If $C$ is a homotopy $\{0, \dots, t\}$-word for some $t \geq 0$, and $X = \bigoplus_{i \in \mathbb{Z}} X^i$ and $Y = \bigoplus_{i \in \mathbb{Z}} Y^i$ are $R$-submodules of $e_{t(C)}M$ and $e_{h(C)}M$ respectively, then $Y^n \cap CX = Y^n \cap CX^{n+\mu_C(t)}$ for each $n \in \mathbb{Z}$.*

*Proof.* Let $H_s = H(l_s^{-1}r_s)$ for each $s \in \{1, \dots, t\}$. If $C$ is trivial both sets are just $Y^n \cap X^n$ and so there is nothing to prove. So we may assume $t > 0$ and $C = l_1^{-1}r_1 \dots l_t^{-1}r_t$.

If the claim holds for when $t = 1$ then for $t > 1$ we have

$$\begin{aligned} Y^n \cap CX &= Y^n \cap l_1^{-1}r_1(\dots(l_t^{-1}r_tX)\dots)) = Y^n \cap l_1^{-1}r_1(M^{n+H_1} \cap (\dots((l_t^{-1}r_tX))\dots)) \\ &= Y^n \cap l_1^{-1}r_1(M^{n+H_1} \cap (l_2^{-1}r_2(M^{n+H_1+H_2} \cap (\dots((l_t^{-1}r_tX))\dots)) \\ &= Y^n \cap l_1^{-1}r_1(M^{n+H_1} \cap (\dots(M^{n+H_1+\dots+H_{t-1}} \cap (l_t^{-1}r_tX^{n+H_1+\dots+H_t}))\dots)) \\ &= Y^n \cap l_1^{-1}r_1(M^{n+\mu_C(1)} \cap (\dots(M^{n+\mu_C(t-1)} \cap (l_t^{-1}r_tX^{n+\mu_C(t)}))\dots)) \end{aligned}$$

So it suffices to prove $Y^n \cap l^{-1}rX = Y^n \cap l^{-1}rX^{n+\mu_C(1)}$ where $C = l^{-1}r$. Let $g : U \to V$ be a graded $R$-module map of degree $m \in \mathbb{Z}$ and $(g)$ be the linear relation $\{(u, g(u)) \mid u \in U\}$ from $U$ to $V$. For all $i \in \mathbb{Z}$ we have $V^i \cap (g)U^{i-m} = V \cap (g)U^{i-m}$, $U^{i-m} \cap (g)^{-1}V^i = U^{i-m} \cap (g)^{-1}V$ and $V^i \cap (g)U^{i-m} \subseteq V^i \cap (g)U$. For $v \in V^i \cap (g)U$ there is some $\sum_j u_j \in \bigoplus_{j\in\mathbb{Z}} U^j$ for which $\sum_{j\in\mathbb{Z}} g(u_j) = v$.

Since $g(u_i) \in V^{i+m}$ we have $v = g(u_{n-m})$, which shows $V^n \cap (g)U^{n-m} \supseteq V^n \cap (g)U$. Thus when $C = \gamma^{-1}d_a$ we have $Y^n \cap \gamma^{-1}(Y^n \cap d_aX) = Y^n \cap \gamma^{-1}(Y^n \cap d_aX^{n-1})$ and so $Y^n \cap \gamma^{-1}d_aX = Y^n \cap \gamma^{-1}d_aX^{n-1}$, and when $C = d_a^{-1}\gamma$ we have $Y^n \cap d_a^{-1}(Y^{n+1} \cap \gamma X) = Y^n \cap d_a^{-1}(Y^{n+1} \cap \gamma X^{n+1})$ and so $Y^n \cap d_a^{-1}\gamma X = Y^n \cap d_a^{-1}\gamma X^{n+1}$. □

**Lemma 2.2.4.** *Let $B$ and $D$ be homotopy words with head $v$ such that $B^{-1}D$ is a homotopy word.*

(i) $B^+(M) \cap D^+(M) \cap e_v\mathrm{rad}(M) \subseteq (B^+(M) \cap D^-(M)) + (B^-(M) \cap D^+(M))$,

(ii) $(B^-(M) + D^+(M) \cap B^+(M)) \cap e_v\mathrm{rad}(M) \subseteq (B^-(M) + D^-(M) \cap B^+(M))$, *and*

(iii) $B^+(M) \cap D^\pm(M) + e_v\mathrm{rad}(M) = (B^+(M) + e_v\mathrm{rad}(M)) \cap (D^\pm(M) + e_v\mathrm{rad}(M))$.

*Proof.* For each $\delta \in \{\pm 1\}$, if it exists let $x_\delta$ denote the arrow with head $v$ and sign $\delta$. If such an arrow doesn't exist let $x_\delta = 0$.

(i) Since $\Lambda$ is semilocal we have $\mathrm{rad}(M) = \mathrm{rad}(\Lambda)M$. So for any $m \in e_v\mathrm{rad}(M)$ there are some $m_1, m_{-1} \in M$ for which $m = x_{-1}m_{-1} + x_1m_1$. By lemma 2.1.29 we have that $x_1m_1 \in B^-(M)$ and $x_{-1}m_{-1} \in D^-(M)$.

So, if additionally $m \in B^{+}(M) \cap D^{+}(M)$ we have $x_1 m_1 \in D^{+}(M) \cap B^{-}(M)$ as $x_1 m_1 = m - x_{-1}m_{-1}$ and $D^{-}(M) \subseteq D^{+}(M)$. By symmetry we have $x_{-1}m_{-1} \in B^{+}(M) \cap D^{-}(M)$.

(ii) If $m \in (B^{-}(M) + D^{+}(M) \cap B^{+}(M)) \cap e_v\mathrm{rad}(M)$ we can write $m = x_1 m_1 + x_{-1}m_{-1}$ for some $m_1, m_{-1} \in M$ as above. By definition we also have $m = m' + m''$ where $m' \in B^{-}(M)$ and $m'' \in D^{+}(M) \cap B^{+}(M)$. Again $x_1 m_1 \in B^{-}(M)$, $x_{-1}m_{-1} \in D^{-}(M)$ and $x_{-1}m_{-1} = m' + m'' - x_1 m_1$ which lies in $B^{+}(M)$.

(iii) Clearly $B^{+}(M) \cap D^{\pm}(M) + e_v\mathrm{rad}(M)$ is contained in the intersection of $B^{+}(M) + e_v\mathrm{rad}(M)$ and $D^{\pm}(M) + e_v\mathrm{rad}(M)$. As above, we can write any element $m$ from this intersection as; $m_B + x_{-1}m_{-1} + x_1 m_1$ for $m_B \in B^{+}(M)$ and $m_{\pm 1} \in M$, and as $m_D + x_{-1}m'_{-1} + x_1 m'_1$ for $m_D \in D^{\pm}(M)$ and $m'_{\pm 1} \in M$. As above we have $x_1 m_1, x_1 m'_1 \in B^{-}(M)$ and $x_{-1}m_{-1}, x_{-1}m'_{-1} \in D^{-}(M) \subseteq D^{\pm}(M)$. Writing $m_B + x_1 m_1 - x_1 m'_1$ as the sum of $x_{-1}m'_{-1} - x_{-1}m_{-1}$ and $m_D$ shows $m = (m_B + x_1 m_1 - x_1 m'_1) + (x_1 m'_1 + x_{-1}m_{-1})$ is an element of $B^{+}(M) \cap D^{\pm}(M) + e_v\mathrm{rad}(M)$. □

**Definition 2.2.5.** (NOTATION: $\bar{C}^{\pm}$) Let $\bar{C}^{\pm}(M) = C^{\pm}(M) + e_v\mathrm{rad}(M)$ for any homotopy $I$-word $C$ where $I \subseteq \mathbb{N}$.

**Corollary 2.2.6.** *Let $B$ and $D$ be homotopy words with head $v$ such that $B^{-1}D$ is a homotopy word. Then*

$$\begin{gathered} F^{+}_{B,D,n}(M) + e_v\mathrm{rad}(M^n) = e_v M^n \cap \bar{B}^{+}(M) \cap \bar{D}^{+}(M), \\ F^{-}_{B,D,n}(M) + e_v\mathrm{rad}(M^n) = e_v M^n \cap ((\bar{B}^{+}(M) \cap \bar{D}^{-}(M)) + (\bar{B}^{-}(M) \cap \bar{D}^{+}(M))), \\ \textit{and } G^{\pm}_{B,D,n}(M) + e_v\mathrm{rad}(M^n) = e_v M^n \cap (\bar{B}^{-}(M) + \bar{D}^{\pm}(M) \cap \bar{B}^{+}(M)). \end{gathered}$$

*Proof.* By lemma 2.2.4 (iii) we have

$$B^{+}(M) \cap D^{\pm}(M) + e_v\mathrm{rad}(M) = (B^{+}(M) + e_v\mathrm{rad}(M)) \cap (D^{\pm}(M) + e_v\mathrm{rad}(M)) \quad (\star_{B,D,\pm})$$

Considering the inherited grading on the $R$-submodules $\bar{C}^{\pm}(M)$, $C^{\pm}(M)$, and $e_v\mathrm{rad}(M)$ of $M = \bigoplus_{i \in \mathbb{Z}} M^i$ we have $e_v M^n \cap \bar{C}^{\pm}(M) = e_v M^n \cap C^{\pm}(M) + e_v\mathrm{rad}(M^n)$. This together with $(\star_{B,D,+})$, $(\star_{B,D,-})$ and $(\star_{D,B,-})$ yields the first two equalities.

Similarly we have

$$\begin{aligned}
G^{\pm}_{B,D,n}(M) + e_v\mathrm{rad}(M^n) &= e_vM^n \cap (B^-(M) + D^{\pm}(M) \cap B^+(M)) + e_v\mathrm{rad}(M^n)\\
&= e_vM^n \cap (B^-(M) + D^{\pm}(M) \cap B^+(M) + e_v\mathrm{rad}(M))\\
&= e_vM^n \cap (B^-(M) + e_v\mathrm{rad}(M) + D^{\pm}(M) \cap B^+(M) + e_v\mathrm{rad}(M))\\
&= e_vM^n \cap (B^-(M) + e_v\mathrm{rad}(M)) + e_vM^n \cap (D^{\pm}(M) \cap B^+(M) + e_v\mathrm{rad}(M)),
\end{aligned}$$

as required. □

Recall $k$ is the residue field $R/\mathfrak{m}$.

**Lemma 2.2.7.** *Let $n \in \mathbb{Z}$ and $v$ be a vertex. For each object $M$ of $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ let $\mathcal{F}^-(M) \subseteq \mathcal{F}^+(M)$ be $R$-submodules of $e_vM$ such that $\mathcal{F}^+(M) \cap e_v\mathrm{rad}(M) \subseteq \mathcal{F}^-(M)$ and $\mathrm{im}(f|_{\mathcal{F}^{\pm}(M)}) \subseteq \mathcal{F}^{\pm}(N)$ for any morphism $f : M \to N$ in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$. For $M$ and $f$ as above and each $m \in e_vM^n \cap \mathcal{F}^+(M)$ let*

$$\begin{aligned}
\mathcal{F}_n(M) &= e_vM^n \cap \mathcal{F}^+(M)/e_vM^n \cap \mathcal{F}^-(M),\\
\bar{\mathcal{F}}^{\pm}(M) &= \mathcal{F}^{\pm}(M) + e_v\mathrm{rad}(M),\\
\bar{\mathcal{F}}_n(M) &= e_vM^n \cap \bar{\mathcal{F}}^+(M)/e_vM^n \cap \bar{\mathcal{F}}^-(M),\\
\mathcal{F}_n([f])(m + e_vM^n \cap \mathcal{F}^-(M)) &= f^n(m) + e_vN^n \cap \mathcal{F}^-(N)\\
\bar{\mathcal{F}}_n([f])(m + e_vM^n \cap \bar{\mathcal{F}}^-(M)) &= f^n(m) + e_vN^n \cap \bar{\mathcal{F}}^-(N)
\end{aligned}$$

*Then $\mathcal{F}_n, \bar{\mathcal{F}}_n : \mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k\text{-}\mathbf{Mod}$ are additive functors such that $\mathcal{F}_n \simeq \bar{\mathcal{F}}_n$.*

*Proof.* Since $e_v\mathfrak{m}M \subseteq e_v\mathrm{rad}(\Lambda)M = e_v\mathrm{rad}(M)$ by corollary 1.1.25 (ii), we have that $\mathfrak{m}M^n \cap \mathcal{F}^+(M) \subseteq \mathcal{F}^-(M)$ and so $\mathcal{F}_n(M)$ is a $k$-vector space. If $f$ is null-homotopic then $f^n = d_N^{n-1}s^n + s^{n+1}d_M^n$ for some graded $\Lambda$ module homomorphism $s : M \to N$ of degree 1, and hence $\mathrm{im}(f^n) \subseteq \mathrm{im}(d_N^{n-1}s^n) + \mathrm{im}(s^{n+1}d_M^n) \subseteq \mathrm{rad}(N^n)$ since $M$ and $N$ are homotopically minimal complexes of projectives. This gives

$$f^n(m) \in \mathrm{im}(f^n|_{\mathcal{F}^+(M)}) \cap e_v\mathrm{rad}(N^n) \subseteq e_vN^n \cap \mathcal{F}^+(N) \cap e_v\mathrm{rad}(N) \subseteq e_vN^n \cap \mathcal{F}^-(N)$$

This shows $\mathcal{F}_n$ is a functor $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k\text{-}\mathbf{Mod}$.

Similarly $\bar{\mathcal{F}}_n$ is also an additive functor $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k\text{-}\mathbf{Mod}$ so it suffices to define a natural isomorphism $\mathcal{F}_n \to \bar{\mathcal{F}}_n$. For each complex $M$ with radical images define the linear map

$$\alpha_M : e_v M^n \cap \mathcal{F}^+(M)/e_v M^n \cap \mathcal{F}^-(M) \to e_v M^n \cap \bar{\mathcal{F}}^+(M)/e_v M^n \cap \bar{\mathcal{F}}^-(M)$$

by sending $m + e_v M^n \cap \mathcal{F}^-(M)$ to $m + e_v M^n \cap \bar{\mathcal{F}}^-(M)$. As $\mathcal{F}^-(M) \subseteq \bar{\mathcal{F}}^-(M)$ we have that $\alpha_M$ is well defined. Since $\bar{\mathcal{F}}^+(M) = \mathcal{F}^+(M) + e_v\text{rad}(M)$ and $e_v\text{rad}(M) \subseteq \bar{\mathcal{F}}^-(M)$ by definition, $\alpha_M$ is surjective. If $\alpha_M(m + \mathcal{F}^-(M)) = 0$ then $m$ lies in the intersection of $\mathcal{F}^-(M) + e_v\text{rad}(M)$ and $\mathcal{F}^+(M)$. By Dedekind's modular law this means $m \in \mathcal{F}^-(M)$ as $e_v\text{rad}(M) \cap \mathcal{F}^+(M) \subseteq \mathcal{F}^-(M)$ which proves $\alpha_M$ is one-to-one and hence an isomorphism. For any morphism $f : M \to N$ of complexes the image of $m + e_v M^n \cap \mathcal{F}^-(M)$ under $\alpha_N \mathcal{F}_n(f)$ is $\alpha_N(f^n(m) + \mathcal{F}^-(N)) = \bar{\mathcal{F}}_n(f)(m + \bar{\mathcal{F}}^-(M))$ which is the image of $m + e_v M^n \cap \mathcal{F}^-(M)$ under $\bar{\mathcal{F}}_n(f) \circ \alpha_M$. Hence the collection $\alpha_M$ defines a natural isomorphism $\alpha$. $\square$

We now use lemma 2.2.4 to apply lemma 2.2.7 in case $\mathcal{F} = F_{B,D,n}$.

**Corollary 2.2.8.** *$F_{B,D,n}$, $\bar{F}_{B,D,n}$, $G_{B,D,n}$, and $\bar{G}_{B,D,n}$ all define naturally isomorphic additive functors $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k\text{-}\mathbf{Mod}$.*

*Proof.* Fix an arbitrary complex $M$ in $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$. The inclusion $e_v\text{rad}(M) \cap F^+_{B,D}(M) \subseteq F^-_{B,D}(M)$ holds by lemma 2.2.4 (i) and the inclusion $e_v\text{rad}(M) \cap G^+_{B,D}(M) \subseteq G^-_{B,D}(M)$ holds by lemma 2.2.4 (ii).

Clearly $e_v\text{rad}(M) \cap \bar{F}^+_{B,D}(M) = e_v\text{rad}(M)$, $e_v\text{rad}(M) \cap \bar{G}^+_{B,D}(M) = e_v\text{rad}(M)$, and $\bar{F}^-_{B,D}(M) \supseteq e_v\text{rad}(M) \subseteq \bar{G}^-_{B,D}(M)$. These inclusions, together with lemma 2.2.7 and corollary 2.1.20, show that $F_{B,D,n}$, $\bar{F}_{B,D,n}$, $G_{B,D,n}$, and $\bar{G}_{B,D,n}$ all define additive functors $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k\text{-}\mathbf{Mod}$, and that there exist natural isomorphisms $F_{B,D,n} \to \bar{F}_{B,D,n}$ and $G_{B,D,n} \to \bar{G}_{B,D,n}$.

The linear map $\beta_M : F^+_{B,D,n}(M)/F^-_{B,D,n}(M) \to G^+_{B,D,n}(M)/G^-_{B,D,n}(M)$ sending $m + F^-_{B,D,n}(M)$ to $m + G^-_{B,D,n}(M)$ is well defined since $F^\pm_{B,D}(M) \subseteq G^\pm_{B,D}(M)$. Furthermore $G^+_{B,D}(M) = F^+_{B,D}(M) + B^-(M)$ and $B^-(M) \subseteq G^-_{B,D}(M)$ so $\beta_M$ is onto.

Since $G^{-}_{B,D}(M) = F^{-}_{B,D}(M) + B^{-}(M)$, $\beta_M(m + F^{-}_{B,D}(M)) = 0$ gives

$$m \in (F^{-}_{B,D,n}(M) + e_v M^n \cap B^{-}(M)) \cap F^{+}_{B,D,n}(M) = F^{-}_{B,D,n}(M) + e_v M^n \cap B^{-}(M) \cap F^{+}_{B,D,n}(M)$$

so $m \in F^{-}_{B,D,n}(M)$ as $B^{-}(M) \cap F^{+}_{B,D}(M) \subseteq F^{-}_{B,D}(M)$ which proves $\beta_M$ is an isomorphism. Finally for any morphism $f : M \to N$ of complexes $\beta_N$ sends the coset $f^n(m) + F^{-}_{B,D,n}(N)$ to $f^n(m) + G^{-}_{B,D,n}(N)$ which is the image of $m + G^{-}_{B,D,n}(M)$ under the map $G_{B,D,n}(f)$. This shows $\beta_N F_{B,D,n}(f) = G_{B,D,n}(f)\beta_M$ which means $\beta$ defines a natural isomorphism $F_{B,D,n} \to G_{B,D,n}$ as required. □

**Corollary 2.2.9.** *If $M$ is a complex of finitely generated projectives then $F_{B,D,n}(M)$ and $G_{B,D,n}(M)$ are finite-dimensional vector spaces over $k = R/\mathfrak{m}$.*

*Proof.* Since $e_v \mathfrak{m} M \subseteq e_v \mathrm{rad}(\Lambda) M = e_v \mathrm{rad}(M)$ by corollary 1.1.25 (ii), $F^{+}_{B,D,n}(M)/e_v \mathrm{rad}(M^n) \cap F^{+}_{B,D,n}(M)$ and $e_v M^n / e_v \mathrm{rad}(M^n)$ are $k$-vector spaces and there is a surjective $k$-linear map $F^{+}_{B,D,n}(M)/e_v \mathrm{rad}(M^n) \cap F^{+}_{B,D,n}(M) \to F_{B,D,n}(M)$.

As $F^{+}_{B,D,n}(M) \subseteq e_v M^n$ and $e_v \mathrm{rad}(M^n) \cap F^{+}_{B,D,n}(M) \subseteq e_v \mathrm{rad}(M^n)$ there is a vector space embedding of $F^{+}_{B,D,n}(M)/e_v \mathrm{rad}(M^n) \cap F^{+}_{B,D,n}(M)$ into $e_v M^n / e_v \mathrm{rad}(M^n)$.

By assumption there is a surjective $\Lambda$-module homomorphism $\theta : F \to M^n$ where $F = \bigoplus_{i=1}^{t} \Lambda$ for some $t > 0$. Let $\varphi$ and $\psi$ be the $R$-module maps defined by the restrictions of $\theta$ to the domains $e_v F$ and $e_v \mathrm{rad}(F)$ respectively. Then $\mathrm{im}(\varphi) = e_v M^n$ and $\mathrm{im}(\psi) = e_v \mathrm{rad}(\Lambda) M^n$ as $\theta$ is surjective.

This shows that there is a surjective $k$-linear map $e_v F / e_v \mathrm{rad}(F) \to e_v M^n / e_v \mathrm{rad}(M^n)$. By corollary 1.1.25 (i) $e_v F / e_v \mathrm{rad}(F) \simeq \bigoplus_{i=1}^{t} e_v \Lambda / e_v \mathrm{rad}(\Lambda) \simeq k^t$ in $k$-**mod**.

Altogether we have

$$\dim_k(e_v M^n / e_v \mathrm{rad}(M^n)) \geq \dim_k(F^{+}_{B,D,n}(M)/e_v \mathrm{rad}(M^n) \cap F^{+}_{B,D,n}(M))$$

and so $t \geq \dim_k(F_{B,D,n}(M))$ as required. □

The next definition (2.2.10) and lemma (2.2.11) were motivated by remark 1.4.40.

**Definition 2.2.10.** (NOTATION: ${}^{\infty}E^{\infty}$) Let $C = B^{-1}D$ be a periodic $\mathbb{Z}$-word of period $p$, say $D = E^{\infty}$ and $B = (E^{-1})^{\infty}$ for some $\{0, \ldots, p\}$-word $E$. In this case we shall write $C = {}^{\infty}E^{\infty}$.

(NOTATION: $E(n)$) Recall definition 1.4.29. If $E$ is a homotopy $\{0, \ldots, p\}$-word with $\mu_E(p) = 0$ we can consider the linear relation $E(n) = \{(m, m') \in e_vM^n \oplus e_vM^n \mid m \in Em'\}$ on $e_vM^n$.

**Lemma 2.2.11.** *Let $n \in \mathbb{Z}$ and $B^{-1}D$ be a periodic homotopy $\mathbb{Z}$-word. Then $E(n)^{\sharp} = F^{+}_{B,D,n}(M)$ and $E(n)^{\flat} = F^{-}_{B,D,n}(M)$ where $D = E^{\infty}$ and $B = (E^{-1})^{\infty}$. Furthermore, there is a k-vector space automorphism $\theta^{E(n)}$ of $E(n)^{\sharp}/E(n)^{\flat}$ defined by setting $\theta^{E(n)}(m + E(n)^{\flat}) = m' + E(n)^{\flat}$ iff $m' \in E(n)^{\sharp} \cap (E(n)^{\flat} + E(n)m)$.*

*Proof.* Assuming $C = B^{-1}D$ has period $p$, $E$ must be a homotopy $\{0, \ldots, p\}$-word $E$. Let $v = h(E)$. Clearly $\bigcup e_vM^n \cap E^l0 \subseteq e_vM^n \cap \bigcup D_{\leq pl}0$ where the unions run over all $l \in \mathbb{N}$, and so $E(n)' \subseteq e_vM^n \cap D^-(M)$. For $m \in e_vM^n \cap D^-(M)$ we have that $m \in D_{\leq r}0$ for some $r \in \mathbb{N}$. Choosing a multiple $sp > r$ gives $e_vM^n \cap D_{\leq sp}0 = e_vM^n \cap E^s0$ and so by corollary 2.2.3 $m \in \bigcup_{l\in\mathbb{N}} e_vM^n \cap E^l0_{M^n} = E(n)'$. Hence $E(n)' = e_vM^n \cap D^-(M)$ and we have $(E^{-1})(n)' = e_vM^n \cap B^-(M)$ by symmetry.

If $m \in e_vM^n \cap D^+(M)$ then by definition there is a sequence $(m_i)$ where $m_0 = m \in M^n$ and $m_t \in l^{-1}_{t+1}r_tm_{t+1} \cap e_{v_D(t)}M$ for each $t \geq 0$. By 2.2.3 we can assume $m_t \in M^{n+\mu_D(t)}$ for each $t \in \mathbb{N}$. As $\mu_D(sp) = 0$ for each integer $s$, we have a sequence $m_0, m_p, m_{2p}, \cdots \in e_vM^n$ satisfying $m_{ip} \in Em_{(i+1)p}$ for each $i \in \mathbb{N}$ and hence $m \in E(n)''$ by definition.

Conversely if $m' \in E(n)''$ then there is a sequence $(m'_i)$ where $m'_0 = m'$ and $m'_t \in Em'_{t+1}$ for each $t \geq 0$. Hence if $E = l_1^{-1}r_1 \ldots l_p^{-1}r_p$ then for each $t \in \mathbb{N}$ there are elements $m_{pt+j} \in e_{v_D(j)}M$ for each $j \in \{0, \ldots, p-1\}$ satisfying $m_0 = m'_0$, $m_{pt} = m'_t$ and $m_{pt+j-1} \in l_j^{-1}r_j$ whenever $j > 0$. By corollary 2.2.3 we can assume $m_{pt+j} \in e_{v_D(j)}M^{n+\mu_D(j)}$ and so $m' = m'_0 = m_0 \in E(n)'$ since $\mu_D(j) = \mu_D(pt + j)$ for each $j \in \{0, \ldots, p-1\}$ and each $t \in \mathbb{N}$. This shows $E(n)'' = e_vM^n \cap D^+(M)$ and we have $(E^{-1})(n)'' = e_vM^n \cap B^+(M)$ by symmetry. Together we have $E(n)^{\sharp} = F^{+}_{B,D,n}(M)$ and $E(n)^{\flat} = F^{-}_{B,D,n}(M)$.

By applying lemma 1.4.31 to the relation $V = E(n)$ on $e_v M^n$ this definition of $\theta^{E(n)}$ gives an $R$-module automorphism of $E(n)^\sharp/E(n)^\flat$. Since $E(n)^\sharp/E(n)^\flat = F_{B,D}(M)$ is a vector space by corollary 2.2.26 this means $\theta^{E(n)}$ is a $k$-vector space automorphism. □

Recall $k[T, T^{-1}]$-$\mathbf{Mod}_{k\text{-}\mathbf{mod}}$ is the full subcategory of $k[T, T^{-1}]$-$\mathbf{Mod}$ consisting of finite-dimensional modules.

**Corollary 2.2.12.** *Let $C = B^{-1}D$ be a periodic homotopy $\mathbb{Z}$-word. Then $F_{B,D,n}(M)$ is a $k[T, T^{-1}]$-module. Consequently $F_{B,D,n}$, $\bar{F}_{B,D,n}$, $G_{B,D,n}$, and $\bar{G}_{B,D,n}$ all define naturally isomorphic (additive) functors $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k[T, T^{-1}]\text{-}\mathbf{Mod}$. Furthermore these functors take objects in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$ to objects in $k[T, T^{-1}]\text{-}\mathbf{Mod}_{k\text{-}\mathbf{mod}}$.*

*Proof.* Recalling the proof of lemma 2.2.11, for each complex $M$ let $E_M$ denote the linear relation $E(n)$ on $e_v M^n$ and let $\theta_M^E$ be the automorphism $\theta^{E(n)}$ of $E_M^\sharp/E_M^\flat$. Lemma 2.2.11 gives an action of $T$ on $F_{B,D,n}(M)$ by $T(m + F_{B,D,n}^-(M)) = \theta_M^E(m + F_{B,D,n}^-(M))$ making $F_{B,D,n}(M)$ into a $k[T, T^{-1}]$-module.

For an arbitrary map $g : M \to N$ of complexes, $m \in E_M^\sharp$ and $m' \in E_M^\sharp \cap (E_M^\flat + E_M m)$ we have $g(m) \in E_N^\sharp$ and $g(m') \in E_N^\sharp \cap (E_N^\flat + E_N g(m))$ by corollary corollary.3.4. This shows

$$\theta_N^E(F_{B,D,n}(g)(m + E_M^\flat)) = \theta_N^E(g(m) + E_N^\flat))$$
$$= g(m') + E_N^\flat = F_{B,D,n}(g)(m' + E_M^\flat) = (F_{B,D,n}(g)(\theta_M^E(m + E_M^\flat))$$

and therefore the action of $T$ commutes with $F_{B,D,n}(g)$ which shows $F_{B,D,n}$ defines an additive functor $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k[T, T^{-1}]\text{-}\mathbf{Mod}$.

Let $(-)$ be the forgetful functor $k[T, T^{-1}]\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$. In what follows we show that if $\mathcal{F} : \mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k\text{-}\mathbf{Mod}$ is an additive functor and there is a natural isomorphism $\omega : (-)F_{B,D,n} \to \mathcal{F}$ then there is an action of $T$ giving $\mathcal{F}(M)$ the structure of a $k[T, T^{-1}]$-module and making $\omega_M$ linear over $k[T, T^{-1}]$. By corollary 2.2.8 this will show $F_{B,D,n}$, $\bar{F}_{B,D,n}$, $G_{B,D,n}$, and $\bar{G}_{B,D,n}$ all define naturally isomorphic additive functors $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k[T, T^{-1}]\text{-}\mathbf{Mod}$.

Define the action of $T$ on $\mathcal{F}(M)$ by extending the assignment $T^l(\bar{m}) = \omega_M(T^l(\omega_M^{-1}(\bar{m}))$ $k$-linearly for each $l \in \mathbb{Z}$ and $\bar{m} \in F(M)$. By construction this action gives $\mathcal{F}(M)$ the structure of a $k[T, T^{-1}]$-module and $\omega_M$ is $k[T, T^{-1}]$-linear.

Hence $\omega$ defines a natural isomorphism of functors $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \to k[T, T^{-1}]\text{-}\mathbf{Mod}$. By corollary 2.2.9 $F_{B,D,n}$ sends complexes $M$ of finitely generated projectives (with radical images) to finite dimensional $k$-vector spaces $F_{B,D,n}(M)$. Thus the functors $F_{B,D,n}$, $\bar{F}_{B,D,n}$, $G_{B,D,n}$, and $\bar{G}_{B,D,n}$ all take objects in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$ to objects in $k[T, T^{-1}]\text{-}\mathbf{Mod}_{k\text{-}\mathbf{mod}}$. $\square$

**Definition 2.2.13.** (Refined Functors for Complexes) By a *refined functor* we will mean $F_{B,D,n}$ or $G_{B,D,n}$ for some $n$, $B$ and $D$ as above. As we saw in definitions 1.4.39 and 1.4.41, the codomain category of a refined functor is $k[T, T^{-1}]\text{-}\mathbf{Mod}_{k\text{-}\mathbf{Mod}}$ or $k\text{-}\mathbf{Mod}$ depending on whether or not $B^{-1}D$ is periodic.

### 2.2.2 Natural Isomorphisms.

We now book-keep to give an equivalence relation on a set of triples $(B, D, n)$ so that the refined functors $F_{B,D,n}$ and $F_{B',D',n'}$ are naturally isomorphic whenever $(B, D, n)$ and $(B', D', n')$ are equivalent. Later (lemmas 2.3.20 and 2.3.21) we shall see the converse.

**Definition 2.2.14.** (Axis, notation: $a_{B,D}$) If $C = B^{-1}D$ is a homotopy word we define the *axis* $a_{B,D}$ *of* $(B, D)$ as the unique integer satisfying $C_{\leq a_{B,D}} = B^{-1}$ and $C_{>a_{B,D}} = D$.

**Lemma 2.2.15.** *Suppose $B$ and $D$ are words such that $C = B^{-1}D$ is a word. Then,*

(i) *if $C$ is a $\mathbb{Z}$-word then $a_{B,D} = 0$,*

(ii) *if $C$ is a $\mathbb{N}$-word or a $-\mathbb{N}$-word then $a_{D,B} = -a_{B,D}$, and*

(iii) *if $C$ is a $\{0, \ldots, t\}$-word for $t \in \mathbb{N}$ then $a_{D,B} = t - a_{B,D}$.*

*Proof.* (i) Here $B^{-1}$ must be a homotopy $-\mathbb{N}$-word and $D$ must be a homotopy $\mathbb{N}$-word. Writing $B^{-1} = \ldots l_0^{-1} r_0$ and $D = l_1^{-1} r_1 \ldots$ gives $C = \ldots l_0^{-1} r_0 \mid l_1^{-1} r_1 \ldots$ by definition.

(ii) If $C$ is a $\mathbb{N}$-word then $B$ is finite. We can assume $a_{B,D} \neq 0$ as otherwise $B$ must be trivial in which case $D^{-1} = C^{-1} = (C^{-1})_{\leq 0}$ and so $a_{D,B} = 0$. Let $a = a_{B,D}$. So $B = l_1^{-1} r_1 \ldots l_a^{-1} r_a$ and so $C^{-1} = D^{-1} r_a^{-1} l_a \ldots r_1^{-1} l_1$ which shows $a_{D,B} = -a$ as $C$ is a $-\mathbb{N}$-word. If $C$ is a $-\mathbb{N}$-word then $C^{-1}$ is an $\mathbb{N}$-word and $a_{B,D} = -(-a_{B,D}) = -a_{D,B}$ by the above case.

(iii) We can assume $a_{B,D} \neq 0$ as otherwise $B$ must be trivial in which case $D^{-1} = C^{-1} = (C^{-1})_{\leq t}$ and so $a_{D,B} = t$. Let $a = a_{B,D}$. So $B = l_1^{-1} r_1 \ldots l_a^{-1} r_a$ and so $C^{-1} = D^{-1} r_a^{-1} l_a \ldots r_1^{-1} l_1$ which shows $a_{D,B} = t - a$ as required. □

**Definition 2.2.16.** (Equivalent Homotopy Words, notation: $\Sigma$) Consider the set $\Sigma$ consisting of all triples $(B, D, n)$ where $B^{-1}D$ is a homotopy word (equivalently $(B, D) \in \underline{\mathcal{W}}_{v,\delta} \times \underline{\mathcal{W}}_{v,-\delta}$ by proposition 2.1.13) and $n$ is an integer. Fix $(B, D, n)$ and $(B', D', n')$ from $\Sigma$ and let $C = B^{-1}D$ and $C' = B'^{-1}D'$. Recall that if $C$ is not a homotopy $\mathbb{Z}$-word any shift of $C$ is $C$. We say the homotopy words $C$ and $C'$ are *equivalent* when $C'$ is either a shift $C[m]$ of $C$ or $C'$ is a shift $C^{-1}[m]$ of $C^{-1}$ for some $m \in \mathbb{Z}$.

The following result is an analogue of [21, Lemma 2.1].

**Lemma 2.2.17.** *Fix $(B, D, n)$ and $(B', D', n')$ from $\Sigma$ such that $C = B^{-1}D$ and $C' = B'^{-1}D'$ are equivalent. Then exactly one of the following statements hold*

(i) $C' = C$ *which is not a homotopy $\mathbb{Z}$-word,*

(ii) $C' = C^{-1}$ *which is not a homotopy $\mathbb{Z}$-word, or*

(iii) $C' = C^{\pm 1}[m]$ *which is a homotopy $\mathbb{Z}$-word.*

*Proof.* Since $C$ and $C'$ are equivalent we have $C' = C[m]$ or $C' = C^{-1}[m]$ for some $m \in \mathbb{Z}$. If $C'$ is not a homotopy $\mathbb{Z}$-word then we are in case (i) or (ii), and if it is we are in case (iii). So it suffices to prove that (i), (ii) and (iii) are all mutually exclusive.

Since $\gamma \neq d_{\mathrm{l}(\gamma)}$ and $\gamma^{-1}d_{\mathrm{l}(\gamma)}d_{\mathrm{l}(\gamma)}^{-1}\gamma$ is not a homotopy word for any $\gamma \in \mathbf{P}$ we have $C_i \neq C_i^{-1}$ and $C_i^{-1} \neq C_{i+1}$ for all $i$. The lemma now follows observing that homotopy $\mathbb{N}$-words and homotopy $-\mathbb{N}$-words cannot coincide, and by following the proof of [21, Lemma 2.1], word-for-word. $\square$

The equivalence relation $\sim$ on triples $(B, D, n) \in \Sigma$ is given in definition 2.2.18. In corollaries 2.2.24 and 2.2.26 we adapt lemmas 1.4.44 and 1.4.45 from the setting of words to the setting of homotopy words. For arbitrary $(B, D, n), (B', D', n') \in \Sigma$ with $(B, D, n) \sim (B', D', n')$ the value of $n' - n$ should be controlled. To see this consider the case $B = B'$, $D = D'$ and $C = B^{-1}D$ is a homotopy $\{0, \dots, t\}$-word. Since $(B, D, n) \sim (B', D', n')$ we should have $P(C)[n] \simeq P(C)[-n']$, which is impossible if $n - n' > t$. To control $n' - n$ in this scenario we introduce some more notation.

**Definition 2.2.18.** (NOTATION: $r(B, D; B', D')$) For $(B, D, n), (B', D', n') \in \Sigma$ we define an integer $r(B, D; B', D')$ by

$$r(B, D; B', D') = \begin{cases} \mu_C(a_{B',D'}) - \mu_C(a_{B,D}) & (\text{if } C' = C \text{ is not a homotopy } \mathbb{Z}\text{-word}) \\ \mu_C(a_{D',B'}) - \mu_C(a_{B,D}) & (\text{if } C' = C^{-1} \text{ is not a homotopy } \mathbb{Z}\text{-word}) \\ \mu_C(\pm m) & (\text{if } C' = C^{\pm 1}[m] \text{ is a homotopy } \mathbb{Z}\text{-word}) \end{cases}$$

Note that this integer is well defined using lemma 2.2.17.

(Equivalence Relation $\sim$ on $\Sigma$) We introduce a relation $\sim$ on $\Sigma$ by setting $(B, D, n) \sim (B', D', n')$ when $B^{-1}D$ and $B'^{-1}D'$ are equivalent and $n' - n = r(B, D; B', D')$.

**Lemma 2.2.19.** *Let* $(B, D, n)$, $(B', D', n')$, $(B'', D'', n'') \in \Sigma$ *and* $C = B^{-1}D$, $C' = B'^{-1}D'$ *and* $C'' = B''^{-1}D''$. *Then*

(i) *if* $C$ *and* $C'$ *are equivalent then* $r(B, D; B', D') = -r(B', D'; B, D)$, *and*

(ii) *if* $C$, $C'$, *and* $C''$ *are all equivalent then*

$$r(B, D; B'', D'') = r(B, D; B', D') + r(B', D'; B'', D'')$$

*Consequently the above relation* $\sim$ *on* $\Sigma$ *defines an equivalence relation.*

*Proof.* In what follows we make use of lemmas 1.3.33 and 2.2.15 without reference.

(i) If $C' = C$ is not a homotopy $\mathbb{Z}$-word then $\mu_C(a_{B',D'}) - \mu_C(a_{B,D}) = -(\mu_{C'}(a_{B,D}) - \mu_{C'}(a_{B',D'}))$. If $C' = C^{-1}$ is a homotopy $\{0, \ldots, t\}$-word then $a_{D',B'} = t - a_{B',D'}$ and $a_{B,D} = t - a_{D,B}$ and therefore $\mu_C(a_{B,D}) = \mu_{C'}(a_{D,B}) - \mu_{C'}(t)$ and $\mu_C(a_{D',B'}) = \mu_{C'}(a_{B',D'}) - \mu_C(t)$.

This shows $\mu_C(a_{B,D}) - \mu_C(a_{D',B'}) = \mu_{C'}(a_{D,B}) - \mu_{C'}(a_{B',D'})$ and so as $C = C'^{-1}$ we have $r(B, D; B', D') = -r(B', D'; B, D)$. If $C' = C^{-1}$ is a homotopy $\pm\mathbb{N}$-word then $a_{D',B'} = -a_{B',D'}$ and $a_{B,D} = -a_{D,B}$ and therefore $\mu_C(a_{B,D}) = \mu_{C'}(a_{D,B})$ and $\mu_C(a_{D',B'}) = \mu_{C'}(a_{B',D'})$ which as above is sufficient.

If $C' = C^{\pm 1}[m]$ is a homotopy $\mathbb{Z}$-word for some $m \in \mathbb{Z}$ then $C = (C'[-m])^{\pm 1} = C'^{\pm 1}[\mp m]$ and we have $\mu_{C'^{\pm 1}[\mp m]}(\pm m) = -\mu_{C'}(\pm m)$ by writing both sides as $\mu_{C'^{\pm 1}}(\pm m + \mp m) - \mu_{C'^{\pm 1}}(\mp m)$, as required.

(ii) We consider different cases for $r(B, D; B', D')$, and in each case consider different cases for $r(B', D'; B'', D'')$. Note that one of $C$, $C'$, or $C''$ is a homotopy $\mathbb{Z}$-word iff they all are. Suppose $C' = C$ is not a homotopy $\mathbb{Z}$-word.

If $C'' = C'$ (resp. $C'' = C'^{-1}$) then the result is clear after writing $\mu_{C'}(a_{B'',D''}) - \mu_C(a_{B,D})$ (resp. $\mu_{C'}(a_{D'',B''}) - \mu_C(a_{B,D})$) as the sum of $\mu_{C'}(a_{B'',D''}) - \mu_C(a_{B',D'})$ (resp. $\mu_{C'}(a_{D'',B''}) - \mu_C(a_{B',D'})$) and $\mu_{C'}(a_{B',D'}) - \mu_C(a_{B,D})$. Suppose instead $C' = C^{\pm 1}[m]$ is a homotopy $\mathbb{Z}$-word so that $r(B, D; B', D') = \mu_C(\pm m)$.

If $C'' = C'^{\pm 1}[m']$ then $C'' = C[m' \pm m]$ and as $\mu_{C^{\pm 1}[m]}(\pm m') = \mu_C(m' \pm m) - \mu_C(\pm m)$ the result follows. Otherwise $C'' = C'^{\mp 1}[m']$ and so $C'' = C^{-1}[m' \mp m]$ and as $\mu_{C^{\pm 1}[m]}(\mp m') = \mu_C(-(m' \mp m)) - \mu_C(\pm m)$ again the result follows.

For reflexivity, $C$ is equivalent to itself and $n - n = 0 = r(B, D; B, D)$ since $\mu_C(0) = 0$ by definition. For symmetry suppose $C$ and $C'$ are equivalent homotopy words and $r(B, D; B', D') = n' - n$.

Then $C'$ and $C$ are equivalent homotopy words and $r(B', D'; B, D) = n - n'$ by (i). For transitivity suppose $C$ and $C'$ are equivalent homotopy words and $r(B, D; B', D') = n' - n$; and suppose also $C'$ and $C''$ are equivalent homotopy words and $r(B', D'; B'', D'') = n'' - n'$. Then by (ii) $r(B, D; B'', D'') = n' - n + n'' - n' = n'' - n$ as required. $\square$

**Definition 2.2.20.** (Equivalence Classes) Recall that $\Sigma = \underline{\mathcal{W}}_{v,1} \times \underline{\mathcal{W}}_{v,-1} \times \mathbb{Z}$. We let $\Sigma(s)$ be the set of all $(B, D, n) \in \Sigma$ where $B^{-1}D$ is not a periodic homotopy $\mathbb{Z}$-word, and $\Sigma(b)$ the set of $(B, D, n) \in \Sigma$ where $B^{-1}D$ is a periodic homotopy $\mathbb{Z}$-word.

(Notation: $\overline{\Sigma}$, $\overline{\Sigma(s)}$, $\overline{\Sigma(b)}$) Note that for $(B, D, n) \sim (B', D', n')$, $(B, D, n)$ lies in $\Sigma(s)$ (resp. $\Sigma(b)$) iff $(B', D', n')$ does too. So $\sim$ restricts to an equivalence relation $\sim_s$ (resp. $\sim_b$ on $\Sigma(s)$ (resp. $\Sigma(b)$). Let $\overline{\Sigma} = \Sigma/\sim$, $\overline{\Sigma(s)} = \Sigma(s)/\sim_s$ and $\overline{\Sigma(b)} = \Sigma(b)/\sim_b$.

(Chosen Representatives) Let $\mathcal{I}(s)$ denote a chosen collection of representatives $(B, D, n) \in \Sigma(s)$, one for each class $\overline{(B, D, n)} \in \overline{\Sigma(s)}$. Similarly define the subset $\mathcal{I}(b) \subseteq \Sigma(b)$ by choosing one representative $(B, D, n) \in \Sigma(b)$ for each class $\overline{(B, D, n)} \in \overline{\Sigma(b)}$. Let $\mathcal{I} = \mathcal{I}(s) \sqcup \mathcal{I}(b)$.

We now look at the symmetry in the definition of $F_{B,D,n}$. The statement and the proof of the following lemma were both found by adaptating [21, Lemma 7.1] from the setting of words to the setting of homotopy words.

**Lemma 2.2.21.** *Let $B$ and $D$ be homotopy words such that $C = B^{-1}D$ is a homotopy word. Then for any $n \in \mathbb{Z}$*

(i) *if $C$ is a non-periodic $\mathbb{Z}$-word then $F_{B,D,n} \simeq F_{D,B,n}$ and*

(ii) *if $C$ is a periodic $\mathbb{Z}$-word then $F_{B,D,n} \simeq \mathrm{res}_\iota F_{D,B,n}$.*

*Proof.* (i) Here $F_{B,D,n} = F_{D,B,n}$ as $F^+_{B,D,n}(M) = F^+_{D,B,n}(M)$ and $F^-_{B,D,n}(M) = F^-_{D,B,n}(M)$.

(ii) Suppose $C = {}^\infty E^\infty$ for some word $E = l_1^{-1}r_1 \dots l_p^{-1}r_p$. Then $D^{-1}B = C^{-1}$ is ${}^\infty(E^{-1})^\infty$. For each complex $M$ of projectives with radical images we define a map $\tau_M : F_{B,D,n}(M) \to \mathrm{res}_\iota F_{D,B,n}(M)$ by sending $m + E(n)^\flat$ to $m + (E^{-1})(n)^\flat$.

Since $E(n)^\flat = (E^{-1})(n)^\flat$ and $E(n)^\sharp = (E^{-1})(n)^\sharp$ as sets this is a well defined vector space isomorphism. By lemma 2.2.11 the action of $T$ on $F_{B,D,n}(M)$ is the automorphism $\theta_M^{E(n)}$, and the action of $T$ on $e_vM^n \cap (E^{-1})^\sharp / e_vM^n \cap (E^{-1})^\flat$ is the automorphism $\theta_M^{(E^{-1})(n)}$. Hence the action of $T$ on $\mathrm{res}_\iota F_{D,B,n}(M)$ is $(\theta_M^{(E^{-1})(n)})^{-1}$.

For any $m + E(n)^\flat$ we have $\theta_n^E(m + E(n)^\flat) = m' + E(n)^\flat$ iff $m' \in E(n)^\sharp \cap (E(n)^\flat + Em)$ which holds iff $m \in (E^{-1})(n)^\sharp \cap ((E^{-1})(n)^\flat + E^{-1}m')$. This shows $\theta_M^{(E^{-1})(n)} = (\theta_M^{E(n)})^{-1}$ and hence $(\theta_M^{(E^{-1})(n)})^{-1}(m + (E^{-1})(n)^\flat) = \theta_M^{E(n)}(m + (E)(n)^\flat)$ which proves $\tau_M(T(m + E(n)^\flat)) = T(\tau_M(m + (E^{-1})(n)^\flat))$ and therefore $\tau_M$ is an isomorphism of $k[T, T^{-1}]$-modules.

Furthermore if $f : M \to N$ is some map of complexes then for each $m + E(n)^\flat \in F_{B,D,n}(M)$ we have $\mathrm{res}_\iota(F_{D,B,n}(f))(m + E(n)^\flat) = f^n(m) + E(n)^\flat$ and so $\mathrm{res}_\iota(F_{D,B,n}(f))(\tau_M(m + E(n)^\flat)) = \tau_M(F_{B,D,n}(f)(m + E(n)^\flat))$ which shows $\tau$ defines the appropriate natural isomorphism. □

The next lemma is essentially [55, p.25, Lemma] with minor adjustments.

**Lemma 2.2.22.** *Let $C$ and $E$ be homotopy words such that $(d_{\mathrm{l}(\gamma)}^{-1}\gamma C)^{-1}E$ is a homotopy word for some $\gamma \in \mathbf{P}$. Then for any $n \in \mathbb{Z}$ the functors $G_{C,\gamma^{-1}d_{\mathrm{l}(\gamma)}E,n}$ and $G_{d_{\mathrm{l}(\gamma)}^{-1}\gamma C,E,n-1}$ are naturally isomorphic.*

*Proof.* Let $u = t(\gamma)$, $v = h(\gamma)$, $B = d_{\mathrm{l}(\gamma)}^{-1}\gamma C$ and $D = \gamma^{-1}d_{\mathrm{l}(\gamma)}E$. We start by defining some isomorphisms between the vector spaces $G_{B,E,n-1}(M)$ and $G_{C,D,n}(M)$. In case $B^{-1}E$ is a periodic $\mathbb{Z}$-word we will show these isomorphisms are $k[T, T^{-1}]$-linear. In both cases we show they define a natural isomorphism between $G_{C,\gamma^{-1}d_{\mathrm{l}(\gamma)}E,n}$ and $G_{d_{\mathrm{l}(\gamma)}^{-1}\gamma C,E,n-1}$. For each complex $M$ of projectives with radical images define the vector spaces $L^{-}(M) \subseteq L^{+}(M) \subseteq e_v M^n$ by

$$L^{\pm}(M) = e_v M^n \cap (\gamma C^{-}(M) + d_{\mathrm{l}(\gamma)}E^{\pm}(M) \cap \gamma C^{+}(M)).$$

For $L(M) = L^{+}(M)/L^{-}(M)$ consider the $R$-linear maps $\nu_M : G_{C,D,n}(M) \to L(M)$ sending $m+G^{-}_{C,D,n}(M)$ to $\gamma m+L^{-}(M)$ and $\eta_M : G_{B,E,n-1}(M) \to L(M)$ sending $m+G^{-}_{B,E,n-1}(M)$ to $d^{n-1}_{\mathrm{l}(\gamma),M}(m) + L^{-}(M)$. It is straightforward to show they are well-defined. We now show these maps are isomorphisms of $R$-modules. In doing so we will show $\mathfrak{m}L^{+}(M) \subseteq \ker(\eta_M^{-1}) = 0$ and hence that $L(M)$ is a vector space over $k = R/\mathfrak{m}$. For any $m + L^{-}(M) \in L(M)$ we have $m = \gamma m' + d^{n-1}_{\mathrm{l}(\gamma),M}(m'')$ where $d^{n-1}_{\mathrm{l}(\gamma),M}(m'') = \gamma m'''$ for $m' \in C^{-}(M)$, $m'' \in E^{+}(M)$ and $m''' \in C^{+}(M)$. This shows

$$\gamma m' + \gamma m''' + L^{-}(M) = m + L^{-}(M) = m - \gamma m' + L^{-}(M) = d^{n-1}_{\mathrm{l}(\gamma),M}(m'') + L^{-}(M)$$

and so $m+L^{-}(M)$ may be written as $\nu_M(m'+m'''+G^{-}_{C,D,n}(M))$ or $\eta_M(m''+G^{-}_{B,E,n+1}(M))$, so $\eta_M$ and $\nu_M$ are both surjective. If $\gamma q \in L^{-}(M)$ for some $q \in G^{+}_{C,D,n}(M)$ then $\gamma(q - q') \in d_{\mathrm{l}(\gamma)}E^{-}(M) \cap C^{+}(M)$ for some $q' \in C^{-}(M)$ and so writing $q$ as the sum of $q'$ and $q - q' \in D^{-}(M) \cap C^{+}(M)$ shows $q \in G^{-}_{C,D,n}(M)$ and so $\nu_M$ is injective.

Similarly if $d^{n}_{\mathrm{l}(\gamma),M}(r) \in L^{-}(M)$ for some $r \in G^{+}_{B,E,n-1}(M))$ then $d^{n}_{\mathrm{l}(\gamma),M}(r - r'') \in \gamma C^{-}(M)$ for some $r'' \in E^{-}(M) \cap B^{+}(M)$.

So, writing $r$ as the sum of $r''$ and $r - r'' \in D^{-}(M) \cap C^{+}(M)$ shows $r \in G^{-}_{C,D,n}(M)$ which proves $\eta_M$ is injective. Suppose $f : M \to N$ is an arbitrary morphism of complexes. Fix an element $x+G^{-}_{C,D,n}(M)$ from $G_{C,D,n}(M)$ and suppose $\eta_M^{-1}\nu_M$ sends $x+G^{-}_{C,DE,n}(M)$ to $x' + G^{-}_{B,E,n-1}(M)$.

This means $\gamma x - d^{n-1}_{\mathrm{l}(\gamma),M}(x') \in L^{-}(M)$ and so by corollary 2.1.20 we have

$$\gamma f^{n}(x) - d^{n-1}_{\mathrm{l}(\gamma),N}(f^{n-1}(x')) = f^{n}(\gamma x) - f^{n}(d^{n-1}_{\mathrm{l}(\gamma),M}(x')) = f^{n}(\gamma x - d^{n-1}_{\mathrm{l}(\gamma),M}(x')) \in L^{-}(N)$$

This means $G_{B,E,n-1}(f)\eta_M^{-1}\nu_M$ and $\eta_N^{-1}\nu_N G_{C,D,n}(f)$ both send $x + G^{-}_{C,DE,n}(M)$ to $f^{n-1}(x') + G^{-}_{B,E,n-1}(N)$ and so the collection $\eta_M^{-1}\nu_M : G_{C,D,n}(M) \to G_{B,E,n-1}(M)$ defines a natural isomorphism $G_{C,D,n} \to G_{B,E,n-1}$ assuming $B^{-1}E$ is a not a periodic $\mathbb{Z}$-word (as in this case the co-domain of $G_{C,D,n}$ and $G_{B,E,n-1}$ is a full subcategory of $k$-**Mod**).

So we assume $B^{-1}E$ is a periodic $\mathbb{Z}$-word of period $p$. Writing $B^{-1}E = {}^{\infty}P^{\infty}$ where $P = l_1^{-1}r_1 \dots l_p^{-1}r_p$ gives $l_p = \gamma$, $r_p = d_{\mathrm{l}(\gamma)}$, and $C^{-1}D = A^{\infty}$ where $A = l_p^{-1}r_p l_1^{-1}r_1 \dots l_{p-1}^{-1}r_{p-1}$ if $p > 1$ and $A = P$ if $p = 1$. Hence $P = P_{<p}\gamma^{-1}d_{\mathrm{l}(\gamma)}$ and $A = \gamma^{-1}d_{\mathrm{l}(\gamma)}P_{<p}$. In corollary 2.2.12 we defined automorphisms $\theta_M^{P(n-1)}$ of $F_{B,E,n-1}(M) = P^{\sharp}_{n-1}/P^{\flat}_{n-1}$ and $\theta_M^{A(n)}$ of $F_{C,D,n}(M) = A^{\sharp}_n/A^{\flat}_n$ giving $P^{\sharp}_{n-1}/P^{\flat}_{n-1}$ and $A^{\sharp}_n/A^{\flat}_n$ the structure of $k[T,T^{-1}]$-modules.

By the proofs of corollaries 2.2.8 and 2.2.12, there is an action of $T$ on $(G_{B,E,n-1}(M)$ given by $\beta_{n-1}T\beta_{n-1}^{-1})$ and on $(G_{C,\gamma^{-1}d_{\mathrm{l}(\gamma)}E,n}(M)$ given by $\beta_n T\beta_n^{-1})$ for natural isomorphisms $\beta_{n-1} : F_{B,E,n-1}(M) \to G_{B,E,n-1}(M)$ and $\beta_n : F_{C,D,n}(M) \to G_{C,D,n}(M)$. We now show $\eta_M^{-1}\nu_M$ is $k[T,T^{-1}]$-linear.

Fix an element $m_0 + F^{-}_{C,D,n}(M)$ of $A^{\sharp}_n/A^{\flat}_n$ for some $m_0 \in A^{\sharp}_n$. This means there is some $r_0 \in P^{\sharp}_{n-1}$ for which $m_0 \in \gamma^{-1}d_{\mathrm{l}(\gamma)}r_0$. Recall $\theta_M^{A(n)}(m_0 + A^{\flat}_n) = m' + A^{\flat}_n$ for some $m' \in A^{\sharp}_n \cap (A^{\flat}_n + A_n m_0)$. So there is some $m'' \in A^{\flat}_n$ for which $m' - m'' \in A_n m$. Let $m_{-1} = m' - m''$. Since $m_{-1} \in A^{\sharp}_n$ there is some $r_{-1} \in P^{\sharp}_{n-1}$ with $r_{-1} \in P_{<p}m_0$ and where $d^{n-1}_{\mathrm{l}(\gamma),M}(r_{-1}) = \gamma m_{-1}$. As $m''$ lies in $F^{-}_{C,D,n}(M) \subseteq G^{-}_{C,D,n}(M)$ we have the map $\eta_M^{-1}\nu_M$ sends $m_{-1} + G^{-}_{C,D,n}(M)$ to $r_{-1} + G^{-}_{B,E,n-1}(M)$.

Furthermore as $r_{-1} \in P_{<p}m_0$ and $m_0 \in \gamma^{-1}d_{\mathrm{l}(\gamma)}r_0$ the element $r_{-1}$ must lie in both $P^{\sharp}_{n-1}$ and $P_{<p}\gamma^{-1}d_{\mathrm{l}(\gamma)}r_0 = P_{n-1}r_0$ so by construction $\theta_M^{P(n-1)}(r_0 + P^{\flat}_{n-1}) = r_{-1} + P^{\flat}_{n-1}$. Also as $m_0 \in \gamma^{-1}d_{\mathrm{l}(\gamma)}r_0$ we have $\eta_M^{-1}\nu_M(m_0 + G^{-}_{C,D,n}(M)) = r_0 + G^{-}_{B,E,n-1}(M)$. Applying $\theta_M^{P(n-1)}\beta_{n-1}^{-1}$ to this equation gives $\beta_n^{-1}(r_{-1} + G^{-}_{B,E,n-1}(M))$ on the right hand side which is sufficient as $r_{-1} + G^{-}_{B,E,n-1}(M) = \eta_M^{-1}(\nu_M(m' + G^{-}_{C,D,n}(M)))$ and $\beta_n(m' + A^{\flat}_n) = m' + G^{-}_{C,D,n}(M)$ and thus $\theta_M^{P(n-1)}\beta_{n-1}^{-1}\eta_M^{-1}\nu_M\beta_n = \beta_n^{-1}\eta_M^{-1}\nu_M\beta_n\theta_M^{A(n)}$. $\square$

**Corollary 2.2.23.** *For any $n \in \mathbb{Z}$, any homotopy words $C$, $D = l_1^{-1}r_1 \dots l_s^{-1}r_s$, and $E$ such that $C^{-1}DE$ is a homotopy word: the functors $G_{C,DE,n}$ and $G_{D^{-1}C,E,n+\mu_D(s)}$ are naturally isomorphic.*

*Proof.* Suppose the corollary holds when $s = 1$. Iterating this assumption gives

$$G_{r_1^{-1}l_1C,\dots l_s^{-1}r_sE,n+\mu_D(1)} \simeq \dots \simeq G_{r_s^{-1}l_s\dots r_1^{-1}l_1C,E,n+\mu_D(s)}$$

and so $G_{C,DE,n} \simeq G_{D^{-1}C,E,n+\mu_D(s)}$. So it is enough to choose any letters $l$ and $r$ such that $C^{-1}l^{-1}rE$ is a homotopy word and show the functors $G_{C,l^{-1}rE,n}$ and $G_{r^{-1}lC,E,n+t}$ are isomorphic where $t = H(l) + H(r)$. By lemma 2.2.22 is enough to show $G_{C,d_{1(\gamma)}^{-1}\gamma E,n} \simeq G_{\gamma^{-1}d_{1(\gamma)}C,E,n+1}$ for any $\gamma \in \mathbf{P}$ with $C^{-1}d_{1(\gamma)}^{-1}\gamma E$ a word.

