COHERENT SHEAVES ON THE PROJECTIVE LINE, WORKING SEMINAR, BIELEFELD 18.10.2018

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1. (Quasi)coherent sheaves of $\mathcal{O}_X$-modules

We work over an algebraically closed field k. A variety X comes equipped with its Zariski topology and for every open set U we have a k-algebra $\mathcal{O}_X(U)$ of regular functions on U. Basic references are Hartshorne [8], Kempf [9], Mumford [10].

A *presheaf* $\mathcal{F}$ *of* $\mathcal{O}_X$*-modules* consists of
- for each open set U of X, an $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$,
- for each inclusion $V \subseteq U$, a restriction map $r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ of $\mathcal{O}_X(U)$-modules, where $\mathcal{F}(V)$ is considered as an $\mathcal{O}_X(U)$ module via the map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$,
- such that $r^V_W r^U_V = r^U_W$ for $W \subseteq V \subseteq U$ and $r^U_U = id$.

There is a natural category of presheaves.

$\mathcal{F}$ is a *sheaf* if for any open covering U_i of an open subset U
- $f \in \mathcal{F}(U)$ is uniquely determined by its restrictions $f_i \in \mathcal{F}(U_i)$.
- Any collection of $f_i \in \mathcal{F}(U_i)$, which agree on all pairwise intersections $U_i \cap U_j$, arise by restriction from some $f \in \mathcal{F}(U)$.

$\mathcal{F}$ is *quasicoherent* if for any inclusion of affine open subsets $V \subseteq U$ of X, the natural map $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism. This means that $\mathcal{F}$ is determined by the $\mathcal{O}_X(U_i)$-modules $\mathcal{F}(U_i)$ for an affine open cover U_i of X.

$\mathcal{F}$ is *coherent* if also $\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module for U affine open, or equivalently for U running through an affine open cover of X.

2. The projective line

$\mathbb{P}^1 = \{[a : b] : a, b \in k, \text{ not both zero}\}/[a : b] \sim [\lambda a : \lambda b]$ for $\lambda \neq 0$. It is identified with $k \cup \{\infty\}$ where $\lambda \in k$ corresponds to $[1 : \lambda]$ and $\infty = [1 : 0]$.

The sets $D(f) = \{[a : b] : f(a, b) \neq 0\}$ ($0 \neq f \in k[x, y]$ a homogeneous polynomial), are a base of open sets for the Zariski topology.

It turns out that the non-empty open sets are exactly the complements of finite sets.

For a non-empty open set U, $\mathcal{O}_{\mathbb{P}^1}(U)$ is the ring of rational functions f/g with $f, g \in k[x, y]$ homogeneous of the same degree and g non-vanishing on U.

The standard affine open covering is $\mathbb{P}^1 = U_0 \cup U_1$ where
- $U_0 = \{[a : b] : a \neq 0\} = \{[1 : b/a] : a \neq 0\} \cong \mathbb{A}^1$,
- $U_1 = \{[a : b] : b \neq 0\} = \{[a/b : 1] : b \neq 0\} \cong \mathbb{A}^1$.

A coherent sheaf on $\mathbb{P}^1$ is given by a triple (M_0, M_1, θ)
- M_0 is a f.g. module for $\mathcal{O}(U_0) = k[s]$ where $s = y/x$,
- M_1 is a f.g. module for $\mathcal{O}(U_1) = k[s^{-1}]$
- θ is an isomorphism of modules for $\mathcal{O}(U_0 \cap U_1) = k[s, s^{-1}]$,

$$\theta : k[s, s^{-1}] \otimes_{k[s^{-1}]} M_1 \to k[s, s^{-1}] \otimes_{k[s]} M_0$$

A morphism $\phi : (M_0, M_1, \theta) \to (M_0', M_1', \theta')$ is given by module maps $\phi_i : M_i \to M_i'$ giving a commutative square

$$\begin{array}{ccc} k[s, s^{-1}] \otimes_{k[s]} M_0 & \xrightarrow{1\otimes\phi_0} & k[s, s^{-1}] \otimes_{k[s]} M_0' \\ \theta \uparrow & & \uparrow \theta' \\ k[s, s^{-1}] \otimes_{k[s^{-1}]} M_1 & \xrightarrow[1\otimes\phi_1]{} & k[s, s^{-1}] \otimes_{k[s^{-1}]} M_1' \end{array}$$

3. Basic properties

$\mathcal{O}_X$ itself is a coherent sheaf of $\mathcal{O}_X$-modules.

