More Lectures on

Representations of Quivers

by

William Crawley-Boevey

These are the notes for eight lectures, continuing the series I gave last term. I had several aims, and have only partially succeeded in achieving any of them.

(1) My first aim was to cover some more basics of the representation theory of quivers. In particular I wanted to cover the wild case, when the quiver Q is not Dynkin or Euclidean.

First, I give a very quick treatment of Auslander-Reiten Theory, and in particular determine the different types of connected components of the AR quiver for a path algebra.

Second, I cover the theorems of Baer and Kerner on the asymptotic behaviour of the translate in the wild case, and I prove that the components of the dimension vector of $\tau^{-r}P(i) \longrightarrow \infty$ as $r \longrightarrow \infty$. This assertion (due to Dlab and Ringel) was used by Baer to prove her results, but we need her results to prove this fact, so some care is needed.

Third, I cover some of the properties and constructions associated with tilting modules and perpendicular categories (but no tilting theory). I prove Hoshino's bound on the quasi-length of a regular module without self extensions, and Ringel's characterization of the quivers with a regular tilting module.

(2) My second aim was to cover some of the recent work of Kerner and his student Lukas. In §5 I define the quasi-period of a regular component, and cover some of the elementary properties of the quasi-period. This suffices to put their results in context. Their proofs, however, often use tilted algebras, and I have not developed enough machinery for this. I content myself with stating their results.

(3) My third aim was to cover the recent work of Schofield. I have not covered much of the geometry of representations of quivers, for example Kac's Theorem that the dimension vectors of indecomposable representations are the roots of the Kac-Moody algebra, so I have had to content myself with Schofield's algorithm for computing the dimension vectors of indecomposables without self extensions. I think that the statement given here is a little neater than Schofield's original.

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§1. Irreducible maps and Auslander-Reiten quivers

Throughout, k is an algebraically closed field, A is a f.d. k-algebra, and we consider f.d. left A-modules. In this section there is no further restriction on A.

RECALL. If $f:X\longrightarrow Y$ then

f is <u>split mono</u> (sm) if $hf = 1_X$ for some $h:Y\longrightarrow X$. Inclusion of a summand.

f is <u>split epi</u> (se) if $fg = 1_Y$ for some $g:Y\longrightarrow X$. Projection onto a summand.

DEFINITIONS.

If X is indecomposable, then $f:X\longrightarrow Y$ is <u>irreducible</u> if it is not sm, and it satisfies (*) $f = hg \Rightarrow g$ is sm or h is se. A <u>source map for</u> X is an irreducible map $f:X\longrightarrow E$ such that any map $X\longrightarrow Z$ which is not sm factors through f.

If Y is indecomposable, then $f:X\longrightarrow Y$ is <u>irreducible</u> if it is not se, and it satisfies (*). A <u>sink map for</u> Y is an irreducible map $f:F\longrightarrow Y$ such that any map $Z\longrightarrow Y$ which is not se factors through f.

REMARKS.

(1) If X,Y are both indecomposable the two notions of an irreducible map $X\longrightarrow Y$ agree. I don't define irreducible maps when X,Y both decompose.

(2) By definition, if Y is indecomposable, then $0\longrightarrow Y$ is irreducible. If Y is simple projective then this is even a sink map.

(3) Other people use the same notion of source and sink maps, but define irreducible maps differently, so that $0\longrightarrow Y$ is not irreducible. They have problems, since then a sink map need not be irreducible.

LEMMA.

(1) If X is indecomposable projective then X has unique maximal proper submodule M and $M\hookrightarrow X$ is sink.

(2) If X is indecomposable injective then X has unique simple submodule S and $X\twoheadrightarrow X/S$ is source.

PROOF (1) If X has maximal proper submodules $M \neq M'$ then $M+M'=X$, so $M \oplus M' \twoheadrightarrow X$, so X is summand of $M \oplus M'$, a contradiction by Krull-Schmidt. If $f: M \longrightarrow X$ factors as hg then $\mathrm{Im}(h) = M$ or X, and then g is sm or h is se. If $Z \longrightarrow X$ is not se, then not epi, so image contained in M, so it factors through f.

THEOREM. There is a bijection

τ: {non-projective indecomposables}$\longrightarrow${non-injective indecomposables}

such that if X is non-projective indecomposable there is an exact sequence

$$\xi: 0 \longrightarrow \tau X \xrightarrow{f} E \xrightarrow{g} X \longrightarrow 0$$

with f a source map for τX and g a sink map for X. In case A is hereditary we have $\tau = D\mathrm{Ext}^1(-,A)$. Such a sequence ξ is called an <u>Auslander-Reiten sequence</u> or an <u>almost split sequence</u>.

PROOF. We only prove this in case A is hereditary, using results from last term. We know that the translation functor τ induces a bijection, also denoted τ. Define a linear map $\phi: \mathrm{End}(X) \longrightarrow k$ via $\phi(\mathrm{rad}\, \mathrm{End}(X))=0$, $\phi(1_X)=1$. Let $\xi \in \mathrm{Ext}^1(X, \tau X) \cong D\mathrm{End}(X)$ correspond to ϕ. Since $\xi \neq 0$, it follows that g is not se and f is not sm.

If $h: Z \longrightarrow X$ is not se, then it factors through g, for h gives a commutative square

$$\begin{array}{ccc} \mathrm{Ext}^1(X,\tau X) & \longrightarrow & \mathrm{Ext}^1(Z,\tau X) \\ \| & & \| \\ D\mathrm{Hom}(X,X) & \longrightarrow & D\mathrm{Hom}(X,Z) \end{array}$$

and the bottom map sends ϕ to 0 since if $\alpha \in \mathrm{Hom}(X,Z)$ then $\phi(h\alpha)=0$. Thus the pullback of ξ splits.

Now g is irreducible: suppose g factors as $E \xrightarrow{s} Z \xrightarrow{h} X$ with s not sm and h not se. Now h=gt with $t: Z \longrightarrow E$ and $g=g(ts)$. We get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau X & \xrightarrow{f} & E & \xrightarrow{g} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle ts} & & \| & & \\ 0 & \longrightarrow & \tau X & \xrightarrow{f} & E & \xrightarrow{g} & X & \longrightarrow & 0 \end{array}$$

Since s is not sm, the map ts is not an iso, so ψ not iso. Thus $\psi^r=0$, so $(ts)^r$ factors through g, say $(ts)^r=ug$ for some $u: X \longrightarrow E$. Now gug = g, and since g is epi we have $gu = 1_X$, a contradiction.

