GEOMETRY OF REPRESENTATIONS OF ALGEBRAS

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Contents

These lecture notes are about the variety Mod(A,r) of r-dimensional modules for an associative algebra A, and to a lesser extent about the variety Alg(n) of n-dimensional associative algebras. My aim was to cover a number of different topics, showing how these varieties have been used to study algebras and their modules. I place special emphasis on representations of quivers, that is, modules for path algebras.

I begin with the notion of a variety, quickly going through the definitions, and illustrating them with examples from representations of algebras. Among the results that I cover from algebraic geometry are the fact that Grassmannians are projective varieties, and Chevalley's theorems about semicontinuous functions and constructible sets.

My first topic concerns degenerations of modules. I prove some necessary and sufficient conditions for the existence of a degeneration between two modules, and then prove a beautiful result of Bongartz describing the degenerations for directed algebras.

The second topic is Geiß's theorem that degenerations of algebras of wild representation type are wild. Actually, this theorem is trivial, but it was not spotted for a long time, and the assertion was not expected, so I still think it is an important contribution.

My third topic is Kac's theorem on the dimension vectors of indecomposable representations of quivers. This theorem is now quite old (published in 1980), but I was keen to work through the proof. In these notes I go through the geometry part quite carefully, but I only sketch the part which involves reducing to finite fields.

I did not have time for the final topic, general representations of quivers, but have included a section in these notes which mentions some of the results, and also some of the open problems.

Throughout these notes the setting is as follows.

- K is an algebraically closed field of arbitrary characteristic.
- A is an associative K-algebra with 1, finitely generated as a K-algebra (and often finite dimensional).
- All modules are finite dimensional left modules.

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§1. Varieties

In this section we recall the definition of a variety, and give two examples arising from representations of algebras. The main example, Mod(A,r), is deferred until the next section.

DEFINITIONS.

- $\mathbb{A}^n = K^n$ with its Zariski topology, so closed sets are defined by the vanishing of collections of polynomials in $K[X_1,..,X_n]$.
- $X \subseteq \mathbb{A}^n$ is <u>locally closed</u> if it is open in its closure, or equivalently if it is the intersection of an open and a closed set.
- The set of <u>regular maps</u> on a locally closed subset $X \subseteq \mathbb{A}^n$ is

$$\mathcal{O}(X) = \left\{ \theta : X \longrightarrow K \;\middle|\; \begin{array}{l} \text{Each } x \in X \text{ has nhd } U \text{ in } \mathbb{A}^n \text{ with } \theta|_{U \cap X} = f/g \\ f,g \in K[X_1,..,X_n],\ g \text{ nonvanishing on } U \end{array} \right\}$$

- $\mathbb{P}^n = \mathbb{P}(K^{n+1})$ = 1-d subspaces of K^{n+1}. The closed subsets of $\mathbb{P}^n$ are defined by the vanishing of collections of homogeneous polynomials in $K[X_0,..,X_n]$.
- The set of regular maps on a locally closed subset $X \subseteq \mathbb{P}^n$ is

$$\mathcal{O}(X) = \left\{ \theta : X \longrightarrow K \;\middle|\; \begin{array}{l} \text{Each } x \in X \text{ has nhd } U \text{ in } \mathbb{P}^n \text{ with } \theta|_{U \cap X} = f/g \\ f,g \in K[X_0,..,X_n],\ f,g \text{ homog, same deg, } g \neq 0 \text{ on } U \end{array} \right\}$$

- A (<u>quasiprojective</u>) <u>variety</u> is a locally closed subset X of $\mathbb{A}^n$ or $\mathbb{P}^n$, with its topology and knowledge of $\mathcal{O}(U)$ for all U open in X.
- A <u>morphism</u> $\phi : X \longrightarrow Y$ is a continuous map such that for all open $U \subseteq Y$ and all regular $\theta : U \longrightarrow K$ the composition $\phi^{-1}(U) \xrightarrow{\phi} U \xrightarrow{\theta} K$ is regular.

- An <u>affine variety</u> is one isomorphic to a closed subset of $\mathbb{A}^n$ (an isomorphism is a morphism with an inverse, which is not the same as a bijective morphism).
- A <u>projective variety</u> is one isomorphic to a closed subset of $\mathbb{P}^n$.

- A topological space X is <u>irreducible</u> if $X \neq \varnothing$ and $X = Y \cup Z$ with Y and Z closed $\Rightarrow$ Y=X or Z=X. Equivalently any non-empty open subset is dense. Any variety has a decomposition into irreducible components (maximal irreducible closed subsets). For some people irreducibility is included in the definition of a variety, but that is not convenient for us.

- $X \times Y$ has the structure of variety, but this is NOT with the product topology. Instead $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$. A product of irreducible varieties is irreducible.

ALGEBRAS

- $\mathrm{Bil}(n) = \{\text{bilinear maps } m: K^n \times K^n \longrightarrow K^n\} \cong \mathbb{A}^{n^3}$.
- $\mathrm{Ass}(n) = \{\text{associative bilinear } m\}$ is a closed subset of $\mathrm{Bil}(n)$, so it is an affine variety.
- $\mathrm{Alg}(n) = \{\text{associative bilinear } m \text{ which have a } 1\}$.

THEOREM.
1. $\mathrm{Alg}(n)$ is an open subset of $\mathrm{Ass}(n)$.
2. The map $\mathrm{Alg}(n) \longrightarrow K^n$, $m \longmapsto$ the 1 for m, is a regular map.
3. $\mathrm{Alg}(n)$ is an affine variety.

PROOF. Let A be a f.d. associative algebra, not necessarily with 1. Let $l_a, r_a : A \longrightarrow A$ be left and right multiplication by $a \in A$.

Exercise: A has a 1 $\Leftrightarrow$ there is some $a \in A$ with l_a and r_a invertible, and in this case the 1 is $l_a^{-1}(a)$.

(1) The set $D_a = \{m \in \mathrm{Ass}(n) \mid \det(l_a^m)\det(r_a^m) \neq 0\}$ is open in $\mathrm{Ass}(n)$, and $\mathrm{Alg}(n) = \bigcup_a D_a$ by the exercise.

(2) On D_a the map is equal to $m \longmapsto [l_a^m]^{-1}(a)$ which is a quotient of polynomial functions on $\mathrm{Bil}(n)$. The denominator is $\det(l_a^m)$ which is nonvanishing on D_a.

(3) Because of (2) there are maps both ways showing that

$$\mathrm{Alg}(n) \cong \{(m,a) \in \mathrm{Ass}(n) \times K^n \mid a \text{ is } 1 \text{ for } m\},$$

and the RHS is a closed subset of $\mathrm{Ass}(n)\times K^n$, so is affine.

REMARKS.

1. $GL(n)$ acts on $\mathrm{Alg}(n)$ by conjugation, and the orbits are the isomorphism classes of algebras.
2. The structure of $\mathrm{Alg}(n)$ is known for small n. For example $\mathrm{Alg}(4)$ has 5 irreducible components, of dimensions 15, 13, 12, 12, 9. See Gabriel's article in SLN 488.

SUBMODULES

If M is an A-module, then

$$\mathrm{Gr}_A\binom{M}{n} = \{n\text{-dimensional submodules of } M\}.$$

In case A = K we write just $\mathrm{Gr}\binom{M}{n}$. This is the usual Grassmannian of n-dimensional subspaces of a vector space M.

THEOREM. The Plücker map $\mathrm{Gr}\binom{M}{n}\longrightarrow\mathbb{P}(\Lambda^n M)$ sending a subspace U to $\Lambda^n U$ is 1-1, and has closed image, so that $\mathrm{Gr}\binom{M}{n}$ is a projective variety.

LEMMA. If $0\neq x\in\Lambda^n M$ then $x^\perp := \{y\in M \mid x\wedge y=0\}$ has dimension $\leq n$, and if it has dimension n, then $x\in\Lambda^n(x^\perp)$.

PROOF. Let $x^\perp$ have basis $e_1,..,e_r$, and extend it to a basis $e_1,..,e_m$ of M. Write

$$x = \textstyle\sum_{i_1<\dots<i_n} x_{i_1\dots i_n}\, e_{i_1}\wedge\dots\wedge e_{i_n}.$$

Now

$$e_{i_1}\wedge\dots\wedge e_{i_n}\wedge e_k = \begin{cases} \pm \text{ basis element of } \Lambda^{n+1}M & (\text{all } i_j\neq k) \\ 0 & (\text{else}) \end{cases}$$

and you get distinct basis elements of $\Lambda^{n+1}M$ in this way, so the condition $x\wedge e_k=0$ for $k\leq r$ implies that the nonzero coefficients $x_{i_1..i_n}$ must have some $i_j=k$. Thus the nonzero $x_{i_1..i_n}$ involve all of $1,..,r$, so $r\leq n$. Moreover, if $r=n$ then $x = x_{12..r}\, e_1\wedge\dots\wedge e_n \in \Lambda^n(x^\perp)$.

PROOF OF THE THEOREM.

The Plücker map is 1-1 since if $U \subseteq M$ has dimension n and $0 \neq x \in \Lambda^n U$ then $U = x^\perp$. Namely, $\dim x^\perp \leq n$ by the lemma, but $U \subseteq x^\perp$ since $\Lambda^{n+1} U = 0$.

By the lemma the image of the Plücker map is

$$\{\langle x\rangle \in \mathbb{P}(\Lambda^n M) \mid \dim x^\perp = n\} = \{\langle x\rangle \in \mathbb{P}(\Lambda^n M) \mid \operatorname{rank}(x\wedge - : M \longrightarrow \Lambda^{n+1} M) \leq m-n\}.$$

This is closed, since the condition that a matrix has rank $\leq r$ is equivalent to the vanishing of all $(r+1)\times(r+1)$ minors, and each minor is a homogeneous polynomial in the entries of the matrix).

COROLLARY. $\mathrm{Gr}_A\binom{M}{n}$ is a projective variety.

PROOF. If multiplication by $a \in A$ induces an isomorphism on M then it induces a morphism $a' : \mathrm{Gr}\binom{M}{n} \longrightarrow \mathrm{Gr}\binom{M}{n}$. Now

$$\mathrm{Gr}_A\binom{M}{n} = \{U \in \mathrm{Gr}\binom{M}{n} \mid a'(U) = U \ \forall \text{ a induces an isomorphism on } M\},$$

so it is closed. (To show that $a(U) \subseteq U$ it suffices to show that $(a-\lambda 1)(U) \subseteq U$ for some $\lambda \in K$, and for general λ the element $a-\lambda 1$ induces an isomorphism on M.)