We have $G_{C,d_{1(\gamma)}^{-1}\gamma E,n} \simeq F_{C,d_{1(\gamma)}^{-1}\gamma E,n}$ and $F_{\gamma^{-1}d_{1(\gamma)}C,E,n+1} \simeq G_{\gamma^{-1}d_{1(\gamma)}C,E,n+1}$ by corollaries 2.2.8 and 2.2.12 and by lemma 2.2.21 we have

$$F_{C,d_{1(\gamma)}^{-1}\gamma E,n} \simeq \mathrm{res}_\iota F_{d_{1(\gamma)}^{-1}\gamma E,C,n},$$
$$\mathrm{res}_\iota F_{E,\gamma^{-1}d_{1(\gamma)}C,n+1} \simeq F_{\gamma^{-1}d_{1(\gamma)}C,E,n+1}$$

if $C^{-1}DE$ is periodic, and otherwise

$$F_{C,d_{1(\gamma)}^{-1}\gamma E,n} \simeq F_{d_{1(\gamma)}^{-1}\gamma E,C,n},$$
$$F_{E,\gamma^{-1}d_{1(\gamma)}C,n+1} \simeq F_{\gamma^{-1}d_{1(\gamma)}C,E,n+1}$$

By corollaries 2.2.8 and 2.2.12 and by lemma 2.2.22 we have $G_{E,\gamma^{-1}d_{1(\gamma)}C,n} \simeq G_{d_{1(\gamma)}^{-1}\gamma E,C,n-1}$. This gives $F_{d_{1(\gamma)}^{-1}\gamma E,C,n} \simeq F_{E,\gamma^{-1}d_{1(\gamma)}C,n+1}$, as required. □

**Corollary 2.2.24.** *Suppose elements $(B, D, n)$ and $(B', D', n')$ from $\mathcal{I}$ are equivalent.*

(i) *If $C$ is not a $\mathbb{Z}$-word then $G_{B,D,n} \simeq G_{B',D',n'}$.*

(ii) *If $C$ is a $\mathbb{Z}$-word and $C' = C[m]$ for some $m \in \mathbb{Z}$ then $G_{B,D,n} \simeq G_{B',D',n'}$.*

(iii) *If $C$ is a $\mathbb{Z}$-word and $C' = C^{-1}[m]$ for some $m \in \mathbb{Z}$ then $G_{B,D,n} \simeq \mathrm{res}_\iota G_{B',D',n'}$ if $C$ is periodic and $G_{B,D,n} \simeq G_{B',D',n'}$ otherwise.*

*Proof.* (i) Suppose firstly $C = C'$. Without loss of generality we may assume $D = ED'$ and so $E^{-1}B = B'$ for some finite word $E = l_1 r_1^{-1} \dots l_t^{-1} r_t$. This gives $\mu_C(a_{B',D'}) = \mu_C(a_{B,D}) + \mu_E(t)$ and so $n + \mu_E(t) = n'$.

By corollary 2.2.23 we have that $G_{B,D,n} = G_{B,l_1 r_1^{-1} \dots l_t^{-1} r_t D',n}$ is naturally isomorphic to $G_{r_t^{-1} l_t \dots r_1^{-1} l_1 B, D', n+\mu_E(t)} = G_{B',D',n'}$. Otherwise we have $C' = C^{-1}$. The case when $C$ is a $-\mathbb{N}$-word will follow from the case where $C$ is an $\mathbb{N}$-word and by the symmetry of the claim and the relation $\sim$ on $\mathcal{I}$. Hence we can assume $C = C'^{-1}$ is a (finite or $\mathbb{N}$)-word and hence $B$ and $D'$ are finite words.

By definition $B^{\pm 1}$ is a $\{0, \dots, a_{B,D}\}$-word, $D'^{\pm 1}$ is a $\{0, \dots, a_{D',B'}\}$-word, $\mu_{D'^{-1}}(a_{D',B'}) = \mu_C(a_{D',B'})$ and $\mu_{B^{-1}}(a_{B,D}) = \mu_C(a_{B,D})$. Letting $b = \mu_{B^{-1}}(a_{B,D})$ and $d = \mu_{D'^{-1}}(a_{D',B'})$ by corollary 2.2.23 we have $G_{B,D,n} \simeq G_{\underline{1}_{v,\delta}, B^{-1}D, n-b}$ and $G_{\underline{1}_{v,\delta}, D'^{-1}B', n-b} \simeq G_{D',B',n+d-b}$. As $G_{\underline{1}_{v,\delta}, B^{-1}D, n-b} = G_{\underline{1}_{v,\delta}, D'^{-1}B', n-b}$ and $n + d - b = n'$ this shows $G_{B,D,n} \simeq G_{D',B',n'}$. By corollary 2.2.8 $G_{D',B',n'} \simeq F_{D',B',n'}$ and $G_{B',D',n'} \simeq F_{B',D',n'}$, and $F_{D',B',n'} \simeq F_{B',D',n'}$ follows by lemma 2.2.21. Altogether this shows $G_{B,D,n} \simeq G_{B',D',n'}$.

(ii) The case where $m \leq 0$ will follow from the case for $m \geq 0$ and the symmetry of the claim and the relation $\sim$ on $\mathcal{I}$. Assuming $m \geq 0$ we have $B = (C_{\leq 0})^{-1}$, $D = (C_{\leq m})_{>0} C_{>m}$, $B' = ((C_{\leq m})_{>0})^{-1} (C_{\leq 0})^{-1}$ and $D' = C_{>m}$. So by corollary 2.2.23 we have that $G_{B,D,n} \simeq G_{B',D',n'}$ as $\mu_{C'[-m]}(m) = \mu_C(m)$ by lemma 1.3.33 (iii).

(iii) Let $B'' = D$, $D'' = B$, $C'' = C^{-1}$ and $n'' = n$. By corollary 2.2.12 we have $G_{B,D,n} \simeq F_{D'',B'',n''}$ and $F_{B'',D'',n''} \simeq G_{B'',D'',n''}$. By lemma 2.2.21 $F_{D'',B'',n''} \simeq \operatorname{res}_\iota F_{B'',D'',n''}$ if $C$ is periodic and $F_{D'',B'',n''} \simeq F_{B'',D'',n''}$ otherwise. Since $C''[m] = C'$ and $n' - n'' = \mu_{C''}(m)$ by lemma 1.3.33, by part (ii) above in the case for $\mathbb{Z}$-words we have $G_{B'',D'',n''} \simeq G_{B',D',n'}$ and hence $\operatorname{res}_\iota G_{B'',D'',n''} \simeq \operatorname{res}_\iota G_{B',D',n'}$. Altogether this shows $G_{B,D,n} \simeq \operatorname{res}_\iota G_{B',D',n'}$ if $C$ is periodic and $G_{B,D,n} \simeq G_{B',D',n'}$ otherwise. $\square$

### 2.2.3 Constructive Functors.

Recall Kaplansky [42, p.372, (1)] showed that every projective $R$-module is free since $R$ is local. Hence $R$-**Proj** is the category of free $R$-modules, and $R$-**proj** is the full subcategory of finitely generated free modules.

**Definition 2.2.25.** (CONSTRUCTIVE FUNCTORS FOR COMPLEXES, NOTATION: $S_{B,D,n}$) For $(B, D, n) \in \mathcal{I}(s)$ we define a functor $S_{B,D,n} : R\text{-}\mathbf{Proj} \to \mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ as follows. On objects, $S_{B,D,n}$ sends a free $R$-module $V$ to the complex $S_{B,D,n}(V)$ whose homogeneous component is $P^i(C)[\mu_C(a_{B,D}) - n] \otimes_R V$ and whose differential is $d^i_{P(C)[\mu_C(a_{B,D})-n]} \otimes 1_V$ in degree $i \in \mathbb{Z}$. Hence $S_{B,D,n}(V)$ may be considered the direct sum of copies of $P(C)$ (shifted by $\mu_C(a_{B,D}) - n$) indexed by a (possibly infinite) $R$-basis of $V$.

For an $R$-linear map $f : V \to V'$ and bases $\{v_\lambda \mid \lambda \in \Omega\}$ for $V$ and $\{v'_{\lambda'} \mid \lambda' \in \Omega'\}$ for $V'$, write $\mathrm{l}(v_\lambda) = \sum a_{\lambda',\lambda} v_{\lambda'}$ for scalars $a_{\lambda',\lambda} \in R$ for each $\lambda, \lambda' \in \Omega$. Let $\underline{b}_{i,C}$ denote the coset of $e_{v_C(i)}$ in the summand $\Lambda e_{v_C(i)}$ of $P(C)[\mu_C(a_{B,D}) - n]$, and let $\underline{b}_{i,\lambda,C} = \underline{b}_{i,C} \otimes v_\lambda$ for each $i \in I_C$ and $\lambda \in \Omega$. Similarly for each $\lambda' \in \Omega'$ let $\underline{b}'_{i,\lambda',C} = \underline{b}_{i,C} \otimes v'_{\lambda'}$ for each $i \in I_C$. Define $S_{B,D,n}$ on morphisms by extending the assignment $S_{B,D,n}(f)(\underline{b}_{i,\lambda,C}) = \sum_{\lambda'} a_{\lambda',\lambda} \underline{b}'_{i,\lambda',C}$ linearly over $\Lambda$. Note that if $B^{-1}D$ has controlled homogeny then $S_{B,D,n}$ defines a functor into $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{proj})$ upon restriction to $R$-**proj**. The converse also holds.

For $(B, D, n) \in \mathcal{I}(b)$ we have $a_{B,D} = 0$. Furthermore the free $R$-modules $V$ and $V'$ have the additional structure of left $R[T, T^{-1}]$-modules and the $R$-linear map $f : V \to V'$ above is additionally $R[T, T^{-1}]$-linear. Hence $T$ defines automorphisms $\varphi_V : V \to V$ and $\varphi_{V'} : V' \to V'$ satisfying $f\varphi_V = \varphi_{V'} f$. Suppose $C$ is periodic of period $p$. Recall $R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$ is the full subcategory of $R[T, T^{-1}]$-**Mod** consisting of $R[T, T^{-1}]$-modules which are free as $R$-modules. Define the functor $S_{B,D,n} : R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}} \to \mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ on objects by $S_{B,D,n}(V) = P(C, V)[-n]$.

The formula $S_{B,D,n}(f)(\underline{b}_{i,\lambda,C}) = \sum_{\lambda' \in \Omega'} a_{\lambda',\lambda} \underline{b}'_{i,\lambda',C}$ gives $S_{B,D,n}(f)(\underline{b}_{i-p,\lambda,C}) = T(\sum_{\lambda'} a_{\lambda',\lambda} \underline{b}'_{i,\lambda',C})$ and consequently $S_{B,D,n}(f)(T\underline{b}_{i,\lambda,C}) = T(S_{B,D,n}(f)(\underline{b}_{i,\lambda,C}))$. Hence $S_{B,D,n}(f)$ defines a $\Lambda \otimes_R R[T, T^{-1}]$-module morphism $P^i(C)[-n] \to P^i(C)[-n']$ for each $i \in \mathbb{Z}$ which defines a morphism of complexes $P(C, V)[-n] \to P(C, V')[-n']$.

The proof of the following result will be written with the notation introduced above.

**Corollary 2.2.26.** *Suppose elements $(B, D, n)$ and $(B', D', n')$ from $\mathcal{I}$ are equivalent. Then there is a bijection $\omega : I_{C'} \to I_C$ defining a morphism of complexes $\theta : P(C')[\mu_{C'}(a_{B',D'}) - n'] \to P(C)[\mu_C(a_{B,D}) - n]$.*

*Furthermore if $C$ is a periodic $\mathbb{Z}$-word of period $p$ we have $(\theta(\underline{b}_{i,C'}T) = \underline{b}_{\omega(i),C}T$ when $C' = C[m]$) and $(\theta(\underline{b}_{i,C'}T^{-1}) = \underline{b}_{\omega(i),C}T$ when $C' = C^{-1}[m]$). Consequently the following statements hold.*

(i) *If $C$ is not a periodic homotopy $\mathbb{Z}$-word then $S_{B,D,n} \simeq S_{B',D',n'}$.*

(ii) *If $I_C = \mathbb{Z}$, $C$ is periodic and $C' = C[m]$ (for some $m \in \mathbb{Z}$) then $S_{B,D,n} \simeq S_{B',D',n'}$.*

(iii) *If $I_C = \mathbb{Z}$, $C$ is periodic and $C' = C^{-1}[m]$ then $S_{B,D,n} \simeq S_{B',D',n'}\,\mathrm{res}_\iota$.*

Unlike the proof of corollary 2.2.24 it will be useful here to treat the cases $(B, D, n), (B', D', n') \in \mathcal{I}(s)$ and $(B, D, n), (B', D', n') \in \mathcal{I}(b)$ separately.

*Proof.* If $C' = C$ then $I_C = I_{C'}$ and we let $\omega$ be the identity map. Instead suppose $C' = C^{-1}$. If $C$ is finite then $I_C = \{0, \dots, t\}$ for some $t \in \mathbb{N}$ in which case $I_{C^{-1}} = I_C$, $v_{C^{-1}}(i) = v_C(t - i)$ and $\mu_{C^{-1}}(i) = \mu_C(t - i) - \mu_C(t)$ for each $i \in I_{C^{-1}}$ by lemma 1.3.33. Here $a_{B',D'} = t - a_{D',B'}$ by lemma 2.2.15 and so $\mu_{C'}(a_{B',D'}) = \mu_{C^{-1}}(m - a_{D',B'}) = \mu_C(a_{D',B'}) - \mu_C(m)$.

This shows $S_{B',D',n'}(R) = P(C^{-1})[\mu_C(a_{B,D}) - n - \mu_C(t)]$. By corollary 1.3.43 there is an isomorphism of complexes $\theta$ as above where the bijection $\omega$ is defined by $\omega(i) = t - i$. If $C$ is infinite it is enough to let $\omega$ be defined by $\omega(i) = -i$ using a similar argument to the above replacing $t$ with 0.

Suppose now $C' = C[m]$ is a homotopy $\mathbb{Z}$-word. As $\mu_{C'}(a_{B',D'}) - n' = -n - \mu_{C'[-m]}(m)$ and $\mu_{C'[-m]}(m) = \mu_C(m)$, by corollary 1.3.43 there is an isomorphism of complexes $\theta$ from $S_{B',D',n'}(R) = P(C[m])[-n - \mu_C(m)]$ to $P(C)[-n] = S_{B,D,n}(R)$ as above where the bijection $\omega$ is defined by $\omega(i) = m + i$. If $C$ is periodic of period $p$ then $\underline{b}_{i,C[m]}T = \underline{b}_{i-p,C[m]} = \underline{b}_{i-p+m,C}$ and so $\theta(\underline{b}_{i,C'}T) = \underline{b}_{\omega(i),C}T$.

Instead suppose $C' = C^{-1}[m]$ is a homotopy $\mathbb{Z}$-word. Again by corollary 1.3.43 it is enough to let $\omega(i) = -(i+m)$, and similarly if $C$ is periodic of period $p$ the equations $\underline{b}_{i,C'}T^{-1} = \underline{b}_{i+p,C^{-1}[m]} = \underline{b}_{-(i+m)-p,C}$ show $\theta(\underline{b}_{i,C'}T^{-1}) = \underline{b}_{\omega(i),C}T$.

(i) Here $(B, D, n) \in \mathcal{I}(s)$. Given a free $R$-module $V$ with $R$-basis $\{v_\lambda \mid \lambda \in \Omega\}$ letting $\sigma_V^r = \theta^r \otimes_R 1_V$ defines a $\Lambda$-module map $\sigma_V^r$ from $P^r(C')[\mu_{C'}(a_{B',D'}) - n'] \otimes_R V$ to $P^r(C)[\mu_C(a_{B,D}) - n] \otimes_R V$ which sends $\underline{b}_{i,\lambda,C'} = \underline{b}_{i,C'} \otimes v_\lambda$ to $\underline{b}_{\omega(i),\lambda,C} = \underline{b}_{\omega(i),C} \otimes v_\lambda$. Furthermore this is a morphism of complexes since $\underline{b}^{\pm}_{\omega(i),C} \otimes v_\lambda = (\theta^r \otimes 1_V)(\underline{b}^{\pm}_{i,C'} \otimes v_\lambda)$ for any $i \in I_{C'}$. So for each ($\lambda \in \Omega$, $g \in \mathbb{Z}$, and $i \in \mu_{C'}^{-1}(g + \mu_{C'}(a_{B',D'}) - n')$) we have

$$(\theta \otimes 1_{V'})(S_{B',D',n'}(f)(\underline{b}_{i,\lambda,C'})) = (\theta \otimes 1_{V'})(\textstyle\sum_{\lambda'} a_{\lambda',\lambda}\underline{b}'_{i,\lambda',C}) = (\theta \otimes 1_V)(\sum_{\lambda'} a_{\lambda',\lambda}\underline{b}'_{i,C} \otimes v'_{\lambda'})$$
$$= \textstyle\sum_{\lambda'} a_{\lambda',\lambda}\theta(\underline{b}'_{i,C'}) \otimes 1_{V'}(v'_{\lambda'}) = \sum_{\lambda'} a_{\lambda',\lambda}\underline{b}'_{\omega(i),C} \otimes v'_{\lambda'} = \sum_{\lambda'} a_{\lambda',\lambda}\underline{b}'_{\omega(i),\lambda',C} = S_{B',D',n'}(f)(\underline{b}_{\omega(i),\lambda,C})$$

which altogether shows $(\sigma_{V'} \circ S_{B',D',n'}(f))(\underline{b}_{i,\lambda,C'}) = S_{B',D',n'}(f)(\sigma_V(\underline{b}_{i,\lambda,C'}))$ and so $\sigma$ defines a natural isomorphism.

(ii) Here $(B, D, n) \in \mathcal{I}(b)$. For a $R[T, T^{-1}]$-module $V$ with free $R$-basis $\{v_\lambda \mid \lambda \in \Omega\}$ since $\theta(\underline{b}_{i,C'}T) = \underline{b}_{\omega(i),C}T$ the map $\theta^r$ is $R[T, T^{-1}]$-balanced. Hence as $d_{P(C,V)} = d_{P(C)} \otimes 1_V$ writing $\sigma_V^r = \theta^r \otimes_{R[T,T^{-1}]} 1_V$ defines a $\Lambda$-module map from $P^r(C', V) = P^r(C')[-n'] \otimes_{R[T,T^{-1}]} V$ to $P^r(C)[-n] \otimes_{R[T,T^{-1}]} V = P^r(C, V)$ sending $\underline{b}_{i,s,C'} = \underline{b}_{i,C'} \otimes v_s$ to $\underline{b}_{\omega(i),s,C} = \underline{b}_{\omega(i),C} \otimes v_s$. As in part (i), $\sigma_{V'} \circ S_{B',D',n'}(f) = S_{B',D',n'}(f) \circ \sigma_V$ for any $R[T, T^{-1}]$-module map $f : V \to V'$. So again $\sigma$ defines a natural isomorphism $S_{B',D',n'} \to S_{B,D,n}$.

(iii) Similarly since $\theta(\underline{b}_{i,C'}T^{-1}) = \underline{b}_{\omega(i),C}T$ letting $\sigma_V^r = \theta^r \otimes_{R[T,T^{-1}]} 1_{\mathrm{res}_\iota V}$ defines a $\Lambda$-module map from $P^r(C', \mathrm{res}_\iota V)$ to $P^r(C, \mathrm{res}_\iota V)$ giving a natural isomorphism $S_{B'',D'',n''} \simeq S_{B',D',n'}$ where $B'' = D$, $D'' = B$, $C'' = C^{-1}$ and $n'' = n' - \mu_{C''}(m)$. In this notation we have $\mu_{C'[-m]}(m) = \mu_{C''}(m)$ by lemma 1.3.33 (iii) and so $n'' = n$. Hence $(B, D, n) \sim (B'', D'', n'')$ and as $C''[m] = C'$ by part (ii) we have $S_{B'',D'',n''} \simeq S_{D'',B'',n''} = S_{B,D,n}$. Altogether we have $S_{B,D,n} \simeq S_{B',D',n'}$. □

## 2.3 Evaluation on Complexes.

In this section we will apply our refined functors to the complexes $M$ which are of the form $P(C)$ or $P(C,V)$. Recalling corollary 2.2.6, to evaluate a refined functor on a string or band complex it will be enough to provide a description of $A^{\pm}(M)+\mathrm{rad}(M)$ for any (finite or $\mathbb{N}$)-homotopy word $A$.

**Assumption:** In section 2.3 we let $C$ be any homotopy word.

We will adapt [21, Lemma 8.1] for our purposes, which will require some technical results that we have collected together. We start by giving a convenient way of writing elements in the modules $P(C)$ and $P(C,V)$.

### 2.3.1 Coefficients.

**Definition 2.3.1.** (TRANSVERSALS) Choosing a lift $s \in R$ for each $\bar{s} \in R/\mathfrak{m}$ defines a (fixed, yet arbitrary) *transversal* $S$ of $\mathfrak{m}$, that is, a subset of $R$ for which $S \cap (r+\mathfrak{m})$ contains precisely one element for each $r \in R$. To simplify proofs we assume $0 \in S$ and so $\{0\} = S \cap \mathfrak{m}$. This does not change any result in what follows. Let $k$ be the field $R/\mathfrak{m}$.

**Example 2.3.2.** Recall the $p$-adic integers $\widehat{\mathbb{Z}}_p$ from example 1.1.1. Here $\{0,\dots,p-1\}$ is a transversal of $p\widehat{\mathbb{Z}}_p$. For the power series ring $k[[t]]$ the field $k$ defines a transversal of $(t)$.

**Definition 2.3.3.** (NOTATION: $\mathbf{P}[i]$, $\eta_i$, $r_{\sigma,i}$) Recall that for each $i \in I$ the symbol $\underline{b}_i$ denotes the coset of $e_{v_C(i)}$ in the summand $\Lambda e_{v_C(i)}$ of $P^{\mu_C(i)}(C)$, and hence $P(C)$ is generated as a $\Lambda$-module by the elements $\underline{b}_i$. For each $i \in I$ let $\mathbf{P}[i] = \mathbf{P}(v_C(i) \rightarrow)$, (see definition 1.1.7) the set of all non-trivial paths $\sigma \notin (\rho)$ with tail $v_C(i)$.

If $S$ is a fixed transversal of $\mathfrak{m}$ in $R$ we can write any $r \in R$ as $r = \eta + z$ for some $\eta \in S$ and some $z \in \mathfrak{m}$ gives $z\underline{b}_i \in \bigoplus_{a \in \mathbf{A}(v_C(i)\rightarrow)} \Lambda a$ (recall $\Lambda$ is pointwise rad-nilpotent modulo $\mathfrak{m}$). Hence any $m \in P(C)$ can be written as $m = \sum_i (\eta_i \underline{b}_i + \sum_{\sigma \in \mathbf{P}[i]} r_{\sigma,i}\sigma\underline{b}_i)$ where (for each $i$): $\eta_i \in S$ and $r_{\sigma,i} \in R$; $r_{\sigma,i} = 0$ for all but finitely many $\sigma$; and $\eta_j = r_{j,\sigma} = 0$ for all but finitely many $j$.

(NOTATION: $\psi_t$, $\lceil m \rceil$, $\lfloor m \rfloor$, $\mathbf{P}[x,i]$, $\lceil m_{x,t}$, $\lfloor m_{x,t}$, $m\rceil_{x,t}$, $m\rfloor_{x,t}$, $\dagger$) Fix some arbitrary $t \in I$ and let $\psi_t$ denote the $\Lambda$-module epimorphism $P(C) = \bigoplus_i \Lambda e_{v_C(i)} \to \Lambda e_{v_C(t)}$ sending $m = \sum_i m_i \underline{b}_i$ to $m_t$. For $m = \sum_i (\eta_i \underline{b}_i + \sum_{\sigma \in \mathbf{P}[i]} r_{\sigma,i}\sigma\underline{b}_i)$ as above let $\lceil m \rceil = \sum_i \eta_i \underline{b}_i$ and $\lfloor m \rfloor = \sum_{i,\sigma} r_{\sigma,i}\sigma\underline{b}_i$. This gives $\psi_t(\gamma \lceil m \rceil) = \eta_t \gamma$ and $\psi_t(\gamma \lfloor m \rfloor) = \sum_\sigma r_{\sigma,t}\gamma\sigma$. For any arrow $x$ write $\mathbf{P}[x,i]$ for the subset of $\mathbf{P}[i]$ consisting of all $\sigma$ with $\mathrm{l}(\sigma) = x$, and let $[m_{x,t} = \lceil m_{x,t} + \lfloor m_{x,t}$ and $m]_{x,t} = m\rceil_{x,t} + m\rfloor_{x,t}$ where

$$(\dagger)\quad \begin{aligned}
m\rceil_{x,t} &= \begin{cases} \eta_{t+1}\alpha & (\text{if } t+1 \in I \text{ and } l_{t+1}^{-1} r_{t+1} = \alpha^{-1} d_x) \\ 0 & (\text{otherwise}) \end{cases} \\
\lceil m_{x,t} &= \begin{cases} \eta_{t-1}\beta & (\text{if } t-1 \in I \text{ and } l_t^{-1} r_t = d_x^{-1} \beta) \\ 0 & (\text{otherwise}) \end{cases} \\
m\rfloor_{x,t} &= \begin{cases} \sum_{\sigma \in \mathbf{P}[x,t+1]} r_{\sigma,t+1}\sigma\kappa & (\text{if } t+1 \in I \text{ and } l_{t+1}^{-1} r_{t+1} = \kappa^{-1} d_{\mathrm{l}(\kappa)}) \\ 0 & (\text{otherwise}) \end{cases} \\
\lfloor m_{x,t} &= \begin{cases} \sum_{\sigma \in \mathbf{P}[x,t-1]} r_{\sigma,t-1}\sigma\zeta & (\text{if } t-1 \in I \text{ and } l_t^{-1} r_t = d_{\mathrm{l}(\zeta)}^{-1} \zeta) \\ 0 & (\text{otherwise}) \end{cases}
\end{aligned}$$

**Example 2.3.4.** Recall the complete gentle algebra $\Lambda = kQ/(\rho)$ from example 1.2.12, where $\rho = \{ba, cb, ac, sr, ts, rt\}$ and $Q$ is the quiver

$$\begin{tikzcd}
1 \arrow[dd, "b"'] & & 3 \arrow[dl, "t"'] \\
& 0 \arrow[ul, "a"'] \arrow[dr, "r"] & \\
2 \arrow[ur, "c"] & & 4 \arrow[uu, "s"']
\end{tikzcd}$$

In this case the ground ring $R$ is just the field $k$, and so $k$ is a transversal. Recall the homotopy word $C$ with $[C] = [s][t][c^{-1}]$ from example 1.3.35. The associated string complex $P(C)$ was depicted by

$$\begin{tikzcd}
 & & & & & \underline{b}_2 \arrow[dl, "a"] \arrow[dr, "r"'] \arrow[dddll, bend right, "d_t"'] \arrow[dddrr, bend left, "d_c"] & & & \\
 & & & & a\underline{b}_2 & & r\underline{b}_2 & & \\
 & & \underline{b}_1 \arrow[dr, "t"'] \arrow[dddl, bend right, "d_s"'] & & & & & & \underline{b}_3 \arrow[dl, "c"'] \\
 & & & t\underline{b}_1 \arrow[dr, "a"'] & & & & c\underline{b}_3 \arrow[dl, "r"'] & \\
\underline{b}_0 \arrow[dr, "s"'] & & & & at\underline{b}_1 & & rc\underline{b}_3 & & \\
 & s\underline{b}_0 & & & & & & &
\end{tikzcd}$$

Write an arbitrary element $m \in P(C)$ as

$$m = \begin{cases} \eta_0\underline{b}_0 + r_{s,0}s\underline{b}_0 + \\ \eta_1\underline{b}_1 + r_{t,1}t\underline{b}_1 + r_{at,1}at\underline{b}_1 + \\ \eta_2\underline{b}_2 + r_{a,2}a\underline{b}_2 + r_{r,2}r\underline{b}_2 + \\ \eta_3\underline{b}_3 + r_{c,3}c\underline{b}_3 + r_{rc,3}rc\underline{b}_3 \end{cases}$$

Hence $m\rceil_{s,0} = \eta_1 s$; $m\rceil_{s,0} = 0$ for $x \neq s$; and $\lceil m_{x,0} = m\rfloor_{x,0} = \lfloor m_{x,0} = 0$ for any $x$. Similarly: $m\rceil_{x,1} = \eta_1 t$ ; $m\rceil_{s,0} = 0$ for $x \neq t$; $m\rfloor_{a,1} = r_{a,2}at$; $m\rfloor_{x,1} = 0$ for $x \neq a$; $\lceil m_{x,1} = \lfloor m_{x,1} = 0$ for any $x$; $\lceil m_{x,2} = \lfloor m_{x,2} = m\rceil_{x,2} = m\rfloor_{x,2} = 0$; $m\rceil_{x,3} = m\rfloor_{x,3} = 0$ ; $\lceil m_{c,3} = \eta_2 c$; $\lceil m_{x,3} = 0$ for $x \neq c$; $\lfloor m_{c,3} = r_{r,2}rc$; and $\lfloor m_{x,3} = 0$ for $x \neq c$.

For a vertex $v$ recall the sum $\sum_{a\in\mathbf{A}(v\to)} \Lambda a$ is direct because $\Lambda$ is a quasi-bounded (gentle, and hence string) algebra over $R$. For any arrow $y$ with tail $v$ let $\theta_y : \bigoplus_{a\in\mathbf{A}(v\to)} \Lambda a \to \Lambda y$ be the canonical $\Lambda$-module epimorphism.

Since $P(C) = \bigoplus_i \Lambda e_{v_C(i)}$ the sum $\sum yP(C)$ over all arrows $y$ is direct. Let $\iota_x : xP(C) \to \bigoplus yP(C)$ and $\pi_x : \bigoplus yP(C) \to xP(C)$ respectively denote the natural $R$-module inclusions and projections. For $m' \in P(C)$ write $m' = \sum_i \eta'_i \underline{b}_i + \sum_i \sum_{\sigma \in \mathbf{P}[i]} r'_{i,\sigma} \sigma \underline{b}_i$ as above.

**Lemma 2.3.5.** *For any arrow $x$, $t \in I$ and any elements $m = \sum_i (\eta_i \underline{b}_i + \sum_{\sigma \in \mathbf{P}[i]} r_{\sigma,i} \sigma \underline{b}_i)$ and $m' = \sum_i \eta'_i \underline{b}_i + \sum_i \sum_{\sigma \in \mathbf{P}[i]} r'_{i,\sigma} \sigma \underline{b}_i$ in $P(C)$: the following statements hold.*

(i) $\psi_t(d_{x,P(C)}(\lceil m \rceil)) = \lceil m_{x,t} + m \rceil_{x,t}$ *and* $\psi_t(d_{x,P(C)}(\lfloor m \rfloor)) = \lfloor m_{x,t} + m \rfloor_{x,t}$.

(ii) *If* $\gamma \in \mathbf{P}[x,t]$ *and* $m \in \gamma^{-1} d_x m'$ *then* $\theta_{\mathrm{f}(\gamma)}(\psi_t(d_{x,P(C)}(m'))) - \eta_t \gamma \in \gamma \mathrm{rad}(\Lambda)$.

(iii) *If* $l_{t+1}^{-1} r_{t+1} = \gamma^{-1} d_{\mathrm{l}(\gamma)}$ *then* $\theta_{\mathrm{f}(\gamma)}(m]_{\mathrm{l}(\gamma),t}) - \eta_{t+1}\gamma \in \gamma \mathrm{rad}(\Lambda)$ *and* $\theta_{\mathrm{f}(\gamma)}([m_{x,t}) = 0$.

(iv) *If* $l_t^{-1} r_t = d_{\mathrm{l}(\gamma)}^{-1} \gamma$ *then* $\theta_{\mathrm{f}(\gamma)}([m_{\mathrm{l}(\gamma),t}) - \eta_{t-1}\gamma \in \gamma \mathrm{rad}(\Lambda)$ *and* $\theta_{\mathrm{f}(\gamma)}(m]_{x,t}) = 0$.

**Example 2.3.6.** We check the formulas in the statement of lemma 2.3.5 (i) (above) are consistent with the calculations performed in example 2.3.4. Note that for $m$ as before we have

$$\lceil m \rceil = \eta_0 \underline{b}_0 + \eta_1 \underline{b}_1 + \eta_2 \underline{b}_2 + \eta_3 \underline{b}_3,$$
$$\lfloor m \rfloor = r_{s,0} s \underline{b}_0 + r_{t,1} t \underline{b}_1 + r_{at,1} at \underline{b}_1 + r_{a,2} a \underline{b}_2 + r_{r,2} r \underline{b}_2 + r_{c,3} c \underline{b}_3 + r_{rc,3} rc \underline{b}_3$$

Hence for each arrow $x$ and each $i \in I$ we have

$$\psi_i(d_{x,P(C)}(\lceil m \rceil)) = \psi_i(\iota_x(\pi_x(s\eta_1 \underline{b}_0 + t\eta_2 \underline{b}_1 + c\eta_2 \underline{b}_3)))$$
$$= \left\{ \begin{array}{cc} s\eta_1 = \lceil m_{s,0} + m \rceil_{s,0} & (\text{if } i = 0 \text{ and } x = s) \\ t\eta_2 = \lceil m_{s,1} + m \rceil_{s,1} & (\text{if } i = 1 \text{ and } x = t) \\ c\eta_2 = \lceil m_{s,3} + m \rceil_{s,3} & (\text{if } i = 3 \text{ and } x = s) \\ 0 & (\text{otherwise}) \end{array} \right\}$$

and similarly

$$\begin{aligned}\psi_i(d_{x,P(C)}(\lfloor m \rfloor)) &= \psi_i(d_{x,P(C)}(r_{s,0}s\underline{b}_0 + r_{t,1}t\underline{b}_1 + r_{at,1}at\underline{b}_1 + r_{a,2}a\underline{b}_2 + r_{r,2}r\underline{b}_2 + r_{c,3}c\underline{b}_3)) \\ &= \psi_i(\iota_x(\pi_x(r_{a,2}at\underline{b}_1 + r_{r,2}rc\underline{b}_3))) \\ &= \left\{\begin{array}{ll} r_{a,2}at = \lfloor m_{c,1} + m \rfloor_{c,1} & (\text{if } i = 1 \text{ and } x = a) \\ r_{r,2}rc = \lfloor m_{c,3} + m \rfloor_{c,3} & (\text{if } i = 3 \text{ and } x = r) \\ 0 & (\text{otherwise}) \end{array}\right\}\end{aligned}$$

*Proof of lemma 2.3.5.* (i) Note that $d_{x,P(C)}(\sum \eta_i \underline{b}_i) = \sum_i \eta_i \iota_x(\pi_x(\underline{b}_i^- + \underline{b}_i^+))$, and if $\underline{b}_i^\pm$ defines a non-zero element in $\Lambda\underline{b}_t$ then $i = t \mp 1$. By straightforward case analysis this shows $\psi_t(d_{x,P(C)}(\lceil m \rceil)) = \lceil m_{x,t} + m \rceil_{x,t}$.

By lemma 2.1.2 (i) we have $d_{x,P(C)}(\sum_{i,\sigma} r_{\sigma,i}\sigma\underline{b}_i) = \sum_{i,(\sigma,z,i)} r_{\sigma,i}\sigma d_{z,P(C)}(\underline{b}_i)$ for each $i$ where the triples $(\sigma, z, i)$ run through all $\sigma \in \mathbf{P}[x, i]$ and all arrows $z$ with $\mathrm{f}(\sigma)z \notin (\rho)$. For any such triple $(\sigma, z, i)$ we have $\sigma d_{P(C)}(\underline{b}_i) = \sigma d_{z,P(C)}(\underline{b}_i)$ and so it follows that $\psi_t(d_{x,P(C)}(\lfloor m \rfloor)) = \lfloor m_{x,t} + m \rfloor_{x,t}$.

(ii) Since $\sum_\sigma r_{\sigma,t+1}\sigma \in \mathrm{rad}(\Lambda)$ we have $\theta_{\mathrm{f}(\gamma)}(m) - \eta_{t+1}\gamma \in \gamma\mathrm{rad}(\Lambda)$ and so applying $\theta_{\mathrm{f}(\gamma)}\psi_t$ to either side of $\gamma m = d_{x,P(C)}(m')$ gives $\theta_{\mathrm{f}(\gamma)}\psi_t(d_{x,P(C)}(m')) - \eta_t\gamma = \sum_{\sigma:\mathrm{f}(\sigma)=\mathrm{f}(\gamma)} r_{\sigma,t}\gamma\sigma$, as required.

(iii) If $l_{t+1}^{-1}r_{t+1} = \gamma^{-1}d_x$ then $\lceil m_{x,t} = 0$ unless $l_t^{-1}r_t = d_{\mathrm{l}(\zeta)}^{-1}\zeta$ in which case $\mathrm{f}(\gamma) \neq \mathrm{f}(\zeta)$ since $d_{\mathrm{l}(\zeta)}^{-1}\zeta\gamma^{-1}d_x$ is a word. Furthermore $m\rceil_{\mathrm{l}(\gamma),t+1} = \eta_{t+1}\gamma$ and $m\rfloor_{\mathrm{l}(\gamma),t+1} = \sum_{\sigma\in\mathbf{P}[\mathrm{l}(\gamma),t+1]} r_{\sigma,t+1}\sigma\gamma \in \mathrm{rad}(\Lambda)\gamma\cap\mathrm{l}(\gamma)\Lambda$ which is contained in $\Lambda\mathrm{f}(\gamma)\cap\gamma\mathrm{rad}(\Lambda)$ by corollary 1.2.14 (v). The proof of (iv) is similar, or may be deduced by applying (iii) to the homotopy word $D = C^{-1}$. $\square$

**Definition 2.3.7.** (NOTATION: $C(i,\delta)$, $C(i,\delta)_s$ and $d_i(C,\delta)$) Recall definition 2.1.11. For each $i$ the words $C_{>i}$ and $(C_{\leq i})^{-1}$ have head $v_C(i)$ and opposite sign by proposition 2.1.13. For $\delta \in \{\pm 1\}$ let $C(i,\delta)$ denote the one with sign $\delta$. If $C(i,\delta) = C_{>i}$ then let $d_i(C,\delta) = 1$, and otherwise $C(i,\delta) = (C_{\leq i})^{-1}$ in which case we let $d_i(C,\delta) = -1$.

Note that for any $s \in I_{C(i,\delta)}$ such that $s+1 \in I_{C(i,\delta)}$ we have

$$C(i,\delta)_s = \begin{Bmatrix} (C_{>i})_s & (\text{if } d=1) \\ ((C_{\leq i})^{-1})_s & (\text{if } d=-1) \end{Bmatrix}$$

$$= \begin{Bmatrix} (l_{i+1}^{-1} r_{i+1} l_{i+2}^{-1} r_{i+2} \dots)_s & (\text{if } d=1) \\ (r_i^{-1} l_i r_{i-1}^{-1} l_{i-1} \dots)_s & (\text{if } d=-1) \end{Bmatrix} = \begin{Bmatrix} l_{i+s}^{-1} r_{i+s} & (\text{if } d=1) \\ r_{i-s+1}^{-1} l_{i-s+1} & (\text{if } d=-1) \end{Bmatrix}$$

**Corollary 2.3.8.** *Let $d = d_i(C,\delta)$. For any $m \in P(C)$,*

(i) *if $n-1, n \in I_{C(i,\delta)}$ and $m \in C(i,\delta)_n m'$ then $\eta_{i+d(n-1)} = \eta'_{i+dn}$, and*

(ii) *if $I_{C(i,\delta)} = \{0,\dots,h\}$ and $C(i,\delta)\underline{1}_{u,\epsilon} = C(i,\delta)$ for a vertex $u$ and $\epsilon \in \{\pm 1\}$ then $m \in \underline{1}_{u,\epsilon}^{-}(P(C))$ implies $\eta_{i+dh} = 0$.*

*Consequently $m \in C(i,\delta)^{-}(P(C))$ implies $\eta_i = 0$.*

*Proof.* (i) Let $C(i,\delta)_n = \gamma^{-1} d_{\mathrm{l}(\gamma)}$ and $x = \mathrm{l}(\gamma)$ so $\gamma \in \mathbf{P}[x, i+d(n-1)]$. By lemma 2.3.5 (i) and (ii) (with $t = i+d(n-1)$) we have $\theta_{\mathrm{f}(\gamma)}(\psi_{i+d(n-1)}(d_{x,P(C)}(m'))) = \theta_{\mathrm{f}(\gamma)}([m'_{x,i+d(n-1)}) + \theta_{\mathrm{f}(\gamma)}(m']_{x,i+d(n-1)})$ and $\theta_{\mathrm{f}(\gamma)}(\psi_{i+d(n-1)}(d_{x,P(C)}(m'))) - \eta_{i+d(n-1)}\gamma \in \gamma\mathrm{rad}(\Lambda)$ respectively. If $d=1$ then $l_{i+n}^{-1} r_{i+n} = \gamma^{-1} d_{\mathrm{l}(\gamma)}$ and by lemma 2.3.5 (i) and (iii) (where $t = i+n-1$) we have $\theta_{\mathrm{f}(\gamma)}(\psi_{i+n-1}(d_{x,P(C)}(m'))) - \eta'_{i+n}\gamma \in \gamma\mathrm{rad}(\Lambda)$. If instead $d=-1$ then $l_{i-n+1}^{-1} r_{i-n+1} = d_{\mathrm{l}(\gamma)}^{-1}\gamma$ and by lemma 2.3.5 (i) and (iv) (where $t = i-n+1$) we have $\theta_{\mathrm{f}(\gamma)}(\psi_{i-n+1}(d_{x,P(C)}(m'))) - \eta'_{i-n}\gamma \in \gamma\mathrm{rad}(\Lambda)$.

In either case $(\eta'_{i+dn} - \eta_{i+d(n-1)})\gamma \in \gamma\mathrm{rad}(\Lambda)$, and if $\eta'_{i+dn} - \eta_{i+d(n-1)}$ lies outside $\mathfrak{m}$ then it is a unit, in which case $\gamma\Lambda = \gamma\mathrm{rad}(\Lambda)$ which contradicts lemma 1.1.17 (ii). Hence $\eta'_{i+dn} - \eta_{i+d(n-1)} \in \mathfrak{m}$ and as $S$ is a transversal in $R$ with respect to $\mathfrak{m}$ this means $\eta'_{i+dn} = \eta_{i+d(n-1)}$. For the case where $C(i,\delta)_n = \gamma^{-1} d_{\mathrm{l}(\gamma)}$ the proof is similar and omitted. Here, when we apply lemma 2.3.5 (ii) we exchange $m$ and $m'$. When we apply lemma 2.3.5 (i), (iii) and (iv) we set $t = i+dn$, $t = i-n$ and $t = i+n$ respectively.

(ii) It suffices to prove $\eta_{i+dh} \in \mathfrak{m}$ since $S \cap \mathfrak{m} = 0$. If there is no $\beta \in \mathbf{P}$ for which $\underline{1}_{u,\epsilon}\beta^{-1} d_{\mathrm{l}(\beta)}$ is a word then $\underline{1}_{u,\epsilon}^{-}(P(C)) \subseteq \mathrm{rad}(P(C))$ and so $\psi_{i+dn}(m) \in \mathrm{rad}(\Lambda e_{v_C(i+dn)})$.

Since $\eta_{i+dn} = \psi_{i+dn}(m - \lfloor m \rfloor)$ this gives $\eta_{i+dn} \in \mathfrak{m}$ as $\Lambda e_{v_C(i+dn)}$ is local. Suppose instead there is some $\beta \in \mathbf{P}$ for which $\underline{1}_{u,\epsilon}\beta^{-1}d_{\mathrm{l}(\beta)}$ is a word. By definition $m \in \gamma^{-1}d_{\mathrm{l}(\gamma)}m'$ for some $m' \in P(C)$ and some $\gamma \in \mathbf{P}$ such that $C(i,\delta)\gamma^{-1}d_{\mathrm{l}(\gamma)}$ is a word.

By lemma 2.3.5 (i) $\psi_{i+dh}(d_{x,P(C)}(m')) = [m'_{x,i+dh} + m']_{x,i+dh}$. Since $i + d(h+1) \notin I$, $d = 1$ implies $m']_{x,i+h} = 0$ and $d = -1$ implies $[m'_{x,i-h} = 0$. If $d = 1$ and $[m'_{x,i+h} \neq 0$ then $i + h - 1 \in I$ and $l^{-1}_{i-h+1}r_{i-h+1} = \tau^{-1}d_y$ which means $d_y^{-1}\tau\gamma^{-1}d_{\mathrm{l}(\gamma)}$ is a word and hence $\theta_{\mathrm{f}(\gamma)}([m'_{x,i+h}) = 0$. Similarly $\theta_{\mathrm{f}(\gamma)}(m']_{x,i-h}) = 0$ when $d = -1$ , and altogether this gives $\theta_{\mathrm{f}(\gamma)}(\psi_{i+dh}(d_{x,P(C)}(m'))) = 0$ so $\eta_{i+dh}\gamma \in \gamma\mathrm{rad}(\Lambda)$ by lemma 2.3.5 (ii). As above we can conclude $\eta_{i+dh} \in \mathfrak{m}$ by lemma 1.1.17 (ii) which completes the proof of (ii).

We now show $m \in C(i,\delta)^{-}(P(C))$ implies $\eta_i = 0$. Choose $h \geq 0$ such that ($C(i,\delta)$ is infinite and $m \in C(i,\delta)_{\leq h}0$) or ($I_{C(i,\delta)} = \{0,\ldots,h\}$ and $C(i,\delta)\underline{1}_{u,\epsilon} = C(i,\delta)$ for a vertex $u$ and $\epsilon \in \{\pm 1\}$). So we have elements $m_j = \sum_i \eta_{i,j}\underline{b}_i + \sum_i \sum_{\sigma \in \mathbf{P}[i]} r_{\sigma,i,j}\sigma\underline{b}_i$ from $P(C)$ where $m_0 = m$ and $m_j \in C(i,\delta)_{j+1}m_{j+1}$ whenever $j < h$. By assumption when $C(i,\delta)$ is infinite, or by (ii) when $C(i,\delta)$ is finite, we have $\eta_{i+dh,h} = 0$. Applying (i) to each natural number $j \leq h-1$ gives $\eta_{i+dj,j} = \eta_{i+d(j+1),j+1}$ and together this shows $\eta_i = \eta_{i,0} = \eta_{i+d,1} = \cdots = \eta_{i+d(h-1),h-1} = 0$. $\square$

**Corollary 2.3.9.** *Let $d = d_i(C,\delta)$. For any $m \in P(C)$:*

(i) *if $n-1, n \in I_{C(i,\delta)}$ then $\underline{b}_{i+d(n-1)} \in C(i,\delta)_n\underline{b}_{i+dn}$; and*

(ii) *if $I_{C(i,\delta)} = \{0,\ldots,h\}$ and $C(i,\delta)\underline{1}_{u,\epsilon} = C(i,\delta)$ for a vertex $u$ and $\epsilon \in \{\pm 1\}$ then $\underline{b}_{i+dh} \in \underline{1}^{+}_{u,\epsilon}(P(C))$.*

*Consequently $\underline{b}_i \in C(i,\delta)^{+}(P(C))$.*

*Proof.* (i) If $C(i,\delta)_n = \gamma^{-1}d_{\mathrm{l}(\gamma)}$ for some $\gamma \in \mathbf{P}$ then ($d = 1$ and $l^{-1}_{i+n}r_{i+n} = \gamma^{-1}d_{\mathrm{l}(\gamma)}$) or ($d = -1$ and $r^{-1}_{i+1-n}l_{i+1-n} = \gamma^{-1}d_{\mathrm{l}(\gamma)}$). In either case $d_{\mathrm{l}(\gamma),P(C)}(\underline{b}_{i+dn}) = \gamma\underline{b}_{i+d(n-1)}$ and so $\underline{b}_{i+d(n-1)} \in \gamma^{-1}d_{\mathrm{l}(\gamma)}\underline{b}_{i+dn} = C(i,\delta)_n\underline{b}_{i+dn}$. The case for $C(i,\delta)_n = d^{-1}_{\mathrm{l}(\gamma)}\gamma$ is similar.

(ii) It is enough to assume $C(i,\delta)d_x^{-1}x$ is a word for some arrow $x$ as otherwise $e_uP(C) = \underline{1}^{+}_{u,\epsilon}(P(C))$. By definition $i + d(h+1) \notin I$, $d = 1$ implies $d_{P(C)}(\underline{b}_{i+dh}) = \underline{b}^{-}_{i+h}$ and $d = -1$ implies $d_{P(C)}(\underline{b}_{i-h}) = \underline{b}^{+}_{i-h}$.

Furthermore if $l_{i+dh}^{-1}r_{i+dh} = \beta^{-1}d_{\mathrm{l}(\beta)}$ then $\mathrm{l}(\beta) \neq x$ and so $d_{P(C)}(\underline{b}_{i+dh}) = 0$ which shows $\underline{b}_{i+dh} \in d_x^{-1}x0 \subseteq \underline{1}_{u,\epsilon}^{+}(P(C))$. We now show $\underline{b}_i \in C(i,\delta)^{+}(P(C))$.

Suppose $I_{C(i,\delta)} = \{0,\dots,h\}$ and $C(i,\delta)\underline{1}_{u,\epsilon} = C(i,\delta)$ for a vertex $u$ and $\epsilon \in \{\pm 1\}$. By iterating (i) $h$-times we have $\underline{b}_i \in C(i,\delta)_1\underline{b}_{i+d}$, $\underline{b}_{i+d} \in C(i,\delta)_2 b_{i+2d}$ all the way through to $\underline{b}_{i+d(h-1)}C(i,\delta)_h\underline{b}_{i+dh}$. So by (ii) $\underline{b}_i \in C(i,\delta)\underline{b}_{i+dh}$ which is contained in $C(i,\delta)\underline{1}_{u,\epsilon}^{+}(P(C)) = C(i,\delta)^{+}(P(C))$. Now suppose $C(i,\delta)$ is a homotopy $\mathbb{N}$-word. If $C(i,\delta) = p_1^{-1}q_1p_2^{-1}q_2\dots$ then $C(i,\delta)_n = p_n^{-1}q_n$ for each $n \geq 0$. Again there is a sequence $\underline{b}_i \in C(i,\delta)_1\underline{b}_{i+d}$, $\underline{b}_{i+d} \in C(i,\delta)_2\underline{b}_{i+2d}$, and so on: which means $\underline{b}_i \in C(i,\delta)^{+}(P(C))$ by definition. □

**Definition 2.3.10.** (NOTATION: $\varphi_t$) Let $V$ be a $R[T,T^{-1}]$-module which is a free $R$-module with chosen basis $\{v_\lambda \mid \lambda \in \Omega\}$. Suppose also $C$ is a periodic homotopy $\mathbb{Z}$-word of period $p$, say $C = {}^{\infty}E^{\infty}$ for a homotopy word $E = l_1^{-1}r_1\dots l_p^{-1}r_p$. By lemma 1.3.47 there is a $\Lambda$-module isomorphism $\kappa : P(C,V) \to \bigoplus_{i=0}^{p-1}\Lambda e_{v_C(i)}\otimes_R V$. Fixing $t \in \{0,\dots,p-1\}$ let $\varphi_t : P(C,V) \to \Lambda e_{v_C(t)} \otimes_R V$ be the composition of $\omega_t\kappa$ where $\omega_t : \bigoplus_{i=0}^{p-1}\Lambda e_{v_C(i)} \otimes_R V \to \Lambda e_{v_C(t)} \otimes_R V$ is the natural projection.

**Lemma 2.3.11.** *For any $m \in P(C)$ and $v \in V$:*

(i) *if $x$ is an arrow then $d_{x,P(C,V)}(m\otimes v) = d_{x,P(C)}(m)\otimes v$; and*

(ii) *if $t$ is an integer with $0 \leq t \leq p-1$ then*

$$\varphi_t(d_{P(C,V)}(m\otimes v)) = \begin{cases} \psi_t(d_{P(C)}(m))\otimes v & (\text{if } 0 < t < p-1) \\ \psi_0(d_{P(C)}(m))\otimes v + \psi_p(d_{P(C)}(m))\otimes T^{-1}v & (\text{if } t = 0) \\ \psi_{-1}(d_{P(C)}(m))\otimes Tv + \psi_{p-1}(d_{P(C)}(m))\otimes v & (\text{if } t = p-1) \end{cases}$$

*Proof.* Let $\iota'_x : xP(C,V) \to \bigoplus yP(C,V)$ and $\pi'_x : \bigoplus yP(C,V) \to xP(C,V)$ respectively denote the natural $R$-module inclusions and projections. Hence $d_{x,P(C,V)}(m\otimes v) = \iota'_x(\pi'_x(d_{P(C)}(m)\otimes v))$. If it exists, let $x'$ be the arrow distinct from $x$ but with the same head. If no such arrow exists let $x' = 0$. This gives $d_{P(C)}(m) = m^{+} + m^{-}$ for some $m^{+} \in xP(C)$ and $m^{-} \in x'P(C)$. This means $m^{+}\otimes v \in xP(C,V)$ and $m^{-}\otimes v \in x'P(C,V)$ and so $d_{x,P(C,V)}(m\otimes v) = d_{x,P(C)}(m)\otimes v$ since

$$\begin{aligned}\iota'_x(\pi_x(d_{P(C)}(m)\otimes v)) &= \iota'_x(\pi'_x((m^+ + m^-)\otimes v)) = m^+\otimes v\\ &= \iota'_x(\pi'_x(m^+))\otimes v + \iota'_x(\pi'_x(m^-))\otimes v = \iota'_x(\pi'_x(m^+ + m^-))\otimes v\end{aligned}$$

Writing $d_{P(C)}(m) = \sum_{j\in\mathbb{Z}}\xi_j\underline{b}_j$ where $\xi_j \in \Lambda$ gives $\xi_j = 0$ whenever $j < -1$ or $j > p$. Note that for $t \in \{0,\dots,p-1\}$ we have $t+ps < -1$ iff $s < -1$ or ($s = -1$ and $t < p-1$), and $t+ps > p$ iff $s > 1$ or ($s = 1$ and $t > 0$). Recall $\kappa$ sends a pure tensor $\sum_j h_j\otimes v$ (where $h_j \in \Lambda e_{v_C(j)}$) to $\sum_{i=0}^{p-1}\sum_{s\in\mathbb{Z}} h_{i+ps}T^s\otimes T^{-s}v$. Altogether the above gives

$$\begin{aligned}&\varphi_t(d_{P(C,V)}(m\otimes v)) = \varphi_t(d_{P(C)}(m)\otimes v) = \varphi_t(\textstyle\sum_j \xi_j\underline{b}_j\otimes v)\\ &= \omega_t(\textstyle\sum_i\sum_{s\in\mathbb{Z}}\xi_{t+ps}\underline{b}_{i+ps}T^s\otimes T^{-s}v) = \xi_{t-p}\otimes Tv + \xi_t\otimes v + \xi_{t+p}\otimes T^{-1}v\\ &= \begin{cases} \xi_t\otimes v & (\text{if } 0 < t < p-1)\\ \xi_0\otimes v + \xi_p\otimes T^{-1}v & (\text{if } t = 0)\\ \xi_{-1}\otimes Tv + \xi_{p-1}\otimes v & (\text{if } t = p-1)\end{cases}\end{aligned}$$

which completes the proof. □

**Definition 2.3.12.** (NOTATION: $\underline{b}_{i,\lambda}$, $\eta_{i,\lambda}$, $r_{\sigma,i,\lambda}$) Recall that for each $\lambda \in \Omega$ and each integer $i$ with $0 \le i \le p-1$ we let $\underline{b}_{i,\lambda} = \underline{b}_i\otimes v_\lambda$. As in definition 2.3.3 any $q \in P(C,V)$ can be written uniquely as $q = \sum_{i,\lambda} q_{i,\lambda}\underline{b}_{i,\lambda}$ where (for each $\lambda$ and $i$) $q_{i,\lambda} = \eta_{i,\lambda} + \sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i,\lambda}\sigma$, $\eta_{i,\lambda} \in S$ and $\{r_{\sigma,i,\lambda} \mid \sigma \in \mathbf{P}[i]\}$ is a finite support subset of $R$. This means $\varphi_t(q) = \sum_\lambda q_{t,\lambda}$.

(NOTATION: $\lceil q\rceil$, $\lfloor q\rfloor$, $\lceil q_{x,t}$, $\lfloor q_{x,t}$, $q\rceil_{x,t}$, $q\rfloor_{x,t}$, $m^\lambda$) As above we now define $\lceil q\rceil = \sum_{i,\lambda}\eta_{i,\lambda}\underline{b}_{i,\lambda}$ and $\lfloor q\rfloor = \sum_{i,\lambda}\sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i,\lambda}\sigma\underline{b}_{i,\lambda}$, and for any arrow $x$ let $[q_{x,t} = \lceil q_{x,t} + \lfloor q_{x,t}$ and $q]_{x,t} = q\rceil_{x,t} + q\rfloor_{x,t}$ where

$$q\rceil_{x,t} = \begin{cases}\sum_\lambda \eta_{t+1,\lambda}\alpha\otimes v_\lambda & (\text{if } 0 \le t < p-1 \text{ and } l_{t+1}^{-1}r_{t+1} = \alpha^{-1}d_x)\\ \sum_\lambda \eta_{0,\lambda}\alpha\otimes Tv_\lambda & (\text{if } t = p-1 \text{ and } l_p^{-1}r_p = \alpha^{-1}d_x)\\ 0 & (\text{otherwise})\end{cases}$$

$$\lceil q_{x,t} = \begin{cases}\sum_\lambda \eta_{t-1,\lambda}\beta\otimes v_\lambda & (\text{if } 0 < t \le p-1 \text{ and } l_t^{-1}r_t = d_x^{-1}\beta)\\ \sum_\lambda \eta_{p-1,\lambda}\beta\otimes T^{-1}v_\lambda & (\text{if } t = 0 \text{ and } l_0^{-1}r_0 = d_x^{-1}\beta)\\ 0 & (\text{otherwise})\end{cases}$$

and

$$q\rfloor_{x,t} = \begin{cases} \sum_\lambda \sum_{\sigma\in\mathbf{P}[x,t+1]} r_{\sigma,t+1,\lambda}\sigma\kappa \otimes v_\lambda & (\text{if } 0 \le t < p-1 \text{ and } l_{t+1}^{-1}r_{t+1} = \kappa^{-1}d_{\mathrm{l}(\kappa)}) \\ \sum_\lambda \sum_{\sigma\in\mathbf{P}[x,0]} r_{\sigma,0,\lambda}\sigma\kappa \otimes Tv_\lambda & (\text{if } t = p-1 \text{ and } l_p^{-1}r_p = \kappa^{-1}d_{\mathrm{l}(\kappa)}) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\lfloor q_{x,t} = \begin{cases} \sum_\lambda \sum_{\sigma\in\mathbf{P}[x,t-1]} r_{\sigma,t-1}\sigma\zeta \otimes v_\lambda & (\text{if } 0 < t \le p-1 \text{ and } l_t^{-1}r_t = d_{\mathrm{l}(\zeta)}^{-1}\zeta) \\ \sum_\lambda \sum_{\sigma\in\mathbf{P}[x,p-1]} r_{\sigma,p-1,\lambda}\sigma\zeta \otimes T^{-1}v_\lambda & (\text{if } t = 0 \text{ and } l_0^{-1}r_0 = d_{\mathrm{l}(\zeta)}^{-1}\zeta) \\ 0 & (\text{otherwise}) \end{cases}$$

(NOTATION: ✡) In what follows we use (✡) to refer to these definitions.

(NOTATION: $m^\lambda$) For $q$ and $\lambda$ as abover define the element $m^\lambda = \sum_{i=0}^{p-1} q_{i,\lambda}\underline{b}_i$ of $P(C)$.

(NOTATION: $\phi_y$) For any vertex $v$ recall the sum $\sum_{a\in\mathbf{A}(v\to)} \Lambda a$ is direct and so $(\bigoplus_{a\in\mathbf{A}(v\to)} \Lambda a) \otimes_R V \simeq \bigoplus_{a\in\mathbf{A}(v\to)} \Lambda a \otimes_R V$. For any arrow $y$ with tail $v$ let $\phi_y : (\bigoplus_{a\in\mathbf{A}(v\to)} \Lambda a) \otimes_R V \to \Lambda a \otimes_R V$ be the natural $\Lambda$-module projection.