Theorem. The quasicoherent sheaves form a Grothendieck category—an abelian category with enough injectives, but in general no projectives.
The coherent sheaves form an abelian subcategory $\operatorname{coh} X$. For a projective variety the Hom spaces are finite dimensional.

The *global sections* of $\mathcal{F}$ are $\Gamma(X, \mathcal{F}) := \mathcal{F}(X) \cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{F})$. Its derived functors are *cohomology* $H^i(X, \mathcal{F}) \cong \operatorname{Ext}^i(\mathcal{O}_X, \mathcal{F})$.

There is a *tensor product* with $(\mathcal{F} \otimes_{O_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ for affine open U. Also *symmetric* and *exterior powers*.

There is a *sheaf Hom* with $\mathcal{H}\text{om}(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$ for affine open U. Taking global sections gives the usual Hom space.

4. Locally free sheaves

$\mathcal{F}$ is *locally free of rank* n if X has an open covering U_i such that each $\mathcal{F}|_{U_i} \cong (\mathcal{O}_{U_i})^n$. If $\mathcal{F}$ is coherent and U_i is an affine open covering, $\mathcal{F}$ is locally free of rank n iff each $\mathcal{F}(U_i)$ is a projective $\mathcal{O}_X(U_i)$-module of rank n.

A *vector bundle of rank* n on X is a variety E with a morphism $\pi : E \to X$ and the structure of an n-dimensional vector space on each fibre $E_x = \pi^{-1}(x)$, satisfying the local triviality condition that X has an open cover U_i and isomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times k^n$ compatible with the projections to U_i and the vector space structure on the fibres.

Theorem. There is an equivalence of categories between vector bundles and locally free sheaves. To a vector bundle E corresponds its sheaf of sections

$$\mathcal{E}(U) = \{s : U \to E : \pi s = id\}.$$

which becomes an $\mathcal{O}_X$-module via $(fs)(u) = f(u)s(u)$ and $(s + s')(u) = s(u) + s'(u)$ using the vector space structure.

There is a duality on locally free sheaves given by $\mathcal{F}^\vee = \mathcal{H}\text{om}(\mathcal{F}, \mathcal{O}_X)$.

An *invertible sheaf* is a locally free sheaf of rank 1. Their isomorphism classes form a group $\text{Pic}(X)$ under tensor product.

5. Torsion sheaves

A coherent sheaf $\mathcal{F}$ is *torsion* if all $\mathcal{F}(U)$ for U affine open are torsion modules. For a curve this means they are f.d. modules.

For $\mathbb{P}^1$, given a non-zero homogeneous polynomial $f \in k[x, y]$, there is a torsion sheaf $\mathcal{S}_f$ given by $M_0 = k[s]/(f(1, s))$, $M_1 = k[s^{-1}]/(f(s^{-1}, 1))$, $\theta = id$.

The indecomposable torsion sheaves on a curve are classified by points $x \in X$ and a positive integer n. For $[a : b] \in \mathbb{P}^1$ it is $\mathcal{S}_f$ for $f(x, y) = (bx - ay)^n$. e.g. for $[1 : 0]$ this is $M_0 = k[s]/(s^n)$, $M_1 = 0$.

Lemma. Every coherent sheaf on a non-singular curve is the direct sum of a torsion sheaf and a locally free sheaf.

Proof. Every coherent $\mathcal{F}$ has maximal torsion subsheaf $\mathcal{T}$. For a non-singular curve $\mathcal{F}/\mathcal{T}$ is locally free and the exact sequence $0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{F}/\mathcal{T} \to 0$ is split. For an affine curve we know this, since its coordinate ring is a Dedekind domain. In general we can cover X with two open affines, with $\mathcal{F}$ torsion-free on one of them, and then a splitting of the sequence on the other affine easily gives a splitting globally.