Thus g is a sink map. Similarly f is a source map.

COROLLARY. Any indecomposable X has a source and a sink map which are unique up to isomorphism.

PROOF. Uniqueness means that if $X\longrightarrow E$ and $X\longrightarrow E'$ are sources, then there is an iso $E\longrightarrow E'$ making a commutative triangle. This follows from the definitions. The existence of a sink follows from the lemma and theorem. The existence of a source follows from the lemma and the theorem applied to $\tau^{-1}X$.

PROPERTIES OF IRREDUCIBLE MAPS.

(1) An irreducible map $f:X\longrightarrow Y$ is mono or epi.

PROOF. Otherwise it factors through $\mathrm{Im}(f)$.

(2) (Exercise) The cokernel of an irreducible mono is indecomposable.

(3) Suppose X is indecomposable and $X\longrightarrow E$ is a source. The irreducible maps starting at X are the compositions $X\longrightarrow E\xrightarrow{se}Y$.

PROOF. If $X\longrightarrow Y$ is irreducible, it factors through source $X\longrightarrow E$, and the map $E\longrightarrow Y$ must then be se. To show a composition is irreducible, show that if $\begin{pmatrix} f \\ g \end{pmatrix}:X\longrightarrow Y\oplus U$ is irreducible then so is $f:X\longrightarrow Y$. Namely, if f factors through V then $\begin{pmatrix} f \\ g \end{pmatrix}$ factors through $V\oplus U$.

(4) Suppose Y is indecomposable and $F\longrightarrow Y$ is sink. The irreducible maps ending at Y are the compositions $X\xrightarrow{sm}F\longrightarrow Y$.

DEFINITION. If X,Y are indecomposable, set

$$\mathrm{rad}(X,Y) = \{f:X\longrightarrow Y \text{ not an isomorphism}\}$$
$$\mathrm{rad}^2(X,Y) = \{f:X\longrightarrow Y \text{ which factor as } hg \text{ with } g \text{ not sm and } h \text{ not se}\}.$$

LEMMA. $\mathrm{rad}^2(X,Y) \subseteq \mathrm{rad}(X,Y)$ and these are subspaces of $\mathrm{Hom}(X,Y)$.

PROOF. If hg is an iso then g is sm and h is se, so we get the inclusion. We have $\mathrm{rad}(X,Y)=\mathrm{rad}\,\mathrm{End}(X)$ if $X\cong Y$, and otherwise $\mathrm{rad}(X,Y)=\mathrm{Hom}(X,Y)$, so it is always a subspace. Let $f_i\in\mathrm{rad}^2(X,Y)$, so $f_i=h_ig_i$ factors through a module Z_i with g_i not sm and h_i not se. Since X,Y are indecomposable it follows that $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}:X\longrightarrow Z_1\oplus Z_2$ is not sm and $(h_1\ h_2):Z_1\oplus Z_2\longrightarrow Y$ not se. Now the factorization

$$f_1+f_2 = (h_1\ h_2)\begin{bmatrix} g1 \\ g2 \end{bmatrix}$$

shows that $\mathrm{rad}^2(X,Y)$ is a subspace.

DEFINITION. $\mathrm{irr}(X,Y) = \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$.

THEOREM. Let X,Y be indecomposable. The following numbers are equal.

(1) $\dim \mathrm{irr}(X,Y)$.

(2) The multiplicity of X as summand of F with $F \longrightarrow Y$ sink map.

(3) The multiplicity of Y as summand of E with $X \longrightarrow E$ source map.

PROOF OF (1)=(2). Let $(f_0\ f_1\ ..\ f_r): Z\oplus X^r \longrightarrow Y$ be a sink map, with Z having no summand isomorphic to X. Clearly $f_i \in \mathrm{rad}(X,Y)$ for $i\geq 1$. We show that $f_1,..,f_r$ give a basis of $\mathrm{irr}(X,Y)$.

Span: say $\phi: X \longrightarrow Y$ belongs to $\mathrm{rad}(X,Y)$. Now ϕ factors through sink map, so can write $\phi = \sum f_i g_i$. For $i\geq 1$ we have $g_i \in \mathrm{End}(X)$, so there are $\lambda_i \in k$ with $g_i - \lambda_i 1_X \in \mathrm{rad}\,\mathrm{End}(X)$. Modulo $\mathrm{rad}^2(X,Y)$ we have

$$\phi = \sum f_i g_i = \textstyle\sum_{i\geq 1} f_i g_i = \sum_{i\geq 1} f_i(g_i - \lambda_i 1_X) + \lambda_i f_i = \sum_{i\geq 1} \lambda_i f_i.$$

Independent: say $\sum_{i\geq 1} \lambda_i f_i \in \mathrm{rad}^2(X,Y)$ with not all $\lambda_i=0$. The map $X \longrightarrow Z\oplus X^r$ sending x to $(\lambda_i x)$ is split mono, so the composition with the sink map is irreducible. Contradiction.

CONSEQUENCES. Let X,Y be indecomposable.

(1) Knowing sources and sinks we can compute all $\mathrm{irr}(X,Y)$. Conversely, if we know all $\mathrm{irr}(X,Y)$ we recover sources and sinks. Namely, X has source and sink maps

$$X \longrightarrow \bigoplus_{Z \text{ indec}} Z^{\dim \mathrm{irr}(X,Z)} \quad \text{and} \quad \bigoplus_{Z \text{ indec}} Z^{\dim \mathrm{irr}(Z,X)} \longrightarrow X$$

(2) If X is non-projective, then $\dim \mathrm{irr}(\tau X,Y) = \dim \mathrm{irr}(Y,X)$.

PROOF. Have source map $\tau X \longrightarrow E$ and sink map $E \longrightarrow X$.

(3) If X,Y are non-projective, then $\dim \mathrm{irr}(\tau X,\tau Y) = \dim \mathrm{irr}(X,Y)$.

DEFINITION. The Auslander-Reiten quiver Γ of A has vertices Γ_0 the set of isoclasses [X] of indecomposable modules (possibly an infinite set), and arrows Γ_1, with the number of arrows from [X] to [Y] being dim irr(X,Y).