**Lemma 2.3.13.** *For any arrow $x$, any integer $t$ with $0 \le t \le p-1$, and any elements $q = \sum_{i,\lambda} \eta_{i,\lambda} + \sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i,\lambda}\sigma$ and $q' = \sum_{i,\lambda} \eta'_{i,\lambda} + \sum_{\sigma\in\mathbf{P}[i]} r'_{\sigma,i,\lambda}\sigma$ from $P(C,V)$: the following statements hold.*

(i) $\varphi_t(d_{x,P(C,V)}(\lceil q\rceil)) = \lceil q_{x,t} + q\rceil_{x,t}$ *and* $\varphi_t(d_{x,P(C,V)}(\lfloor q\rfloor)) = \lfloor q_{x,t} + q\rfloor_{x,t}$.

(ii) *If $\gamma \in \mathbf{P}[x,t]$ and $q \in \gamma^{-1}d_x q'$ then*

$$\phi_{\mathrm{f}(\gamma)}(\varphi_t(d_{x,P(C,V)}(q'))) - \sum_\lambda \eta_{t,\lambda}\gamma \otimes v_\lambda \in \gamma\mathrm{rad}(\Lambda) \otimes_R V$$

(iii) *If $l_{t+1}^{-1}r_{t+1} = \gamma^{-1}d_x$ then $\phi_{\mathrm{f}(\gamma)}(\lceil q_{x,t}) = 0$, and if also $t < p-1$ then*

$$\phi_{\mathrm{f}(\gamma)}(q\rceil_{x,t}) - \sum_\lambda \eta_{t+1,\lambda}\gamma \otimes v_\lambda \in \gamma\mathrm{rad}(\Lambda) \otimes_R V$$

(iv) *If $l_t^{-1}r_t = d_x^{-1}\gamma$ then $\phi_{\mathrm{f}(\gamma)}(q\rfloor_{x,t}) = 0$ and if also $0 < t$ then*

$$\phi_{\mathrm{f}(\gamma)}(\lfloor q_{x,t}) - \sum_\lambda \eta_{t-1,\lambda}\gamma \otimes v_\lambda \in \gamma\mathrm{rad}(\Lambda) \otimes_R V$$

(v) *If* $l_0^{-1}r_0 = d_x^{-1}\gamma$ *then* $\phi_{\mathrm{f}(\gamma)}([q_{x,0}) - \sum_\lambda \eta_{p-1,\lambda}\gamma \otimes T^{-1}v_\lambda \in \gamma\mathrm{rad}(\Lambda)\otimes_R V$.

(vi) *If* $l_p^{-1}r_p = \gamma^{-1}d_x$ *then* $\phi_{\mathrm{f}(\gamma)}(q]_{x,p-1}) - \sum_\lambda \eta_{0,\lambda}\gamma \otimes Tv_\lambda \in \gamma\mathrm{rad}(\Lambda)\otimes_R V$.

*Proof.* Since $C$ is periodic we have $l_0^{-1}r_0 = l_p^{-1}r_p$. Using this together with the formulae in (†) and (✡), we have; $\sum_\lambda \lceil m^\lambda{}_{x,p} \otimes T^{-1}v_\lambda = \lceil q_{x,0}$, $\sum_\lambda \lfloor m^\lambda{}_{x,p} \otimes T^{-1}v_\lambda = \lfloor q_{x,0}$, $\lceil q_{x,t} = \sum_\lambda \lceil m^\lambda{}_{x,t} \otimes v_\lambda$ and $\sum_\lambda \lfloor m^\lambda{}_{x,t} \otimes v_\lambda = \lfloor q_{x,t}$ for $0 < t \leq p-1$, $\sum_\lambda m^\lambda\rfloor_{x,t} \otimes v_\lambda = q\rfloor_{x,t}$ and $\sum_\lambda m^\lambda\rceil_{x,t} \otimes v_\lambda = q\rceil_{x,t}$ for $0 \leq t < p-1$, $\sum_\lambda m^\lambda\rfloor_{x,0} \otimes Tv_\lambda = q\rfloor_{x,p}$ and $\sum_\lambda m^\lambda\rceil_{x,0} \otimes Tv_\lambda = q\rceil_{x,p}$. Writing $q_{i,\lambda} = 0$ for all integers $i$ with $i < 0$ or $i > p$ gives $m^\lambda = \sum_{i\in\mathbb{Z}} q_{i,\lambda}\underline{b}_i$. Hence $m^\lambda\rceil_{x,p} = 0$ as $q_{p+1,\lambda} = 0$ and $\lceil m^\lambda{}_{x,0} = 0$ as $q_{p+1,\lambda} = 0$. Similarly $m^\lambda\rfloor_{x,p} = \lfloor m^\lambda{}_{x,0} = 0$.

(i) and (ii) By lemma 2.3.11, the above, and lemma 2.3.5 (i) we have

$$\begin{aligned}
\varphi_0(d_{x,P(C,V)}(\lceil q\rceil)) &= \textstyle\sum_\lambda \psi_0(d_{x,P(C)}(\lceil m^\lambda\rceil)) \otimes v_\lambda + \psi_p(d_{x,P(C)}(\lceil m^\lambda\rceil)) \otimes T^{-1}v_\lambda \\
&= \textstyle\sum_\lambda(\lceil m^\lambda{}_{x,0} + m^\lambda\rceil_{x,0}) \otimes v_\lambda + (\lceil m^\lambda{}_{x,p} + m^\lambda\rceil_{x,p}) \otimes T^{-1}v_\lambda \\
&= \textstyle\sum_\lambda \lceil m^\lambda{}_{x,p} \otimes T^{-1}v_\lambda + \sum_\lambda m^\lambda\rceil_{x,0} \otimes v_\lambda = \lceil q_{x,0} + q\rceil_{x,0}
\end{aligned}$$

Similarly we can prove $\varphi_t(d_{x,P(C,V)}(\lceil q\rceil)) = \lceil q_{x,t} + q\rceil_{x,t}$ for $0 < t \leq p-1$ and $\varphi_t(d_{x,P(C,V)}(\lfloor q\rfloor)) = \lfloor q_{x,t} + q\rfloor_{x,t}$ for $0 \leq t \leq p-1$. This gives (i). The proof of (ii) is similar to the proof of corollary 2.3.5 (ii), using that $\phi_{\mathrm{f}(\gamma)}(\varphi_t(\sum_{i,\lambda}\sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i,\lambda}\gamma\sigma\underline{b}_{i,\lambda})) \in \gamma\mathrm{rad}(\Lambda)\otimes_R V$.

(iii), (iv), (v) and (vi) For (iii) we have $[q_{x,t} = 0$ unless $l_t^{-1}r_t = d_{\mathrm{l}(\zeta)}^{-1}\zeta$ in which case

$$[q_{x,t} = \begin{cases} \sum_\lambda(\eta_{t-1,\lambda} + \sum_{\sigma\in\mathbf{p}(x,t-1)} r_{\sigma,t-1,\lambda}\sigma)\zeta \otimes v_\lambda & (\text{if } t \neq 0) \\ \sum_\lambda(\eta_{p-1,\lambda} + \sum_{\sigma\in\mathbf{p}(x,t-1)} r_{\sigma,p-1,\lambda}\sigma)\zeta \otimes T^{-1}v_\lambda & (\text{if } t = 0) \end{cases}$$

which for any $t$ lies in $\Lambda\zeta \otimes_R V$. As $l_t^{-1}r_t l_{t+1}^{-1}r_{t+1} = d_{\mathrm{l}(\zeta)}^{-1}\zeta\gamma^{-1}d_x$ is a word in this case $\mathrm{f}(\zeta) \neq \mathrm{f}(\gamma)$ which gives $\phi_{\mathrm{f}(\gamma)}([q_{x,t}) = 0$. Since $t < p-1$ we have $q]_{x,t} = \sum_\lambda(\eta_{t+1,\lambda}\gamma + \sum_{\sigma\in\mathbf{p}(x,t+1)} r_{\sigma,t,\lambda}\sigma\gamma) \otimes v_\lambda$ and so using corollary 1.2.14 (iii) as we did in the proof of lemma 2.3.5 (iii) completes the proof of (iii). The proof of (iv) is similar, and omitted. If $l_p^{-1}r_p = d_x^{-1}\gamma$ then $l_0^{-1}r_0 = d_x^{-1}\gamma$ and so $\phi_{\mathrm{f}(\gamma)}([q_{x,0}) = \sum_\lambda(\eta_{p-1,\lambda} + \sum_{\sigma\in\mathbf{P}[x,p-1]} r_{\sigma,p-1,\lambda}\sigma)\gamma \otimes T^{-1}v_\lambda$ which gives (v), and the proof of (vi) is similar. □

**Lemma 2.3.14.** *Let $i \in \{0,\dots,p-1\}$, $\mu \in \Omega$ and $d = d_i(C,\delta)$. If $q \in C(i,\delta)_n q'$ for an integer $n$ with $1 \leq n \leq p$ and any elements $q = \sum_{i,\lambda} \eta_{i,\lambda} + \sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i,\lambda}\sigma$ and $q' = \sum_{i,\lambda} \eta'_{i,\lambda} + \sum_{\sigma\in\mathbf{P}[i]} r'_{\sigma,i,\lambda}\sigma$ from $P(C,V)$: the following statements hold.*

(i) *If ($i+n<p$, $d=1$) or ($i-n+1>0$, $d=-1$) then $\eta_{i+d(n-1),\mu} = \eta'_{i+dn,\mu}$.*

(ii) *If ($i+n>p$, $d=1$) or ($i-n+1<0$, $d=-1$), then $\eta_{i+d(n-p-1),\mu} = \eta'_{i+d(n-p),\mu}$.*

(iii) *If $i-n+1=0$ and $d=-1$ then $\{\eta_{0,\lambda} \mid \lambda\in\Omega\} = \{0\}$ iff $\{\eta'_{p-1,\lambda} \mid \lambda\in\Omega\} = \{0\}$.*

(iv) *If $i+n=p$ and $d=1$ then $\{\eta'_{0,\lambda} \mid \lambda\in\Omega\} = \{0\}$ iff $\{\eta_{p-1,\lambda} \mid \lambda\in\Omega\} = \{0\}$.*

*Proof.* Note that in general we have $i+n>0$, $i+n-p\leq p-1$, $p-1>i-n$, and $i-n+p\geq 0$. There is some $\gamma$ such that $C(i,\delta)_n = \gamma^{-1}d_{\mathrm{l}(\gamma)}$ or $C(i,\delta)_n = d^{-1}_{\mathrm{l}(\gamma)}\gamma$ and in either case we let $x = \mathrm{l}(\gamma)$.

(i) Let $C(i,\delta)_n = \gamma^{-1}d_x$ so $\gamma\in\mathbf{P}[x, i+d(n-1)]$. By lemma 2.3.13 (i) and (ii) (where $t = i+d(n-1)$) we have $\phi_{\mathrm{f}(\gamma)}(\varphi_{i+d(n-1)}(d_{x,P(C,V)}(q'))) = \phi_{\mathrm{f}(\gamma)}([q'_{x,i+d(n-1)}) + \phi_{\mathrm{f}(\gamma)}(q']_{x,i+d(n-1)})$ and $\phi_{\mathrm{f}(\gamma)}(\varphi_{i+d(n-1)}(d_{x,P(C,V)}(q'))) - \sum_\lambda \eta_{i+d(n-1),\lambda}\gamma\otimes v_\lambda \in \gamma\mathrm{rad}(\Lambda)\otimes_R V$ respectively.

If $i+n<p$ and $d=1$ then $l^{-1}_{i+n}r_{i+n} = \gamma^{-1}d_x$ and by lemma 2.3.13 (i) and (iii) (where $t = i+n-1$) we have $\theta_{\mathrm{f}(\gamma)}(\psi_{i+n-1}(d_{x,P(C,V)}(q'))) - \sum_\lambda \eta'_{i+n,\lambda}\gamma\otimes v_\lambda \in \gamma\mathrm{rad}(\Lambda)\otimes_R V$. If $0 < i-n+1$ and $d=-1$ then $l^{-1}_{i-n+1}r_{i-n+1} = d_x^{-1}\gamma$ and by lemma 2.3.13 (i) and (iv) (where $t = i-n+1$) we have $\theta_{\mathrm{f}(\gamma)}(\psi_{i-n+1}(d_{x,P(C,V)}(q'))) - \sum \eta'_{i-n,\lambda}\gamma\otimes v_\lambda \in \gamma\mathrm{rad}(\Lambda)\otimes_R V$.

In either case (of $d$) the above shows $\sum_\lambda(\eta'_{i+dn,\lambda} - \eta_{i+d(n-1),\lambda})\gamma\otimes v_\lambda$ lies in $\gamma\mathrm{rad}(\Lambda)\otimes_R V$, and since $V$ is a free $R$-module we have $\gamma\mathrm{rad}(\Lambda)\otimes_R V \simeq \gamma\mathrm{rad}(\Lambda)^{(\Omega)}$ and so $\eta'_{i+dn,\lambda} - \eta_{i+d(n-1),\lambda}\gamma \in \gamma\mathrm{rad}(\Lambda)$ for each $\lambda\in\Omega$.

As in the proof of corollary 2.3.8 (i) this shows $\eta'_{i+dn,\lambda} = \eta_{i+d(n-1),\lambda}$ as otherwise we contradict lemma 1.1.17 (ii). The proof in case $C(i,\delta)_n = d_x^{-1}\gamma$ is similar and omitted. Here, when we apply lemma 2.3.13 (ii) we exchange $q$ and $q'$. When we apply lemma 2.3.13 (i) and (ii) we set $t = i+dn$, and when we apply lemma 2.3.13 (iii) and (iv) we set $t = i-n$ and $t = i+n$ respectively.

(ii) Note here that $p-1>i+n-p-1\geq 0$ and $C(i,\delta)_{n-p}=C(i,\delta)_n$ when $d=1$, since $C$ is periodic of period $p$. Similarly $0<i-n+p+1\leq p-1$ and $C(i,\delta)_{n+p}=\gamma^{-1}d_{\mathrm{l}(\gamma)}$ when $d=-1$. The proof from here is similar to the proof of (i), and we proceed as above in the following way. In case $C(i,\delta)_n=\gamma^{-1}d_{\mathrm{l}(\gamma)}$ we let $t=i+d(n-p-1)$ when we apply lemma 2.3.13 (i) and (ii), and we let $t=i+n-p-1$ and $t=i-n+p+1$ when applying lemma 2.3.13 (iii) (for $d=-1$) and (iv) (for $d=1$) respectively. In case $C(i,\delta)_n=d^{-1}_{\mathrm{l}(\gamma)}\gamma$ we let $t=i+d(n-p)$ when we apply lemma 2.3.13 (i) and (ii). When we apply lemma 2.3.13 (ii) we exchange $q$ and $q'$. We let $t=i-n+p$ and $t=i+n-p$ when applying lemma 2.3.13 (iii) (for $d=-1$) and (iv) (for $d=1$) respectively.

(iii) and (iv) For (iii) and (iv) note that $C(i,\delta)_n=(l_p^{-1}r_p)^d$ since $C$ is periodic of period $p$. Suppose $C(i,\delta)_n=d_x^{-1}\gamma$ and so $l_p^{-1}r_p=\gamma^{-1}d_x$ which gives $\phi_{\mathrm{f}(\gamma)}(q]_{x,p-1})-\sum_\lambda \eta_{0,\lambda}\gamma\otimes Tv_\lambda\in\gamma\mathrm{rad}(\Lambda)\otimes_R V$ by lemma 2.3.13 (vi) and $\phi_{\mathrm{f}(\gamma)}([q_{x,p-1})=0$ by lemma 2.3.13 (iii) (where $t=p-1$). Here $\gamma\in\mathbf{P}[x,p-1]$ so by lemma 2.3.13 (i) and (ii) (where $q$ and $q'$ are exchanged in (ii)) we have $\phi_{\mathrm{f}(\gamma)}(\varphi_{p-1}(d_{x,P(C,V)}(q)))=\phi_{\mathrm{f}(\gamma)}([q_{x,p-1})+\phi_{\mathrm{f}(\gamma)}(q]_{x,p-1})$ and $\phi_{\mathrm{f}(\gamma)}(\varphi_{p-1}(d_{x,P(C,V)}(q)))-\sum_\lambda\eta'_{p-1,\lambda}\gamma\otimes v_\lambda\in\gamma\mathrm{rad}(\Lambda)\otimes_R V$ respectively. Altogether we have $\sum_\lambda\eta_{0,\lambda}\gamma\otimes Tv_\lambda-\sum_\lambda\eta'_{p-1,\lambda}\gamma\otimes v_\lambda\gamma\mathrm{rad}(\Lambda)\otimes_R V$ and as above the result follows from lemma 1.1.17 (ii) and the fact that $T$ is an automorphism of $V$. As for the proof of parts (i) and (ii), the proof for (iii) in case $C(i,\delta)_n=\gamma^{-1}d_x$ is similar, and so is the proof of part (iv). Hence they are omitted. $\square$

**Lemma 2.3.15.** *If $i$ is an integer with $0\leq i\leq p-1$ and $d=d_i(C,\delta)$ then*

(i) $\underline{b}_{i,\lambda}\in C(i,\delta)^+(P(C,V))$, *and*

(ii) $q\in C(i,\delta)^-(P(C,V))$ *implies* $\eta_{i,\lambda}=0$ *for each* $\lambda\in\Omega$.

*Proof.* (i) If $C(i,\delta)_n=\gamma^{-1}d_{\mathrm{l}(\gamma)}$ we have $\underline{b}_{i+d(n-1)}\in\gamma^{-1}d_{\mathrm{l}(\gamma)}\underline{b}_{i+dn}$ by corollary 2.3.9 in the case of $\mathbb{Z}$-words, and so $\gamma\underline{b}_{i+d(n-1)}=d_{\mathrm{l}(\gamma),P(C)}(\underline{b}_{i+dn})$ which shows $\gamma\underline{b}_{i+d(n-1),\lambda}=d_{\mathrm{l}(\gamma),P(C,V)}(\underline{b}_{i+dn,\lambda})$. This gives $\gamma\underline{b}_{i+d(n-1)}\otimes v_\lambda=d_{\mathrm{l}(\gamma),P(C)}(\underline{b}_{i+dn})\otimes v_\lambda$ and so $\gamma\underline{b}_{i+d(n-1),\lambda}=d_{\mathrm{l}(\gamma),P(C,V)}(\underline{b}_{i+dn,\lambda})$ by lemma 2.3.11. This gives $\underline{b}_{i+d(n-1),\lambda}\in\gamma^{-1}d_{\mathrm{l}(\gamma)}\underline{b}_{i+dn,\lambda}$. Similarly if $C(i,\delta)_n=d^{-1}_{\mathrm{l}(\gamma)}\gamma$ then $\underline{b}_{i+d(n-1)}\in d^{-1}_{\mathrm{l}(\gamma)}\gamma\underline{b}_{i+dn}$ by corollary 2.3.9 and as above this shows $\underline{b}_{i+d(n-1),\lambda}\in d^{-1}_{\mathrm{l}(\gamma)}\gamma\underline{b}_{i+dn,\lambda}$. Hence $\underline{b}_{i+d(n-1),\lambda}\in C(i,\delta)_n\underline{b}_{i+dn,\lambda}$ for any natural number $n$.

As $C(i,\delta)$ is an $\mathbb{N}$-word, by the above there is a sequence $\underline{b}_{i,\lambda} \in C(i,\delta)_1\underline{b}_{i+d,\lambda}$, $\underline{b}_{i+d,\lambda} \in C(i,\delta)_2\underline{b}_{i+2d,\lambda}$ and so on, which shows $\underline{b}_{i,\lambda} \in C(i,\delta)^+(P(C))$ by definition.

(ii) Since $q \in C(i,\delta)^-(P(C,V))$ there is some $r \in \mathbb{N}$ for which $q \in C(i,\delta)_{\leq r}0$. So we have elements $q_j = \sum_{s=0}^{p-1}\sum_\lambda(\eta_{s,\lambda,j} + \sum_{\sigma\in\mathbf{p}(i)} r_{\sigma,s,\lambda,j}\sigma)\underline{b}_{s,\lambda} \in P(C,V)$ where $q = q_0$ and $q_j \in C(i,\delta)_j q_{j+1}$ whenever $j \leq r-1$. Since $C$ is periodic of period $p$ the homotopy words $D_1 = C(i,\delta)_{p-i+1}\dots C(i,\delta)_p C(i,\delta)_1 \dots C(i,\delta)_{p-i}$ and $D_{-1} = C(i,\delta)_{i+2}\dots C(i,\delta)_p C(i,\delta)_1 \dots C(i,\delta)_{i+1}$ are cyclic. By definition $q \in E_d D_d^r 0$ where $E_1 = C(i,\delta)_1 \dots C(i,\delta)_{p-i}$ and $E_{-1} = C(i,\delta)_1 \dots C(i,\delta)_{i+1}$.

Suppose $d = 1$. Fix a natural number $h \leq j$. As $C$ is periodic of period $p$ we have $q_{ph+p-i-1} \in C(i,\delta)_{p-i} q_{ph+p-i}$ and so $\{\eta_{0,\lambda,ph+p-i} \mid \lambda \in \Omega\} = \{0\}$ iff $\{\eta_{p-1,\lambda,ph+p-i+1} \mid \lambda \in \Omega\} = \{0\}$ by lemma 2.3.14 (iv). For each integer $n$ with $0 < n < p-i$ we have $q_{n-1+ph} \in C(i,\delta)_n q_{n+ph}$ again as $C$ is periodic, and so by lemma 2.3.14 (i) this gives $\eta_{i+n-1,\lambda,n-1+ph} = \eta_{i+n,\lambda,n+ph}$ for each $\lambda \in \Omega$. Assuming $h > 0$, for each integer $n$ with $p > n > p-i$ we have $q_{n-1+p(h-1)} \in C(i,\delta)_n q_{n+p(h-1)}$ and so by lemma 2.3.14 (ii) this gives $\eta_{i+n-p-1,\lambda,n+p(h-1)-1} = \eta_{i+n-p,\lambda,n+p(h-1)}$ for each $\lambda \in \Omega$. Note $q_{pr+p-i} = 0$ which gives $\sum_\lambda \eta_{s,\lambda,pr+p-i}\underline{b}_{s,\lambda} \in \mathrm{rad}(\Lambda e_{v_C(s)} \otimes_R V)$ and so $\eta_{s,\lambda,pr+p-i} \in \mathfrak{m}$ for each $s$ and each $\lambda$. Hence we have $0 = \eta_{0,\lambda,pr+p-i} = \eta_{0,\lambda,pr-i}$ for each $\lambda \in \Omega$ by applying the above to $h = r$. Assuming $r > 0$, letting $h = r-1$ again gives $\eta_{0,\lambda,pr-i} = \eta_{i,\lambda,p(r-1)} = \eta_{0,\lambda,p(r-1)-i}$ for each $\lambda \in \Omega$. Iterating this argument yields $0 = \eta_{i,\lambda,p} = \eta_{1,\lambda,p-i+1} = \eta_{0,\lambda,p-i}$ and $\eta_{0,\lambda,p-i} = \eta_{p-1,\lambda,p-i-1} = \dots = \eta_{i,\lambda,0} = \eta_{i,\lambda}$ for each $\lambda \in \Omega$. Hence $\{\eta_{i,\lambda} \mid \lambda \in \Omega\} = \{0\}$ as required. The case for $d = -1$ is similar, uses lemma 2.3.14 (iii), and is omitted. □

The results above will be used repeatedly in what follows.

### 2.3.2 Refining Complexes.

Recall that for elements in $P(C)$ we use the notation $m = \sum_i(\eta_i\underline{b}_i + \sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i}\sigma\underline{b}_i)$ where $\eta_i \in S$, $r_{\sigma,i} \in R$, $\eta_i = r_{\sigma,i} = 0$ for all but finitely many $i \in I$, and $r_{\sigma,i} = 0$ for all but finitely many $\sigma$. This statement and proof of the following is essentially [21, Lemma 8.1].

**Corollary 2.3.16.** *For $A \in \underline{\mathcal{W}}_{v,\delta}$ and a homotopy $I$-word $C$ let $(I, A, +)$ (resp. $(I, A, -)$) be the set of $i \in I$ such that $v_C(i) = v$ and $C(i, \delta) \leq A$ (resp. $C(i, \delta) < A$). Then*

$$A^{\pm}(P(C)) + e_v\mathrm{rad}(P(C)) = \textstyle\sum_{i\in(I,A,\pm)} R\underline{b}_i + e_v\mathrm{rad}(P(C))$$

*Proof.* We will use corollary 2.1.20 (ii) without reference. By corollary 2.3.9 we have $\underline{b}_i \in C(i, \delta)^+(P(C))$ for each $i \in I$. So if $\sum_i \eta_i\underline{b}_i$ lies in the span of all $\underline{b}_i$ with $i \in (I, A, +)$ then $\sum_i \eta_i\underline{b}_i \in A^+(P(C))$. If we also have $\eta_i = 0$ for $C(i, \delta) = A$ then $\sum_i \eta_i\underline{b}_i \in A^-(P(C))$ by proposition 2.1.30. This gives the containment $A^{\pm}(P(C)) \supseteq \sum_{i\in(I,A,\pm)} R\underline{b}_i$. Given $m \in A^{\pm}(P(C))$ we can write $m = \sum_i \eta_i\underline{b}_i + m'$ where $\eta_i \in S$, $m' \in e_v\mathrm{rad}(P(C))$. For $i \in I$ such that $v_C(i) \neq v$ we have $\eta_i = 0$ since $\sum_i \eta_i\underline{b}_i \in e_vP(C)$. If $m \in A^+(P(C))$ then given any $i \in I$ with $C(i, \delta) > A$ we have $m \in C(i, \delta)^-(P(C))$ which gives $\eta_i = 0$ by corollary 2.3.8. Similarly $m \in A^-(P(C))$ implies $\eta_i = 0$ given $i \in I$ with $C(i, \delta) \geq A$. This shows $A^{\pm}(P(C)) \subseteq \sum_{i\in(I,A,\pm)} R\underline{b}_i + e_v\mathrm{rad}(P(C))$ as required. □

Recall that for elements in $P(C, V)$ (where $V$ is an $R[T, T^{-1}]$-module which is free over $R$, say with $R$-basis $\Omega$) we use the notation $q = \sum_{i,\lambda}(\eta_{i,\lambda} + \sum_{\sigma\in\mathbf{p}(i)} r_{\sigma,i,\lambda}\sigma)\underline{b}_{i,\lambda}$ where $\eta_{i,\lambda} \in S$, $r_{\sigma,i,\lambda} \in R$, $\eta_{i,\lambda} = r_{\sigma,i,\lambda} = 0$ for all but finitely many $(i, \lambda) \in \{0, \dots, p-1\} \times \Omega$, and for each pair $(i, \lambda)$ we have $r_{\sigma,i,\lambda} = 0$ for all but finitely many $\sigma$.

**Corollary 2.3.17.** *For $A \in \underline{\mathcal{W}}_{v,\delta}$ and any periodic $\mathbb{Z}$-word $C$ of period $p$ let $(p, A, +)$ (resp. $(p, A, -)$) be the set of integers $i$ such that $0 \leq i \leq p-1$, $v_C(i) = v$ and $C(i, \delta) \leq A$ (resp. $C(i, \delta) < A$). Then*

$$A^{\pm}(P(C, V)) + e_v\mathrm{rad}(P(C, V)) = \textstyle\sum_{\lambda,i\in I(A,\pm,p)} R\underline{b}_{i,\lambda} + e_v\mathrm{rad}(P(C, V))$$

*Proof.* We again use corollary 2.1.20 (ii) without reference, as in the proof of corollary 2.3.16. We have $\underline{b}_{i,\lambda} \in C(i,\delta)^{+}(P(C,V))$ for each $\lambda$ by lemma 2.3.15 (i). So if $\sum_i \eta_{i,\lambda}\underline{b}_{i,\lambda}$ lies in the span of all $\underline{b}_i$ with $i \in (p,A,+)$ then $\sum_i \eta_{i,\lambda}\underline{b}_{i,\lambda} \in A^{+}(P(C,V))$. If we also have $\eta_{i,\lambda} = 0$ for $C(i,\delta) = A$ then $\sum_i \eta_i\underline{b}_i \in A^{-}(P(C,V))$ by proposition 2.1.30. This gives the containment $A^{\pm}(P(C,V)) \supseteq \sum_{\lambda,i\in(p,A,\pm)} R\underline{b}_{i,\lambda}$.

Given $m \in A^{\pm}(P(C,V))$ we can write $m = \sum_{\lambda,i} \eta_{i,\lambda}\underline{b}_{i,\lambda} + m'$ where $\eta_{i,\lambda} \in S$ , $m' \in e_v\mathrm{rad}(P(C,V))$. For $i$ such that $v_C(i) \neq v$ we have $\eta_{i,\lambda} = 0$ since $\sum_{\lambda i} \eta_{i,\lambda}\underline{b}_{i,\lambda} \in e_vP(C,V)$. If $m \in A^{+}(P(C,V))$ then given any $i$ with $C(i,\delta) > A$ we have $m \in C(i,\delta)^{-}(P(C))$ which gives $\eta_{i,\lambda} = 0$ by lemma 2.3.15 (ii). Similarly $m \in A^{-}(P(C,V))$ implies $\eta_{i,\lambda} = 0$ given $i$ with $C(i,\delta) \geq A$. This shows $A^{\pm}(P(C,V)) \subseteq \sum_{\lambda,i\in(p,A,+)} R\underline{b}_{i,\lambda} + e_v\mathrm{rad}(P(C,V))$ as required. $\square$

**Definition 2.3.18.** Recall from section 3.2 there is a quotient functor : $\mathcal{C}(\Lambda\text{-}\mathbf{Proj}) \to \mathcal{K}(\Lambda\text{-}\mathbf{Proj})$ sending any complex to itself and any morphism $f : M \to N$ of complexes to the homotopy equivalence class $[f]$.

(NOTATION: $\Xi$) We let $\Xi$ denote the restriction of $q$ to the full subcategory $\mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj})$ of $\mathcal{C}(\Lambda\text{-}\mathbf{Proj})$, considered as a functor into $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj})$.

For the next result we require an ordering on the functors $G_{B,D,n}$. To do so we recall the total order given on homotopy words by definition 2.1.26. For each vertex $v$ order the pairs of homotopy words in $\underline{\mathcal{W}}_{v,1} \times \underline{\mathcal{W}}_{v,-1}$ by setting $(B,D) < (B',D')$ whenever $B < B'$ or $B = B'$ and $D < D'$. See [55, p.26, second Lemma] for the case of words.

**Lemma 2.3.19.** *Suppose $V$ is a free $R$-module, $n \in \mathbb{Z}$, $B$ and $D$ are homotopy words such that $C = B^{-1}D$ is a homotopy word. Then the following statements hold.*

(i) *The map $\Phi_V : k \otimes_R V \to \bar{F}_{B,D,n}(P)$ given by $(r + \mathfrak{m}) \otimes v \mapsto r\underline{b}_{a_{B,D}} \otimes v + \bar{F}^{-}_{B,D,n}(P)$ is a $k$-linear embedding where $P = \Xi(S_{B,D,n}(V))$.*

(ii) *If $C$ isn't periodic then $\Phi$ gives a natural transformation $k \otimes_R - \to \bar{F}_{B,D,n}\,\Xi\, S_{B,D,n}$.*

(iii) *If $C$ is periodic then $\Phi$ induces a natural transformation $k[T,T^{-1}] \otimes_{R[T,T^{-1}]} - \to \bar{F}_{B,D,n}\,\Xi\, S_{B,D,n}$.*

*Proof.* (i) Let $i = a_{B,D}$. As $V$ is a free $R$-module with $R$-basis say $\{v_\lambda \mid \lambda \in \Omega\}$ we have a $k$-basis of $k \otimes_R V$ given by $\{(1+\mathfrak{m}) \otimes v_\lambda \mid \lambda \in \Omega\}$. Hence elements of $k \otimes_R V$ have the form $\sum_\lambda (s_\lambda + \mathfrak{m}) \otimes v_\lambda$ and such an element is 0 iff $s_\lambda \in \mathfrak{m}$ for each $\lambda$, which implies $s_\lambda \underline{b}_i \otimes v_\lambda \in \mathrm{rad}(P(C)) \otimes V \subseteq \bar{F}^-_{B,D,n}(P)$. Hence $\Phi_V$ is well defined and it is clear that $\Phi_V$ is $k$-linear.

Since $V$ is free given $\sum_\lambda s_\lambda \underline{b}_i \otimes v_\lambda \in \bar{F}^-_{B,D,n}(P)$ we must have $s_\lambda \underline{b}_i \in \bar{F}^-_{B,D,n}(P)$ for each $\lambda$. By corollaries 2.3.16 and 2.3.17 there must be elements $r_{j,\lambda} \in R$ where $j$ runs through the union of $(i,1,+) \cap (i,-1,-)$ and $(i,1,-) \cap (i,-1,+)$ and where $s_\lambda \underline{b}_i - \sum_j r_{j,\lambda} \underline{b}_j \in \mathrm{rad}(P(C))$. Since $i \notin (i,-1,-) \cup (i,1,-)$ if $s_\lambda \notin \mathfrak{m}$ then $s_\lambda$ is a unit in which case $\Lambda e_{v_C(i)} \subseteq \mathrm{rad}(\Lambda e_{v_C(i)})$ which is a contradiction. Thus $\Phi_V$ is an embedding.

(ii) In this case $k \otimes_R -$ and $\bar{F}_{B,D,n}\,\Xi S_{B,D,n}$ define functors $R$-**Proj** $\to$ $k$-**Mod**. Given $f$ is a morphism of free $R$-modules, say from $V$ with $R$-basis $\{v_\lambda \mid \lambda \in \Omega\}$ to $V'$ with $R$-basis $\{v'_{\lambda'} \mid \lambda' \in \Omega'\}$, recall (from section 2.2.3) that $S_{B,D,n}(f)(\underline{b}_i \otimes v_\lambda) = \sum_{\lambda'} a_{\lambda',\lambda} \underline{b}_i \otimes v'_{\lambda'}$ where for each $\lambda$ one has $\mathrm{l}(v_\lambda) = \sum_{\lambda'} a_{\lambda',\lambda} v_{\lambda'}$ for some $a_{\lambda',\lambda} \in R$. Note $k \otimes_R V$ and $k \otimes_R V'$ respectively have $k$-bases $\{1 \otimes v_\lambda \mid \lambda \in \Omega\}$ and $\{1 \otimes v'_{\lambda'} \mid \lambda' \in \Omega'\}$.

By the above and by definition; $\Phi_V(1 \otimes v_\lambda) = \underline{b}_i \otimes v_\lambda$ for any $\lambda$, $\Phi_{V'}(1 \otimes v_{\lambda'}) = \underline{b}_i \otimes v_{\lambda'}$ for any $\lambda'$, $(k \otimes_R f)(1 \otimes v_\lambda) = 1 \otimes \mathrm{l}(v_\lambda)$ and $\bar{F}_{B',D',n'}([S_{B,D,n}(f)])(v) = S_{B,D,n}(f)(v) + \bar{F}^-_{B',D',n'}(S_{B,D,n}(V))$ for any $v \in V$. Altogether we have $\sum_{\lambda'} a_{\lambda',\lambda} \underline{b}_i \otimes v'_{\lambda'} + \bar{F}^-_{B',D',n'}(S_{B,D,n}(V)) = \Phi_{V'}(1 \otimes \sum_{\lambda'} a_{\lambda',\lambda} v_{\lambda'})$ and so $(\bar{F}_{B',D',n'}(S_{B,D,n}(f)) \circ \Phi_V)(1 \otimes v_\lambda) = (\Phi_{V'} \circ k \otimes_R f)((1 \otimes v_\lambda)$ which shows $\Phi$ is a natural transformation.

(iii) Any element of $k \otimes_R R[T,T^{-1}]$ is of the form $(1+\mathfrak{m}) \otimes z(T)$ since any sum $x$ of non-zero pure tensors satisfies

$$x = \textstyle\sum_{s=1}^d (\lambda_s + \mathfrak{m}) \otimes f_s(T) = \sum_{s=1}^d (1+\mathfrak{m}) \otimes \lambda_s f_s(T) = (1+\mathfrak{m}) \otimes (\sum_{s=1}^d \lambda_s \otimes f_s(T)) \;.$$

The algebra map $g : k \times R[T,T^{-1}] \to k[T,T^{-1}]$ defined by $g(\lambda + \mathfrak{m}, \sum_{t\in\mathbb{Z}} \mu_t T^t) = \sum_{t\in\mathbb{Z}} (\lambda\mu_t + \mathfrak{m}) T^t$ is balanced over $R$, hence $g$ induces a map of abelian groups $h : k \otimes_R R[T,T^{-1}] \to k[T,T^{-1}]$ where $h((\lambda+\mathfrak{m}) \otimes \sum_{t\in\mathbb{Z}} \mu_t T^t) = \sum_{t\in\mathbb{Z}} (\lambda\mu_t + \mathfrak{m}) T^t$. Clearly $h$ is a surjective homomorphism of right $R[T,T^{-1}]$-modules.

If $h((1+\mathfrak{m})\otimes\sum_{t\in\mathbb{Z}}\mu_t T^t)=0$ then $\mu_t\in\mathfrak{m}$ for each $t$ which means $(1+\mathfrak{m})\otimes\sum_{t\in\mathbb{Z}}\mu_t T^t=\sum_{t\in\mathbb{Z}}(\mu_t+\mathfrak{m})\otimes T^t=0$ and so $h$ is an isomorphism. Altogether we have the natural isomorphisms

$$\begin{aligned}k[T,T^{-1}]\otimes_{R[T,T^{-1}]}-&\simeq(k\otimes_R R[T,T^{-1}])\otimes_{R[T,T^{-1}]}-\\&\simeq(k\otimes_R -)\circ(R[T,T^{-1}]\otimes_{R[T,T^{-1}]}-)\simeq k\otimes_R -\end{aligned}$$

as functors $R[T,T^{-1}]$-$\mathbf{Mod}_{R\text{-}\mathbf{Proj}}\to k[T,T^{-1}]$-$\mathbf{Mod}$. In the proof of part (ii) we showed that $\bar{F}_{B',D',n'}([S_{B,D,n}(f)])\Phi_V=\Phi_{V'}(k\otimes_R f)$ whenever $f:V\to V'$ is a morphism of free $R$-modules. Again, in section 2.2.3 we saw that when $f:V\to V'$ is also a morphism of $R[T,T^{-1}]$-modules, $\bar{F}_{B',D',n'}([S_{B,D,n}(f)])$ is a $k[T,T^{-1}]$-linear map. By the above $k[T,T^{-1}]\otimes_{R[T,T^{-1}]}-\ \simeq\ k\otimes_R -$ and $\bar{F}_{B,D,n}\,\Xi\,S_{B,D,n}$ define functors $R[T,T^{-1}]$-$\mathbf{Mod}_{R\text{-}\mathbf{Proj}}\to k[T,T^{-1}]$-$\mathbf{Mod}$, and so it suffices to show $\Phi_V$ is $k[T,T^{-1}]$-linear. If $V$ has an $R$-basis $\{v_\lambda\mid\lambda\in\Omega\}$, writing $Tv_\lambda=\sum_\mu a_{\mu\lambda}v_\mu$ for $a_{\mu\lambda}\in R$ with finite support over $\mu$ gives $T((r+\mathfrak{m})\otimes v_\lambda)=\sum_\mu a_{\mu\lambda}(r+\mathfrak{m})\otimes v_\mu$ which is sent to $\sum_\mu a_{\mu\lambda}\underline{b}_i\otimes v_\mu$ under $\Phi_V$. Since we are tensoring over $R$ one has $\sum_\mu a_{\mu\lambda}\underline{b}_i\otimes v_\mu=\underline{b}_i\otimes\sum_\mu a_{\mu\lambda}v_\mu$ and so $\Phi_V(T(r+\mathfrak{m})\otimes v_\lambda)=T\Phi_V((r+\mathfrak{m})\otimes v_\lambda)$ as required. $\square$

We can now evaluate our refined functors on string complexes.

**Lemma 2.3.20.** *Let $n,n'\in\mathbb{Z}$ and for some homotopy words $B$ and $D$ let $C=B^{-1}D$ be an homotopy $I$-word which is not a periodic homotopy $\mathbb{Z}$-word. If $B'$ and $D'$ are homotopy words such that $C'=B'^{-1}D'$ is a homotopy word, then*

(i) *for any $i\in I$ we have $a_{C(i,1),C(i,-1)}=i$ and*

$$\bar{F}^{+}_{C(i,1),C(i,-1),n}(P(C)[\mu_C(i)-n])=\bar{F}^{-}_{C(i,1),C(i,-1),n}(P(C)[\mu_C(i)-n])+R\underline{b}_i,$$

(ii) *if $C'=C$ and $n-n'=\mu_C(a_{B,D})-\mu_C(a_{B',D'})$ there is a natural isomorphism $k\otimes_R-\simeq F_{B',D',n'}\,\Xi\,S_{B,D,n}$, and*

(iii) *if $(B,D,n)$ is not equivalent to $(B',D',n')$ then $\bar{F}_{B',D',n'}(P(C)[\mu_C(a_{B,D})-n])=0$.*

*Proof.* (i) $a_{C(i,1),C(i,-1)} = i$ is clear by definition. Let $P = P(C)[\mu_C(i) - n]$, $v = v_C(i)$ and $I(C(i,\delta), \pm) = (i, \delta, \pm)$. By corollary 2.3.16 $\underline{b}_i \in \bar{F}^+_{C(i,1),C(i,-1),n}(P)$ which shows $\bar{F}^+_{C(i,1),C(i,-1),n}(P)$ contains $\bar{F}^-_{C(i,1),C(i,-1),n}(P) + Rb_i$. Now let $m \in \bar{F}^+_{C(i,1),C(i,-1),n}(P)$.

By assumption and by corollary 2.3.16 we may write $m = \sum_j \eta_j \underline{b}_j + m_0$ for $\eta_j \in S$ and some $m_0 \in e_v\text{rad}(P)$ where $\eta_j = 0$ whenever $C(j,1) > C(i,1)$, or $C(j,-1) > C(i,-1)$. Since $m \in P^n = P^{\mu_C(i)}(C)$ we also have $\eta_j = 0$ for any $j \in I$ with $\mu_C(j) \neq \mu_C(i)$.

So $\sum_j \eta_j \underline{b}_j$ lies in $\sum_t R\underline{b}_t$ where $t$ runs through $(i,1,+) \cap (i,-1,+)$. If we let $(i,\delta,=)$ be the set of $j \in I$ with $C(j,\delta) = C(i,\delta)$ then $(i,1,+) \cap (i,-1,+)$ is the union of the sets $(i,1,+) \cap (i,-1,-)$, $(i,1,-) \cap (i,-1,+)$ and $(i,1,=) \cap (i,-1,=)$. So by corollary 2.3.16 $\sum_j \eta_j \underline{b}_j$ lies in $F^-_{C(i,1),C(i,-1),n}(P) + \sum_t R\underline{b}_t$ where $t$ runs through all $j \in (i,1,=) \cap (i,-1,=)$ with $\mu_C(j) = \mu_C(i)$.

Suppose there is an integer $t$ which satisfies $0 \leq t \leq p-1$, $C(t,1) = C(i,1)$, $C(t,-1) = C(i,-1)$ and $\mu_C(t) = \mu_C(i)$. If $C_{>i} = (C_{\leq t})^{-1}$ and $(C_{\leq i})^{-1} = C_{>t}$ then $C[t] = C^{-1}[i]$ which means $C$ is a shift of its inverse, contradicting lemma 2.2.17. Hence $C_{>i} = C_{>t}$ and $C_{\leq i} = C_{\leq t}$ which shows $C = C[t-i]$.

Applying lemma 1.3.33 (iii) twice yields $\mu_C(t-i) = 0$. This shows that $t \neq i$ and $C$ is a periodic homotopy $\mathbb{Z}$-word with period $t-i$, which contradicts that $p$ was minimal, since $t - i < p$. Altogether we have shown that $\sum_j \eta_j \underline{b}_j$ lies in $F^-_{C(i,1),C(i,-1),n}(P) + R\underline{b}_i$, as required.

(ii) By part (i) we have $i = a_{B,D}$ and so $B = C(i,-1)$ and $D = C(i,1)$, and also any element of $S_{B,D,n}(V)$ may be written as the coset of a sum of pure tensors $\sum_{t=1}^n r_t\underline{b}_i \otimes v_t + \bar{F}^-_{B,D,n}(\Xi(S_{B,D,n}(V)))$ for some $v_1, \ldots, v_n \in V$. Hence the $k$-linear embedding $\Phi_V$ from lemma 2.3.19 (i) is surjective, and so the natural transformation $\Phi$ defines in lemma 2.3.19 (ii) is a natural isomorphism.

By assumption $(B, D, n) \sim (B', D', n')$. This means $F_{B',D',n'} \simeq \bar{F}_{B,D,n}$ by corollary 2.2.8 and corollary 2.2.24 (i). By the above this gives $F_{B',D',n'} \,\Xi\, S_{B,D,n} \simeq \bar{F}_{B,D,n} \,\Xi\, S_{B,D,n} \simeq k \otimes_R -$. Note this natural isomorphism is defined between functors $R$-**Proj** $\to$ $k$-**Mod**.

(iii) It is enough to show $\bar{F}_{B',D',n'}(P(C)[\mu_C(a_{B,D}) - n]) = 0$.

Exchanging $B'$ and $D'$ if necessary we can assume $s(B') = 1$ and $s(D') = -1$. Let $P = P(C)[\mu_C(a_{B,D}) - n]$ and suppose $\bar{F}_{B',D',n'}(P) \neq 0$. It suffices to show this implies $(B, D, n)$ is equivalent to $(B', D', n')$. By corollary 2.3.16 the subspaces $\bar{G}^{\pm}_{B',D',n'}(P)$ are spanned by sets of elements of the form $b_i$ together with $\mathrm{rad}(P)$. Hence as $\bar{F}_{B',D',n'}$ and $\bar{G}_{B',D',n'}$ are naturally isomorphic there must be some $i \in I$ for which $\underline{b}_i$ lies in $\bar{G}^{+}_{B',D',n'}(P)$ but outside $\bar{G}^{-}_{B',D',n'}(P)$ as otherwise $\bar{G}_{B',D',n'}(P) = 0$. By part (i) we know that $b_i$ lies in $\bar{G}^{+}_{C(i,1),C(i,-1),n}(P)$ but outside $\bar{G}^{-}_{C(i,1),C(i,-1),n}(P)$. Note $v_C(i) = h(D')$ by the above.

If $B' \neq C(i,1)$ then $B' < C(i,1)$ or $B' > C(i,1)$ by lemma 2.1.27. By proposition 2.1.30, if $B' < C(i,1)$ then $\bar{G}^{-}_{C(i,1),C(i,-1),n}(P) \supseteq \bar{G}^{+}_{B',D',n'}(P)$ and if $B' > C(i,1)$ then $\bar{G}^{+}_{C(i,1),C(i,-1),n}(P) \subseteq \bar{G}^{-}_{B',D',n'}(P)$. Neither of these inclusions are possible by the existence of $\underline{b}_i$, so we must have that $B' = C(i,1)$. Since $(B', D') \neq (C(i,1), C(i,-1))$ we must have $D \neq C(i,-1)$ and so similarly to the above we have $\bar{G}^{-}_{C(i,1),C(i,-1),n}(P) \supseteq \bar{G}^{+}_{B',D',n'}(P)$ if $D' < C(i,-1)$ and the reverse inclusion otherwise. Altogether we have a contradiction assuming $(B', D') \neq (C(i,1), C(i,-1))$ hence $B'^{-1}D' = C$ and $i = a_{B',D'}$by part (i). Furthermore considering $\underline{b}_i$ lies in both $P^{\mu_C(i)}(C)$ and $P^{n'}$ we must have that $\mu_C(i) = n' + \mu_C(a_{B,D}) - n$ and so $(B, D, n)$ is equivalent to $(B', D', n')$ as required. □

To evaluate refined functors on complexes of the form $P(C, V)$ we will use a similar argument.

**Lemma 2.3.21.** *Let $n, n' \in \mathbb{Z}$ and for some cyclic word $E = l_1^{-1} r_1 \dots l_p^{-1} r_p$ let $B = (E^{-1})^{\infty}$ and $D = E^{\infty}$ such that $C = B^{-1}D$ is a periodic $\mathbb{Z}$-word of period $p$. Fix some words $B'$ and $D'$ such that $C' = B'^{-1}D'$ is a word. Then,*

(i) *for any $i \in \{0, \dots, p-1\}$ we have $a_{C(i,1),C(i,-1)} = i$ and*

$$\bar{F}^{+}_{C(i,1),C(i,-1),n}(P(C,V)[\mu_C(i) - n]) = \bar{F}^{-}_{C(i,1),C(i,-1),n}(P(C,V)[\mu_C(i) - n]) + \textstyle\sum_{\lambda} R\underline{b}_{i,\lambda},$$

(ii) *if $C' = C[m]$ and $n - n' = \mu_C(m)$ for some $m \in \mathbb{Z}$ then there is a natural isomorphism $k[T, T^{-1}] \otimes_{R[T,T^{-1}]} - \simeq F_{B',D',n'} \Xi S_{B,D,n}$, and*

(iii) *if $(B, D, n)$ is not equivalent to $(B', D', n')$ then $\bar{F}_{B',D',n'}(P(C,V)[-n]) = 0$.*

*Proof.* (i) Let $P = P(C,V)[\mu_C(i) - n]$, and $I(C(i,\delta),\pm,p) = (i,\delta,\pm)$ and $v = v_C(i)$. The equality $a_{C(i,1),C(i,-1)} = i$ is clear by definition. By corollary 2.3.17 $\bar{F}^{+}_{C(i,1),C(i,-1),n}(P)$ contains $\bar{F}^{-}_{C(i,1),C(i,-1),n}(P) + \sum_\lambda R\underline{b}_{i,\lambda}$. Now suppose $m \in \bar{F}^{+}_{C(i,1),C(i,-1),n}(P)$. By assumption we may write $m = \sum_\lambda \sum_{j=1}^{p} \eta_{j,\lambda}\underline{b}_{i,\lambda} + m_0$ for scalars $\eta_{j,\lambda} \in S$ and some $m_0 \in e_v\mathrm{rad}(P)$. By corollary 2.3.17 if $j \in I$ satisfies ($\mu_C(j) \neq \mu_C(i)$, or $C(j,1) > C(i,1)$, or $C(j,-1) > C(i,-1)$) then $\eta_{j,\lambda} = 0$ for each $\lambda \in \Omega$. Note also that for any $j$ with $0 \leq j \leq p-1$ and any $\delta \in \{\pm 1\}$, $C(j,\delta) \leq C(i,\delta)$ and $C(j,-\delta) < C(i,-\delta)$ together imply $\underline{b}_{j,\lambda} \in C(j,\delta)^{+}(P(C,V)) \cap C(j,-\delta)^{-}(P(C,V))$ by corollary 2.3.17 and proposition 2.1.30. If we let $(i,\delta,=)$ be the set of $j$ with $C(j,\delta) = C(i,\delta)$ then $(i,1,+) \cap (i,-1,+)$ is the union of the sets $(i,1,+) \cap (i,-1,-)$, $(i,1,-) \cap (i,-1,+)$ and $(i,1,=) \cap (i,-1,=)$.

Altogether this shows $m \in \sum_{\lambda,j} R\underline{b}_{j,\lambda} + \bar{F}^{-}_{C(i,1),C(i,-1),n}(P(C)[\mu_C(i) - n])$ where $j$ runs through the elements of $(i,1,=) \cap (i,-1,=)$ for which $\mu_C(j) = \mu_C(i)$. It is enough to suppose $t$ with $0 \leq t \leq p-1$ satisfies $C(t,1) = C(i,1)$, $C(t,-1) = C(i,-1)$, and $\mu_C(t) = \mu_C(i)$; and show $t = i$. If $C_{>i} = (C_{\leq t})^{-1}$ and $(C_{\leq i})^{-1} = C_{>t}$ then $C[t] = C^{-1}[i]$ which means $C$ is a shift of its inverse, contradicting [21, Lemma 2.1]. Hence $C_{>i} = C_{>t}$ and $C_{\leq i} = C_{\leq t}$ which shows $C = C[t-i]$. By lemma 1.3.33 (iii) (applied twice) we have $\mu_C(t-i) = 0$. So if $t \neq i$ then we contradict that $C$ is periodic of period $p > t - i$ this is impossible.

(ii) As in the proof of part (ii) of lemma 2.3.11, by part (i) the $k$-linear embedding $\Phi_V$ from lemma 2.3.19 (i) is surjective. So the natural transformation $\Phi$ defined in lemma 2.3.19 (iii) is a natural isomorphism. Again the result follows by the above, corollary 2.2.8 and corollary 2.2.24 (i). Note this natural isomorphism is defined between functors $R[T,T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}} \to k[T,T^{-1}]\text{-}\mathbf{Mod}$.

(iii) Let $P = P(C,V)[-n]$. Exchanging $B'$ and $D'$ if necessary we can assume $s(B') = 1$ and $s(D') = -1$. Assuming $\bar{F}_{B',D',n'}(P) \neq 0$ it suffices to show this implies $(B,D,n)$ is equivalent to $(B',D',n')$. By corollary 2.3.17 the $R$-modules $\bar{G}^{\pm}_{B',D',n'}(P)$ are generated by elements of the form $\underline{b}_{i,\lambda}$ together with $e_v\mathrm{rad}(P)$. Hence as $\bar{F}_{B',D',n'}$ and $\bar{G}_{B',D',n'}$ are naturally isomorphic there must be some integer $i$ with $0 \leq i \leq p-1$ and $\lambda \in \Omega$ for which $\underline{b}_{i,\lambda}$ lies in $\bar{G}^{+}_{B',D',n'}(P)$ but not $\bar{G}^{-}_{B',D',n'}(P)$ as otherwise $\bar{G}_{B',D',n'}(P) = 0$.

By part (i) we know that $\underline{b}_{i,\lambda}$ lies in $\bar{G}^{+}_{C(i,1),C(i,-1),n}(P[\mu_C(i)])$ but not $\bar{G}^{-}_{C(i,1),C(i,-1),n}(P[\mu_C(i)])$. Hence we must have that $n = n'$. Note $v_C(i) = h(D')$ by the above. We suppose that $(B', D') \neq (C(i,1), C(i,-1))$ toward finding a contradiction, and proceed with case analysis.

If $B' \neq C(i,1)$ then $B' < C(i,1)$ or $B' > C(i,1)$ by lemma 2.1.27. By proposition 2.1.30, if $B' < C(i,1)$ then $\bar{G}^{-}_{C(i,1),C(i,-1),n}(P[\mu_C(i)])$ contains $\bar{G}^{+}_{B',D',n'}(P)$ and if $B' > C(i,1)$ then $\bar{G}^{+}_{C(i,1),C(i,-1),n}(P[\mu_C(i)])$ is contained in $\bar{G}^{-}_{B',D',n'}(P)$. Since either of these inclusions is impossible by the existence of $\underline{b}_{i,\lambda}$ we must have that $B' = C(i,1)$.

Since $(B', D') \neq (C(i,1), C(i,-1))$ we must have $D \neq C(i,-1)$ and so similarly to the above we have that $\bar{G}^{-}_{C(i,1),C(i,-1),n}(P[\mu_C(i)])$ contains $\bar{G}^{+}_{B',D',n'}(P)$ if $D' < C(i,-1)$ and the reverse inclusion otherwise. Altogether we have a contradiction and so $(B', D') = (C(i,1), C(i,-1))$ and hence $B'^{-1}D' = C[i]$. This gives $e_v P^n(C,V)[\mu_C(i) - n] = e_v P^{n'}(C,V)[-n]$ so we must have that $\mu_C(i) = n + \mu_C(i) - n = n' - n$ which shows $(B, D, n)$ is equivalent to $(B', D', n')$. In the case $s(B') = -1$ and $s(D') = 1$ we similarly find $B'^{-1}D' = C^{-1}[i]$ and $\mu_C(-i) = n' - n$. $\square$

## 2.4 Compactness and Covering.

**Assumption:** In section 2.4 we fix an object $M^\bullet$ in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$ with underlying $\Lambda$-module $M$, and so $M^i$ is finitely generated and $\mathrm{im}(d^i_M) \subseteq \mathrm{rad}(M^{i+1})$ for all $i$.

Crawley-Boevey [21] considered modules over *string algebras* such as $k[x,y]/(xy)$. The modules this author classifies must satisfy various finiteness conditions. He proves modules under the said conditions must satisfy so-called *covering properties* of the refined functors $F_{w,w'}$ where $w$ and $w'$ are words (not homotopy words).

For an example of this property see lemma 1.4.49. These properties were then used to classify such modules via the functorial filtration method (see the proof of [21, Theorem 1.3] and [21, Lemma 10.6]). The results in section 2.4 were found by adapting results from [21, §10], and appear to be new. The main result of this section is the following adaptation of lemma 1.4.49.

**Lemma 2.4.1.** (Covering Properties) *Fix a vertex $v$, an integer $r$ and some $\delta \in \{\pm 1\}$. Suppose $U$ is an $R$-submodule of $e_v M^r$ for which $e_v\mathrm{rad}(M^r) \subseteq U$.*

(i) (One-Sided Functors) *If $H$ is a linear variety in $e_v M^r$ and $m \in H \setminus U$, then there is a homotopy word $C \in \underline{\mathcal{W}}_{v,\delta}$ such that $H \cap (U + m)$ meets $C^+(M)$ but not $C^-(M)$.*

(ii) (Refined functors) *If $m \in e_v M^r \setminus U$ then there are words $B \in \underline{\mathcal{W}}_{v,\delta}$ and $D \in \underline{\mathcal{W}}_{v,-\delta}$ such that $U + m$ meets $G^+_{B,D,r}(M)$ but not $G^-_{B,D,r}(M)$.*

### 2.4.1 Linear Compactness.

**Assumption:** In what follows the topology we refer to will be the $\mathfrak{m}$-adic topology. Recall a base of open sets for an $R$-module $N$ with this topology is

$$\{m + \mathfrak{m}^n U \mid U \text{ is an } R\text{-submodule of } N,\, m \in N \text{ and } n \in \mathbb{N}\}$$

where $\mathfrak{m}^0 U = U$. Any $R$-module homomorphism is continuous in the $\mathfrak{m}$-adic topology.

We now recall and use the notion of linear compactness following Zelinsky [66].

**Definition 2.4.2.** (NOTATION: $\subseteq_{\mathrm{c}}$) Let $L$ be a subset of an $R$-module $N$. We write $L \subseteq_{\mathrm{c}} N$ iff $L$ is closed.

[66, p.80] (LINEAR VARIETIES AND COMPACTNESS) We say $L$ is a *linear variety* if $L = U + m \subseteq_{\mathrm{c}} N$ for some $R$-submodule $U$ of $N$. We say $N$ is *linearly compact* if any collection of linear varieties in $N$ with the finite intersection property must have a non-void intersection.

**Example 2.4.3.** For any integer $n > 0$ the $R$-module $R/\mathfrak{m}^n$ has a composition series

$$0 \subsetneq \mathfrak{m}^{n-1}R/\mathfrak{m}^n \subsetneq \cdots \subsetneq \mathfrak{m}R/\mathfrak{m}^n \subsetneq R/\mathfrak{m}^n$$

Note each quotient is simple as $\mathfrak{m}$ is the maximal ideal of $R$. Hence $R/\mathfrak{m}^n$ is an artinian $R$-module (for example see [2, 11.1, Proposition]. In particular $R/\mathfrak{m}^n$ has the minimum condition on closed submodules, and so it is linearly compact by [66, p.81, Proposition 5]. Since $\Lambda$ is a complete gentle algebra $R$ is $\mathfrak{m}$-adically complete, and so $R$ (as a module over itself) is isomorphic to the inverse limit of a system of linearly compact $R$-modules. By [66, p.81, Proposition 4] this means $R$ is a linearly compact $R$-module.

The use of example 2.4.3 in the proof of lemma 2.4.4 below should motivate the assumption (throughout chapter 2) that $R$ is $\mathfrak{m}$-adically complete.