6. The sheaves $\mathcal{O}(i)$ on $\mathbb{P}^1$

The sheaf $\mathcal{O}(i)$ for $i \in \mathbb{Z}$ is given by $M_0 = k[s]$, $M_1 = k[s^{-1}]$, and $\theta : k[s, s^{-1}] \to k[s, s^{-1}]$ is multiplication by s^i.

Lemma. $\text{Hom}(\mathcal{O}(i), \mathcal{O}(i + d)) \cong k[x, y]_d$ the homogeneous polynomials of degree d.

Proof. If $f \in k[x,y]$ is homogeneous of degree d, then $(f/x^d = f(1,s), f/y^d = f(s^{-1},1))$ defines a map $\mathcal{O}(i) \to \mathcal{O}(i+d)$. (The cokernel is $\mathcal{S}_f$.) It is easy to see that this gives a bijection.

Lemma. - $\mathcal{O}(0) \cong \mathcal{O}_{\mathbb{P}^1}$, $\mathcal{O}(i) \otimes \mathcal{O}(j) \cong \mathcal{O}(i+j)$, $\mathcal{O}(i)^\vee \cong \mathcal{O}(-i)$.
- Up to isomorphism these are the only invertible sheaves.
- $\mathcal{O}(-1)$ corresponds to the *universal subbundle* $E = \{([a,b],(c,d)) \in \mathbb{P}^1 \times k^2 : ad = bc\}$.

Proof. For the universal subbundle
$\mathcal{E}(U_0) \cong k[s]f$ where $f : U_0 \to E$, $[a:b] \mapsto ([a:b],(1,b/a))$.
$\mathcal{E}(U_1) \cong k[s^{-1}]g$ where $g : U_1 \to E$, $[a:b] \mapsto ([a:b],(a/b,1))$.
On $U_0 \cap U_1$, $(sg)([a:b]) = s([a:b])g([a:b]) = (b/a)([a:b],(a/b,1)) = ([a:b],(1,b/a)) = f([a:b])$.

Birkhoff-Grothendieck Theorem [7]. Every locally free sheaf on $\mathbb{P}^1$ is isomorphic to a direct sum of copies of the $\mathcal{O}(i)$.

Proof. Since projectives over $k[s]$ and $k[s^{-1}]$ are free, a locally free sheaf of rank n is given by $M_0 = k[s]^n$, $M_1 = k[s^{-1}]^n$ and an element of $\mathrm{GL}_n(k[s,s^{-1}])$.

Birkhoff factorization [5]: any such matrix factorizes as ADB with $A \in \mathrm{GL}_n(k[s])$, D diagonal and $B \in \mathrm{GL}_n(k[s^{-1}])$.

The sheaf is then isomorphic to that given by D, a direct sum of invertible sheaves.

Lemma. $\dim \mathrm{Ext}^1(\mathcal{O}(i),\mathcal{O}(j)) = \dim k[x,y]_{i-j-2}$.
In particular $\dim \mathrm{Ext}^1(\mathcal{O},\mathcal{O}(-2)) = 1$, represented by the exact sequence $0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^2 \to O \to 0$ given by (x,y), $(y,-x)$.

Proof. We consider an exact sequence $0 \to \mathcal{O}(j) \to \mathcal{F} \to \mathcal{O}(i) \to 0$, so

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & k[s] & \xrightarrow{\binom{1}{0}} & k[s]^2 & \xrightarrow{(0\,1)} & k[s] & \longrightarrow & 0 \\
0 & \longrightarrow & k[s,s^{-1}] & \longrightarrow & k[s,s^{-1}]^2 & \longrightarrow & k[s,s^{-1}] & \longrightarrow & 0 \\
 & & {\scriptstyle s^j}\big\uparrow & & {\scriptstyle \binom{a\,b}{c\,d}}\big\uparrow & & {\scriptstyle s^i}\big\uparrow & & \\
0 & \longrightarrow & k[s,s^{-1}] & \longrightarrow & k[s,s^{-1}]^2 & \longrightarrow & k[s,s^{-1}] & \longrightarrow & 0 \\
0 & \longrightarrow & k[s^{-1}] & \xrightarrow[\binom{1}{0}]{} & k[s^{-1}]^2 & \xrightarrow[(0\,1)]{} & k[s^{-1}] & \longrightarrow & 0
\end{array}
$$

The middle matrix must have the form $\begin{pmatrix} s^j & b \\ 0 & s^i \end{pmatrix}$ with $b \in k[s,s^{-1}]$.