A translation quiver is a quiver Γ with subsets $\Gamma_p, \Gamma_i \subseteq \Gamma_0$ and a bijection $\tau: \Gamma_0 \setminus \Gamma_p \longrightarrow \Gamma_0 \setminus \Gamma_i$ with the property that if $x, y \in \Gamma_0$ and $x \notin \Gamma_p$ then the number of arrows $\tau x \longrightarrow y$ = the number of arrows $y \longrightarrow x$.

The AR quiver becomes a translation quiver by defining
Γ_p = the set of vertices [X] with X projective
Γ_i = the set of vertices [X] with X injective
$\tau : \Gamma_0 \setminus \Gamma_p \longrightarrow \Gamma_0 \setminus \Gamma_i$ given by $\tau[X] = [\tau X]$

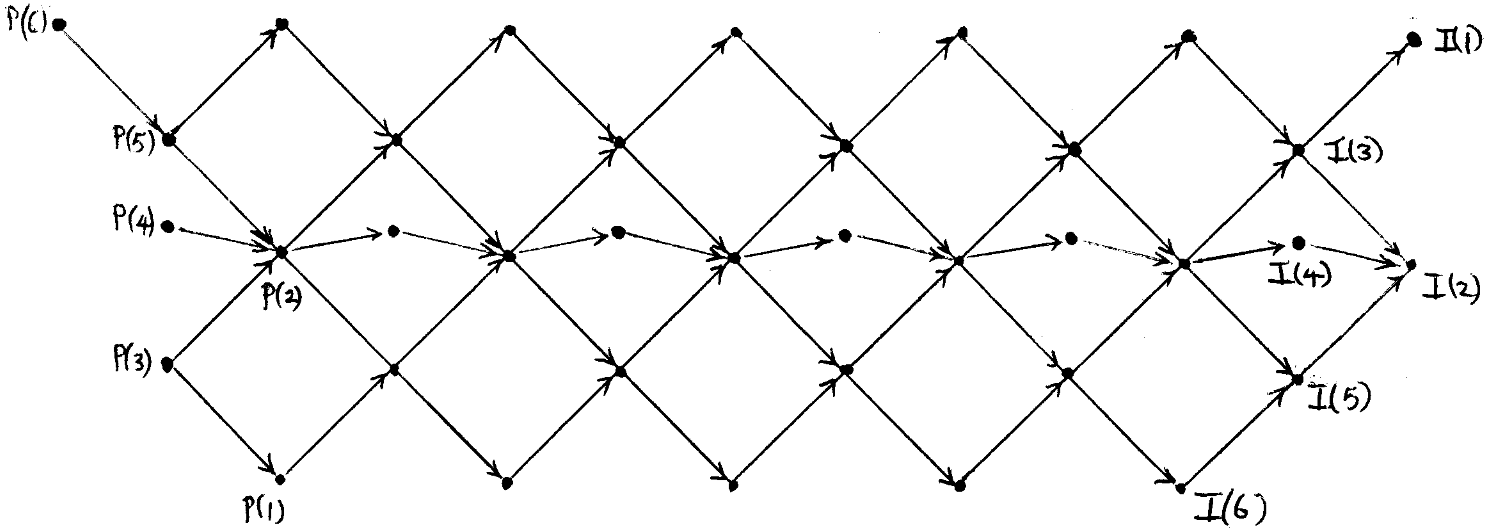
SOME PROPERTIES.

. Only finitely many arrows start or stop at each vertex.
. If x is non-projective, some arrow terminates at x.
. There are no loops (since any irreducible map is mono or epi).
. There are only finitely many projective or injective vertices.

EXAMPLES. See the next pages for some typical examples of AR quivers in the special case when the algebra has finite representation type, so the AR quiver is actually finite. Only the first one comes from a path algebra.

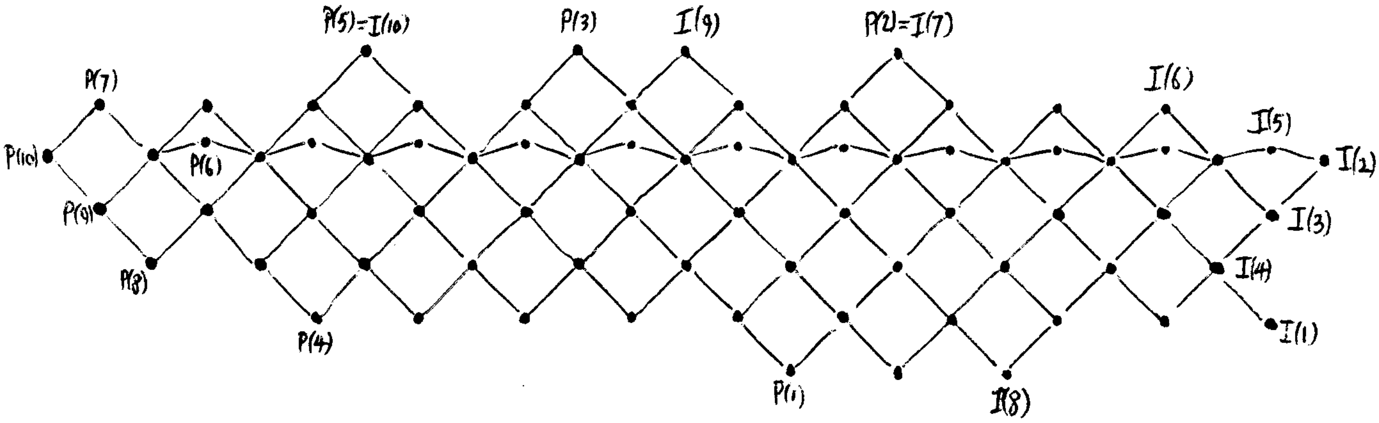
Some finite Auslander-Reiten quivers

$A = kQ$ $\qquad Q =$
$$\begin{tikzcd} 1 \arrow[dr] & & 2 \arrow[dl] \arrow[d] \arrow[dr] & \\ & 3 & 4 & 5 \arrow[dr] \\ & & & & 6 \end{tikzcd}$$



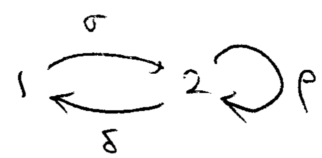
$$A = kQ/(\beta_{i,j}\beta_{i+1,j},\ \beta_{i+1,j+1}\alpha_{i+1,j} - \alpha_{i,j}\beta_{i+1,j})$$

