Exceptional Sequences of Representations of Quivers

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ABSTRACT. We show that the braid group acts transitively on the set of exceptional sequences of representations of a quiver.

At the 1992 Canadian Mathematical Society Annual Seminar, A. N. Rudakov lectured on exceptional sequences of vector bundles for $\mathbb{P}^2$, and more generally for Del Pezzo surfaces. This led us to consider the corresponding theory for representations of quivers. There is the notion of a complete exceptional sequence of representations of a quiver, and there is an action of the braid group on the set of such sequences. We show that this action is transitive. The proof uses a theorem of A. Schofield.

Exceptional sequences and the action of the braid group were discovered in the Moscow school of vector bundles, see [1,2,3]. The natural setting is in the context of triangulated categories, and this is described in [1].

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Occasionally there is an action of the braid group on the set of exceptional sequences for an abelian category. For example the argument of [2, §3.3] shows that this holds if the spaces \( \text{Ext}^1(X,Y) \) are naturally f.d. vector spaces for \( i \geq 0 \), \( \text{Ext}^1(X,Y) = 0 \) for \( i \geq 3 \), and whenever \( \text{Ext}^2(X,Y) \neq 0 \) and \( \text{Ext}^1(X,X) = \text{Ext}^1(Y,Y) = 0 \) we have \( \dim \text{Ext}^2(X,Y) < \dim \text{Hom}(Y,X) \).

Of course this applies to the category of representations of a quiver, since in this case \( \text{Ext}^1(X,Y) = 0 \) for \( i \geq 2 \). We do not need to quote this fact, however, since for quiver representations, the existence of a braid group action follows quite easily from properties of perpendicular categories.

Let \( k \) be an algebraically closed field, let \( Q \) be a quiver with no oriented cycles, and let \( kQ \) be the path algebra. By an exceptional representation we mean a finite dimensional left \( kQ \)-module \( X \) with \( \text{End}(X) = k \) and \( \text{Ext}(X,X) = 0 \).

By an exceptional sequence \( E = (X_1, \ldots, X_r) \) of length \( r \) we mean a sequence of exceptional representations satisfying \( \text{Hom}(X_j, X_i) = \text{Ext}(X_j, X_i) = 0 \) for \( 1 \leq i < j \leq r \). We say that an exceptional sequence is a complete sequence if it has length equal to the number of vertices of \( Q \).

If \( C \) is a collection of representations, recall that the perpendicular categories are defined by

\[
\L C = \{ M \in kQ\text{-mod} \mid \text{Hom}(M,X) = \text{Ext}(M,X) = 0 \text{ for all } X \in C \}
\]

\[
C^\perp = \{ M \in kQ\text{-mod} \mid \text{Hom}(X,M) = \text{Ext}(X,M) = 0 \text{ for all } X \in C \}.
\]

In particular we can use this notion for an exceptional sequence \( E = (X_1, \ldots, X_r) \). Now \( E^\perp \) and \( E^\perp \) may be calculated by induction on the length of \( E \), and hence one can show that if \( Q \) has \( n \) vertices then \( E^\perp \) and \( E^\perp \) are equivalent to the categories of representations of quivers \( Q(E^\perp) \) and \( Q(E^\perp) \) with \( n-r \) vertices and no oriented cycles, see for example
[4, Theorem 2.3]. Moreover the functors from \( kQ(E^\perp) \)-mod and \( kQ(\perp E) \)-mod to \( kQ \)-mod are exact and induce isomorphisms on both \( \text{Hom} \) and \( \text{Ext} \). Thus we can talk about simple objects of \( E^\perp \), exceptional sequences for \( E^\perp \), complete sequences for \( E^\perp \), etc.

**Lemma 1.** Any exceptional sequence \( (X_1, \ldots, X_a, Z_1, \ldots, Z_c) \) can be enlarged to a complete sequence \( (X_1, \ldots, X_a, Y_1, \ldots, Y_b, Z_1, \ldots, Z_c) \).

**Proof.** It suffices to find a complete sequence \( (Y_1, \ldots, Y_b, Z_1, \ldots, Z_c) \) for \( \perp (X_1, \ldots, X_a) \), so we may assume that \( a=0 \). Next it suffices to find a complete sequence \( (Y_1, \ldots, Y_b) \) for \( (Z_1, \ldots, Z_c)^\perp \) so we may assume that \( c=0 \). Now the indecomposable projective, injective, or simple representations all give complete sequences when suitably ordered.

**Lemma 2.** If \( E=(X_1, \ldots, X_n) \) and \( F=(Y_1, \ldots, Y_n) \) are complete sequences which differ in at most one place, say \( X_j \cong Y_j \) for \( j \neq i \), then also \( X_i \cong Y_i \).