**Lemma 2.4.4.** *Let $i \in \mathbb{Z}$, $v$ be a vertex, then*

(i) *the module $e_vM^i$ is linearly compact for any vertex $v$,*

(ii) *if $U \subseteq_{\mathrm{c}} e_vM^i$ with $e_v\mathfrak{m}^nM^i \subseteq U$ for some $n > 0$ then $U + m \subseteq_{\mathrm{c}} e_vM^i$, and*

(iii) *$\{m\} = 0 + m \subseteq_{\mathrm{c}} e_vM^i$ for any $m \in e_vM^i$.*

*Proof.* (i) By example 2.4.3 $R$ is linearly compact as an $R$-module. By [66, p.6, Proposition 1] this means any finitely generated free $R$-module is linearly compact, so by [66, p. 81, Proposition 2] any finitely generated $R$-module is linearly compact.

As $M^i$ is a finitely generated $\Lambda$-module $e_v M^i$ is a finitely generated $R$-module by corollary 1.1.25 (ib), because $Q$ is finite and $R$ is noetherian.

(ii) For (ii) and (iii) it is enough (say by [51, p.98, Corollary 17.7]) to choose a limit point $l$ of $U + m$ and show $l \in U + m$. By definition, any open neighborhood of $l$ meets $U + m$ somewhere other than $l$. That is, for any $t \geq 0$ there is some $u_t \in U$ such that $u_t + m \neq l$ and $u_t + m \in l + \mathfrak{m}^t M$. In particular, there is some $u_{n+1} \in U$ and some $x_{n+1} \in \mathfrak{m}^{n+1}M$ with $u_{n+1} + m = l + x_{n+1}$. Since $e_v(u_{n+1} + m - l) = u_{n+1} + m - l$ we have that $x_{n+1}$ lies in $e_v M \cap \mathfrak{m}^{n+1}M \subseteq e_v \mathfrak{m}^n M$ which is contained in $U$, and so $l = (u_{n+1} - x_{n+1}) + m \in U + m$ as required.

(iii) By definition, any open neighborhood of $l$ contains $m$. That is, for any $t \geq 0$ we have $m \in l + e_v \mathfrak{m}^t M$ and so $m - l \in e_v \mathfrak{m}^t M$. This shows $m - l \in \bigcap_{t \geq 0} e_v \mathfrak{m}^t M^i = 0$ by corollary 1.1.25 (iii). □

**Corollary 2.4.5.** *Let $i \in \mathbb{Z}$ , $\gamma \in \mathbf{P}$ and $\alpha$ be an arrow. If $U \subseteq e_{t(\gamma)} M^i$, $V \subseteq e_{h(\alpha)} M^i$ and $W \subseteq e_{h(\gamma)} M^i$ are closed submodules then: $\gamma U \subseteq e_{h(\gamma)} M^i$, $d_\alpha V \subseteq e_{h(\alpha)} M^{i+1}$, $d_\alpha^{-1} V \cap e_{h(\alpha)} M^{i-1} \subseteq e_{h(\alpha)} M^{i-1}$ and $\gamma^{-1} W \cap e_{t(\gamma)} M^i \subseteq e_{t(\gamma)} M^i$ all define closed submodules.*

*Proof.* The restrictions $d^{i-1}_{\alpha,M} : e_v M^{i-1} \to e_v M^i$ and $d^i_{\alpha,M} : e_v M^i \to e_v M^{i+1}$ of $d_{\alpha,M}$ define $R$-module maps. We may also consider the $R$-module map $\gamma_\times : e_{t(\gamma)} M^i \to e_{h(\gamma)} M^i$ sending $m$ to $\gamma m$. By lemma 2.4.4 $e_{t(\gamma)} M^i$, $e_{h(\alpha)} M^i$ and $e_{h(\gamma)} M^i$ are linearly compact. It suffices to prove that, if $z : X \to Y$ is a homomorphism of linearly compact $R$-modules and $X' \subseteq X$ and $W \subseteq Y$ are closed submodules, then $f^{-1}(Y') \subseteq X$ and $z(X') \subseteq Y$ are closed.

Since $z$ is an $R$-module map the image and pre-image are submodules of $Y$ and $X$ respectively, and furthermore $z$ is continuous in the $\mathfrak{m}$-adic topology. This shows $z^{-1}(Y')$ is closed. As $X$ is linearly compact and $X'$ is a closed submodule, $X'$ must be linearly compact by [66, p.81, Proposition 3]. By [66, p.81, Proposition 2] linearly compact modules $X'$ are sent to linearly compact modules $z(X')$. Since we are using the $\mathfrak{m}$-adic topology the base of open neighborhoods of zero in $Y$ are submodules, so linearly compact submodules such as $z(X')$ of $Y$ are closed by [66, p.82, Proposition 7]. □

**Corollary 2.4.6.** *Let $C$ be a homotopy $\{0,\dots,t\}$-word and $N$ be a submodule of $M$ such that $M^{i+\mu_C(t)}\cap N\subseteq M^{i+\mu_C(t)}$ is closed. Then $e_{h(C)}M^i\cap CN\subseteq e_{h(C)}M^i$ is closed.*

*Proof.* Let $v=h(C)$. We have $e_vM^i\cap CN=e_vM^i\cap C(M^{i+\mu_C(t)}\cap N)$ by corollary 2.2.3. By iteration it is enough to assume $t=1$, and so $C=(d^{-1}_{\mathrm{l}(\gamma)}\gamma)^{\pm 1}$. By corollary 2.4.5 if $C=d^{-1}_{\mathrm{l}(\gamma)}\gamma$ then $\gamma N$ is closed. Applying corollary 2.4.5 again shows $d^{-1}_{\mathrm{l}(\gamma)}\gamma N\cap e_vM^i$ is also closed. The case $C=\gamma^{-1}d_{\mathrm{l}(\gamma)}$ is similar. □

**Lemma 2.4.7.** (Realisation) *Let $t>0$, $i\in\mathbb{Z}$ and $C$ be a homotopy $I$-word with $I\subseteq\mathbb{N}$. Then*

(i) *for $m\in e_{h(\gamma)}M^i$ and $\gamma\in\mathbf{P}$, if $(\gamma^{-1}d_{\mathrm{l}(\gamma)})^{\pm 1}m\cap e_{t(\gamma)}M^{i\pm 1}$ is non-empty then it is a linear variety,*

(ii) *for any $m\in M^{i+\mu_C(t-1)}\cap\bigcap_{n\in I,\,n\geq t}(C_{>t-1})_{\leq n}M$ we have*

$$M^{i+\mu_C(t)}\cap r_t^{-1}l_tm\cap\bigcap_{n\in I,\,n\geq t+1}(C_{>t})_{\leq n}M\neq\emptyset$$

*and* (iii) *if $I=\mathbb{N}$ and $S\subseteq e_vM^i$ then $S\cap C^+(M^\bullet)=\bigcap_{n\geq 0}S\cap C_{\leq n}M$.*

*Proof.* (i) We will show that if $P=\gamma^{-1}d_{\mathrm{l}(\gamma)}m\cap e_{t(\gamma)}M^{i+1}$ is non-empty then it is a linear variety. The other case will follow similarly. $P$ is closed by lemma 2.4.4. Since $P\neq\emptyset$ choose $x\in P$. We now show $P=P'+x$ where $P'=e_{t(\gamma)}M^{i+1}\cap\gamma^{-1}0$. By definition $P'+x\subseteq P$. Conversely for any $x'\in P$ we have that $x'-x\in e_{t(\gamma)}M^{i+1}\cap\gamma^{-1}0$ as $\gamma x'=d_{\mathrm{l}(\gamma),M}(m)=\gamma x$.

(ii) For each $n\geq t$ we have $m\in(C_{>t-1})_{\leq n}M$ and so there is some $u_n\in M^{i+\mu_C(t)}\cap r_t^{-1}l_tm$ for which $u_n\in(C_{>t})_{\leq n}M$. By corollary 2.4.5 (ii) $M^{i+\mu_C(t)}\cap r_t^{-1}l_tm$ is a closed coset of $e_{v_C(t)}M^{i+\mu_C(t)}$. By part (i) above each of the subsets $M^{i+\mu_C(t)}\cap(C_{>t})_{\leq n}M$ are closed submodules. Consider the collection $\Delta$ consisting of all $M^{i+\mu_C(t)}\cap(C_{>t})_{\leq n}M$ where $n\geq t$, together with the set $M^{i+\mu_C(t)}\cap r_t^{-1}l_tm$ which is a linear variety by part (i). Let $V_n=M^{i+\mu_C(t)}\cap r_t^{-1}l_tm\cap(C_{>t})_{\leq n}M$ for each $n\geq 0$. As the collection $\{(C_{>t})_{\leq n}M\mid n\geq t\}$ forms a chain the intersection of finitely many elements from $\Delta$ is contained in some $V_{n'}$.

As $V_{n'}$ contains $u_{n'}$ this shows the collection $\Delta$ satisfies the finite intersection property. Hence the intersection $M^{i+\mu_C(t)} \cap r_t^{-1} l_t m \cap \bigcap_{n\geq 0}(C_{>t})_{\leq n} M$ of all the sets in $\Delta$ is non-empty, since $e_{h(C)} M^{i+\mu_C(t)}$ is linearly compact by lemma 2.4.4.

(iii) We can assume $S$ is non-empty. Clearly $C^+(M^\bullet) \subseteq \bigcap_{n\geq 0} C_{\leq n} M$ by definition and so it is enough to pick $s \in S$ such that $s \in C_{\leq n} M$ for all $n \geq 0$ and show $s \in C^+(M^\bullet)$. Let $C = l_1^{-1} r_1 l_2^{-1} r_2 \ldots$ and suppose for an arbitrary but fixed $i > 0$ there is some element $s_{i-1} \in M^{j+\mu_C(i-1)} \cap \bigcap_{n>i-1}(C_{>i-1})_{\leq n} M$. Using part (ii) there is an element $s_i \in M^{j+c(i-1)} \cap \bigcap_{n>i-1}(C_{>i-1})_{\leq n} M$ for which $s_{i-1} \in l_i^{-1} r_i s_i$. Setting $s_0 = s$ defines an element of $M^{i+\mu_C(0)} \cap \bigcap_{n>0}(C_{>0})_{\leq n} M$ and by the above this iteratively defines a sequence $s_0, s_1, s_2, \cdots \in M$ for which $s = s_0$ and $s_i \in l_{i+1}^{-1} r_{i+1} s_{i+1}$ for each $i \in \mathbb{N}$. Hence $s = s_0$ must lie in $C^+(M^\bullet)$. $\square$

### 2.4.2 Covering Properties.

Let us now use the above. The next result was adapted from [21, Lemma 10.3].

**Lemma 2.4.8.** (Weak covering property) *Fix a vertex $v$, an integer $r$ and some $\delta \in \{\pm 1\}$. For any non-empty subset $S$ of $e_v M^r$ which does not meet* $\mathrm{rad}(M)$ *there is a homotopy word $C \in \underline{\mathcal{W}}_{v,\delta}$ such that either:*

(i) *$C$ is finite and $S$ meets $C^+(M^\bullet)$ but not $C^-(M^\bullet)$, or*

(ii) *$C$ is an $\mathbb{N}$-word and $S$ meets $C_{\leq n}M$ but not $C_{\leq n}\mathrm{rad}(M)$ for each $n \geq 0$.*

*Proof.* We assume (a) is false. So for any finite homotopy word $B \in \underline{\mathcal{W}}_{v,\delta}$ either $S \cap B^+(M^\bullet) = \emptyset$ or $S \cap B^-(M^\bullet) \neq \emptyset$. We refer to this assumption as $(\star_B)$ for any finite homotopy word $B$.

Assuming $(\star_B)$ for every possible $B$ it suffices to construct an $\mathbb{N}$-word $C$ iteratively from $C_{\leq 0} = \underline{1}_{v,\delta}$ and show that $S$ meets $C_{\leq n}M$ but not $C_{\leq n}\mathrm{rad}(M)$ for each $n \geq 0$. Clearly this holds when $n = 0$ as $S$ meets $\underline{1}_{v,\delta}M = S$ but not $\underline{1}_{v,\delta}\mathrm{rad}(M) = e_v\mathrm{rad}(M)$.

Assuming $S$ meets $C_{\leq m}M$ but not $C_{\leq m}\mathrm{rad}(M)$ for some arbitrary fixed $m \geq 0$ it suffices to choose letters $l_{m+1}$ and $r_{m+1}$ such that $S$ meets $C_{\leq m}l_{m+1}^{-1}r_{m+1}M$ but not $C_{\leq m}l_{m+1}^{-1}r_{m+1}\mathrm{rad}(M)$. We proceed via case analysis.

Consider the case where $S$ meets $(C_{\leq m})^-(M^\bullet)$. Suppose there does not exist any arrow $y$ for which $C_{\leq m}y^{-1}d_y$ is a homotopy word. Without loss of generality we can assume there are two arrows $\alpha_+$ and $\alpha_-$ with head $t(C_{\leq m})$, in which case $(C_{\leq m})^-(M^\bullet) = C_{\leq m}(\mathrm{im}(d_{\alpha_-,M}) + \alpha_+ M)$.

This means $(C_{\leq m})^-(M^\bullet) \subseteq C_{\leq m}\mathrm{rad}(M)$ which does not meet $S$, contradicting our assumption that (a) is false. So there exists some arrow $y$ for which $C_{\leq m}y^{-1}d_y$ is a word. As $S$ meets $(C_{\leq m})^-(M^\bullet)$ by definition there is some $\gamma \in \mathbf{P}$ of minimal length for which $S$ meets $C_{\leq m}\gamma^{-1}d_{\mathrm{l}(\gamma)}M$. Letting $l_{m+1} = \gamma$ and $r_{m+1} = d_{\mathrm{l}(\gamma)}$ it is sufficient (for case (i)) to show $S$ does not meet $C_{\leq m}\gamma^{-1}d_\gamma\mathrm{rad}(M)$.

If $\gamma$ is an arrow then $\gamma^{-1}d_{\gamma}\mathrm{rad}(M) = e_{t(\gamma)}\mathrm{rad}(M)$ by corollary 2.1.9 and so $S$ does not meet $C_{\leq m}\gamma^{-1}d_{\gamma}\mathrm{rad}(M)$ by (the inductive) assumption. So we can assume $\gamma = \mathrm{l}(\gamma)\alpha$ for some $\alpha \in \mathbf{P}$.

By corollary 2.1.9 (ii) we have $\alpha^{-1}d_{\alpha}M \cap \alpha^{-1}\mathrm{l}(\gamma)^{-1}\mathrm{l}(\gamma)d_{\mathrm{l}(\alpha)}M$ and by lemma 2.1.2 we have $\mathrm{l}(\gamma)d_{\mathrm{l}(\alpha)}M = d_{\mathrm{l}(\gamma)}\mathrm{l}(\gamma)M = d_{\mathrm{l}(\gamma)}\mathrm{rad}(M)$. The minimality of the length of $\gamma$ shows that $S$ does not meet $C_{\leq m}\alpha^{-1}d_{\mathrm{l}(\alpha)}M$ and altogether this shows $S$ does not meet $C_{\leq m}\gamma^{-1}d_{\mathrm{l}(\gamma)}\mathrm{rad}(M)$.

Consider instead the case where $S$ does not meet $(C_{\leq m})^{-}(M^{\bullet})$. This means $S$ does not $(C_{\leq m})^{+}(M^{\bullet})$ by $(\star_{C_{\leq m}})$. Hence there is some arrow $x$ for which $C_{\leq m}d_x^{-1}x$ is a homotopy word as otherwise $(C_{\leq m})^{+}(M^{\bullet}) = C_{\leq m}M$ which meets $S$ by (the inductive) assumption. By the definition of $d_{x,M}$ any element of $e_{h(x)}M$ gets sent to $xM$, and so $d_x^{-1}xM = e_{h(x)}M$ and so $S$ meets $C_{\leq m}M = C_{\leq m}d_x^{-1}xM$. Consider the set $L$ of all $\lambda \in \mathbf{P}$ for which $C_{\leq m}d_x^{-1}\lambda$ is a word.

If $L$ is infinite then by lemma 2.1.19 we have $\bigcap_{\lambda} C_{\leq m}d_x^{-1}\lambda M = (C_{\leq m})^{+}(M^{\bullet})$ which does not meet $S$. By corollary 2.1.10 $C_{\leq m}d_x^{-1}\lambda M \subseteq C_{\leq m}d_x^{-1}\lambda' M$ when $\lambda$ is longer than $\lambda'$ so there is some maximal length $\mu \in L$ for which $S$ meets $C_{\leq m}d_x^{-1}\mu M$. As $L$ is infinite $\eta\mu \in L$ for some arrow $\eta$ in which case $C_{\leq m}d_x^{-1}\mu\mathrm{rad}(M) = C_{\leq m}d_x^{-1}\mu\eta M$ which does not meet $S$ by construction. In this case it is sufficient to let $l_{m+1} = d_{\mathrm{l}(\mu)}$ and $r_m = \mu$. Otherwise $L$ is finite with longest path $\mu'$ in which case let $l_{m+1} = d_{\mathrm{l}(\mu')}$ and $r_m = \mu'$ since $C_{\leq m}d_x^{-1}\mu'\mathrm{rad}(M) = C_{\leq m}d_x^{-1}0$ which equals $(C_{\leq m})^{+}(M^{\bullet})$ by lemma 2.1.19. □

*Proof of lemma 2.4.1.* (i) Let $S = H \cap (U + m)$. If it exists, any element $m_0 \in S \cap \mathrm{rad}(M)$ satisfies $m_0 = u + m$ for some $u \in U$. Since $m_0 \in e_v\mathrm{rad}(M^r) \subseteq U$ this gives the contradiction $m = m_0 - u \in U$.

So $S \cap \mathrm{rad}(M) = \emptyset$ and therefore by lemma 2.4.8 there is a homotopy word $C$ such that either $C$ is finite and $S \cap C^{+}(M^{\bullet}) \neq \emptyset = S \cap C^{-}(M^{\bullet})$, or $C$ is a homotopy $\mathbb{N}$-word and for all $n \geq 0$ we have $S \cap C_{\leq n}M \neq \emptyset = S \cap C_{\leq n}\mathrm{rad}(M)$. If $C$ is finite there is nothing to prove, so suppose otherwise. The collection $\Delta = \{S \cap C_{\leq n}M \mid n \geq 0\}$ consists of linear varieties by corollary 2.4.6.

The intersection of a finite collection $S \cap C_{\leq n(1)}M, \dots, S \cap C_{\leq n(d)}M \in \Delta$ of these linear varieties is $S \cap C_{\leq n(i)}M \neq \emptyset$, where $n(i)$ is maximal among $n(1), \dots, n(d)$. Thus $\Delta$ has the finite intersection property. By lemma 2.4.4 $e_v M^r$ is linearly compact, and so $\bigcap_{n\geq 0} S \cap C_{\leq n}M \neq \emptyset$. This shows $S \cap C^+(M) \neq \emptyset$ by lemma 2.4.7. Since $S \cap C_{\leq n}\mathrm{rad}(M) = \emptyset$ for all $n \geq 0$ we have $S \cap C^-(M^\bullet) \subseteq \bigcup S \cap C_{\leq n}\mathrm{rad}(M) = \emptyset$.

(ii) By (i) with $H = e_v M^r$ there is a word $B \in \underline{\mathcal{W}}_{v,\delta}$ such that $U + m$ meets $B^+(M)$ but not $B^-(M)$. So there is some $m' \in B^+(M)$ for which $m' = u + m$ for some $u \in U$. Note that $m' \notin U + e_v M^r \cap B^-(M)$ as otherwise $m' = u' + m''$ for some $m'' \in e_v M^r \cap B^-(M)$ and $u' \in U$ in which case $m'' = u - u' + m$ contradicting the fact $(U + m) \cap B^-(M)$ is empty. We may apply part (i) to the $R$-submodule $U' = U + e_v M^r \cap B^-(M)$, the subset $H' = e_v M^r \cap B^+(M)$ of $e_v M^r$ and $m' \in H' \setminus U'$. Doing so gives some word $D \in \underline{\mathcal{W}}_{v,-\delta}$ for which $H' \cap (U' + m')$ meets $D^+(M)$ but not $D^-(M)$. This gives some $u' \in U$ and some $y \in B^-(M) \cap e_v M^r$ for which $x = u' + y + m' \in B^+(M) \cap D^+(M)$. So $-y + x = (u' + u) + m$ which defines an element of $G^+_{B,D,r}(M) \cap (U + m)$.

Suppose for a contradiction there exists some $x' \in G^-_{B,D,r}(M) \cap (U + m)$. Then there is some $u'' \in U$, $a^- \in B^-(M) \cap e_v M^r$ and $a^+ \in B^+(M) \cap D^-(M) \cap e_v M^r$ for which $x' = u'' + m = a^- + a^+$. Writing $a^+$ as $u'' - u - a^- + m'$ defines an element from $U + B^-(M) \cap e_v M^r + m'$ and hence the intersection of $H' \cap (U' + m')$ and $D^-(M)$ which is impossible. Hence $G^-_{B,D,r}(M) \cap (U + m)$ is empty but $G^+_{B,D,r}(M) \cap (U + m)$ is non-empty. $\square$

## 2.5 Mapping Properties.

If $R$ is a field $k$ the ring $\Lambda$ is an Assem-Skowroński gentle algebra by corollary 1.2.11. In this setting we have discussed in section 1.4.4 how functorial filtrations have been used to classify modules in terms of words. Of key importance were two *mapping properties*: namely lemmas 1.4.48 and 1.4.50. By lemma 1.4.48 (iii): there is a homomorphism $\kappa_{u,u',M} : S_{u,u'}(F_{u,u'}(M)) \to M$ of $\Lambda$-modules such that $F_{u,u'}(\kappa_{u,u',M})$ is an isomoprhism (for appropriate words $u$ and $u'$ and $\Lambda$-modules $M$). By lemma 1.4.50: a $\Lambda$-module homomorphism $\theta : N \to M$ must be an isomorphism provided $F_{w,w'}(\theta)$ is an isomorphism for each pair $(w, w')$ of words (for appropriate $\theta$). In section 2.5 we give analogous mapping properties for homotopy words.

### 2.5.1 Local Mapping Properties.

In this section we state and prove an analogue of lemma 1.4.48 (iii). For this we require the following book keeping.

**Assumption:** In section 2.5.1 we let $C$ be some homotopy $I$-word where $C = B^{-1}D$ for homotopy words $B$ and $D$. We write $C = \dots l_i^{-1} r_i \dots$ provided it is non-trivial, to short hand the notation given by definition 1.3.26. If $B$ is non-trivial write $B = l_{B,1}^{-1} r_{B,1} \dots$ and similarly $D = l_{D,1}^{-1} r_{D,1} \dots$

**Lemma 2.5.1.** *Let $j \in I$. If $t = a_{B,D} - j$ then:*

(i) $v_C(j) = v_B(t)$ *if* $t \geq 0$ *and* $v_C(j) = v_D(-t)$ *otherwise; and*

(ii) $\mu_C(j) - \mu_C(i) = \mu_B(t)$ *if* $t \geq 0$ *and* $\mu_C(j) - \mu_C(i) = \mu_D(-t)$ *otherwise.*

*Proof.* (i) Let $i = a_{B,D}$. By definition we have $B = (C_{\leq i})^{-1}$ and $D = C_{>i}$ which gives ($B = r_i^{-1} l_i \dots$ or $B = \underline{1}_{h(C),s(C)}$) and ($D = l_{i+1}^{-1} r_{i+1} \dots$ or $D = \underline{1}_{h(C),s(C)}$). In the notational convention stated before the lemma, $l_{B,t} = r_{i-t+1}$ and $r_{B,t} = l_{i-t+1}$ for $t, t+1 \in I_B$. Hence for $i - j \geq 0$ we have $t(l_{B,i-j+1}) = t(r_{i-(i-j+1)+1})$ and thus $v_B(i - j) = v_C(j)$. If $B$ is trivial then $I$ is bounded below by 0 and $i = 0$.

This shows $-j \geq 0$ and $j \geq 0$ so $j = 0$ and $v_B(-j) = v_C(j)$. Similarly $l_{D,t} = l_{i+t}$ and $r_{B,t} = r_{i+t}$ for $t, t+1 \in I_D$, and hence for $j - i \geq 0$ we have $t(l_{D,j-i+1}) = t(l_{i+j-i+1})$ and so $v_D(j-i) = v_C(j)$. If $D$ is trivial then $I$ is bounded above by $j$ and $i = j$. So $v_D(j-i) = v_D(0) = v_C(i) = v_C(j)$.

(ii) Suppose $I$ is finite or $\mathbb{N}$, and so $B$ is an $\{0, \dots, i\}$-word. If $i \geq j$ then by definition $\mu_C(j) = \mu_{B^{-1}}(j)$ which equals $\mu_B(i-j) - \mu_B(i)$ by lemma 1.3.33 (iii). So when $j = i$ we have $-\mu_C(i) = \mu_B(i)$ and so $\mu_C(j) - \mu_C(i) = \mu_B(j-i)$. If instead $j > i$ then $\mu_C(j) = \mu_{B^{-1}}(i) + \mu_D(j-i)$ by definition and so $\mu_C(j) - \mu_C(i) = \mu_{B^{-1}}(i) + \mu_D(j-i) - \mu_C(i)$ which equals $\mu_D(j-i)$.

Suppose $I = -\mathbb{N}$. Then by lemma 1.3.33 (i) and (iii) we have $I_{C^{-1}} = \{-i \mid i \in I = -\mathbb{N}\} = \mathbb{N}$ and $\mu_{C^{-1}}(-j) = \mu_C(j)$ for $j \in -\mathbb{N}$. Since $C^{-1}$ is a homotopy $\mathbb{N}$-word, by the above we have $\mu_C(j) - \mu_C(i) = \mu_{C^{-1}}(-j) - \mu_{C^{-1}}(-i)$ which is $\mu_D(-i - -j) = \mu_D(j-i)$ when $j \geq i$ and $\mu_B(i-j)$ otherwise. When $I = \mathbb{Z}$ we have $i = 0$ and $\mu_{B^{-1}}(j) = \mu_B(-j)$ by lemma 1.3.33 (iii). This gives $\mu_C(j) - \mu_C(i) = \mu_C(j)$ which is $\mu_D(j-0)$ when $j \geq 0$ and $\mu_{B^{-1}}(j) = \mu_B(-j)$ when $-j \geq 0$, as required. $\square$

The proof of the next lemma follows the same idea as [21, Lemma 8.3].

**Lemma 2.5.2.** *Let $(B, D, n) \in \mathcal{I}(s)$ and $C = B^{-1}D$. Let $M$ be an object in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj})$. Then for some basis $\mathcal{B} = \{\bar{u}_\lambda \mid \lambda \in \Omega\}$ of $F_{B,D,n}(M)$ there is a morphism of complexes $\theta_{B,D,n,M} : \bigoplus_\lambda P(C)[\mu_C(a_{B,D}) - n] \to M$ such that $F_{B,D,n}(\theta_{B,D,n,M})$ is an isomorphism.*

*Proof.* Let $i = a_{B,D}$. For $j \in I$ note $\underline{b}_{j,\lambda}$ lies in degree $\mu_C(j) - \mu_C(i) + n$, and furthermore $d_{\bigoplus_\lambda P(C)[\mu_C(i)-n]}(\underline{b}_{j,\lambda}) = \underline{b}^+_{j,\lambda} + \underline{b}^-_{j,\lambda}$ where; $\underline{b}^+_{j,\lambda} = 0$ unless ($j+1 \in I$ and $l^{-1}_{j+1} r_{j+1} = d^{-1}_{\mathrm{l}(\alpha)}\alpha$) in which case $\underline{b}^+_{j,\lambda} = \alpha \underline{b}_{j+1,\lambda}$, and $\underline{b}^-_{j,\lambda} = 0$ unless ($j-1 \in I$ and $l^{-1}_j r_j = \beta^{-1} d_{\mathrm{l}(\beta)}$) in which case $\underline{b}^-_{j,\lambda} = \beta \underline{b}_{j-1,\lambda}$.

For each $\lambda \in \Omega$ we can choose a lift $u_\lambda \in F^+_{B,D,n}(M) \setminus F^-_{B,D,n}(M)$ of $\bar{u}_\lambda$. Since $u_\lambda \in e_v M^n \cap B^+(M)$ by corollary 2.2.3 there is $u^B_{s,\lambda} \in e_{v_B(s)} M^{n+\mu_B(s)}$ for each $s \in I_B$ where $u^B_{0,\lambda} = u_\lambda$ and $u^B_{s-1,\lambda} \in l^{-1}_{B,s} r_{B,s} u^B_{s,\lambda}$ given $s - 1 \in I_B$.

Similarly there exists $u^D_{t,\lambda} \in e_{v_D(t)}M^{n+\mu_D(t)}$ for each $t \in I_D$ where $u^D_{0,\lambda} = u_\lambda$ and $u^D_{t-1,\lambda} \in l^{-1}_{D,t} r_{D,t} u^D_{t,\lambda}$ given $t-1 \in I_D$. Set $u_{j,\lambda} = u^B_{i-j,\lambda}$ whenever $j \leq i$ and $u_{j,\lambda} = u^D_{j-i,\lambda}$ whenever $j \geq i$.

By lemma 2.5.1 (i) and (ii) we have $u_{j,\lambda} \in e_{v_C(j)}M^{n+\mu_C(j)-\mu_C(i)}$ for any $j$, so letting $\theta_{B,D,n,M}(\underline{b}_{j,\lambda}) = u_{j,\lambda}$ defines a degree 0 graded $\Lambda$-module map $\theta_{B,D,n,M} : \bigoplus_{\mathcal{B}} P(C)[\mu_C(i)-n] \to M$. Recalling the proof of lemma 2.5.1 (i), $l_{B,s} = r_{i-s+1}$ and $r_{B,s} = l_{i-s+1}$ for $s, s+1 \in I_B$ and hence $i \geq j$ implies $u_{j-1,\lambda} \in l^{-1}_j r_j u_{j,\lambda}$. Similarly considering $D$ where $i < j$ we again have $u_{j-1,\lambda} \in l^{-1}_j r_j u_{j,\lambda}$.

Furthermore $d_M(u_{j,\lambda}) = u^{+1}_{j,\lambda} + u^{-1}_{j,\lambda}$ where $u^{\pm 1}_{j,\lambda} = \sum_{\sigma^{\pm 1}} d_{\sigma^{\pm 1},M}(u_{j,\lambda})$ and $\sigma^{\pm 1}$ runs through the set of arrows with head $v_C(j)$ and sign $\pm 1$ (which has at most one element). Writing $s((C_{\leq j})^{-1}) = q$ gives $s(C_{>j}) = -q$ by proposition 2.1.13.

By case analysis we now show $u^{-q}_{j,\lambda} = \theta_{B,D,n,M}(\underline{b}^-_{j,\lambda})$. Suppose $\underline{b}^-_{j,\lambda} = 0$. If $\sigma^{-q}$ does not exist then $u^{-q}_{j,\lambda} = 0$ and there is nothing to prove. So we can assume $\sigma^{-q} = \alpha$ exists. Note $s(d^{-1}_\alpha) = q = -s(C)$ and $h(d^{-1}_\alpha) = h(C)$. If $j-1 \notin I$ then $j = \min I = 0$ which means $B = \underline{1}_{h(C),-s(C)}$ and so $B^+(M) = d^{-1}_\alpha 0$ as required. Otherwise $j-1 \in I$ and $l^{-1}_j r_j = d^{-1}_{\mathrm{l}(\tau)}\tau$ for some $\tau \in \mathbf{P}$ since $\underline{b}^-_{j,\lambda} = 0$. Here $s(\mathrm{f}(\tau)^{-1}) = s((C_{\leq j})^{-1}) = q$ and so $\tau^{-1} d_{\mathrm{l}(\tau)} M \subseteq d^{-1}_\alpha \alpha 0$ by corollary 2.1.10 (iv) which again gives $u_{j,\lambda} \in d^{-1}_\alpha 0$.

Suppose instead $\underline{b}^-_{j,\lambda} \neq 0$, and so $j-1 \in I$ and $l^{-1}_j r_j = \beta^{-1} d_{\mathrm{l}(\beta)}$ for some $\beta \in \mathbf{P}$. In this case $\underline{b}^-_{j,\lambda} = \beta \underline{b}_{j-1,\lambda}$ and $s(\mathrm{l}(\beta)) = -s(d^{-1}_{\mathrm{l}(\beta)}) = -s((C_{\leq j})^{-1}) = -q$ which means $\mathrm{l}(\beta) = \sigma^{-q}$ and $u_{j-1,\lambda} \in l^{-1}_j r_j u_{j,\lambda} = \beta^{-1} d_{\mathrm{l}(\beta)} u_{j,\lambda}$. So here we have $d_{\mathrm{l}(\beta),M}(u_{j,\lambda}) = \beta u_{j-1,\lambda} = \beta \theta_{B,D,n,M}(\underline{b}_{j-1,\lambda})$ and so $u^{-q}_{j,\lambda} = \theta_{B,D,n,M}(\underline{b}^-_{j,\lambda})$. Similarly one can consider the different possibilities for $l^{-1}_{j+1} r_{j+1}$ and prove $u^{+q}_{j,\lambda} = \theta_{B,D,n,M}(\underline{b}^+_{j,\lambda})$. For this the cases $j+1 \notin I$ and $j+1 \in I$ are separated, similar to the above.

Together this gives $\theta_{B,D,n,M}(\underline{b}^+_{j,\lambda}) + \theta_{B,D,n,M}(\underline{b}^-_{j,\lambda}) = u^+_{j,\lambda} + u^-_{j,\lambda}$ and so $\theta_{B,D,n,M}(\underline{b}^+_{j,\lambda} + \underline{b}^-_{j,\lambda})) = d_M(u_{j,\lambda})$. This shows $\theta_{B,D,n,M}$ is a morphism of complexes. By lemma 2.3.20 (i) the set of elements $\bar{\underline{b}}_{i,\lambda} = \underline{b}_{i,\lambda} + F^-_{B,D,n}(P(C)[\mu_C(i)-n])$ where $\lambda$ runs through $\mathcal{B}$ define a basis of the $k$-vector space $F_{B,D,n}(\bigoplus_\lambda P(C)[\mu_C(i)-n])$. Since $\theta_{B,D,n,M}(\underline{b}_{i,\lambda}) = u_{i,\lambda}$ we have that $F_{B,D,n}(\theta_{B,D,n,M})(\bar{b}_{i,\lambda}) = \bar{u}_{i,\lambda} = \bar{u}_\lambda$ so $F_{B,D,n}(\theta_{B,D,n,M})$ is an isomorphism. $\square$

**Remark 2.5.3.** (Automorphisms and Reductions) Recall that, by lemma 1.4.31, any linear relation $V$ on an $R$-module $M$ defines an $R[T, T^{-1}]$-module $V^\sharp/V^\flat$ where the action of $T$ is given by setting $T(m + V^\flat) = m' + V^\flat$ iff $m' \in V^\sharp \cap (V^\flat + Vm)$.

Recall from definition 1.4.32 that a *reduction* of $V$ is a pair $(U, g)$ where $U$ is an $R[T, T^{-1}]$-module which is free over $R$ and $g : U \to M$ is an $R$-module map for which $V^\sharp = \mathrm{im}(g) + V^\flat$ and $g(Tu) \in Vg(u)$ for each $u \in U$. Recall that a reduction $(U, g)$ *meets in* $\mathfrak{m}$ if the pre-image $g^{-1}(V^\flat)$ is contained in $\mathfrak{m}U$.

If $R$ is a field recall that $V$ is *split* if there is an $R$-linear subspace $W$ of $M$ such that $V^\sharp = W \oplus V^\flat$ and $\#Vm \cap W = 1$ for each $m \in W$. Recall that, by corollary 1.4.33, if $(U, f)$ is a reduction of a relation $V$ on $M$ which meets in $\mathfrak{m} = 0$, then $V$ is split.

Suppose now $C = B^{-1}D$ is a periodic homotopy $\mathbb{Z}$-word (say $D = E^\infty$ and $B = (E^{-1})^\infty$ for some homotopy $\{0, \dots, p\}$-word $E$) and let $n \in \mathbb{Z}$ be arbitrary. Recall the linear relation $E(n) = \{(m, m') \in e_vM^n \oplus e_vM^n \mid m \in Em'\}$ on $e_vM^n$ and by lemma 2.2.11 we have $E(n)^\sharp = F^+_{B,D,n}(M)$ and $E(n)^\flat = F^-_{B,D,n}(M)$.

**Lemma 2.5.4.** *Let $(B, D, n) \in \mathcal{I}(b)$ and so $C = B^{-1}D$ is a periodic $\mathbb{Z}$-word of period $p > 0$. Let $M$ be an object of $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj})$ such that $F_{B,D,n}(M)$ has finite dimension $d$ over $k = R/\mathfrak{m}$.*

*Then there is an object $U$ of $R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$ with rank $d$ over $R$ and a morphism $\theta_{B,D,n,M} : P(C, U)[-n] \to M$ of complexes such that $F_{B,D,n}(\theta_{B,D,n,M})$ is an isomorphism.*

*Proof.* Let $F_{B,D,n}(M) = V$. By definition $B = (E^{-1})^\infty$ and $D = E^\infty$ where $E = l_1^{-1}r_1 \dots l_p^{-1}r_p$ is a cyclic homotopy $\{0, \dots, p\}$-word. Note that $F^+_{B,D,n}(M) = E(n)^\sharp$ and $F^-_{B,D,n}(M) = E(n)^\flat$ which means $E(n)^\sharp/E(n)^\flat = V$.

By the second part of corollary 2.2.12 this is a finite-dimensional $k[T, T^{-1}]$-module, and so by lemma 1.4.34 there is a reduction $(U, g)$ of $E(n)$ which meets in $\mathfrak{m}$ and where $U$ is finitely generated as an $R$-module. Choose an $R$-basis $u_1, \dots, u_d$ of $U$.

Since $(U, g)$ is a reduction we have $\mathrm{im}(g) \subseteq E(n)^\sharp$ and so $g(u_i) \in F^+_{B,D,n}(M)$ for each $i$ by lemma 2.2.11.

Similarly $g(Tu_i) \in Eg(u_i)$ so there are elements $v_{0,i}, \dots, v_{p,i} \in M$ for $1 \le i \le d$ where $v_{j,i} \in e_{v_E(j)} M^{n+\mu_E(j)}$ for each $j$ (by corollary 2.2.3), $v_{p,i} = g(u_i)$, $v_{0,i} = g(Tu_i)$, and $v_{j-1,i} \in l_j^{-1} r_j v_{j,i}$ given $j > 0$.

By lemma 1.3.47, to define a $\Lambda$-module map $\theta_{B,D,n,M} : P(C,V)[-n] \to M$ it is enough to extend $\theta_{B,D,n,M}(\underline{b}_j \otimes \bar{v}_i) = v_{j,i}$ linearly over $\Lambda$, where $\bar{v}_i = g(u_i) + F^-_{B,D,n}(M)$ for $1 \le i \le d$ and $0 \le j \le p-1$. Note there is some $a_{l,i} \in R$ for $1 \le l \le d$ satisfying $Tu_i = \sum_l a_{l,i} u_l$ since $u_1, \dots, u_d$ is an $R$-basis of $U$.

Applying $g$ gives $v_{0,i} = \sum_l a_{l,i} v_{p,l}$. Recalling lemma 1.4.31, since $g(Tu_i) \in E(n)^\sharp \cap Eg(u_i) \subseteq E(n)^\sharp \cap (E(n)^\flat + Eg(u_i))$ we have $T(g(u_i) + F^-_{B,D,n}(M)) = g(Tu_i) + F^-_{B,D,n}(M)$ and so $T\bar{v}_i = \sum_l a_{l,i} \bar{v}_l$.

Since $b_j \otimes \bar{u}_i \in P^{\mu_C(j)+n}(C,V)[-n]$ and $u_{j,i} \in M^{n+\mu_C(j)}$ the map $\theta_{B,D,n,M}$ is homogeneous of degree 0. We now check that $\theta_{B,D,n,M}$ is a morphism of complexes.

We proceed following similar steps to the proof of lemma 2.5.2, but there are minor complications to consider. For arbitrary $j$ and $w = v_E(j)$ we have $d_M(v_{j,i}) = v^{+1}_{j,i} + v^{-1}_{j,i}$ by lemma 2.1.2 where we let $v^{\pm 1}_{j,i} = \sum_{\sigma^{\pm 1}} d_{\sigma^{\pm 1},M}(v_{j,i})$ where $\sigma^{\pm 1}$ runs through the set of arrows with head $w$ and sign $\pm 1$ (which has at most one element).

As in the proof of lemma 2.5.2 let $s((E_{\le j})^{-1}) = q$ and so $s(E_{>j}) = -q$. In what follows we prove $v^{-q}_{j,i} = \theta_{B,D,n,M}(\underline{b}^-_j \otimes \bar{v}_i)$ by case analysis, separating the cases $j = 0$ and $j \neq 0$. Similarly one can show $v^{+q}_{j,i} = \theta_{B,D,n,M}(\underline{b}^+_j \otimes \bar{v}_i)$ after separating cases $j \neq p-1$ and $j = p-1$.

If $l_j^{-1} r_j = d^{-1}_{\mathrm{l}(\tau)} \tau$ for some $\tau \in \mathbf{P}$ then $\underline{b}^-_j = 0$ so $\underline{b}^-_j \otimes \bar{v}_i = 0$. Again $v_{j,i} \in r_j^{-1} l_j v_{j-1,i}$ as $s(\mathrm{f}(\tau)^{-1}) = q$ and so $v_{j,i} \in \tau^{-1} d_{\mathrm{l}(\tau)} M$ which means $v_{j,i} \in d_\alpha^{-1} 0$ for $j \neq 0$ and any arrow $\alpha$ with head $u$ and sign $-q$ by corollary 2.1.10 (iv). This shows $v^{-q}_{j,i} = 0 = \theta_{B,D,n,M}(\underline{b}^-_j \otimes \bar{v}_i)$. For $j = 0$ we have $\underline{b}^-_0 = 0$ and $l_p^{-1} r_p = d^{-1}_{\mathrm{l}(\tau)} \tau$ so $\underline{b}^-_p = 0 = v^{-q}_{p,i}$. This gives $\sum_{\sigma^{-q}} d_{\sigma^{-q},M}(\sum_l a_{l,i} v_{p,l}) = \sum_{\sigma^{-q}} \sum_l a_{l,i} d_{\sigma^{-q},M}(v_{p,l})$ and so $v^{-q}_{0,i} = 0 = \theta_{B,D,n,M}(\underline{b}^-_0 \otimes \bar{v}_i)$.

Otherwise $l_j^{-1} r_j = \beta^{-1} d_{\mathrm{l}(\beta)}$ for some $\beta \in \mathbf{P}$. In this case where $j \neq 0$ one has $\underline{b}^-_j \otimes \bar{v}_i = \beta \underline{b}_{j-1,i}$ and again $s(\mathrm{l}(\beta)) = -q$ which means $\mathrm{l}(\beta) = \sigma^{-q}$ and $v_{j-1,i} \in \beta^{-1} d_{\mathrm{l}(\beta)} v_{j,i}$.

As in the proof of lemma 2.5.2 this gives $v^{-q}_{j,i} = \beta v_{j-1,i} = \theta_{B,D,n,M}(\beta \underline{b}_{j-1,i}) = \theta_{B,D,n,M}(\underline{b}^{-}_{j} \otimes \bar{v}_i)$. For $j = 0$ we have $l^{-1}_{p} r_p = \beta^{-1} d_{\mathrm{l}(\beta)}$ and therefore

$$\begin{aligned}\underline{b}^{-}_{0} \otimes \bar{v}_i = \beta \underline{b}_{-1,i} = \beta \underline{b}_{-1} \otimes \bar{v}_i = \beta \underline{b}_{p-1} T \otimes \bar{v}_i = \beta \underline{b}_{p-1} \otimes T\bar{v}_i \\ = \beta \underline{b}_{p-1} \otimes \textstyle\sum_l a_{l,i} \bar{v}_l = \sum_l a_{l,i} (\beta \underline{b}_{p-1} \otimes \bar{v}_l) = \sum_l a_{l,i} \beta \underline{b}_{p-1,l} = \sum_l a_{l,i} \underline{b}^{-}_{p,l}\end{aligned}$$

So, by the above we have

$$\begin{aligned}\theta_{B,D,n,M}(\underline{b}^{-}_{0} \otimes \bar{v}_l) = \theta_{B,D,n,M}(\textstyle\sum_l a_{l,i} \underline{b}^{-}_{p,l}) = \sum_l a_{l,i} \theta_{B,D,n,M}(\underline{b}^{-}_{p,l}) = \sum_l a_{l,i} v^{-q}_{p,l} \\ = \textstyle\sum_l a_{l,i} \sum_{\sigma^{-q}} d_{\sigma^{-q},M}(v_{p,l}) = \sum_{\sigma^{-q}} d_{\sigma^{-q},M}(\sum_l a_{l,i} u_{p,l}) = \sum_{\sigma^{-q}} d_{\sigma^{-q},M}(v_{0,\lambda}) = v^{-q}_{0,l}\end{aligned}$$

Similarly one can consider the different possibilities for $l^{-1}_{j+1} r_{j+1}$ and prove $v^{+q}_{j,l} = \theta_{B,D,n,M}(\underline{b}^{+}_{j} \otimes \bar{v}_l)$. For this the cases $j \neq p-1$ and $j = p-1$ are separated, similar to the above. Together we have that $\theta_{B,D,n,M}(\underline{b}^{+}_{s} \otimes \bar{v}_l + \underline{b}^{-}_{s} \otimes \bar{v}_l) = v^{+}_{s,l} + v^{-}_{s,l}$ and thus $\theta_{B,D,n,M}(d_{P(C,V)[-n]}(\underline{b}_{s,l})) = d_M(\theta_{B,D,n,M}(\underline{b}_{s,l}))$ whenever $0 \leq s \leq p-1$. Write $\bar{\underline{b}}_{p,l}$ for the coset $\underline{b}_{p,l} + F^{-}_{B,D,n}(P(C,V)[-n])$.

By lemma 2.3.21 (ii) the elements $\bar{\underline{b}}_{p,1}, \ldots, \bar{\underline{b}}_{p,d}$ give a $k$-basis of $F_{B,D,n}(P(C,V)[-n]) \cong k \otimes_R V$. Since $F_{B,D,n}(\theta_{B,D,n,M})(\bar{\underline{b}}_{p,l}) = \bar{v}_{p,l}$, to prove $F_{B,D,n}(\theta_{B,D,n,M})$ is an isomorphism we need only show $\bar{v}_{p,1}, \ldots, \bar{v}_{p,d}$ is a $k$-linearly independent subset of $V = F_{B,D,n}(M)$. If we have $\sum_l \lambda_l \bar{v}_{p,l} = 0$ in $V$ for some $\lambda_l \in k$ then writing $\lambda_l = r_l + \mathfrak{m}$ for $r_l \in R$ gives $\mathrm{l}(\sum_l r_l u_l) \in F^{-}_{B,D,n}(M) = E(n)^{\flat}$ by definition. Since $(U, f)$ meets in $\mathfrak{m}$ we have $\sum_l r_l u_l \in \mathfrak{m}U = \bigoplus_{l=1}^{d} \mathfrak{m} u_l$ and so $r_l \in \mathfrak{m}$ for each $l$, as required. $\square$

### 2.5.2 Global Mapping Properties.

In this section we state and prove an analogue for part (ii) of [15, p.163, Proposition].

**Lemma 2.5.5.** *Let $\theta : P \to M$ be a morphism in $\mathcal{K}_{\min}(\Lambda\textbf{-Proj})$, and suppose $M^i$ is finitely generated for each $i$. Suppose $F_{B,D,n}(\theta)$ is surjective for each $n \in \mathbb{Z}$ and each pair of words $(B, D)$ for which $B^{-1}D$ is a word. Then $\theta^i$ is surjective for each $i$.*

*Proof.* For a contradiction suppose that $\theta^i$ is not surjective for some $i \in \mathbb{Z}$. Since $\mathrm{rad}(M^i)$ is a superfluous submodule of $M^i$ if $\mathrm{im}(\theta^i) + \mathrm{rad}(M^i) = M^i$ then $\mathrm{im}(\theta^i) = M^i$ which by assumption is impossible. So we have $e_v\mathrm{im}(\theta^i) + e_v\mathrm{rad}(M^i) \neq e_vM^i$ for some vertex $v$. Hence $e_v\mathrm{im}(\theta^i) + e_v\mathrm{rad}(M^i)$ is contained in a maximal $R$-submodule $U$ of $e_vM^i$. Since $e_v\mathrm{rad}(M^i) \subseteq U$ and $U \neq e_vM^i$, by the covering property for refined functors (lemma 2.4.1 (ii)) for some element $m \in e_vM^i \setminus U$ there are homotopy words $B \in \underline{\mathcal{W}}_{v,\delta}$ and $D \in \underline{\mathcal{W}}_{v,-\delta}$ for which ($B^{-1}D$ is a homotopy word and) $U + m$ meets $G^{+}_{B,D,i}(M)$ but not $G^{-}_{B,D,i}(M)$. So there is some $u \in U$, $a \in B^{-}(M)$ and $b \in B^{+}(M) \cap D^{+}(M)$ such that $u + m = a + b$. Since $u, m \in e_vM^i$ we may assume $a, b \in e_vM^i$. Note that $G_{B,D,i}(\theta)$ is onto and sends $x + G^{-}_{B,D,i}(P)$ to $\theta^i(x) + G^{-}_{B,D,i}(M)$. So there is some $x \in G^{+}_{B,D,i}(P)$ for which $\theta^i(x) - b \in G^{-}_{B,D,i}(M)$ and therefore $b = \theta^i(x) + c + d$ for some $c \in B^{-}(M)$ and $d \in B^{+}(M) \cap D^{-}(M)$. Since $\theta^i(x) \in e_v\mathrm{im}(\theta^i) \subseteq U$ we have by construction an element $u - \theta^i(x) + m = (a + c) + d$ of $(U + m) \cap G^{-}_{B,D,i}(M)$ which is impossible. $\square$

**Assumption:** In what follows in this section we fix some notation. Let $\mathcal{S}$ and $\mathcal{B}$ be index sets, $\{t(\sigma), s(\beta) \mid \sigma \in \mathcal{S}, \beta \in \mathcal{B}\}$ be a collection of integers, $\{V^{\beta} \mid \beta \in \mathcal{B}\}$ be a set of objects from $R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$ and $\{A(\sigma), E(\beta) \mid \sigma \in \mathcal{S}, \beta \in \mathcal{B}\}$ be a set of homotopy words, where each $A^{\sigma}$ is non-periodic and each $E^{\beta}$ is periodic of period $p_{\beta}$. Consider a direct sum of complexes of the form

$$N = \left(\bigoplus\nolimits_{\sigma\in\mathcal{S}} P(A(\sigma))[-t(\sigma)]\right) \oplus \left(\bigoplus\nolimits_{\beta\in\mathcal{B}} P(E(\beta), V^{\beta})[-s(\beta)]\right) \quad (\text{✡})$$

The following is the analogue of [21, Lemma 9.4].

**Lemma 2.5.6.** *Let $\theta : N \to M$ be a morphism in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj})$ such that $\bar{F}_{B,D,n}(\theta)$ is injective for each $(B, D, n) \in \mathcal{I}$. Then $\theta^i$ is injective for each $i \in \mathbb{Z}$.*

*Proof.* Assume there is some $h \in \mathbb{Z}$ for which $\theta^h$ is not injective. By corollary 1.1.25 $\Lambda$ is semilocal and so $\mathrm{rad}(N^h) = \mathrm{rad}(\Lambda)N^h$ and $\mathrm{rad}(M^h) = \mathrm{rad}(\Lambda)M^h$ by [48, (24.7) Theorem, p.349]. Since $\Lambda/\mathrm{rad}(\Lambda)$ is semisimple, $N^h/\mathrm{rad}(N^h)$ is an injective $\Lambda/\mathrm{rad}(\Lambda)$-module. Hence the induced map $\bar{\theta}^h : N^h/\mathrm{rad}(N^h) \to M^h/\mathrm{rad}(M^h)$ is not injective, as otherwise it must be a section which would mean $\theta^h$ was injective by [40, Lemma 2.2, p.218]. Thus $\bar{\theta}^h$ is not injective, and so there is a vertex $v$ and a non-zero element $n \in e_v N^h \setminus e_v\mathrm{rad}(N^h)$ for which $\theta^h(n) \in e_v\mathrm{rad}(M^h)$.

Since $N$ is a direct sum, there is a finite subset $\Gamma = \{A(1), \dots, A(m), E(1), \dots, E(q)\}$ of $\{A(\sigma), E(\beta) \mid \sigma \in \mathcal{S}, \beta \in \mathcal{B}\}$ for which $n$ lies in the direct sum of $\bigoplus_{\sigma=1}^{m} P^{h-t(\sigma)}(A(\sigma))$ and $\bigoplus_{\beta=1}^{q} P^{h-s(\beta)}(E(\beta), V^{\beta})$.

Hence we now assume $\mathcal{S} = \{1, \dots, m\}$ and $\mathcal{B} = \{1, \dots, q\}$. For each $\sigma$ and $\beta$ let; $\langle\sigma\rangle = I_{A(\sigma)}$, $J_\sigma = P(A(\sigma))[-t(\sigma)]$, $\langle\beta\rangle' = \{0, \dots, p_\beta - 1\}$, $L_\beta = P(E(\beta), V^\beta)[-s(\beta)]$ and let $\Omega(\beta)$ be an $R$-basis for the free $R$-module $V^\beta$. Hence there is some $x \in e_v\mathrm{rad}(N^h)$ and (transversal) scalars $\eta_{\sigma,i}, \eta_{\beta,j,\lambda} \in S$ for which

$$n = \sum_{\sigma=1}^{m} \sum_{i\in\langle\sigma\rangle} \eta_{q,i}\underline{b}_i^\sigma + \sum_{\beta=1}^{q} \sum_{j\in\langle\beta\rangle'} \sum_{\lambda\in\Omega(\beta)} \eta_{\beta,j,\lambda}\underline{b}_{j,\lambda}^\beta + x$$

and for each $\sigma$ there is some $i[\sigma] \in \langle\sigma\rangle$ for which $v_{A(\sigma)}(i\,[\sigma]) = v$ and $\mu_{A(\sigma)}(i\,[\sigma]) = h - t(\sigma)$, and for each $\beta$ there is some $j\,[\beta] \in \langle\beta\rangle'$ for which $v_{E(\beta)}(j\,[\beta]) = v$ and $\mu_{E(\beta)}(j\,[\beta]) = h - s(\beta)$. For each $\delta \in \{\pm 1\}$ let $A(\sigma)(i\,[\sigma]\,,\delta) = A(\sigma, \delta)$ and $E(\beta)(j\,[\beta]\,,\delta) = E(\beta, \delta)$. Since $n \notin e_v\mathrm{rad}(N^h)$ for all $\sigma$ and $\beta$ we can assume $\eta_{\sigma,i[\sigma]} \neq 0$ and $\eta_{\beta,j[\beta],\lambda} \neq 0$ for some $\lambda \in \Omega(\beta)$. By lemma 2.3.20 (i) and lemma 2.3.21 (i) we respectively have

$$\bar{F}^{+}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma) = \bar{F}^{-}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma) + R\underline{b}_{i[\sigma]} \text{ and}$$
$$\bar{F}^{+}_{E(\beta,1),E(\beta,-1),h}(L_\beta) = \bar{F}^{-}_{E(\beta,1),E(\beta,-1),h}(L_\beta) + \textstyle\sum_\lambda R\underline{b}_{j[\beta],\lambda}.$$

The first equation shows $\eta_{\sigma,i[\sigma]}\underline{b}^\sigma_{i[\sigma]}$ lies in $\bar{F}^{+}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma) \subseteq \bar{G}^{+}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma)$.

We also have $\eta_{\sigma,i[\sigma]}\underline{b}^{\sigma}_{i[\sigma]} \notin \bar{F}^{-}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma)$ by lemma 2.3.20 (ii). It is straightforward to show that $\bar{F}^{-}_{A(\sigma,1),A(\sigma,-1),h}(J_q)$ contains the intersection of $\bar{F}^{+}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma)$ and $\bar{G}^{-}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma)$, which means $\eta_{\sigma,i[\sigma]}\underline{b}^{\sigma}_{i[\sigma]} \in \bar{G}^{+}_{A^{\sigma}_{1},A^{\sigma}_{-1},h}(J_\sigma)$ and $\eta_{\sigma,i[\sigma]}\underline{b}^{\sigma}_{i[\sigma]} \notin \bar{G}^{-}_{A^{\sigma}_{1},A^{\sigma}_{-1},h}(J_\sigma)$. A similar argument using lemma 2.3.21 (ii) shows $\eta_{\beta,j[\beta],\lambda}\underline{b}_{j[\beta],\lambda} \in \bar{G}^{+}_{E(\beta,1),E(\beta,-1),h}(L_\beta)$ and $\eta_{\beta,j[\beta],\lambda}\underline{b}_{j[\beta],\lambda} \notin \bar{G}^{-}_{E(\beta,1),E(\beta,-1),h}(L_\beta)$.

Note that for each $\delta \in \{\pm 1\}$ we have $A(\sigma,\delta), E(\beta,\delta) \in \underline{\mathcal{W}}_{v,\delta}$ for $1 \le \sigma \le m$ and $1 \le \beta \le q$. After reordering we can assume that $A(\sigma,1) \le A(\sigma',1)$ for $\sigma \le \sigma'$ and $E(\beta,1) \le E(\beta',1)$ for $\beta \le \beta'$. Let $B$ be the largest homotopy word of $A(m,1)$ and $E(q,1)$. If $B = A(m,1)$ let $D$ be the largest homotopy word $A(\sigma',-1)$ among $A(1,-1),\dots,A(m,-1)$ for which $A(\sigma',1) = A(m,1)$. Otherwise $B = E(q,1)$ and let $D$ be the largest homotopy word $E(\beta',-1)$ among $E(1,-1),\dots,E(q,-1)$ for which $E(\beta',1) = E(q,1)$.

Note that $B$ and $D$ have the same head and opposite signs, which means $B^{-1}D$ is a homotopy word by proposition 2.1.13. By construction, proposition 2.1.30, and corollary 2.1.20 we have $\bar{G}^{+}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma) \subseteq \bar{G}^{+}_{B,D,h}(N)$ for any $\sigma$ and $\bar{G}^{+}_{E(\beta,1),E(\beta,-1),h}(L_\beta) \subseteq \bar{G}^{+}_{B,D,h}(N)$ for any $\beta$. So $n \in \bar{G}^{+}_{B,D,h}(N)$. If $n \in \bar{G}^{-}_{B,D,h}(N)$ and $(B,D) = (A(\sigma',1),A(\sigma',-1))$ then

$$\textstyle\sum_{i\in\langle\sigma'\rangle}\eta_{\sigma',i}\underline{b}^{\sigma'}_{i} = n - \sum_{\sigma=1,\,\sigma\neq\sigma'}^{m}\sum_{i\in\langle\sigma\rangle}\eta_{q,i}\underline{b}^{\sigma}_{i} + \sum_{\beta=1}^{q}\sum_{j\in\langle\beta\rangle'}\sum_{\lambda\in\Omega(\beta)}\eta_{\beta,j,\lambda}\underline{b}^{\beta}_{j,\lambda} + x$$

which by the above lies in $\bar{G}^{+}_{A(\sigma,1),A(\sigma,-1),h}(J_\sigma) \cap \bar{G}^{-}_{A(\sigma',1),A(\sigma',-1),h}(N)$ and thus gives the contradiction $\sum_{i\in\langle\sigma'\rangle}\eta_{\sigma',i}\underline{b}^{\sigma'}_{i} \in \bar{G}^{-}_{A(\sigma,1),A(\sigma,-1),h}(J_{\sigma'})$. Similarly if $(B,D) = (E(\beta',1),E(\beta',-1))$ we come to the contradiction $\sum_{j\in\langle\beta'\rangle'}\sum_{\lambda\in\Omega(\beta')}\eta_{\beta',j,\lambda}\underline{b}^{\beta'}_{j,\lambda} \in \bar{G}^{-}_{E(\beta,1),E(\beta,-1),h}(L_{\beta'})$. So we must have that $n \in \bar{G}^{+}_{B,D,h}(N)\setminus\bar{G}^{-}_{B,D,h}(N)$ . Since $\bar{G}^{-}_{B,D,h}(\theta)$ sends the coset of $n$ to the coset of $\theta^{h}(n) \in e_v\mathrm{rad}(M^h) \subseteq \bar{G}^{-}_{B,D,h}(M)$ we conclude that $\bar{G}_{B,D,h}(\theta)$ is not injective, as required. $\square$

### 2.5.3 Direct Sums of String and Band Complexes.

Here we look at some fruit of the labour involved in defining a functorial filtration.