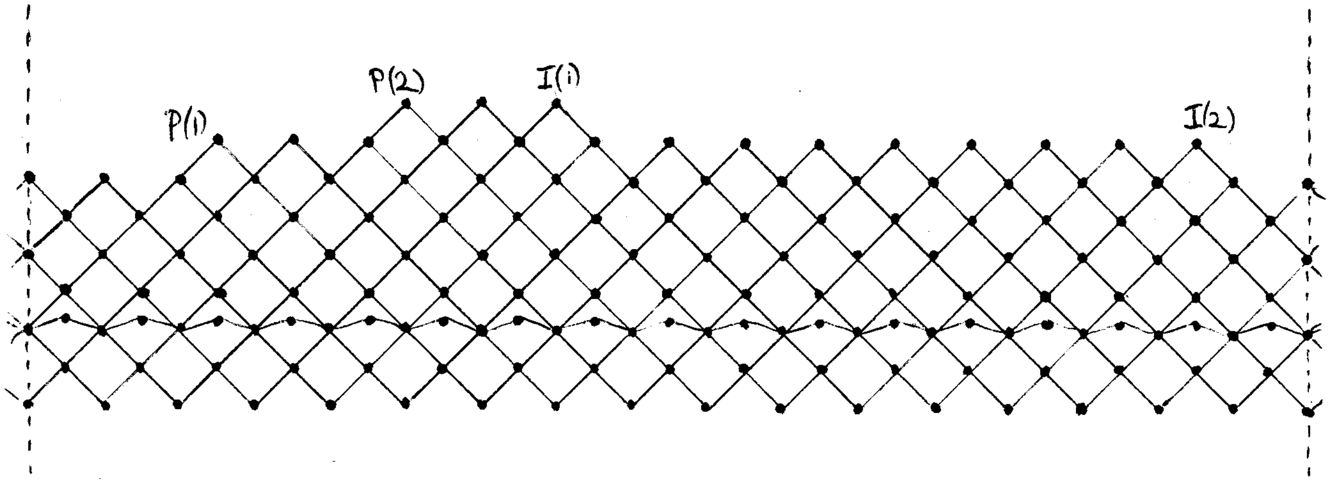
$Q =$
$$\begin{tikzcd} & & 1 \arrow[d, "\beta_{43}"] \\ 2 \arrow[r, "\alpha_{31}"] \arrow[d, "\beta_{31}"] & 3 \arrow[r, "\alpha_{32}"] \arrow[d, "\beta_{32}"] & 4 \arrow[d, "\beta_{33}"] \\ 5 \arrow[r, "\alpha_{21}"] \arrow[d, "\beta_{21}"] & 6 \arrow[r, "\alpha_{22}"] \arrow[d, "\beta_{22}"] & 7 \arrow[d, "\beta_{23}"] \\ 8 \arrow[r, "\alpha_{11}"] & 9 \arrow[r, "\alpha_{12}"] & 10 \end{tikzcd}$$



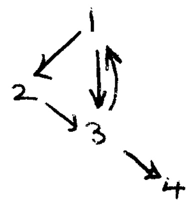
The arrows go from left to right.
τ translates one place to the left.

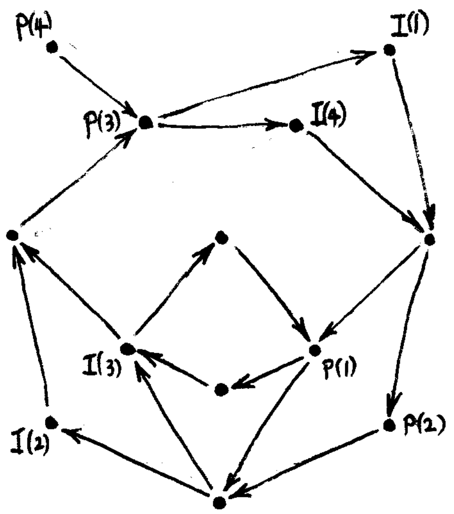
$A = kQ/(\rho^2\sigma, \delta\rho, \delta\sigma, \sigma\delta - \rho^4)$ $\quad Q =$





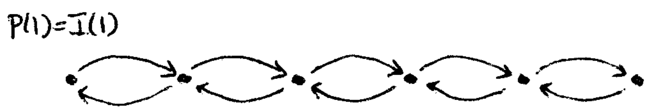
$A = kQ/$(all products of two arrows) $\quad Q =$





$\tau I(1) = P(4)$

$A = k[T]/(T^6)$



$\tau X = X$ for X nonprojective

§2. Preprojective, regular and preinjective modules

In this section $A = kQ$ with Q a <u>connected</u> quiver without oriented cycles. Last term we introduced the notions in the title in case Q is Euclidean. This time we also cover the wild case.

DEFINITIONS.

. X is <u>preprojective</u> if $\tau^i X=0$ for $i>>0$. If X is indecomposable, then $X=\tau^{-r}P(j)$ for uniquely determined $r\geq 0$, j.

. X is <u>preinjective</u> if $\tau^{-i}X=0$ for $i>>0$. If X is indecomposable, then $X=\tau^{r}I(j)$ for uniquely determined $r\geq 0$, j.

. X is <u>regular</u> if no preprojective or preinjective summand.

PROPERTIES.

(1) Let X, Y be indecomposable.

If Y is preprojective and X is not, then $\mathrm{Hom}(X,Y)=\mathrm{Ext}^1(Y,X)=0$.

If Y is preinjective and X is not, then $\mathrm{Hom}(Y,X)=\mathrm{Ext}^1(X,Y)=0$.

PROOF. As before.

(2) A submodule of a preprojective is preprojective.

A submodule of a regular has no preinjective summand.

A quotient of a preinjective is preinjective.

A quotient of a regular has no preprojective summand.

(3) If X, Y are indecomposable and $X\longrightarrow Y$ is irreducible, then X, Y have the same type.

PROOF. It suffices to prove that X preprojective $\Leftrightarrow$ Y preprojective. The implication $\Leftarrow$ is easy. If X is preprojective, either Y projective, or there is an irreducible map $\tau Y\longrightarrow X$, so τY is preprojective. Either way Y is preprojective.