**Proof.** Passing to \( \perp (X_1, \ldots, X_{i-1}) \) we may suppose that \( i=1 \). Now passing to \( (X_1, \ldots, X_{i-1})^{\perp} \) we may suppose that \( Q \) has only one vertex. But then it has only one exceptional representation.

If \( E \) is an exceptional sequence, let \( C(E) \) be the smallest full subcategory of \( kQ-\text{mod} \) which contains \( E \) and is closed under extensions, kernels of epis, and cokernels of monos.
Lemma 3. If $E=(X_1, \ldots, X_n)$ is a complete sequence, then $C(E) = kQ\text{-mod}$.

Proof. By induction on $n$, the number of vertices of $Q$. If $n=0$ there is nothing to prove, so suppose that $n>0$. Let $X=X_n$. Now $X_1, \ldots, X_{n-1}$ is a complete sequence for $X^\perp$, so by the induction we have $C(X_1, \ldots, X_{n-1}) = X^\perp$, and hence $C(E)$ contains $X$ and $X^\perp$.

Suppose that $X$ is non-projective. The Bongartz completion of $X$ is a tilting module $T=X \otimes Y$ with $Y \in X^\perp$. For each projective module $P$ there is an exact sequence $0 \to P \to T' \to T'' \to 0$ with $T', T'' \in \text{add}(T)$, so $C(E)$ contains the projectives. Since any representation has a projective resolution it follows that $C(E)=kQ\text{-mod}$.

Suppose that $X$ is projective, say corresponding to vertex $i$, so that $X^\perp$ consists of the representations which are zero at $i$. Thus $C(E)$ contains the simples $S_j$ corresponding to vertices $j \neq i$. There is also an exact sequence $0 \to \text{rad} X \to X \to S_i \to 0$ with $\text{rad} X \in X^\perp$ so that $S_i \in C(E)$. Thus $C(E)=kQ\text{-mod}$.

Lemma 4. Let $E=(X_1, \ldots, X_r)$ be an exceptional sequence.

1. $C(E) = ^1(E^\perp) = ^1F$ where $F$ is any complete sequence for $E^\perp$.

2. $C(E) = (^1E)^\perp = G^\perp$ where $G$ is any complete sequence for $^1E$.

Proof. We have $C(E) \subseteq ^1(E^\perp) \subseteq ^1F$ since $^1(E^\perp)$ contains $E$ and is closed under extensions, kernels of epis and cokernels of monos. Now $E$ is a complete sequence for $^1F$, so we have $C(E)=^1F$ by Lemma 3. Part (2) is the same.
**Lemma 5.** If $E$ is an exceptional sequence of length $r$ then $C(E)$ is equivalent to the category of representations of a quiver $Q(E)$ with $r$ vertices and no oriented cycles. Moreover the functor $kQ(E)\text{-mod} \rightarrow kQ\text{-mod}$ is exact and induces isomorphisms on both Hom and Ext.

**Lemma 6.** If $(X,Y)$ is an exceptional sequence then there are unique representations $R_Y X$ and $L_X Y$ with the property that $(Y,R_Y X)$ and $(L_X Y,X)$ are exceptional sequences in $C(X,Y)$.

The next result is due to Schofield [5].

**Lemma 7.** If $X$ is exceptional and not simple then there is an exceptional sequence $(X,Y)$ such that $X$ is not a simple object of $C(X,Y)$.

**Proof.** We may suppose that $X$ is sincere – otherwise we can pass to the support of $X$. We work by induction on the number $n$ of vertices of $Q$. Now $n \geq 2$ since $X$ is not simple. If $n=2$ there is an exceptional sequence $(X,Y)$, we have $C(X,Y) = kQ\text{-mod}$ and by assumption $X$ is not simple. If $n > 2$, let $Y$ be a simple object of $\mathcal{X}$. Now $X$ is sincere as an object of $\mathcal{X}^\perp$ by [4, Lemma 4.2], and since $\mathcal{X}^\perp$ is equivalent to the representations of a quiver with $n-1$ vertices, the induction applies.

**Lemma 8.** Let $E = (X_1, \ldots, X_r)$ be an exceptional sequence and let $1 \leq i < r$.

1. $(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, Y, X_{i+2}, \ldots, X_r)$ is an exceptional sequence in $C(E)$ if and only if $Y \cong R_{X_{i+1}} X_i$.

2. $(X_1, X_2, \ldots, X_{i-1}, Z, X_i, X_{i+2}, \ldots, X_r)$ is an exceptional...
sequence in C(E) if and only if \( Z \cong L_{X_{i+1}} X_i \).

