THE DELIGNE-SIMPSON PROBLEM

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Abstract. Given k similarity classes of invertible matrices, the Deligne-Simpson problem asks to determine whether or not one can find matrices in these classes whose product is the identity and with no common invariant subspace. The first author conjectured an answer in terms of an associated root system, and proved one implication in joint work with Shaw. In this paper we prove the other implication, thus confirming the conjecture.

1. Introduction

Given conjugacy classes $C_1, \ldots, C_k$ in $\mathrm{GL}_n(\mathbb{C})$, the Deligne-Simpson problem asks to determine whether or not there is a solution to the equation

$$A_1 A_2 \ldots A_k = 1 \qquad (A_i \in C_i)$$

which is irreducible in the sense that the A_i have no non-trivial common invariant subspace. This is motivated by the problem of classifying systems of linear ordinary differential equations in the complex domain in terms of their local monodromies; see [12] for more background and motivation. In [2] we have given a conjectural solution to this problem, and in [7] we have proved one implication. Here we prove the other implication, thus confirming the conjecture.

In order to fix the conjugacy classes, we choose a *weight sequence* $\mathbf{w} = (w_1, \ldots, w_k)$ with $w_i \geq 1$ and a collection of complex numbers $\xi = (\xi_{ij})$ $(1 \leq i \leq k,\ 1 \leq j \leq w_i)$ with

$$(A_i - \xi_{i1}1)(A_i - \xi_{i2}1) \ldots (A_i - \xi_{i,w_i}1) = 0$$

for $A_i \in C_i$. For example w_i can be the degree of the minimal polynomial of A_i, and $\xi_{i1}, \ldots, \xi_{i,w_i}$ its roots, in some order. The conjugacy classes are then determined by n, $\mathbf{w}$, ξ and the numbers

$$n_{ij} = \operatorname{rank}(A_i - \xi_{i1}1)(A_i - \xi_{i2}1) \ldots (A_i - \xi_{ij}1)$$

for $A_i \in C_i$, for $1 \leq i \leq k$ and $1 \leq j < w_i$.

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The numbers n_{ij} can best be understood in terms of a suitable root system. Associated to a quiver Q with vertex set I there is root system (consisting of real and imaginary roots, positive or negative) contained in the root lattice

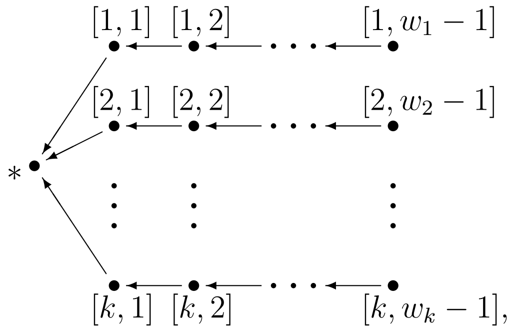
$$\Gamma_Q = \bigoplus_{v \in I} \mathbb{Z}\alpha_v,$$

where the α_v may be regarded as symbols, see for example [11]. The Euler form $\langle -, - \rangle_Q$ on Γ_Q is given by

$$\langle \sum_v n_v \alpha_v, \sum_v n'_v \alpha_v \rangle_Q = \sum_{v \in I} n_v n'_v - \sum_{a \in Q} n_{t(a)} n'_{h(a)},$$

its symmetrization is $(\alpha, \beta)_Q = \langle \alpha, \beta \rangle_Q + \langle \beta, \alpha \rangle_Q$, and the corresponding quadratic form is $q_Q(\alpha) = \langle \alpha, \alpha \rangle_Q$. We define $p_Q(\alpha) = 1 - q_Q(\alpha)$.

Let $Q_{\mathbf{w}}$ be the star-shaped quiver



with vertex set $I_{\mathbf{w}} = \{*\} \cup \{[i,j] : 1 \le i \le k, 1 \le j < w_i\}$. To simplify notation we write a subscript $[i,j]$ as ij, we write $\Gamma_{\mathbf{w}}$ for the root lattice $\Gamma_{Q_{\mathbf{w}}}$, and $p_{\mathbf{w}}$ for the function $p_{Q_{\mathbf{w}}}$. Thus

$$p_{\mathbf{w}}(\alpha) = 1 - n_*^2 - \sum_{i=1}^{k} \sum_{j=1}^{w_i - 1} n_{ij}^2 + \sum_{i=1}^{k} n_* n_{i1} + \sum_{i=1}^{k} \sum_{j=1}^{w_i - 2} n_{ij} n_{i,j+1}$$

For $\alpha = \sum n_v \alpha_v \in \Gamma_{\mathbf{w}}$, we define

$$\xi^{[\alpha]} = \prod_{i=1}^{k} \prod_{j=1}^{w_i} \xi_{ij}^{n_{i,j-1} - n_{ij}} \text{ and } \xi * [\alpha] = \sum_{i=1}^{k} \sum_{j=1}^{w_i} \xi_{ij} (n_{i,j-1} - n_{ij}),$$

with the convention that $n_{i0} = n_*$ and $n_{i,w_i} = 0$.

The conjugacy classes C_i determine an element $\alpha_C \in \Gamma_{\mathbf{w}}$, with $n_* = n$ and n_{ij} as given above. Conversely $\mathbf{w}$, ξ, and α_C determine the C_i. Our main result is as follows.

Theorem 1.1. *For there to be an irreducible solution to $A_1 \dots A_k = 1$ with matrices $A_i \in C_i$ it is necessary and sufficient that $\alpha = \alpha_C$ be a positive root, $\xi^{[\alpha]} = 1$, and $p_{\mathbf{w}}(\alpha) > p_{\mathbf{w}}(\beta) + p_{\mathbf{w}}(\gamma) + \dots$ for any nontrivial decomposition of α as a sum of positive roots $\alpha = \beta + \gamma + \dots$ with $\xi^{[\beta]} = \xi^{[\gamma]} = \dots = 1$.*

The sufficiency is proved in [7]. In section 7 we prove necessity. This result was already announced in [3], but the argument given here is slightly different to the one envisaged at that time: instead of adapting the argument in [1] for parabolic bundles equipped with connections, we base our argument on [6].

2. Weighted projective lines and parabolic bundles

Let $\mathbb{X}$ be the weighted projective line over $\mathbb{C}$ consisting of the projective line $X = \mathbb{P}^1$, a collection $D = (a_1, a_2, \dots, a_k)$ of distinct marked points in $\mathbb{P}^1$, and a weight sequence $\mathbf{w} = (w_1, \dots, w_k)$ with $w_i \geq 1$ (where $w_i = 1$ is equivalent to the point a_i being unmarked).

Let $\operatorname{Coh}\mathbb{X}$ be the category of coherent sheaves on $\mathbb{X}$ defined by Geigle and Lenzing [8]. We call its objects *parabolic sheaves*. This is an abelian category, and we use the name *subsheaf* for a sub-object in this category. Every parabolic sheaf is the direct sum of a torsion parabolic sheaf and a torsion-free parabolic sheaf. There is one simple torsion parabolic sheaf S_a supported at each point point $a \notin D$, and at the point a_i there are simple torsion parabolic sheaves S_{ij} $(0 \leq j < w_i)$ with exact sequences

$$0 \to \mathcal{O}(j\vec{x}_i) \to \mathcal{O}((j+1)\vec{x}_i) \to S_{ij} \to 0$$

see [8, (2.5.2)]. As in [16, §5.1], we identify the Grothendieck group $K_0(\operatorname{Coh}\mathbb{X})$ with $\hat{\Gamma} = \Gamma \oplus \mathbb{Z}\partial$, where Γ is the root lattice for the quiver $Q_{\mathbf{w}}$, with

$$[\mathcal{O}(k\vec{c})] = \alpha_* + k\partial, \quad [S_a] = \partial, \quad [S_{ij}] = \begin{cases} \alpha_{ij} & (j \neq 0) \\ \partial - \sum_{\ell=1}^{w_i-1} \alpha_{i\ell} & (j = 0). \end{cases}$$

The *dimension vector* $\underline{\dim}\,\mathcal{E} \in \Gamma$ and degree $\deg_{\mathbb{P}^1}\mathcal{E} \in \mathbb{Z}$ are defined by

$$[\mathcal{E}] = \underline{\dim}\,\mathcal{E} + (\deg_{\mathbb{P}^1}\mathcal{E})\partial.$$

The Euler form $\langle -,- \rangle_{\mathbb{X}} : \Gamma \times \Gamma \to \mathbb{Z}$ with

$$\langle [\mathcal{E}], [\mathcal{F}] \rangle_{\mathbb{X}} = \dim \operatorname{Hom}(\mathcal{E},\mathcal{F}) - \dim \operatorname{Ext}^1(\mathcal{E},\mathcal{F})$$

is given as follows.

Lemma 2.1. *For $\alpha = \sum_{v\in I} n_v\alpha_v$, $\alpha' = \sum_{v\in I} n'_v\alpha_v \in \Gamma$ and $k, k' \in \mathbb{Z}$,*

$$\langle \alpha + k\partial, \alpha' + k'\partial \rangle_{\mathbb{X}} = \langle \alpha, \alpha' \rangle_{Q_{\mathbf{w}}} + k'n_* - kn'_*.$$

Proof. Straightforward. □

Torsion-free parabolic sheaves on $\mathbb{X}$ are also called *parabolic bundles* on $\mathbb{P}^1$, and following Lenzing [13], they may be identified with collections $\mathcal{E} = (E, E_{ij})$ consisting of an (algebraic or holomorphic) vector bundle E on $\mathbb{P}^1$ and flags of vector subspaces

$$E_{a_i} = E_{i0} \supseteq E_{i1} \supseteq \cdots \supseteq E_{i,w_i-1} \supseteq E_{i,w_i} = 0$$

of the fibres E_{a_i} at each of the marked points a_i. As in [4], we make the identification in such a way that

$$\underline{\dim}\,\mathcal{E} = n_*\alpha_* + \sum_{i=1}^{k}\sum_{j=1}^{w_i-1} n_{ij}\alpha_{ij} \in \Gamma,$$

with $n_* = \operatorname{rank} E$ and $n_{ij} = \dim E_{ij}$, and $\deg_{\mathbb{P}^1}\mathcal{E}$ is the usual degree of the vector bundle E. Observe that the dimension vector of a parabolic bundle is necessarily *strict*, meaning that $n_* \geq n_{i1} \geq n_{i2} \geq \cdots \geq n_{i,w_i-1} \geq 0$. We denote by $\operatorname{Par}\mathbb{X}$ the category of parabolic bundles on $\mathbb{X}$. Morphisms between parabolic bundles correspond to vector bundle homomorphisms which respect the flags.

We write $\Omega = \Omega^1_{\mathbb{P}^1}(\log D)$ for the sheaf of differential 1-forms on $\mathbb{P}^1$ with logarithmic poles on D. It is a line bundle on $\mathbb{P}^1$. If $\theta \in \operatorname{Hom}(F, E\otimes\Omega)$, then there are residue maps $\operatorname{Res}_{a_i}\theta : F_{a_i} \to E_{a_i}$.