Equivalence of two such extensions (given by b, b') is map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(j) & \longrightarrow & \mathcal{F}_b & \longrightarrow & \mathcal{O}(i) & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \| & & \\ 0 & \longrightarrow & \mathcal{O}(j) & \longrightarrow & \mathcal{F}_{b'} & \longrightarrow & \mathcal{O}(i) & \longrightarrow & 0 \end{array}$$

The map h is given by matrices which must have the form

$$\begin{pmatrix} 1 & f(s) \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & g(s^{-1}) \\ 0 & 1 \end{pmatrix}$$

Thus we get a commutative square

$$\begin{array}{ccc} k[s,s^{-1}]^2 & \xrightarrow{\left(\begin{smallmatrix} 1 & f(s) \\ 0 & 1 \end{smallmatrix}\right)} & k[s,s^{-1}]^2 \\ {\scriptstyle \left(\begin{smallmatrix} s^j & b \\ 0 & s^i \end{smallmatrix}\right)} \uparrow & & \uparrow {\scriptstyle \left(\begin{smallmatrix} s^j & b' \\ 0 & s^i \end{smallmatrix}\right)} \\ k[s,s^{-1}]^2 & \xrightarrow[\left(\begin{smallmatrix} 1 & g(s^{-1}) \\ 0 & 1 \end{smallmatrix}\right)]{} & k[s,s^{-1}]^2 \end{array}$$

Thus $b' = b + s^i f(s) - s^j g(s^{-1})$. The coefficients of s^n with $j < n < i$ are invariant. The number of these is $\dim k[x,y]_{i-j-2}$.

Note the following Birkhoff factorization for $\lambda \neq 0$,

$$\begin{pmatrix} s^{-2} & \lambda s^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda^{-1} & \lambda^{-1}s \end{pmatrix} \begin{pmatrix} s^{-1} & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s^{-1} & \lambda \end{pmatrix}.$$

7. The sheaf of differentials

If A is a commutative k-algebra, the *module of Kähler differentials* Ω_A is the A-module generated by symbols da ($a \in A$) subject to
- $d(a+b) = da + db$,
- $d(ab) = adb + bda$,
- $d\lambda = 0$ for $\lambda \in k$.

Example. $\Omega_{k[x]} = k[x]dx$, for $d(f(x)) = f'(x)dx$.

The *sheaf of differentials* Ω_X is the coherent sheaf with $\Omega_X(U) = \Omega_{\mathcal{O}_X(U)}$ for U affine open.

Lemma. $\Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$.

Proof. Let $\mathcal{F} = \Omega_{\mathbb{P}^1}$. $\mathcal{F}(U_0) = k[s]ds$, $\mathcal{F}(U_1) = k[s^{-1}]d(s^{-1})$. Then θ sends the generator $d(s^{-1})$ of $\mathcal{F}(U_1)$ to

$$d(s^{-1}) = -s^{-2}ds = -s^{-2}.(\text{the generator of } \mathcal{F}(U_0)).$$

If X is non-singular of dimension n then:
- Ω_X is locally free of rank n, it corresponds to the *cotangent bundle* of X.
- $\Omega_X^\vee$ corresponds to the *tangent bundle* of X. For $\mathbb{P}^1$ it is $\mathcal{O}(2)$.
- The *canonical bundle* is the top exterior power $\omega_X = \wedge^n \Omega_X$. For $\mathbb{P}^1$ it is $\Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$.

8. Serre duality

Theorem. For a non-singular projective curve and coherent $\mathcal{F}, \mathcal{G}$,

$$\operatorname{Ext}^1(\mathcal{F}, \mathcal{G}) \cong D \operatorname{Hom}(\mathcal{G}, \mathcal{F} \otimes \omega). \qquad (D = \operatorname{Hom}_k(-, k))$$

It is usually stated for $\mathcal{F} = \mathcal{O}_X$ and maybe $\mathcal{G}$ locally free. I don't know a good proof of the version here.