**Assumption:** As in (✡) from section 2.5.2 we let $N$ be a direct sum of complexes of the form $P(A(\sigma))[-t(\sigma)]$ and $P(E(\beta), V^{\beta})[-s(\beta)]$. This means $\sigma$ and $\beta$ run through index sets $\mathcal{S}$ and $\mathcal{B}$ respectively, each $A(\sigma)$ is a non-periodic homotopy word, and each $E(\beta)$ is a periodic homotopy word.

**Definition 2.5.7.** (NOTATION: $\overline{\mathcal{W}}$, $\underline{\mathcal{W}}(s)$, $\underline{\mathcal{W}}(b)$) Let $\overline{\mathcal{W}}$ denote the set of all equivalence classes of homotopy words. For each equivalence class $\overline{C}$ of $\overline{\mathcal{W}}$ we choose one representative $C$ and one pair of words $(B, D)$ for which $B^{-1}D = C$. Let $\underline{\mathcal{W}}$ be the set of these chosen pairs $(B, D)$. Let $\underline{\mathcal{W}}(s)$ (resp. $\underline{\mathcal{W}}(b)$) be the subset of $\underline{\mathcal{W}}$ consisting of all pairs $(B, D)$ for which $B^{-1}D$ is not a periodic homotopy $\mathbb{Z}$-word (resp. $B^{-1}D$ is a periodic homotopy $\mathbb{Z}$-word).

For convenience we recall some notation from definition 2.2.20. We let $\Sigma = \underline{\mathcal{W}}_{v,1} \times \underline{\mathcal{W}}_{v,-1} \times \mathbb{Z}$; $\Sigma(s)$ be the set of $(B, D, n) \in \Sigma$ with $B^{-1}D$ not periodic; and $\Sigma(b)$ be the set of $(B, D, n) \in \Sigma$ with $B^{-1}D$ periodic. Recall that for $(B, D, n), (B', D', n') \in \Sigma$, given $C = B^{-1}D$ and $C' = B'^{-1}D'$ we write $(B, D, n) \sim (B', D', n')$ provided $n' - n = r(B, D; B', D')$ where

$$r(B, D; B', D') = \begin{cases} \mu_C(a_{B',D'}) - \mu_C(a_{B,D}) & \text{(if } C' = C \text{ is not a homotopy } \mathbb{Z}\text{-word)} \\ \mu_C(a_{D',B'}) - \mu_C(a_{B,D}) & \text{(if } C' = C^{-1} \text{ is not a homotopy } \mathbb{Z}\text{-word)} \\ \mu_C(\pm m) & \text{(if } C' = C^{\pm 1}[m] \text{ is a homotopy } \mathbb{Z}\text{-word)} \end{cases}$$

Recall that: $\sim$ restricts to an equivalence relation $\sim_s$ (resp. $\sim_b$) on $\Sigma(s)$ (resp. $\Sigma(b)$); $\overline{\Sigma} = \Sigma/\sim$; $\overline{\Sigma(s)} = \Sigma(s)/\sim_s$; and $\overline{\Sigma(b)} = \Sigma(b)/\sim_b$. Recall $\mathcal{I}(s)$ (resp. $\mathcal{I}(b)$) is a fixed a chosen collection of representatives $(B, D, n)$ in $\Sigma(s)$ (resp. $\Sigma(b)$), one for each class $\overline{(B, D, n)}$ from $\overline{\Sigma(s)}$ (resp. $\overline{\Sigma(b)}$).

(NOTATION: $A(\sigma, \delta)$) For each $\delta \in \{\pm 1\}$ let $A(\sigma, \delta)$ be the truncated word $A(\sigma)(0, \delta)$: the unique word in $\{(A(\sigma)_{\leq 0})^{-1}, A(\sigma)_{0<}\}$ with sign $\delta$.

(NOTATION: $\mathcal{S}(B,D,n)$, $\mathcal{B}^{\pm}(B,D,n)$) For $(B,D,n)\in\mathcal{I}(s)$ let $\mathcal{S}(B,D,n)$ be the set of $\sigma\in\mathcal{S}$ such that $(B,D,n)\sim((A(\sigma,1),A(\sigma,-1),t(\sigma))$. By definition this means $\mathcal{S}(B,D,n)$ is the set of all $\sigma$ such that $t(\sigma)-n=r(B,D;A(\sigma,1),A(\sigma,-1))$. For $(B,D,n)\in\mathcal{I}(b)$ let $\mathcal{B}^{\pm}(B,D,n)$ be the set of $\beta\in\mathcal{B}$ such that $s(\beta)-n=\mu_{B^{-1}D}(\pm m)$ and $E(\beta)=(B^{-1}D)^{\pm 1}[m]$ for some $m\in\mathbb{Z}$.

(NOTATION: $S^{n}_{B,D,n}(V)$, $V^{\beta}_{\pm}$) Let $B$ and $D$ be homotopy words where $C=B^{-1}D$ is a homotopy word. If $(B,D)\in\underline{\mathcal{W}}(s)$ let $V$ be an object of $R$-**Proj**. Otherwise $(B,D)\in\underline{\mathcal{W}}(b)$, and let $V$ be an object of $R[T,T^{-1}]$-$\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$. Then $S_{B,D,n}(V)$ is a complex of projective $\Lambda$-modules. For $n\in\mathbb{Z}$ let $S^{n'}_{B,D,n}(V)$ denote the module in degree $n'$ of $S_{B,D,n}(V)$. For $\beta\in\mathcal{B}(B,D,n)^{+}$ let $\bar{V}^{\beta}_{+}=k\otimes_{R[T,T^{-1}]}V^{\beta}$. For $\beta\in\mathcal{B}(B,D,n)^{-}$ let $\bar{V}^{\beta}_{-}=k\otimes_{R[T,T^{-1}]}\operatorname{res}_{\iota}V^{\beta}$, and recall $\operatorname{res}_{\iota}V^{\beta}$ is defined by swapping the actions of $T$ and $T^{-1}$ on $V^{\beta}$.

**Lemma 2.5.8.** *Let $B$, $B'$, $D$ and $D'$ be homotopy words such that $C=B^{-1}D$ and $C'=B'^{-1}D'$ are homotopy words. Let $n$ and $n'$ be integers.*

(i) *If $n\neq n'$ then $\mathcal{S}(B,D,n)\cap\mathcal{S}(B,D,n')=\emptyset=\mathcal{B}(B,D,n)^{\pm}\cap\mathcal{B}(B,D,n')^{\pm}$.*

(ii) *$\bigcup_{t\in\mathbb{Z}}\mathcal{S}(B,D,t)$ is the set of $\sigma\in\mathcal{S}$ where $A(\sigma)$ and $C$ are equivalent.*

(iii) *$\bigcup_{n\in\mathbb{Z}}\mathcal{B}(B,D,t)^{+}\cup\mathcal{B}(B,D,t)^{-}$ is the set of $\beta\in\mathcal{B}$ where $E(\beta)$ and $C$ are equivalent.*

(iv) *If $B^{-1}D\nsim B'^{-1}D'$ then $\mathcal{S}(B,D,n)\cap\mathcal{S}(B',D',n)=\emptyset=\mathcal{B}(B,D,n)^{\pm}\cap\mathcal{B}(B',D',n)^{\pm}$.*

*Proof.* (i) If $\sigma\in\mathcal{S}(B,D,n)\cap\mathcal{S}(B,D,n')$ then $(B,D,n)\sim(B,D,n')$ by transitivity and so $n'-n=0$. Similarly, $\mathcal{B}(B,D,n)^{\pm}\cap\mathcal{B}(B,D,n')^{\pm}\neq\emptyset$ implies $n=n'$.

(ii) and (iii) Clearly if $\sigma\in\mathcal{S}(B,D,n)$ then $A(\sigma)$ must be equivalent to $C$. Conversely if $A(\sigma)$ is equivalent to $C$ then $(B,D,n)\sim(A(\sigma,1),A(\sigma,-1),t(\sigma))$ where $n=t(\sigma)-r(B,D;A(\sigma,1),A(\sigma,-1))$. A similar argument justifies the respective statements about $\bigcup_{n\in\mathbb{Z}}\mathcal{B}(B,D,n)^{+}$ and $\bigcup_{n\in\mathbb{Z}}\mathcal{B}(B,D,n)^{-}$.

(iv) If $\sigma\in\mathcal{S}(B,D,n)\cap\mathcal{S}(B',D',n)$ (or $\beta\in\mathcal{B}(B,D,n)\cap\mathcal{B}(B',D',n)$) then $(B,D,n)\sim(B',D',n)$ by transitivity which means $B^{-1}D$ is equivalent to $B'^{-1}D'$ and $r(B,D,B',D')=0$. This is only true provided $(B,D)=(B',D')$. $\square$

The following statement and argument for the proof is essentially [21, Theorem 9.1].

**Theorem 2.5.9.** *If $n' \in \mathbb{Z}$ and $(B', D') \in \underline{\mathcal{W}}$ then*

(i) *if $(B', D') \in \underline{\mathcal{W}}(s)$ there are $\dim_k(F_{B',D',n'}(N))$ elements in $\sigma \in \mathcal{S}(B', D', n')$, and*

(ii) *if $(B', D') \in \underline{\mathcal{W}}(b)$ then $F_{B',D',n'}(N) \simeq (\bigoplus_{\beta_+} V_+^{\beta+}) \oplus (\bigoplus_{\beta_-} V_-^{\beta-})$ where $\beta_\pm$ runs through $\mathcal{B}(B', D', n')^\pm$.*

*If $(B, D) \in \mathcal{W}$ and $C = B^{-1}D$ then*

(iii) *if $(B, D) \in \underline{\mathcal{W}}(s)$ there are $\sum_{n' \in \mathbb{Z}} \dim_k(F_{B,D,n'}(N))$ elements $\widehat{\sigma} \in \mathcal{S}$ where $A(\widehat{\sigma}) = C$ or $A(\widehat{\sigma}) = C^{-1}$, and*

(iv) *if $(B, D) \in \underline{\mathcal{W}}(b)$ then $\bigoplus_{n' \in \mathbb{Z}} F_{B,D,n'}(N) \simeq (\bigoplus_{\widehat{\beta}_+} V_+^{\widehat{\beta}+}) \oplus (\bigoplus_{\widehat{\beta}_-} V_-^{\widehat{\beta}-})$ where $\widehat{\beta}_\pm$ runs through all $\widehat{\beta} \in \mathcal{B}$ where $E(\widehat{\beta})$ is a shift of $C^{\pm 1}$.*

*Proof.* By lemma 2.1.21 (iv) and (v), for any $(B, D, n) \in \mathcal{I}$ the refined functor $F_{B,D,n}$ preserves small coproducts.

(i), (ii) By lemma 2.1.21 (iv) and (v) the refined functor $F_{B',D',n'}$ preserves small coproducts. This together with lemma 2.3.20 (iii) shows $F_{B',D',n'}(N) \simeq \bigoplus_{\sigma \in \mathcal{S}} F_{B',D',n'}(P(A(\sigma))[-t(\sigma)])$ as $B'^{-1}D'$ is not periodic. If $\sigma \in \mathcal{S}(B', D', n')$ then $(A(\sigma, 1), A(\sigma, -1), t(\sigma)) \sim (B', D', n')$ and so by corollaries 2.2.8 and 2.2.24 we have $F_{B',D',n'}(P(A(\sigma))[-t(\sigma)]) \simeq \bar{F}_{B',D',n'}(S_{B',D',n'}(R))$ which is isomorphic to $R \otimes_k k \simeq k$ by lemma 2.3.20 (ii). Otherwise $\sigma \notin \mathcal{S}(B, D, n)$ and so as above $F_{B,D,n}(P(A(\sigma))[-t(\sigma)]) = 0$ by lemma 2.3.20 (iii). Altogether $F_{B,D,n}(N) \simeq \bigoplus_{\sigma \in \mathcal{S}(B,D,n)} k$ which has dimension $\#\mathcal{S}(B, D, n)$.

For (ii), as above by lemmas 2.1.21 and 2.3.21 (iii) we have $F_{B',D',n'}(N) \simeq \bigoplus_{\sigma \in \mathcal{B}} F_{B',D',n'}(P(E(\beta), V^\beta)[-s(\beta)])$. If $\beta \in \mathcal{B}(B, D, n)^\pm$ then by corollaries 2.2.12 and 2.2.24 and lemma 2.3.21 (ii) we have $F_{B',D',n'}(P(E(\beta), V^\beta)[-s(\beta)]) \simeq \bar{V}_\pm^\beta$ as above. If $\beta \notin \mathcal{B}(B, D, n)^+ \cup \mathcal{B}(B, D, n)^-$ then $F_{B',D',n'}(P(E(\beta), V^\beta)[-s(\beta)]) = 0$ by lemma 2.3.21 (iii). As above this shows $F_{B',D',n'}(N) \simeq (\bigoplus_{\beta_+} \bar{V}_+^{\beta+}) \oplus (\bigoplus_{\beta_-} \bar{V}_-^{\beta-})$ where $\beta_\pm$ runs through $\mathcal{B}(B', D', n')^\pm$.

(iii), (iv) By lemma 2.5.8 (i) $\sum_{n\in\mathbb{Z}} \#\,\mathcal{S}(B,D,n) = \#\bigcup_{n\in\mathbb{Z}} \mathcal{S}(B,D,n)$ and so $\sum_{n\in\mathbb{Z}} \dim(F_{B,D,n}(N)) = \#\{\widehat{\sigma} \in \mathcal{S} \mid B^{-1}D \sim A(\widehat{\sigma})\}$ by part (i) and lemma 2.5.8 (ii). The proof for part (iv) is similar, but one uses part (ii) and lemma 2.5.8 (i) and (iii). $\square$

## 2.6 Completing the Proof.

In section 2.6 we complete the proofs of the main results in this thesis: theorem 2.0.1, theorem 2.0.4 and theorem 2.0.5. After this section we see some applications of these theorems.

**Definition 2.6.1.** (Full, Faithful, Dense, Representation Equivalence) If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, we say: $F$ *reflects isomorphisms* if, for each arrow $\alpha$ from $\mathcal{A}$, if $F(\alpha)$ is an isomorphism then $\alpha$ is an isomorphism; $F$ is *full* if, for any objects $X$, $Y$ in $\mathcal{A}$ and any arrow $\varphi : F(X) \to F(Y)$ in $\mathcal{B}$, there is an arrow $\beta : X \to Y$ in $\mathcal{A}$ for which $F(\beta) = \varphi$; $F$ is *dense* if given any object $Z$ in $\mathcal{B}$, there is an object $X$ in $\mathcal{A}$ for which $F(X) \simeq Z$; and $F$ is a *representation equivalence* if it is full, dense, and reflects isomorphisms.

**Proposition 2.6.2.** *Let $\mathcal{A}$ be a full subcategory of $R$-**Proj** and $\overline{\mathcal{A}}$ be the full subcategory of $k$-**Mod** consisting of all vector spaces isomorphic to $k \otimes_R M$ for some $R$-module $M$ in $\mathcal{A}$. Then $k \otimes_R - : R$-**Proj** $\to k$-**Mod** restricts to a representation equivalence $k \otimes_R -| : \mathcal{A} \to \overline{\mathcal{A}}$.*

*Proof.* By construction $k \otimes_R -|$ is dense. For the duration of the proof we let 0 and 1 (resp. $\overline{0}$ and $\overline{1}$) denote the additive and multiplicative identity elements in $R$ (resp. $k$). Choose a transversal $S$ such that $S \cap \overline{1} = \{1\}$ and $S \cap \overline{0} = \{0\}$. Let $\{m_i\}_{i\in I}$ and $\{n_j\}_{j\in J}$ be $R$-bases for objects $M$ and $N$ from $\mathcal{C}$ respectively, and for each $i$ and $j$ let $\{\overline{m}_i\}_i$ and $\{\overline{n}_j\}_j$ be the $k$-bases of $M$ and $N$ defined by $\overline{m}_i = \overline{1} \otimes m_i$ and $\overline{n}_j = \overline{1} \otimes n_j$.

Now let $g : k \otimes_R N \to k \otimes_R M$ be an arbitrary $k$-linear map. For each $j$ let $g(\overline{n}_j) = \sum_i \overline{t}_{ji}\overline{m}_i$ where $\overline{t}_{ji} \in k$ and $\overline{t}_{ji} = \overline{0}$ for all but finitely many $i$. Since $S \cap \overline{0} = \{0\}$ this means $t_{ji} = 0$ for all but finitely many $i$, where $t_{ij} \in R$ is chosen such that $\{t_{ij}\} = S \cap \overline{t}_{ij}$ for each $i$ and $j$.

Let $\beta : N \to M$ be the $R$-module homomorphism given by $\beta(n_j) = \sum_i t_{ji}m_i$ for each $j$. Since $M$ and $N$ are objects in $\mathcal{A}$, which is a full subcategory of $R$-**Proj**, $\beta$ is an arrow in $\mathcal{A}$. By definition $k \otimes_R -|\,(\beta) = g$ and so $k \otimes_R -|$ is full.

It remains to show $k \otimes_R -|$ reflects isomorphisms. Recall (from definition 3.1.38) that, for a ring $\Gamma$ with a complete set of idempotents $\{\mathrm{e}_i\}_i$, a $\Gamma$-module is quasi-free if it is a direct sum of modules of the form $R\mathrm{e}_i$. Hence an $R$-module is finitely generated and quasi-free iff it is free and of finite rank. If $M$ is any free $R$-module then $\overline{M} \simeq k \otimes_R M$ where $\overline{M} = M/\mathrm{rad}(M)$.

Hence, by lemma 3.1.39 it suffices to: let $f : R^m \to R^n$ be a homomorphism of finitely generated free $R$-modules; assume the map $\bar{f} : k^m \to k^n$ is an isomorphism; and show $f$ must have been an isomorphism. Let $A = (a_{ij})$ be the $n \times m$ matrix with entries from $R$ that defines $f$.

Write $\bar{A} = (\bar{a}_{ij})$ for the $n \times m$ matrix with entries from $k$ where $\bar{a}_{ij} = a_{ij} + \mathfrak{m}$ for each $i$ and $j$. Since $\bar{f}$ is an isomorphism we have $n = m$, and the determinant $\det(\bar{A})$ is a non-zero element of the field $k$. This means $\det(A)$ is an element of the ring $R$ which lies outside the maximal ideal $\mathfrak{m}$. This means $\det(A)$ is a unit in $R$, and so $A$ has an inverse in the matrix ring over $R$, and therefore $f$ is an isomorphism. $\square$

**Definition 2.6.3.** (Preserving Small Coproducts) If $\mathcal{A}$ and $\mathcal{B}$ have *small* (that is, set indexed) coproducts, we say a functor $F : \mathcal{A} \to \mathcal{B}$ *preserves small coproducts* if, for each collection $\{X_j\}_{j\in\mathcal{J}}$ of objects in $\mathcal{A}$, there are isomorphisms $\sigma_X : F(\bigoplus_{j\in\mathcal{J}} X_j) \to \bigoplus_{j\in\mathcal{J}} F(X_j)$, such that $\sigma_Y F(\bigoplus_{j\in\mathcal{J}} f_j) = (\bigoplus_{j\in\mathcal{J}} F(f_j))\sigma_X$ for each collection of arrows $\{f_j : X_j \to Y_j\}_{j\in\mathcal{J}}$ in $\mathcal{A}$.

We now fix some notation until the end of the proof of lemma 2.6.5. The reader is referred to the appendix for various definitions and notation. The next definition appears to be new, although it was motivated directly from parts (a), (b), (c) and (d) of lemma 1.4.38.

**Definition 2.6.4.** Let $\mathfrak{M}$ be an abelian category with small coproducts, and let $\mathfrak{N}$ be an abelian subcategory. Suppose $\mathfrak{M}$ and $\mathfrak{N}$ have all of their radicals (see 3.1.8). Recall (3.2.20) $\mathcal{P}_{\mathfrak{M}}$ (resp. $\mathcal{P}_{\mathfrak{N}}$) is the full subcategory of $\mathfrak{M}$ (resp. $\mathfrak{N}$) consisting of the projective objects. Recall (3.2.24) $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{M}})$, $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}})$, $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{N}})$ and $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$ are the full subcategories of $\mathcal{C}(\mathcal{P}_{\mathfrak{M}})$, $\mathcal{K}(\mathcal{P}_{\mathfrak{M}})$, $\mathcal{C}(\mathcal{P}_{\mathfrak{N}})$ and $\mathcal{K}(\mathcal{P}_{\mathfrak{N}})$ consisting of homotopically minimal complexes.

Recall $\Xi : \mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{M}}) \to \mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}})$ is the restriction of the canonical quotient functor $\mathcal{C}(\mathfrak{M}) \to \mathcal{K}(\mathfrak{M})$ (see [64, p.370, Proposition 10.1.2]). Let $\mathscr{I}$ be an index set. For each $i \in \mathscr{I}$ let $\mathfrak{A}_i$ and $\mathfrak{X}_i$ be additive categories with arbitrary coproducts, and let $S_i : \mathfrak{A}_i \to \mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{M}})$ and $F_i : \mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}}) \to \mathfrak{X}_i$ be functors. So far this gives

$$\mathfrak{A}_i \xrightarrow{\quad S_i \quad} \mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{M}}) \xrightarrow{\Xi} \mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}}) \xrightarrow{\quad F_i \quad} \mathfrak{X}_i$$

(Detecting Functors) We say that the collection of functors $\{(S_i, F_i) \mid i \in \mathscr{I}\}$ *detects the objects in* $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$ if, for each $i \in \mathscr{I}$:

(FFI) the functor $F_i \Xi S_i$ is a representation equivalence;

(FFII) $F_j \Xi S_i \simeq 0$ for each $j \in \mathscr{I}$ with $j \neq i$;

(FFIII) $F_i$ preserves small coproducts;

(FFIV) for every object $M$ in $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$ there is an object $A_{i,M}$ in $\mathfrak{A}_i$ and a map $\gamma_{i,M} : \Xi(S_i(A_{i,M})) \to M$ in $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}})$ such that $F_i(\gamma_{i,M})$ is an isomorphism;

and given a morphism $\theta : N \to M$ in $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{M}})$;

(FFV) if $M$ lies in $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{N}})$ and $F_i(\Xi(\theta))$ is epic for all $i \in \mathcal{I}$ then $\theta^n$ is epic for all $n$;

(FFVI) and if $N = \bigoplus_{i \in \mathscr{I}} S_i(A_i)$ for $A_i$ in $\mathfrak{A}_i$ and $F_i(\Xi(\theta))$ is monic for each $i \in \mathcal{I}$ then $\theta^n$ is monic for each $n$.

**Lemma 2.6.5.** *If $\{(S_i, F_i) \mid i \in \mathscr{I}\}$ detects the objects in $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$ then:*

(i) *any object $M$ of $\mathcal{K}(\mathcal{P}_{\mathfrak{N}})$ is isomorphic to $\bigoplus_{i \in \mathscr{I}} \Xi(S_i(A_{i,M}))$;*

(ii) *the indecomposable objects in $\mathcal{K}(\mathcal{P}_{\mathfrak{N}})$ are of the form $\Xi(S_i(A))$ where $i \in \mathscr{I}$ and $A$ is an indecomposable object in $\mathfrak{A}_i$;*

(iii) *if $A$ is an indecomposable object in $\mathfrak{A}_i$ (for some $i \in \mathscr{I}$) then $\Xi(S_i(A))$ is an indecomposable object in $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}})$; and*

(iv) *for $i, j \in \mathscr{I}$ and non-zero objects $A$ and $A'$ from $\mathfrak{A}_i$ and $\mathfrak{A}_j$ we have $\Xi(S_i(A)) \simeq \Xi(S_j(A'))$ iff $i = j$ and $A \simeq A'$.*

*Proof.* Note that by corollary 3.2.25 it suffices to prove that any object $M$ of $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$ is isomorphic to $\bigoplus_{i\in\mathscr{I}}\Xi(S_i(A_{i,M}))$. To avoid repetition we now fix some notation for the duration of the proof. Any direct sum $M\oplus N$ of objects $M$ and $N$ in an additive category $\mathcal{C}$ comes equipped with monomorphisms $\iota_M : M\to M\oplus N$ and $\iota_N : Y\to M\oplus N$, and epimorphisms $\pi_M : M\oplus N\to M$ and $\pi_N : M\oplus N\to N$, which satisfy $\iota_M\pi_M+\iota_N\pi_N=\mathrm{id}_{M\oplus N}$, $\pi_M\iota_M=\mathrm{id}_M$, $\pi_N\iota_N=\mathrm{id}_N$, and $\pi_M\iota_N=\pi_N\iota_M=0$.

(i) For each $l\in\mathscr{I}$ let $\iota_l : N_l\to\bigoplus_{i\in\mathscr{I}}N_i$ be the canonical monomorphism of the coproduct, where $N_i=\Xi(S_i(A_{i,M}))$ for each $i$ (which exist by (FFIV)). Consider the collection of arrows $\{f_i : X_i\to Y_i\}_i$ in $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}})$ given by: $X_l=N_l$ and $X_i=0$ for $i\neq l$; $Y_i=N_i$ for all $i$; and $f_l=\mathrm{id}$ and $f_i=0$ for $i\neq l$. By (FFII) and (FFIII) (in the notation of definition 2.6.3) there are isomorphisms $\sigma_X : F_l(N_l)\to F_l(N_l)$ and $\sigma_Y : F_l(\bigoplus_{i\in\mathscr{I}}N_i)\to F_l(N_l)$ such that $\sigma_Y F_l(\iota_l)=F_l(\mathrm{id})\sigma_X$ and hence $F_l(\iota_l)$ is an isomorphism. Let $\pi_l : \bigoplus_{i\in\mathscr{I}}N_i\to N_l$ be the canonical monomorphism of the product. Since $\pi_l\iota_l=\mathrm{id}$ we have that $F_l(\pi_l)$ is the inverse to $F_l(\iota_l)$, and hence an isomorphism.

Applying (FFIV) and the universal property of the coproduct defines a unique map $\theta : \Xi(\bigoplus_{i\in\mathscr{I}}S_i(A_{i,M}))\to M$ satisfying $\theta\iota_i=\gamma_{i,M}$ for each $i\in\mathscr{I}$. Since $\Xi$ is dense there is an object $L$ in $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{N}})$ and an isomorphism $\psi : \Xi(L)\to M$ in $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$ . Since $\Xi$ is full we have a morphism $\varphi : \bigoplus_{i\in\mathcal{I}}S_i(A_{i,M})\to L$ in $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{N}})$ for which $\Xi(\varphi)=\psi^{-1}\theta$. By the above we have $F_i(\theta)=F_i(\gamma_{i,M})F_i(\pi_i)$ which is an isomorphism for each $i\in\mathcal{I}$, and so $F_i(\Xi(\varphi))$ is an isomorphism for each $i\in\mathscr{I}$. By (FFV) and (FFVI) this means $\varphi^n$ is an isomorphism for each $n\in\mathbb{Z}$ and so $\theta$ is an isomorphism. Clearly $\Xi$ preserves small coproducts and so this means $M\simeq\bigoplus_{i\in\mathscr{I}}\Xi(S_i(A_{i,M}))$ as required.

(ii) Again by corollary 3.2.25 it suffices to prove the corresponding statement about $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$. We let $M$ be an indecomposable object of $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{N}})$. By part (i) we have that $M\simeq\bigoplus_{i\in\mathscr{I}}\Xi(S_i(A_{i,M}))$ and so $\Xi(S_i(A_{i,M}))=0$ apart from when $i=t$ for some $t\in\mathscr{I}$. Hence $M\simeq\Xi(S_t(A_{t,M}))$. It suffices to prove that $A_{t,M}$ is an indecomposable. Suppose there are objects $X$ and $Y$ of $\mathfrak{A}_t$ for which $A_{t,M}=X\oplus Y$. This shows $M\simeq\Xi(S_t(X))\oplus\Xi(S_t(Y))$ and so $\Xi(S_t(X))=0$ without loss of generality. This means $F_t(\Xi(S_t(\mathbf{0})))$ is an isomorphism where $\mathbf{0} : X\to 0$ in $\mathfrak{A}_t$. Since $F_t\Xi S_t$ reflects isomorphisms by (FFI), $X=0$.

(iii) If $F_i(\Xi(S_i(A)) = 0$ then $A = 0$ since $F_i\Xi S_i$ is dense and reflects isomorphisms by (FFI). Hence $\Xi(S_i(A)) \neq 0$, and we suppose $\Xi(S_i(A)) = X' \oplus Y'$ for objects $X'$ and $Y'$ of $\mathcal{K}_{\min}(\mathcal{P}_{\mathfrak{M}})$. Since $\Xi$ is the quotient functor there must be objects $X$ and $Y$ in $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{M}})$ such that $S_i(A) = X \oplus Y$, $\Xi(X) = X'$ and $\Xi(Y) = Y'$. Hence $F_i(\Xi(S_i(A))) \simeq F_i(X') \oplus F_i(Y')$ and as representation equivalences preserve indecomposables we have $F_i(X') = 0$ without loss of generality.

For $j \in \mathscr{I}$ with $j \neq i$ we have $F_j(X') \oplus F_j(Y') = F_j(\Xi(S_i(A))) = 0$ by (FFII). This means that for any $j \in \mathscr{I}$ we have that $F_j(\Xi(\iota_X)) = 0$ and so

$$\begin{aligned}\mathrm{id}_{F_j(\Xi(Y))} = F_j(\Xi(\mathrm{id}_Y)) = F_j(\Xi(\pi_Y))F_j(\Xi(\iota_Y)), \text{ and } \mathrm{id}_{F_j(\Xi(S_i(A)))} = F_j(\Xi(\mathrm{id}_{X\oplus Y}))\\ = F_j(\Xi(\iota_X))F_j(\Xi(\pi_X)) + F_j(\Xi(\iota_Y))F_j(\Xi(\pi_Y)) = F_j(\Xi(\iota_Y))F_j(\Xi(\pi_Y)).\end{aligned}$$

This means $F_j(\Xi(\pi_Y))$ is an isomorphism with inverse $F_j(\Xi(\iota_Y))$. Since $\pi_Y$ is an arrow in $\mathcal{C}_{\min}(\mathcal{P}_{\mathfrak{M}})$ of the form $S_i \to Y$ this means $\pi_Y^n$ is a monomorphism for each $n \in \mathbb{Z}$ by (FFVI). Since $\pi_Y^n$ is split epic for each $n \in \mathbb{Z}$ this means $\pi_Y^n$ is an isomorphism for each $n \in \mathbb{Z}$, and so $\pi_Y$ is an isomorphism. Hence we have that $X' = 0$.

(iv) It is clear that if $i = j$ and $A \simeq A'$ then $S_i(A) \simeq S_j(A')$ since functors preserve isomorphisms. Suppose now $S_i(A) \simeq S_j(A')$ for some $i, j \in \mathscr{I}$ and objects $A$ of $\mathfrak{A}_i$ and $A'$ of $\mathfrak{A}_j$. If $i \neq j$ then $F_j(\Xi(S_j(A'))) \simeq F_j(\Xi(S_i(A))) = 0$ and so $A' = 0$ which is a contradiction. Hence $i = j$ and $F_i(\Xi(S_i(A))) \simeq F_j(\Xi(S_j(A')))$ and so $A \simeq A'$ as required. $\square$

Let us now verify the hypotheses of lemma 2.6.5 in our setting. Recall that if $(B, D, n) \in \mathcal{I}$ then the constructive functors $S_{B,D,n}$ have the form

$$\begin{array}{lr} R\text{-}\mathbf{Proj} \longrightarrow \mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj}) & (\text{if } (B, D, n) \in \mathcal{I}(s)) \\ R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}} \longrightarrow \mathcal{C}_{\min}(\Lambda\text{-}\mathbf{Proj}) & (\text{if } (B, D, n) \in \mathcal{I}(b)) \end{array}$$

and the refined functors $F_{B,D,n}$ have the form

$$\begin{array}{lll} \mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \longrightarrow k\text{-}\mathbf{Mod} & & (\text{if } (B,D,n)\in\mathcal{I}(s)) \\ \mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj}) \longrightarrow k[T,T^{-1}]\text{-}\mathbf{Mod} & & (\text{if } (B,D,n)\in\mathcal{I}(b)) \end{array}$$

The following proposition is analogous to [15, p.163, Proposition].

**Proposition 2.6.6.** *Let* $\mathfrak{M}=\Lambda\text{-}\mathbf{Mod}$, $\mathfrak{N}=\Lambda\text{-}\mathbf{mod}$ *and* $\mathscr{I}=\mathcal{I}=\mathcal{I}(s)\sqcup\mathcal{I}(b)$*; and for* $i=(B,D,n)\in\mathcal{I}$ *let*

$$(\mathfrak{A}_i,\mathfrak{X}_i)=\begin{cases} (R\text{-}\mathbf{Proj},k\text{-}\mathbf{Mod}) & (\textit{if } B^{-1}D \textit{ is not periodic}) \\ (R[T,T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}},k[T,T^{-1}]\text{-}\mathbf{Mod}) & (\textit{if } B^{-1}D \textit{ is periodic}) \end{cases}$$

*Then the collection* $\{(S_{B,D,n},F_{B,D,n})\mid(B,D,n)\in\mathcal{I}\}$ *detects the objects in* $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$.

*Proof.* (FFI) Suppose $(B,D,n)$ lies in $\mathcal{I}(s)$ (resp. $\mathcal{I}(b)$). Then by lemma 2.3.20 (ii) (resp. lemma 2.3.21 (ii)) the functor $F_{B,D,n}\Xi S_{B,D,n}$ is naturally isomorphic to $k\otimes_R-:R\text{-}\mathbf{Proj}\to k\text{-}\mathbf{Mod}$ (resp. $k\otimes_{R[T,T^{-1}]}-:R[T,T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}\to k[T,T^{-1}]\text{-}\mathbf{Mod}$). By proposition 2.6.2 this functor is a representation equivalence.

(FFII) For distinct $(B,D,n),(B',D',n')\in\mathcal{I}$ we have $(B,D,n)\nsim(B',D',n')$ since $\mathcal{I}$ was defined by taking representatives of equivalence classes. If $(B,D,n)$ lies in $\mathcal{I}(s)$ then for any free $R$-module $V$ we have $\bar{F}_{B',D',n'}(P(C)[\mu_C(a_{B,D})-n]\otimes_R V)=0$ by lemma 2.3.20 (iii) where $C=B^{-1}D$. This shows $F_{B',D',n'}\Xi S_{B,D,n}=0$ since $\bar{F}_{B',D',n'}$ and $F_{B',D',n'}$ are naturally isomorphic by corollary 2.2.12. If $(B,D,n)$ lies in $\mathcal{I}(b)$ then similarly, using lemma 2.3.21 (iii), this shows $F_{B',D',n'}\Xi S_{B,D,n}=0$.

(FFIII) Fix a collection $\{X_j\}_{j\in\mathcal{J}}$ of objects in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj})$. By lemma 2.1.21 (iv) and (v), for any homotopy $I$-word $C$ with $I\subseteq\mathbb{N}$ we have $C^{\pm}(\bigoplus_j X_j)=\bigoplus_j C^{\pm}(X_j)$. Hence any $x\in F^{+}_{B,D,n}(\bigoplus_{j\in\mathcal{J}}X_j)$ lies in $B^{+}(\bigoplus_j X_j)=\bigoplus_j B^{+}(X_j)$ and $D^{+}(\bigoplus_j X_j)=\bigoplus_j D^{+}(X_j)$. Letting $x=\sum_j x_j$ where $x_j\in X_j$ shows $x_j\in B^{+}(X_j)\cap D^{+}(X_j)$. So we have $F^{+}_{B,D,n}(\bigoplus_{j\in\mathcal{J}}X_j)=\bigoplus_{j\in\mathcal{J}}F^{+}_{B,D,n}(X_j)$, and similarly we can show $F^{-}_{B,D,n}(\bigoplus_{j\in\mathcal{J}}X_j)=\bigoplus_{j\in\mathcal{J}}F^{-}_{B,D,n}(X_j)$.

Hence the map $\sigma_X : F_{B,D,n}(\bigoplus_{j\in\mathcal{J}} X_j) \to \bigoplus_{j\in\mathcal{J}} F_{B,D,n}(X_j)$ defined by sending any $x + F^-_{B,D,n}(\bigoplus_{j\in\mathcal{J}} X_j) \in F_{B,D,n}(\bigoplus_{j\in\mathcal{J}} X_j)$ to $\sum_j x_j + F^-_{B,D,n}(X_j)$ is well-defined and bijective. Clearly this map is $R$-linear (and hence $k$-linear). Now let $\{f_j : X_j \to Y_j \mid j \in \mathcal{J}\}$ be a collection of arrows in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj})$. By definition we have

$$\sigma_Y(F_{B,D,n}(\bigoplus_j f_j)(x + F^-_{B,D,n}(\bigoplus_{j\in\mathcal{J}} X_j))) = \sigma_Y(\sum_j f_j(x_j) + F^-_{B,D,n}(X))$$
$$= \sum_j (f_j(x_j) + F^-_{B,D,n}(X_j)) = (\bigoplus_{j\in\mathcal{J}} F_{B,D,n}(f_j))(\sigma_X(x + F^-_{B,D,n}(\bigoplus_{j\in\mathcal{J}} X_j)))$$

(FFIV) Let $M$ be an object in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$. By corollary 2.2.9 $F_{B,D,n}(M)$ is a finite-dimensional $k$-vector space. If $(B, D, n)$ lies in $\mathcal{I}(s)$, by lemma 2.5.2 there is a free $R$-module $V = R^{\dim(F_{B,D,n}(M))}$ and a morphism $\theta_{B,D,n} : S_{B,D,n}(V) \to M$ for which $F_{B,D,n}(\theta_{B,D,n})$ is an isomorphism. Similarly if $(B, D, n)$ lies in $\mathcal{I}(b)$ then by lemma 2.5.4 there is an object $V$ of $R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$ and a morphism $\theta_{B,D,n} : S_{B,D,n}(V) \to M$ for which $F_{B,D,n}(\theta_{B,D,n})$ is an isomorphism.

(FFV), (FFVI) Let $\theta : N \to M$ be an arrow in the category $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{Proj})$. If $M$ is a complex in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$ and $F_{B,D,n}(\theta)$ is epic for all $(B, D, n) \in \mathcal{I}$ then $\theta^n$ is epic for each $n \in \mathbb{N}$ by lemma 2.5.5. If instead $N$ is a direct sum of string and band complexes $F_{B,D,n}(\theta)$ is monic for all $(B, D, n) \in \mathcal{I}$ then $\theta^n$ is monic for each $n \in \mathbb{N}$ by lemma 2.5.6. □

*Proof of theorems 2.0.1 and 2.0.4.* Parts (i), (ii) and (iii) of theorem 2.0.1 are precisely parts (i), (ii) and (iii) of lemma 2.6.5, after applying proposition 2.6.6. Theorem 2.0.4 similarly follows by lemma 2.6.5 (iv), definitions 2.2.16 and 2.2.18. □

*Proof of theorem 2.0.5.* Let $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{B}$ and $\mathcal{B}'$ be index sets.

For each $\sigma \in \mathcal{S}$ and $\sigma' \in \mathcal{S}'$ let $t(\sigma)$ and $u(\sigma')$ be integers and let $A(\sigma)$ and $B(\sigma)$ be homotopy words which are not periodic homotopy $\mathbb{Z}$-words. For each $\beta \in \mathcal{B}$ and $\beta' \in \mathcal{B}'$ let $s(\beta)$ and $r(\beta')$ be integers, let $E(\beta)$ and $D(\beta)$ be periodic homotopy $\mathbb{Z}$-words and let $V^\beta$ and $U^{\beta'}$ be indecomposable objects in $R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{Proj}}$.

Now suppose $N \simeq N'$ in the category $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$ where

$$N = \left(\bigoplus_{\sigma\in\mathcal{S}} P(A(\sigma))[-t(\sigma)]\right) \oplus \left(\bigoplus_{\beta\in\mathcal{B}} P(E(\beta), V^{\beta})[-s(\beta)]\right)$$

and

$$N' = \left(\bigoplus_{\sigma'\in\mathcal{S}'} P(B(\sigma'))[-u(\sigma')]\right) \oplus \left(\bigoplus_{\beta'\in\mathcal{B}'} P(D(\beta'), U^{\beta'})[-r(\beta')]\right)$$

For any $(B, D, n) \in \mathcal{I}$ we have $F_{B,D,n}(N) \simeq F_{B,D,n}(N')$ which are finite-dimensional as $k$-vector spaces by corollary 2.2.9. Recall that for any word $C \in \mathcal{S} \cup \mathcal{S}'$ we let $C(\sigma, \delta)$ be the unique word in $\{(C(\sigma)_{\leq 0})^{-1}, C(\sigma)_{0<}\}$ with sign $\delta \in \{\pm 1\}$. For $(B, D, n) \in \mathcal{I}(s)$ recall $\mathcal{S}(B, D, n)$ (resp. $\mathcal{S}'(B, D, n)$) is the set of $\sigma \in \mathcal{S}$ (resp. $\sigma' \in \mathcal{S}'$) such that $(B, D, n)$ is equivalent to $((A(\sigma, 1), A(\sigma, -1), t(\sigma))$ (resp. $((B(\sigma', 1), B(\sigma', -1), u(\sigma')))$.

For $(B, D, n) \in \mathcal{I}(b)$ recall $\mathcal{B}^{\pm}(B, D, n)$ (resp. $\mathcal{B}'^{\pm}(B, D, n)$) is the set of $\beta \in \mathcal{B}$ (resp. $\beta' \in \mathcal{B}'$) such that $s(\beta) - n = \mu_{B^{-1}D}(\pm m)$ (resp. $r(\beta') - n = \mu_{B^{-1}D}(\pm m)$ and $D(\beta') = (B^{-1}D)^{\pm 1}[m]$) for some $m \in \mathbb{Z}$. For $\beta \in \mathcal{B}(B, D, n)^{+}$ (resp. $\beta' \in \mathcal{B}'(B, D, n)^{+}$) recall $\bar{V}^{\beta}_{+} = k \otimes_{R[T,T^{-1}]} V^{\beta}$ (resp. $\bar{U}^{\beta'}_{+} = k \otimes_{R[T,T^{-1}]} U^{\beta'}$). For $\beta \in \mathcal{B}(B, D, n)^{-}$ (resp. $\beta' \in \mathcal{B}'(B, D, n)^{-}$) recall $\bar{V}^{\beta}_{-} = k \otimes_{R[T,T^{-1}]} \operatorname{res}_{\iota} V^{\beta}$ (resp. $\bar{U}^{\beta'}_{-} = k \otimes_{R[T,T^{-1}]} \operatorname{res}_{\iota} U^{\beta'}$) and recall this is defined by swapping the actions of $T$ and $T^{-1}$.

For any $(B, D, n) \in \mathcal{I}$ we let $C = B^{-1}D$ and define a function $\varphi_{B,D,n}$ as follows. If $(B, D, n) \in \mathcal{I}(s)$ then as $F_{B,D,n}(N) \simeq F_{B,D,n}(N')$ we have that $\dim(F_{B,D,n}(N)) = \dim(F_{B,D,n}(N'))$ and so $\#\mathcal{S}(B, D, n) = \#\mathcal{S}'(B, D, n)$ by theorem 2.5.9 (i). Hence there exists a bijection $\mathcal{S}(B, D, n) \to \mathcal{S}'(B, D, n)$, and we let $\varphi^{s}_{B,D,n}$ be this bijection. Note that if $\varphi^{s}_{B,D,n}(\sigma) = \sigma'$ then $P(A(\sigma))[-t(\sigma)] \simeq P(C)[-n]$ and $P(B(\sigma'))[-u(\sigma')] \simeq P(C)[-n]$ and so $P(A(\sigma))[-t(\sigma)] \simeq P(B(\sigma'))[-u(\sigma')]$. Suppose instead $(B, D, n) \in \mathcal{I}(b)$. Then as $F_{B,D,n}(N) \simeq F_{B,D,n}(N')$, by theorem 2.5.9 (ii) we have that

$$(\bigoplus_{\beta_{+}} \bar{V}^{\beta_{+}}_{+}) \oplus (\bigoplus_{\beta_{-}} \bar{V}^{\beta_{-}}_{-}) \simeq (\bigoplus_{\beta'_{+}} \bar{U}^{\beta'_{+}}_{+}) \oplus (\bigoplus_{\beta'_{-}} \bar{U}^{\beta'_{-}}_{-})$$

where $\beta_{\pm}$ (resp. $\beta'_{\pm}$) runs through $\mathcal{B}(B, D, n)^{\pm}$ (resp. $\mathcal{B}'(B, D, n)^{\pm}$). Note this is a $k[T, T^{-1}]$-module isomorphism of indecomposable finite-dimensional modules.

Hence by Krull-Remak-Schmidt property for $k[T,T^{-1}]$-$\mathbf{Mod}_{k\text{-}\mathbf{mod}}$ there is a bijection

$$\varphi^b_{B,D,n} : \mathcal{B}(B,D,n)^- \cup \mathcal{B}(B,D,n)^+ \to \mathcal{B}'(B,D,n)^- \cup \mathcal{B}'(B,D,n)^+$$

which is isomorphism preserving between the summands of $F_{B,D,n}(N)$ and $F_{B,D,n}(N')$. For $\beta \in \mathcal{B}(B,D,n)^+$ (resp. $\beta' \in \mathcal{B}'(B,D,n)^+$) let $V^\beta_+ = V^\beta$ (resp. $U^{\beta'}_+ = U^{\beta'}$). For $\beta \in \mathcal{B}(B,D,n)^-$ (resp. $\beta' \in \mathcal{B}'(B,D,n)^-$) let $V^\beta_- = \mathrm{res}_\iota V^\beta$ (resp. $U^{\beta'}_- = \mathrm{res}_\iota U^{\beta'}$).

Note that if $\varphi^b_{B,D,n}(\beta_\delta) = \beta'_\delta$ (for some $\delta \in \{\pm\}$) then $P(E(\beta),V^\beta)[-s(\beta)] \simeq P(C,V^\beta_\delta)[-n]$, $P(D(\beta'),U^{\beta'})[-r(\beta')] \simeq P(C,U^{\beta'}_\delta)[-n]$ and $\bar{V}^\beta_\delta \simeq \bar{U}^{\beta'}_\delta$. Since $\bar{V}^\beta_\delta \simeq \bar{U}^{\beta'}_\delta$ we have $V^\beta_\delta \simeq U^{\beta'}_\delta$ by proposition 2.6.2, and so $P(E(\beta),V^\beta)[-s(\beta)] \simeq P(D(\beta'),U^{\beta'})[-r(\beta')]$. Suppose instead $\varphi^b_{B,D,n}(\beta_1) = \beta'_{-1}$. Then $P(E(\beta),V^\beta)[-s(\beta)] \simeq P(C,V^\beta)[-n]$, $P(D(\beta'),U^{\beta'})[-r(\beta')] \simeq P(C^{-1},U^{\beta'})[-n]$ and $\bar{V}_+ \simeq \bar{U}_-$. Since $\bar{V}^\beta_+ \simeq \bar{U}^{\beta'}_-$ we have by $V^\beta_+ \simeq U^{\beta'}_- = \mathrm{res}_\iota U^{\beta'}$ proposition 2.6.2. This gives $\mathrm{res}_\iota V^\beta \simeq U^{\beta'}$ and so

$$P(E(\beta),V^\beta)[-s(\beta)] \simeq P(C,V^\beta)[-n] \simeq P(C^{-1},\mathrm{res}_\iota V^\beta)[-n] \simeq P(D(\beta'),U^{\beta'})[-r(\beta')]$$

Similarly $P(E(\beta),V^\beta)[-s(\beta)] \simeq P(D(\beta'),U^{\beta'})[-r(\beta')]$ if $\varphi^b_{B,D,n}(\beta_{-1}) = \beta'_1$.

Define the function $\varphi : \mathcal{S} \cup \mathcal{B} \to \mathcal{S}' \cup \mathcal{B}'$ by setting $\varphi(\alpha) = \alpha'$ if ($\alpha \in \mathcal{S}(B,D,n)$ and $\alpha' = \varphi^s_{B,D,n}(\alpha)$) or ($\alpha \in \mathcal{B}(B,D,n)^+ \cup \mathcal{B}(B,D,n)^-$ and $\alpha' = \varphi^b_{B,D,n}(\alpha)$). By lemma 2.5.8 and by construction, $\varphi$ is a bijection which preserves isomorphism classes of the indecomposable complexes arising in the decompositions $N$ and $N'$. $\square$

## 2.7 Further Remarks.

### 2.7.1 Application: Derived Categories.

We now apply the classification above in section 2.7.1. Recall definition 3.3.15.

**Assumption:** In section 2.7.1 we consider a direct sum of string and band complexes of the form

$$N = \left(\bigoplus_{\sigma\in\mathcal{S}} P(A(\sigma))[-t(\sigma)]\right) \oplus \left(\bigoplus_{\beta\in\mathcal{B}} P(E(\beta), V^{\beta})[-s(\beta)]\right)$$

as in (✡) from section 2.5.2.

**Proposition 2.7.1.** *In the notation above, the following statements hold.*

*$N$ has finitely generated homogeneous components iff:*

(ia) *$A(\sigma)$ has controlled homogeny for each $\sigma$;*

(ib) *$V^{\beta}$ is finitely generated as an $R$-module for each $\beta$;*

*and for all $n \in \mathbb{Z}$,*

(ic) *there are finitely many $\sigma \in \mathcal{S}$ such that $P^{n+t(\sigma)}(A(\sigma)) \neq 0$, and*

(id) *there are finitely many $\beta \in \mathcal{B}$ such that $P^{n+s(\beta)}(E(\beta), V) \neq 0$.*

*$N$ is bounded above (resp. below) iff there exists $n \in \mathbb{Z}$ such that:*

(iia) *$\mu_{A(\sigma)}(j) + t(\sigma) \leq n$ (resp. $\mu_{A(\sigma)}(j) + t(\sigma) \geq n$) for each $\sigma \in \mathcal{S}$ and $j \in I_{A(\sigma)}$; and*

(iib) *$\mu_{E(\beta)}(j) + s_{\beta} \leq n$ (resp. $\mu_{E(\beta)}(j) + s(\beta) \geq n$) for each $\beta \in \mathcal{B}$ and $j \in \mathbb{Z}$.*

*Proof.* Suppose $N$ lies in $\mathcal{K}_{\min}(\Lambda\text{-}\mathbf{proj})$.

Suppose for a contradiction there is some $\sigma \in \mathcal{S}$ such that $A(\sigma)$ does not have controlled homogeny. If $A(\sigma)$ is an $I$-word then there is some $l \in \mathbb{Z}$ and a sequence $(i(n) \mid n \geq 0) \in I^{\mathbb{N}}$ such that $\mu_{A(\sigma)}(i(n)) = l$ for each $n$. Hence there is a chain of embeddings $\bigoplus_{n\geq 0} \Lambda \underline{b}_{i(n)} \to P^{l-t(\sigma)}(A(\sigma))[-t(\sigma)] \to N$.

Recall $\Lambda$ is a noetherian ring by corollary 1.1.25 (ii). Since $N^{l-t_\sigma}$ is a finitely generated $\Lambda$-module, $\bigoplus_n \Lambda \underline{b}_{i(n)}$ is also finitely generated. Suppose $\bigoplus_n \Lambda \underline{b}_{i(n)} = \sum_{m=1}^{d} \Lambda g_m$ for some $g_1, \dots, g_d \in \bigoplus_n \Lambda \underline{b}_{i(n)}$. For each $m$ we have $g_m = \sum_{n\geq 0}^{r(m)} \lambda_{m,n} \underline{b}_{i(n)}$ for some $r(1), \dots, r(d) \in \mathbb{N}$ and some $\lambda_{m,n} \in \Lambda$. This gives the contradiction $\underline{b}_{i(r)} \notin \bigoplus_n \Lambda \underline{b}_{i(n)}$ where $r = \max\{r(1), \dots, r(d)\} + 1$. Hence (ia) holds.

Similarly if some $V^\beta$ has an $R$-basis $\Omega$ which is infinite, then there is a chain of $\Lambda$-module monomorphisms $\bigoplus_{\lambda\in\Omega} \Lambda \underline{b}_0 \to P^{-s(\beta)}(E(\beta), V)[-s(\beta)] \to N^{-s(\beta)}$ which contradicts that $N^{-s(\beta)}$ is finitely generated. Finding this contradiction uses lemma 1.3.47. Hence each $V^\beta$ must have a finite $R$-basis. Hence (ib) holds.

For any $n \in \mathbb{Z}$, if $P(A(\sigma))[-t(\sigma)]^n \neq 0$ for infinitely many $\sigma \in \mathcal{S}$ then we can find a contradiction as we did for (ia). Similarly there must be finitely many $\beta \in \mathcal{B}$ such that $P(E(\beta))[-s(\beta)]^n \neq 0$: and so (ic) and (id) hold.

Now suppose (ia), (ib), (ic) and (id) hold and $n$ is an arbitrary intger. Note that

$$N^n = \left(\bigoplus_{\sigma\in\mathcal{S}^n} P^n(A(\sigma))[-t(\sigma)]\right) \oplus \left(\bigoplus_{\beta\in\mathcal{B}^n} P^n(E(\beta), V^\beta)[-s(\beta)]\right)$$

where we let $\mathcal{S}^n$ (resp. $\mathcal{B}^n$) be the set of $\sigma \in \mathcal{S}$ (resp. $\beta \in \mathcal{B}$) where $P^{n-t(\sigma)}(A(\sigma)) \neq 0$ (resp. $P^{n-s(\beta)}(E(\beta), V^\beta) \neq 0$). By (ia) (resp. (ib)) $\mathcal{S}^n$ (resp. $\mathcal{B}^n$) is a finite set, and for each $\sigma \in \mathcal{S}$ (resp. $\beta \in \mathcal{B}$) the $\Lambda$-module is finitely generated by (ic) (resp. (id) and lemma 1.3.47). It is straightforward to show that $N$ is bounded above (resp. below) iff conditions (iia) and (iib) hold. $\square$

Thus we have described when a complex $N$ (of the above form) lies in $\mathcal{K}^-(\Lambda\text{-}\mathbf{proj})$. Recall that $\mathcal{D}^-(\Lambda\text{-}\mathbf{mod}) \simeq \mathcal{K}^-(\Lambda\text{-}\mathbf{proj})$ by corollary 3.3.28.

**Corollary 2.7.2.** (Description of $\mathcal{D}^-(\Lambda\text{-}\mathbf{mod})$) *Any indecomposable object in* $\mathcal{K}^-(\Lambda\text{-}\mathbf{proj})$ *is isomorphic to a complex of the form*

(i) $P(A)[-t]$ *where $A$ has controlled homogeny and* $\mathrm{im}(\mu_A(i))$ *is bounded above; or*

(ii) $P(E, V)[-s]$ *where $V$ is indecomposable and lies in* $R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{proj}}$.

**Remark 2.7.3.** (FULL CYCLES OF ZERO-RELATIONS) Recall definition 1.5.28: that a *full cycle of zero-relations* $\alpha_1 \dots \alpha_n$ is a cycle in $Q$ which is not the product of shorter cycles in $Q$, and for which $\alpha_n \alpha_1 \in (\rho)$ and $\alpha_i \alpha_{i+1} \in (\rho)$ for each $i \in \{1, \dots, n\}$ such that $i + 1 \leq n$.

**Definition 2.7.4.** (INVERSE/DIRECT ACYCLIC) A homotopy $\mathbb{N}$-word $C$ is said to be *inverse acyclic* (resp. *direct acyclic*) if $C$ has the form $(\alpha_n^{-1} d_{\alpha_n} \dots \alpha_1^{-1} d_{\alpha_1})^\infty$ (resp. $(d_{\alpha_n}^{-1} \alpha_n \dots d_{\alpha_1}^{-1} \alpha_1)^\infty$) for some some full cycle $\alpha_1 \dots \alpha_n$ of zero-relations.

**Lemma 2.7.5.** *Suppose $N$ lies in $\mathcal{K}^-(\Lambda\text{-}\mathbf{proj})$. Then $N$ has bounded cohomology iff there is some $t > 0$ such that:*

(i) *for each $\beta \in \mathcal{B}$ and $j \in I_{E(\beta)}$ we have $\mu_{E(\beta)}(j) + s(\beta) \geq -t$;*

(ii) *for each $\sigma \in \mathcal{S}$ with $I_{A(\sigma)} \supseteq \mathbb{N}$ the word $A(\sigma)_{>t}$ is inverse acyclic; and*

(iii) *for each $\sigma \in \mathcal{S}$ with $I_{A(\sigma)} \supseteq -\mathbb{N}$ the word $(A(\sigma)_{<-t})^{-1}$ is inverse acyclic.*

We can now describe the indecomposable objects in the bounded derived category. As above, recall that $\mathcal{D}^b(\Lambda\text{-}\mathbf{mod}) \simeq \mathcal{K}^{b,-}(\Lambda\text{-}\mathbf{proj})$ by corollary 3.3.28. The following corollary generalises a classification due to Bekkert and Merklen (see theorem 1.5.29) to all complete gentle algebras.

**Corollary 2.7.6.** (DESCRIPTION OF $\mathcal{D}^b(\Lambda\text{-}\mathbf{mod})$) *Any indecomposable object in $\mathcal{K}^{-,b}(\Lambda\text{-}\mathbf{proj})$ is isomorphic to a complex of the form*

(i) *$P(A)[-t]$ where $I_A \supseteq \mathbb{N}$ (resp. $I_A \supseteq -\mathbb{N}$) implies $A_{>t}$ (resp. $(A_{<-t})^{-1}$) is inverse acyclic for some $t > 0$; or*

(ii) *$P(E, V)[-t]$ where $V$ is an indecomposable object in $R[T, T^{-1}]\text{-}\mathbf{Mod}_{R\text{-}\mathbf{proj}}$.*

### 2.7.2 Kernels in String Complexes.

Our aim in section 2.7.2 is to give a proof of lemma 2.7.5. To begin with we calculate the Kernel of the differential maps for string complexes.

**Definition 2.7.7.** (Kernel Part, notation: $\kappa(i)$) Let $C$ be a homotopy $I$-word. For each $i \in I$ we define the path $\kappa(i)$ by

(a) $\kappa(i) = e_{v_C(i)}$ if $(i-1 \notin I$ or $l_i^{-1}r_i = d_z^{-1}\tau)$ and $(i+1 \notin I$ or $l_{i+1}^{-1}r_{i+1} = \gamma^{-1}d_y)$,

(b) $\kappa(i) = \mathrm{f}(\tau)$ if $(i-1 \in I$ and $l_i^{-1}r_i = d_z^{-1}\tau)$ and $(i+1 \in I$ and $l_{i+1}^{-1}r_{i+1} = d_y^{-1}\gamma)$,

(c) $\kappa(i) = \mathrm{f}(\gamma)$ if $(i-1 \in I$ and $l_i^{-1}r_i = \tau^{-1}d_z)$ and $(i+1 \in I$ and $l_{i+1}^{-1}r_{i+1} = \gamma^{-1}d_y)$,

(d) $\kappa(i) = \beta$ if $(i-1 \notin I$ and $\beta y \notin \mathbf{P})$ and $(i+1 \in I$ and $l_{i+1}^{-1}r_{i+1} = d_y^{-1}\gamma)$,

(e) $\kappa(i) = \alpha$ if $(i-1 \in I$ and $l_i^{-1}r_i = \tau^{-1}d_z)$ and $(i+1 \notin I$ and $\alpha z \notin \mathbf{P})$, and

(f) $\kappa(i) = 0$ (if $i-1 \in I$ and $l_i^{-1}r_i = \tau^{-1}d_z)$ and $(i+1 \in I$ and $l_{i+1}^{-1}r_{i+1} = d_y^{-1}\gamma)$.

Note that for any $i \in I$ exactly one of the ((a), (b), (c), (d), (e) and (f)) is true.

(Full, Left/Right Arms, Left/Right Peripheral Arms) We say that the $i^{\text{th}}$ *kernel part* is: *full* in case (a); a *left* (resp. *right*) *arm* in case (b) (resp. (c)); a *left* (resp. *right*) *peripheral arm* in case (d) (resp. (e)); and 0 in case (f).