Geigle and Lenzing introduce a twist $\mathcal{E}(\vec{\omega})$, giving Serre duality on $\operatorname{Coh}\mathbb{X}$ in the form

$$\operatorname{Ext}^1(\mathcal{E},\mathcal{F}) \cong \operatorname{Hom}(\mathcal{F},\mathcal{E}(\vec{\omega}))^*.$$

Lemma 2.2. *Given parabolic bundles* $\mathcal{E} = (E, E_{ij})$ *and* $\mathcal{F} = (F, F_{ij})$, *we can identify* $\operatorname{Hom}(\mathcal{F},\mathcal{E}(\vec{\omega}))$ *with the set*

$$H_{\mathcal{F},\mathcal{E}} = \{\theta \in \operatorname{Hom}(F, E\otimes\Omega) : (\operatorname{Res}_{a_i}\theta)(F_{i,j}) \subseteq E_{i,j+1} \quad (1\leq i\leq k, 0\leq j< w_i)\}.$$

Proof. In the notation of [8, §2.2], we have $\vec{\omega} = (k-2)\vec{c} - \sum_{i=1}^k \vec{x}_i$, so $\mathcal{E}(\vec{\omega}) = (E\otimes\Omega)(-\sum_{i=1}^k \vec{x}_i)$.

If $\mathcal{E}$ is torsion-free, then so is $\mathcal{E}(\vec{\omega})$. Now the twist operation $\mathcal{E}(\vec{x}_i)$ of [8, §1.7] can be understood as in [13, §4.1] and [5, §2] as the operation of rotating a cycle.

The fibre E_a at a point $a \in X$ is $E\otimes_{\mathcal{O}_{X,a}}(\mathcal{O}_{X,a}/\mathfrak{m}_{X,a})$. We write $i_a(V)$ for the skyscraper sheaf given by a finite-dimensional vector space V at the point a. It is a coherent sheaf isomorphic to the direct sum of $\dim V$ copies of the simple torsion sheaf at a. There is a natural map $E \to i_a(E_a)$. Given any subspace V of E_a we get an exact sequence

$$0 \to E^{a,V} \to E \to i_a(E_a/V) \to 0.$$

and $E^{a,0} \cong E(-a)$.

The parabolic structure of $\mathcal{F}$ at a_i is given by subspaces F_{ij}, and it corresponds to a cycle of vector bundles

$$F^{a_i,0} \to F^{a_i,F_{i,w_i-1}} \to \dots F^{a_i,F_{i2}} \to F^{a_i,F_{i1}} \to F \cong F^{a_i,0}(a_i)$$

Now $\mathcal{E}(\vec{\omega})$ is given by tensoring E with Ω and rotating the cycle at each marked point, so at a_i it is given by a cycle of vector bundles

$$E^{a_i,E_{i1}}(-a_i)\otimes\Omega \to E^{a_i,0}\otimes\Omega \to E^{a_i,E_{i,w_i-1}}\otimes\Omega \to \cdots \to E^{a_i,E_{i1}}\otimes\Omega$$

Now a morphism from $\mathcal{F}$ to $\mathcal{E}(\vec{\omega})$ is given by a morphism between these cycles. Since the maps in the cycles are monomorphisms, such a morphism is determined by a morphism $F \to E \otimes \Omega$, subject to the condition that it sends $F^{a_i,F_{ij}}$ into $E^{a_i,E_{i,j+1}} \otimes \Omega$ for all j. This is the condition that $(\operatorname{Res}_{a_i} \theta)(F_{i,j}) \subseteq E_{i,j+1}$. □

3. Connections on parabolic bundles

Let $\mathbb{X}$ be a weighted projective line of weight type $\mathbf{w} = (w_1, \dots, w_k)$. Let $\zeta = (\zeta_{ij})$ ($1 \le i \le k$, $1 \le j \le w_i$) be a collection of complex numbers.

We say that ζ is *non-resonant* if the numbers ζ_{ij} for fixed i never differ by a non-zero integer, that is $\zeta_{ij} - \zeta_{i,j'} \notin \mathbb{Z} \setminus \{0\}$ for $1 \le i \le k$ and $1 \le j, j' \le w_i$.

In [5] we have constructed functorial exact sequences

$$0 \to \mathcal{E}(\vec{\omega}) \to D_\zeta(\mathcal{E}) \to \mathcal{E} \to 0$$

for $\mathcal{E}$ a parabolic sheaf on $\mathbb{X}$, and we define a ζ-*connection* on $\mathcal{E}$ to be a section $s : \mathcal{E} \to D_\zeta(\mathcal{E})$.

Recall that a parabolic bundle on $\mathbb{X}$ can be identified with a tuple (E, E_{ij}) consisting of a vector bundle E on $\mathbb{P}^1$ and flags of subspaces E_{ij} of the fibres E_{a_i}. In this setting we make a different definition.

Recall that Ω denotes the sheaf of differential 1-forms on $\mathbb{P}^1$ with logarithmic poles on D. Given a vector bundle E on X, recall that a *logarithmic connection on* E is a homomorphism of sheaves of abelian groups

$$\nabla : E \to E \otimes \Omega$$

satisfying Leibnitz's rule. It has residues $\operatorname{Res}_{a_i} \nabla \in \operatorname{End}(E_{a_i})$. The following is a special case of [15, Corollaire 3], where, as mentioned in [2, §7], we use the opposite sign convention to Mihai for residues.

Lemma 3.1. *If E is a vector bundle on $\mathbb{P}^1$ and $\nabla : E \to E \otimes \Omega$ is a logarithmic connection, then*

$$\sum_{i=1}^{k} \operatorname{tr}(\operatorname{Res}_{a_i} \nabla) = -\deg E.$$

By a ζ-connection on (E, E_{ij}), we mean a logarithmic connections ∇ on E satisfying

$$(\operatorname{Res}_{a_i} \nabla - \zeta_{ij} 1)(E_{i,j-1}) \subseteq E_{ij}$$

for all $1 \le i \le k$ and $1 \le j \le w_i$. In [5] we have proved the following result.

Lemma 3.2. *For a parabolic bundle $\mathcal{E}$ on $\mathbb{X}$ there is a 1-1 correspondence between the ζ-connections on $\mathcal{E}$ in the sense of a section $s : \mathcal{E} \to D_\zeta(\mathcal{E})$ and in the sense of a logarithmic connection ∇ on E.*

We denote by $\operatorname{CohConn}_\zeta \mathbb{X}$ the category of pairs $(\mathcal{E}, s)$ where $\mathcal{E}$ is a parabolic sheaf on $\mathbb{X}$ and $s : \mathcal{E} \to D_\zeta(\mathcal{E})$ is a ζ-connection on $\mathcal{E}$. Given $(\mathcal{E}, s)$ in $\operatorname{CohConn}_\zeta \mathbb{X}$, we say that a subsheaf $\mathcal{F}$ of $\mathcal{E}$ is *invariant* provided that the composition

$$\mathcal{F} \to \mathcal{E} \xrightarrow{s} D_\zeta(\mathcal{E}) \to D_\zeta(\mathcal{E}/\mathcal{F})$$

is zero, so that the restriction of s to $\mathcal{F}$ factors through the morphism $D_\zeta(\mathcal{F}) \to D_\zeta(\mathcal{E})$, where $\mathcal{F} \to \mathcal{E}$ is the inclusion morphism. In this case the induced morphism $\mathcal{F} \to D_\zeta(\mathcal{F})$ is a ζ-connection on $\mathcal{F}$, and there is an induced morphism $\overline{s} : \mathcal{E}/\mathcal{F} \to D_\zeta(\mathcal{E}/\mathcal{F})$ which is a ζ-connection on $\mathcal{E}/\mathcal{F}$.

We say that a pair $(\mathcal{E}, s)$ in $\operatorname{CohConn}_\zeta \mathbb{X}$ is *irreducible* if $\mathcal{E}$ is non-zero and it has has no non-zero proper invariant subsheaves. We say that a pair $(\mathcal{E}, s)$ with $\mathcal{E}$ non-zero a parabolic bundle is *weakly irreducible* if it has no non-zero proper invariant subsheaves such that the quotient is a parabolic bundle.

We denote by $\operatorname{ParConn}_\zeta \mathbb{X}$ the category of pairs $(\mathcal{E}, \nabla)$ consisting of a parabolic bundle $\mathcal{E}$ on $\mathbb{X}$ and a ζ-connection ∇. By the discussion above, this can be identified with the full subcategory of $\operatorname{CohConn}_\zeta \mathbb{X}$ consisting of the pairs $(\mathcal{E}, s)$ where $\mathcal{E}$ is torsion-free.

Lemma 3.3. *If $\mathcal{E}$ is a parabolic bundle on $\mathbb{X}$, then there exists $(\mathcal{E}, \nabla) \in \operatorname{ParConn}_\zeta \mathbb{X}$ if and only if $\deg_{\mathbb{P}^1} \mathcal{E}' + \zeta * [\underline{\dim}\, \mathcal{E}'] = 0$ for all indecomposable direct summands $\mathcal{E}'$ of $\mathcal{E}$. In particular, if there exists $(\mathcal{E}, \nabla)$, then $\deg_{\mathbb{P}^1} \mathcal{E} + \zeta * [\underline{\dim}\, \mathcal{E}] = 0$.*

Proof. This is [2, Theorem 7.1]. Alternatively, the last part follows directly from Lemmas 3.1 and 3.2. □

Lemma 3.4. *The invariant subsheaves $\mathcal{F} = (F, F_{ij})$ of an object $(\mathcal{E}, \nabla) \in \operatorname{ParConn}_\zeta \mathbb{X}$ are given by subsheaves F of E with $\nabla(F) \subseteq F \otimes \Omega$, and subspaces F_{ij} satisfying*

$$(\operatorname{Res}_{a_i} \nabla - \zeta_{ij})(F_{i,j-1}) \subseteq F_{ij}$$

for all i, j. In particular, if F is a subsheaf of E with $\nabla(F) \subseteq F \otimes \Omega$, one can take $F_{ij} = F_{a_i} \cap E_{ij}$.