(4) The category of regular modules is closed under images of maps and extensions. The functors τ and τ^- act as inverse equivalences on this category.

LEMMA. If $\tau^{-m}P(i)$ and $\tau^{-r}P(j) \neq 0$, then

$$\dim \operatorname{irr}(\tau^{-m}P(i),\tau^{-r}P(j)) = \begin{cases} \text{number of arrows } j\longrightarrow i & (\text{if } m=r) \\ \text{number of arrows } i\longrightarrow j & (\text{if } r=m+1) \\ 0 & (\text{else}) \end{cases}$$

PROOF. $P(j)=Ae_j$ has unique largest proper submodule with basis the non-trivial paths starting at j, so this submodule is isomorphic to $\oplus_{\rho: j\longrightarrow i} P(i)$. The inclusion is a sink map, so $\operatorname{irr}(P(i),P(j))$ has dimension the number of arrows from j to i in Q.

If $r=0$, for there to be non-zero map we must have $m=0$, so the assertion holds in this case. If $r\neq 0$ then

$$\dim \operatorname{irr}(\tau^{-m}P(i),\tau^{-r}P(j)) = \dim \operatorname{irr}(\tau^{-(r-1)}P(j),\tau^{-m}P(i))$$

and the result follows by induction on $r+m$.

LEMMA. The following are equivalent

(1) Q is Dynkin.

(2) Some $P(i)$ is preinjective.

(3) There is m with $\tau^{m}X=0$ for all X.

(4) Every module is preprojective.

(5) Some indecomposable is both preprojective and preinjective.

PROOF.

(1)⇒(2) If $\tau^{-r}P(i)\neq 0$ for all $r\geq 0$, then these modules are non-isomorphic indecomposables. But Q has finite representation type.

(2)⇒(3) If $j\longrightarrow k$ is an arrow in Q then $\operatorname{irr}(P(k),P(j))\neq 0$. Since $P(i)$ is preinjective and Q is connected, all projectives are preinjective. Thus $\tau^{-m}A=0$ some m. Now $\forall$ X we have $\tau^{m}X\cong \operatorname{Hom}(A,\tau^{m}X)\cong \operatorname{Hom}(\tau^{-m}A,X)=0$.

(3)⇒(1) Every indecomposable has the form $\tau^{-r}P(j)$ with $0\leq r<m$ and $1\leq j\leq n$. Thus finite representation type.

(3)⇒(4) empty.

(4)⇒(5) take an injective.

(5)⇒(2) If X is both, then $\tau^{r}X \cong P(i)$ and $\tau^{-s}X=0$, so $\tau^{-(s+r)}P(i)=0$.

LEMMA. Suppose Q not Dynkin and X is indecomposable.

(1) $\dim \tau^{r}X\longrightarrow\infty$ as $r\longrightarrow\infty$ ⇔ X preinjective or (Q wild and X regular).

(2) $\dim \tau^{-r}X\longrightarrow\infty$ as $r\longrightarrow\infty$ ⇔ X preprojective or (Q wild and X regular).

PROOF. We know the behaviour for Q Euclidean, so suppose Q is wild. If X is not preprojective and dim $\tau^r X$ does not tend to ∞, then some dimension vector must repeat, say

$$\underline{\dim}\ \tau^r X = \underline{\dim}\ \tau^{r+s} X = \alpha \neq 0.$$

Now $c^s\alpha = \alpha$. Thus $\beta=\alpha+c\alpha+\ldots+c^{s-1}\alpha$ is c-invariant, so $\beta\in\mathrm{rad}(q)$, so Q is Euclidean by a lemma from last term.

THEOREM. Suppose Q is wild and X is indecomposable regular with sink map $f:E\longrightarrow X$. If E is decomposable then it can be decomposed as $E=E_1\oplus E_2$ with the E_i indecomposable and dim E_1 < dim X < dim E_2.

PROOF. There is an AR sequence $\xi: 0\longrightarrow\tau X\longrightarrow E\longrightarrow X\longrightarrow 0$. Let $E = \oplus_{i=1}^k E_i$ with the E_i indecomposable.

(1) If $k\geq 2$ then $\theta:E_1\oplus E_2\longrightarrow X$ is epi. Else mono, since irreducible. Now

$$2 \dim \tau X \leq \textstyle\sum_{i=1}^2 \dim \tau E_i + \dim E_i < \dim X + \dim \tau X.$$

The first inequality follows from the fact that the modules E_i belong to AR sequences of the form $0\longrightarrow\tau E_i\longrightarrow\tau X\oplus C_i\longrightarrow E_i\longrightarrow 0$; the second from fact that θ and $\tau\theta:\tau E_1\oplus\tau E_2\longrightarrow\tau X$ are mono since τ is left exact. Thus dim τX < dim X. Since $\tau\theta$ mono the same argument shows dim $\tau^2 X$ < dim τX, etc. Contradiction.

(2) $k\leq 2$. Suppose otherwise. Now $E_1\oplus E_2\longrightarrow X$ is epi, so

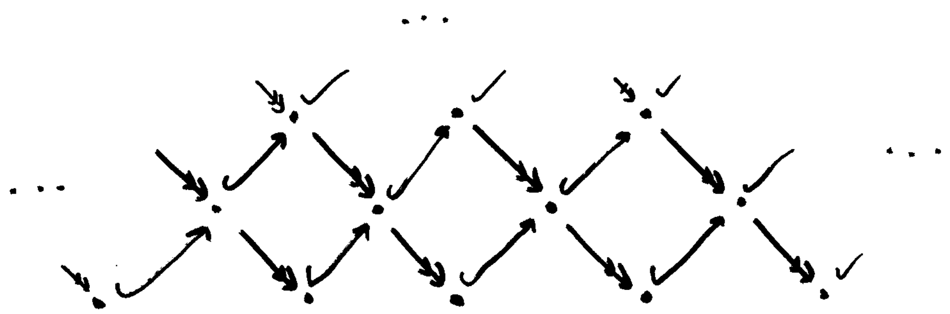
$$\dim E_3 \leq \dim X + \dim \tau X - \dim E_1\oplus E_2 < \dim \tau X,$$

so $\tau X\longrightarrow E_3$ is epi. Similarly $\tau X\longrightarrow E_i$ is epi $\forall$ i. Thus $(\tau X)^2\twoheadrightarrow X$. Now τX has sink map $\tau E\longrightarrow X$ and τE is the sum of $k\geq 3$ terms, so $(\tau^2 X)^2\twoheadrightarrow\tau X$, etc. Thus for all $r\geq 1$ there is s with $(\tau^r X)^s\twoheadrightarrow X$. We can take s=dim X. Now apply the right exact functor τ^{-r}, to get $X^s\twoheadrightarrow\tau^{-r}X$. Thus dim $\tau^{-r}X$ bounded, contradicting the previous lemma.