SCHEMES

More general than a variety is a K-scheme. I don't want to define what a scheme is, but only make some observations. For an introduction to schemes which explains the functor of points, see D. Eisenbud and J. Harris, "Why schemes".

- A scheme can be described by its functor of points, a functor (commutative K-algebras)$\longrightarrow$Sets.
- Affine schemes are those which are representable, so isomorphic to a functor $\mathrm{Hom}_{K\text{-alg}}(R,-)$.
- There is the notion of an algebraic scheme. In the affine case we want R to be a f.g. algebra over K.
- There is the notion of a reduced scheme. In the affine case we want R to have no non-zero nilpotent elements.

FACT. Any algebraic scheme $\underline{X}$ gives a variety $\underline{X}(K)$. This defines a 1-1 correspondence between reduced algebraic schemes and varieties.

EXAMPLES.

- $\underline{GL}(n)(R) = GL(n,R)$ is an affine algebraic reduced scheme.
- $\underline{alg}(n)(R) =$ associative R-algebra structures on R^n with 1. This is an affine, algebraic, scheme, in general non-reduced.
- $\underline{Gr}\binom{M}{n}(R) =$ R-module summands of $M \otimes_K R$ of rank n. This is a projective, algebraic reduced scheme.

§2. Varieties of modules

In this section we define the variety of modules, and give some examples.

DEFINITION. Let A be a f.g. associative K-algebra with 1. If $r\in\mathbb{N}$ then

$$\begin{aligned}\mathrm{Mod}(A,r) &= \{\text{left } A\text{-module structures on } K^r\} \\ &= \{K\text{-algebra maps } A\longrightarrow M_r(K)\}.\end{aligned}$$

GENERALIZATION. Fix a complete set $(e_1,..,e_n)$ of orthogonal idempotents in A (not necessarily primitive).

- Thus $e_i e_j=\delta_{ij}e_i$ and $\sum e_i=1$.
- If M is any A-module then $M=\oplus_{i=1}^n e_i M$.
- The <u>dimension</u> <u>vector</u> of M is the vector $\alpha\in\mathbb{N}^n$ with $\alpha_i=\dim e_i M$.
- For $\alpha\in\mathbb{N}^n$ set

$$\begin{aligned}\mathrm{Mod}(A,\alpha) &= \left\{\begin{matrix}\text{left } A\text{-module structures on } K^{\alpha 1}\oplus\ldots\oplus K^{\alpha n} \text{ with} \\ e_i \text{ acting as projection onto i-th factor}\end{matrix}\right\} \\ &= \left\{\begin{matrix}K\text{-algebra maps } A\longrightarrow M_r(K) \text{ sending} \\ e_i \text{ to the projection matrix}\end{matrix}\right\} \quad (\text{where } r=\textstyle\sum \alpha_i).\end{aligned}$$

- Note that $\mathrm{Mod}(A,\alpha)$ depends on the set of idempotents $(e_1,..,e_n)$.

LEMMA. $\mathrm{Mod}(A,\alpha)$ is naturally an affine variety.

PROOF. Fix a surjective homomorphism $\theta:K\langle X_1,..,X_N\rangle\longrightarrow A$ with kernel I. Here $K\langle X_1,...,X_N\rangle$ is the free associative algebra, so each $p\in K\langle X_1,..,X_N\rangle$ is a non-commutative polynomial in $X_1,..,X_N$. Thus we can evaluate p on an N-tuple of square matrices to get square matrix.

Choose q_i with $e_i = \theta(q_i)$. Let $r = \sum \alpha_i$. Then

$$\mathrm{Mod}(A,\alpha) = \left\{(M_1,..,M_N)\in M_r(K)^N \;\middle|\; \begin{matrix} p(M_1,..,M_N)=0 \ \forall\ p\in I \text{ and} \\ q_i(M_1,..,M_N) = \text{proj. matrix}\end{matrix}\right\}$$

This is a closed subset of $M_r(K)^N$ so an affine variety. We leave it as an exercise to show that you get an isomorphic variety if you choose a different map θ, so is natural.

REMARKS

1. If A is f.d. then the inclusion

   $$\mathrm{Mod}(A,r) = \{\theta:A\otimes K^r \longrightarrow K^r \mid \theta \text{ is an action of } A\} \subseteq \mathrm{Hom}_K(A\otimes_K K^r, K^r)$$

   endows Mod(A,r) with the same structure as an affine variety.
2. $\underline{\mathrm{Mod}}(A,r)(R)$ = $A\otimes_K R$-mod structures on R^r = K-algebra maps $A \longrightarrow M_r(R)$. This is an affine scheme. In interesting cases $\underline{\mathrm{Mod}}(A,r)$ will be reduced, or we can ask questions which don't depend on its being reduced. Because of this we only use Mod(A,r).

DEFINITIONS.

- $x\in\mathrm{Mod}(A,\alpha)$ gives an A-module with dim vector α which we denote K_x. Each A-module M with dimension vector α is isomorphic to some K_x.

- If α,β are dimension vectors we define

  $$\mathrm{Hom}(\alpha,\beta) = \left\{\begin{matrix}\text{linear maps } K^{\alpha 1}\oplus..\oplus K^{\alpha n} \longrightarrow K^{\beta 1}\oplus..\oplus K^{\beta n} \\ \text{sending each } K^{\alpha i} \text{ into } K^{\beta i}\end{matrix}\right\} \cong \prod_i \mathrm{Hom}(K^{\alpha i},K^{\beta i}).$$

  If $x\in\mathrm{Mod}(A,\alpha)$ and $y\in\mathrm{Mod}(A,\beta)$ then $\mathrm{Hom}(K_x,K_y) \subseteq \mathrm{Hom}(\alpha,\beta)$.

- We define $\mathrm{End}(\alpha) = \mathrm{Hom}(\alpha,\alpha)$ and $\mathrm{GL}(\alpha) = \mathrm{Aut}(\alpha) = \prod_i \mathrm{GL}(\alpha_i)$.

- $\mathrm{GL}(\alpha)$ acts on $\mathrm{Mod}(A,\alpha)$ by conjugation. If $g\in\mathrm{GL}(\alpha)$ then g can be considered as a block-diagonal element of GL(r) $(r=\sum\alpha_i)$, and the action is $g(M_1,..,M_N)=(gM_1g^{-1},..,gM_Ng^{-1})$ for $(M_1,..,M_N) \in \mathrm{Mod}(A,\alpha)$.

- We have $K_x \cong K_y \Leftrightarrow$ x and y are in the same orbit under $\mathrm{GL}(\alpha)$. We denote by $\mathcal{O}_M$ the orbit of modules isomorphic to M.

EXERCISE. Show that $\mathrm{Stab}_{\mathrm{GL}(\alpha)}(x) \cong \mathrm{Aut}_A(K_x)$.

EXAMPLES

1. If A is commutative then $\underline{\mathrm{Mod}}(A,1)$ is the affine scheme defined by A, and Mod(A,1) is the affine variety with with regular functions

A/(nilpotents).

2. Commuting matrices. $\mathrm{Mod}(K[X,Y],r) = \{(M,N) \mid M,N\in M_r(K) \text{ and } MN=NM\}$. This is irreducible by M. Gerstenhaber, On dominance and varieties of commuting matrices, Ann. Math. 73 (1961), 324-348.

3. Matrices. $\mathrm{Mod}(M_n(K),n) = \{K\text{-algebra maps } M_n(K) \longrightarrow M_n(K)\}$

$= \mathrm{Aut}(M_n(K))$ since $M_n(K)$ is a simple algebra

$= PGL_n(K)$

since all automorphisms of $M_n(K)$ are inner, for example by the Skolem-Noether Theorem.

4. Quivers. A quiver Q is a finite directed graph (maybe with loops, cycles and multiple arrows). It has vertex set $Q_0=\{1,..,n\}$, and arrow set Q_1. Each arrow has head at the vertex h(a) and tail at t(a). We draw

$$h(a) \bullet \xleftarrow{a} \bullet\ t(a).$$

- A non-trivial path is a sequence $a_m \ldots a_1$ with $h(a_i)=t(a_{i+1})$. Pictorially

$$\bullet \xleftarrow{a_m} \bullet \longleftarrow \ldots \xleftarrow{a_1} \bullet.$$

- There is a trivial path e_i for each vertex i. • The path algebra KQ has basis the paths, and multiplication given by the composition of paths, or zero if they are incompatible. It is a f.g. associative algebra.
- $(e_1,..,e_n)$ are a complete set of orthogonal idempotents. We always use this set of idempotents when we consider $\mathrm{Mod}(KQ,\alpha)$.
- KQ-modules correspond to representations of Q, which are specified by giving a vector space X_i for each vertex i and a linear map $X_a:X_i \longrightarrow X_j$ for each arrow $a:i\longrightarrow j$.
- The dimension vector of a representation X is the vector α with $\alpha_i = \dim X_i$.
- Because of the correspondence above we have

$$\mathrm{Mod}(KQ,\alpha) = \prod_{\text{arrows } i\longrightarrow j} \mathrm{Hom}(K^{\alpha i},K^{\alpha j}).$$

- If $x\in\mathrm{Mod}(KQ,\alpha)$ and $y\in\mathrm{Mod}(KQ,\beta)$ then

$$\mathrm{Hom}_{KQ}(K_x,K_y) = \{(\phi_i)\in\mathrm{Hom}(\alpha,\beta) \mid y_a\phi_i = \phi_j x_a \text{ for all } a:i\longrightarrow j\}.$$

5. Determinental varieties and complexes.

- Let Q be the quiver

$$1 \bullet \xrightarrow{a} \bullet\ 2$$

so that $\mathrm{Mod}(KQ,\alpha) = \mathrm{Hom}(K^{\alpha 1},K^{\alpha 2})$. A representation X of Q is determined up to isomorphism by $\underline{\dim}$ X and rank X_a, so the orbits in $\mathrm{Mod}(KQ,\alpha)$ are $\mathcal{O}_r=\{x\in\mathrm{Hom}(K^{\alpha 1},K^{\alpha 2})|\mathrm{rank}\ x=r\}$ with $r\le\min\{\alpha_1,\alpha_2\}$. The r^{th} determinental variety is $\overline{\mathcal{O}_r}=\{x\in\mathrm{Hom}(K^{\alpha 1},K^{\alpha 2})|\mathrm{rank}\ x\le r\}$.