Proof. Clear. □

Lemma 3.5. *If ζ is non-resonant, then $(\mathcal{E}, \nabla) \in \operatorname{ParConn}_\zeta \mathbb{X}$ is irreducible if and only if it is weakly irreducible.*

Proof. If it is irreducible, it is certainly weakly irreducible. Conversely, suppose it is weakly irreducible, but not irreducible. Then there is a non-zero proper invariant subsheaf $\mathcal{E}'$ of $(\mathcal{E}, \nabla)$. Since $\mathcal{E}$ is torsion-free, so is $\mathcal{E}'$. Let $(\mathcal{E}', \nabla')$ be the corresponding element of $\operatorname{ParConn}_\zeta \mathbb{X}$. Since ζ is non-resonant, the morphism $(\mathcal{E}', \nabla') \to (\mathcal{E}, \nabla)$ has constant rank, see [2, Theorem 6.1]. It follows that the quotient $\mathcal{E}/\mathcal{E}'$ is torsion-free. But this contradicts weak irreducibility. □

Lemma 3.6. *If* $(\mathcal{E}, \nabla)$ *is in* $\operatorname{ParConn}_\zeta \mathbb{X}$*, and there is a decomposition* $\mathcal{E} = \mathcal{E}^1 \oplus \mathcal{E}^2$ *with* $\operatorname{Ext}^1(\mathcal{E}^2, \mathcal{E}^1) = 0$*, then* $\mathcal{E}^1$ *is an invariant subsheaf for* $(\mathcal{E}, \nabla)$*.*

Proof. It is rather easy to see this for $(\mathcal{E}, s) \in \operatorname{CohConn}_\zeta \mathbb{X}$. The composition

$$\mathcal{E}^1 \to \mathcal{E} \xrightarrow{s} D_\zeta(\mathcal{E}) \to D_\zeta(\mathcal{E}^2) \to \mathcal{E}^2$$

is the same as the inclusion of $\mathcal{E}^1$ in $\mathcal{E}$ followed by the projection onto $\mathcal{E}^2$, so it is zero. Thus the map $\mathcal{E}^1 \to D_\zeta(\mathcal{E}^2)$ factors through $\mathcal{E}^2(\vec{\omega})$. But then it is zero, as $\operatorname{Hom}(\mathcal{E}^1, \mathcal{E}^2(\vec{\omega})) \cong D \operatorname{Ext}^1(\mathcal{E}^2, \mathcal{E}^1) = 0$ by Serre duality.

For later comparison, however, we write it out in terms of parabolic bundles. The underlying bundle of $\mathcal{E}$ is $E^1 \oplus E^2$ and the parabolic structure is $E_{ij} = E^1_{ij} \oplus E^2_{ij}$. The connection takes block form

$$\begin{pmatrix} \nabla_{11} & \nabla_{12} \\ \nabla_{21} & \nabla_{22} \end{pmatrix}$$

where $\nabla_{pp} : E^p \to E^p \otimes \Omega$ is a logarithmic connection on E^p and $\nabla_{pq} : E^q \to E^p \otimes \Omega$ is a homomorphism of bundles for $p \neq q$. Moreover the residues $\operatorname{Res}_{a_i} \nabla : E^1_{a_i} \oplus E^2_{a_i} \to E^1_{a_i} \oplus E^2_{a_i}$ take block form

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

where $R_{pq} = \operatorname{Res}_{a_i} \nabla_{pq} : E^q_{a_i} \to E^p_{a_i}$. By assumption $(R - \zeta_{ij} 1)(E_{i,j-1}) \subseteq E_{ij}$ which implies that

$$(R_{pp} - \zeta_{ij} 1)(E^p_{i,j-1}) \subseteq E^p_{ij}$$

and

$$R_{pq}(E^q_{i,j-1}) \subseteq E^p_{ij}$$

for $p \neq q$. Thus by Lemma 2.2, ∇_{21} defines a homomorphism of parabolic bundles from $\mathcal{E}^1$ to $\mathcal{E}^2(\vec{\omega})$, so it must be zero. Thus $\mathcal{E}^1$ is an invariant subsheaf for $(\mathcal{E}, \nabla)$. □

4. The tubular case

We begin with some standard definitions which are valid for all $\mathbb{X}$, but particularly useful in the tubular case. Let $\mathbb{X}$ be a weighted projective line of weight type $\mathbf{w} = (w_1, \ldots, w_k)$. Let w be the least common multiple of the components of $\mathbf{w}$. We write $\deg_{\mathbb{X}} \mathcal{E}$ for the degree of a parabolic sheaf in the sense of Geigle and Lenzing [8, Proposition 2.8],

$$\deg_{\mathbb{X}} \mathcal{E} = w \deg_{\mathbb{P}^1} \mathcal{E} + \sum_{i=1}^{k} \sum_{j=1}^{w_i - 1} n_{ij} w / w_i \in \mathbb{Z}$$

where the n_{ij} are given by

$$\underline{\dim}\,\mathcal{E} = n_*\alpha_* + \sum_{i=1}^{k}\sum_{j=1}^{w_i-1} n_{ij}\alpha_{ij}.$$

The *slope* of a parabolic sheaf $\mathcal{E}$ is $\mu(\mathcal{E}) = \deg_{\mathbb{X}} \mathcal{E} / \operatorname{rank} \mathcal{E}$, see [14, §2.5], so parabolic bundles have slope in $\mathbb{Q}$, and the torsion parabolic sheaves have slope ∞. A parabolic sheaf $\mathcal{E}$ is *semistable* (respectively *stable*) if for every non-zero proper subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (respectively $\mu(\mathcal{F}) < \mu(\mathcal{E})$).

In the rest of this section we assume that $\mathbb{X}$ is of *tubular* type, or *virtual genus* 1, meaning that $\sum_{i=1}^{k} 1/w_i = k-2$. Assuming that the w_i are non-increasing, the possible weight types are $(2,2,2,2)$, $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$. Observe that the quiver $Q_{\mathbf{w}}$ is extended Dynkin, and let 0 be an extending vertex. With the w_i non-increasing, we have $w = w_1$ and can take $0 = [1, w_1 - 1]$. Let δ be the minimal positive imaginary root for $Q_{\mathbf{w}}$. Thus the coefficient of α_* in δ is w. The representation theory of tubular weighted projective lines has been worked out by Geigle and Lenzing [8] and Lenzing and Meltzer [14].

Lemma 4.1. *(i) For a parabolic sheaf $\mathcal{E}$ we have*

$$\deg_{\mathbb{X}} \mathcal{E} = \langle \delta - w\partial, [\mathcal{E}]\rangle_{\mathbb{X}} = w \deg_{\mathbb{P}^1} \mathcal{E} + w \operatorname{rank} \mathcal{E} + \langle \delta, \underline{\dim}\,\mathcal{E}\rangle_{Q_{\mathbf{w}}}.$$

In particular, if $\underline{\dim}\,\mathcal{E} \in \mathbb{Z}\delta$, then $\deg_{\mathbb{X}} \mathcal{E} \in \mathbb{Z}w$.

(ii) For parabolic bundles $\mathcal{E}$ and $\mathcal{E}'$ of ranks r and r' we have

$$w^2\langle [\mathcal{E}], [\mathcal{E}']\rangle_{\mathbb{X}} = wrr'(\mu(\mathcal{E}') - \mu(\mathcal{E})) + \langle w\,\underline{\dim}\,\mathcal{E} - r\delta, w\,\underline{\dim}\,\mathcal{E}' - r'\delta\rangle_{Q_{\mathbf{w}}}.$$

Proof. Straightforward. □

The indecomposables of a given slope $q \in \mathbb{Q}$ are semistable, and form a uniserial abelian category whose Auslander-Reiten components are tubes [8, Theorem 5.6]. It follows that $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) = 0$ if $\mathcal{E}$ and $\mathcal{F}$ are indecomposable with $\mu(\mathcal{E}) > \mu(\mathcal{F})$. The simple objects in this category are the stable parabolic sheaves of slope q. Moreover this category is equivalent to the category of torsion parabolic sheaves [14, Theorem 4.4], so the ranks of the inhomogeneous tubes are equal to the components of $\mathbf{w}$.

Lemma 4.2. *Suppose T is an inhomogeneous tube of slope $q \in \mathbb{Q}$ and $\mathcal{E}$ is the direct sum of one copy of each stable parabolic bundle in T.*

(i) $\underline{\dim}\,\mathcal{E} = k\delta$, where k is a positive integer.

(ii) The dimension vectors of the stable parabolic bundles in T are linearly independent.

(iii) If $q \in w\mathbb{Z}$ then $k = 1$.

(iv) Any indecomposable parabolic sheaf of slope $< q$ has a non-zero map to $\mathcal{E}$ and any indecomposable parabolic sheaf of slope $> q$ has a non-zero map from $\mathcal{E}$.

Proof. (i) We have $\mathcal{E} \cong \mathcal{E}(\vec{\omega})$. Then

$$\langle [\mathcal{F}], [\mathcal{E}] \rangle_{\mathbb{X}} = \langle [\mathcal{F}], [\mathcal{E}(\vec{\omega})] \rangle_{\mathbb{X}} = -\langle [\mathcal{E}], [\mathcal{F}] \rangle_{\mathbb{X}}.$$

Thus $([\mathcal{F}], [\mathcal{E}]) = 0$ for all $\mathcal{F}$. If $[\mathcal{E}] = \alpha + s\partial$ and $[\mathcal{F}] = \alpha' + s'\partial$ this says that $(\alpha', \alpha) = 0$ for all α'. Thus $\alpha = k\delta$, some k. Of course k is a positive integer since $k\delta_* = \alpha_* = \operatorname{rank} \mathcal{E} > 0$.

(ii) Suppose some linear combination is zero, say $\sum_i m_i \underline{\dim}\, \mathcal{S}_i = 0$, with i running through $\mathbb{Z}/r\mathbb{Z}$, where r is the rank of the tube, and $\mathcal{S}_{i+1} = \mathcal{S}_i(\vec{\omega})$. Since the rank of a parabolic sheaf is the $*$-component of its dimension vector, we have

$$\sum_i m_i \,(w\, \underline{\dim}\, \mathcal{S}_i - (\operatorname{rank} \mathcal{S}_i)\delta) = 0.$$

Thus by the lemma, we have $\sum_i m_i \langle [\mathcal{S}_i], [\mathcal{S}_j] \rangle_{\mathbb{X}} = 0$ for all j. Now

$$\langle [\mathcal{S}_i], [\mathcal{S}_j] \rangle_{\mathbb{X}} = \begin{cases} 1 & (i = j) \\ -1 & (i = j+1) \\ 0 & (\text{otherwise}). \end{cases}$$

so $m_j - m_{j+1} = 0$, so the m_i are all equal. But now the m_i are zero by (i).

(iii) Let $\mathcal{F}$ be a stable parabolic bundle in T and let $\alpha = \underline{\dim}\, \mathcal{F}$. Then $\langle [\mathcal{F}], [\mathcal{F}] \rangle_{\mathbb{X}} = 1$ so α is a positive real root in Γ. We have $q \operatorname{rank} \mathcal{F} = \langle \delta, \alpha \rangle_{Q_{\mathbf{w}}} + w \operatorname{rank} \mathcal{F} + w \deg_{\mathbb{P}^1} \mathcal{F}$.