Our main result in section 2.7.2 is the following.

**Corollary 2.7.8.** *Let $C$ be a homotopy $I$-word. For any $n \in \mathbb{Z}$ we have* $\ker(d^n_{P(C)}) = \bigoplus_{i \in \mu_C^{-1}(n)} \Lambda\kappa(i)\underline{b}_i$.

Before we see a proof we recycle some examples.

**Example 2.7.9.** Recall the gentle algebra $kQ/(\rho)$ and the homotopy word $C$ with $[C] = [s][t][c^{-1}]$ from example 2.3.4.

The associated string complex $P(C)$ was depicted by

$$
\begin{tikzcd}
 & & & & & \underline{b}_2 \arrow[dl, "a"'] \arrow[dr, "r"] \arrow[dddll, bend right, "d_t"'] \arrow[dddrr, bend left, "d_c"] & & & \\
 & & & & a\underline{b}_2 & & r\underline{b}_2 & & \\
 & & \underline{b}_1 \arrow[dr, "t"'] \arrow[dddl, bend right, "d_s"'] & & & & & & \underline{b}_3 \arrow[dl, "c"'] \\
 & & & t\underline{b}_1 \arrow[dr, "a"'] & & & & c\underline{b}_3 \arrow[dl, "r"'] & \\
\underline{b}_0 \arrow[dr, "s"'] & & & & at\underline{b}_1 & & rc\underline{b}_3 & & \\
 & s\underline{b}_0 & & & & & & &
\end{tikzcd}
$$

By definition: $\kappa(0) = e_4$, $\kappa(1) = \mathrm{f}(t) = t$, $\kappa(2) = 0$ and $\kappa(3) = e_2$. This means that $0^{\text{th}}$ and $3^{\text{rd}}$ kernel parts are full, the $1^{\text{st}}$ is a right arm, and the $2^{\text{nd}}$ is 0. By corollary 2.7.8 we have $\ker(d^0_{P(C)}) = \Lambda\underline{b}_0$, $\ker(d^{-1}_{P(C)}) = \Lambda t\underline{b}_1 \oplus \Lambda\underline{b}_3$ and $\ker(d^{-2}_{P(C)}) = 0$.

**Example 2.7.10.** Recall the complete gentle algebra $k[[x,y]]/(xy)$ and the $\mathbb{N}$-word

$$C = x^{-2}d_xy^{-1}d_yx^{-2}d_xd_y^{-1}y^3d_x^{-1}xy^{-1}d_yx^{-2}d_xy^{-1}d_yx^{-2}d_x\dots$$

from example 1.3.28. For integers $i$ with $0 \leq i \leq 8$ we have $\kappa(0) = \kappa(5) = e_v$, $\kappa(1) = \kappa(4) = \kappa(7) = y$, $\kappa(2) = \kappa(6) = \kappa(8) = x$, and $\kappa(3) = 0$.

Hence

$$\begin{array}{cc}
\vdots & \vdots \\
\ker(d^{-5}_{P(C)}) = \Lambda y\underline{b}_9 & \ker(d^{-4}_{P(C)}) = \Lambda x\underline{b}_8 \\
\ker(d^{-3}_{P(C)}) = 0 \oplus \Lambda y\underline{b}_7 & \ker(d^{-2}_{P(C)}) = \Lambda x\underline{b}_2 \oplus \Lambda y\underline{b}_4 \oplus \Lambda x\underline{b}_6 \\
\ker(d^{-1}_{P(C)}) = \Lambda y\underline{b}_1 \oplus \Lambda\underline{b}_5 & \ker(d^{0}_{P(C)}) = \Lambda\underline{b}_0
\end{array}$$

**Remark 2.7.11.** The proof of corollary 2.7.8 will involve some technical results which use calculations from section 2.3. Let $C$ be a homotopy $I$-word, $i \in I$ and let $x$ be an arrow. For convenience we recall some notation introduced in definitions 1.2.16 and 2.3.3. The set $\mathbf{P}x^{-1}$ (resp. $x^{-1}\mathbf{P}$) consists of paths $p \in \mathbf{P}$ with $px \in \mathbf{P}$ (resp. $xp \in \mathbf{P}$). There is a canonical $R$-linear surjection $\pi_x : \bigoplus_{y\in\mathbf{A}(\rightarrow h(x))} y\Lambda \to x\Lambda$ given by $\pi_x(\sum_y y\lambda_y) = x\lambda_x$. The set $\mathbf{P}[i]$ consists of all non-trivial paths $\sigma \in \mathbf{P}$ with tail $v_C(i)$. The subset $\mathbf{P}[x, i]$ of $\mathbf{P}[i]$ consists of all $\sigma$ with $\mathrm{l}(\sigma) = x$. Fix $n \in \mathbb{Z}$. Any element of $P^n(C)$ can be written as

$$m = \textstyle\sum_{i\in\mu_C^{-1}(n)}(\eta_i\underline{b}_i + \sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i}\sigma\underline{b}_i) \text{ with scalars } \eta_i, r_{\sigma,i} \text{ from } R.$$

Since $\Lambda$ is point-wise rad-nilpotent modulo $\mathfrak{m}$ we have $\mathfrak{m}\Lambda e_v \subseteq \mathrm{rad}(\Lambda e_v)$ for each vertex $v$. So we can assume each $\eta_i$ lies in a subset $S$ of $R$ chosen such that $\#S \cap (r + \mathfrak{m}) = 1$ for all $r \in R$. Let $t \in I$. The elements $m\rceil_{x,t}$, $\lceil m_{x,t}$, $m\rfloor_{x,t}$ and $\lfloor m_{x,t}$ in $P(C)$ were defined by

$$m\rceil_{x,t} = \begin{cases} \eta_{t+1}\alpha & (\text{if } t+1 \in I \text{ and } l_{t+1}^{-1}r_{t+1} = \alpha^{-1}d_x) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\lceil m_{x,t} = \begin{cases} \eta_{t-1}\beta & (\text{if } t-1 \in I \text{ and } l_{t}^{-1}r_{t} = d_x^{-1}\beta) \\ 0 & (\text{otherwise}) \end{cases}$$

$$m\rfloor_{x,t} = \begin{cases} \sum_{\sigma\in\mathbf{P}[x,t+1]} r_{\sigma,t+1}\sigma\kappa & (\text{if } t+1 \in I \text{ and } l_{t+1}^{-1}r_{t+1} = \kappa^{-1}d_{\mathrm{l}(\kappa)}) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\lfloor m_{x,t} = \begin{cases} \sum_{\sigma\in\mathbf{P}[x,t-1]} r_{\sigma,t-1}\sigma\zeta & (\text{if } t-1 \in I \text{ and } l_{t}^{-1}r_{t} = d_{\mathrm{l}(\zeta)}^{-1}\zeta) \\ 0 & (\text{otherwise}) \end{cases}$$

**Assumption:** We use the notation from remark 2.7.11 without reference for the statements and proofs of lemmas 2.7.12 and 2.7.14 and for the proof of corollary 2.7.8.

Recall corollary 1.1.17 (iii): for any $p, p' \in \mathbf{P}$, if $\Lambda p = \Lambda p'$ and $\mathrm{f}(p) = \mathrm{f}(p')$ (resp. $p\Lambda = p'\Lambda$ and $\mathrm{l}(p) = \mathrm{l}(p')$) then $p = p'$.

**Lemma 2.7.12.** *Let $C$ be a homotopy $I$-word and $t \in \mu_C^{-1}(n)$.*

(i) *If $t-1 \in I$ and $l_t^{-1}r_t = \gamma^{-1}d_y$, then ($\eta_t = 0 = \sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma$ iff $e_{h(a)}m\big]_{a,t-1} = 0$ for any arrow $a$).*

(ii) *If $t+1 \in I$ and $l_{t+1}^{-1}r_{t+1} = d_z^{-1}\tau$ then ($\eta_t = 0 = \sum_{\sigma\in\mathbf{P}z^{-1}} r_{\sigma,t}\sigma$ iff $\big[e_{h(a)}m_{a,t+1} = 0$ for any arrow $a$).*

*Proof.* (i) Note that for any arrow $a \neq y$ we have $e_{h(a)}m\rceil_{a,t-1} = 0$ by definition.

Suppose $\eta_t = \sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma = 0$, and so for each arrow $a$ with $h(a) = h(y)$ we have $e_{h(a)}m\rceil_{a,t-1} \in R\eta_t\gamma = 0$. If $h(a) \neq h(y)$ then $a \neq y$ which was the case considered above, and so for any arrow $a$ we have $e_{h(a)}m\rceil_{a,t-1} = 0$. Since $h(y) = v_C(t)$ and $\mathrm{l}(\gamma) = y$, for any $\sigma \in \mathbf{P}[t]$ with $\sigma \notin \mathbf{P}y^{-1}$ we have $r_{\sigma,t}\sigma\gamma = 0$. So, writing $\mathbf{P}[t]$ as the union of $\mathbf{P}y^{-1}$ and $\mathbf{P}[t] \setminus \mathbf{P}y^{-1}$ gives $\sum_{\sigma\in\mathbf{P}[t]} r_{\sigma,t}\sigma\gamma = 0$. Since $\pi_a(e_{h(a)}(\sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma)\gamma) = \sum_{\sigma\in\mathbf{P}[a,t]} r_{\sigma,t}\sigma\gamma$ we have $e_{h(a)}m\rfloor_{a,t-1} = 0$ for any arrow $a$. Consequently $e_{h(a)}m\big]_{a,t-1} = e_{h(a)}m\rceil_{a,t-1} + e_{h(a)}m\rfloor_{a,t-1} = 0$ for any arrow $a$.

Conversely suppose $e_{h(a)}m\big]_{a,t-1} = 0$ for any arrow $a$. Again when $a \neq y$ we always have $e_{h(a)}m\rceil_{a,t-1} = 0$ and so $0 = e_{h(a)}m\rfloor_{a,t-1} = \sum_{\sigma\in\mathbf{P}[a,t]} r_{\sigma,t}\sigma\gamma$ which gives $\sum_{\sigma\in\mathbf{P}[a,t]\cap\mathbf{P}y^{-1}} r_{\sigma,t}\sigma\gamma = 0$. Since $e_{h(y)}m\big]_{y,t-1} = 0$ we have $\eta_t\gamma = -\sum_{\sigma\in\mathbf{P}[y,t]} r_{\sigma,t}\sigma\gamma \in \mathrm{rad}(\Lambda e_{v_C(t)}\gamma)$ and so $\eta_t \in \mathfrak{m}$ by corollary 1.1.17 (iii). Since $\eta_t$ lies in a transversal $S$ with $S \cap \mathfrak{m} = 0$ we have $\eta_t = 0$ and hence $\sum_{\sigma\in\mathbf{P}[y,t]} r_{\sigma,t}\sigma\gamma = 0$ and so $\sum_{\sigma\in\mathbf{P}[y,t]\cap\mathbf{P}y^{-1}} r_{\sigma,t}\sigma\gamma = 0$. Since $\mathbf{P}y^{-1} = \mathbf{P}[t]\cap\mathbf{P}y^{-1}$ is the union of $\mathbf{P}[y,t]\cap\mathbf{P}y^{-1}$ and $\bigcup_{a\neq y}\mathbf{P}[a,t]\cap\mathbf{P}y^{-1}$, altogether we have

$$(\textstyle\sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma)\gamma = \sum_{a\neq y}\sum_{\sigma\in\mathbf{P}[a,t]\cap\mathbf{P}y^{-1}} r_{\sigma,t}\sigma\gamma + \sum_{\sigma\in\mathbf{P}[y,t]\cap\mathbf{P}y^{-1}} r_{\sigma,t}\sigma\gamma \ = 0$$

which shows $\sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma = 0$ by corollary 1.2.18. The proof of (ii) is similar, and omitted. $\square$

**Remark 2.7.13.** Let us now recall some more notation as we did in remark 2.7.11. For $t \in I$ the map $\psi_t : \bigoplus_{j\in I} \Lambda e_{v_C(j)} \to \Lambda e_{v_C(t)}$ is given by $\psi_t(\sum_j \lambda_j e_{v_C(j)}) = \lambda e_{v_C(t)}$. For an arrow $x$ the map $\theta_x : \bigoplus_{z\in \mathbf{A}(t(x)\to)} \Lambda \to \Lambda x$ is given by $\theta_x(\sum_z \lambda_z z) = \lambda_x x$. For $m$ as above we let $\lceil m \rceil = \sum_i \eta_i \underline{b}_i$ and $\lfloor m \rfloor = \sum_{i,\sigma} r_{\sigma,i}\sigma\underline{b}_i$. This gave $\psi_t(\gamma \lceil m \rceil) = \eta_t\gamma$ and $\psi_t(\gamma \lfloor m \rfloor) = \sum_\sigma r_{\sigma,t}\gamma\sigma$ by lemma 2.3.5 (i). Provided $l_{t+1}^{-1}r_{t+1} = \gamma^{-1}d_{\mathrm{l}(\gamma)}$ this also gave: $\theta_{\mathrm{f}(\gamma)}(m]_{\mathrm{l}(\gamma),t}) - \eta_{t+1}\gamma \in \gamma\mathrm{rad}(\Lambda)$ and $\theta_{\mathrm{f}(\gamma)}([m_{x,t}) = 0$ by lemma 2.3.5 (iii); and $(\eta_t = 0 = \sum_{\sigma\in\mathbf{P}z^{-1}} r_{\sigma,t}\sigma$ iff $\left[e_{h(a)}m_{a,t+1} = 0\right.$ for any arrow $a)$ by lemma 2.7.12.

**Lemma 2.7.14.** *Let $C$ be a homotopy $I$-word and $n \in \mathbb{Z}$. Then $m \in \ker(d^n_{P(C)})$ iff $m = \sum_{i\in\mu_C^{-1}(n)}(\eta_i\underline{b}_i + \sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i}\sigma\underline{b}_i)$ where*

(i) *$\eta_t = 0 = \sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma$ for any $t \in \mu_C^{-1}(n)$ with $t-1 \in I$ and $l_t^{-1}r_t = \gamma^{-1}d_y$, and*

(ii) *$\eta_t = 0 = \sum_{\sigma\in\mathbf{P}z^{-1}} r_{\sigma,t}\sigma$ for any $t \in \mu_C^{-1}(n)$ with $t+1 \in I$ and $l_{t+1}^{-1}r_{t+1} = d_z^{-1}\tau$.*

*Proof.* Note that $d^n_{P(C)}(m) = \sum_v d^n_{P(C)}(e_v m)$ which is 0 precisley when $d^n_{P(C)}(e_v m) = 0$ for all vertices $v$. By lemma 2.1.2 one has $d_{P(C)}(e_v m) = 0$ iff $d_{x,P(C)}(e_{h(x)}m) = 0$ for all arrows $x$ with head $v$. Since $m \in P^n(C)$ we have $d_{x,P(C)}(e_{h(x)}m) = 0$ iff $\psi_s(d_{x,P(C)}(e_{h(x)}m)) = 0$ for all $s \in \mu_C^{-1}(n+1)$. Hence by lemma 2.3.5 (i) we have that $m \in \ker(d^n_{P(C)})$ iff $\left[e_{h(a)}m_{a,s} + e_{h(a)}m\right]_{a,s} = 0$ for all arrows $a$ and $s \in \mu_C^{-1}(n+1)$.

Suppose $\psi_s(d_{a,P(C)}(_{h(a)}m)) \neq 0$ for some arrow $a$ and some $s \in \mu_C^{-1}(n+1)$, and so $\left[e_{h(a)}m_{a,s} \neq 0\right.$ or $\left.e_{h(a)}m\right]_{a,s} \neq 0$. If $\left[e_{h(a)}m_{a,s} \neq 0\right.$ then $s-1 \in I$ and $l_s^{-1}r_s = d_z^{-1}\tau$. By lemma 2.7.12 we also have $\eta_{s-1} \neq 0$ or $\sum_{\sigma\in\mathbf{P}z^{-1}} r_{\sigma,s-1}\sigma \neq 0$. Writing $t = s-1$ gives $\mu_C(t) = n$ and $l_{t+1}^{-1}r_{t+1} = d_z^{-1}\tau$, and therefore condition (ii) doesn't hold. Otherwise $\left[e_{h(a)}m_{a,s} = 0\right.$ and so $\left.e_{h(a)}m\right]_{a,s} \neq 0$ which means $s+1 \in I$ and $l_s^{-1}r_s = \gamma^{-1}d_y$, and again writing $t = s+1$ shows condition (i) doesn't hold. This shows that if conditions (i) and (ii) both hold, then $m \in \ker(d^n_{P(C)})$.

Now suppose $m \in \ker(d^n_{P(C)})$, and so $\left[e_{h(a)}m_{a,s} + e_{h(a)}m\right]_{a,s} = 0$ for all arrows $a$ and $s \in \mu_C^{-1}(n+1)$. To show condition (i) holds we start by assuming $t \in \mu_C^{-1}(n)$, $t-1 \in I$ and $l_t^{-1}r_t = \gamma^{-1}d_y$. This gives $\left.e_{h(a)}m\right]_{a,t-1} = \theta_{\mathrm{f}(\gamma)}(\left.e_{h(a)}m\right]_{a,t-1})$ and by lemma 2.3.5 (iii) we have $\theta_{\mathrm{f}(\gamma)}(\left[e_{h(a)}m_{a,t-1}\right.) = 0$.

By definition we have $\mu_C(t-1) = n+1$ and so for all arrows $a$ we get

$$e_{h(a)}m\big]_{a,t-1} = \theta_{\mathrm{f}(\gamma)}(e_{h(a)}m\big]_{a,t-1}) = \theta_{\mathrm{f}(\gamma)}(\big[e_{h(a)}m_{a,t-1} + e_{h(a)}m\big]_{a,t-1}) = 0$$

By lemma 2.7.12 this shows $\eta_t = \sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma = 0$. We can similarly show condition (ii) holds, by using lemmas 2.3.5 (iv) and 2.7.12. □

*Proof of corollary 2.7.8.* Let $i \in \mu_C^{-1}(n)$. If the $i^{\text{th}}$ kernel part is full then $d_{P(C)}(\underline{b}_i) = 0$ by definition. If the $i^{\text{th}}$ kernel part is a left arm then $d_{P(C)}(\mathrm{f}(\tau)\underline{b}_i) = \mathrm{f}(\tau)\underline{b}_i^- \in \Lambda\mathrm{f}(\tau)z\underline{b}_{i-1} = 0$. Similarly if it is a right arm then $d_{P(C)}(\mathrm{f}(\gamma)\underline{b}_i) = 0$. If it is a left peripheral arm then $d_{P(C)}(\beta\underline{b}_i) = \beta\underline{b}_i^+ = \beta\gamma\underline{b}_{i+1} = 0$ . Again similarly if it is a right peripheral arm then $d_{P(C)}(\alpha\underline{b}_i) = 0$. Altogether this shows $\ker(d^n_{P(C)}) \supseteq \bigoplus_{i\in\mu_C^{-1}(n)} \Lambda\kappa(i)\underline{b}_i$.

Now let $m \in \ker(d^n_{P(C)})$, and write $m$ as $\sum_i m_i\underline{b}_i$ where we have $m_i = \eta_i + \sum_{\sigma\in\mathbf{P}[i]} r_{\sigma,i}\sigma$ (for $\eta_i$ and $r_{\sigma,i}$ as before) for each $i$. By lemma 2.7.14, for any $t \in \mu_C^{-1}(n)$ we have $\eta_t = 0 = \sum_{\sigma\in\mathbf{P}w^{-1}} r_{\sigma,t}\sigma$ when $t-1 \in I$ and $l_t^{-1}r_t = \omega^{-1}d_w$, and $\eta_t = 0 = \sum_{\sigma\in\mathbf{P}x^{-1}} r_{\sigma,t}\sigma$ when $t+1 \in I$ and $l_{t+1}^{-1}r_{t+1} = d_x^{-1}\chi$. It suffices to show $m_i \in \Lambda\kappa(i)$ for all $i$.

If the $i^{\text{th}}$ kernel part is 0 then ($i-1 \in I$ and $l_i^{-1}r_i = \tau^{-1}d_z$) and ($i+1 \in I$ and $l_{i+1}^{-1}r_{i+1} = d_y^{-1}\gamma$). By assumption this gives $\eta_i = 0 = \sum_{\sigma\in\mathbf{P}z^{-1}} r_{\sigma,t}\sigma$ and $\eta_i = 0 = \sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma$ respectively (using lemma 2.7.14 as above). Since $\mathbf{A}(\to v_C(i)) = \{y, z\}$ for any path $\sigma \in \mathbf{P}[i]$ we have either $\mathrm{f}(\sigma)y \notin (\rho) \ni \mathrm{f}(\sigma)z$ or $\mathrm{f}(\sigma)z \notin (\rho) \ni \mathrm{f}(\sigma)y$ (as $(Q,\rho)$ satisfies gentle conditions) which shows $m_i = 0$.

If ($i+1 \in I$ and $l_{i+1}^{-1}r_{i+1} = d_y^{-1}\gamma$) then $\eta_i = 0 = \sum_{\sigma\in\mathbf{P}y^{-1}} r_{\sigma,t}\sigma$ as above. Assuming $\lambda_i \neq 0$ means $\lambda_i = \sum_{\sigma\in\mathbf{P}x^{-1}} r_{\sigma,i}\sigma$ for some $x \in \mathbf{A}(\to v_C(i))$ with $xy \in (\rho)$. If the $i^{\text{th}}$ kernel part is a left arm then $\mathrm{f}(\tau)y \in (\rho)$ and as $(Q,\rho)$ satisfies gentle conditions we have $x = \mathrm{f}(\tau)$, which shows $\lambda_i \in \Lambda\kappa(i)$. If the $i^{\text{th}}$ kernel part is a left peripheral arm then ($i-1 \notin I$ and $\beta y \in (\rho)$) which again means $x = \beta$ and so $m_i \in \Lambda\kappa(i)$. The proof is similar in case the $i^{\text{th}}$ kernel part is a right arm or a right peripheral arm, which altogether proves $\ker(d^n_{P(C)}) \subseteq \bigoplus_{i\in\mu_C^{-1}(n)} \Lambda\kappa(i)\underline{b}_i$, as required. □

We can now see the desired proof of lemma 2.7.5.

*Proof of lemma 2.7.5.* We are assuming the direct sum $N$ of string and band complexes in $\mathcal{K}^{-}_{\min}(\Lambda\text{-}\mathbf{proj})$ has bounded cohomology. We only prove (i) and (ii). The proof of (iii) is similar to the proof of (ii).

(i) Suppose $M$ is a right bounded complex of projective modules. Then $M$ is K-projective by example 3.3.18. Suppose also the cohomology of $M$ is 0 in every degree. Then the zero map from $M$ to the zero complex is a quasi-isomorphism, and hence a homotopy equivalence by lemma 3.3.19.

This shows any indecomposable object in $\mathcal{K}^{-}_{\min}(\Lambda\text{-}\mathbf{proj})$ must have non-zero cohomology in some degree. Any band complex is bounded and (indecomposable, and hence) non-zero by theorem 2.0.1. Hence any band complex has non-zero cohomology in some degree. Since N is a right bounded complex with bounded cohomology, the complex $\bigoplus_{\beta\in\mathcal{B}} P(E(\beta), V^{\beta})[-s(\beta)]$ (consisting of the band complexes arising in $N$) must be a bounded complex of projectives.

(ii) Fix $\sigma \in \mathcal{S}$. Here we assume $I_{A(\sigma)} \supseteq \mathbb{N}$. Suppose firstly that there is a sequence $(i_n \mid n \in \mathbb{N}) \in I^{\mathbb{N}}$ such that the $i_n^{\text{th}}$ kernel part is full for each $n$. If $\{\mu_{A(\sigma)}(i_n) \mid n \in \mathbb{N}\}$ is bounded then $A(\sigma)$ doesn't have controlled homogeny, and so $N^r$ is not finitely generated for some $r$, contradicting the assumption that $N$ lies in $\mathcal{K}^{\mp}(\Lambda\text{-}\mathbf{proj})$. Hence $\{\mu_{A(\sigma)}(i_n) \mid n \in \mathbb{N}\}$ is unbounded. Without loss of generality suppose $N$ lies in $\mathcal{K}^{+}(\Lambda\text{-}\mathbf{proj})$, and so $\{\mu_{A(\sigma)}(i_n) \mid n \in \mathbb{N}\}$ does not have an upper bound.

This means there is a subsequence $(i_{n(r)} \mid r \in \mathbb{N})$ of $(i_n \mid n \in \mathbb{N})$ such that $\mu_{A(\sigma)}(i_{n(r)}) < \mu_{A(\sigma)}(i_{n(r+1)})$ for all $r$. Since $N$ lies in $\mathcal{K}^{\mp}_{\min}(\Lambda\text{-}\mathbf{proj})$ we have $\mathrm{im}(d_N) \subseteq \mathrm{rad}(N)$ and hence $\underline{b}_{i_{n(t)}} \notin \mathrm{im}(d_N^{\mu_{A(\sigma)}(i_{n(t)})+t(\alpha)})$. By corollary 2.7.8 we have $\underline{b}_{i_{n(t)}} \in \ker(d_N^{\mu_{A(\sigma)}(i_{n(t)})+t(\alpha)})$ for each $t$, which contradicts that $N$ has bounded cohomology.

Hence we have shown that there are no sequences $(i_n \mid n \in \mathbb{N}) \in I^{\mathbb{N}}$ such that the $i_n^{\text{th}}$ kernel part is full for each $n$. Choose $l \in I$ such that the $i^{\text{th}}$ kernel part is not full for all $i > l$. This means $A(\sigma)_{>l} = d^{-1}_{\mathrm{l}(\gamma_1)}\gamma_1 d^{-1}_{\mathrm{l}(\gamma_2)}\gamma_2 \dots$ for a sequence of paths $\gamma_j \in \mathbf{P}$ where $\mathrm{f}(\gamma_j)\mathrm{l}(\gamma_{j+1}) \notin \mathbf{P}$ for each $j \geq 1$. Now choose $q \in \mathbb{Z}$ such that $H^p(P(A(\sigma))[-t(\sigma)] = 0$ for all $p > q$. Choose $t > l$ such that $\mu_{A(\sigma)}(i) > q + t(\sigma)$ for each $i > t$.

If there is some $j > t - l$ where $\gamma_j$ has length greater than 1, then $d_{P(A(\sigma))[-t(\sigma)]}(\underline{b}_{l+j}) = \gamma_j \underline{b}_{l+j+1}$ and so $\mathrm{f}(\gamma_j)\underline{b}_{l+j+1} \notin \mathrm{im}(d_N)$. By corollary 2.7.8 we have $\mathrm{f}(\gamma_j)\underline{b}_{l+j+1} \in \ker(d_N)$, which contradicts that $H^p(P(A(\sigma))[-t(\sigma)] = 0$ for all $p > q$.

Hence $\gamma_j$ is an arrow for each $j > t - l$. Now let $\alpha_h = \gamma_{j+h}$ for each integer $h > 0$. Since $Q$ is finite there is some $h > 0$ such that $\alpha_h = \alpha_{h+n}$ for some $n > 0$, which means $\alpha_h = \alpha_{h+n}$ for each $h > 0$ since $(Q, \rho)$ satisfies gentle conditions. Altogether we have $A(\sigma)_{>t} = ((\alpha_n^{-1} d_{\alpha_n} \dots \alpha_1^{-1} d_{\alpha_1})^{-1})^{\infty}$, as required. $\square$

### 2.7.3 Singularity Categories.

**Assumption:** In section 2.7.3 we let $\Gamma$ be a unital (left and right) noetherian ring.

We now explain how some of the calculations above seem consistent with a theorem (2.7.21) of Kalck.

**Definition 2.7.15.** [13, Definition 1.2.2] (SINGULARITY CATEGORIES) By a *perfect complex* we mean an object $P$ of $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ such that $P^i$ is a projective module for each $i \in \mathbb{Z}$. The *singularity category* (or[1], the *stable derived category*) of $\Gamma$ is denoted $\mathcal{D}_{\text{sing}}(\Gamma)$, and defined as follows.

The objects in $\mathcal{D}_{\text{sing}}(\Gamma)$ are the same as the objects in the derived category $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$. For objects $X$ and $Y$ of $\mathcal{D}_{\text{sing}}(\Gamma)$ we consider the set $\text{Hom}_{-\text{perf}\rightarrow}(X,Y)$ of all $\overline{(s,f)} \in \text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})}(X,Y)$ such that $\overline{(s,f)} = \overline{(s'',f'')} \circ \overline{(s',f')}$ for some $\overline{(s'',f'')} \in \text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})}(P,Y)$ and $\overline{(s',f')} \in \text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})}(X,P)$ where $P$ is a perfect complex. We then define homomorphisms in $\mathcal{D}_{\text{sing}}(\Gamma)$ by setting

$$\text{Hom}_{\mathcal{D}_{\text{sing}}(\Gamma)}(X,Y) = \text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})}(X,Y)/\text{Hom}_{-\text{perf}\rightarrow}(X,Y)$$

(ORBIT CATEGORIES) [44, §1] Let $t \in \mathbb{Z}$ and consider the automorphism $[t]$ on $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$. The *orbit category* of $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$ with respect to $[t]$ is denoted $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})/[t]$, and defined as follows. The objects in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})/[t]$ are the same as the objects in $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})$, bounded complexes of finitely generated $\Gamma$-modules. For objects $X$ and $Y$ of $\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})/[t]$ we let

$$\text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})/[t]}(X,Y) = \bigoplus\nolimits_{i\in\mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})}(X,Y[ti])$$

where the composition of $f = \sum_i f_i \in \text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})/[t]}(X,Y)$ and $g = \sum_i g_i \in \text{Hom}_{\mathcal{D}^b(\Gamma\text{-}\mathbf{mod})/[t]}(Y,Z)$ is defined as $gf = \sum h_i$ where for each $i$ we let $h_i = \sum_{l\in\mathbb{Z}}[tl](g_{i-l})f_l$.

[1] In the terminology of Buchweitz.

**Example 2.7.16.** Let $\Gamma$ be the field $k$, considered as a (one-dimensional) $k$-algebra. By example 1.5.11 indecomposable objects in $\mathcal{D}^b(k\text{-}\mathbf{mod})$ are complexes of the form

$$k(j) = \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where $k$ (considered as a one-dimensional $k$ vector space) lies in degree $j \in \mathbb{Z}$. This shows that every object in $\mathcal{D}^b(k\text{-}\mathbf{mod})$ is perfect, and so $\mathcal{D}_{\text{sing}}(k)$ is trivial. For $j, j' \in \mathbb{Z}$ suppose $j = j' + tn$ for some $n \in \mathbb{Z}$. Let $f = \sum f_i$ where $f_i = 0$ unless $i = j$, in which case we let $f_j = \text{id}_{k(j)}$ (in $\mathcal{D}^b(k\text{-}\mathbf{mod})$) This defines a homomorphism

$$f \in \text{Hom}_{\mathcal{D}^b(k\text{-}\mathbf{mod})}(k(j), k(j)) = \text{Hom}_{\mathcal{D}^b(k\text{-}\mathbf{mod})}(k(j), k(j')[-nt])$$
$$\subseteq \bigoplus_{i\in\mathbb{Z}} \text{Hom}_{\mathcal{D}^b(k\text{-}\mathbf{mod})}(k(j), k(j')[ti]) = \text{Hom}_{\mathcal{D}^b(k\text{-}\mathbf{mod})/[t]}(k(j), k(j'))$$

Similarly there is a homomorphism $g \in \text{Hom}_{\mathcal{D}^b(k\text{-}\mathbf{mod})/[t]}(k(j'), k(j))$ where $g_i = 0$ unless $i = j'$, in which case we let $g_{j'} = \text{id}_{k(j')}$. By definition $gf = \sum_i h_i$ and we have $h_i = 0$ unless $i = j + j'$ in which case $h_{j+j'} = [tj]g_{j'}f_j$ which is the identity on $k(j)$ in $\mathcal{D}^b(k\text{-}\mathbf{mod})/[t]$. Similarly we can show $fg$ is the identity on $k(j')$ in $\mathcal{D}^b(k\text{-}\mathbf{mod})/[t]$ and hence $k(j) \simeq k(j')$. Similary one can show that, if $t \nmid j - j'$ then $\text{Hom}_{\mathcal{D}^b(k\text{-}\mathbf{mod})/[t]}(k(j), k(j')) = 0$. Clearly the objects $k(j)$ are all indecomposable, and so we have that $\{k(0), \ldots, k(t-1)\}$ constitutes a complete list of pairwise non-isomorphic indecomposable objects in $\mathcal{D}^b(k\text{-}\mathbf{mod})/[t]$.

**Definition 2.7.17.** (Acyclic Complexes) The *acyclic category* of $\Gamma$ is denoted $\mathcal{K}_{\text{acyc}}(\Gamma)$, and defined to be the full subcategory of the homotopy category $\mathcal{K}(\Gamma\text{-}\mathbf{mod})$ consisting of complexes $P$ where $P^i$ is a projective module and $H^i(P) = 0$ for all $i \in \mathbb{Z}$.

**Proposition 2.7.18.** *If $M$ is an indecomposable object in $\mathcal{K}_{\text{acyc}}(\Gamma)$ then $M \simeq P(C)$ where $C$ has the form ${}^{\infty}(\alpha_n^{-1}d_{\alpha_n} \ldots \alpha_1^{-1}d_{\alpha_1})^{\infty}$ for some full cycle $\alpha_1 \ldots \alpha_n$ of zero-relations.*

*Proof.* By theorems 2.0.1 and 2.0.5 $M$ is isomorphic to a shift of a string or band complex. As we saw in the proof of lemma 2.7.5, any (shift of any) band complex has non-zero cohomology in some degree, and so $M \simeq P(C)$ for some homotopy word $C$. By corollary 2.7.8 we have $\text{im}(d^{n-1}_{P(C)}) = \bigoplus_{i\in\mu_C^{-1}(n)} \Lambda\kappa(i)\underline{b}_i$ for any $n \in \mathbb{Z}$.

As in the proof of lemma 2.7.5, by case analysis one can show that there is no $i \in I$ such that the $i^{\text{th}}$ kernel part is full. Similarly one can then show that ($C_{0<}$ is direct acyclic and $(C_{\leq 0})^{-1}$ is inverse acyclic) or ($C_{0<}$ is inverse acyclic and $(C_{\leq 0})^{-1}$ is direct acyclic). $\square$

We are ready to state part of a theorem due to Buchweitz.

**Theorem 2.7.19.** [13, Theorem 4.1.1] *If the* $\Gamma$*-modules* ${}_{\Gamma}\Gamma$ *and* $\Gamma_{\Gamma}$ *have finite injective dimension then there is an equivalence of categories* $\mathcal{K}_{\text{acyc}}(\Gamma) \to \mathcal{D}_{\text{sing}}(\Gamma)$*.*

To apply this theorem we require the following theorem due to Geiß and Reiten.

**Theorem 2.7.20.** [31, 3.4 Theorem] *If* $\Gamma$ *is an Assem-Skowroński gentle algebra over* $k$ *then* ${}_{\Gamma}\Gamma$ *and* $\Gamma_{\Gamma}$ *have finite injective dimension.*

Hence to study $\mathcal{D}_{\text{sing}}(\Gamma)$ we study $\mathcal{K}_{\text{acyc}}(\Gamma)$. Let us see why this theorem has relevance in the context of gentle algebras.

**Theorem 2.7.21.** [41, p.3, Theorem 2.5 (b)] *Let* $\Gamma$ *be an Assem-Skowroński gentle algebra over* $k$*. Then there is a (triangle) equivalence of categories*

$$\mathcal{D}_{\text{sing}}(\Gamma) \simeq \prod_{c \in \mathcal{C}(\Lambda)} \mathcal{D}^b(k\text{-}\mathbf{mod})/[L(c)]$$

*where*

(i) $\mathcal{C}(\Lambda)$ *is the set of equivalence classes of repetition-free cyclic paths in* $\Lambda$ *with full relations, and*

(ii) $L(c)$ *denotes the length of any path providing a representative for the class* $c$*.*

## Final Thoughts and Conjectures.

In chapter 1 we introduced quasi-bounded special biserial algebras, and restricted our focus to quasi-bounded string algebras, quasi-bounded gentle algebras and complete gentle algebras. These notions were based on their counterparts from the study of finite-dimensional algebras, which leads the author to the following question.

**Problem 2.7.22.** What known results about Pogorzały-Skowroński special biserial algebras (resp. Butler-Ringel string algebras, resp. Assem-Skowroński gentle algebras) generalise to all quasi-bounded special biserial (resp. string, resp. gentle) algebras?

The main results of this thesis seem to suggest that the answer to this question is non-empty. At the end of chapter 2 (2.7) we considered consequences of our main results, in the context of derived categories. We conjecture that there is more to explore using descriptions such as corollary 2.7.8.

**Conjecture 2.7.23.** *There is a way to calculate the cohomology of any string or band complex in terms of string modules and band modules. Furthermore, this calculation can be used to classify finitely generated modules.*

There is already positive evidence toward this conjecture.

**Theorem 2.7.24.** [16, Theorem 2.8] *Let $\Lambda$ be an Assem-Skowroński gentle algebra over an algebraically closed field. Then the cohomology complex of a string or band complex is a direct sum of string and band modules.*

Chapter 2 finished with a discussion about singularity categories. Theorems 2.7.19, 2.7.20 and 2.7.21 should together motivate the following conjecture.

**Conjecture 2.7.25.** *There is a class of complete gentle algebras $\Lambda$, strictly containing the Assem-Skowroński gentle algebras, such that*

(i) ${}_{\Lambda}\Lambda$ *and* $\Lambda_{\Lambda}$ *have finite injective dimension, and*

(ii) *there is an equivalence of categories* $\mathcal{K}_{\mathrm{acyc}}(\Lambda) \to \prod_{c\in\mathcal{C}(\Lambda)} \mathcal{D}^b(k\text{-}\mathbf{mod})/[L(c)]$.

# Chapter 3

# Appendix.

## 3.1 Abelian Categories.

We recall some notions from homological algebra. We follow: books by Aluffi [1] and Freyd [26] and papers by Bergman [10] and Krause [45] for ideas about abelian categories; and the books by Gelfand and Manin [33], Neeman [52], Weibel [64] and Zimmerman [67] for ideas about triangulated categories.

**Assumption:** In section 3.1 by a *category* $\mathcal{C}$ we mean an additive and locally small category.

**Remark 3.1.1.** Let $\mathcal{A}$ be a *pre-abelian* category, that is an *additive* catgeory (see [1, p.561, Definition 1.1]) where every arrow $\theta : X \to Y$ has a *kernel* $\mathrm{k}(\theta) : \ker(\theta) \to X$ and a *cokernel* $\mathrm{c}(\theta) : Y \to \mathrm{coker}(\theta)$ (see [1, p.561, Definition 1.2]). We write $\mathrm{im}(\theta) \to Y$ for the kernel of its cokernel, which we call the *image*. Dually we write $X \to \mathrm{coim}(\theta)$ for the cokernel of the kernel of $\theta$, which we call the *coimage* (see [1, p.572, Definition 1.15]).

Since $\theta \mathrm{k}(\theta) = 0$ the universal property of the coimage gives a unique morphism $\varphi : \mathrm{coim}(\theta) \to Y$ such that $\varphi \mathrm{c}(\mathrm{k}(\theta)) = \theta$. Since $\mathrm{c}(\theta)\varphi \mathrm{c}(\mathrm{k}(\theta)) = 0$ and $\mathrm{c}(\mathrm{k}(\theta))$ is epic (consider the dual of [1, p.562, Lemma 1.4]) we have $\mathrm{c}(\theta)\varphi = 0$ and so the universal property of the image gives a morphism $\overline{\theta} : \mathrm{coim}(\theta) \to \mathrm{im}(\theta)$ for which $\theta = \mathrm{k}(\mathrm{c}(\theta))\overline{\theta}\mathrm{c}(\mathrm{k}(\theta))$.

Altogether this gives a commutative diagram of the form

$$\begin{tikzcd} \ker(\theta) \arrow[r, "\mathrm{k}(\theta)"] & X \arrow[r, "\theta"] \arrow[d, "\mathrm{c}(\mathrm{k}(\theta))"'] & Y \arrow[r, "\mathrm{c}(\theta)"] & \mathrm{coker}(\theta) \\ & \mathrm{coim}(\theta) \arrow[r, "\overline{\theta}"] & \mathrm{im}(\theta) \arrow[u, "\mathrm{k}(\mathrm{c}(\theta))"'] & \end{tikzcd}$$

Now suppose $\psi : Y \to Z$ is another arrow such that $\psi\theta = 0$. Since $\mathrm{k}(\mathrm{c}(\theta)) = \theta \circ \mathrm{c}(\mathrm{k}(\theta)) \circ \overline{\theta}^{-1}$ we have $\psi \mathrm{k}(\mathrm{c}(\theta)) = 0$ and so the universal property of the kernel of $\psi$ gives a unique arrow $h(\theta, \psi) : \mathrm{im}(\theta) \to \ker(\psi)$ for which $\mathrm{k}(\mathrm{c}(\theta)) = \mathrm{k}(\psi)h(\theta, \psi)$, and hence $h(\theta, \psi)$ is monic. This construction and notation will be used repeatedly in what follows. We say the sequence $X \xrightarrow{\theta} Y \xrightarrow{\psi} Z$ is *exact at* $Y$ if $h(\theta, \psi)$ is epic, or equivalently, if $\mathrm{c}(\theta)\mathrm{k}(\psi) = 0$. A longer sequence is said to be *exact* if it is exact at every object which is not at the start or the end of the sequence.

**Assumption:** In section 3.1 we assume $\mathcal{A}$ is an *abelian* category (see [1, p.564, Definition 1.6]) and use the notation of remark 3.1.1. This means that

(a) if $\theta$ is monic (equivalently [1, p.574, Exercise 1.9] if $\mathrm{k}(\theta)$ is $0 \to X$) then $\mathrm{c}(\mathrm{k}(\theta))$ is an isomorphism; and

(b) if $\theta$ is epic (equivalently if $\mathrm{c}(\theta)$ is $Y \to 0$) then $\mathrm{k}(\mathrm{c}(\theta))$ is an isomorphism (see [64, Definition 1.2.2]).

Under this assumption (that $\mathcal{A}$ is an abelian category) $\overline{\theta}$ must be an isomorphism (see [1, p.570, Theorem 1.13]). In practice $\mathcal{A}$ will be $\Gamma$-**Mod** for some ring $\Gamma$.

### 3.1.1 Sums and Intersections.

We now introduce some terminology following from Freyd [26].

**Definition 3.1.2.** [26, pp.19, 20 and 42]. For monics $m : M \to X$ and $n : N \to X$ we write $m \leq n$ (and sometimes $M \subseteq N$) and say $m$ *is contained in* $n$, or $n$ *contains* $m$, provided there is an arrow $a : M \to N$ for which $m = na$. In this case $a$ must be monic and uniquely determined.

Given a set $I$ and a monic $\alpha_i : T_i \to X$ for each $i \in I$ we say a monic $\beta : T \to X$ is a *supremum* (resp. *infimum*) of the $\alpha_i$'s provided $\alpha_i \leq \beta$ (resp. $\beta \leq \alpha_i$) for each $i$, and for any monic $\gamma : T' \to X$ with $\alpha_i \leq \gamma$ (resp. $\gamma \leq \alpha_i$) for each $i$, we have $\beta \leq \gamma$ (resp. $\gamma \leq \beta$).

If $m \leq n$ and $n \leq m$ then the map $a$ from above is an isomorphism, in which case we write $m \sim n$ and say $m$ and $n$ are *equivalent*. By a *subobject* of $X$ we mean the equivalence class $\underline{m}$ of a monic arrow $m : M \to X$. We say $X$ is *simple* if there are exactly two distinct subobjects of $X$, namely the equivalence classes $\underline{\mathrm{id}}$ and $\underline{0}$ of the identity $\mathrm{id} : X \to X$ and $0 \to X$. A subobject $\underline{m}$ of $X$ (represented by $m : M \to X$) is called *maximal* provided the cokernel $\mathrm{coker}(m)$ is simple.

**Lemma 3.1.3.** (see [64, p.426, Exercise A.4.4]) *Let $I$ be a set and $m_i : A_i \to A$ be a monic arrow for each $i \in I$. Then*

(i) *if $\mathcal{A}$ is cocomplete a supremum $\Sigma : \sum A_j \to A$ of the $m_i$'s exists and is unique, and*

(ii) *if $\mathcal{A}$ is complete an infimum $\cap : \bigcap A_j \to A$ of the $m_i$'s exists and is unique.*

**Definition 3.1.4.** Recall the notation used in the statement of lemma 3.1.3. We say *$A$ is the sum over the $A_i$'s*, and write $A \approx \sum A_j$, provided $\Sigma$ is epic. We say *$A$ is the intersection over the $A_i$'s*, and write $A \approx \bigcap A_j$, provided $\cap$ is epic.

### 3.1.2 Splitting and Projectivity.

**Definition 3.1.5.** (Superfluous Subobjects) Let $\mathcal{A}$ be an abelian category. A subobject $\underline{s}$ of $X$ (given by $s : T \to X$) is said to be *superfluous* if, for every subobject $\underline{t}$ of $X$ (given by $t : T \to X$), if $S + T \approx X$ then $\underline{t} = \underline{\mathrm{id}}$.

**Lemma 3.1.6.** *If $f : X \to Y$ is epic, then $f$ is essential if and only* $\ker(f)$ *defines a supurfluous subobject* $\underline{\mathrm{k}(f)}$ *of $X$.*

**Remark 3.1.7.** Recall an arrow $r : V \to W$ (resp. $s : W \to V$) is called a *retraction* (resp. *section*) if $rs = \mathrm{id}$ for some arrow $s : W \to V$ (resp. $r : V \to W$). In this case consider the direct sums $\ker(r) \oplus W$ and $W \oplus \mathrm{coker}(s)$ together with their canonical arrows giving the diagrams

$$\begin{tikzcd} \ker(r) \arrow[r, shift left, "\iota_{\mathrm{k}}"] & \ker(r) \oplus W \arrow[l, shift left, "\pi_{\mathrm{k}}"] \arrow[r, shift left, "p_w"] & W \arrow[l, shift left, "i_w"] \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[r, shift left, "j_w"] & W \oplus \mathrm{coker}(s) \arrow[l, shift left, "q_w"] \arrow[r, shift left, "\pi_{\mathrm{c}}"] & \mathrm{coker}(s) \arrow[l, shift left, "\iota_{\mathrm{c}}"] \end{tikzcd}$$

Since $rs = \mathrm{id}$ we have $r(\mathrm{id} - sr) = 0$ and so the universal property of the kernel gives an arrow $t : V \to \ker(r)$ such that $\mathrm{id} - sr = \mathrm{k}(r)t$. This gives $\mathrm{k}(r)t\mathrm{k}(r) = \mathrm{k}(r)$ and so $t\mathrm{k}(r) = \mathrm{id}$ as $\mathrm{k}(r)$ is monic. Similarly $\mathrm{k}(r)ts = 0$ and thus $ts = 0$.This gives an isomorphism $\alpha : V \to \ker(r) \oplus W$ given by $\alpha = i_w r + \iota_{\mathrm{k}} t$ with inverse $\beta = sp_w + \mathrm{k}(r)\pi_{\mathrm{k}}$. Dually, as $(\mathrm{id} - sr)s = 0$ there is an arrow $u : \mathrm{coker}(s) \to V$ such that $\mathrm{id} - sr = u\mathrm{c}(s)$, $\mathrm{c}(s)u = \mathrm{id}$ (as $\mathrm{c}(s)$ is epic) and $ru = 0$. Again there is an isomorphism $\gamma : V \to W \oplus \mathrm{coker}(s)$ given by $\gamma = j_w r + \iota_{\mathrm{c}}\mathrm{c}(s)$ with inverse $\delta = sq_w + u\pi_{\mathrm{c}}$.

We aim to generalise well-known results about projective covers, such that those given in the book by Lam [48, Section 24], to the setting of an abelian category.

**Definition 3.1.8.** (Essential Epics, Projective Covers) An epic arrow $f : X \to Y$ in $\mathcal{A}$ is said to be *essential* if, for any arrow $r : R \to X$, if $fr$ is epic then $r$ is epic. A *projective cover* of $X$ consists of a pair $(P, \phi)$ where $P$ is a projective object and $\phi : P \to X$ is an epic arrow which is essential.

For example if $R$ is a projective object then $(R, \mathrm{id})$ is a projective cover of $R$.

(Enough Covers, All Its Radicals) We say $\mathcal{A}$ *has enough projective covers* if there is a projective cover $(P, \phi)$ of any object $X$ in $\mathcal{A}$. We say that an object $X$ of $\mathcal{A}$ *has a radical* if the collection of all maximal subobjects $X_i$ forms a set (indexed by $i \in I$)[1] and the product $\prod_{i \in I} \mathrm{coker}(X_i \to X)$ exists in $\mathcal{A}$.

(Notation: If $X$ has a radical in $\mathcal{A}$ then the *radical* of $X$ is denoted $\cap_X : \mathrm{rad}(X) \to X$ and defined as the infimum $\bigcap X_i \to X$ of the monics $X_i \to X$. We say $\mathcal{A}$ *has all of its radicals* if every object in $\mathcal{A}$ has a radical.

**Example 3.1.9.** Let $\Lambda$ be a noetherian ring, and so $\Lambda$-**mod** is an abelian subcategory of $\Lambda$-**Mod** which has all of its radicals.

Consequently (see for example [48, 24.16 Theorem]) if $\Lambda$ is semiperfect (see example 3.1.36) every finitely generated $\Lambda$-module has a projective cover, and so $\Lambda$-**mod** has enough projective covers.

**Lemma 3.1.10.** [2, 17.17 Lemma][2] *Let $\varphi : L \to X$ be an epic arrow where $L$ is projective, and let $(P, \phi)$ be a projective cover of $X$. Then there is a retraction $\theta : L \to P$ such that $\phi\theta = \varphi$.*

**Example 3.1.11.** Recall (say from [48, p. 336, Definition 23.1]) that $\Lambda$ is *perfect* if it is semilocal and $\mathrm{rad}(\Lambda)$ is $T$-nilpotent (that is, for any sequence $a_1, a_2, \dots \in \mathrm{rad}(\Lambda)$ there is some $n \geq 1$ with $a_1 \dots a_n = 0 = a_n \dots a_1$).

Consequently (see for example [48, 24.18 Theorem]) every $\Lambda$-module has a projective cover, and so $\Lambda$-**Mod** has all of its projective covers.

**Proposition 3.1.12.** *If $\mathcal{A}$ has all of its radicals and $f : X \to Y$ is an arrow then there is a monic arrow $f_{\mathrm{rad}} : \mathrm{rad}(X) \to \mathrm{rad}(Y)$ for which $\cap_Y f_{\mathrm{rad}} = f \cap_X$.*

The next result is a key step in proving corollary 3.2.25, which is a vital reduction used in chapter 2.

---

[1] Note that this condition holds provided $\mathcal{A}$ has a generator (see [26, p. 69, Proposition 3.35]).

[2] The statement in [2] is for the case $\mathcal{A} = \Lambda$-**Mod** for a ring $\Lambda$.

Due to its importance we give a proof which is dual to the discussion before [45, Lemma B.1].

**Lemma 3.1.13.** *Let $\mathcal{A}$ be an abelian category which has enough projective covers and all of its radicals. If $d : L' \to L$ is an arrow between projectives there exist isomorphisms $\alpha : L \to R \oplus W$ and $\alpha' : L' \to R \oplus W'$ such that $\alpha d = (\mathrm{id} \oplus v)\alpha'$ and $v : W' \to W$ is an arrow with* $\mathrm{im}(\nu) \subseteq \mathrm{rad}(W)$.

*Proof.* Let $X = \mathrm{coker}(d)$ and $\varphi = \mathrm{c}(d)$. Let $(W, \phi)$ be a projective cover of $X$, and so by lemma 3.1.10 there is a retraction $\theta : L \to W$ such that $\phi\theta = \varphi$. Let $\lambda$ be the section for which $\theta\lambda = \mathrm{id}$.

By remark 3.1.7 there is an arrow $t : L \to \ker(\theta)$ such that $t\mathrm{k}(\theta) = \mathrm{id}$ and $t\lambda = 0$, and an isomorphism $\alpha : L \to \ker(\theta) \oplus W$ given by $\alpha = i_w\theta + \iota_\mathrm{k} t$ (with inverse $\beta = \lambda p_w + \mathrm{k}(\theta)\pi_\mathrm{k}$) where we label the canonical maps of the direct sum $\ker(\theta) \oplus W$ as

$$\ker(\theta) \underset{\pi_\mathrm{k}}{\overset{\iota_\mathrm{k}}{\rightleftarrows}} \ker(\theta) \oplus W \underset{i_w}{\overset{p_w}{\rightleftarrows}} W$$

Since $\mathrm{c}(d)\mathrm{k}(\theta) = \phi\theta\mathrm{k}(\theta) = 0$ by the universal property of the kernel $\mathrm{k}(\mathrm{c}(d)) : \mathrm{im}(d) \to L$ there is a unique arrow $\eta : \ker(\theta) \to \mathrm{im}(d)$ for which $\mathrm{k}(\theta) = \mathrm{k}(\mathrm{c}(d))\eta$. Note that $t\mathrm{k}(\mathrm{c}(d))\eta = \mathrm{id}$ and so the composition $\tau = t\mathrm{k}(\mathrm{c}(d))$ is a retraction. In particular $td = \tau\bar{d}\mathrm{c}(\mathrm{k}(d))$ is an epimorphism. Considering the arrow $\eta$ and the epic $\bar{d}\mathrm{c}(\mathrm{k}(d))$ by the projectivity of $\ker(\theta)$ there is an arrow $\varepsilon : \ker(\theta) \to L'$ with $\eta = \bar{d}\mathrm{c}(\mathrm{k}(d))\varepsilon$. Since $t\mathrm{k}(\mathrm{c}(d))\eta = \mathrm{id}$ we have $td\varepsilon = \mathrm{id}$ and so $\varepsilon$ is a section.

By remark 3.1.7 there is an arrow $u : \mathrm{coker}(\varepsilon) \to L'$ with $\mathrm{c}(\varepsilon)u = \mathrm{id}$ and $tdu = 0$, and an isomorphism $\beta' = \varepsilon q_\theta + u\pi_\mathrm{c}$ (with inverse $\alpha' = j_\theta td + \iota_\mathrm{c}\mathrm{c}(\varepsilon)$) where we label the canonical maps of the direct sum $\ker(\theta)\oplus$ as

$$\ker(\theta) \underset{q_\theta}{\overset{j_\theta}{\rightleftarrows}} \ker(\theta) \oplus \mathrm{coker}(\varepsilon) \underset{\iota_\mathrm{c}}{\overset{\pi_\mathrm{c}}{\rightleftarrows}} \mathrm{coker}(\varepsilon)$$

So far we have the following diagram in $\mathcal{A}$.

$$
\begin{tikzcd}
L' \arrow[rrr, "d"] \arrow[dr, bend left, "\mathrm{c}(\mathrm{k}(d))"] \arrow[ddd, shift left, "\alpha'"] & & & L \arrow[r, "\mathrm{c}(d)"] \arrow[dr, "\theta"'] \arrow[ddl, shift left, "t"] \arrow[ddd, shift left, "\alpha"] & \mathrm{coker}(d) \\
 & \mathrm{coim}(d) \arrow[r, "\bar{d}"] & \mathrm{im}(d) \arrow[ur, bend left, "\mathrm{k}(\mathrm{c}(d))"] & & W \arrow[u, "\phi"'] \\
 & \mathrm{coker}(\varepsilon) \arrow[uul, "u"] & \ker(\theta) \arrow[u, "\eta"] \arrow[uull, "\varepsilon"] \arrow[uur, "\mathrm{k}(\theta)"] & & \\
\ker(\theta) \oplus \mathrm{coker}(\varepsilon) \arrow[uuu, shift left, "\beta'"] & & & \ker(\theta) \oplus W \arrow[uuu, shift left, "\beta"] &
\end{tikzcd}
$$

Let $R = \ker(\theta)$ and $W' = \mathrm{coker}(\varepsilon)$, and let $\Xi : R \oplus W' \to R \oplus W$ be the arrow defined by the composition $\Xi = \alpha d \beta'$. From the above we have $d\varepsilon = \mathrm{k}(\mathrm{c}(d))\eta = \mathrm{k}(\theta)$ and so $\theta d \varepsilon = 0$. We also have $tdu = 0$ and $td\varepsilon = \mathrm{id}$, which shows $\Xi = i_w \theta d u \pi_{\mathrm{c}} + \iota_{\mathrm{k}} q_\theta$. Let $\nu = \theta du$. Since $\phi\nu = 0$ the monic $h(\nu, \phi)$ gives $\mathrm{im}(\nu) \subseteq \ker(\phi)$. Since $(P, \phi)$ is a projective cover by lemma 3.1.6 $\underline{\mathrm{k}(\phi)}$ defines a superfluous subobject of $W$. This means that $\mathrm{k}(\phi) \leq m$ for every maximal subobject $m : M \to W$ and so $\mathrm{im}(\nu) \subseteq \mathrm{rad}(W)$ as required. $\square$

For the purposes of discussing theorem 1.5.22 we need to introduce the stable module category. To this end we follow Happel [34] and Zimmerman [67].

**Definition 3.1.14.** (Extension Closure) Let $\mathcal{B}$ be a full and additive subcategory of an abelian category $\mathcal{A}$. We say $\mathcal{B}$ is *extension-closed* if, for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, if $X$ and $Z$ are objects in $\mathcal{B}$, then so too is $Y$.

[34, p.10] (Proper Monics And Epics) Let $\mathcal{S}$ denote the class of exact sequences $s = 0 \to X \to Y \to Z \to 0$ where $X$, $Y$ and $Z$ are objects in $\mathcal{B}$. If $f : X \to Y$ is a monomorphism (resp. $g : Y \to Z$ is an epimorphism) such that $s = 0 \to X \to Y \to Z \to 0$ lies in $\mathcal{B}$ then we say *$s$ begins with $f$* (resp. *ends with $g$*). Suppose $\mathcal{B}$ is extension-closed in $\mathcal{A}$. A monomorphism $f : X \to Y$ (resp. epimorphism $g : Y \to Z$) is called *proper* if there is some $s \in \mathcal{S}$ such that $s$ begins with $f$ (resp. ends with $g$).

($\mathcal{S}$-PROJECTIVES INJECTIVES) Suppose $\mathcal{B}$ is extension-closed in $\mathcal{A}$. An object $X$ in $\mathcal{B}$ is called *$\mathcal{S}$-projective* if, for all proper epimorphisms $\pi : M \to N$ and any morphism $n : X \to N$, there is an arrow $m : X \to M$ such that $\pi m = n$. $X$ is called *$\mathcal{S}$-injective* if the dual property holds: for all proper monomorphisms $\iota : M \to N$ and any morphism $m : M \to X$ there is an arrow $n : N \to X$ such that $m = n\iota$. We say $\mathcal{B}$ *has enough $\mathcal{S}$-projectives* if, given any object $M$ in $\mathcal{B}$ there is a proper epimorphism $P \to M$ where $P$ is $\mathcal{S}$-projective in $\mathcal{B}$. We say $\mathcal{B}$ *has enough $\mathcal{S}$-injectives* if, given any object $M$ in $\mathcal{B}$ there is a proper monomorphism $M \to I$ where $I$ is $\mathcal{S}$-injective in $\mathcal{B}$.

(FROBENIUS CATEGORIES) Suppose $\mathcal{B}$ is extension-closed in $\mathcal{A}$. For $\mathcal{B}$ and $\mathcal{S}$ as above we call $\mathcal{B}$ *Frobenius* if $\mathcal{B}$ has enough $\mathcal{S}$-projectives, $\mathcal{B}$ has enough $\mathcal{S}$-injectives and the $\mathcal{S}$-projectives coincide with the $\mathcal{S}$-injectives (that is, each $\mathcal{S}$-projective is $\mathcal{S}$-injective, and each $\mathcal{S}$-injective is $\mathcal{S}$-projective).