- More generally let Q be the quiver

$$\overset{1}{\bullet} \xrightarrow{a_1} \cdots \longrightarrow \overset{m}{\bullet} \xrightarrow{a_m} \overset{m+1}{\bullet},$$

and let $I = \langle a_{i+1}a_i\rangle \le KQ$. Then

$$\mathrm{Mod}(KQ/I,\alpha) = \{x \in \textstyle\prod_{i=1}^{m} \mathrm{Hom}(K^{\alpha i},K^{\alpha i+1}) \mid x_{i+1}x_i=0 \text{ for } 1\le i<m\}.$$

- The Buchsbaum-Eisenbud variety of complexes is

$$W(r_1,..,r_m) = \{x\in\mathrm{Mod}(KQ/I,\alpha)|\mathrm{rank}\ x_i\le r_i\}.$$

If $r_{i-1}+r_i\le\alpha_i$ this variety is the closure of an orbit, and in this case it is a normal, Cohen-Macaulay variety. See papers of Kempf and of De Concini and Strickland.

- Remark: knowing that closures of orbits are normal is important. For example, for Schubert varieties this leads to the Demazure character formula.

6. Preprojective algebras.

- Let Q be a quiver without loops. Let Q′ the quiver obtained by adding a reverse arrow $a^*:j\longrightarrow i$ for each arrow $a:i\longrightarrow j$, and let

$$A = KQ' / \left(\textstyle\sum_{a\in Q} [a,a^*] \right).$$

- The relevant variety is

$$\Lambda_\alpha = \left\{x \in \mathrm{Mod}(A,\alpha) \;\middle|\; \begin{array}{l}\text{Each non-trivial path in}\\ KQ' \text{ acts nilpotently on } K_x\end{array}\right\}$$

(The condition is automatic if Q is Dynkin)

- Each irreducible component of Λ_α has dimension 1/2 dim $\mathrm{Mod}(KQ',\alpha)$. See Lusztig, J. Amer. Math. Soc. 4 (1991). This paper uses perverse sheaves on Mod(KQ,α) to study canonical basis of quantized enveloping algebras.

- If Q is Euclidean then there is a corresponding Dynkin diagram, and a corresponding finite subgroup G of SU(2). In work of Kronheimer the variety Mod(A,α) is related to the Kleinian singularity $\mathbb{C}[X,Y]^G$. An algebraic explanation seems to be that the skew group algebra $\mathbb{C}[X,Y]*G$ is Morita equivalent to A.

§3. Chevalley's Theorems and applications

In this section we derive Chevalley's Theorems from the simplest version, and give some consequences.

DEFINITION.

- The <u>dimension</u> dim X of a topological space X, is the largest n such that there is a chain $X_0 \subset X_1 \subset \ldots \subset X_n$ of distinct non-empty irreducible closed subsets of X. (dim $\emptyset = -\infty$).
- Observe that if $X \subseteq Y$ then dim X $\leq$ dim Y. This is strict if Y is irreducible and X is closed.
- The <u>local dimension</u> at $x \in X$ is $\dim_x X = \min\{\dim U | U \text{ nhd of } x\}$.

FACTS from commutative algebra.

1. dim $\mathbb{A}^n$ = dim $\mathbb{P}^n$ = n (so varieties have dimension).
2. If $U \neq \emptyset$ is open in an irreducible variety X then dim U = dim X.
3. If X,Y are irreducible varieties then dim $X \times Y$ = dim X + dim Y.

CONSEQUENCES.

- If X_i are locally closed in Y then $\dim \bigcup_{i=1}^n X_i = \max\{\dim X_i\}$.
- $\dim_x X = \max\{\dim Z | Z \text{ is an irreducible cpt of } X \text{ containing } x\}$.

The next result also follows from commutative algebra. For a proof, see Mumford's Red book. We spend the rest of this section deriving corollaries.

MAIN LEMMA. If $\pi: X \longrightarrow Y$ is a dominant morphism of irreducible varieties, ie $\overline{\pi(X)}=Y$, then any irreducible component of a fibre $\pi^{-1}(y)$ has dimension at least dim X - dim Y. Moreover there is an open $\emptyset \neq U \subseteq Y$ with $\dim \pi^{-1}(u) = \dim X - \dim Y$ for all $u \in U$.

DEFINITION. A subset of a variety is <u>constructible</u> if it is a finite union of locally closed subsets. Constructibility is closed under finite unions and intersections, under complements, and under inverse images. An example of a constructible set which is not locally closed is $\mathbb{A}^2 \backslash \{x\text{-axis}\} \cup \{\text{origin}\} = \{(x,y) | x=yz \text{ for some } z\}$.

THEOREM 1. If $\pi:X\longrightarrow Y$ is a morphism of varieties then $\pi(X)$ is constructible. More generally π sends constructible sets to constructible sets.

SKETCH.

- Work by induction on dim X.
- We may assume X is irreducible.
- We may assume that $Y = \overline{\pi(X)}$ so Y irreducible and π is dominant.
- By the main lemma, $\pi(X)$ contains a non-empty open subset U of Y.
- Now $\pi(X) = U \cup \pi(X\backslash\pi^{-1}U)$ and $\pi(X\backslash\pi^{-1}U)$ is constructible since $\dim (X\backslash\pi^{-1}U) < \dim X$.

EXAMPLE. $\mathrm{Ind}(A,\alpha) = \{x\in\mathrm{Mod}(A,\alpha) | K_x \text{ indecomposable}\}$ is constructible, since its complement is $\bigcup_{\alpha=\beta+\gamma,\beta,\gamma\neq 0} \mathrm{Im}(\phi_{\beta,\gamma})$ where

$$\phi_{\beta,\gamma}:GL(\alpha)\times\mathrm{Mod}(A,\beta)\times\mathrm{Mod}(A,\gamma)\longrightarrow\mathrm{Mod}(A,\alpha),\ (g,x,y) \longmapsto g(x\oplus y).$$

UPPER SEMICONTINUOUS FUNCTIONS

$f:X\longrightarrow\mathbb{Z}$ is <u>upper</u> <u>semicontinuous</u> if $\{x\in X | f(x)\geq n\}$ is closed for all $n\in\mathbb{Z}$.

THEOREM 2. If $\pi:X\longrightarrow Y$ is a morphism of varieties then the function $x \longmapsto \dim_x \pi^{-1}(\pi(x))$ is upper semicontinuous.

SKETCH. Let $Z(\pi,n) = \{x | \dim_x\pi^{-1}\pi(x)\geq n\}$.

- We prove $Z(\pi,n)$ is closed by induction on dim X.
- We may assume X is irreducible, for if $X=\bigcup X_i$ is the decomposition into irreducible components, then $Z(\pi,n) = \bigcup Z(\pi|_{X_i},n)$.
- We may assume that $Y = \overline{\pi(X)}$ so Y is irreducible and π is dominant.
- If $n \leq \dim X-\dim Y$ then $Z(\pi,n) = X$ by the main lemma, so it is closed.
- If $n > \dim X-\dim Y$ then $Z(\pi,n) = Z(\pi|_{X\backslash\pi^{-1}(U)},n)$. Now $Z(\pi|_{X\backslash\pi^{-1}(U)},n)$ is closed in $X\backslash\pi^{-1}(U)$ by induction and $X\backslash\pi^{-1}(U)$ is closed in X.

SPECIAL CASE. Suppose X is a variety, V vector space, and we are given subsets $V_x \subseteq V$ for all $x \in X$. Suppose that

- each V_x is a cone in V, that is, it contains 0, and is closed under scalar multiplication.
- $\{(x,v) | v \in V_x\}$ is locally closed in $X \times V$.

Then the map $x \longmapsto \dim V_x$ is upper semicontinuous.

PROOF. Use the morphism $\{(x,v) | v \in V_x\} \longrightarrow X$. The fibre over x is V_x. Also, since V_x is a cone, every irreducible component of V_x contains 0, so $\dim_0 V_x = \dim V_x$.

APPLICATIONS.

1. The map $\mathrm{Mod}(A,\alpha) \times \mathrm{Mod}(A,\beta) \longrightarrow \mathbb{N}$, $(x,y) \longmapsto \dim \mathrm{Hom}_A(K_x,K_y)$ is upper semicontinuous. It suffices to observe that

$$\{(x,y,\theta) | \theta \in \mathrm{Hom}_A(K_x,K_y)\} \subseteq \mathrm{Mod}(A,\alpha) \times \mathrm{Mod}(A,\beta) \times \mathrm{Hom}(\alpha,\beta)\}$$

   is closed.

2. Thus also $\mathrm{Mod}(A,\alpha) \longrightarrow \mathbb{N}$, $x \longmapsto \dim \mathrm{End}_A(K_x)$ upper semicontinuous.

3. Let us say that $\theta \in \mathrm{End}(W)$ is equipotent if all eigenvalues of θ are equal. This is a closed condition, for if

$$\det(t1-\theta) = t^n + nc_1 t^{n-1} + \binom{n}{2} c_2 t^{n-2} + \ldots,$$

   then θ is equipotent $\Leftrightarrow$ $c_r = c_1^r$ for all r.

- $\mathrm{Equi}(K_x) = \{$equipotent endomorphisms of $K_x\}$ is a cone, so the function $\mathrm{Mod}(A,\alpha) \longrightarrow \mathbb{N}$, $x \longmapsto \dim \mathrm{Equi}(K_x)$ is upper semicontinuous.
- This gives another proof that $\mathrm{Ind}(A,\alpha)$ is constructible, for

$$\mathrm{Ind}(A,\alpha) = \{x | \mathrm{End}(K_x) = \mathrm{Equi}(K_x)\} = \bigcup_r \{x | \dim \mathrm{End}(K_x) \le r, \dim \mathrm{Equi}(K_x) \ge r\}$$

   and each term in the union is locally closed.

GROUP ACTIONS

- Let G be an algebraic group acting on a variety X.
- For simplicity we suppose G is an irreducible variety (one usually says that G is a "connected" algebraic group.)

LEMMA.

- Each orbit Gx is locally closed and irreducible.
- $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$.
- $\overline{Gx} \setminus Gx$ is a union of orbits of dimension $< \dim Gx$.

PROOF.