Now $\alpha - \delta$ is not zero, since α is a real root, so it a root, hence positive or negative. If it is positive, then by *loc. cit.* there is an indecomposable parabolic sheaf $\mathcal{G}$ with $[\mathcal{G}] = \alpha - \delta + s\partial$ where $s = \deg \mathcal{F} + w - q$. Then $\mathcal{G}$ has slope

$$\mu(\mathcal{G}) = (\langle \delta, \alpha - \delta \rangle_{\mathbb{X}} + w(\operatorname{rank} \mathcal{F} - w) + ws)/(\operatorname{rank} \mathcal{F} - w) = q.$$

Moreover $\langle \mathcal{G}, \mathcal{F} \rangle_{\mathbb{X}} = \langle \alpha, \alpha \rangle_{\mathbb{X}} = 1$, so there is a non-zero map $\mathcal{G} \to \mathcal{F}$. This is impossible.

Thus $\alpha - \delta$ is a negative root. Thus by *loc. cit.* there is an indecomposable parabolic sheaf $\mathcal{G}$ with $[\mathcal{G}] = \delta - \alpha + t\partial$ where $t = \deg \mathcal{F} + q - w$. This $\mathcal{G}$ has slope

$$\mu(\mathcal{G}) = (\langle \delta, \delta - \alpha \rangle_{\mathbb{X}} + w(w - \operatorname{rank} \mathcal{F}) + wt)/(w - \operatorname{rank} \mathcal{F}) = q.$$

Moreover $\langle [\mathcal{G}], [\mathcal{F}] \rangle_{\mathbb{X}} = -1 = \langle [\mathcal{F}], [\mathcal{G}] \rangle_{\mathbb{X}}$. Thus there are non-split extensions of $\mathcal{F}$ on top of the top of $\mathcal{G}$, and of the socle of $\mathcal{G}$ on top of $\mathcal{F}$. It follows from the uniserial structure of the parabolic sheaves of slope q that $\mathcal{F} \oplus \mathcal{G}$ must involve all of the stable parabolic sheaves in the tube T. By construction $\underline{\dim}\, \mathcal{F} \oplus \mathcal{G} = \delta$. Thus the dimension vector of $\mathcal{E}$ is at most δ, and hence equal to δ.

(iv) We use that if $\underline{\dim}\, \mathcal{E} = k\delta$ and $\mu(\mathcal{E}) = q$ then

$$\begin{aligned} \langle [\mathcal{E}], [\mathcal{F}] \rangle_{\mathbb{X}} &= \langle k\delta + (\deg_{\mathbb{P}^1} \mathcal{E})\partial, \underline{\dim}\, \mathcal{F} + (\deg_{\mathbb{P}^1} \mathcal{F})\partial \rangle_{\mathbb{X}} \\ &= k(\mu(\mathcal{F}) - q) \operatorname{rank} \mathcal{F} \end{aligned}$$

and similarly for $\langle[\mathcal{F}],[\mathcal{E}]\rangle_{\mathbb{X}}$. □

Lemma 4.3. *If $\mathcal{F}$ is an exceptional parabolic bundle on the mouth of a tube in the Auslander-Reiten quiver of* $\operatorname{Coh}\mathbb{X}$*, then $\mathcal{F}^{\perp}$ is equivalent to the category of finite dimensional representations of an extended Dynkin quiver.*

Proof. By a theorem of Hübner and Lenzing the perpendicular category is equivalent to the category of finite dimensional A-modules for finite dimensional hereditary algebra A; we just need to see that A is tame hereditary. Let $\mathcal{F}$ have slope q. The coherent sheaves of slope q form a tubular family, the tubes not containing $\mathcal{F}$ all belong to $\mathcal{F}^{\perp}$, and the intersection of the tube T containing $\mathcal{F}$ with $\mathcal{F}^{\perp}$ is a tube of rank 1 less than T. Thus the Auslander-Reiten quiver of A contains tubes, so A must have a tame hereditary factor. Now since any indecomposable A-module is either in a tube, or has a non-zero map to or from a fixed tube, it follows that A is connected. □

Lemma 4.4. *Suppose $(\mathcal{E},\nabla)\in\operatorname{ParConn}_{\zeta}\mathbb{X}$ and $\underline{\dim}\,\mathcal{E}=hm\delta$ with $m\geq 1$ and $h\geq 2$. Suppose that $s=\zeta*[m\delta]\in\mathbb{Z}$. Then $(\mathcal{E},\nabla)$ is not irreducible.*

Proof. First suppose that $\mathcal{E}$ has indecomposable summands of different slopes, or, all indecomposable summands of $\mathcal{E}$ have the same slope, but they belong to at least two different tubes. In this case there is a non-trivial decomposition $\mathcal{E}=\mathcal{E}^1\oplus\mathcal{E}^2$ with $\operatorname{Ext}^1(\mathcal{E}^2,\mathcal{E}^1)=0$, so Lemma 3.6 applies.

Thus we may suppose that all indecomposable summands of $\mathcal{E}$ have the same slope q and belong the same tube T. Choose a different inhomogeneous tube of slope q, and an exceptional parabolic bundle $\mathcal{F}$ on the mouth of the tube. Then $\mathcal{F}^{\perp}$ is equivalent to the category of representations of an extended Dynkin quiver Q'. By [5], the category of objects $(\mathcal{G},s)\in\operatorname{CohConn}_{\zeta}\mathbb{X}$ with $\mathcal{G}\in\mathcal{F}^{\perp}$ is equivalent to the category of representations of the deformed preprojective algebra $\Pi^{\lambda}(Q')$ for some λ. In particular $(\mathcal{E},\nabla)$ corresponds to such a representation, call it M.

Consider an indecomposable uniserial object $\mathcal{E}'$ in T having as composition factors one copy of each of the simples on the mouth of T. We must have $\underline{\dim}\,\mathcal{E}'=a\delta$ for some a. Since it has slope q, it has degree $a(q-w)$, so $aq\in\mathbb{Z}$.

By Lemma 3.3, we have $\deg_{\mathbb{P}^1}\mathcal{E}=-hs\in h\mathbb{Z}$. Thus there is a class $\phi=m\delta-s\partial\in K_0(\operatorname{Coh}\mathbb{X})$ with $h\phi=[\mathcal{E}]$. By [14, Theorem 4.6(iv)] there is an indecomposable object $\mathcal{E}''$ in T with $[\mathcal{E}'']=\phi$. Now clearly we must have $m\delta$ a multiple of $a\delta$. Thus a divides m.

Since the dimension vector of $\mathcal{E}$ is a multiple of δ, it must involve each simple in T with the same multiplicity, so $\underline{\dim}\,\mathcal{E}=b\,\underline{\dim}\,\mathcal{E}'=ba\delta$ for some $b\geq 1$. Thus $ba=hm$. It follows that $b\geq 2$.

Now $\mathcal{E}'$ corresponds to a representation M' of Q' of dimension vector δ', the minimal imaginary root δ' for Q'. Then M corresponds to a representation of dimension vector $b\delta'$. Since M exists as a $\Pi^\lambda(Q)$-module, we must have $\lambda \cdot b\delta' = 0$. Thus also $\lambda \cdot \delta' = 0$.

Then by [1, Theorem 1.2] there can be no simple module M. (All that is needed is the argument of Case (I) in [1, §10].) Also could use Crawley-Boevey and Hubery [6]. □

For later use we need the following.

Lemma 4.5. *If $\mathcal{E} \oplus \mathcal{E}'$ has dimension vector a multiple of δ, where $\mathcal{E}$ and $\mathcal{E}'$ are direct sums of indecomposables in disjoint sets of tubes of the same slope, then the dimension vectors of $\mathcal{E}$ and $\mathcal{E}'$ are also multiples of δ.*

Proof. By Lemma 4.1 we have $\langle[\mathcal{E}], [\mathcal{E} \oplus \mathcal{E}']\rangle_{\mathbb{X}} = 0$. Since $\mathcal{E}$ and $\mathcal{E}'$ are in different tubes, $\langle[\mathcal{E}], [\mathcal{E}']\rangle_{\mathbb{X}} = 0$. Thus $\langle[\mathcal{E}], [\mathcal{E}]\rangle_{\mathbb{X}} = 0$, so $w\,\underline{\dim}\,\mathcal{E} - (\operatorname{rank}\mathcal{E})\delta$ is a multiple of δ, hence so is $\underline{\dim}\,\mathcal{E}$. □

5. The extended tubular case

Let $\mathbb{X}$ be a weighted projective line of weight type $\mathbf{w} = (w_1, \dots, w_k)$. In this section we assume that $\mathbb{X}$ is of *extended tubular* type, meaning that $\mathbf{w}$ is one of $(3, 2, 2, 2)$, $(4, 3, 3)$, $(5, 4, 2)$ and $(7, 3, 2)$. Thus $\mathbf{w}' = (w_1 - 1, w_2, \dots, w_k)$ is of tubular type. Let $\mathbb{X}'$ be the weighted projective line of weight type $\mathbf{w}'$ with the same marked points as $\mathbb{X}$. We denote by δ the minimal positive imaginary root for $Q_{\mathbf{w}'}$, considered as a dimension vector for $Q_{\mathbf{w}}$. We write w' for the least common multiple of the components of $\mathbf{w}'$. We write μ for the slope function for $\mathbb{X}'$.

To shorten notation, we write ∞ and 0 for the vertices $[1, w_1 - 1]$ and $[1, w_1 - 2]$ of $Q_{\mathbf{w}}$. Note that 0 is an extending vertex for the extended Dynkin quiver $Q_{\mathbf{w}'}$. Thus we write S_∞ and S_0 for the simple torsion parabolic sheaves S_{1,w_1-1} and S_{1,w_1-2} in $\operatorname{Coh}\mathbb{X}$. Thus by [8, (2.5.4)] we have $S_\infty(\vec{\omega}) \cong S_0$ and $S_{1,0}(\vec{\omega}) \cong S_\infty$.

There is a notion of *reduction of weight* considered by Geigle and Lenzing [9, §9]. It gives the following.

Lemma 5.1. *There are functors*

$$\operatorname{Coh}\mathbb{X} \underset{i}{\overset{f}{\rightleftarrows}} \operatorname{Coh}\mathbb{X}'$$

satisfying

(i) i is left adjoint to f and $fi \cong 1$ and f is exact.

(ii) i is fully faithful and exact and gives an equivalence between $\operatorname{Coh}\mathbb{X}'$ and $C = S_{1,0}^{\perp} = {}^{\perp}S_\infty$.

(iii) $S_0 \in C$ so it is isomorphic to $i(S_0')$ for some simple torsion parabolic sheaf S_0' fo $\mathbb{X}'$.

(iv) i preserves rank, sends torsion parabolic sheaves to torsion parabolic sheaves and parabolic bundles to parabolic bundles.

(v) Considering parabolic bundles as vector bundles on $\mathbb{P}^1$ equipped with flags of subspaces of the fibres, i extends the parabolic structure by zero at the vector space E_∞ and f forgets the vector space E_∞.

(vi) If $\mathcal{E}$ is a parabolic sheaf on $\mathbb{X}$, then $f(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is isomorphic to a direct sum of copies of S_∞. Thus, for any parabolic sheaf $\mathcal{E}$ on $\mathbb{X}$, the cokernel of the natural map $if(\mathcal{E}) \to \mathcal{E}$ is isomorphic to a direct sum of copies of S_∞.