(3) If k=2 then $E_i\longrightarrow X$ are not both mono. If so, then $\tau X\longrightarrow E_1$ is mono, so $\tau X\hookrightarrow E_1\hookrightarrow X$. Get $\tau^r X\hookrightarrow X$ $\forall r$ as τ left exact. Impossible.

(4) If k=2 then $E_i\longrightarrow X$ are not both epi. Else $\tau X\twoheadrightarrow X$. Get $X\twoheadrightarrow\tau^{-r}X$ $\forall r$. Impossible.

COROLLARY. If Q is wild then a connected component of the AR quiver consisting of regular modules has shape $\mathbb{Z}A_\infty$:



PROOF. Let X be an indecomposable in the component whose sink map $E \longrightarrow X$ has E indecomposable. For example one can take X of minimal dimension in the component. Define $[r]X$ via $[1]X=X$ and by letting $[r]X$ be the unique indecomposable with an irreducible epi $[r]X \twoheadrightarrow [r-1]X$. We leave it as an exercise to show that any indecomposable in the component has form $\tau^i [r]X$.

DEFINITIONS. Let X be indecomposable regular.

. X is <u>quasi-simple</u> if the sink map $E \longrightarrow X$ has E indecomposable, ie X is in the bottom row of the diagram.

. If X is quasi-simple, then $[i]X$ is the module constructed above, with a sequence of irreducible epis $[i]X \twoheadrightarrow [i-1]X \twoheadrightarrow \dots \twoheadrightarrow [1]X=X$.

. The <u>quasi-length</u> of an indecomposable is the number of the row containing the module, so $[i]X$ has quasi-length i.

. The <u>quasi-top</u> of an indecomposable is the quasi-simple reached after a sequence of irreducible epis, so $[i]X$ has quasi-top X.

REMARK. If Q is Euclidean then any regular component of the AR quiver is a tube. If X is regular uniserial with regular socle S and maximal proper regular submodule Y then $Y \hookrightarrow X$ and $X \twoheadrightarrow X/S$ are clearly irreducible.

EXERCISES.

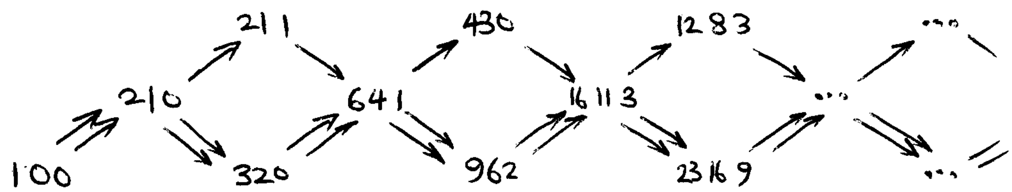
(1) Any connected component of the AR quiver has only finitely many indecomposables of each dimension.

(2) If Q is not Dynkin then the AR quiver has infinitely many connected components.

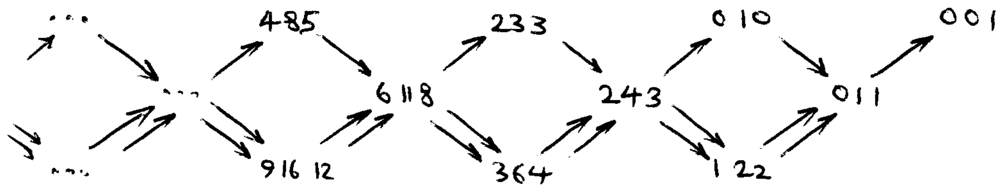
EXAMPLE. See the next pages.