- Gx is the image of the map $G \longrightarrow X$, $g \longmapsto gx$, so $\overline{Gx}$ is irreducible and Gx is constructible. It follows that there is $\emptyset \neq U \subseteq Gx$, U open in $\overline{Gx}$.
- Now $GU = \bigcup_{g\in G} gU$ is contained in Gx and G-stable, so equals Gx. Each gU is open in $\overline{Gx}$, so GU is open in $\overline{Gx}$. Thus Gx is locally closed.
- Now, since G is irreducible, so is Gx.
- The fibres of $G \longrightarrow Gx$ are cosets of $\mathrm{Stab}(x)$, so all have the same dimension. By the main lemma, $\dim \mathrm{Stab}(x) = \dim G - \dim Gx$.
- The last statement is clear.

LEMMA. The map $x \longmapsto \dim \mathrm{Stab}(x)$ is upper semicontinuous. Therefore,

- the set $X_{(\le s)} = \{x\in X \mid \dim Gx \le s\}$ is closed, and
- the set $X_{(s)} = \{x\in X \mid \dim Gx = s\}$ is locally closed.

PROOF. Let $Z = \{(g,x)\in G\times X \mid gx=x\}$ and let $\pi : Z \longrightarrow X$ be the projection. Now

$$\dim_{(1,x)} \pi^{-1}\pi(1,x) = \dim_1 \mathrm{Stab}(x) = \dim \mathrm{Stab}(x)$$

since for a group each point looks the same.

§4. Degenerations of modules

We prove some general results about degenerations of modules. Then we study K[X] and directing algebras.

Recall that $\mathcal{O}_M$ denotes the orbit in Mod(A,α) of points x with $K_x \cong M$. We have $\dim GL(\alpha) - \dim \mathcal{O}_M = \dim Stab(x) = \dim Aut(M) = \dim End(M)$

DEFINITION. M <u>degenerates</u> to N if $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$. This is a partial order, for if M degenerates to N and $M \not\cong N$ then $\dim \mathcal{O}_N < \dim \mathcal{O}_M$ by the lemma about group actions.

LEMMA. If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact then M degenerates to $L \oplus N$.

PROOF. For simplicity we do the case of Mod(A,r). An element $x \in Mod(A,r)$ is defined by matrices $x_a \in M_r(K)$ where a runs through a set of generators of A. Now there is $x \in \mathcal{O}_M$ in which each matrix x_a has the form $\begin{pmatrix} y_a & w_a \\ 0 & z_a \end{pmatrix}$ with $K_y \cong N$, $K_z \cong L$. For $t \in K$, define an element x^t via

$$x_a^t = \begin{pmatrix} y_a & tw_a \\ 0 & z_a \end{pmatrix}.$$

For $t \neq 0$, x^t is the conjugation of x by $\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \in GL(r)$, so $x^t \in Mod(A,r)$, and moreover $x^t \in \mathcal{O}_M$. Thus $x^0 \in \overline{\mathcal{O}_M}$, and of course $K_{x^0} \cong L \oplus N$.

THEOREM. $\overline{\mathcal{O}_M}$ contains a unique orbit of semisimple modules. It follows that $\mathcal{O}_M$ is closed $\Leftrightarrow$ M is semisimple.

PROOF. $\overline{\mathcal{O}_M}$ contains $\mathcal{O}_{gr\ M}$ by the lemma, so we need to prove uniqueness.

- If M is an A-module and $a \in A$, then the characteristic polynomial is defined by $char.pol_M(a) = \det(tI - \ell_a)$ where $\ell_a : M \longrightarrow M$ is multiplication by a.
- If $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$ then $char.pol_N(a) = char.pol_M(a)$ for all $a \in A$. This holds because the coefficients of char.pol(a) define a regular map $mod(A,\alpha) \longrightarrow \mathbb{A}^r$ where $r = \sum \alpha_i$.
- If $char.pol_M(a) = char.pol_N(a)$ for all $a \in A$ then the simples have the same multiplicities in M and N, for if S is simple, then

$$[M:S] = \frac{1}{\dim S} \min_{a\in \mathrm{Ann}(S)} \mathrm{ord}_{t=0}\ \mathrm{char.pol}_M(a).$$

where $\mathrm{ord}_{t=0}$ denotes the order of the zero at t=0. (For a proof, we may assume that M is semisimple, next that M is faithful. Now A is semisimple and the result is easy.)

REMARK. The opposite extreme is to determine the open orbits $\mathcal{O}_M$. The following implications hold.

$$\mathrm{Ext}^1(M,M)=0 \Leftrightarrow \mathcal{O}_M \text{ open subscheme of } \underline{\mathrm{Mod}}(A,\alpha) \Rightarrow \mathcal{O}_M \text{ open in } \mathrm{Mod}(A,\alpha)$$

The last implication has a converse if $\mathrm{Mod}(A,\alpha)$ is reduced, for example for A=KQ.

PARTIAL ORDERINGS

DEFINITION. Write $M \le_{ext} N$ for the reflexive and transitive relation generated by $M\le_{ext} L\oplus N$ if there is an exact seq. $0\longrightarrow L\longrightarrow M\longrightarrow N\longrightarrow 0$.

- By the lemma $M\le_{ext} N$ implies that M degenerates to N.
- It follows that $\le_{ext}$ is a partial order.

DEFINITION. Write $M \le_{hom} N$ if $\dim \mathrm{Hom}(X,M)\le \dim \mathrm{Hom}(X,N)$ for all modules X.

- The function $\dim \mathrm{Hom}(-,M)$ determines M up to isomorphism. (For a proof one can reduce to the case when A is finite dimensional, when it is a theorem of Auslander. Alternatively, see K. Bongartz, A generalization of a theorem of M. Auslander, Bull. London Math. Soc., 21 (1989), 255-256.)
- It follows that $\le_{hom}$ is a partial order.
- If M degenerates to N then $M \le_{hom} N$ by upper semicontinuity (the set $\{U | \dim \mathrm{Hom}(X,U)\ge \dim \mathrm{Hom}(X,M)\}$ is closed, contains $\mathcal{O}_M$, so $\mathcal{O}_N$).

REMARK. The general problem is to describe degenerations. We have

$$M \le_{ext} N \Rightarrow M \text{ degenerates to } N \Rightarrow M \le_{hom} N.$$

Thus the best possible case is when $M \le_{hom} N \Rightarrow M \le_{ext} N$. This doesn't hold for all algebras A, but for some algebras it does hold.

THEOREM. $M \le_{hom} N \Rightarrow M \le_{ext} N$ for r-dimensional K[T]-modules.

PROOF. M and N decompose into generalized eigenspaces

$$M = \oplus_{t\in K} M_t, \quad N = \oplus_{t\in K} N_t.$$

The conditions $M \le_{hom} N$ and dim M = dim N imply that dim M_t = dim N_t and $M_t \le_{hom} N_t$ for all t. Thus we may suppose that $M=M_t$ and $N=N_t$. Without loss of generality t=0, so T acts nilpotently on M and N.

Now M is described by a partition $\mu=(\mu_1,\mu_2,..)$ of r, and also by the corresponding Young frame, a diagram whose i^{th} row has length μ_i, for example

□□□□□□
□□□□□□
□□□□
□□

Explicitly the diagram has one column of length i for each summand of the form $K[T]/(T^i)$, or equivalently μ_i is the number of summands $K[T]/(T^j)$ with $j \ge i$ (also the dimension of the i^{th} layer in the socle series).

Let N be described by the partition ν. Now $M \le_{hom} N$ implies that

$$\dim \mathrm{Hom}(K[T]/T^i, M) \le \dim \mathrm{Hom}(K[T]/T^i, N) \text{ for all } i,$$

so $\mu_1+..+\mu_i \le \nu_1+..+\nu_i$ for all i, that is $\mu \le \nu$ in the dominance ordering. Now the dominance ordering is generated by the following moves: $\mu \le \nu$ if the diagram for ν is obtained from that of μ by moving a corner block from a column of length j to a column further to the right of length i<j. For example

μ	□□□□□□ □□□□□□ □□□□ □■	$\le$	□□□□□□ □□□□□□ □□□□■ □	ν

Now we have exact sequences

$$0 \longrightarrow K[T]/T^{i+1} \longrightarrow K[T]/T^i \oplus K[T]/T^j \longrightarrow K[T]/T^{j-1} \longrightarrow 0.$$

for each such move, so $\mu \le \nu$ implies that $M \le_{ext} N$.

Reformulating this, we obtain the

COROLLARY (Gerstenhaber-Hesselink). For $A,B \in M_n(K)$ the following statements are equivalent

- A degenerates to B under the conjugation action of $GL_n(K)$
- $\operatorname{rank}(A-tI)^r \geq \operatorname{rank}(B-tI)^r$ for all $t \in K$ and $r \in \mathbb{N}$.

PROOF. Consider A and B as n-dimensional K[T]-modules. The numerical condition is now $A \leq_{hom} B$.

PREPROJECTIVE MODULES

DEFINITION. A <u>path</u> of A-modules is a sequence $X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n$ of non-zero non-isomorphisms between indecomposables. Write $X_0 \twoheadrightarrow X_n$.

- An indecomposable module X is <u>preprojective</u> if there are no infinite paths ending at X. An arbitrary module is <u>preprojective</u> if all indecomposable summands are preprojective.

THEOREM (Bongartz). If N is preprojective and $M \leq_{hom} N$ then $M \leq_{ext} N$.

SPECIAL CASE. If A is representation-directed, ie every module is preprojective (eg KQ with Q Dynkin), then M degenerates to N $\Leftrightarrow M \leq_{hom} N$. This is combinatorial since A has only finitely many indecomposable modules.

Some cases of KQ with Q Dynkin were solved before Bongartz, for example the following orientation of D_n

$$\begin{tikzcd}[row sep=small] \bullet \arrow[dr] & & & \\ & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \\ \bullet \arrow[ur] & & & \end{tikzcd}$$

was solved by Abeasis and Del Fra, Adv. Math 52(1984), 81-172. I suppose that their work takes 90 pages since they use the same brute force method we used for K[T].

OPEN PROBLEM. Show that the equivalence M degenerates to N $\Leftrightarrow M \leq_{hom} N$ hold for path algebras of Euclidean quivers. The Kronecker quiver $\bullet \rightrightarrows \bullet$ has been dealt with by Bongartz.

PROPERTIES OF PREPROJECTIVE MODULES.