(STABLE CATEGORIES) The stable category $\underline{\mathcal{B}}$ of $\mathcal{B}$ is defined as follows. The objects in $\underline{\mathcal{B}}$ are the same as the objects in $\mathcal{B}$. For objects $X$ and $Y$ in $\underline{\mathcal{B}}$ (and hence $\mathcal{B}$) consider the set -P-$\mathrm{Hom}_{\mathcal{B}}(X, Y)$ of $f \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$ such that $f = gh$ where the domain of $g$ (codomain of $h$) is $\mathcal{S}$-projective. It is straightforward to show that -P-$\mathrm{Hom}_{\mathcal{B}}(X, Y)$ is closed under addition, and so we let $\mathrm{Hom}_{\underline{\mathcal{B}}}(X, Y)$ be the quotient, denoted $\underline{\mathrm{Hom}}_{\mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{B}}(X, Y)/$-P-$\mathrm{Hom}_{\mathcal{B}}(X, Y)$. To see that composition is well defined see the first part of [67, Remark 5.1.2].

### 3.1.3 Diagram Chasing and Homology.

In [10] Bergman presented an idea which captures the notion of diagram chasing in an abelian category. For application later we shall require some well-known results, such as [64, Snake Lemma 1.3.2] and [64, Horseshoe Lemma 2.2.8], the proofs of which use this idea. These results follow from [10, 1.7, Salamander Lemma], which we now outline.

**Definition 3.1.15.** For an object $A$ of $\mathcal{A}$ a *diagram of squares through* $A$ is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\gamma"] & P \arrow[d, "\beta"] & \\ Q \arrow[r, "\delta"] & A \arrow[r, "\mu"] \arrow[d, "\eta"] & M \arrow[d, "\rho"] \\ & N \arrow[r, "\lambda"] & Y \end{tikzcd}$$

where $\mu\delta = 0$ and $\eta\beta = 0$. Let $\psi = \rho\mu$ and $\theta = \delta\gamma$.

**Remark 3.1.16.** Fix a diagram of squares through an object $A$ in $\mathcal{A}$ using the notation of definition 3.1.15. By the universal property of the kernel there are arrows $b : \mathrm{im}(\theta) \to \mathrm{im}(\beta)$, $d : \mathrm{im}(\theta) \to \mathrm{im}(\delta)$, $m : \ker(\mu) \to \ker(\psi)$ and $n : \ker(\eta) \to \ker(\psi)$ for which $\mathrm{k}(\mathrm{c}(\delta))d = \mathrm{k}(\mathrm{c}(\theta)) = \mathrm{k}(\mathrm{c}(\beta))b$, $\mathrm{k}(\mu) = \mathrm{k}(\psi)m$ and $\mathrm{k}(\eta) = \mathrm{k}(\psi)n$. Recall (from 3.1.1) there are monics $h(\delta, \mu) : \mathrm{im}(\delta) \to \ker(\mu)$ and $h(\beta, \eta) : \mathrm{im}(\beta) \to \ker(\eta)$ for which $\mathrm{k}(\mu)h(\delta, \mu) = \mathrm{k}(\mathrm{c}(\delta))$ and $\mathrm{k}(\eta)h(\beta, \eta) = \mathrm{k}(\mathrm{c}(\beta))$. Let $x = h(\delta, \mu)$ and $y = h(\beta, \eta)$.

Let $i_\delta : \mathrm{im}(\delta) \to \mathrm{im}(\beta) + \mathrm{im}(\delta)$ and $i_\beta : \mathrm{im}(\beta) \to \mathrm{im}(\beta) + \mathrm{im}(\delta)$ be the canonical monics of the sum $\mathrm{im}(\beta) + \mathrm{im}(\delta)$ for which $\mathrm{k}(\mathrm{c}(\beta)) = \Sigma_{\beta,\delta} i_\beta$ and $\mathrm{k}(\mathrm{c}(\delta)) = \Sigma_{\beta,\delta} i_\delta$ (where $\Sigma_{\beta,\delta} : \mathrm{im}(\beta) + \mathrm{im}(\delta) \to A$ is the supremum of $\mathrm{k}(\mathrm{c}(\beta))$ and $\mathrm{k}(\mathrm{c}(\delta))$). Let $j_\mu : \ker(\mu) \cap \ker(\eta) \to \ker(\mu)$ and $j_\eta : \ker(\mu) \cap \ker(\eta) \to \ker(\eta)$ be the canonical monics of the intersection $\ker(\mu) \cap \ker(\eta)$ for which $\mathrm{k}(\mu)j_\mu = \cap_{\mu,\eta} = \mathrm{k}(\eta)j_\eta$ (where $\cap_{\mu,\eta} : \ker(\mu) \cap \ker(\eta) \to A$ is the infimum of $\mathrm{k}(\mu)$ and $\mathrm{k}(\eta)$).

Since $\cap_{\mu,\eta}$ is an infimum there must be a unique monic $\theta' : \mathrm{im}(\theta) \to \ker(\mu) \cap \ker(\eta)$ for which $j_\mu\theta' = xd$ and $j_\eta\theta' = yb$. Similarly there must be a unique monic $\psi' : \mathrm{im}(\delta) + \mathrm{im}(\beta) \to \ker(\psi)$ for which $\psi' i_\delta = mx$ and $\psi' i_\beta = ny$.

**Definition 3.1.17.** (see [10, Definition 1.1])

Consider a diagram of squares through $A$ in the notation from definition 3.1.15 above.

(Vertical/Horizontal Homology, Notation: $A\curlywedge$, $A\prec$) The *vertical homology* and *horizontal homology* at $A$ are defined as the quotients $A\curlywedge := \mathrm{coker}(y) = \ker(\eta)/\mathrm{im}(\beta)$ and $A\prec := \mathrm{coker}(x) = \ker(\mu)/\mathrm{im}(\delta)$ respectively.

(Receptor/Donor, Notation: $^{\square}A$, $A\curlywedge$, $A\prec$) The *receptor* and *donor* at $A$ are defined as the quotients $^{\square}A := \mathrm{coker}(\theta') = \ker(\mu) \cap \ker(\eta)/\mathrm{im}(\theta)$ and $A_{\square} := \mathrm{coker}(\psi') = \ker(\psi)/(\mathrm{im}(\delta) + \mathrm{im}(\beta))$ respectively.

**Lemma 3.1.18.** (Notation: $A[\square,\prec]$, $A[\prec,\square]$, $A[\square,\curlywedge]$, $A[\curlywedge,\square]$)

[10, Lemma 1.2] *Given a diagram of squares through A in the notation from definition 3.1.15 and remark 3.1.16, there are arrows $A[\square,\prec]$, $A[\prec,\square]$, $A[\square,\curlywedge]$ and $A[\curlywedge,\square]$ defining the following commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd}
& {^{\square}A} \arrow[dl, "{A[\square,\prec]}"'] \arrow[dr, "{A[\square,\curlywedge]}"] & \\
A\prec \arrow[dr, "{A[\prec,\square]}"'] & & A\curlywedge \arrow[dl, "{A[\curlywedge,\square]}"] \\
& A_{\square} &
\end{tikzcd}$$

*Proof.* Since $\mathrm{c}(y)j_\eta\theta' = \mathrm{c}(y)yb = 0$ there is a unique arrow $y' : \mathrm{coker}(\theta') \to \mathrm{coker}(y)$ for which $y'\mathrm{c}(\theta') = \mathrm{c}(y)j_\eta$. Similarly there is a unique arrow $x' : \mathrm{coker}(\theta') \to \mathrm{coker}(x)$ for which $x'\mathrm{c}(\theta') = \mathrm{c}(x)j_\mu$. Since $\mathrm{c}(\psi')mx = \mathrm{c}(\psi')\psi' i_\beta = 0$ there is a unique arrow $x'' : \mathrm{coker}(x) \to \mathrm{coker}(\psi')$ for which $\mathrm{c}(\psi')m = x''\mathrm{c}(x)$. Similarly there is a unique arrow $y'' : \mathrm{coker}(y) \to \mathrm{coker}(\psi')$ for which $\mathrm{c}(\psi')n = y''\mathrm{c}(y)$.

This gives $\mathrm{k}(\psi)mj_\mu = \cap_{\mu,\eta} = \mathrm{k}(\psi)nj_\eta$ and $\mathrm{k}(\psi)$ is a monic so $mj_\mu = nj_\eta$. Hence $x''x'\mathrm{c}(\theta') = x''\mathrm{c}(x)j_\mu = \mathrm{c}(\psi')mj_\mu$ and similarly $y''y'\mathrm{c}(\theta') = \mathrm{c}(\psi')nj_\eta$ and so as $\mathrm{c}(\theta')$ is epic we have $x''x' = y''y'$.

We let $A[\square,\prec] = x'$, $A[\prec,\square] = x''$, $A[\square,\curlywedge] = y'$ and $A[\curlywedge,\square] = y''$. $\square$

**Definition 3.1.19.** For an arrow $\mu : A \to M$ in $\mathcal{A}$ a *horizontal digram of squares through* $\mu$ is a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\gamma"] & P \arrow[r, "\xi"] \arrow[d, "\beta"] & B \arrow[d, "\zeta"] & \\
Q \arrow[r, "\delta"] & A \arrow[r, "\mu"] \arrow[d, "\eta"] & M \arrow[r, "\sigma"] \arrow[d, "\rho"] & W \arrow[d, "\tau"] \\
& N \arrow[r, "\lambda"] & Y \arrow[r, "\nu"] & L
\end{tikzcd}$$

where $\mu\delta = 0$, $\eta\beta = 0$, $\sigma\mu = 0$, $\rho\varphi = 0$, $\chi\alpha = 0$ and $\nu\lambda = 0$. Recall $\theta = \beta\alpha$ and $\psi = \lambda\eta$. We also let $\chi = \zeta\xi$ and $\varphi = \nu\rho$.

**Proposition 3.1.20.** *Given a horizontal diagram of squares through* $\mu$ *in the notation above, there exist arrows* $\mathrm{i}(\beta, \zeta)$, $\mathrm{k}(\eta, \rho)$ *and* $H^{\curlywedge}(\mu)$ *such that the diagram*

$$\begin{tikzcd}
\mathrm{im}(\beta) \arrow[r, "h(\beta,\eta)"] \arrow[d, "\mathrm{i}(\beta,\zeta)"'] & \ker(\eta) \arrow[r, "\mathrm{c}(h(\beta,\eta))"] \arrow[d, "\mathrm{k}(\eta,\rho)"'] & \mathrm{coker}(h(\beta,\eta)) \arrow[d, "H^{\curlywedge}(\mu)"] \\
\mathrm{im}(\zeta) \arrow[r, "h(\zeta,\rho)"'] & \ker(\rho) \arrow[r, "\mathrm{c}(h(\zeta,\rho))"'] & \mathrm{coker}(h(\zeta,\rho))
\end{tikzcd}$$

*is commutative. Furthermore,* $H^{\curlywedge}(\mu)$ *is the unique such arrow making the right hand square commute.*

Note that a horizontal diagram of squares of this form defines two separate diagrams of squares around $A$ and $M$.

**Lemma 3.1.21.** [10, Lemma 1.4] *Given a horizontal diagram of squares through* $\mu$ *in the notation above, there is an arrow* $\langle \mu, \curlywedge \rangle : A_{\square} \to {}^{\square}M$ *such that:*

(i) *the following diagram commutes*

$$\begin{tikzcd}
 & {}^{\square}A \arrow[dl, "A{[\square,\prec]}"'] \arrow[dr, "A{[\square,\curlywedge]}"] & & & & {}^{\square}M \arrow[dl, "M{[\square,\curlywedge]}"] \arrow[dr, "M{[\square,\prec]}"] & \\
A_{\prec} \arrow[dr, "A{[\prec,\square]}"'] & & A_{\curlywedge} \arrow[dl, "A{[\curlywedge,\square]}"'] \arrow[rrr, "H^{\curlywedge}(\mu)"] & & & M_{\curlywedge} \arrow[dr, "M{[\curlywedge,\square]}"'] & & M_{\prec} \arrow[dl, "M{[\prec,\square]}"] \\
 & A_{\square} \arrow[uurrrr, bend right, "\langle\mu,\curlywedge\rangle"] & & & & & M_{\square} &
\end{tikzcd}$$

(ii) $\langle \mu, \curlywedge \rangle\, A[\prec, \square] = 0$ *and* $\mathrm{c}(A[\prec, \square])\mathrm{k}(\langle \mu, \curlywedge \rangle) = 0$*;*

(iii) $M[\square, \prec]\, \langle \mu, \curlywedge \rangle = 0$ *and* $\mathrm{c}(\langle \mu, \curlywedge \rangle)\mathrm{k}(M[\square, \prec]) = 0$*;*

(iv) $M[\curlywedge, \square]M[\square, \curlywedge]\, \langle \mu, \curlywedge \rangle = 0$ *and* $\mathrm{c}(M[\square, \curlywedge]\, \langle \mu, \curlywedge \rangle)\mathrm{k}(M[\curlywedge, \square]) = 0$*; and*

(v) $\langle \mu, \curlywedge \rangle\, A[\curlywedge, \square]A[\square, \curlywedge] = 0$ *and* $\mathrm{c}(A[\square, \curlywedge])\mathrm{k}(\langle \mu, \curlywedge \rangle\, A[\curlywedge, \square]) = 0$.

We can now yield a dual statement by taking advantage of some symmetry. For an arrow $\eta : A \to N$ in $\mathcal{A}$ a *vertical digram of squares through* $\eta$ is a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\gamma"] & P \arrow[d, "\beta"] & \\
Q \arrow[r, "\delta"] \arrow[d, "\upsilon"] & A \arrow[r, "\mu"] \arrow[d, "\eta"] & M \arrow[d, "\rho"] \\
C \arrow[r, "\varrho"] & N \arrow[r, "\lambda"] \arrow[d, "\kappa"] & Y \arrow[d, "\vartheta"] \\
 & V \arrow[r, "\varsigma"] & K
\end{tikzcd}$$

where $\vartheta\rho = 0$, $\kappa\eta = 0$, $\lambda\varrho = 0$, $\mu\delta = 0$, $\eta\beta = 0$ and $\upsilon\gamma = 0$. Note that this is just a reflection of the diagram above in the diagonal axis. There are dual statements of lemma 3.1.21 and proposition 3.1.20. We now show how one uses these results.

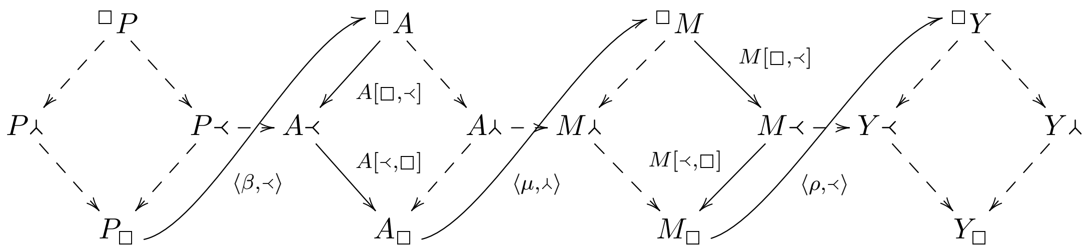
**Corollary 3.1.22.** *Given a horizontal diagram of squares through* $\mu : A \to M$*, together with vertical diagrams of squares through* $\beta : P \to A$ *and* $\rho : M \to Y$*, yielding a diagram of the form*

$$
\begin{tikzcd}
D \arrow[r] \arrow[d] & H \arrow[d] & & \\
X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & B \arrow[d] & \\
Q \arrow[r] & A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & W \arrow[d] \\
& N \arrow[r] & Y \arrow[r] \arrow[d] & L \arrow[d] \\
& & K \arrow[r] & J
\end{tikzcd}
$$

*there is an exact sequence of the form*

$$P_{\square} \longrightarrow A{\prec} \longrightarrow A_{\square} \longrightarrow {}^{\square}M \longrightarrow M{\prec} \longrightarrow {}^{\square}Y$$

*given by (certain compositions of) the solid arrows in the schema*




There is a dual result to corollary 3.1.22. Corollary 3.1.22, together with this dual result, are the first and second parts of [10, 1.7 Salamander Lemma] respectively. [10, Corollary 2.1] and [10, Corollary 2.2] are special cases of these results.

Using these special cases, Bergman gives proofs of the following three results.

**Lemma 3.1.23.** [67, Schanuel's Lemma 1.8.12] *If $P$ and $P'$ are projective objects in $\mathcal{A}$ and $\alpha : P \to M$ and $\alpha' : P' \to M$ are epic arrows in $\mathcal{A}$ then* $\ker(\alpha) \oplus P' \simeq P \oplus \ker(\alpha')$.

**Lemma 3.1.24.** [64, Snake Lemma 1.3.2] *Suppose we have a commutative diagram in $\mathcal{A}$ with exact rows, and of the form*

$$\begin{tikzcd} & A' \arrow[r, "w'"'] \arrow[d, "\alpha"'] & B' \arrow[r, "x'"'] \arrow[d, "\beta"'] & C' \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "w"] & B \arrow[r, "x"] & C & \end{tikzcd}$$

*Then there is an arrow $\delta : \ker(\gamma) \to \operatorname{coker}(\alpha)$ such that the canonical sequence*

$$\ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \longrightarrow \operatorname{coker}(\alpha) \longrightarrow \operatorname{coker}(\beta) \longrightarrow \operatorname{coker}(\gamma)$$

*is exact.*

**Theorem 3.1.25.** [64, Theorem 1.3.1] *If*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*is an exact sequence in $\mathcal{C}(\mathcal{A})$ then there is a collection of arrows $\omega_n : H_n(Z) \to H_{n+1}(Z)$ ($n \in \mathbb{Z}$) such that*

$$\cdots \longrightarrow H_{n-1}(Z) \xrightarrow{\quad\omega_{n-1}\quad} H_n(X) \xrightarrow{\quad H_n(f)\quad} H_n(Y) \xrightarrow{\quad H_n(g)\quad} H_n(Z) \xrightarrow{\omega_n} \cdots$$

*is an infinite exact sequence in $\mathcal{A}$.*

For a proof of this result see [64, pp.13,14]. Note that the arrows $\omega_n$ are usually described as the *induced arrows for the long exact sequence on homology.*

### 3.1.4 Adjunctions, Limits and Colimits.

We recall some notions about limits and colimits, and then extend them using notions introduced by Spaltenstein [60] for later use.

**Remark 3.1.26.** Fix a category $\mathcal{B}$, a small category $I$ and write $\mathcal{B}^I$ for the category of functors from $I$ to $\mathcal{B}$ (see [64, Functor Categories 1.6.4]). We say $\mathcal{B}$ *has all limits* (resp. *colimits*) *of shape* $I$ if the limit (resp. colimit) of every diagram of shape $I$ exists in $\mathcal{B}$.

In this case there is a functor $\lim : \mathcal{B}^I \to \mathcal{B}$ (resp. $\mathrm{colim} : \mathcal{B}^I \to \mathcal{B}$) taking any diagram to its limit (resp. colimit). By [64, Application 2.6.7, Exercise 2.6.4 and Variation 2.6.9] there is a diagonal functor $\Delta : \mathcal{B} \to \mathcal{B}^I$ for which, when lim (resp. colim) exists, there is an adjoint pair $(\Delta, \lim)$ (resp. $(\mathrm{colim}, \Delta)$). In this case, by [64, Theorem 2.6.1] if $\mathcal{B}$ is abelian then lim is left exact when it exists, and colim is right exact when it exists. We say $\mathcal{B}$ *has all its limits* (resp. *colimits*) if for every small category $I$ it has all limits (resp. colimits) of shape $I$.

**Definition 3.1.27.** [64, Definition 2.6.13] Let $I$ be a small category with $ob(I) \neq \emptyset$. We say $I$ is *filtered* (resp. *cofiltered*) if for every pair of objects $X$ and $Y$

(a) for any objects $i$ and $j$ there is an object $k$ and arrows $i \to k$ and $j \to k$ (resp. $k \to i$ and $k \to j$), and

(b) for any arrows $u$ and $v$ of the form $i \to j$ there is an arrow $w : j \to k$ (resp. $t : h \to i$) with $wu = wv$ (resp. $ut = vt$).

A *filtered colimit* (resp. *cofiltered limit*) is the colimit (resp. limit) of a diagram of shape $I$ where $I$ is some filtered (resp. cofiltered) category.

**Example 3.1.28.** If $I$ is a partially ordered set we can consider the category (which is also denoted $I$) whose objects are elements of $I$, and where there is a unique arrow $i \to j$ whenever $i \leq j$. Because of this uniqueness, condition (b) holds from definition 3.1.27. Hence $I$ is filtered (resp. cofiltered) iff every pair of elements in $I$ has an upper (a lower) bound in $I$.

**Definition 3.1.29.** We call a partially ordered set $I$ *directed* if it is filtered when considered as a category. In this case, a *direct* (resp. *inverse*) *$I$-system* in a category $\mathcal{B}$ is a diagram in $\mathcal{B}$ of shape $I$ (resp. $I^{op}$)[3]. If it exists, the *direct limit* (resp. *inverse limit*) of a direct system $F : I \to \mathcal{B}$ (resp. inverse system $G : I^{op} \to \mathcal{B}$) is the colimit $\mathrm{colim}(F)$ (resp. limit $\lim(G)$) of this diagram. In this case we change notation to $\varinjlim(F) := \mathrm{colim}(F)$ (resp. $\varprojlim(F) := \lim(F)$). We say $\mathcal{B}$ *has all its direct* (resp. *inverse*) *limits* if for every directed partially ordered set $I$ it has all colimits (resp. limits) of shape $I$.

We say that a covariant functor $G : \mathcal{Y} \to \mathcal{Z}$ *preserves small direct* (resp. *inverse*) *limits* if for every direct system $F : I \to \mathcal{Y}$ (resp. inverse system $F : I^{op} \to \mathcal{Y}$ ) whose direct limit $\varinjlim(F)$ (resp. inverse limit $\varprojlim(F)$) exists in $\mathcal{Y}$, the direct limit $\varinjlim(GF)$ (resp. inverse limit $\varprojlim(GF)$) exists in $\mathcal{Z}$, and there is an isomorphism $G(\varinjlim(F)) \to \varinjlim(GF)$ (resp. $G(\varprojlim(F)) \to \varprojlim(GF)$).

We say that a contravariant functor $G : \mathcal{Y} \to \mathcal{Z}$ *transforms small direct* (resp. *inverse*) *limits into small inverse* (resp. *direct*) *limits* if for every direct system $F : I \to \mathcal{Y}$ (resp. inverse system $F : I^{op} \to \mathcal{Y}$ ) whose direct limit $\varinjlim(F)$ (resp. inverse limit $\varprojlim(F)$) exists in $\mathcal{Y}$, the inverse limit $\varprojlim(GF)$ (resp. direct limit $\varinjlim(GF)$) exists in $\mathcal{Z}$, and there is an isomorphism $G(\varinjlim(F)) \to \varprojlim(GF)$ (resp. $G(\varprojlim(F)) \to \varinjlim(GF)$).

[3] that is, a functor of the form $I \to \mathcal{B}$ (resp. $I^{op} \to \mathcal{B}$)

### 3.1.5 Rings with Enough Idempotents.

**Definition 3.1.30.** [27, p.95] (Unital, Local And Colocal Modules) Let $\Gamma$ be a ring, possibly without a unit. We say a $\Gamma$-module $M$ is *unital* if $\Gamma M = M$. We say $M$ is *local* (resp. *colocal*) if it has a unique maximal (resp. simple) submodule $M'$.

(Local And Simple Idempotents) An idempotent $e \in \Gamma$ is called *left* (resp. right) *local* if the left (resp. right) $\Gamma$-module $\Gamma e$ (resp. $e\Gamma$) is local with unique maximal submodule is $\mathrm{rad}(\Gamma)e$ (resp. $e\mathrm{rad}(\Gamma)$). We say $e$ is *left* (resp. *right*) *simple* if it is left (resp. right) local and $\mathrm{rad}(\Gamma)e = 0$ (resp. $e\mathrm{rad}(\Gamma) = 0$). An idempotent which is left and right local is called *local*. An idempotent which is left and right simple is called *simple*.

(Complete Sets Of Idempotents) We say $\Gamma$ has *enough idempotents* if there is a set $E = \{e_i \mid i \in I\}$ of idempotents for which $\Gamma = \bigoplus_i \Gamma e_i = \bigoplus_i e_i\Gamma$. In this case we call $E$ a *complete set of* (orthogonal) *idempotents*. Furthermore, if each idempotent in $E$ has a given property we give the same name to $E$. For example we call $E$ *left local* if each idempotent in $E$ is left local. We write $\Gamma$-**MOD** for the category of all modules and $\Gamma$-**Mod** for the full subcategory of $\Gamma$-**MOD** consisting of unital modules.

(Local Units) Note that if $\Gamma$ has a complete set of idempotents $E$ (as above) then any $\alpha \in \Gamma$ can be written as a finite sum $\alpha = \sum_{i,j\in I} \alpha_{ij}$ where $\alpha_{ij} \in e_i\Gamma e_j$ for each $i, j \in I$. Let $I(\alpha)$ be the finite subset of $I$ consisting of all $l \in I$ such that $\alpha_{il} \neq 0$ for some $i$ or $\alpha_{lj} \neq 0$ for some $j$. By definition we have $\alpha_{ij} = e_\alpha \alpha_{ij} = \alpha_{ij} e_\alpha$ for any $i, j \in I$ where $e_\alpha = \sum_{l\in I(\alpha)} e_l$. More generally for elements $a_1, \ldots, a_d \in \Gamma$ for each $t$ with $1 \leq t \leq d$ we have $a_t = ea_t = a_t e$ where we let $I(a_1, \ldots, a_d) = \bigcup_{t=1}^{d} I(a_t)$ and $e = \sum_{l\in I(a_1,\ldots,a_d)} e_l$. We call $e$ the *local unit* for the elements $a_1, \ldots, a_d$.

**Example 3.1.31.** Let $Q$ be any quiver. Note that $RQ = \bigoplus_v RQe_v = \bigoplus_v e_v RQ$ where $v$ runs through all the vertices. Hence $RQ$ is a ring with enough idempotents, and so any element $z \in RQ$ defines a local unit.

**Definition 3.1.32.** (Radical, Socle) By the *radical* $\mathrm{rad}(M)$ (resp. *socle* $\mathrm{soc}(M)$ ) of a module $M$ we mean the intersection (resp. sum) of its maximal (resp. simple) submodules.

(Essential And Superfluous Submodules, Notation: $\leqslant$, $\leqslant_{\mathrm{s}}$, $\leqslant_{\mathrm{e}}$) We write $N \leqslant M$ to mean $N$ is a submodule of $M$. In this case $N$ will be called *superfluous* (resp. *essential*) if, for any other proper (resp. non-zero) $L \leqslant M$ we have $N + L \neq M$ (resp. $N \cap L \neq 0$), in which case we write $N \leqslant_{\mathrm{s}} M$ (resp. $N \leqslant_{\mathrm{e}} M$).

To proceed we require some results about modules over a (possibly non-unital) ring $\Gamma$ with enough idempotents. The first of these (below) should shed light on how to adapt proofs for unital rings to (more generally) rings with enough idempotents.

**Lemma 3.1.33.** [2, 5.19] *Let $\Gamma$ be a ring with a complete set of idempotents $\{e_i \mid i \in I\}$ and let $M$ be a $\Gamma$-module. For any $L \leqslant M$ we have $L \leqslant_{\mathrm{e}} M$ iff for any non-zero $m \in M$ there is some $\gamma \in \Gamma$ for which $\gamma m \in L$ and $\gamma m \neq 0$.*

*Proof.* Suppose $L \leqslant_{\mathrm{e}} M$ and fix $m \in M$ with $m \neq 0$. Since $m \in M = \Gamma M$ there must by some $\gamma \in \Gamma$ and some $m' \in M$ for which $\gamma m' = m$. Let $e$ be a local unit for $\gamma$, and so $e\gamma = \gamma$. This means $m = e\gamma m' = em$ and so $\Gamma m \neq 0$ which gives $\Gamma m \cap L \neq 0$ since $L \leqslant_{\mathrm{e}} M$. Conversely if there is some $P \leqslant M$ with $P \neq 0$ and $L \cap P = 0$ then there is some $p \in P$ with $p \neq 0$ and we must have $\gamma p = 0$ for any $\gamma \in \Gamma$ such that $\gamma p \in L$. $\square$

In the citations above, and for each part of lemma 3.1.34 below, we refer to the book by Anderson and Fuller [2]. In this book any ring is assumed to be unital. These results have been collected together for one reason: the proof from [2] easilly adapts to a proof for the case where $\Gamma$ has enough idempotents with few or no adjustments. Lemma 3.1.33 is an example of this adaptation.

**Lemma 3.1.34.** *Let $\Gamma$ be a ring with a complete set of orthogonal idempotents $\{e_i \mid i \in I\}$. Then the following statements hold.*

*If $M$ is any $\Gamma$-module then*

(ia) [2, 9.7 and 9.13] $\mathrm{rad}(M) = \sum_{N \leqslant_{\mathrm{s}} M} N$ *and* $\mathrm{soc}(M) = \bigcap_{N \leqslant_{\mathrm{e}} M} N$,

(ib) [2, 5.17.] *if $N_1, \ldots, N_m \leqslant M$ are superfluous (resp. essential) then so is $\sum_{i=1}^{n} N_i$ (resp. $\bigcap_{i=1}^{n} N_i$),* (ic) [2, 9.8 and 9.14] *if* $f : M \to N$ *is a* $\Gamma$*-module homomorphism then* $f(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$ *and* $f(\mathrm{soc}(M)) \subseteq \mathrm{soc}(N)$,

(id) $M$ *is finitely generated iff there is an epimorphism* $\bigoplus_{i\in J} \Gamma e_i \to M$ *for some* $J = \bigsqcup_{r=1}^{n} I(r)$ *where each* $I(r) \subseteq I$ *is finite.*

*If* $\{M_a \mid a \in A\}$ *is a set of unital* $\Gamma$*-modules, then*

(iia) [2, 16.11] *we have that* ($\bigoplus M_a$ *is projective iff each* $M_a$ *is projective) and that* ($\prod_a M_a$ *is injective iff each* $M_a$ *is injective),*

(iib) [2, 9.19] $\mathrm{rad}(\bigoplus M_a) = \bigoplus \mathrm{rad}(M_a)$ *and* $\mathrm{soc}(\bigoplus M_a) = \bigoplus \mathrm{soc}(M_a)$,

*and if* $N_a \leqslant M_a$ *for each* $a \in A$, *then*

(iiia) [2, 6. Exercises, 2 (1)] $\bigoplus M_a/N_a \simeq \bigoplus M_a/\bigoplus N_a$,

(iiib) [2, 5.20] *if* $A$ *is finite then* $\bigoplus N_a \leqslant_{\mathrm{s}} \bigoplus M_a$ *(resp.* $\bigoplus N_i \leqslant_{\mathrm{e}} \bigoplus M_i$*) iff* $N_i \leqslant_{\mathrm{s}} M_i$ *(resp.* $N_i \leqslant_{\mathrm{e}} M_i$*)* $\forall i \in I$,

(iiic) [2, 9.2] *if* $M_a = M$ *and* $N_a$ *is simple for each* $a$, *then any* $N \leqslant \sum_a N_a$ *gives some* $B \subseteq A$ *such that the sum* $N + \sum_{b\in B} N_b$ *is direct and equals* $\sum_a N_a$.

We now adapt the proof of [48, (24.7) Theorem] (see also [2, 17.4. Proposition]) to rings with enough idempotents. Unlike the results above, we give a proof. This is because non-trivial steps are required.

**Lemma 3.1.35.** [2, 17.2, 17.10 and 17.14] *Let* $\Gamma$ *be a ring with a complete set of idempotents* $E = \{e_i \mid i \in I\}$.

(i) *Every unital* $\Gamma$*-module is the epimorphic image of a free module.*

(ii) *If* $P$ *is a unital* $\Gamma$*-module then (*$P$ *is projective iff it is a summand of a free module).*

(iii) *If* $P$ *is a unital projective* $\Gamma$*-module then* $\mathrm{rad}(P) = \mathrm{rad}(\Gamma)P \subsetneq P$.

*Proof.* (i) Let $X$ denote the underlying set of some unital $\Gamma$-module $M$. Setting $\pi(\mu) = \sum_{x\in X} \mu_x x$ for any $\mu = \sum_{x\in X} \mu_x \in \Gamma^{(X)} = \bigoplus_x \Gamma$ (where $\mu_x = 0$ for all but finitely many $x$) gives a well-defined $\Gamma$-module homomorphism $\pi : \Gamma^{(X)} \to M$.

The map $\pi$ is onto in this case, because any $m = \Gamma M$ satisfies $m = \gamma m'$ for some $\gamma \in \Gamma$ and $m' \in M$: and so $\pi(\mu) = m$ where $\mu = \sum_{x\in X} \mu_x$ is defined by setting $\mu_{m'} = \gamma$ and $\mu_x = 0$ for $x \neq m'$.

(ii) Since $\Gamma = \bigoplus_i \Gamma e_i$, $\Gamma$ is unital and projective considered as a module over itself by [65, 49.2 (3)]. Now suppose $P$ is a unital projective $\Gamma$-module. By (i) $P = \mathrm{im}(\pi)$ where $\pi : \Gamma^{(X)} \to P$ is some epimorphism. Since $P$ is projective $\pi$ is a retraction and so $P$ summand of a free $\Gamma$-module $F$, say $F = P' \oplus K$ where $P' \simeq P$.

Since $\Gamma$ has enough idempotents any element $x \in \mathrm{rad}(\Gamma)$ defines a local unit $e$ (such that $x = xe$) and so $\mathrm{rad}(\Gamma) \subseteq \mathrm{rad}(\Gamma)\Gamma$. We always have the reverse inclusion, and so if $F = \Gamma^{(Y)}$ for some set $Y$ then $\mathrm{rad}(F) = \mathrm{rad}(\Gamma)F$, and so $\mathrm{rad}(P') \oplus \mathrm{rad}(K) = \mathrm{rad}(\Gamma)P' \oplus \mathrm{rad}(\Gamma)K$ . This shows $\mathrm{rad}(P') = \mathrm{rad}(\Gamma)P'$ and so $\mathrm{rad}(P) = \mathrm{rad}(\Gamma)P$. Showing $\mathrm{rad}(\Gamma)P \subsetneq P$ will be more involved. For a contradiction assume $P$ has no maximal submodule and so $P = \mathrm{rad}(P)$.

Since $P$ is a summand of $F$ there is a section $\sigma : P \to F$, a set $X$ and a subset $T \subseteq I \times X$ such that $F = \bigoplus_{t\in T} \Gamma e_t$ where we write $e_t$ for $e_i$ whenever $t = (i, x)$. Let $\pi : F \to P$ be the retraction for $\sigma$ (and so $\pi\sigma = \mathrm{id}$). For each $t$ we have $\pi(e_t) \in P = \mathrm{rad}(P)$ and so $\sigma(\pi(e_t)) \in \mathrm{rad}(F)$ by proposition 3.1.12. Since $\mathrm{rad}(F) = \bigoplus_{t\in T} \mathrm{rad}(\Gamma)e_t$ for each $t$ we have a finite subset $V_t$ of $T$ and elements $r_{st} \in \mathrm{rad}(\Gamma)e_s$ for each $s \in V_t$ such that $\sigma(\pi(e_t)) = \sum_{s\in V_t} r_{st}$. Since $P$ is non-trivial choose $0 \neq p \in P$ and write $\sigma(p) = \sum_{u\in U} \gamma_u$ for $\gamma_u \in \Gamma e_u$ and some finite subset $U$ of $T$. Writing $\sigma(p) = \sigma(\pi(\sigma(p)))$ shows

$$\textstyle\sum_{u\in U} \gamma_u = \sigma(\pi(\sum_{u\in U} \gamma_u e_u))) = \sum_{u\in U} \gamma_u \sigma(\pi(e_u)) = \sum_{u\in U} \gamma_u \sum_{s\in V_u} r_{su}.$$

Since $U$ is finite and (hence) the union $\bigcup_{u\in U} V_u$ is finite we can find a common finite subset $L$ of elements $l, m \in T$ such that: $\gamma_l = \gamma_u$ if $l = u$ for some $u \in U$, and otherwise $\gamma_l = 0$; and $r_{lm} = r_{st}$ if $m \in U$ and $l = s$ for some $s \in V_m$, and $r_{lm} = 0$ otherwise. Since $p \neq 0$ we can identify $L$ with $\{1, \ldots, n\}$ for some $n > 0$ which (together with the above) gives $\sum_{l=1}^n \gamma_l = \sum_{l=1}^n \gamma_l \sum_{m=1}^n r_{lm}$. Let $e$ be the local unit for the elements $\gamma_1, \ldots, \gamma_n, r_{11}, \ldots, r_{nn} \in \Gamma$ and consider the unital ring $e\Gamma e$ with unit $e$.

We now have a system of linear equations in $e\Gamma e$ given by $\sum_{m=1}^{n} \gamma_l(e\delta_{lm} - r_{lm}) = 0$ (where $\delta_{lm} = 1$ iff $l = m$, and otherwise $\delta_{lm} = 0$) for each $l$. We have defined a matrix $A = (a_{lm})$ where $a_{lm} = e\delta_{lm} - r_{lm}$. The matrix $A$ defines a unit in the matrix ring $\mathbb{M}_n(e\Gamma e)$ by [48, (4.5) Corollary. (A)], and by the above we have $A(\gamma_1, \ldots, \gamma_n)^t = 0$. Altogether we have $\gamma_l = 0$ for each $l$ and so $p = 0$ which is a contradiction. As required we have shown $P = \mathrm{rad}(P)$ is impossible. □

**Example 3.1.36.** Recall (for example [48, p.336, Definition 23.1]) that a unital ring $\Omega$ is *semiperfect* if it is *semilocal* (that is, $\Omega/\mathrm{rad}(\Omega)$ is a semisimple ring) and if idempotents in $\Omega/\mathrm{rad}(\Omega)$ lift to idempotents in $\Omega$ (that is, for any element $e \in \Omega$ with $e^2 - e \in \mathrm{rad}(\Omega)$ there is some idempotent $f \in \Omega$ such that $e - f \in \mathrm{rad}(\Omega)$). This is equivalent to either one of the following statements

(i) Every finitely generated $\Omega$-module has a projective cover (see definition [48, (24.9) Definition] and [48, (24.16) Theorem]).

(ii) $\Omega$ has a complete set of orthogonal local idempotents $\{e_1, \ldots, e_n\}$ (see [48, (23.6) Theorem]).

Under any of these assumptions every finitely generated projective $\Omega$-module is a direct sum of modules of the form $\Omega e_i$ (see [48, (24.14) Corollary]).

In paragraph 3, line 21 of [27, p.95] Fuller noted that the equivalence of (i) and (ii) from example 3.1.36 has been generalised to rings with a complete set of local idempotents. This generalisation is due to Harada: (i) and (ii) in example 3.1.36 generalise respectively to 1') and 2) in [36, Theorem 2].

**Lemma 3.1.37.** *Let $\Gamma$ be a ring with a complete set of local idempotents $E = \{e_i \mid i \in I\}$. Then $\Gamma e_i$ is indecomposable and projective for each $i$, and any indecomposable projective $\Gamma$-module is isomorphic to one of these.*

In the proof below we freely use results about module categories from the books by Anderson and Fuller [2] and Wisbauer [65].

*Proof.* Since $\Gamma = \bigoplus_i \Gamma e_i$ each $\Gamma e_i$ is a summand of a free module, and projective by lemma 3.1.35. Suppose $\Gamma e_i = M \oplus N$ for some submodules $M$ and $N$ of $\Gamma e_i$. Since $e_i$ is local the quotient $\Gamma e_i/\mathrm{rad}(\Gamma e_i)$ is simple.

Since $M \oplus N/\mathrm{rad}(M \oplus N) \simeq (M/\mathrm{rad}(M)) \oplus (N/\mathrm{rad}(N))$ by lemma 3.1.34 (iib) and (iiia) without loss of generality we must have $N/\mathrm{rad}(N) = 0$. Since $N$ is a summand of the free module $\Gamma$, $N$ is projective by [65, 49.2 (3)], and since $N = \mathrm{rad}(N)$ we must have $N = 0$ by lemma 3.1.35.

Let $\overline{\Gamma} = \Gamma/\mathrm{rad}(\Gamma)$. As a $\Gamma$-module we have $\overline{\Gamma} \simeq \bigoplus_i \overline{\Gamma} e_i$ by lemma 3.1.34 (iib) and (iiia) and lemma 3.1.35. Hence $\overline{E} = \{e_i + \mathrm{rad}(\Gamma) \mid i \in I\}$ is a complete set of simple idempotents in $\overline{\Gamma}$. Let $P$ be an indecomposable projective $\Gamma$-module and so $\overline{P} = P/\mathrm{rad}(P)$ is a $\overline{\Gamma}$-module. We can always find a free $\overline{\Gamma}$-module $\overline{F}$ together with a $\overline{\Gamma}$-module epimorphism $\theta : \overline{F} \to \overline{P}$. As $\overline{F}$ is free there is a set $T$ for which $\overline{F} = \bigoplus_{(i,t)\in I\times T} \overline{\Gamma} e_i$.

By lemma 3.1.34 (iiic) the exact sequence $0 \to \ker(\theta) \to \overline{F} \to \overline{P} \to 0$ yields a subset $S \subseteq I \times T$ and an isomorphism $\overline{P} \simeq \bigoplus_{(i,t)\in S} \overline{\Gamma} e_i$. Since $P$ is indecomposable it is non-trivial and so by lemma 3.1.35 we have that $\overline{P} \neq 0$ as $P$ is projective. Hence there is some element $(i,t) \in S$ for which $\overline{\Gamma} e_i \neq 0$.

Recall that by [65, 49.7], if $A$ is any left ideal of $\Gamma$ then ($A \subseteq \mathrm{rad}(\Gamma)$ iff $Ae \subseteq_{\mathrm{e}} \Gamma e$ for every idempotent $e \in \Gamma$. In particular, we have that $\mathrm{rad}(\Gamma)e_i$ is a superfluous submodule of $\Gamma e_i$, and as $e_i$ is local this means the canonical epimorphism $\Gamma e_i \to \overline{\Gamma} e_i$ is a projective cover of $\Gamma$-modules.

We also have an epimorphism of $\Gamma$-modules $P \to \overline{\Gamma} e_i$ which, by lemma 3.1.10, defines an isomorphism $P \simeq \overline{\Gamma} e_i \oplus P'$ for some submodule $P'$ of $P$. Since $P$ is indecomposable and $\overline{\Gamma} e_i \neq 0$ we must have $P \simeq \overline{\Gamma} e_i$ as required. □

**Definition 3.1.38.** Let $\Gamma$ be a ring with a complete set $E$ of local idempotents $e_i$.

(Notation: $\bar{f}$) If $f : N \to N'$ is any homomorphism of $\Gamma$-modules then $f(n) \in \mathrm{rad}(N')$ for any $n \in \mathrm{rad}(N)$ by [65, 21.6 (1i)], and hence there is an induced homomorphism $\bar{f} : \overline{N} \to \overline{N'}$ where $\overline{N} = N/\mathrm{rad}(N)$, $\overline{N'} = N'/\mathrm{rad}(N')$ and $\bar{f}(n + \mathrm{rad}(N)) = f(n) + \mathrm{rad}(N')$ for each $n + \mathrm{rad}(N) \in \overline{N}$.

(Reflects Modulo The Radical) We say a full subcategory $\mathcal{C}$ of $\Gamma$-**MOD** *reflects monomorphisms* (resp. *epimorphisms*, resp. *isomorphisms*) *modulo the radical* if for any homomorphism $f : N \to N'$ in $\mathcal{C}$, if $\bar{f}$ is a monomorphism (resp. epimorphism, resp. isomorphism) then $f$ is a monomorphism (resp. epimorphism, resp. isomorphism).

(Quasi-Free Modules) [65, §49] We say a module is *quasi-free* if it is a direct sum of modules of the form $\Gamma e_i$. We write $\Gamma$-**Quas** (resp. $\Gamma$-**quas**) for the full subcategory of $\Gamma$-**Mod** (resp. $\Gamma$-**mod**) consisting of quasi-free modules.

**Lemma 3.1.39.** (see [40, Lemma 2.2, p.218]) *Let $\Gamma$ be a ring with a complete set of local idempotents $E = \{e_i \mid i \in I\}$. If $\Gamma$-**quas** reflects monomorphisms (resp. epimorphisms, resp. isomorphisms) modulo the radical then so does $\Gamma$-**Proj**.*

For the proof of lemma 3.1.39 we need the following.

**Lemma 3.1.40.** *Let $\Gamma$ be a ring with a complete set of local idempotents $E = \{e_i \mid i \in I\}$. Then any quasi-free $\Gamma$-module is the union (and hence direct limit in $\Gamma$-**Mod**) of its finitely generated quasi-free summands.*

*Proof.* Let $F = \bigoplus_{x \in X} \Gamma e_x$ where $X$ is some subset of a disjoint union of copies of $I$. Let $\mathcal{Z}$ be the set of finitely generated direct summands $Z$ of $F$. If $m \in F$ then there is some $i(1), \dots, i(n) \in I$ and some $\gamma_1, \dots, \gamma_n \in \Gamma$ for which $m = \sum_{j=1}^{n} \gamma_j e_{i(j)}$. This shows $m$ lies in $\bigoplus_{j=1}^{n} \Gamma e_{i(j)}$ and hence $F$ is the union of its finitely generated quasi-free summands. Write $\iota_X$ for the inclusion of any $X \in \mathcal{X}$ into $F$. Write $\iota_{X,Y}$ for the inclusion of any $X \in \mathcal{Z}$ into some other $Y \in \mathcal{Z}$. It is clear that $\iota_X = \iota_Y \iota_{X,Y}$.

Now assume there is an object $G$ and arrows $\alpha_X : X \to G$ for each $X \in \mathcal{Z}$ such that $\alpha_X = \alpha_Y \iota_{X,Y}$. To show $\langle F, \iota_X \rangle$ is direct limit it suffices to show there is a unique homomorphism $\alpha : F \to G$ such that $\alpha \iota_X = \alpha_X$. Let $m \in F$. Consider the set $\mathcal{Z}(m)$ of $M \in \mathcal{Z}$ for which $m \in M$. We have already seen that $m$ lies in some $M \in \mathcal{Z}$, and so $\mathcal{Z}(m)$ is empty. Let $F(m) = \bigcap_{M \in \mathcal{Z}(m)} M$.

Define the map $\alpha : F \to G$ by setting $\alpha(m) = \alpha_{F(m)}(m)$ for each $m \in F$. Note that $\alpha_{F(m)}(m) = \alpha_X(\iota_{F(m),X}(m))$ and so $\alpha(\iota_X(m)) = \alpha_X(m)$ which means $\alpha \iota_X = \alpha_X$.

To show $\alpha$ is unique, suppose $\beta : F \to G$ satisfies $\beta\iota_X = \alpha_X$ for each $X \in \mathcal{Z}$. Then for each $m \in F$ we have $\beta(m) = \beta(\iota_{F(m)}(m)) = \alpha_{F(m)}(m) = \alpha(m)$. □

*Proof of lemma 3.1.39.* Let $f : M \to N$ be an arrow in $\Gamma$-**Proj** such that $\bar{f} : \overline{M} \to \overline{N}$ is a epimorphism. We will prove $f : M \to N$ is a epimorphism assuming $\Gamma$-**proj** reflects epimorphisms modulo the radical.

The proof that ($\bar{f}$ is a monomorphism implies $f$ is a monomorphism) in case $\Gamma$-**proj** reflects monomorphisms modulo the radical is similar, and omitted. It shall follow from these two cases that ($\bar{f}$ is an isomorphism implies $f$ is an isomorphism) in case $\Gamma$-**quas** reflects isomorphisms modulo the radical.

Recall $\overline{M} = M/\mathrm{rad}(M)$, $\overline{N} = N/\mathrm{rad}(N)$ and $\bar{f}(m + \mathrm{rad}(M)) = f(m) + \mathrm{rad}(N)$. By lemma 3.1.35 $M$ and $N$ are direct summands of free modules. Let $C$ be a $\Gamma$-module such that $M \oplus C$ is free. Note that $\bar{f} \oplus \mathrm{id}_{\overline{C}} : \overline{M} \oplus \overline{C} \to \overline{N} \oplus \overline{C}$ is a monomorphism (resp. epimorphism), and it suffices to show the map $f \oplus \mathrm{id}_C : M \oplus C \to M \oplus C$ is injective (resp. surjective).

Hence we can assume $M$ is free. Similarly we can assume $N$ is free. Thus $f : M \to N$ is a homomorphism of (quasi-)free $\Gamma$-modules such that $\bar{f} : \overline{M} \to \overline{N}$ is a monomorphism (resp. epimorphism). Note that $\overline{M}$ and $\overline{N}$ are direct sums of modules of the form $\overline{\Gamma}e_i$. Hence $\overline{M}$ and $\overline{N}$ are semisimple modules. This means $\bar{f} : \overline{M} \to \overline{N}$ is a section (resp. retraction). Let $\bar{g} : \overline{N} \to \overline{M}$ be the left (resp. right) inverse of $\bar{f}$, and so $\bar{g}\bar{f}$ (resp. $\bar{f}\bar{g}$) is the identity map on $\overline{M}$ (resp. $\overline{N}$).

Suppose we are in the case where $\bar{f} : \overline{M} \to \overline{N}$ is a retraction, and let $\bar{g}$ be a right inverse of $\bar{f}$. Since $N$ is quasi-free write $N = \bigoplus_{y\in Y} \Gamma e_y$ for some subset $Y$ of a disjoint union of copies of $I$. Similarly write $M = \bigoplus_{x\in X} \Gamma e_x$ for some subset $X$ of a disjoint union of copies of $I$. We want to prove $f : M \to N$ is an epimorphism. Let $n \in N$ and $S$ be a finite subset of $Y$ such that $n$ lies in the submodule $L = \bigoplus_{y\in S} \Gamma e_y$ of $N$. For each $y \in Y$ we have $\bar{g}(e_y) = \sum_{x\in X} \bar{\mu}_{x,y} e_x$ for elements $\bar{\mu}_{x,y} \in \overline{\Gamma}$. Consider the finite set $V$ of $x \in X$ such that $\bar{\mu}_{x,y} \neq 0$ for some $y \in S$. Since $\overline{M}$ is semisimple there is a finite subset $T$ of $V$ such that the image of $\bar{g}$ upon restriction to $\overline{L}$ is $\bigoplus_{x\in T} \overline{\Gamma}e_x$. Let $\overline{P} = \bigoplus_{x\in T} \overline{\Gamma}e_x$.

Let $\bar{g}| : \overline{L} \to \overline{P}$ be the restriction of $\bar{g}$ to $\overline{L}$. Let $P = \bigoplus_{x \in T} \Gamma e_x$. Since $\bar{g}|$ is onto for any $x \in T$ there is an element $\bar{l}_x \in \overline{L}$ such that $\bar{g}(\bar{l}_x) = e_x + \mathrm{rad}(P)$. Let $\bar{l}_x = \sum_{y \in S} \bar{\gamma}_{x,y} e_y$ for some elements $\bar{\gamma}_{x,y} \in \overline{\Gamma}$. This means $\bar{f}(e_x + \mathrm{rad}(P)) = \sum_{y \in S} \bar{\gamma}_{x,y} e_y$, and so $f(e_x) - \sum_{y \in S} \gamma_{x,y} e_y \in \bigoplus_{y \in S} \mathrm{rad}(\Gamma) e_y$ for some lifts $\gamma_{x,y}$ of $\bar{\gamma}_{x,y}$. Hence $f(e_x) \in L$ for each $x \in T$.

This means the restriction $f|$ of $f$ to $P$ defines a homomorphism $P \to L$. Since $\bar{f}$ is a left inverse of $\bar{g}$, the restriction $\bar{f}\big| : \overline{P} \to \overline{L}$ is a left inverse of $\bar{g}|$. This shows $f| : P \to L$ is an epimorphism of finitely generated quasi-free modules, since $\Gamma$-**quas** reflects epimorphisms. Hence there is some $m \in P \subseteq M$ such that $f(m) = f|(m) = n$, and so $f$ is surjective. $\square$

## 3.2 Homotopy Categories.

**Assumption:** In section 3.2 let $\mathcal{A}$ be a complete and cocomplete abelian category.

### 3.2.1 Complexes.

**Definition 3.2.1.** (COCHAIN COMPLEXES) A (*cochain*) *complex* $X$ *in* $\mathcal{A}$ is a collection of arrows $d_X^i : X^i \to X^{i+1}$ ($i \in \mathbb{Z}$) in $\mathcal{A}$ such that $d_X^{i+1} d_X^i = 0$ for all $i$ (see [67, p. 300, Definitions 3.5.1 and 3.5.2]). Note that Weibel [64] uses the convention of chain complexes. The maps $d_X^i$ will be called the *differentials* of $X$.

(NOTATION: $\mathcal{C}(\mathcal{A})$) Let $\mathcal{C}(\mathcal{A})$ be the category whose objects are complexes in $\mathcal{A}$, and where $f \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$ is given by collection of arrows $f^i : X^i \to Y^i$ ($i \in \mathbb{Z}$) in $\mathcal{A}$ such that $f^{i+1} d_X^i = d_Y^i f^i$ for all $i$.

(CATEGORIES OF COMPLEXESR, NOTATION:) For a full subcategory $\mathcal{X}$ of $\mathcal{A}$ let $\mathcal{C}(\mathcal{X})$ be the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of complexes $X$ where $X^i$ is an object in $\mathcal{X}$ for all $i$. We use $\mathcal{C}^{\pm}(\mathcal{X})$ be the full subcategory of $\mathcal{C}(\mathcal{X})$ consisting of complexes $X$ such that $X^i = 0$ for $\pm i \ll 0$. Let $\mathcal{C}^{\emptyset}(\mathcal{X}) = \mathcal{C}(\mathcal{X})$ and let $\mathcal{C}^b(\mathcal{X})$ be the full subcategory of $\mathcal{C}(\mathcal{X})$ consisting of objects $X$ which lie in both $\mathcal{C}^+(\mathcal{X})$ and $\mathcal{C}^-(\mathcal{X})$.

(TRANSLATION FUNCTOR, NOTATION: [1]) There is an automoprhism [1] of $\mathcal{C}(\mathcal{A})$ defined as follows. For any complex $X$ the image $X[1]$ of $X$ under [1] is given by setting $X[1]^n = X^{n-1}$ and $d_{X[1]}^n = d_X^{n-1}$. For any $f \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$ the image $f[1]$ of $f$ under [1] is given by setting $f[1]^n = f^{n-1}$. Let $[-1]$ be the inverse of [1], and [0] be the identity functor on $\mathcal{C}(\mathcal{A})$. For each $t > 0$ let $[t] = [1] \circ \cdots \circ [1]$ (the composition of [1] with itself $t$ times) and $[-t] = [-1] \circ \cdots \circ [-1]$ (the composition of [1] with itself $t$ times).

**Remark 3.2.2.** Let $f : X \to Y$ be an arrow in $\mathcal{C}(\mathcal{A})$. Since $f^{n+1} d_X^n \mathrm{k}(f^n) = 0$ there is an arrow $d_{\ker(f)}^n : \ker(f^n) \to \ker(f^{n+1})$ for which $d_X^n \mathrm{k}(f^n) = \mathrm{k}(f^{n+1}) d_{\ker(f)}^n$. Hence there is a complex $\ker(f)$ and a morphism of complexes $\mathrm{k}(f) : \ker(f) \to X$. Similarly we may define $\mathrm{coker}(f)$ and an arrow $\mathrm{c}(f) : Y \to \mathrm{coker}(f)$.

Another iteration of this construction gives objects $\mathrm{coim}(f)$ and $\mathrm{im}(f)$ and arrows $\mathrm{c}(\mathrm{k}(f)) : X \to \mathrm{coim}(f)$, $\mathrm{k}(\mathrm{c}(f)) : \mathrm{im}(f) \to Y$ and $\bar{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$ where $f = \mathrm{k}(\mathrm{c}(f))\bar{f}\mathrm{c}(\mathrm{k}(f))$ and $\bar{f}$ is an isomorphism of complexes. This together with the proof of [64, Theorem 1.2.3] shows $\mathcal{C}(\mathcal{A})$ is abelian.

Sometimes it will be easier to define a complex in terms of the infinite diagram the complex defines in $\mathcal{A}$.

**Definition 3.2.3.** [64, Truncations 1.2.7 and 1.2.8] Let $X$ be a complex and $n$ be an integer. The *truncations* of $X$ to the *left* (resp. right) are denoted $\tau_{\leq n}X$ and $\sigma_{\leq n}X$ (resp. $\tau_{>n}$ and $\sigma_{>n}$). They are defined as follows:

$$\begin{tikzcd}
\tau_{\leq n}X : & \cdots \arrow[r] & X^{n-2} \arrow[r, "d_X^{n-2}"] & X^{n-1} \arrow[d, "\mathrm{c}(\mathrm{k}(d_X^{n-1}))"'] & & \ker(d_X^n) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
& & & \mathrm{coim}(d_X^{n-1}) \arrow[rr, "\overline{d_X^{n-1}}"] & & \mathrm{im}(d_X^{n-1}) \arrow[u, "h(d_X^{n-1}, d_X^n)"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\tau_{>n}X : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathrm{coim}(d_X^n) \arrow[dr, "\overline{d_X^n}"'] & X^{n+1} \arrow[r, "d_X^{n+1}"] & X^{n+2} \arrow[r] & \cdots \\
& & & & & \mathrm{im}(d_X^n) \arrow[u, "\mathrm{k}(\mathrm{c}(d_X^n))"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma_{\leq n}X : & \cdots \arrow[r] & X^{n-2} \arrow[r, "d_X^{n-2}"] & X^{n-1} \arrow[r, "d_X^{n-1}"] & X^n \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma_{>n} : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X^{n+1} \arrow[r, "d_X^{n+1}"] & X^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{array}{cccccccc}
\text{degree :} & \cdots & n-2 & n-1 & n & n+1 & n+2 & \cdots
\end{array}$$

The truncations of the form $\tau_{\leq n}$ and $\tau_{>n}$ are called *good* truncations. The truncations of the form $\sigma_{\leq n}$ and $\sigma_{>n}$ are called *brutal* truncations.

**Remark 3.2.4.** For any object $X$ in $\mathcal{C}(\mathcal{A})$ and any $n \in \mathbb{Z}$ there is an arrow $h(d_X^{n-1}, d_X^n) : \mathrm{im}(d_X^{n-1}) \to \ker(d_X^n)$ for which $\mathrm{k}(d_X^n)h(d_X^{n-1}, d_X^n) = \mathrm{k}(\mathrm{c}(d_X^{n-1}))$.

Since $d^n_Y f^n \mathrm{k}(d^n_X) = f^{n+1} d^n_X \mathrm{k}(d^n_X) = 0$ there is an arrow $\theta^n : \ker(d^n_X) \to \ker(d^n_Y)$ for which $\mathrm{k}(d^n_Y)\theta^n = f^n \mathrm{k}(d^n_X)$. Dually there is an arrow $\psi^{n-1} : \mathrm{im}(d^{n-1}_X) \to \mathrm{im}(d^{n-1}_Y)$ for which $\mathrm{k}(\mathrm{c}(d^{n-1}_Y))\psi^n = f^n \mathrm{k}(\mathrm{c}(d^{n-1}_X))$.

Altogether we have $\theta^n h(d^{n-1}_X, d^n_X) = h(d^{n-1}_Y, d^n_Y)\psi^{n-1}$ which shows $\mathrm{c}(h(d^{n-1}_Y, d^n_Y))\theta^n h(d^{n-1}_X, d^n_X) = 0$. If we let $H^n(X) = \mathrm{coker}(h(d^{n-1}_X, d^n_X))$ and $H^n(Y) = \mathrm{coker}(h(d^{n-1}_Y, d^n_Y))$ this means there is an arrow $H^n(f) : H^n(X) \to H^n(Y)$ for which $H^n(f)\mathrm{c}(h(d^{n-1}_X, d^n_X)) = h(d^{n-1}_Y, d^n_Y)\theta^n$.

**Definition 3.2.5.** By remark 3.2.4 there is an additive functor $H^n : \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ called the $n^{th}$ *cohomology* functor (see [1, p.596, Lemma 3.4] or [67, p.301, Remark 3.5.7] in case $\mathcal{A} = \Lambda$-**Mod**). Note that $X$ is exact at $X^i$ iff $H^i(X) = 0$, and if this holds for all $i$ we say $X$ is *exact*. We call an arrow $f : X \to Y$ in $\mathcal{C}(\mathcal{A})$ a *quasi-isomorphism* provided $H^n(f)$ is an isomorphism (in $\mathcal{A}$) for all $n \in \mathbb{Z}$.