Proof. The functor f is φ_* of [9, Theorem 9.5]. As remarked there, it has a left adjoint, which we denote by i. □

Lemma 5.2. *We have $\operatorname{Ext}^1(i(\mathcal{F}'), \mathcal{E}) \cong \operatorname{Ext}^1(\mathcal{F}', f(\mathcal{E}))$ for $\mathcal{E} \in \operatorname{Coh}\mathbb{X}$ and $\mathcal{F}' \in \operatorname{Coh}\mathbb{X}'$. In particular, i induces an isomorphism on Ext^1.*

Proof. For parabolic bundles we can interpret this as follows. Given an exact sequence $0 \to \mathcal{E} \to \mathcal{G} \to i(\mathcal{F}') \to 0$, the sequence of vector spaces for the vector space ∞ is $0 \to E_\infty \to G_\infty \to 0 \to 0$, so $E_\infty \cong G_\infty$, and the rest of the data determines an exact sequence $0 \to f(\mathcal{E}) \to f(\mathcal{G}) \to \mathcal{F}' \to 0$.

[In general, for any sheaf $\mathcal{E}$ on $\mathbb{X}$ we have an exact sequence $0 \to if(\mathcal{E}) \to \mathcal{E} \to \tau S^n \to 0$ where $n = \dim \operatorname{Hom}(\mathcal{E}, \tau S) = \dim \operatorname{Ext}^1(S, \mathcal{E})$.]

Given an exact sequence $0 \to \mathcal{E} \to \mathcal{G} \to i(\mathcal{F}') \to 0$, applying f gives an exact sequence $0 \to f(\mathcal{E}) \to f(\mathcal{G}) \to \mathcal{F}' \to 0$. This gives a map $\operatorname{Ext}^1(i(\mathcal{F}'), \mathcal{E}) \to \operatorname{Ext}^1(\mathcal{F}', f(\mathcal{E}))$. Given an exact sequence $0 \to f(\mathcal{E}) \to \mathcal{G}' \to \mathcal{F}' \to 0$, applying i and taking the pushout along the morphism $if(\mathcal{E}) \to \mathcal{E}$ gives an exact sequence $0 \to \mathcal{E} \to \mathcal{G} \to i(\mathcal{F}') \to 0$. This gives a map $\operatorname{Ext}^1(\mathcal{F}', f(\mathcal{E})) \to \operatorname{Ext}^1(i(\mathcal{F}'), \mathcal{E})$. It is easy to see that these are inverses.

Last part is actually obvious, since the functor i identifies $\operatorname{Coh}\mathbb{X}'$ with a full extension-closed subcategory of $\operatorname{Coh}\mathbb{X}$, so i induces an isomorphism on Ext^1. □

Lemma 5.3. *Say $(\mathcal{E}, \nabla) \in \operatorname{ParConn}_\zeta \mathbb{X}$ and $\underline{\dim}\,\mathcal{E} = \alpha_\infty + h\delta$ with $h \geq 2$. We suppose that $s = \zeta * [\delta] \in \mathbb{Z}$ and $\zeta * [\alpha_\infty] = 0$. Then $q = \mu(f(\mathcal{E}))$ is in $\mathbb{Z}$. We decompose*

$$f(\mathcal{E}) = \mathcal{E}^{<q} \oplus \mathcal{E}^{=q} \oplus \mathcal{E}^{>q}$$

where each term is a direct sum of indecomposables with slopes in the indicated ranges. If

$$0 \to \mathcal{F} \to \mathcal{E}^{>q} \to S_0' \to 0$$

is an exact sequence in $\operatorname{Coh}\mathbb{X}'$, then $\mu(\mathcal{F}') \geq q$ for any non-zero direct summand $\mathcal{F}'$ of $\mathcal{F}$.

Proof. Because of the existence of ∇, we have

$$\deg_{\mathbb{P}^1} \mathcal{E} = -\zeta * [\underline{\dim}\,\mathcal{E}] = -hs.$$

Now this is just the usual degree of the vector bundle E. so it is also $\deg_{\mathbb{P}^1} \mathcal{E}'$. Thus by Lemma 4.1 we have

$$q = \mu(f(\mathcal{E})) = [w'(\deg_{\mathbb{P}^1} \mathcal{E}) + w' \operatorname{rank} \mathcal{E}]/(\operatorname{rank} \mathcal{E}) = w' - s.$$

Thus $q \in \mathbb{Z}$. Suppose $\mu(\mathcal{F}') < q$. Take the pushout of the exact sequence defining $\mathcal{F}$ along the projection $\mathcal{F} \to \mathcal{F}'$ to get

$$0 \to \mathcal{F}' \to \mathcal{G} \to S_0' \to 0.$$

There is an epimorphism from $\mathcal{E}^{>q} \to \mathcal{G}$, so by the Hom properties of slopes in the tubular case, we must have $\mu(\mathcal{G}) > q$. Let $m = \operatorname{rank} \mathcal{F}' = \operatorname{rank} \mathcal{G}$. Then $\deg_{\mathbb{X}'} \mathcal{F}' < mq$, $\deg_{\mathbb{X}'} \mathcal{G} > mq$ and $\deg_{\mathbb{X}'} \mathcal{G} - \deg_{\mathbb{X}'} \mathcal{F}' = \deg_{\mathbb{X}'} S_0' = 1$. This is impossible since mq, $\deg_{\mathbb{X}'} \mathcal{F}'$ and $\deg_{\mathbb{X}'} \mathcal{G}$ are integers. □

Lemma 5.4. *Suppose that* $(\mathcal{E}, \nabla) \in \operatorname{ParConn}_\zeta \mathbb{X}$ *and* $\underline{\dim}\, \mathcal{E} = \alpha_\infty + h\delta$ *with* $h \geq 2$, $\zeta * [\alpha_\infty] = 0$ *and* $\zeta * [\delta] \in \mathbb{Z}$. *If* $f(\mathcal{E})$ *has indecomposable summands of different slopes, then* $(\mathcal{E}, \nabla)$ *is not irreducible.*

Proof. Let $\mathcal{E}' = f(\mathcal{E}) \in \operatorname{Coh} \mathbb{X}'$. Let $q = \mu(\mathcal{E}')$. This gives $\mathcal{E}' = \mathcal{E}^1 \oplus \mathcal{E}^2 \oplus \mathcal{E}^3$ where $\mathcal{E}^1 = (\mathcal{E}')^{>q}$, $\mathcal{E}^2 = (\mathcal{E}')^{=q}$ and $\mathcal{E}^3 = (\mathcal{E}')^{<q}$. By assumption $\mathcal{E}^1$ and $\mathcal{E}^3$ are non-zero. This gives a decomposition of the vector bundle $E = E^1 \oplus E^2 \oplus E^3$. Let $E_{ij}^r = E_{a_i}^r \cap E_{ij}$. Then $E_{ij} = E_{ij}^1 \oplus E_{ij}^2 \oplus E_{ij}^3$ except at the ∞ vertex.

Now $(\operatorname{Res}_{a_1} \nabla - \zeta_\infty)(E_0) \subseteq E_\infty$, so $\phi = (\operatorname{Res}_{a_1} \nabla - \zeta_\infty)|_{E_0}$ considered as a map $E_0 \to E_0$ has rank ≤ 1. Using the decomposition of E_0 it becomes a 3 by 3 block matrix.

$$\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}.$$

Thus ϕ_{31} has rank ≤ 1. Let $\mathcal{F}$ be the parabolic bundle obtained from $\mathcal{E}^1$ by replacing the subspace E_0^1 by $\operatorname{Ker} \phi_{31}$. By Lemma 5.3, for any summand $\mathcal{F}'$ of $\mathcal{F}$ we have $\mu(\mathcal{F}') \geq q$. Thus $\operatorname{Hom}(\mathcal{F}, \mathcal{E}^3(\vec{\omega})) = 0$.

But now ∇_{31} defines a map of parabolic bundles from $\mathcal{F}$ to $\mathcal{E}^3(\vec{\omega})$, so $\nabla_{31} = 0$. In particular $\phi_{31} = 0$. Now since the matrix above has rank ≤ 1, we get $\phi_{21} = 0$ or $\phi_{32} = 0$.

If $\phi_{21} = 0$, then ∇_{21} defines a map of parabolic bundles from $\mathcal{E}^1$ to $\mathcal{E}^2(\vec{\omega})$. But $\operatorname{Hom}(\mathcal{E}^1, \mathcal{E}^2(\vec{\omega})) = 0$. Thus $\nabla_{21} = 0$. Since $\phi_{21} = 0$ and $\phi_{31} = 0$ we have $\phi(E_0^1) \subseteq E_0^1$, either zero, or a 1-dimensional subspace. Let $\mathcal{G}$ be the parabolic bundle on $\mathbb{X}$ obtained from $\mathcal{E}^1$ by taking this as the subspace for the vertex ∞. Then $\mathcal{G}$ is an invariant subsheaf for $(\mathcal{E}, \nabla)$.

If $\phi_{32} = 0$, then ∇_{32} defines a map of parabolic bundles from $\mathcal{E}^2$ to $\mathcal{E}^3(\vec{\omega})$. But $\operatorname{Hom}(\mathcal{E}^2, \mathcal{E}^3(\vec{\omega})) = 0$. Thus $\nabla_{32} = 0$. Since $\nabla_{31} = 0$ and $\nabla_{32} = 0$ it follows that $\mathcal{E}^1 \oplus \mathcal{E}^2$, extended to a parabolic bundle $\mathcal{G}$ on $\mathbb{X}$ by the subspace $\phi(E_0^1 \oplus E_0^2)$ defines an invariant subsheaf of $(\mathcal{E}, \nabla)$. □

Lemma 5.5. *Suppose that* $(\mathcal{E},\nabla) \in \mathrm{ParConn}_\zeta \mathbb{X}$ *and* $\underline{\dim}\,\mathcal{E} = \alpha_\infty + h\delta$ *with* $h \geq 2$, $\zeta*[\alpha_\infty] = 0$ *and* $\zeta*[\delta] \in \mathbb{Z}$. *If all indecomposable summands of* $f(\mathcal{E})$ *have the same slope* q, *then* $(\mathcal{E},\nabla)$ *is not irreducible; it has a proper non-zero invariant subsheaf* $\mathcal{F}$ *such that all indecomposable summands of* $f(\mathcal{F})$ *and* $f(\mathcal{E}/\mathcal{F})$ *have slope* q.

Proof. We prove this by induction on h. Let $\mathcal{E}' = f(\mathcal{E}) \in \mathrm{Coh}\,\mathbb{X}'$. Let $s = \zeta * [\delta]$.