- If X is indecomposable preprojective then End X = K, for otherwise there is infinite path $..\xrightarrow{f}X\xrightarrow{f}X\xrightarrow{f}X$.
- If X is indecomposable, M is preprojective and $\mathrm{Hom}(X,M)\neq 0$ then X is preprojective.
- $X\twoheadrightarrow Y$ is a partial order on the indecomposable preprojectives.
- $\mathrm{Ext}^1(Y,X)\neq 0 \Rightarrow X\twoheadrightarrow Y$ (Otherwise there is a non-split extension

$$0\longrightarrow X\xrightarrow{f}E_1\oplus..\oplus E_n\xrightarrow{g}Y\longrightarrow 0,$$

where the middle term has been decomposed into indecomposable summands E_i. Now if any component of f or g is zero or an isomorphism, the sequence splits. Thus there is path $X\longrightarrow E_1\longrightarrow Y$.)
- $\mathrm{Ext}^1(X,X)=0$ for X indecomposable preprojective.

PROOF OF THE THEOREM.

1. We fix N and prove it for all M by induction on dim $\mathcal{O}_M$. If $M\cong N$ then nothing to do, so suppose $M\not\cong N$. Now M is preprojective, for if U is an indecomposable summand of M then

$$0 \neq \dim \mathrm{Hom}(U,M) \leq \dim \mathrm{Hom}(U,N),$$

so U is preprojective.

2. There is a map $\theta:M\longrightarrow N$ such that no indecomposable summand X of $\mathrm{Ker}\ \theta$ is a summand of M.

PROOF. Write $M = \oplus_{i=1}^{r} U_i^{(n_i)}$ with the U_i indecomposable and non-isomorphic. Since $\twoheadrightarrow$ is a partial order on preprojectives, we may assume that $\mathrm{Hom}(U_i,U_j)=0$ for $i<j$. Let $M_j = \oplus_{i\leq j} U_i^{(n_i)}$. We define $\theta|_{M_j}$ by induction on j. Now $\theta|_{M_{j-1}}$ induces a map $\mathrm{Hom}(U_j,M_{j-1})\longrightarrow\mathrm{Hom}(U_j,N)$, say with image I_j. Now

$$\dim \mathrm{Hom}(U_j,N) \geq \dim \mathrm{Hom}(U_j,M) = \dim \mathrm{Hom}(U_j,M_{j-1})+n_j \geq \dim I_j+n_j.$$

Thus there are maps $\theta_{j1},..,\theta_{jn_j}\in\mathrm{Hom}(U_j,N)$ which are linearly independent modulo I_j. Use these to define $\theta|_{M_j}$.

Let X be an indecomposable summand of M contained in $\mathrm{Ker}\ \theta$. Let $f_{ip}:X\longrightarrow U_i$ be the composition of the inclusion $X\hookrightarrow M$ and the projection of M onto the p^{th} copy of U_i. Since X is a summand, some

f_{ip} is invertible, say f_{jq}. Thus $X \cong U_j$, so $f_{ip}=0$ for $i>j$ and each of the maps

$$f_{jp}f_{jq}^{-1} : U_j \longrightarrow U_j$$

is scalar multiplication. Now $X \subseteq \operatorname{Ker} \theta$, so $\sum \theta_{ip}f_{ip} = 0$. Thus

$$\textstyle\sum_p \theta_{jp}f_{jp}f_{jq}^{-1} = - \sum_{i<j,p} \theta_{ip}f_{ip}f_{jq}^{-1} \in I_j,$$

which contradicts the construction of θ.

3. M and N have the same dimension, and $M \not\cong N$, so $\operatorname{Ker} \theta \neq 0$. Let X be an indecomposable summand of $\operatorname{Ker}(\theta)$ which is maximal with respect to $\twoheadrightarrow$. Let $Y=M/X$, so $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$.

4. $M \leq_{hom} X \oplus Y \leq_{hom} N$.

PROOF. We need $\dim \operatorname{Hom}(V,M) \leq \dim \operatorname{Hom}(V,X\oplus Y) \leq \dim \operatorname{Hom}(V,N)$ for all indecomposable modules V. Now we have a long exact sequence

$$0 \longrightarrow \operatorname{Hom}(V,X) \longrightarrow \operatorname{Hom}(V,M) \longrightarrow \operatorname{Hom}(V,Y) \longrightarrow \operatorname{Ext}^1(V,X).$$

If $\operatorname{Ext}^1(V,X)=0$ then $\dim \operatorname{Hom}(V,X\oplus Y) = \dim \operatorname{Hom}(V,M) \leq \dim \operatorname{Hom}(V,N)$ as required, so suppose $\operatorname{Ext}^1(V,X)\neq 0$. By observations above $X \twoheadrightarrow V$ and $V \not\cong X$ so that $\operatorname{Hom}(V,X)=0$. If Z is a complement to X in $\operatorname{Ker}(\theta)$ then also $\operatorname{Hom}(V,Z)=0$ by the choice of X. Now there is an exact sequence $0 \longrightarrow Z \longrightarrow Y \longrightarrow N$, so $0 \longrightarrow \operatorname{Hom}(V,Z) \longrightarrow \operatorname{Hom}(V,Y) \longrightarrow \operatorname{Hom}(V,N)$ is exact, but the first term is zero. Thus

$$\dim \operatorname{Hom}(V,M) \leq \dim \operatorname{Hom}(V,X\oplus Y) = \dim \operatorname{Hom}(V,Y) \leq \dim \operatorname{Hom}(V,N),$$

as required.

5. We have an exact sequence $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$, so M degenerates to $X\oplus Y$. Also $X\oplus Y \not\cong M$, for otherwise the sequence

$$0 \longrightarrow \operatorname{Hom}(Y,X) \longrightarrow \operatorname{Hom}(M,X) \longrightarrow \operatorname{Hom}(X,X) \longrightarrow 0$$

is exact on the right by dimensions, so X is a summand of M, which is impossible. Thus $\dim \mathcal{O}_{X\oplus Y} < \dim \mathcal{O}_M$. Now $X\oplus Y \leq_{hom} N$ so by induction $X\oplus Y \leq_{ext} N$. Also $M \leq_{ext} X\oplus Y$. Thus $M \leq_{ext} N$.

§5. Representation type of algebras

We prove Geiß's Theorem that degenerations of wild algebras are wild.

THE VARIETY ALGMOD

- Let $\mathrm{Algmod}(n,r) = \{(x,y)\in\mathrm{Alg}(n)\times\mathrm{Hom}_K(K^n,M_r(K))\,|\,y\in\mathrm{Mod}(K_x,r)\}$ where for $x\in\mathrm{Alg}(n)$ we write K_x for the corresponding algebra.
- This is closed subset, so an affine variety.
- Let $\pi:\mathrm{Algmod}(n,r)\longrightarrow\mathrm{Alg}(n)$ be the projection.
- We have $\pi^{-1}(x) = \mathrm{Mod}(K_x,r)$.
- $GL(r)$ acts on $\mathrm{Algmod}(n,r)$.

THEOREM. $\pi:\mathrm{Algmod}(n,r)\longrightarrow\mathrm{Alg}(n)$ sends $GL(r)$-stable closed subsets to closed subsets.

(A subset X being G-stable just means that $gX\subseteq X$ for all $g\in G$). The theorem is a reformulation of Lemma 3.2 in Gabriel's article in SLN 488. Our proof is simpler since it avoids using semisimple modules. We first need some lemmas.

- Let M be a vector space of dimension m.
- Let $\mathrm{Surj}(M,r) = \{\theta:M\longrightarrow K^r \text{ surjective}\}$.
- $GL(r)$ acts on this.

LEMMA. Let $\sigma:\mathrm{Surj}(M,r)\longrightarrow\mathrm{Gr}\binom{M}{m-r}$ be the map sending θ to $\mathrm{Ker}(\theta)$
1. σ identifies $GL(r)$-orbits in $\mathrm{Surj}(M,r)$ with points in $\mathrm{Gr}\binom{M}{m-r}$.
2. σ is a morphism.
3. σ is locally a projection $U\times GL(r)\longrightarrow U$.

(Thus σ is a fibre bundle, and $\mathrm{Gr}\binom{M}{m-r} = \mathrm{Surj}(M,r)\, //\, GL(r)$).

PROOF. (1) is clear. For $\lambda:K^r\hookrightarrow M$, define
- $V_\lambda = \{\theta\in\mathrm{Surj}(M,r)\,|\,\theta\lambda \text{ is isomorphism}\}$.
- $U_\lambda = \{N\in\mathrm{Gr}\binom{M}{m-r}\,|\,M=N\oplus\mathrm{Im}(\lambda)\}$.

The V_λ are an open covering of $\mathrm{Surj}(M,r)$, the U_λ are an open covering of $\mathrm{Gr}\binom{M}{m-r}$, and σ sends $V_\lambda \longrightarrow U_\lambda$. Using these coverings one can prove (2), but we skip this. Now we have inverse maps

$$V_\lambda \quad \underset{(U,g)\mapsto g\lambda^{-1}p_U}{\overset{\theta\mapsto(\mathrm{Ker}\ \theta,\theta\lambda)}{\rightleftarrows}} \quad U_\lambda \times GL(r)$$

where p_U is the projection $M\longrightarrow \mathrm{Im}(\lambda)$ complementary to U, and $\lambda^{-1}:\mathrm{Im}(\lambda)\longrightarrow K^r$. Thus σ is locally a projection $U_\lambda\times GL(r)\longrightarrow U_\lambda$.

LEMMA. If X is a variety then the projection $X\times \mathrm{Surj}(M,r)\longrightarrow X$ sends $GL(r)$-stable closed subsets to closed subsets.

PROOF. The map factors as $X\times \mathrm{Surj}(M,r) \xrightarrow{(1,\sigma)} X\times \mathrm{Gr}\binom{M}{m-r} \xrightarrow{p} X$.

- Using that σ is locally a projection one can show that $(1,\sigma)$ sends closed $GL(r)$-stable subsets of $X\times \mathrm{Surj}(M,r)$ to closed subsets.
- Since $\mathrm{Gr}\binom{M}{m-r}$ is projective, it is "complete", which means that p sends closed sets to closed sets.

PROOF OF THE THEOREM. Let

$$W = \{(x,\theta)\in \mathrm{Alg}(n)\times \mathrm{Surj}(K^{nr},r) \mid \mathrm{Ker}(\theta) \text{ is } K_x\text{-submodule of } (K_x)^r\}.$$

This is closed subset by same proof that Gr_A is closed in Gr. Now we have a commutative diagram

$$\begin{array}{ccc} W & \hookrightarrow & \mathrm{Alg}(n)\times \mathrm{Surj}(K^{nr},r) \\ \downarrow\rho & & \downarrow \mathrm{proj.} \\ \mathrm{Algmod}(n,r) & \xrightarrow{\pi} & \mathrm{Alg}(n) \end{array}$$

where ρ sends (x,θ) to (x, quotient module structure on K^r). Now ρ is onto, since any r-dimensional A-module is a quotient of A^r. Using the covering V_λ one can show that ρ is a morphism.