For $\mathbb{P}^1$ we computed $\dim \operatorname{Ext}^1(\mathcal{O}(i), \mathcal{O}(j)) = \dim k[x, y]_{i-j-2} = \dim \operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(i-2)) = \dim \operatorname{Hom}(\mathcal{O}(j), \mathcal{O}(i) \otimes \omega)$.

It follows that the category of coherent sheaves for a non-singular projective curve is hereditary since $\operatorname{Ext}^1(\mathcal{F}, -)$ is right exact. Also it has Auslander-Reiten sequences, with the translate given by $\mathcal{F} \otimes \omega$. Get AR quiver.

9. Grothendieck group

The *Grothendieck group* $K_0(\operatorname{coh} X)$ is the $\mathbb{Z}$-module generated by the isomorphism classes $[\mathcal{F}]$ ($\mathcal{F}$ coherent), subject to $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ for $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ exact.

For X a non-singular curve, $K_0(\operatorname{coh} X) \cong \operatorname{Pic}(X) \oplus \mathbb{Z}$.

For $\mathbb{P}^1$ the *rank* and *degree* of a coherent sheaf are defined by $\operatorname{rank} \mathcal{O}(i) = 1$, $\deg \mathcal{O}(i) = i$, $\operatorname{rank} \mathcal{S}_f = 0$, $\deg \mathcal{S}_f = \deg f$, and additively on direct sums. They define an isomorphism $K_0(\operatorname{coh} \mathbb{P}^1) \to \mathbb{Z}^2$, $[\mathcal{F}] \mapsto (\operatorname{rank} \mathcal{F}, \deg \mathcal{F})$.

For a non-singular variety of dimension n, the *Euler form* is the bilinear form

$$\langle -, - \rangle : K_0(\operatorname{coh} X) \times K_0(\operatorname{coh} X) \to \mathbb{Z}, ([\mathcal{F}], [\mathcal{G}]) \mapsto \sum_{i=0}^{n} (-1)^i \dim \operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$$

The *genus* of a non-singular curve is $g = \dim \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = \dim \operatorname{Hom}(\mathcal{O}_X, \omega)$.

Theorem. For a non-singular curve

$$\langle [\mathcal{F}], [\mathcal{G}] \rangle = (\operatorname{rank} \mathcal{F})(\deg \mathcal{G}) - (\deg \mathcal{F})(\operatorname{rank} \mathcal{G}) + (1 - g)(\operatorname{rank} F)(\operatorname{rank} G).$$

With $\mathcal{F} = \mathcal{O}_X$ and Serre duality, this gives the Riemann-Roch Theorem.

10. Serre's Theorem

I don't know a good reference, but see the introduction to [1].

Let X be the projective variety given by a commutative graded k-algebra R satisfying $R = R_0 \oplus R_1 \oplus R_2 \oplus \dots$ with $R_0 = k$ and R is generated by R_1.

Serre's Theorem. $\operatorname{coh} X$ is equivalent to $\operatorname{grmod} R/\operatorname{tors} R$. In particular $\operatorname{coh} \mathbb{P}^1$ is equivalent to $\operatorname{grmod} k[x,y]/\operatorname{tors} k[x,y]$.

Here $\operatorname{grmod} R$ is the category of f.g. $\mathbb{Z}$-graded R-modules and $\operatorname{tors} R$ is the subcategory of f.d. $\mathbb{Z}$-graded R-modules.

Now $A = \operatorname{grmod} R$ is an abelian category and $S = \operatorname{tors} R$ is a *Serre subcategory*, meaning that if $0 \to L \to M \to N \to 0$ is exact, then $M \in S \Leftrightarrow L, N \in S$. By definition the *quotient category* A/S has the same objects as A, with

$$\operatorname{Hom}_{A/S}(M,N) = \varinjlim \operatorname{Hom}_A(M', N/N'),$$

the direct limit taken over all subobjects M' of M and N' of N with $M/M', N' \in S$.

Andrew has notes containing a proof for $\mathbb{P}^1$.