Example $Q = 1 \leftleftarrows 2 \leftarrow 3$

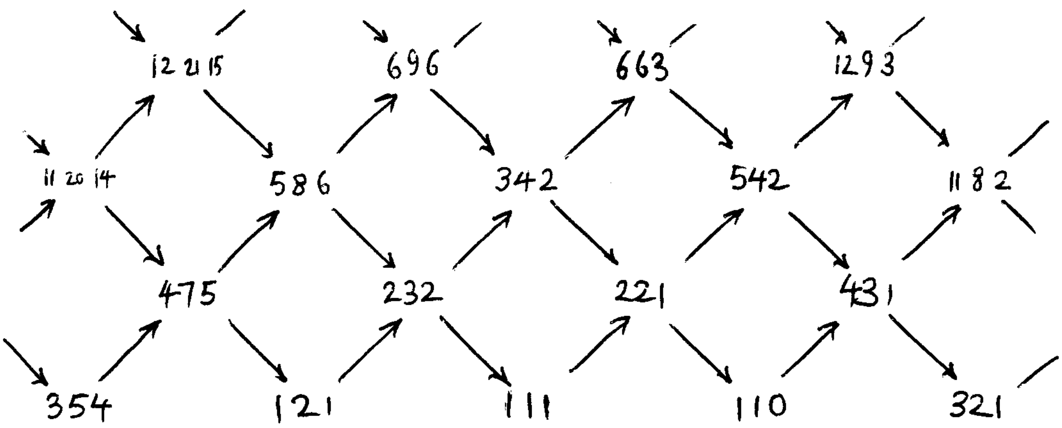
Preprojectives



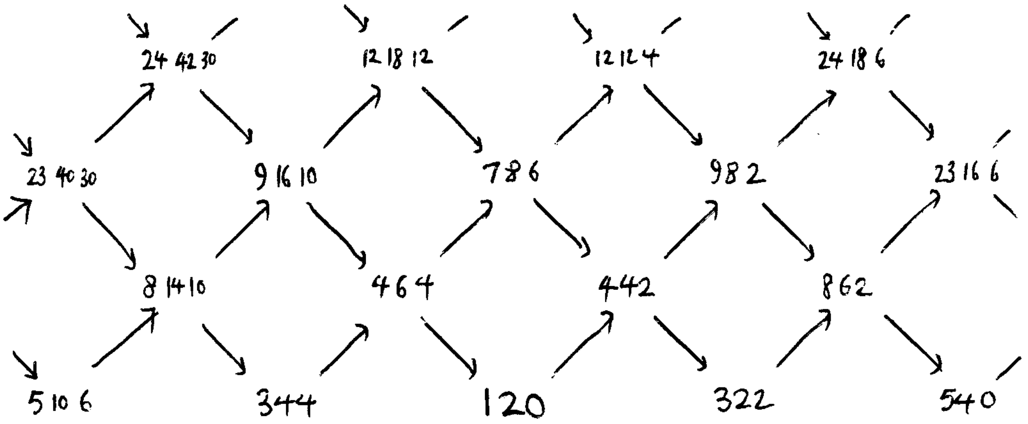
Preinjectives



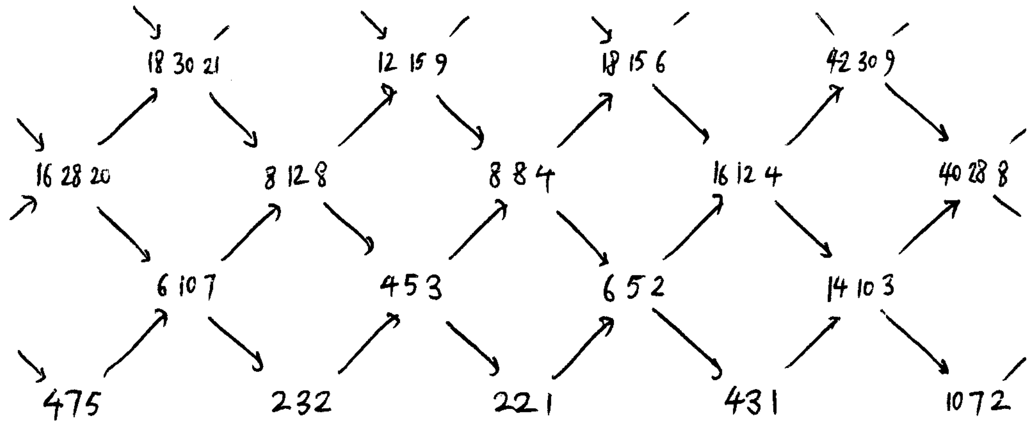
For each $\lambda \in \mathbb{P}^1 k$ there is a component like this



There is only one component of this type



For each $\lambda \in \mathbb{P}^2 k$ there is a component like this



etc.

Taken from C. M. Ringel, Finite dimensional hereditary algebras of wild representation type, Math. Z. 161 (1978) 235–255.

§3. Asymptotic behaviour of the translate.

In this section $A = kQ$ with Q connected wild and without oriented cycles. The first lemma should really be in a different section.

LEMMA. (Without assumption on Q). If $X_1,\dots,X_r$ are non-isomorphic indecomposables and $\mathrm{Ext}^1(X_i,X_j)=0$ $\forall i,j$, then the vectors $\alpha_i=\underline{\dim}\, X_i$ are independent over $\mathbb{Z}$. In particular $r\le n$.

PROOF. Otherwise we have $\sum r_i\alpha_i = \Sigma s_i\alpha_i$ with $r_i,s_i\ge 0$ and not all $r_i=s_i$. Thus $\oplus X_i^{r_i}$ and $\oplus X_i^{s_i}$ are non-isomorphic modules without self extensions and of the same dimension vector. Impossible.

LEMMA. If $X\ne 0$ is regular, there is a non-zero map $\tau^r X\longrightarrow X$ for some $r\ne 0$.

PROOF. We may suppose X indecomposable. The modules $X,\tau^2X,\dots,\tau^{2n}X$ are non-isomorphic indecomposables, so $\mathrm{Ext}^1(\tau^{2i}X,\tau^{2j}X)\ne 0$ some i,j, so $\mathrm{Hom}(\tau^r X,X)\ne 0$ where $r=2j-2i-1$.

LEMMA. If $X\ne 0$ is regular, there is $r\in\mathbb{Z}$ and an exact sequence $0\longrightarrow Z\longrightarrow \tau^r X\longrightarrow Y\longrightarrow 0$ with Y regular and Z having a non-zero preprojective summand. Moreover $\tau^{-i}Z \hookrightarrow \tau^{r-i}X$ for all $i\ge 0$.

PROOF. By induction on $\dim X$.

(1) Suppose there is an exact sequence $0\longrightarrow X'\longrightarrow X\longrightarrow X''\longrightarrow 0$ with X' and X'' non-zero regular. By induction there is an exact sequence $0\longrightarrow Z\longrightarrow \tau^r X'\longrightarrow Y\longrightarrow 0$. Now $Z\longrightarrow \tau^r X'\longrightarrow \tau^r X$ is mono and the cokernel is an extension of Y and X'' so regular. As required.

(2) Suppose there is no such exact sequence. Since $\dim \tau^{-r}X\longrightarrow\infty$ we may assume that $\dim \tau^{-r}X > \dim X$ $\forall$ $r>0$ by replacing X with $\tau^{-i}X$. There is non-zero $f:\tau^s X\longrightarrow X$ for some $s\ne 0$. Now $\mathrm{Im}(f)$ is regular and there is an exact sequence $0\longrightarrow \mathrm{Ker}(f)\longrightarrow \tau^s X\longrightarrow \mathrm{Im}(f)\longrightarrow 0$.

. If Ker(f)=0 and s<0 then $\tau^s X \hookrightarrow X$. A contradiction.

. If Ker(f)=0 and s>0 then τ^s left exact, $\tau^{2s}X \hookrightarrow \tau^s X$, etc. Contradiction.

. If Ker(f) regular then $0 \longrightarrow \tau^{-s}\mathrm{Ker}(f) \longrightarrow X \longrightarrow \tau^{-s}\mathrm{Im}(f) \longrightarrow 0$. Contradiction.

Thus Ker(f) has a nonzero preprojective summand. Finally, since Y is regular we have $0 \longrightarrow \tau^{-i}Z \longrightarrow \tau^{r-i}X \longrightarrow \tau^{-i}Y \longrightarrow 0$ exact.

LEMMA. If $1 \le i, s \le n$ then dim $[\tau^{-r}P(j)]_s$ is unbounded as $r \longrightarrow \infty$.