If $Z \subseteq \mathrm{Algmod}(n,r)$ is $GL(r)$-stable and closed, so is $\rho^{-1}(Z)$. Thus $\rho^{-1}(Z)$ is $GL(r)$-stable closed subset of $\mathrm{Alg}(n)\times \mathrm{Surj}(K^{nr},r)$. Thus $\pi(Z) = \mathrm{proj}.(\rho^{-1}(Z))$ is closed by the lemma.

NUMBER OF PARAMETERS

Let G be a connected algebraic group acting on a variety X.

EXERCISE. If $Y \subseteq X$ is constructible and G-stable, then you can write

$$Y = Z_1 \,\dot{\cup}\, .. \,\dot{\cup}\, Z_n$$

with the Z_i being G-stable irreducible locally closed subsets of X. This decomposition is not unique, but the number of top-dimensional Z_i is the number of top-dimensional irreducible components of $\overline{Y}$, so is unique. The key idea for the proof is that if $Z \subseteq X$ is locally closed and irreducible then the fact that $G \times Z$ is irreducible implies that $\overline{GZ}$ is irreducible.

DEFINITIONS. The <u>number of parameters</u> of G on Y is

$$\mu_G(Y) = \max_s \left(\dim Y \cap X_{(s)} - s\right) = \max_s \left(\dim Y \cap X_{(\le s)} - s\right).$$

The <u>number of top-dimensional families of orbits</u> is

$$t_G(Y) = \textstyle\sum_s \text{ (no. of irred comps of } \overline{Y \cap X_{(s)}} \text{ of dimension } s+\mu_G(Y)).$$

REMARKS.

1. We don't talk much about t_G, but it is well-behaved.
2. If the set of orbits Y/G was a variety, then μ would be its dimension and t would be the number of top-dimensional irreducible components.

PROPERTIES (left as exercises).

1. If $Y_i \subseteq X$ are G-stable, then $\mu(\bigcup Y_i) = \max\{\mu Y_i\}$.
2. $\mu Y = 0 \Leftrightarrow Y$ contains only finitely many orbits, and if so, then tY is the number of orbits.
3. If Y contains a constructible subset Z meeting each orbit then $\mu Y \le \dim Z$.
4. If $f: Z \longrightarrow X$ is a map, and the inverse image of each orbit has dimension $\le d$ then $\mu X \ge \dim Z - d$.

LEMMA. If $\pi:X\longrightarrow Y$ is constant on orbits, and sends G-stable closed subsets of X to closed subsets of Y, then the function $y\longmapsto\mu_G(\pi^{-1}(y))$ is upper semicontinuous.

PROOF. First $y\longmapsto\dim \pi^{-1}(y)$ is upper semicontinuous, since

$$\{y\in Y \mid \dim \pi^{-1}(y)\geq r\} = \pi(\{x\in X \mid \dim_x\pi^{-1}\pi(x)\geq r\}).$$

Now $\pi|_{X_{(\leq s)}}$ sends closed G-stable subsets to closed subsets, and

$$\mu(\pi^{-1}(y)) = \max_s \left(\dim (\pi|_{X_{(\leq s)}})^{-1}(y) - s\right)$$

TAME AND WILD

THEOREM (Drozd). A finite dimensional algebra A is either

Tame: for any r there are A-K[T]-bimodules $M_1,..,M_N$, f.g. free/K[T], such that any indecomposable A-module of dimension $\leq$ r is isomorphic to some $M_i\otimes K[T]/(T-\lambda)$.

Wild: there is an A-K<X,Y>-bimodule M, f.g. free/K<X,Y> such that the functor $M\otimes_{K\langle X,Y\rangle}-$ sends non-isomorphic f.d. K<X,Y>-modules to non-isomorphic A-modules.

The proof is hard.

LEMMA.

1. If A is wild there is s with μ Mod(A,sr) $\geq r^2$ for all r.
2. If A is tame then μ Mod(A,r) $\leq$ r.

PROOF. If M is an A-B-bimodule, free of rank s over B, then after choosing a basis of M over B one obtains a homomorphism $A\longrightarrow M_s(B)$, and this induces a map $\mathrm{Mod}(B,r)\longrightarrow\mathrm{Mod}(A,sr)$ corresponding to the functor $M\otimes_B -$.

(1) We have a map $\mathrm{Mod}(K\langle X,Y\rangle,r)\longrightarrow\mathrm{Mod}(A,sr)$. The inverse image of an orbit is an orbit, so has dimension $\leq$ dim GL(r). Thus

$$\mu\ \mathrm{Mod}(A,sr) \geq \dim \mathrm{Mod}(K\langle X,Y\rangle,r) - \dim GL(r) = 2r^2 - r^2 = r^2.$$

(2) If $1 \le i_1, .., i_k \le N$ is a sequence with $\sum \mathrm{rank}_{K[T]}(M_{ij}) = r$, then $\oplus_j M_{ij} \otimes K[T]/(T-\lambda_j)$ defines a constructible subset of Mod(A,r) of dimension $\le k \le r$. Let Z be the union of these sets over all possible sequences. Then Z meets every orbit so μ Mod(A,r) $\le$ dim Z $\le$ r.

THEOREM (Geiß). A degeneration of a wild algebra is wild.

This is not the original version circulated by Geiß, in which only special degenerations were allowed, but a private communication from him (I had simultaneously proved the general case without the use of Algmod by replacing modules with their projective presentations).

PROOF. By the lemma $\{x \in \mathrm{Alg}(n) | K_x \text{ is wild}\} = \bigcup_r M_r$ where

$$M_r = \{x \in \mathrm{Alg}(n) \mid \mu \ \mathrm{Mod}(K_x, r) > r\}.$$

Now M_r is closed by the properties of Algmod and μ, and it is obviously GL(n)-stable. If $x, y \in \mathrm{Alg}(n)$ and $y \in \overline{GL(n)x}$, then

$$K_x \text{ wild} \Rightarrow x \in M_r \text{ (some r)} \Rightarrow y \in \overline{GL(n)x} \subseteq M_r \Rightarrow K_y \text{ wild},$$

as required.

EXAMPLE. $A = K\langle a,b\rangle/(a^2-bab, b^2-aba, (ab)^2, (ba)^2)$ degenerates to $B = K\langle a,b\rangle/(a^2, b^2, (ab)^2, (ba)^2)$. Now B is known to be tame, so A is tame. This is the only known proof that A is tame. (The degeneration is given as follows. Let $x^t \in \mathrm{Alg}(7)$ have basis 1,a,b,ab,ba,aba,bab and multiplication as indicated, and $a^2=tbab, b^2=taba, (ab)^2=(ba)^2=0$. Then $K_{x^t} \cong A$ for $t \ne 0$, and $K_{x^o} \cong B$.)

REMARK. $\{x \in \mathrm{Alg}(n) | K_x$ finite rep. type$\}$ is open in Alg(n). See Gabriel's article in SLN 488. The proof uses the second Brauer-Thrall conjecture, which is hard, and was not properly proved until much later.

§6. Kac's Theorem

We give part of the proof of Kac's Theorem, and sketch the rest.

- Let Q be a quiver with vertices {1,..,n}.
  The <u>Ringel</u> <u>form</u> is defined by $\langle\alpha,\beta\rangle = \sum \alpha_i\beta_i - \sum_{a:i\to j} \alpha_i\beta_j$
  For KQ-modules have $\dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) = \langle\underline{\dim}\, X, \underline{\dim}\, Y\rangle$.
  The <u>Tits</u> <u>form</u> is $q(\alpha) = \langle\alpha,\alpha\rangle = \dim GL(\alpha) - \dim \mathrm{Mod}(KQ,\alpha)$.
  The corresponding symmetric bilinear form is $(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$.

- $\varepsilon_i\in\mathbb{Z}^n$ is the i^{th} coordinate vector.
  ε_i is a <u>simple</u> <u>root</u> if there is no loop at the vertex i.
  If ε_i is simple, there is a reflection $\sigma_i:\mathbb{Z}^n\longrightarrow\mathbb{Z}^n$, $\alpha\longmapsto\alpha-(\alpha,\varepsilon_i)\varepsilon_i$.
  The <u>Weyl</u> <u>group</u> $W = \langle\sigma_i \mid \varepsilon_i \text{ simple}\rangle \le GL_n(\mathbb{Z})$.
  W preserves (-,-) and q.

- The <u>fundamental</u> <u>region</u> is the set
  $F = \{\alpha\in\mathbb{N}^n \mid \alpha\neq 0,\ \text{support}(\alpha) \text{ connected},\ (\alpha,\varepsilon_i)\le 0\ \forall \text{ (simple) } \varepsilon_i\}$.
  Here support(α) denotes the subquiver of Q, and the word simple is in parentheses since $(\alpha,\varepsilon_i)\le 0$ is automatic if there is a loop at i.

- <u>Real</u> <u>roots</u> = $\{w(\varepsilon_i) \mid w\in W,\ \varepsilon_i \text{ simple}\}$. These have $q(\alpha)=1$.
  <u>Imaginary</u> <u>roots</u> = W(F). These have $q(\alpha)\le 0$.
  (Strictly speaking these are only the positive imaginary roots).

THEOREM (Kac). If $\alpha\in\mathbb{N}^n$ then there is an indecomposable representation of dimension $\alpha \Leftrightarrow \alpha$ is a root. If so, then

- $\mu(\mathrm{Ind}(KQ,\alpha)) = 1-q(\alpha)$
- $t(\mathrm{Ind}(KQ,\alpha)) = 1$.

(where we use the action of $GL(\alpha)$ on $\mathrm{Mod}(KQ,\alpha)$). In particular, for α a real root there is a unique indecomposable representation.

LEMMA A. For $\alpha\in F$ we have $\mu(\mathrm{Ind}(KQ,\alpha)) = 1-q(\alpha)$ and $t(\mathrm{Ind}(KQ,\alpha)) = 1$.
LEMMA B. For ε_i simple and $\alpha\in\mathbb{N}^n$, $\alpha\neq\varepsilon_i$ we have $\mu(\mathrm{Ind}(KQ,\alpha)) = \mu(\mathrm{Ind}(KQ,\sigma_i(\alpha)))$ and $t(\mathrm{Ind}(KQ,\alpha)) = t(\mathrm{Ind}(KQ,\sigma_i(\alpha)))$.