Any complex $X$ gives a complex $H(X)$ of cohomology groups where we let $d^i_{H(X)} = 0$ for all $i$. For another full subcategory $\mathcal{Y}$ of $\mathcal{A}$, and for $\delta, \epsilon \in \{\emptyset, -, +, b\}$, let $\mathcal{C}^{\delta,\epsilon}_{\mathcal{Y}}(\mathcal{X})$ be the full subcatgeory of $\mathcal{C}^{\delta}(\mathcal{X})$ such that $H(X)$ is an object in $\mathcal{C}^{\epsilon}(\mathcal{Y})$. We call the complex $X$ *acyclic* if $H(X)$ is the zero complex.

**Definition 3.2.6.** A *projective resoloution* of an object $X$ in $\mathcal{A}$ is a complex $P$ where $P^n = 0$ for $n > 1$, $P^1 = X$, and for $n \leq 0$ the object $P^n$ is projective and $P$ is exact at $P^n$.

**Lemma 3.2.7.** [33, p.141, 3. Theorem] *If $f : X \to Y$ is an arrow in $\mathcal{A}$ and $P_X$ and $P_Y$ are projective resoloutions of $X$ and $Y$ then there is a morphism $g : P_X \to P_Y$ such that $d^0_{P_Y} g^0 = f d^0_{P_X}$.*

**Lemma 3.2.8.** [64, Horseshoe Lemma 2.2.8] *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{A}$. Let $P_X$ and $P_Z$ be projective resoloutions of $X$ and $Z$ respectively.*

*Then $P_Y = P_X \oplus P_Z$ is a projective resoloution of $Y$, and there is an exact sequence of complexes $0 \to P_X \to P_Y \to P_Z \to 0$ given by $0 \to X \to Y \to Z \to 0$ in degree $0$ and the canonical arrows of the direct sum $P^n_Y = P^n_X \oplus P^n_Z$ in degree $n < 0$.*

### 3.2.2 Double Complexes.

**Definition 3.2.9.** [64, Example 1.2.4] Consider a collection of objects $X^{(i,j)}$ (for $i, j \in \mathbb{Z}$) together with two collections of arrows; $d_{X,c}^{(i,j)} : X^{(i,j)} \to X^{(i+1,j)}$ which we call *column differentials*, and $d_{X,r}^{(i,j)} : X^{(i,j)} \to X^{(i,j+1)}$ which we call *row differentials*. These arrows satisfy the *relations of a double complex*: $d_{X,c}^{(i+1,j)} d_{X,c}^{(i,j)} = 0$, $d_{X,r}^{(i,j+1)} d_{X,r}^{(i,j)} = 0$ and $d_{X,c}^{(i,j+1)} d_{X,r}^{(i,j)} + d_{X,r}^{(i+1,j)} d_{X,c}^{(i,j)} = 0$. The double complex $X$ may be depicted by

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & X^{(i-1,j-1)} \arrow[r, "d_{X,r}^{(i-1,j-1)}"] \arrow[d, swap, "d_{X,c}^{(i-1,j-1)}"] & X^{(i-1,j)} \arrow[r, "d_{X,r}^{(i-1,j)}"] \arrow[d, "d_{X,c}^{(i-1,j)}"] & X^{(i-1,j+1)} \arrow[r] \arrow[d, "d_{X,c}^{(i-1,j+1)}"] & \cdots \\
\cdots \arrow[r] & X^{(i,j-1)} \arrow[r, "d_{X,r}^{(i,j-1)}"] \arrow[d, swap, "d_{X,c}^{(i,j-1)}"] & X^{(i,j)} \arrow[r, "d_{X,r}^{(i,j)}"] \arrow[d, "d_{X,c}^{(i,j)}"] & X^{(i,j+1)} \arrow[r] \arrow[d, "d_{X,c}^{(i,j+1)}"] & \cdots \\
\cdots \arrow[r] & X^{(i+1,j-1)} \arrow[r, "d_{X,r}^{(i+1,j-1)}"] \arrow[d] & X^{(i+1,j)} \arrow[r, "d_{X,r}^{(i+1,j)}"] \arrow[d] & X^{(i+1,j+1)} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Let $\mathcal{C}^2(\mathcal{A})$ denote the category whose objects are double complexes, and where a *homomorphism* $f : X \to Y$ of double complexes consists of a collection of arrows $f^{(i,j)} : X^{(i,j)} \to Y^{(i,j)}$ in $\mathcal{A}$ for which $d_{Y,r}^{(i,j)} f^{(i,j)} = f^{(i,j+1)} d_{X,r}^{(i,j)}$ and $d_{Y,c}^{(i,j)} f^{(i,j)} = f^{(i+1,j)} d_{X,c}^{(i,j)}$. The composition of homomorphisms $f : X \to Y$ and $g : Y \to Z$ is defined by setting $(gf)^{(i,j)} = g^{(i,j)} f^{(i,j)}$.

Considering the first (resp. second) of these relations, the $i^{th}$ row (resp. $j^{th}$ column) defines a complex $X(i,r)$ (resp. $X(j,c)$) given by $(X(i,r))^n = X^{(i,n)}$ and $d_{X(i,r)}^n = d_{X,r}^{(i,n)}$ (resp. $(X(j,c))^m = X^{(m,j)}$ and $d_{X(j,c)}^m = d_{X,c}^{(m,j)}$) for each integer $n$ (resp. $m$). Making use of [64, Sign Trick 1.2.5], the collection of arrows of the form $c(X,i)^n := (-1)^n d_{X,c}^{(i,n)}$ (resp. $r(X,j)^m := (-1)^m d_{X,h}^{(m,j)}$) defines a morphism of complexes $c(X,i) : X(i,r) \to X(i+1,r)$ (resp. $r(X,j) : X(j,c) \to X(j+1,c)$), given by looking at the column (resp. row) differentials whose domain lies in row $i$ (resp. column $j$).

**Proposition 3.2.10.** *There is a functor* $\mathrm{R}$ *(resp.* $\mathrm{C}$*) defining an equivalence of categories* $\mathcal{C}(\mathcal{C}(\mathcal{A})) \to \mathcal{C}^2(\mathcal{A})$ *such that* $\mathrm{R}(X)(i,r) = X^i$ *(resp.* $\mathrm{C}(X)(j,c) = X^j$*) for any object* $X$ *in* $\mathcal{C}(\mathcal{C}(\mathcal{A}))$*.*

*Proof.* We shall prove the existence of the functor R. The proof of the existence of the functor C will be similar. For an object $X$ in $\mathcal{C}(\mathcal{C}(\mathcal{A}))$ let $\mathrm{R}(X)^{(i,j)}$ be the $j^{th}$ homogeneous component of the complex $X^i$ in $\mathcal{C}(\mathcal{A})$. Let $d^{(i,j)}_{\mathrm{R}(X),r} = (-1)^j d^j_{X^i}$ and let $d^{(i,j)}_{\mathrm{R}(X),c} = (d^i_X)^j$, the $j^{th}$ homogeneous component of the arrow $d^i_X : X^i \to X^{i+1}$ in $\mathcal{C}(\mathcal{A})$. By construction $\mathrm{R}(X)(i,r) = X^i$ and it is clear that R defines an equivalence of categories. □

Since $\mathcal{C}(\mathcal{A})$ is abelian, proposition 3.2.10 shows $\mathcal{C}^2(\mathcal{A})$ is also abelian.

Fix $j \in \mathbb{Z}$. Consider the objects $\mathrm{im}(r(X,j-1))$ and $\ker(r(X,j))$ in $\mathcal{C}(\mathcal{A})$ which define subobjects of the complex $X(j,c)$ given by column $j$. From our discussion on cohomology above, for each $i$ the arrows $d^i_{\ker(r(X,j))} : \ker(d^{(i,j)}_{X,r}) \to \ker(d^{(i+1,j)}_{X,r})$ and $d^i_{\mathrm{im}(r(X,j-1))} : \mathrm{im}(d^{(i,j-1)}_{X,r}) \to \mathrm{im}(d^{(i+1,j-1)}_{X,r})$ satisfy

$$d^i_{\ker(r(X,j))} h(d^{(i,j-1)}_{X,r}, d^{(i,j)}_{X,r}) = h(d^{(i+1,j-1)}_{X,r}, d^{(i+1,j)}_{X,r}) d^i_{\mathrm{im}(r(X,j-1))}$$

Since this is true for each $i$ the collection of arrows $h(d^{(n,j-1)}_{X,r}, d^{(n,j)}_{X,r})$ (for $n \in \mathbb{Z}$) defines an arrow $\mathrm{im}(r(X,j-1)) \to \ker(r(X,j))$ in $\mathcal{C}(\mathcal{A})$. Taking the cokernel gives an object $H^j(X(-,r))$ in $\mathcal{C}(\mathcal{A})$ where $H^j(X(-,r))^i = H^j(X(i,r))$ and $d^i_{H^j(X(-,r))} = H^j(d^{(i,j)}_{X,c})$ for each $i \in \mathbb{Z}$.

**Definition 3.2.11.** [64, Definition 5.7.1] Let $M$ be a complex in $\mathcal{C}(\mathcal{A})$. A (*Cartan-Eilenberg*) *projective resoloution* of $M$ is a double complex $P$ such that for each $j$

(a) $P^{(0,j)} = M^j$ and $P^{(i,j)} = 0$ for $i > 1$,

(b) $P^{(i,j)}$ is projective for each $i < 0$,

(d) if $M^j = 0$ then $P(j,c) = 0$,

(e) $\mathrm{im}(r(P,j))$ defines a projective resoloution of $\mathrm{im}(d^j_M)$, and

(f) $H^j(P(-,r))$ defines a projective resoloution of $H^j(M)$.

Hence $P$ has the form

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & P^{(-2,j-1)} \arrow[r, "d_{P,r}^{(-2,j-1)}"] \arrow[d, "d_{P,c}^{(-2,j-1)}"'] & P^{(-2,j)} \arrow[r, "d_{P,r}^{(-2,j)}"] \arrow[d, "d_{X,c}^{(-2,j)}"] & P^{(-2,j+1)} \arrow[r] \arrow[d, "d_{P,c}^{(-2,j+1)}"] & \cdots \\
\cdots \arrow[r] & P^{(-1,j-1)} \arrow[r, "d_{P,r}^{(-1,j-1)}"] \arrow[d, "d_{P,c}^{(-1,j-1)}"'] & P^{(-1,j)} \arrow[r, "d_{P,r}^{(-1,j)}"] \arrow[d, "d_{P,c}^{(-1,j)}"'] & P^{(-1,j+1)} \arrow[r] \arrow[d, "d_{P,c}^{(-1,j+1)}"'] & \cdots \\
\cdots \arrow[r] & M^{j-1} \arrow[r, "d_M^{j-1}"] \arrow[d] & M^{j} \arrow[r, "d_M^{j}"] \arrow[d] & M^{j+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

and we write $P_{-}$ for the *deleted projective resoloution*, the double complex given by

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & P^{(-2,j-1)} \arrow[r, "d_{P,r}^{(-2,j-1)}"] \arrow[d, "d_{P,c}^{(i-1,j-1)}"'] & P^{(-2,j)} \arrow[r, "d_{P,r}^{(-2,j)}"] \arrow[d, "d_{X,c}^{(-2,j)}"] & P^{(-2,j+1)} \arrow[r] \arrow[d, "d_{P,c}^{(-2,j+1)}"] & \cdots \\
\cdots \arrow[r] & P^{(-1,j-1)} \arrow[r, "d_{P,r}^{(-1,j-1)}"] \arrow[d] & P^{(-1,j)} \arrow[r, "d_{P,r}^{(-1,j)}"] \arrow[d] & P^{(-1,j+1)} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Note that definition 3.2.11 differs slightly to the one given by [64, Definition 5.7.1]. To correct this one takes the deleted resoloution.

**Lemma 3.2.12.** [64, Exercise 5.7.1] *Let $M$ be a complex in $\mathcal{C}(\mathcal{A})$.*

*If $P$ is a projective resoloution of $M$ then for each $j$:*

(i) *the complex $P(j,c)$ is a projective resoloution of $M^j$; and*

(ii) *the complex* $\ker(r(P,j))$ *is a projective resoloution of* $\ker(d_M^j)$.

Using lemma 3.2.8 we have the following result.

**Lemma 3.2.13.** [64, Lemma 5.7.2] *Let $\mathcal{A}$ have enough projectives. Then every complex has a projective resoloution.*

**Definition 3.2.14.** Suppose $\mathcal{A}$ is a cocomplete category. Let $X$ be a double complex. For each $n \in \mathbb{Z}$ let $\mathrm{tot}^{\oplus}(X)^n = \bigoplus_{i+j=n} X^{(i,j)}$ and for each $i,j \in \mathbb{Z}$ with $i+j=n$ let $\iota_X^{(i,j)} : X^{(i,j)} \to \mathrm{tot}^{\oplus}(X)^n$ and $\pi_X^{(i,j)} : \mathrm{tot}^{\oplus}(X)^n \to X^{(i,j)}$ be the canonical arrows.

There is an arrow $d^n_{\mathrm{tot}^{\oplus}(X)} : \mathrm{tot}^{\oplus}(X)^n \to \mathrm{tot}^{\oplus}(X)^{n+1}$ (given by the universal property) such that $d^n_{\mathrm{tot}^{\oplus}(X)}\iota_X^{(i,j)} = \iota_X^{(i+1,j)} d_{X,c}^{(i,j)} + \iota_X^{(i,j+1)} d_{X,r}^{(i,j)}$ (for each $i,j \in \mathbb{Z}$ with $i+j=n$). Using the uniqueness of the universal property together with the relations of a double complex one can show $d^{n+1}_{\mathrm{tot}(X)} d^n_{\mathrm{tot}(X)} = 0$ and so $\mathrm{tot}^{\oplus}(X)$ defines a complex of objects in $\mathcal{A}$, provided $\mathcal{A}$ is cocomplete.

Let $f : X \to Y$ is a homomorphism of double complexes and let $n \in \mathbb{Z}$. The collection of arrows $\iota_Y^{(i,j)} f^{(i,j)} : X^{(i,j)} \to \mathrm{tot}^{\oplus}(Y)^n$ (where $(i,j)$ runs through all pairs of integers with $i+j=n$) defines a unique arrow $\mathrm{tot}^{\oplus}(f)^n : \mathrm{tot}^{\oplus}(X)^n \to \mathrm{tot}^{\oplus}(Y)^n$ for which $\mathrm{tot}^{\oplus}(f)^n \iota_X^{(i,j)} = \iota_Y^{(i,j)} f^{(i,j)}$ for each $(i,j)$ (using the universal property). By construction $\mathrm{tot}^{\oplus}(f)^{n+1} d^n_{\mathrm{tot}^{\oplus}(X)} \iota_X^{(i,j)} = d^n_{\mathrm{tot}^{\oplus}(Y)} \mathrm{tot}^{\oplus}(f)^n \iota_X^{(i,j)}$ and hence there is a functor $\mathrm{tot}^{\oplus} : \mathcal{C}^2(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ taking a double complex to its total complex.

If $M$ is a chain complex then $M = \mathrm{tot}^{\oplus}(\mathrm{R}(A_M))$ where $A_M$ is the object of $\mathcal{C}(\mathcal{C}(\mathcal{A}))$ given by concentrating the complex $M$ in degree 0, and R is the functor from proposition 3.2.10. If $\theta : M \to N$ is an arrow in $\mathcal{C}(\mathcal{A})$ then similarly we have $\theta = \mathrm{tot}^{\oplus}(\mathrm{R}(A_\theta))$ where we let $(A_\theta)^0 = \theta$ and $(A_\theta)^n = 0$ for $n \neq 0$. This shows that $\mathrm{tot}^{\oplus}$ is full and dense.

**Lemma 3.2.15.** [64, Exercise 5.7.1] *Let $M$ be a complex and $P$ be a projective resoloution of $M$. Then there is a quasi-isomorphism $\theta : \mathrm{tot}^{\oplus}(P_{-}) \to M$.*

### 3.2.3 Homotopy Category.

**Definition 3.2.16.** We say $f$ and $g$ from $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$ are *homotopy equivalent*, and write $f \sim g$, provided there is a collection of arrows $s^i : X^i \to Y^{i-1}$ ($i \in \mathbb{Z}$) such that $f^i - g^i = d_Y^{i-1} s^i + s^{i+1} d_X^i$ for all $i$. The arrow $f$ is said to be *null homotopic* if $f$ is homotopy equivalent to the zero cochain map $0 \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$. A homotopy equivalence is an arrow $f \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$ such that there is some $g \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(Y, X)$ such that $fg \sim 1_Y$ and $gf \sim 1_X$ . A complex $X$ is said to be *null homotopic* if there is a homotopy equivalence between $X$ and the zero complex 0, or equivalently, if $\mathrm{id} : X \to X$ is null homotopic.

The relation $\sim$ defines an equivalence relation by [1, p.614, Exercise 4.4] which is compatible with addition, and hence there is an additive subgroup of $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$ given by the null homotopic cochain maps. Let $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$ denote the quotient group. Hence equivalence classes $[f] \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$ in have the form $[f] = \{g \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y) \mid f \sim g\}$. By [1, p.614, Exercise 4.6] the assignment $([g], [f]) \mapsto [gf]$ is well defined.

**Definition 3.2.17.** There is a category $\mathcal{K}(\mathcal{A})$, which we call the *homotopy category*, whose objects are complexes in $\mathcal{A}$ and where $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$ are defined above. By the *quotient* $\mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ we refer to the functor given by $X \mapsto X$ and $f \mapsto [f]$.

For $n \in \mathbb{Z}$ there is a *shift* functor $(-)[n] : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ given by $X[n]^i = X^{i+n}$ and $f[n]^i = f^{i+n}$ (see [64, Translations 1.2.9]). For $f \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$ (and for all $i$) let $\mathrm{cone}(f)^i = X^{i+1} \oplus Y^i$ and consider the canonical maps $\pi_X^{i+1} : \mathrm{cone}(f)^i \to X^{i+1}$, $\iota_X^{i+1} : X^{i+1} \to \mathrm{cone}(f)^i$, $\pi_Y^i : \mathrm{cone}(f)^i \to Y^i$ and $\iota_Y^i : Y^i \to \mathrm{cone}(f)^i$ that equip the direct sums in $\mathcal{A}$. If we let $d^i_{\mathrm{cone}(f)} = \iota^Y_{i+1}(d^i_Y \pi^Y_i - f^{i+1}\pi^X_{i+1}) - \iota^X_{i+2} d^{i+1}_X \pi^X_{i+1}$ for each $i$ then $\mathrm{cone}(f)$ is a complex by [1, p.615, Exercise 4.1], which we call the *mapping cone of* $f$. This complex can be depicted by drawing the differentials in matrix notation:

$$\cdots \xrightarrow{\begin{pmatrix} -d_X^{n-1} & 0 \\ -f^{n-1} & d_Y^{n-2} \end{pmatrix}} X^n \oplus Y^{n-1} \xrightarrow{\begin{pmatrix} -d_X^{n} & 0 \\ -f^{n} & d_Y^{n-1} \end{pmatrix}} X^{n+1} \oplus Y^{n} \xrightarrow{\begin{pmatrix} -d_X^{n+1} & 0 \\ -f^{n+1} & d_Y^{n} \end{pmatrix}} \cdots$$

**Example 3.2.18.** If $X$ is a complex write $\alpha_n : X^n \to X^{n+1} \oplus X^n$ and $\beta_n : X^{n+1} \oplus X^n \to X^n$ be the canonical monic and epic arrows that equip the direct sums in $\mathcal{A}$. Then, setting $s^n = -\alpha_n \beta_n$ gives a homotopy equivalence between $\mathrm{id} : X \to X$ and 0. Hence the mapping cone cone(id) is null homotopic (see [64, Exercise 1.5.1] and [64, Example 10.1.5]). This example gives a special case of the following.

**Lemma 3.2.19.** *If $f : X \to Y$ is an arrow in $\mathcal{C}(\mathcal{A})$ then*

[67, Lemma 3.5.32] (i) *$f$ is a quasi-isomorphim iff* $\mathrm{cone}(f)$ *is an acyclic complex, and*

(ii) *if* $\mathrm{id} : \mathrm{cone}(f) \to \mathrm{cone}(f)$ *is null homotopic then $f$ is a homotopy equivalence.*

### 3.2.4 Homotopic Minimality.

**Definition 3.2.20.** Let $\mathcal{P}$ denote the full subcategory of $\mathcal{A}$ consisting of projective objects. For an object $P$ in $\mathcal{P}$ and any $n \in \mathbb{Z}$ consider the complex $\mathrm{D}^n P$ whose homogeneous component is $P$ in degrees $n$ and $n+1$ and zero elsewhere, and where $d^n_{\mathrm{D}^n(P)}$ is the identity on $P$. $\mathrm{D}P$ is homotopy equivalent to zero so it is zero in the homotopy category (for example see [67, p.336, Lemma 3.5.44]).

**Lemma 3.2.21.** *For any complex $M$ in $\mathcal{C}(\mathcal{P})$ and any $n \in \mathbb{Z}$ there exist objects $P|M,n|$ in $\mathcal{P}$ and $M|n|$ in $\mathcal{C}(\mathcal{P})$ together with an isomorphism of complexes $f|M,n| : M \to M|n| \oplus \mathrm{D}^n P|M,n|$ such that*

(i) $M|n|^r = M^r$ *and* $f|M,n|^r = \mathrm{id}$ *for* $r \in \mathbb{Z}$ *with* $r \neq n, n+1$,

(ii) $d^r_{M|n|} = d^r_M$ *for* $r \in \mathbb{Z}$ *with* $r \neq n-1, n, n+1$,

(iii) $\mathrm{im}(d^n_{M|n|}) \subseteq \mathrm{rad}(M|n|^{n+1})$, *and*

(iv) *if* $\mathrm{im}(d^{n\pm 1}_M) \subseteq \mathrm{rad}(M^{n\pm 1+1})$ *then* $\mathrm{im}(d^{n\pm 1}_{M|n|}) \subseteq \mathrm{rad}(M|n|^{n\pm 1+1})$.

*Proof.* We apply lemma 3.1.13 to the arrow $d = d^n_M$ (where $L = M^n$ and $L' = M^{n+1}$). Let $P|M,n| = R$, $M|n|^n = W$, $M|n|^{n+1} = W'$, $f|M,n|^n = \alpha$, $f|M,n|^{n+1} = \alpha'$ and $\nu = d^n_{M|n|}$. This gives isomorphisms $f|M,n|^n : M^n \to M|n|^n \oplus P|M,n|$ and $f|M,n|$ : $M^{n+1} \to M|n|^{n+1} \oplus P|M,n|$ such that $\alpha' d^n_M \alpha^{-1} = d^n_{M|n|} \oplus \mathrm{id}$. Write $\alpha d^{n-1}_M$ as the column $(\gamma, \delta)^t$ and $d^{n+1}_M \alpha'^{-1}$ as the row $(\lambda, \mu)$ for arrows $\gamma : M^{n-1} \to M|n|^n$, $\delta : M^{n-1} \to P|M,n|$, $\lambda : M|n|^{n+1} \to M^{n+1}$ and $\mu : P|M,n| \to M^{n+2}$. So far we have a commutative diagram of the form

$$\begin{tikzcd}
M^{n-1} \arrow[r, "d^{n-1}_M"] \arrow[d, equal] & M^n \arrow[r, "d^n_M"] \arrow[d, "f|M,n|^n"] & M^{n+1} \arrow[r, "d^{n+1}_M"] \arrow[d, "f|M,n|^{n+1}"] & M^{n+2} \arrow[d, equal] \\
M^{n-1} \arrow[r, "(\gamma,\delta)^t"] & M|n|^n \oplus P|M,n| \arrow[r, "d^n_{M|n|}\oplus\mathrm{id}"] & M|n|^{n+1} \oplus P|M,n| \arrow[r, "(\lambda,\mu)"] & M^{n+2}
\end{tikzcd}$$

The commutativity of this diagram gives $d^n_{M_n}\gamma = 0$ and $\delta = 0$ since $\alpha' d^n_M d^{n-1}_M = 0$. Similarly $\lambda d^n_{M_n} = 0$ and $\mu = 0$.

Let $M_n^r = M^r$ and $f|n|^r = \mathrm{id}$ for $r \in \mathbb{Z}$ with $r \neq n, n+1$, and $d^r_{M|n|} = d^r_M$ for $r \in \mathbb{Z}$ with $r \neq n-1, n, n+1$. Letting $d^{n-1}_{M|n|} = (\lambda, 0)$ and $d^{n+1}_{M|n|} = (\gamma, 0)$ defines a complex $M|n|$ and $f|n|$ is the required isomorphism. (i), (ii) and (iii) follow by lemma 3.1.13 and by construction. If $\mathrm{im}(d_M^{n\pm 1}) \subseteq \mathrm{rad}(M^{n\pm 1+1})$ then by propsition 3.1.12 we have

$$\mathrm{im}(d^n_{M|n|}) = \mathrm{im}((\mathrm{id}, 0) \circ (f|M,n|^{n+1} d^n_M (f|M,n|^n)^{-1}) \circ (\mathrm{id}, 0)^t) \subseteq \mathrm{rad}(M|n|^{n+1})$$

as required. $\square$

We switch to the more convenient notation.

For a fixed integer $m \leq -1$ and an object $M$ in $\mathcal{C}(\mathcal{P})$ applying lemma 3.2.21 $-m$-times yields complexes in $\mathcal{C}(\mathcal{P})$ $M_{[0]} = M$, $M_{[-1]} = M|-1|$ up to $M_{[m]} = (M|m+1|)|m|$ (defined iteratively). For each $t \in \mathbb{Z}$ with $m \leq t \leq -1$ let; $P_t = P|M_{[t+1]}, t|$, $N_t = M_{[t]} \oplus \mathrm{D}^t P_t \oplus \cdots \oplus \mathrm{D}^{-1} P_{-1}$, $f_{[t]} = f|M_{[t+1]}, t|$, $g_t = f_{[t]} \oplus \mathrm{id}_{[t]}$ (for $t < -1$ (where $\mathrm{id}_{[t]}$ is the identity on $\mathrm{D}^t P_t \oplus \cdots \oplus \mathrm{D}^{-1} P_{-1}$ ) and $g_{-1} = f_{[-1]}$. So far we have a diagram describing $-m$ consecutive isomorphisms in $\mathcal{C}(\mathcal{P})$, given by the (rows of) downward arrows in the schema

$$\begin{tikzcd}
\cdots \arrow[r] & M^{-4}_{[0]} \arrow[r] \arrow[d, equal] & M^{-3}_{[0]} \arrow[r] \arrow[d, equal] & M^{-2}_{[0]} \arrow[r] \arrow[d, equal] & M^{-1}_{[0]} \arrow[r] \arrow[d] & M^{0}_{[0]} \arrow[r] \arrow[d] & M^{1}_{[0]} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & M^{-4}_{[-1]} \arrow[r] \arrow[d, equal] & M^{-3}_{[-1]} \arrow[r] \arrow[d, equal] & M^{-2}_{[-1]} \arrow[r] \arrow[d] & M^{-1}_{[-1]} \oplus P_{-1} \arrow[r] \arrow[d] & M^{0}_{[-1]} \oplus P_{-1} \arrow[r] \arrow[d, equal] & M^{1}_{[-1]} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & M^{-4}_{[-2]} \arrow[r] \arrow[d, equal] & M^{-3}_{[-2]} \arrow[r] \arrow[d] & M^{-2}_{[-2]} \oplus P_{-2} \arrow[r] \arrow[d] & M^{-1}_{[-2]} \oplus P_{-2} \oplus P_{-1} \arrow[r] \arrow[d, equal] & M^{0}_{[-2]} \arrow[r] \arrow[d, equal] & M^{1}_{[-2]} \arrow[r] \arrow[d, equal] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We shall now define an isomorphism $f_-$ given by composing all the rows. Observe that any column in the diagram above consists of at most two isomorphisms which are not the identity arrow. Let $p_m : M \to N_m$ be the composition $g_m \ldots g_{-1}$. Now let $r \in \mathbb{Z}$ be arbitrary. For $r \leq 0$ let $M^r_- = M^r_{[r-1]}$, $d^r_{M_-} = d^r_{M_{[r-1]}}$ and $f^r_- = p^r_{r-1}$.

Let $N_-^0 = M_-^0$ and $d_{N_-}^0 = (d_{M_-}^0 \;\; 0 \;\; 0)$ and for $r \leq -1$ let $N_-^r = M_-^r \oplus P_{r-1} \oplus P_r$ and

$$d_{N_-}^r = d_{M_-}^r \oplus \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} : N_-^r = M_-^r \oplus P_{r-1} \oplus P_r \longrightarrow M_-^{r+1} \oplus P_r \oplus P_{r+1} = N_-^{r+1}$$

For $r \geq 1$ let $M_-^r = N_-^r = M^r$, $d_{M_-}^r = d_{N_-}^r = d_M^r$ and $f_-^r = \mathrm{id}$. Note that $d_{N_-}^{-1}$ has codomain $M_-^0 \oplus P_{-1}$. By definition (and some simple matrix multiplication) $d_{N_-}^{r+1} d_{N_-}^r = 0$. The next two corollaries are some useful consequences of the lemma above for the updated notation.

**Corollary 3.2.22.** *In the above notation;*

(i) *$M_-$ and $N_-$ define objects in $\mathcal{C}(\mathcal{P})$ such that $M_- \oplus \bigoplus_{m\leq -1} \mathrm{D}^m P_m = N_-$,*

(ii) *$f_-$ defines an isomorphism of complexes from $M$ to $N_-$, and*

(iii) $\mathrm{im}(d_{M_-}^m) \subseteq \mathrm{rad}(M_-^{m+1})$ *for each integer $m \leq -1$.*

A dual result also holds, which will require similar constructions. We omit these details, since they are similar.

**Corollary 3.2.23.** *For any object $M$ in $\mathcal{C}(\mathcal{P})$ there are objects $M_+$ and $N_+$ in $\mathcal{C}(\mathcal{P})$ such that*

(i) *$N_+ = M_+ \oplus \bigoplus_{n\geq 0} \mathrm{D}^n P_n$ for some objects $P_n$ in $\mathcal{P}$,*

(ii) *there is an isomorphism of complexes $f_+ : M \to N_+$,*

(iii) $\mathrm{im}(d_{M_+}^r) \subseteq \mathrm{rad}(M_+^{r+1})$ *for each $r \geq 0$, and*

(iv) *for any $m \leq -1$, if* $\mathrm{im}(d_M^m) \subseteq \mathrm{rad}(M^{m+1})$ *then* $\mathrm{im}(d_{M_+}^m) \subseteq \mathrm{rad}(M_+^{m+1})$.

The following terminology comes from [45, Appendix B].

**Definition 3.2.24.** An object $M$ in $\mathcal{C}(\mathcal{P})$ is called *homotopically minimal* provided $\mathrm{im}(d_M^n) \subseteq \mathrm{rad}(M^{n+1})$ for all $n \in \mathbb{Z}$. Let $\mathcal{C}_{\min}(\mathcal{P})$ and $\mathcal{K}_{\min}(\mathcal{P})$ denote the full subcategories of $\mathcal{C}(\mathcal{P})$ and $\mathcal{K}(\mathcal{P})$ and consisting of homotopically minimal complexes (as in definition 3.2.24).

We now see an important result which was adapted from [45, Proposition B.2].

**Corollary 3.2.25.** *If $\mathcal{A}$ has projective covers and all of its radicals then $\mathcal{K}_{\min}(\mathcal{P})$ is a dense subcategory of $\mathcal{K}(\mathcal{P})$.*

*Proof.* For any collection of objects $P(n)$ in $\mathcal{P}$ (for $n \in \mathbb{Z}$) the complex $P = \bigoplus_{n \in \mathbb{Z}}(D^n P(n))$ has $P(m-1) \oplus P(m)$ in degree $m$ and hence is an object in $\mathcal{K}(\mathcal{P})$ isomorphic to zero. Let $(M_-)_+ = N$. By corollaries 3.2.22 and 3.2.23 there is an isomorphism between $M$ and $N$ in $\mathcal{K}(\mathcal{A})$ and $N$ is a homotopically minimal complex. $\square$

## 3.3 Derived Categories.

### 3.3.1 Localisation.

In order to introduce the derived category we need to define what it means to localise a category. We let $\mathcal{T}$ be any additive category.

**Definition 3.3.1.** [33, p.145] Let $\mathcal{S}$ be a collection of arrows in $\mathcal{T}$. We define an oriented graph $\Gamma(\mathcal{T},\mathcal{S})$ as follows. For each object in $\mathcal{T}$ we define a vertex in $\Gamma(\mathcal{T},\mathcal{S})$ using the same symbol. The set of edges with tail $V$ and head $W$ will consist of the arrows $a : V \to W$ in $\mathcal{T}$, together with an edge labeled $x_s$ for each arrow $s : W \to V$ in $\mathcal{S}$. A *path* in $\Gamma(\mathcal{T},\mathcal{S})$ is given by a symbol of the form $p = p_1 \circ \cdots \circ p_n$ where $n \geq 1$ and $p_i$ is an edge in $\Gamma(\mathcal{T},\mathcal{S})$ for each integer $i$ with $1 \leq i \leq n$ for which the tail of $p_i$ is the head of $p_{i+1}$ whenever $i < n$.

Let $\mathrm{comp}(V,W)$ be the set of all pairs of paths of the form $(ba, b \circ a)$ where $a \in \mathrm{Hom}_{\mathcal{T}}(V,X)$ and $b \in \mathrm{Hom}_{\mathcal{T}}(X,W)$ for some object $X$ in $\mathcal{T}$. For any vertex $V$ of $\Gamma(\mathcal{T},\mathcal{S})$ we let $\mathrm{triv}(V)$ be the set of pairs of paths of the form $(x_r r, \mathrm{id})$ or of the form $(\mathrm{id}, t x_t)$ where $r : V \to R$ (resp. $t : T \to V$) is some arrow in $\mathcal{S}$. We define the relation $R(V,W)$ on the set of all paths in $\Gamma(\mathcal{A},\mathcal{S})$ from $V$ to $W$ by $\mathrm{comp}(V,W)$ if $V \neq W$ and $\mathrm{comp}(V,W) \cup \mathrm{triv}(V)$ if $V = W$. Write $\sim (V,W)$ for the smallest equivalence relation containing the relation $R(V,W)$ and we write $p \sim p'$ for paths $p$ and $p'$ from $V$ to $W$ such that $(p,p') \in R(V,W)$.

The localisation of $\mathcal{T}$ at $\mathcal{S}$ will be denoted $\mathcal{T}[\mathcal{S}^{-1}]$ and defined as follows. The objects in $\mathcal{T}[\mathcal{S}^{-1}]$ will be the same as the vertices in $\Gamma(\mathcal{T},\mathcal{S})$, that is, the same as the objects in $\mathcal{T}$. For objects $V$ and $W$ in $\mathcal{T}[\mathcal{S}^{-1}]$ the collection $\mathrm{Hom}_{\mathcal{T}[\mathcal{S}^{-1}]}(V,W)$ will consist of all equivalence classes of paths in $\Gamma(\mathcal{T},\mathcal{S})$ from $V$ to $W$.

**Lemma 3.3.2.** (Universal property of localisation) *Let $\mathcal{S}$ be a collection of arrows in a category $\mathcal{T}$.*

(i) *There is a functor $\mathrm{Q} : \mathcal{T} \to \mathcal{T}[\mathcal{S}^{-1}]$ such that $\mathrm{Q}(s)$ is an isomorphism for all $s \in \mathcal{S}$.*

(ii) *If $F : \mathcal{T} \to \mathcal{T}'$ is a functor such that $F(s)$ is an isomorphism for any arrow $s \in \mathcal{S}$ then there is a unique functor $G : \mathcal{T}[\mathcal{S}^{-1}] \to \mathcal{T}'$ for which $F = G\mathrm{Q}$.*

We now see some conditions on $\mathcal{T}$ and $\mathcal{S}$ which, when held, yield a neater description of the category $\mathcal{T}[\mathcal{S}^{-1}]$.

**Definition 3.3.3.** [33, p.147, Definition] For $\mathcal{T}$ and $\mathcal{S}$ as above we say:

(a) $\mathcal{S}$ is *closed under identities* if for any object $X$ in $\mathcal{A}$ the arrow $\mathrm{id} : X \to X$ lies in $\mathcal{S}$;

(b) $\mathcal{S}$ is *closed under composition* if the composition $st$ lies in $\mathcal{S}$ for all $s, t \in \mathcal{S}$;

(c) $\mathcal{S}$ *satisfies the right Ore condition* if, for any arrows $f : A \to B$ and $s : B' \to B$ where $s$ lies in $\mathcal{S}$, there are arrows $g : A' \to B'$ and $t : A' \to A$ where $t$ lies in $\mathcal{S}$ and$ft = sg$;

(d) $\mathcal{S}$ *satisfies the left Ore condition* if, for any arrows $a : X \to Y$ and $t : X \to X'$ where $t$ lies in $\mathcal{S}$, there are arrows $b : X' \to Y'$ and $s : Y \to Y'$ where $s$ lies in $\mathcal{S}$ and $sa = bt$;

(e) $\mathcal{S}$ *has the left cancellation property* if, for any arrows $f : A \to B$ and $f' : A \to B$, if there is an arrow $s$ from $\mathcal{S}$ such that $fs = gs$ then there is an arrow $t$from $\mathcal{S}$ such that $tf = tg$; and

(f) $\mathcal{S}$ *has the right cancellation property* if, for any arrows $f : A \to B$ and $f' : A \to B$, if there is an arrow $t$from $\mathcal{S}$ such that $tf = tg$ then there is an arrow $s$ from $\mathcal{S}$ such that $fs = gs$.

[47, 3.1 Calculus of fractions] We say $\mathcal{S}$ *admits a calculus of left fractions* if conditions (a), (b), (d) and (f) all hold. We say $\mathcal{S}$ *admits a calculus of right fractions* if conditions (a), (b), (c) and (e) all hold. We say $\mathcal{S}$ is a *multiplicative system*[4]

**Example 3.3.4.** In the homotopy catery of an abelian category, the collection of quasi isomorphisms forms a multiplicative system. For a proof see [64, Proposition 10.4.1].

**Definition 3.3.5.** Let $\mathcal{S}$ be a multiplicative system in $\mathcal{T}$. For objects $V$, $W$ and $X$ from $\mathcal{T}$ a *(left) roof from $V$ to $W$ through $X$* is a pair $(s, f)$ where $f : X \to W$ is an arrow in $\mathcal{T}$ and $s : X \to V$ is an arrow in $\mathcal{S}$, drawn

$$V \xleftarrow{\quad s \quad} X \xrightarrow{\quad f \quad} W$$

[4] What we call multiplicative systems, Gelfand and Manin [33] call *localisation classes*.

Since $\mathcal{S}$ is closed under identities (definition 3.3.3 (a)) $(\mathrm{id}, \mathrm{id})$ defines a roof from $X$ to $X$ through $X$. Let $(s, f)$ be a roof from $V$ to $W$ through $X$ and let $(s', f')$ be a roof from $V$ to $W$ through $X'$. We say $(s, f)$ *is covered by* $(s', f')$, and write $(s, f) \leq (s', f')$, provided there is an arrow $t : X' \to X$ from $\mathcal{S}$ for which $st = s'$ and $ft = f'$. In this case there is a commutative diagram (which motivates the terminology)

$$\begin{tikzcd}
& X' \arrow[d, "t"] \arrow[ddl, "s'"'] \arrow[ddr, "f'"] & \\
& X \arrow[dl, "s"] \arrow[dr, "f"'] & \\
V & & W
\end{tikzcd}$$

We write $(s, f) \sim (s', f')$ provided there are arrows $t : X'' \to X$ and $t' : X'' \to X'$ from $X$ to $X'$ through (some object) $X''$ such that $s't' = st$ is an arrow in $\mathcal{S}$ and $f'g = ft$. Equivalently there is a roof $(s'', f'')$ from $V$ to $W$ through some object $X''$ which covers both $(s, f)$ and $(s', f')$. In this case we can summarise the situation using either of the commutative diagrams

$$\begin{tikzcd}
& & X'' \arrow[dl, "t"'] \arrow[dr, "t'"] & & \\
& X \arrow[dl, "s"'] \arrow[drrr, "f"'] & & X' \arrow[dlll, "s'"] \arrow[dr, "f'"] & \\
V & & & & W
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "s"'] \arrow[dr, "f"] & \\
V & X'' \arrow[l] \arrow[r] \arrow[u, "t"'] \arrow[d, "t'"] & W \\
& X' \arrow[ul, "s'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

and stipulating that (any one of) the (equal) compositions $X'' \to V$ must define an arrow in $\mathcal{S}$. If $(r, h)$ is a roof from $U$ to $V$ through $Y$ and $(s, f)$ is a roof from $V$ to $W$ through $X$ then, as $\mathcal{S}$ satisfies the right Ore condition (definition 3.3.3 (c)), there are arrows $t : Z \to Y$ and $t' : Z \to X$ such that $t$ is from $\mathcal{S}$ and $st' = ht$. In this case define the *composition* of $(r, h)$ and $(s, f)$ to be $(r, h) \circ (s, f) := (rt, fg)$.

For a proof of the next lemma see [33, p.149].

**Lemma 3.3.6.** *The relation $\sim$ on roofs defines an equivalence relation that respects the composition of roofs.*

**Definition 3.3.7.** We denote the *category of roofs* in $\mathcal{T}$ *with respect to* $\mathcal{S}$ by $\widehat{\mathcal{T}}(\mathcal{S})$, and define it as follows. The objects of $\widehat{\mathcal{T}}(\mathcal{S})$ will be the same as the objects in $\mathcal{T}$. We define the collection of arrows $\mathrm{Hom}_{\widehat{\mathcal{T}}(\mathcal{S})}(V, W)$ to consist of equivalence classes $\overline{(s, f)}$ of roofs $(s, f)$ from $V$ to $W$ through some $X$.

Note that there is a functor $\widehat{\mathrm{Q}} : \mathcal{T} \to \widehat{\mathcal{T}}(\mathcal{S})$ given by by taking an object $X$ in $\mathcal{T}$ to the corresponding object in $\widehat{\mathcal{T}}(\mathcal{S})$, and taking an arrow $f : X \to Y$ in $\mathcal{T}$ to the roof equivalence class $\overline{(\mathrm{id}, f)}$. From the universal property of locaising categories (see lemma 3.3.2) we have the following.

**Lemma 3.3.8.** [33, III.2.8 Lemma] *If $\mathcal{S}$ is a multiplicative system of arrows in a category $\mathcal{T}$ then the categories $\mathcal{T}[\mathcal{S}^{-1}]$ and $\widehat{\mathcal{T}}(\mathcal{S})$ are isomorphic.*

**Proposition 3.3.9.** [33, III.2.10 Proposition] *Let $\mathcal{S}$ be a multiplicative system of arrows in a category $\mathcal{T}$, and let $\mathcal{R}$ be a full subcategory of $\mathcal{T}$. Write $\mathcal{S}_{\mathcal{R}}$ for the class of all arrows in $\mathcal{S}$ between objects in $\mathcal{R}$.*

*Suppose that $\mathcal{S}_{\mathcal{R}}$ is a multiplicative system of arrows in $\mathcal{R}$, and that for every object $Z$ in $\mathcal{R}$ and every morphism $f : Y \to Z$ in $\mathcal{S}$ there is a morphism $g : X \to Y$ such that $fg \in \mathcal{S}_{\mathcal{R}}$. Then the inclusion functor $\widehat{\mathcal{R}}(\mathcal{S}_{\mathcal{R}}) \to \widehat{\mathcal{T}}(\mathcal{S})$ is full and faithful.*

### 3.3.2 Locally Small Compatible Systems.

Although all categories (such as $\mathcal{T}$) are assumed to be locally small, the category $\mathcal{T}[\mathcal{S}^{-1}]$ need not be. This issue is not something easy to sweep under the rug. When localising the homotopy catgeory $\mathcal{K}(\mathcal{A})$ at the multiplicative system of quasi isomorphisms we obtain the derived category $\mathcal{D}(\mathcal{A})$. It is well-known (and shall be seen) that $\mathcal{D}(\mathcal{A})$ is triangulated, so it must be additive and hence locally small. We would like to outline how to verify this. We now see some conditions which, when held by $\mathcal{T}$ and $\mathcal{S}$, force the category $\mathcal{T}[\mathcal{S}^{-1}]$ to be locally small.

By using Von Neumann-Bernays-Gödel (NBG) set theory we have a way of verifying when certain classes are sets. We follow Jech [39, p.5].

**Definition 3.3.10.** Recall that (for any $n \in \mathbb{N}$ with $n \geq 1$) a *formula* $\varphi$ is built from a countably infinite set of variables Var using the atomic symbols $\in$ and $=$ by means of connectives $\wedge$ and $\neg$ and the quatifier $\exists$. We use brackets (, ), to make sentences easier to read. We also shorthand formulas. For example, $\alpha \vee \beta$ means $\neg(\neg\alpha \wedge \neg\beta)$, and $\forall x\alpha$ means $\neg\exists x\neg\alpha$ (where $\alpha$ and $\beta$ are formulas and $x$ is a variable). A variable $x$ which occurs in a formula $\varphi$ is said to *freely* occur if $\varphi$ is not built using a formula of the form $\exists x\alpha$ (and therefore, the building of $\varphi$ cannot involve a formula of the form $\forall x\alpha$). The notation $\varphi(u_1, \ldots, u_n)$ describes a formula $\varphi$ where the free variables occurring in $\varphi$ are among the variables $u_1, \ldots, u_n$, which are assumed to be pairwise distinct, but not all $u_i$ need occur.

If $\varphi(x, u_1, \ldots, u_n)$ is a formula and $p_1, \ldots, p_n$ are sets then the class $C$ *definable from* $\varphi$ *and the parameters* $p_1, \ldots, p_n$ is the collection of sets $a$ for which the statement found by replacing $x, u_1, \ldots, u_n$ with $a, p_1, \ldots, p_n$ (resp.) is true. We use the notation $C = \{a \mid \varphi(a, p_1, \ldots, p_n)\}$. We write $b \in C$ to mean $b$ is a set found in the collection $C$. Two classes $C = \{a \mid \varphi(a, p_1, \ldots, p_n)\}$ and $D = \{b \mid \psi(b, q_1, \ldots, q_m)\}$ are said to be *equal* provided for any set $c$ we have $\varphi(c, p_1, \ldots, p_n) \leftrightarrow \psi(c, q_1, \ldots, q_m)$. In this case we write $C = D$.

**Example 3.3.11.** Let $F : \mathcal{C} \to \mathcal{D}$ be an isomorphism of categories where $\mathcal{C}$ is a locally small catgeory. Let $G : \mathcal{D} \to \mathcal{C}$ be the inverse to $F$. For objects $X$ and $Y$ in $\mathcal{D}$ we consider the subclass $F_{X,Y}$ of $\mathrm{Hom}_{\mathcal{C}}(G(X), G(Y)) \times \mathrm{Hom}_{\mathcal{D}}(X, Y)$ given by the pairs $(f, F(f))$.

$F_{X,Y}$ is a class mapping, and using the schema of replacement axiom we have that $\{F(f) \mid f \in \mathrm{Hom}_{\mathcal{C}}(G(X), G(Y))\}$ is a set because $\mathrm{Hom}_{\mathcal{C}}(G(X), G(Y))$ is a set by assumption. Thus $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ is a set and so $\mathcal{D}$ is also locally small.

**Definition 3.3.12.** [64, 10.3.6] A multiplicative system $\mathcal{S}$ of arrows in a category $\mathcal{T}$ is called *locally small* if for every object $X$ there is a set $\mathcal{S}_X$ of arrows $s \in \mathcal{S}$ of the form $X' \to X$, where for any arrow $f : W \to X$ in $\mathcal{S}$ there is an arrow $g : V \to W$ such that $fg \in S_X$.

**Example 3.3.13.** In the homotopy category of an abelian category, the multiplicative system of quasi isomorphisms (see example 3.3.4) is locally small. A proof of this is given in the proof of [64, Proposition 10.4.4]. In this proof one uses the idea from example 3.3.11.

The respective statement in the following is [47, Lemma 3.3.1].

**Corollary 3.3.14.** [64, Corollary 10.3.11] *Let $\mathcal{S}$ be a locally small multiplicative system in an additive (more generally, locally small) category $\mathcal{T}$. Then $\mathcal{T}[\mathcal{S}^{-1}]$ is additive (resp. locally small).*

**Definition 3.3.15.** For an abelian category $\mathcal{A}$ the *derived category* of $\mathcal{A}$ is denoted $\mathcal{D}(\mathcal{A})$ and defined by the localisation of $\mathcal{K}(\mathcal{A})$ at the class $\mathcal{S}_q$ of quasi isomorphisms in $\mathcal{K}(\mathcal{A})$. By examples 3.3.4 and 3.3.13 $\mathcal{S}_q$ is a locally small multiplicative system in $\mathcal{K}(\mathcal{A})$, and so $\mathcal{D}(\mathcal{A}) \simeq \widehat{\mathcal{K}}(\mathcal{A})(\mathcal{S}_q)$ is a locally small category.

### 3.3.3 K-Projective Complexes.

The final result of this section (corollary 3.3.28) may be summarised by [64, Exercise 10.4.5]. We shall explain some details for the proof, following work of Keller [43] and Spaltenstein [60].

**Definition 3.3.16.** [60, p.125, 0.4] For objects $X$ and $Y$ of $\mathcal{C}(\mathcal{A})$ the associated *hom-complex* is an object $\hom(X,Y)$ of $\mathcal{C}(\mathbf{Ab})$ defined by $\hom(X,Y)^n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i+n})$ for each $n \in \mathbb{Z}$, and for $(f_i)_i \in \hom(X,Y)^n$ setting $d^n_{\hom(X,Y)}((f_i)) = (d_Y^{i+n} f_i - (-1)^n f_{i+1} d_X^i)_i$.

For each $n$ consider the assignment $H^n(\hom(X,Y)) \to \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y[n])$ defined by $(f_i) + \mathrm{im}(d^{n-1}_{\hom(X,Y)}) \mapsto [f]$ (for each $(f_i) \in \ker(d^n_{\hom(X,Y)})$) where we let $f^i = f_i$. If $(f_i) \in \mathrm{im}(d^{n-1}_{\hom(X,Y)})$ then $f : X \to Y[n]$ is null-homotopic, where the homotopy $s$ is given by letting $s^j : X^j \to Y^{j+n-1}$ be the arrow $-f_j$ when $n$ is even, and $f_j$ when $n$ is odd.

Hence this gives a well-defined function. By a similar argument this function is injective, and it is clearly surjective and additive. Hence we have an isomorphism $H^n(\hom(X,Y)) \to \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y[n])$ of abelian groups (this is precisely [60, 0.4 (3)]).

**Definition 3.3.17.** [60, p.127, Definition] We say a complex $M$ of $\mathcal{C}(\mathcal{A})$ is *K-projective* if, for any acyclic complex $A$, the complex $\hom(M,A)$ in $\mathcal{C}(\mathbf{Ab})$ is acyclic. We let $\mathcal{C}_{\mathrm{p}}(\mathcal{A})$ and $\mathcal{K}_{\mathrm{p}}(\mathcal{A})$ denote the full subcategories of $\mathcal{C}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ respectively given by the K-projective complexes.

Together with the above, this means that a complex $M$ is K-projective iff $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(M, A) = 0$ for any acyclic complex $A$. This is the definition Keller uses (see [43, 1.1 Unbounded resoloutions]). By [43, Theorem 1.1 (a)] K-projective complexes are *cellular complexes* in the sense of Weibel (see [64, Exercise 10.4.5]).

**Example 3.3.18.** [60, 3.2 Examples (a)] Recall $\mathcal{A}$ is assumed to have enough projectives. Let $M$ be an object in $\mathcal{C}^-(\mathcal{P})$ where (we recall that) $\mathcal{P}$ is the full subcategory of $\mathcal{A}$ consisting of projective objects.

Without loss of generality to this example we assume $M^n = 0$ for $n > 0$. Let $f : M \to A$ be an arrow in $\mathcal{K}(\mathcal{A})$ where $A$ is acyclic. By dualising the proof given in [33, p.180] one can construct a homotopy equivalence $f \sim 0$ using induction and the definition of a projective object. Hence every object in $\mathcal{C}^-(\mathcal{P})$ is K-projective (see the dual argument to [33, p.180, Proof of III.5.22B, C] for details).

The next lemma generalises [67, Lemma 3.5.44] in light of example 3.3.18.

**Lemma 3.3.19.** *Any quasi-isomorphism between K-projective complexes is a homotopy equivalence.*

**Proposition 3.3.20.** [60, 1.4 Proposition] *For an object $M$ of $\mathcal{C}(\mathcal{A})$ the following are equivalent:*

(i) *$M$ is K-projective;*

(ii) *for any complex $N$ the homomorphism of abelian groups*

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(M, N) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M, N), \quad [f] \mapsto \overline{(\mathrm{id}, f)}$$

*(induced by the localisation functor) is an isomorphism;*

(iii) *for any quasi-isomorphism $s : L \to N$ and any arrow $f : M \to N$ there is an arrow $g : M \to L$ (in $\mathcal{C}(\mathcal{A})$) such that $[sg] = [f]$; and*

(iv) *for any quasi-isomorphism $s : P \to M$ there is an arrow $t : M \to P$ such that $[st] = [\mathrm{id}]$.*

To see an alternative explanation of example 3.3.18, apply the equivalence of (i) and (ii) in proposition 3.3.20 to [64, Corollary 10.4.7]. Recall definitions 3.1.27 and 3.1.29 and example 3.1.28.

**Definition 3.3.21.** [60, 0.5, p.126] We say an exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{C}(\mathcal{A})$ is *semisplit* if $f^n$ is a section (or equivalently, $g^n$ is a retraction) for each $n \in \mathbb{Z}$. If $\mathcal{B}$ is another abelian category, a (covariant or contravariant) functor $F : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ is said to *preserve semisplit sequences* if the image of any semisplit sequence is a semisplit sequence.

**Definition 3.3.22.** [60, Definitions 2.1 (a) and 2.6 (a)] Let $\mathcal{X}$ be a class of objects in $\mathcal{C}(\mathcal{A})$ and let $F : I \to \mathcal{C}(\mathcal{A})$ (resp. $F : I^{op} \to \mathcal{C}(\mathcal{A})$) be a direct (resp. inverse) system. We say $F$ is an $\mathcal{X}$-*special direct system* (resp. an $\mathcal{X}$-*special inverse system*) system if

(a) $I$ is well-ordered,

(b) if $i \in I$ does not have a predecessor then $F(i) = \varinjlim_{h<i} F(h)$ (resp. $F(i) = \varprojlim_{i<j} F(j)$),

and if $i \in I$ has a predecessor $i-1$, then

(c') $F(i-1 \to i)$ (resp. $F(i \to i-1)$) is monic (resp. epic),

(c") the object of $\mathcal{C}(\mathcal{A})$ given by $\mathrm{coker}(F(i-1 \to i))$ (resp. $\ker(F(i \to i-1))$) lies in $\mathcal{X}$, and

(c"') the exact sequence $0 \to F(i-1) \to F(i) \to \mathrm{coker}(F(i-1 \to i)) \to 0$ (resp. $0 \to \ker(F(i \to i-1)) \to F(i) \to F(i-1) \to 0$) is semsplit.

**Definition 3.3.23.** [60, Definitions 2.1 (b) and 2.6 (b)] Let $\mathcal{X}$ be a class of objects in $\mathcal{C}(\mathcal{A})$. We say that $\mathcal{X}$ is *closed under special inverse* (resp. *direct*) *limits* if, for any $\mathcal{X}$-special direct system $F$, the direct limit $\varinjlim(F)$ (resp. inverse limit $\varprojlim(F)$) of $F$ is an object in $\mathcal{X}$.

**Example 3.3.24.** [60, 2.3 Lemma, p.130] The collection of all acyclic complexes in $\mathcal{C}(\mathbf{Ab})$ is closed under special inverse limits.

**Proposition 3.3.25.** [60, 2.7 Proposition] *Let $\mathcal{B}$ be an abelian category and let $\mathcal{X}$ be a class of objects in $\mathcal{C}(\mathcal{B})$ which is closed under special inverse limits. Suppose $\mathcal{A}$ has all its direct limits. Let $F : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ be a contravariant functor which preserves semisplit sequences and transforms direct limits into inverse limits.*

*Then the class of objects $M$ in $\mathcal{C}(\mathcal{A})$ such that $F(M)$ is an object in $\mathcal{X}$ is closed under special direct limits.*

The next result follows by applying the above in case $\mathcal{B} = \mathbf{Ab}$, $\mathcal{X}$ is the class of acyclic complexes and $F$ is the contravariant functor $\hom(-, A)$ (and taking the union of the resulting classes over all acyclic complexes $A$).

**Corollary 3.3.26.** [60, 2.8 Corollary] *If $\mathcal{A}$ has all direct limits then the class $\mathcal{C}_{\mathrm{p}}(\mathcal{A})$ in $\mathcal{C}(\mathcal{A})$ is closed under special direct limits.*

Recall definitions 3.2.3 and 3.2.14. Keller [43, p.14, Appendix: Proof of Theorem 1.1] gives a proof of the following.

**Lemma 3.3.27.** [60, 3.5 Corollary] *Suppose $\mathcal{A}$ has all direct limits and enough projectives. Then every complex $M$ in $\mathcal{C}(\mathcal{A})$ has a projective resoloution $P$ such that $\mathrm{tot}^{\bigoplus}(P_{-})$ is K-projective.*

**Corollary 3.3.28.** *The localisation functor $\mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ restricts to give a commutative diagram of functors*

$$\begin{tikzcd}
\mathcal{K}_{\mathrm{p}}(\mathcal{A}) \arrow[r, "\sim"] & \mathcal{D}(\mathcal{A}) \\
\mathcal{K}^{-}(\mathcal{P}) \arrow[r, "\sim"] \arrow[u, "\cup"] & \mathcal{D}^{-}(\mathcal{A}) \arrow[u, "\cup"'] \\
\mathcal{K}^{-,b}(\mathcal{P}) \arrow[r, "\sim"] \arrow[u, "\cup"] & \mathcal{D}^{b}(\mathcal{A}) \arrow[u, "\cup"']
\end{tikzcd}$$

*where each horizontal arrow is a triangle equivalence.*

*Proof.* We shall only prove that there is a triangle equivalence $\mathcal{K}_{\mathrm{p}}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. The other equivalences are well-known (see [64, Theorem 10.4.8] and [67, Proposition 3.5.43]). Let $\mathcal{T} = \mathcal{K}(\mathcal{A})$. Let $\mathcal{S}$ denote the class $\mathcal{S}_{\mathrm{q}}$ of all quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$ (which is a locally small multiplicative system). Let $\mathcal{R} = \mathcal{K}_{\mathrm{p}}(\mathcal{A})$ and $\mathcal{S}_{\mathcal{R}}$ be the class of quasi-isomorphisms in $\mathcal{R}$. By lemma 3.3.19 every arrow in $\mathcal{S}_{\mathcal{R}}$ is an isomorphism in $\mathcal{R}$ and so by the universal property of the localisation (lemma 3.3.2) the functor $\mathrm{Q} : \mathcal{R} \to \mathcal{R}[\mathcal{S}_{\mathcal{R}}^{-1}]$ is an isomorphism of categories.

The restriction of the homology functor $H^0 : \mathcal{T} \to \mathcal{A}$ to the subcategory $\mathcal{R}$ of $\mathcal{T}$ defines a homological functor $\mathcal{R} \to \mathcal{A}$ by theorem 3.1.25. By the equivalence of (i) and (iv) in proposition 3.3.20 we can now apply proposition 3.3.9, which shows that the inclusion functor $\iota : \mathcal{R}[\mathcal{S}_{\mathcal{R}}^{-1}] \to \mathcal{T}[\mathcal{S}^{-1}]$ is full and faithful. This means the functor $\iota\mathrm{Q} : \mathcal{R} \to \mathcal{T}[\mathcal{S}^{-1}]$ is also full and faithful. By lemmas 3.2.15 and 3.3.27 $\iota\mathrm{Q}$ is also dense, and hence an equivalence of categories. $\square$

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