Case (i). The summands of $\mathcal{E}'$ belong to at least two different tubes. Decompose as $\mathcal{E}' = \mathcal{E}^1 \oplus \mathcal{E}^2$ with the summands of $\mathcal{E}^1$ and $\mathcal{E}^2$ in disjoint sets of tubes. By Lemma 4.5, the dimension vectors of $\mathcal{E}^1$ and $\mathcal{E}^2$ are also multiples of δ, say $h_1\delta$ and $h_2\delta$. Since $\mathcal{E}$ has slope q, we have

$$q \,\mathrm{rank}\,\mathcal{E} = \deg_{\mathbb{X}} \mathcal{E} = w \deg_{\mathbb{P}^1} \mathcal{E} + w\,\mathrm{rank}\,\mathcal{E} + \langle \delta, \underline{\dim}\,\mathcal{E}\rangle_{Q_{\mathbf{w}'}},$$

so $\deg_{\mathbb{P}^1} \mathcal{E} = (q-w)/w.\,\mathrm{rank}\,\mathcal{E}$, and similarly $\deg_{\mathbb{P}^1} \mathcal{E}^i = (q-w)/w.\,\mathrm{rank}\,\mathcal{E}^i$. Since there is a ζ-connection on $\mathcal{E}$ we have $\deg_{\mathbb{P}^1} \mathcal{E} + \zeta * [\underline{\dim}\,\mathcal{E}] = 0$. Then $\mathrm{rank}\,\mathcal{E}^i / \mathrm{rank}\,\mathcal{E} = w'h_i/w'h = h_i/h$, so

$$\begin{aligned}\zeta * [\underline{\dim}\,\mathcal{E}^i] &= \frac{h_i}{h}\zeta * [\underline{\dim}\,\mathcal{E}] = -\frac{h_i}{h}\deg_{\mathbb{P}^1}\mathcal{E} = -\frac{h_i}{h}\frac{q-w}{w}\,\mathrm{rank}\,\mathcal{E} \\ &= -\frac{q-w}{w}\,\mathrm{rank}\,\mathcal{E}^i = -\deg_{\mathbb{P}^1}\mathcal{E}^i.\end{aligned}$$

Considering parabolic bundles in terms of a vector bundle on $\mathbb{P}^1$ with flags of subspaces, we get $E = E^1 \oplus E^2$ and decompositions $E_{ij} = E^1_{ij} \oplus E^2_{ij}$ for all vertices $[i,j]$ except $\infty = [1, w_1 - 1]$. This decomposition gives a block decomposition

$$\nabla = \begin{pmatrix} \nabla_{11} & \nabla_{12} \\ \nabla_{21} & \nabla_{22} \end{pmatrix}$$

where ∇_{pp} is a logarithmic connection $E^p \to E^p \otimes \Omega$ and ∇_{pq} is a homomorphism $E^q \to E^p \otimes \Omega$ for $p \neq q$. The residues of ∇ then take block form

$$\mathrm{Res}_{a_i} \nabla = \begin{pmatrix} \mathrm{Res}_{a_i}\nabla_{11} & \mathrm{Res}_{a_i}\nabla_{12} \\ \mathrm{Res}_{a_i}\nabla_{21} & \mathrm{Res}_{a_i}\nabla_{22} \end{pmatrix}$$

Since ∇ is a ζ-connection on E we have

$$(\mathrm{Res}_{a_i} \nabla - \zeta_{ij} 1)(E_{i,j-1}) \subseteq E_{ij}$$

for all $1 \leq i \leq k$ and $1 \leq j \leq w_i$. In particular, for $j < w_i'$ we have

$$(\mathrm{Res}_{a_i} \nabla_{pp} - \zeta_{ij} 1)(E^p_{i,j-1}) \subseteq E^p_{ij}.$$

Thus, thus considering the case $i = 1$ and $j = w_1' = w_i - 1$, so the vertices $\infty = [1, w_1 - 1]$ and $0 = [1, w_1 - 2]$, the map

$$\phi = \mathrm{Res}_{a_1} \nabla - \zeta_\infty 1 : E_0 \to E_0$$

has image contained in E_∞, so $\operatorname{rank}\phi \leq 1$. Using the decomposition $E_0 = E_0^1 \oplus E_0^2$ we get a block decomposition

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

where ϕ_{pp} is the restriction of $\operatorname{Res}\nabla_{pp} - \zeta_\infty 1$ to E_0^p and for $p \neq q$, ϕ_{pq} is the restriction of $\operatorname{Res}\nabla_{pq}$ to E_0^q.

Now since ∇_{pp} is a logarithmic connection on E^p, we have

$$\sum_{i=1}^{k} \operatorname{tr}(\operatorname{Res}_{a_i}\nabla_{pp}) = -\deg E^p.$$

Now $\dim E_{ij}^p = h_p\delta_{ij}$ for $1 \leq i \leq k$ and $1 \leq j < w_i'$, so with the convention that $\zeta_{i0} = \zeta_*$ and $\zeta_{i,w_i} = 0$, for $i \neq 1$ we have

$$\operatorname{tr}(\operatorname{Res}_{a_i}\nabla_{pp}) = \sum_{j=1}^{w_i'} \zeta_{ij}(h_p\delta_{i,j-1} - h_p\delta_{i,j})$$

and

$$\operatorname{tr}(\operatorname{Res}_{a_1}\nabla_{pp}) = \sum_{j=1}^{w_i'} \zeta_{1j}(h_p\delta_{i,j-1} - h_p\delta_{i,j}) + \operatorname{tr}\phi_{pp}.$$

Thus by the calculations above, $\operatorname{tr}\phi_{pp} = 0$. We have two possible cases.

Case (i)(a) $h_2 = 1$, so $\underline{\dim}\,\mathcal{E}^2 = \delta$. Then $\dim E_0^2 = 1$, so $\phi_{22} = 0$ since it is an endomorphism of trace zero of a 1-dimensional vector space. Thus since $\operatorname{rank}\phi \leq 1$ we get $\phi_{pq} = 0$ for $(p,q) = (1,2)$ or $(2,1)$. But then ∇_{pq} defines a homomorphism $\mathcal{E}^q \to \mathcal{E}^p(\vec{\omega})$. But $\operatorname{Hom}(\mathcal{E}^q, \mathcal{E}^p(\vec{\omega})) = 0$, since the summands of $\mathcal{E}^p$ and $\mathcal{E}^q$ belong to disjoint sets of tubes. Thus $\nabla_{pq} = 0$. This implies that $\mathcal{E}^q$, considered as a parabolic bundle on $\mathbb{X}$ with the vector space $\mathcal{E}_\infty^q = \operatorname{Im}\phi_{qq}$ at vertex ∞, gives an invariant subsheaf of $(\mathcal{E}, \nabla)$.

Case (i)(b) $h_2 > 1$. We define $E_\infty^2 = \operatorname{Im}\phi_{22} \subseteq E_0^2$. This turns $\mathcal{E}^2$ into a parabolic bundle on $\mathbb{X}$, and $(\mathcal{E}^2, \nabla_{22}) \in \operatorname{ParConn}_\zeta \mathbb{X}$. By induction, this pair has an invariant subsheaf $\mathcal{F}$ such that the indecomposable summands of $\mathcal{F}$ and of $\mathcal{E}^2/\mathcal{F}$ have slope q.

Considering $\mathcal{F}$ as a tuple (F, F_{ij}) of a vector bundle F on $\mathbb{P}^1$ and flags of subspaces of its fibres, we have $\nabla_{22}(F) \subseteq F \otimes \Omega$ and $(\operatorname{Res}_{a_i}\nabla_{22} - \zeta_{ij}1)(F_{i,j-1}) \subseteq F_{ij}$.

Choose a complement Z to F_0 in E_0^2. Then E_0 has direct sum decomposition $E_0^1 \oplus F_0 \oplus Z$. With respect to this decomposition ϕ takes block form

$$\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & 0 & \phi_{33} \end{pmatrix}$$

with 0 in the $(3,2)$ position since $\mathcal{F}$ is invariant in $\mathcal{E}^2$. Now $\operatorname{rank}\phi \leq 1$, so either $\phi_{12} = 0$ or $\phi_{31} = 0$.

If $\phi_{12} = 0$, then the restriction of ∇_{12} to $\mathcal{F}$ induces a morphism $\mathcal{F} \to \mathcal{E}^1(\vec{\omega})$. But the indecomposable summands of $\mathcal{F}$ have the same slope as $\mathcal{E}^2$, and so they are in the same tube as the summands of $\mathcal{E}^2$, so there is no such morphism. It follows that $\mathcal{F}$ is an invariant subsheaf for $(\mathcal{E}, \nabla)$.

If $\phi_{31} = 0$, then ∇_{21} induces a morphism $\mathcal{E}^1 \to (\mathcal{E}^2/\mathcal{F})(\vec{\omega})$. But the indecomposable summands of $\mathcal{E}^2/\mathcal{F}$ have the same slope as $\mathcal{E}^2$, and so they are in the same tube as the summands of $\mathcal{E}^2$, so there is no such morphism. It follows that $\mathcal{E}^1 \oplus \mathcal{F}$ is an invariant subsheaf for $(\mathcal{E}, \nabla)$.

Case (ii). All indecomposable summands of $\mathcal{E}'$ belong to the same tube. Choose an inhomogeneous tube T of the same slope which is different from this one. Consider the non-split extension of torsion parabolic sheaves for $\mathbb{X}$,

$$0 \to S_\infty \to R \to S_{1,0} \to 0.$$

The parabolic sheaf R is in ${}^\perp S_\infty$, but the parabolic sheaves S_∞ and $S_{1,0}$ are not in ${}^\perp S_\infty$. Thus $R \cong i(L)$ for some simple torsion parabolic sheaf L for $\mathbb{X}'$. Choose an indecomposable $\mathcal{F}'$ on the mouth of the tube T with $\operatorname{Hom}(\mathcal{F}', L) \neq 0$. We know that this is possible. Thus $\mathcal{E}' \in (\mathcal{F}')^\perp$ and $\operatorname{Ext}^1(\mathcal{F}', \mathcal{F}') = 0$. Let $\mathcal{F} = i(\mathcal{F}')$. By Lemmas 5.1 and 5.2 we have $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = 0$ and $\mathcal{F}^\perp$ is the category of parabolic sheaves $\mathcal{G}$ such that $f(\mathcal{G}) \in (\mathcal{F}')^\perp$. In particular $\mathcal{E} \in \mathcal{F}^\perp$. Since $f(S_\infty) = 0$ we have $S_\infty \in \mathcal{F}^\perp$.

Now consider the category $\mathcal{F}^\perp$. By Hübner and Lenzing [10], [5, Theorem 4.2] this is equivalent to the category of representations of some quiver $Q_{\mathcal{F}}$. Also consider $(\mathcal{F}')^\perp$. This is equivalent to the category of representations of an extended Dynkin quiver $Q_{\mathcal{F}'}$.