PROOF OF THE THEOREM. If α is an imaginary root, then $\alpha=w(\beta)$, $\beta\in F$ and the lemmas give the assertion. If α is real root then $\alpha=w(\varepsilon_j)$ with ε_j a simple root. Now $\mathrm{Ind}(KQ,\varepsilon_j) = \{pt\}$, so $\mu(\mathrm{Ind}(KQ,\varepsilon_j))=0$ and $t(\mathrm{Ind}(KQ,\varepsilon_j))=1$, and Lemma B gives the assertion.

Suppose there is an indecomposable of dimension α and α is not a real root. By Lemma B there is an indecomposable of dimension $w(\alpha)$ for all w, and in particular $w(\alpha)\geq 0$ for all $w\in W$. Choose $\beta = w(\alpha)$ minimal. Since β is made smaller by any reflection, it follows that $(\beta,\varepsilon_i)\leq 0$ for all simple roots ε_i. Now there is an indecomposable of dimension β, so support(β) is connected. Thus $\beta\in F$.

LEMMA A

Suppose $\alpha\in F$, so that $\alpha\geq 0$, $\alpha\neq 0$, support(α) connected and $(\alpha,\varepsilon_i)\leq 0$ $\forall i$. We have to prove that $\mu(\mathrm{Ind}(\alpha)) = 1-q(\alpha)$ and $t(\mathrm{Ind}(\alpha)) = 1$.

LEMMA 1. Either

1. support(α) is Euclidean and $q(\alpha)=0$, or
2. if $\alpha=\beta_1+..+\beta_r$ $(r\geq 2)$ with $\beta_i\geq 0$ non-zero then $q(\alpha) < \sum q(\beta_i)$.

PROOF. We may assume Q=support(α), and so Q is connected. If (2) fails then $\sum (\alpha-\beta_i,\beta_i) = (\alpha,\alpha) - \sum(\beta_i,\beta_i) \geq 0$, so there is $0\leq\beta\leq\alpha$, $\beta\neq 0,\alpha$, with $(\alpha-\beta,\beta)\geq 0$. Now

$$0 \leq (\alpha-\beta,\beta) = \sum_i \underbrace{(\alpha,\varepsilon_i)}_{\leq 0}\beta_i(\alpha_i-\beta_i)/\alpha_i + \frac{1}{2}\sum_{i\neq j} \underbrace{(\varepsilon_i,\varepsilon_j)}_{\leq 0}\alpha_i\alpha_j \left(\frac{\beta i}{\alpha i} - \frac{\beta j}{\alpha j}\right)^2$$

so $\frac{\beta i}{\alpha i} = \frac{\beta j}{\alpha j}$ whenever $(\varepsilon_i,\varepsilon_j)<0$, ie if an arrow connects i——j. Thus α is a multiple of β. Now the first sum implies that $(\alpha,\varepsilon_i)=0$ for all i. This implies that Q is Euclidean and that $q(\alpha)=0$.

IN THE FIRST CASE of Lemma 1 there is a complete classification of the representations of dimension α, and using this one can prove Lemma A. Thus we now assume that the second case of Lemma 1 holds.

LEMMA 2. The general rep of dimension α is indecomposable, ie $\mathrm{Ind}(KQ,\alpha)$ contains a non-empty open subset of $\mathrm{Mod}(KQ,\alpha)$.

PROOF. If $\alpha=\beta+\gamma$ ($\beta,\gamma\neq 0$) there is a map

$$\theta:GL(\alpha)\times\mathrm{Mod}(KQ,\beta)\times\mathrm{Mod}(KQ,\gamma)\longrightarrow\mathrm{Mod}(KQ,\alpha),\ (g,x,y)\longmapsto g(x\oplus y).$$

This map is constant on the orbits of a free action of $H=GL(\beta)\times GL(\gamma)$, so $\dim\overline{\mathrm{Im}(\theta)}\leq\dim\mathrm{LHS}-\dim H$. Using the fact that $q(\alpha)=\dim GL(\alpha)-\dim\mathrm{Mod}(KQ,\alpha)$, one deduces that

$$\dim\mathrm{Mod}(KQ,\alpha)-\dim\overline{\mathrm{Im}(\theta)}\geq q(\beta)+q(\gamma)-q(\alpha)>0,$$

so $\overline{\mathrm{Im}(\theta)}$ is a proper subset of $\mathrm{Mod}(KQ,\alpha)$. The assertion follows.

DEFINITION. Let $\lambda=(\lambda_1,..,\lambda_n)$ with $\lambda_i=(\lambda_i^1,\lambda_i^2,...)$ a partition of α_i.

- $\theta\in\mathrm{End}(\alpha)$ is of <u>type</u> λ if the maps $\theta_i\in\mathrm{End}(K^{\alpha_i})$ are nilpotent of type λ_i (so λ_i^r is the number of Jordan blocks of size $\geq r$).
- The zero map corresponds to the sequence z with $z_i=(\alpha_i,0,...)$.
- Let $N_\lambda=\{\theta|\theta\in\mathrm{End}(\alpha)$ of type $\lambda\}$. It is locally closed.
- If $\theta\in\mathrm{End}(\alpha)$ let $\mathrm{Mod}_\theta=\{x\in\mathrm{Mod}(KQ,\alpha)|\theta\in\mathrm{End}(K_x)\}$

LEMMA 3.

1. If $\theta\in N_\lambda$ then $\dim\mathrm{Mod}_\theta=\sum_{a:i\longrightarrow j}\sum_r\lambda_i^r\lambda_j^r$
2. $\dim N_\lambda=\dim GL(\alpha)-\sum_i\sum_r\lambda_i^r\lambda_i^r$

PROOF. It is easy to check that if $f\in\mathrm{End}(V)$ and $g\in\mathrm{End}(W)$ are nilpotent endomorphisms of type μ and ν, then $\dim\{h:V\longrightarrow W|gh=hf\}=\sum_r\mu^r\nu^r$. (1) follows immediately. For (2) note that N_λ is an orbit for the conjugation action of $GL(\alpha)$ on $\mathrm{End}(\alpha)$, so if $\theta\in N_\lambda$ then

$$\begin{aligned}\dim N_\lambda&=\dim GL(\alpha)-\dim\{g\in GL(\alpha)|g\theta=\theta g\}\\&=\dim GL(\alpha)-\dim\{g\in\mathrm{End}(\alpha)|g\theta=\theta g\}\\&=\dim GL(\alpha)-\textstyle\sum_i\sum_r\lambda_i^r\lambda_i^r.\end{aligned}$$

DEFINITIONS.

- $g=\dim GL(\alpha)$. If $x\in\mathrm{Mod}(KQ,\alpha)$ then its orbit has dimension $g-\dim\mathrm{End}(K_x)$.
- $I=\mathrm{Ind}(KQ,\alpha)=\bigcup_{s<g}I_{(s)}$. Note that $I_{(s)}$ is locally closed in $\mathrm{Mod}(KQ,\alpha)$ by the results about equipotent endomorphisms in §3.

- $B = \{x \in \mathrm{Mod}(KQ,\alpha) \mid K_x$ is a <u>brick</u>, ie $\mathrm{End}(K_x)=K\} = I_{(g-1)}$.
- $N = \{$non-zero nilpotent $\theta \in \mathrm{End}(\alpha)\} = \bigcup_{\lambda\neq z} N_\lambda$.
- $MN = \{(x,\theta) \in \mathrm{Mod}(KQ,\alpha)\times N \mid \theta \in \mathrm{End}(K_x)\} = \bigcup_{\lambda\neq z} MN_\lambda$.
- $I_{(s)}N = \{(x,\theta) \in I_{(s)}\times N \mid \theta \in \mathrm{End}(K_x)\} \subseteq MN$.

LEMMA 4.

1. for $\lambda\neq z$ we have $\dim MN_\lambda < g-q(\alpha)$, so $\dim MN < g-q(\alpha)$.
2. For $s<g-1$ we have $\dim I_{(s)} < s+1-q(\alpha)$, so $\mu(I_{(s)}) < 1-q(\alpha)$.
3. B is non-empty and open in $\mathrm{Mod}(KQ,\alpha)$, so $\mu(B) = 1-q(\alpha)$, $t(B) = 1$.

Now Lemma A follows from (2) and (3) since $\mathrm{Ind}(KQ,\alpha) = B \cup \bigcup_{s<g-1} I_{(s)}$.

PROOF. (1) Let $MN_\lambda \xrightarrow{\pi} N_\lambda$ be the projection. Now $\pi^{-1}(\theta)=\mathrm{Mod}_\theta$ is of constant dimension, so

$$\dim MN_\lambda \le \dim N_\lambda + \dim \mathrm{Mod}_\theta \overset{\text{Lemma3}}{=} g - \textstyle\sum_r q(\lambda^r) \overset{\text{Lemma1}}{<} g - q(\alpha),$$

since $\alpha = \sum_r \lambda^r$, and at least two λ^r are non-zero since $\lambda\neq z$.

(2) If $s<g-1$ and $x \in I_{(s)}$ then K_x is indecomposable and not a brick, so has a non-zero nilpotent endomorphism. Thus the projection $I_{(s)}N \xrightarrow{\pi} I_{(s)}$ is onto. Now

$$\dim \pi^{-1}(x) = \dim \mathrm{End}(K_x)\cap N = \dim \mathrm{rad}\,\mathrm{End}(K_x) = g-s-1.$$

Thus $\dim I_{(s)} = \dim I_{(s)}N-(g-s-1) \le \dim MN-(g-s-1) \le s+1-q(\alpha)$ by (1).

(3) For $s<g-1$ we have

$$\dim I_{(s)} < s+1-g + \dim \mathrm{Mod}(KQ,\alpha) < \dim \mathrm{Mod}(KQ,\alpha),$$

so $\overline{I_{(s)}}$ is a proper closed subset of $\mathrm{Mod}(KQ,\alpha)$. Also, by Lemma 2 the set of decomposable representations is contained in a proper closed subset of $\mathrm{Mod}(KQ,\alpha)$. Thus $B\neq\varnothing$. Also B is open in $\mathrm{Mod}(KQ,\alpha)$ by upper semicontinuity, so it is irreducible. Now $\mu(B) = \dim B-(g-1) = 1-q(\alpha)$ and $t(B)=1$.

LEMMA B. SKETCH

DEFINITION. A representation of Q over an arbitrary field F is <u>absolutely</u> <u>indecomposable</u> if it remains indecomposable as a representation over the algebraic closure $\overline{F}$.