Proof. The stated sequences are exceptional since

\[
R_{X_{i+1}} X_i, L_{X_i} X_{i+1} \in C(X_i, X_{i+1}) \cong \langle (X_i, X_{i+1}) \rangle = \langle (X_i, X_{i+1}) \rangle ^\perp, \quad \text{and} \quad X_1, \ldots, X_{i-1} \in C(X_i, X_{i+1}), X_{i+2}, \ldots, X_r \in C(X_i, X_{i+1}) ^\perp. \quad \text{The uniqueness follows from Lemmas 2 and 5.}
\]

Let \( B_r \) be the braid group on \( r \) strings, so with generators \( \sigma_1, \ldots, \sigma_{r-1} \) where \( \sigma_i \) moves the \( i \)-th over the \( (i+1) \)-th string, and with relations \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( i \neq j \pm 1 \) and \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) for \( 1 \leq i \leq r-2 \). Let \( E_r \) be the set of exceptional sequences of length \( r \) (up to isomorphism).

**Lemma 9.** The assignments

\[
\sigma_i (X_1, \ldots, X_r) = (X_1, X_2, \ldots, X_{i-1}, X_i, X_{i+1}, X_{i+2}, \ldots, X_r)
\]

\[
\sigma_i^{-1} (X_1, \ldots, X_r) = (X_1, X_2, \ldots, X_{i-1}, L_{X_i} X_{i+1}, X_i, X_{i+2}, \ldots, X_r)
\]

define an action of \( B_r \) on \( E_r \).

Proof. We have

\[
\sigma_i^{-1} \sigma_i (X_1, \ldots, X_r) = (X_1, X_2, \ldots, X_{i-1}, Y, X_{i+1}, X_{i+2}, \ldots, X_r)
\]

for some \( Y \in C(X_1, \ldots, X_r) \), so \( Y \cong X_i \) by uniqueness. Thus the stated actions are inverse. To verify the relation

\( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \)

note that \( \sigma_i \sigma_{i+1} \sigma_i (X_1, \ldots, X_r) \) and \( \sigma_{i+1} \sigma_i \sigma_{i+1} (X_1, \ldots, X_r) \) both have the form

\[
(X_1, \ldots, X_{i-1}, X_{i+2}, R_{X_{i+1}} X_i, Y, X_{i+3}, \ldots, X_r)
\]

with \( Y \in C(X_1, \ldots, X_r) \), so they must be equal.
Theorem. If $Q$ has $n$ vertices then the action of $B_n$ on the set $E_n$ of complete sequences is transitive.

Proof. We prove this by induction on $n$. If $n=1$ there is nothing to prove, while if $n=2$ it can be checked since every exceptional representation is preprojective or preinjective. Thus suppose $n>2$. Let $O$ be an orbit for the action of $B_n$.

Let $d$ be the minimum dimension of any representation in any complete sequence in $O$. We show that $d=1$, so for a contradiction suppose otherwise. Let $E$ be a complete sequence in $O$ containing a representation $X$ of dimension $d$. Applying $s_1 s_2 \ldots s_{n-1}$ if necessary we may assume that $E=(X, X_2, \ldots, X_n)$. Let $(X, Y)$ be the exceptional sequence given by Lemma 7. It extends to a complete sequence $F=(X, Y, Y_3, \ldots, Y_n)$. Now $(X_2, \ldots, X_n)$ and $(Y, Y_3, \ldots, Y_n)$ are complete sequences for $\underline{1}X$, so by the induction they are in the same orbit under $B_{n-1}$, and hence $E$ and $F$ are in the same orbit under $B_n$. Now $C(X, Y)$ contains a complete sequence $(S, T)$ with $S$ and $T$ the simple objects of $C(X, Y)$, and $s_1^k F=(S, T, Y_3, \ldots, Y_n)$ for some $k \in \mathbb{Z}$. By assumption $X$ is not simple as an object of $C(X, Y)$, so it involves both $S$ and $T$, and hence $\dim S < \dim X = d$. But $s_1^k F \in O$, contradicting the minimality of $d$.

We have shown that $O$ contains a sequence $E$ which involves a simple $S$, and indeed we may assume that $E=(S, Z_2, \ldots, Z_n)$. Let $P=(P_1, \ldots, P_n)$ be the complete sequence of projectives, with $P_j$ being the projective cover of $S$. As above, passing to $\underline{1}S$ and using the induction, we see that $O$ contains the sequence $F=(S, P_1, \ldots, P_{j-1}, P_{j+1}, \ldots, P_n)$. Now $s_{j-1} \ldots s_2 s_1 F = P$ since the two sides differ in at most one place. Thus $P \in O$, and it follows that the action of $B_n$
on $E_n$ is transitive.

**Corollary.** Exceptional sequences $E,F \in E_r$ are in the same orbit under $B_r$ if and only if $C(E) = C(F)$.

Finally note that if $(X,Y)$ is an exceptional sequence,

$$\dim R_Y X = \pm (\dim X - \langle \dim X, \dim Y \rangle \dim Y)$$

$$\dim L_X Y = \pm (\dim Y - \langle \dim X, \dim Y \rangle \dim X),$$

so the theorem gives a convenient method for producing the real Schur roots for $Q$.

**References**


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