Let U_i be the objects in $(\mathcal{F}')^\perp$ corresponding to simple representations of $Q_{\mathcal{F}'}$. Then $\operatorname{Hom}(\mathcal{F}', U_i) = 0$, so $\operatorname{Hom}(L, U_i) = 0$ since there is an epimorphism from $\mathcal{F}'$ to L. Thus $\operatorname{Hom}(R, i(U_i)) = 0$. Now $\operatorname{Hom}(S_\infty, i(U_i)) = 0$, for otherwise, since $i(U_i))$ is indecomposable, it must be a torsion parabolic sheaf, so uniserial, and then its socle must be S_∞, and so either it is isomorphic to S_∞ isn't in the image of i, or it has R as a subseaf, which isn't possible since $\operatorname{Hom}(R, i(U_i)) = 0$.

Thus the parabolic sheaves S_∞ and $i(U_i)$ are orthogonal bricks. In addition, for any parabolic sheaf $\mathcal{G}$ on $\mathbb{X}$, since i is exact the sheaf $if(\mathcal{G}))$ has a filtration by copies of the $i(U_j)$, and the cokernel of the map $if(\mathcal{G}) \to \mathcal{G}$ is a direct sum of copies of S_∞. It follows that S_∞ and the $i(U_j)$ correspond to the simple representations of $Q_{\mathcal{F}}$.

What does the quiver $Q_{\mathcal{F}}$ look like? The vertices are labelled by ∞ and the same i as occur indexing the U_i. Now $\operatorname{Ext}^1(i(U_j), S_\infty) = 0$ so ∞ is a sink. Now $S_0 \in {}^\perp S_\infty$ since $w_1 > 2$, so S_0 is one of the simples U_j, say U_0.

$$\operatorname{Ext}^1(S_\infty, i(U_j)) \cong D\operatorname{Hom}(i(U_j), S_\infty(\vec{\omega})) = D\operatorname{Hom}(U_j, S_0)$$

which is 1-dimensional if $j = 0$ and otherwise 0. Thus there is one arrow starting from ∞ and going to the vertex 0. On the other hand, $\mathrm{Ext}^1(i(U_j), i(U_k)) \cong \mathrm{Ext}^1(U_j, U_k)$ so quiver obtained by deleting the vertex ∞ is isomorphic to the quiver $Q_{\mathcal{F}'}$.

Now $Q_{\mathcal{F}'}$ has representations corresponding to the parabolic sheaves in all the tubes of slope q other than T. Moreover any other indecomposable representation has a non-zero morphism to or from one of these tubes. Thus $Q_{\mathcal{F}'}$ must be an extended Dynkin quiver.

The vertices of $Q_{\mathcal{F}'}$ are the same i as occur indexing the U_i. In particular vertex 0 corresponds to $U_0 \cong S_0'$, with $i(S_0') \cong S_0$.

We show that 0 is a source in $Q_{\mathcal{F}'}$. If not, there is a vertex i with $\mathrm{Ext}^1(U_i, S_0') \neq 0$. Since S_0' is a simple torsion parabolic sheaf, and U_i is indecomposable, we must have that U_i is torsion, and that its socle is the simple torsion parabolic sheaf which has an extension with S_0'. But this is L. But then $\mathrm{Hom}(L, U_i) \neq 0$, so $\mathrm{Hom}(\mathcal{F}', U_i) \neq 0$, contradicting that $U_i' \in (\mathcal{F}')^\perp$.

We show that 0 is an extending vertex for $Q_{\mathcal{F}'}$. Let δ' be the minimal positive imaginary root for $Q_{\mathcal{F}'}$. Let T' be the direct sum of the stable parabolic sheaves in an inhomogeneous tube of slope q for $\mathbb{X}'$ which is different from T. The summands of T' correspond to regular simple modules in an inhomogeneous tube for $Q_{\mathcal{F}'}$, so T' corresponds to a representation of $Q_{\mathcal{F}'}$ is δ'. Now $q = \mu(\mathcal{E}')$, and as in Lemma 5.3 this is an integer. Thus $\underline{\dim}\, T' = \delta$ by Lemma 4.2. Then since $S_0' = U_0$ corresponds to a simple injective representation of $Q_{\mathcal{F}'}$,

$$\delta_0' = \langle [T'], [U_0] \rangle_{\mathbb{X}'} = \langle \underline{\dim}\, T', \alpha_0 \rangle_{Q_{\mathbf{w}'}} = \langle \delta, \alpha_0 \rangle_{Q_{\mathbf{w}'}} = 1,$$

so 0 is an extending vertex.

Now by [5, Theorem 5.1] the category of parabolic sheaves in $\mathcal{F}^\perp$ equipped with a ζ-connection is equivalent to the category of $\Pi^\lambda(Q_{\mathcal{F}})$-modules, for some λ. In particular, since $\mathcal{E} \in \mathcal{F}^\perp$, $(\mathcal{E}, \nabla)$ corresponds to a module M for $\Pi^\lambda(Q_{\mathcal{F}})$.

Now the restriction of M to $Q_{\mathcal{F}}$ corresponds to $\mathcal{E}$, whose dimension vector is $\alpha_\infty + h\delta$. In $\mathcal{F}^\perp$ we have parabolic sheaves S_∞ and T'. Their dimension vectors are α_∞ and δ. Now T' corresponds to a representation of $Q_{\mathcal{F}}$ of dimension vector δ', and S_∞ corresponds to the simple representation of at vertex ∞ of $Q_{\mathcal{F}}$, so its dimension vector is the simple root α_∞' corresponding to vertex ∞. Thus $\mathcal{E}$ corresponds to a representation of $Q_{\mathcal{F}}$ of dimension vector $\beta' = \alpha_\infty' + h\delta'$.

Moreover $\lambda \cdot \beta' = 0$ since $(\mathcal{E}, \nabla)$ exists. Also $\lambda \cdot \delta' = 0$. Namely, if one takes not T', but the extension T'' of the stables in the tube, one gets an indecomposable parabolic bundle of slope q and dimension vector δ. Now as in the calculation in case (i), $\deg_{\mathbb{P}^1} T'' + \zeta * [\underline{\dim}\, T''] = 0$. Thus there is a ζ-connection ∇'' on T''. Then (T'', ∇'') corresponds to a $\Pi^\lambda(Q_{\mathcal{F}})$-module. Its underlying $Q_{\mathcal{F}}$-representation corresponds to T'', so it has dimension vector δ'. Thus $\lambda \cdot \delta' = 0$. Thus by [1, Theorem 9.1], M cannot be a simple representation of $\Pi^\lambda(Q_{\mathcal{F}})$. In fact by [6, Theorem 8.1] it has a proper non-trivial submodule M' which is regular as a representation of $Q_{\mathcal{F}}$. But then as a representation of $Q_{\mathcal{F}}$, the summands of M' are in the same tube as the summands of M. Thus they correspond to parabolic bundles in the same tube as the summands of $\mathcal{E}$. The same holds for the summands of the quotient. □

6. Special case of DSP

We fix a transversal T to $\mathbb{Z}$ in $\mathbb{C}$, such as $T = \{z \in \mathbb{C} \mid 0 \leq \Re z < 1\}$. We will use the following standard result.

Lemma 6.1. *The assignment* $A \mapsto \exp(-2\pi\sqrt{-1}A)$ *induces a bijection between conjugacy classes of matrices with eigenvalues in* T *and conjugacy classes of invertible matrices.*

As in the introduction, let $C_1, \dots, C_k$ be conjugacy classes in $\mathrm{GL}_n(\mathbb{C})$ corresponding to a weight sequence $\mathbf{w} = (w_1, \dots, w_k)$, complex numbers $\xi = (\xi_{ij})$ $(1 \leq i \leq k,\ 1 \leq j \leq w_i)$ and $\alpha \in \Gamma = \Gamma_{Q_w}$.

Theorem 6.2. *There is no irreducible solution to the DSP in the following cases:*

(a) The quiver $Q_{\mathbf{w}}$ *is extended Dynkin with minimal positive imaginary root* δ*, the dimension vector is* $\alpha = hm\delta$ *with* $m \geq 1$ *and* $h \geq 2$ *and* $\xi^{[\delta]}$ *a primitive* m*th root of unity.*

(b) The quiver is obtained from an extended Dynkin quiver by adjoining another vertex ∞ *to an extending vertex, the dimension vector is* $\alpha = \alpha_\infty + h\delta$ *with* $h \geq 2$ *and* $\xi^{[\alpha_\infty]} = 1$ *and* $\xi^{[\delta]} = 1$.

Proof. Suppose we have an irreducible solution $(g_1, \dots, g_k)$ to the DSP.

Let C_i be the conjugacy classes of the g_i. They are determined by $\mathbf{w}$, ξ and α.

Define ζ non-resonant by $\xi_{ij} = \exp(-2\pi\sqrt{-1}\zeta_{ij})$ and $\zeta_{ij} \in T$. Let D_i be the conjugacy classes with eigenvalues in T with $\exp(-2\pi\sqrt{-1}A_i)$ in C_i for $A_i \in D_i$. The D_i are the conjugacy classes determined by $\mathbf{w}$, ζ and α.

Fix distinct points $D = (a_1, a_2, \dots, a_k)$ in $X = \mathbb{P}^1$ and let $\mathbb{X}$ be the corresponding weighted projective line.

The irreducible solution to the DSP corresponds to a representation of $\pi_1(X \setminus D)$. That is, g_i is the image of a loop around a_i.

Under the equivalence of [2, Theorem 6.1] this corresponds to a vector bundle E and logarithmic connection ∇ whose eigenvalues are in T, meaning that the eigenvalues of the residues $\mathrm{Res}_{a_i}\nabla$ are in T. In fact we have $\mathrm{Res}_{a_i}\nabla \in D_i$.

We define

$$E_{ij} = \mathrm{Im}\,\mathrm{rank}(\mathrm{Res}_{a_i}\nabla - \zeta_{i1}1)(\mathrm{Res}_{a_i}\nabla - \zeta_{i2}1)\dots(\mathrm{Res}_{a_i}\nabla - \zeta_{ij}1).$$

Then $\mathcal{E} = (E, E_{ij})$ becomes a parabolic bundle on $\mathbb{X}$ and ∇ becomes a ζ-connection.

We say that a ζ-connection ∇ on a parabolic bundle $\mathcal{E} = (E, E_{ij})$ is *normal* provided that

$$(\operatorname{Res}_{a_i} \nabla - \zeta_{ij} 1)(E_{i,j-1}) = E_{ij}$$

for all $1 \leq i \leq k$ and $1 \leq j \leq w_i$. By construction, our $(\mathcal{E}, \nabla)$ is normal.

By [2, §6] and the discussion at the start of [2, §7], monodromy gives an equivalence from the category of pairs $(\mathcal{E}, \nabla)$ consisting of a parabolic bundle on $\mathbb{X}$ equipped with a normal ζ-connection ∇ and the category of representations of $\pi_1(X \setminus D)$ of type ξ.

Now $(\mathcal{E}, \nabla) \in \operatorname{ParConn}_\zeta \mathbb{X}$ corresponds to a pair $(\mathcal{E}, s) \in \operatorname{CohConn}_\zeta \mathbb{X}$.

Moreover (E, ∇) is irreducible.