- Let $n(Q,\alpha,p^r)$ be the number of isomorphism classes of absolutely indecomposable representations of Q over $\mathbb{F}_{p^r}$ of dimension α.

LEMMA 1. Let $\mu=\mu(\mathrm{Ind}(KQ,\alpha))$ and $t=t(\mathrm{Ind}(KQ,\alpha))$. For p=char K, or for p >> 0 if char K=0, we have $n(Q,\alpha,p^r)/p^{r\mu}\longrightarrow t$ as $r\longrightarrow\infty$.

IDEA. We use schemes to change characteristic. A (quasi-affine algebraic) $\mathbb{Z}$-<u>scheme</u> is a functor (commutative rings)$\longrightarrow$sets, of form

$$X(R) = \{x\in R^n \mid \text{all } p_\lambda(x)=0, \text{ some } q_\mu(x)\neq 0\}$$

for some families $p_\lambda,q_\mu\in\mathbb{Z}[X_1,..,X_n]$. Clearly X(K) is a variety for any algebraically closed field K.

Theorem. If X is a $\mathbb{Z}$-scheme and char K=0, then dim X(K) = dim $X(\overline{\mathbb{F}_p})$ for p>>0.

Theorem of Lang-Weil. If char K=p, q is a power of p, and $X \subseteq \mathbb{P}^n$ is an irreducible closed subset of dimension d defined by polynomials with coefficients in a finite field $\mathbb{F}_q$, then the number of points of X which can be realized in $\mathbb{P}^n$ by an (n+1)-tuple of elements of $\mathbb{F}_{q^r}$ is $q^{rd} + O(q^{r(d-1/2)})$ as $r\longrightarrow\infty$.

Combining these two facts we obtain the following. Let X be a $\mathbb{Z}$-scheme. Suppose X(K) has dimension d and t top-dimensional irreducible components. For p=char K, or for p>>0 if char K=0, we have $|X(\mathbb{F}_{p^r})|/p^{rd}\longrightarrow t$ as $r\longrightarrow\infty$.

Now there are Chevalley-type results for $\mathbb{Z}$-schemes, so one can study constructible subfunctors, actions of $\mathbb{Z}$-group-schemes on $\mathbb{Z}$-schemes, etc. Now there is a $\mathbb{Z}$-scheme <u>Mod</u>(Q,α) with F-points Mod(FQ,α), there is a constructible subfunctor <u>Ind</u>(Q,α) with F-points the absolutely indecomposable representations, and there is $\mathbb{Z}$-group-scheme <u>GL</u>(α)

acting on $\underline{\mathrm{Mod}}(Q,\alpha)$. The assertion of the lemma follows in a standard way.

LEMMA 2. Let i be a sink in Q, a vertex at which no arrows start. Let Q' be the quiver obtained from Q by reversing all the arrows terminating at i. Then $n(Q,\alpha,p^r) = n(Q',\sigma_i(\alpha),p^r)$ for $\alpha\neq\varepsilon_i$, $\alpha\geq 0$.

SKETCH. There are inverse equivalences

$$\begin{array}{l} \text{Reps } X \text{ of } Q \text{ of dim } \alpha \\ \text{with } \oplus_{a:j\to i}X_j \twoheadrightarrow X_i \end{array} \underset{G}{\overset{F}{\rightleftarrows}} \begin{array}{l} \text{Reps } X' \text{ of } Q' \text{ of dim } \sigma_i(\alpha) \\ \text{with } X'_i \hookrightarrow \oplus_{a:i\to j}X'_j \end{array}$$

given by <u>reflection</u> <u>functors</u> F and G. Here F sends a representation X in which the map $f:\oplus_{a:j\to i}X_j\longrightarrow X_i$ is onto, to the representation X' of Q' with $X'_i = \mathrm{Ker}(f)$ and $X'_k = X_k$ for $k\neq i$, and with maps as in the representation X, except that the map $X'_i\longrightarrow X'_j$ corresponding to an arrow $a:j\longrightarrow i$ in Q is the composite of f with the projection onto X'_j.

Now if $\alpha\neq\varepsilon_i$ and $\alpha\geq 0$ then indecomposable representations of Q and Q' of dimensions α and $\sigma_i(\alpha)$ belong to the indicated subcategories, so there is a 1-1 correspondence between absolutely indecomposable representations of Q and Q'.

LEMMA 3. $n(Q,\alpha,p^r)$ doesn't depend on the orientation of Q.

IDEA. There is a result of Brauer which implies that if G acts on a vector space V over a finite field $\mathbb{F}$ then $|V/G| = |V^*/G|$. This can be used to show that Q and a reorientation Q' of Q have the same number of representations over $\mathbb{F}$ of dimension α. Varying α it follows that Q and Q' have the same number of indecomposables over $\mathbb{F}$ of dim α. Now varying $\mathbb{F}$ and using a Galois Theory argument, one can show that Q and Q' have the same number of absolutely indecomposable representations over $\mathbb{F}$ of dimension α.

§7. General representations: results and open problems

In this section $K = \mathbb{C}$.

SUBREPRESENTATIONS

- Write $\beta \hookrightarrow \alpha$ if the general representation of dimension α has a sub-representation of dimension β.
- Write $\hom(\alpha,\beta)$ and $\mathrm{ext}(\alpha,\beta)$ for the general value of $\dim \mathrm{Hom}(K_x,K_y)$ and $\dim \mathrm{Ext}(K_x,K_y)$ with $(x,y)\in \mathrm{Mod}(KQ,\alpha)\times\mathrm{Mod}(KQ,\beta)$. By upper semicontinuity these are also the minimum values.

THEOREM (Schofield). $\beta \hookrightarrow \alpha \Leftrightarrow \mathrm{ext}(\beta,\alpha-\beta) = 0$.

QUESTION 1. Schofield claims this for all K, but his proof only works in characteristic zero. Is the result true in general?

THEOREM (Schofield). $\mathrm{ext}(\alpha,\beta) = \max\{ - \langle\alpha',\beta\rangle \mid \alpha' \hookrightarrow \alpha \}$.

Combined, these two theorems allow inductive calculations.

If $\beta\le\alpha$ are dimension vectors and M has dimension α, there is a variety

$$\mathrm{Gr}_{\mathbb{C}Q}\binom{M}{\beta} \subseteq X := \prod_{i=1}^{n} \mathrm{Gr}\binom{\mathbb{C}^{\alpha i}}{\beta i}.$$

of subrepresentations of M of dimension β. This subvariety has a <u>fundamental class</u> in $H^*(X,\mathbb{Z})$. For the general representation of dimension α this class is constant, say $c(Q,\alpha,\beta)\in H^*(X,\mathbb{Z})$. We describe below the computation of this element (because of the complexity I have not done any examples). On X there are universal bundles as follows.

- $S_j \subseteq \mathbb{C}^{\alpha j}\times X$ is the j^{th} universal sub-bundle, whose fibre over $(U_i)\in X$ is the subspace $U_j \subseteq \mathbb{C}^{\alpha j}$.
- Q_j is the j^{th} universal quotient bundle, whose fibre over $(U_i)\in X$ is the quotient $\mathbb{C}^{\alpha j}/U_j$.

If E and F are vector bundles, there is a vector bundle $\mathcal{H}om(E,F)$ whose fibres are the linear maps between the fibres of E and F. For a vector bundle $E \longrightarrow X$ the set of global sections $s: X \longrightarrow E$ is denoted by $\Gamma(X,E)$. Now there is a map

$$f_{jk} : \mathrm{Hom}(\mathbb{C}^{\alpha j},\mathbb{C}^{\alpha k}) \longrightarrow \Gamma(X,\mathcal{H}om(S_j,Q_k))$$

$$\theta \longmapsto \text{the section which on the fibre over } (U_i) \text{ is the composition } U_j \hookrightarrow \mathbb{C}^{\alpha j} \xrightarrow{\theta} \mathbb{C}^{\alpha k} \twoheadrightarrow \mathbb{C}^{\alpha k}/U_k.$$

The map f_{jk} is onto (and is usually 1-1 if $j \neq k$). Thus we obtain a map $f: \mathrm{Mod}(KQ,\alpha) \twoheadrightarrow \Gamma(X, E)$ where E is the vector bundle

$$E = \bigoplus_{a:j \longrightarrow k} \mathcal{H}om(S_j,Q_k).$$

Now the zero set of the section f(x) is $\mathrm{Gr}_{\mathbb{C}Q}\binom{K_x}{\beta}$, and by the theory of chern classes it follows that $c(Q,\alpha,\beta)$ is the top chern class of E. Now the cohomology ring of X and the chern classes of the S_j and Q_j are known by Schubert calculus. It is therefore possible to compute the chern classes of E.

SCHUR ROOTS

α is a <u>Schur</u> <u>root</u> if there is brick of dimension α. If so, the general representation of dimension α is brick.

THEOREM (Schofield). α is Schur root $\Leftrightarrow$ $\langle\beta,\alpha\rangle - \langle\alpha,\beta\rangle > 0$ $\forall$ $\beta \hookrightarrow \alpha$, $\beta \neq 0,\alpha$.

If α is a root, since $t(\mathrm{Ind}(KQ,\alpha))=1$, there is unique $e=e(\alpha)$ with

$$\mu(\{x \in \mathrm{Ind}(KQ,\alpha) \mid \dim \mathrm{End}(\alpha)=e\}) = 1-q(\alpha).$$

Moreover $e(\alpha)=1 \Leftrightarrow \alpha$ is Schur root.

QUESTION 2. How can you compute $e(\alpha)$?

RATIONAL INVARIANTS

The field of rational invariants is

$$\mathbb{C}(\alpha) = [\text{function field of } \mathrm{Mod}(\mathbb{C}Q,\alpha)]^{GL(\alpha)}.$$

By a result of Kac, you can compute $\mathbb{C}(\alpha)$ if you know it for Schur roots.

QUESTION 3 (standard).

- Is $\mathbb{C}(\alpha)$ rational, ie is $\mathbb{C}(\alpha) \cong \mathbb{C}(X_1,..,X_n)$ for some n?
- Weaker, is it stably rational, ie is $\mathbb{C}(\alpha)(Y_1,..,Y_m)$ rational for some m?

By Schofield and Le Bruyn, to prove stable rationality it suffices to understand the quiver with one vertex and two loops. Question 3 is connected with questions about moduli spaces of vector bundles on $\mathbb{P}^2$, and the ring of generic matrices. See the survey by Le Bruyn.

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