This comes from a functor $\operatorname{grmod} R \to \operatorname{coh} X$. The functor sends a graded $k[x,y]$-module M to (M_0, M_1, θ) where
- M_0 is the degree 0 part of the graded module $k[x,x^{-1},y] \otimes_{k[x,y]} M$. Naturally a $k[s]$-module, $s = y/x$.
- M_1 is the degree 0 part of the graded module $k[x,y,y^{-1}] \otimes_{k[x,y]} M$. Naturally a $k[s^{-1}]$-module.
- the map θ comes from identifying both $k[s,s^{-1}] \otimes M_i$ with the degree 0 part of the graded module $k[x,x^{-1},y,y^{-1}] \otimes_{k[x,y]} M$.

The grading shift $M(i)_n = M_{i+n}$ on $\operatorname{grmod} R$ corresponds to the tensor product with $\mathcal{O}(i)$.

11. Beilinson's Theorem

Beilinson's result [3, 4] as interpreted by Geigle and Lenzing [6] and Baer [2].

A *tilting sheaf* for a non-singular projective variety X is a coherent sheaf $\mathcal{T}$ with
- $\operatorname{Ext}^i(\mathcal{T},\mathcal{T}) = 0$ for $i > 0$.
- $\mathcal{T}$ generates $D^b(\operatorname{coh} X)$ as a triangulated category.
- $\Lambda := \operatorname{End}(\mathcal{T})^{op}$ has finite global dimension.

Theorem. $\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(1)$ is a tilting sheaf for $\mathbb{P}^1$ and Λ is the Kronecker algebra.

Proof. We have shown that $\mathrm{Ext}^1(\mathcal{T},\mathcal{T})=0$ and the higher Exts are all zero. The exact sequence $0\to\mathcal{O}(i)\to\mathcal{O}(i+1)^2\to\to\mathcal{O}(i+2)\to 0$ shows that the subcategory generated by $\mathcal{T}$ contains $\mathcal{O}(2)$, and then in the same way that it contains all $\mathcal{O}(i)$. Thus it contains $\mathcal{S}_f$, so it is all of $D^b(\mathrm{coh}\,\mathbb{P}^1)$.

One gets a functor $\mathrm{Hom}(\mathcal{T},-):\mathrm{coh}\,X\to\Lambda\text{-mod}$. It has a left adjoint, denoted $\mathcal{T}\otimes_\Lambda -$.

Theorem. They give inverse equivalences

$$R\,\mathrm{Hom}(\mathcal{T},-):D^b(\mathrm{coh}\,X)\rightleftarrows D^b(\Lambda\text{-mod}):\mathcal{T}\otimes^L -$$

Since $\mathrm{coh}\,\mathbb{P}^1$ and the Kronecker algebra are hereditary, any indecomposable object of the derived category is represented by a complex which lives in only one degree. Get familiar picture

References

[1] M. Artin and J. J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (1994), 228–287.

[2] D. Baer, Tilting sheaves in representation theory of algebras, Manuscripta Math. 60 (1988), 323-347.

[3] A. A. Beĭlinson, Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra, Funktsional. Anal. i Prilozhen. 12 (1978), 68-69. English translation: Functional Anal. Appl. 12 (1978), 214–216 (1979).

[4] A. A. Beĭlinson, The derived category of coherent sheaves on $\mathbf{P}^n$, Selected translations, Selecta Math. Soviet. 3 (1983/84), 233–237.

[5] G. D. Birkhoff, Singular points of ordinary linear differential equations, Trans. Amer. Math. Soc. 10 (4) (1909), 436–470.

[6] W. Geigle and H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, in: Singularities, representation of algebras, and vector bundles (Proc. Lambrecht, 1985), G.-M. Greuel and G. Trautmann (eds.), 265–297, Lecture Notes in Math., 1273, Springer, Berlin, 1987.

[7] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, Amer. J. Math. 79 (1957), 121–138.

[8] R. Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977

[9] G. R. Kempf, Algebraic varieties. London Mathematical Society Lecture Note Series, 172. Cambridge University Press, Cambridge, 1993

[10] D. Mumford, The red book of varieties and schemes. Second, expanded edition. Includes the Michigan lectures (1974) on curves and their Jacobians. With contributions by Enrico Arbarello. Lecture Notes in Mathematics, 1358. Springer-Verlag, Berlin, 1999.