Suppose otherwise. Suppose that this has an invariant subsheaf $\mathcal{E}'$ such that the quotient $\mathcal{E}/\mathcal{E}'$ is torsion. Then the sub-object $(\mathcal{E}', s')$ corresponds to a pair $(\mathcal{E}', \nabla')$

This gives a nonzero map $(\mathcal{E}', \nabla') \to (\mathcal{E}, \nabla)$.

Forgetting the parabolic structure it gives a nonzero map of vector bundles equipped with logarithmic connections of type ζ.

Since ζ is non-resonant, by Riemann-Hilbert correspondence we obtain a nonzero map of representations of the fundamental group. But the two representations have the same dimension, and the target repreentation is irreducible, so the map is an isomorphism. Thus the map of vector bundles equipped with logarithmic connections is an isomorphism, see [2, Theorem 6.1].

Thus $(\mathcal{E}, \nabla)$ and (E', ∇') are equal, except that the parabolic structures may differ. We know, however, that that there is an inclusion morphism $\mathcal{E}' \to \mathcal{E}$ which is the identity map on the underlying vector bundle. Thus the spaces E'_{ij} are subspaces of the E_{ij}. But then the fact that ∇ is normal for $\mathcal{E}$ ensures that

$$E_{ij} = (\operatorname{Res}_{a_i} \nabla - \zeta_{ij} 1) \dots (\operatorname{Res}_{a_i} \nabla - \zeta_{i1} 1)(E_{a_i}),$$

while

$$E'_{ij} \supseteq (\operatorname{Res}_{a_i} \nabla - \zeta_{ij} 1) \dots (\operatorname{Res}_{a_i} \nabla - \zeta_{i1} 1)(E_{a_i})$$

and hence $E'_{ij} = E_{ij}$ for all i, j. Thus $\mathcal{E}' = \mathcal{E}$.

—-

Now in case (a) or (b) we are in cases already considered, where we showed that there is no irreducible pair $(\mathcal{E}, s)$ in $\operatorname{CohConn}_\zeta \mathbb{X}$.

In particular, in case (b) we have $\xi^{[\alpha_\infty]} = 1$. Thus $\xi_{1,w_1} = \xi_{1,w_1-1}$. Thus $\zeta_{1,w_1} = \zeta_{1,w_1-1}$. Thus $\zeta * [\alpha_\infty] = 0$.

□

7. Multiplicative preprojective algebras

Let Q be a quiver with vertex set I and let $q \in (\mathbb{C}^*)^I$. Let $\Lambda^q(Q)$ be the corresponding multiplicative preprojective algebra, see [7]. For $\alpha \in \Gamma_Q$, say $\alpha = \sum_{i \in I} n_i \alpha_i$, we define $q^\beta = \prod_{i \in I} q_i^{n_i}$.

Given conjugacy classes $C_1, \dots, C_k$ in $\mathrm{GL}_n(\mathbb{C})$ as in the introduction, we choose $k, \mathbf{w}, \xi$ as there, and obtain a quiver $Q_{\mathbf{w}}$, root lattice $\Gamma_{\mathbf{w}}$ and an element $\alpha_C \in \Gamma_{\mathbf{w}}$. We define $q_C \in (\mathbb{C}^*)^{I_{\mathbf{w}}}$ by $q_* = 1/\prod_{i=1}^k \xi_{i1}$ and $q_{ij} = \xi_{ij}/\xi_{i,j+1}$, so that $\xi^[\alpha] = 1/q^\alpha$ for all $\alpha \in \Gamma_{\mathbf{w}}$. By [7, Lemma 8.3] we have the following.

Lemma 7.1. *There is an irreducible solution to $A_1 \dots A_k = 1$ with matrices $A_i \in C_i$ if and only if there is a simple $\Lambda^{q_C}(Q_{\mathbf{w}})$-module of dimension vector α_C.*

There are reflections $u_v(q)$ as in [7] and reflection functors.
Consider pairs $[q, \alpha]$ with $q \in (K^*)^I$ and $\alpha \in \mathbb{Z}^I$.
The reflection at i is admissible for the pair $[q, \alpha]$ if $q_i \neq 1$.
Equivalence relation $\sim$ is generated by admissible reflections.
$\bar{R}_q^+$ is the set of positive roots with $q^\alpha = 1$.
$\mathbb{N}\bar{R}_q^+$ is the set of sums of elements of $\bar{R}_q^+$.
$\bar{\Sigma}_q$ is the elements $\alpha \in \bar{R}_q^+$ such that there is no decomposition $\alpha = \beta + \gamma + \dots$ with $p(\alpha) > p(\beta) + p(\gamma) + \dots$ and $\beta, \gamma, \dots \in \bar{R}_q^+$.

Theorem 7.2. *If $\alpha \in \mathbb{N}^I$ the $\alpha \in \bar{\Sigma}_q$ if and only if $0 \neq \alpha \in \mathbb{N}\bar{R}_q^+$ and $(\beta, \alpha - \beta) \leq -2$ whenever $\beta, \alpha - \beta$ are nonzero and in $\mathbb{N}\bar{R}_q^+$.*

Proof. This is analogue to [1, Theorem 5.6]. One needs to modify [1, Lemmas 5.1–5.5]. □

Don't know if we need or want analogue of [1, Theorem 5.8].
$\bar{F}_q$ is the set of $\alpha \in \bar{R}_q^+$ with $(\alpha', \epsilon_i) \leq 0$ for any $(q', \alpha') \sim (q, \alpha)$ and any vertex i with $q'_i = 1$.

Lemma 7.3. *If there is a simple Λ^q-module of dimension α then either $[q, \alpha]$ is equivalent to a pair $[q', \alpha']$ with α' the coordinatevector of a loopfree vertex, or $\alpha \in \bar{F}_q$.*

Proof. This is obtained by modifying [1, Lemma 7.4]. Also modify [1, Lemma 7.1, 7.3]. Replace [1, Lemma 7.2] by [7, Lemma 5.1]. □

Theorem 7.4. *If $[q, \alpha]$ is a pair with $\alpha \in \bar{F}_q \setminus \bar{\Sigma}_q$, then after first passing to an equivalent pair, and then passing to the support quiver of α and the corresponding restrictions of q and α, one of the following cases holds:*

(I) Q is extended Dynkin with minimal positive imaginary root δ, the dimension vector is $\alpha = hm\delta$ with $m \geq 1$ and $h \geq 2$ and q^δ a primitive mth root of unity.

(II). Q is a disjoint union of two parts connected by one arrow between vertices i, j, with $\alpha_i = \alpha_j = 1$, and if $\alpha = \beta + \gamma$ where β and γ are the restriction of α to each side of the edge joining i and j, then $q^\beta = q^\gamma = 1$.

(III) Q is a disjoint union of two parts connected by one arrow between vertices j, k with $\alpha_j = 1$, the part containing k is extended

Dynkin with k as an extending vertex and minimal imaginary root δ, the restriction of α to this part is $h\delta$ with $h \geq 2$ and $q^{\alpha_\infty} = 1$ and $q^\delta = 1$.

Proof. First modify [1, Lemmas 8.2-8.15] (8.8 and 8.13 need no change).

For the proof we assume that $\alpha \in \bar{F}_q$.

Say that β is a (-1)-vector for the pair (q, α) if $\beta, \alpha - \beta \in \mathbb{N}\bar{R}_q^+$ and $(\beta, \alpha - \beta) = -1$.

Say that β is a divisor for (q, α) if it is a (-1)-vector, $(\beta, \epsilon_i) \leq 0$ for all i, and $(\alpha - \beta, \epsilon_i) \leq 0$ whenever $(\beta, \epsilon_i) = 0$.

Critical vertex.

This is obtained by modifying [1, Theorem 8.1]. This involves additivity arguments with the characteristic of the field, so would need to be modified.

Suppose that $\alpha \in \bar{F}_q \setminus \bar{\Sigma}_q$. Since α is an imaginary root, $q(\alpha) \leq 0$.

Suppose first that $q(\alpha) = 0$. By passing to an equivalent pair we may assume that α is in the fundamental region. Since $q(\alpha) = 0$ this implies that the support of α is extended Dynkin and α is a multiple of the minimal imaginary root δ. Let m be the smallest positive integer with $(q^\delta)^m = 1$. Since $q^\alpha = 1$, it follows that $\alpha = hm\delta$ for some $h \geq 1$. If $h = 1$ then in any proper decomposition of α as a sum of roots $\alpha = \beta + \gamma + \dots$ in $\bar{R}_q^+$, the terms must be real roots, so $p(\alpha) > p(\beta) + p(\gamma) + \dots$. Thus $\alpha \in \bar{\Sigma}_q$, a contradiction. Thus we have case (I).

Thus we may suppose that $q(\alpha) < 0$. We replace $[q, \alpha]$ by an equivalent pair to ensure that α has support as small as possible. Then we pass to the support quiver Q' of α and the restrictions $[q', \alpha']$ of q and α. Clearly $\alpha' \in \bar{F}_{q'} \setminus \bar{\Sigma}_{q'}$. Observe that by minimality, if $[q'', \alpha'']$ is a pair equivalent to $[q', \alpha']$, then α'' has support Q', and the pair $[q'', \alpha'']$ could be obtained from $[q, \alpha]$ by first applying admissible reflections, and then passing to the support quiver of α''.

By the analogue of [1, Lemma 8.3] there is a (-1)-vector β for $[q', \alpha']$, and hence a divisor β' for some equivalent pair $[q'', \alpha'']$ by the analogue of [1, Lemma 8.4]. Now β' and $\alpha'' - \beta'$ cannot both be sincere by the analogue of [1, Lemma 8.11]. Thus either β' is a non-sincere divisor, or we obtain a non-sincere divisor for some pair equivalent to $[q'', \alpha'']$ on applying the analogue of [1, Lemma 8.4] to the (-1)-vector $\alpha'' - \beta'$ for $[q'', \alpha'']$. Thus case (II) or (III) holds by the analogues of [1, Lemmas 8.14 and 8.15]. $\square$

Theorem 7.5. *If there is a simple $\Lambda^q(Q)$-module of dimension α, then $\alpha \in \bar{\Sigma}_q$.*

Proof. In three special cases we know there is no simple module.

(I) Q is extended Dynkin star-shaped with minimal positive imaginary root δ, the dimension vector is $\alpha = hm\delta$ with $m \geq 1$ and $h \geq 2$ and q^δ a (primitive?) mth root of unity.

(II). There are adjacent vertices i, j on an arm with coefficient 1 in α, and if $\alpha = \beta + \gamma$ where β and γ are thes restriction of α to each side of the edge joining i and j, then $q^\beta = q^\gamma = 1$.

(III) Q is obtained from an extended Dynkin quiver star-shaped quiver by adjoining another vertex ∞ to an extending vertex, the dimension vector is $\alpha = \alpha_\infty + h\delta$ with $h \geq 2$ and $q^{\alpha_\infty} = 1$ and $q^\delta = 1$.

Or case (III) for cycle quiver. Convert to problem about deformed preprojective algebra. □

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