PROOF. We have AR sequence

$$0 \longrightarrow \tau^{-r}P(i) \longrightarrow \bigoplus_{\rho: i \to j} \tau^{-(r+1)}P(j) \oplus \bigoplus_{\rho: j \to i} \tau^{-r}P(j) \longrightarrow \tau^{-(r+1)}P(i) \longrightarrow 0$$

so if dim $[\tau^{-r}P(i)]_s$ is bounded, then so is dim $[\tau^{-r}P(j)]_s$ for any j connected to i. Since Q is connected dim $[\tau^{-r}A]_s$ bounded. Now

$$\dim (\tau^{-r}A)_s = \dim \mathrm{Hom}(P(s), \tau^{-r}A) = \dim \mathrm{Hom}(\tau^{-r}A, I(s)) = \dim \tau^{r}I(s) \longrightarrow \infty.$$

LEMMA. Almost all preprojectives indecomposables are sincere.

PROOF. The modules zero at vertex i correspond to kQ′-modules, where Q′ is the quiver obtained by deleting i. Now any indecomposable kQ′-module has support on a connected component Q″ of Q′. It suffices to prove that only finitely many indecomposable kQ″-modules are preprojective as kQ-modules. Doing this for all Q″, only finitely many indecomposable kQ′-modules are preprojective for kQ. Thus only finitely many indecomposable preprojectives are zero at i. Now doing this for each i gives the assertion.

If Q″ is Dynkin the assertion is clear, so suppose not Dynkin. We construct a preprojective kQ″-module which is not preprojective as kQ-module.

Since Q is connected, some vertex j of Q″ is connected by an arrow to i. Choose an indecomposable preprojective kQ″-module X with dim $X_j \ge 2$. We can regard X as a kQ-module, and it is still a brick without self extensions. Also $(\varepsilon_i, \underline{\dim}\, X) \le -2$. Now $(\varepsilon_i, \underline{\dim}\, X) = -\dim \mathrm{Ext}^1(S(i), X) - \dim \mathrm{Ext}^1(X, S(i))$. Choose a non-split extension Y of X and S(i), one way around or another. Since S(i) and X are bricks, and Hom(S(i),X) = Hom(X,S(i)) = 0 it follows that Y is a brick. Also

$$q(\underline{\dim}\, Y) = q(\varepsilon_i + \underline{\dim}\, X) = 1 + 1 + (\varepsilon_i, \underline{\dim}\, X) \le 0$$

so Y is not preprojective. Let $Z = \tau^{-}_{kQ''}X$ the inverse translate of X as a

kQ''-module, so $\mathrm{Ext}^1_{kQ''}(Z,X)\neq 0$. Thus $\mathrm{Ext}^1_{kQ}(Z,X)\neq 0$. Also $\mathrm{Hom}(Z,S(i))=0$, and it follows from the long exact sequence that $\mathrm{Ext}^1(Z,Y)\neq 0$. Thus Z is not preprojective for kQ. But Z is a preprojective kQ''-module.

Let W be an indecomposable kQ''-module, preprojective as a kQ-module. If $\tau^r_{kQ}W=0$ and $\ldots\longrightarrow W_2\longrightarrow W_1\longrightarrow W_0=W$ is a path in the AR quiver for kQ'' then $\mathrm{Hom}(W_{i+1},W_i)\neq 0$ so $\tau^r_{kQ}W_i=0$. The path must therefore be finite and cannot include Z. Thus W is preprojective for kQ'' and there is no path $Z\longrightarrow..\longrightarrow W$. There are only finitely many possible W with these properties.

TECHNICAL LEMMA. Let $X\neq 0$ be regular. Define

$$D:\mathbb{Z}\times Q_0\longrightarrow \mathbb{N},\ D(r,j) = \dim\, [\tau^r X]_j$$

and partially order $\mathbb{Z}\times Q_0$ via

$$(r,j)\leq(r',j') \Leftrightarrow r<r' \text{ or } r=r' \text{ and } \mathrm{Hom}(P(j),P(j'))\neq 0.$$

Let $(t,m)\in\mathbb{Z}\times Q_0$ and $M=D(t,m)$. Suppose that

a. $(r,j)\geq(t,m) \Rightarrow D(r,j)\geq M$.

b. $(r,j)<(t,m) \Rightarrow D(r,j)>M$ (resp. $D(r,j)>0$).

If $0\longrightarrow Z\longrightarrow\tau^{-r}P(m)\longrightarrow Y\longrightarrow 0$ is exact with $Z,Y\neq 0$, then $\langle \underline{\dim}\, Y,\underline{\dim}\,\tau^{t-r}X\rangle < 0$ (resp. Y is not preprojective).

PROOF. Since $Z\neq 0$ it has a summand $\tau^{-s}P(j)$. Now $\mathrm{Hom}(\tau^{-s}P(j),\tau^{-r}P(m))\neq 0$ so $(s,j)\leq(r,m)$ and this is strict since $Y\neq 0$. Thus $(s+t-r,j)<(t,m)$. Thus $D(s+t-r,j)>M$ (resp. >0) Now

$$\begin{aligned}\langle \underline{\dim}\, Y,\underline{\dim}\,\tau^{t-r}X\rangle &= \langle \underline{\dim}\,\tau^{-r}P(m),\underline{\dim}\,\tau^{t-r}X\rangle - \langle \underline{\dim}\, Z,\underline{\dim}\,\tau^{t-r}X\rangle \\ &= \dim \mathrm{Hom}(\tau^{-r}P(m),\tau^{t-r}X) - \dim \mathrm{Hom}(Z,\tau^{t-r}X) \\ &\leq M - D(s+t-r,j) < 0 \text{ (resp. } < M).\end{aligned}$$

If Y is preprojective it has summand $\tau^{-u}P(k)$ with $(r,m)\leq(u,k)$, so $(t,m)\leq(u+t-r,k)$, and so $D(u+t-r,k)\geq M$. Thus

$$\begin{aligned}\langle \underline{\dim}\, Y,\underline{\dim}\,\tau^{t-r}X\rangle &= \dim \mathrm{Hom}(Y,\tau^{t-r}X) \\ &\geq \dim \mathrm{Hom}(\tau^{-u}P(k),\tau^{t-r}X) = D(u+t-r,k) \geq M.\end{aligned}$$

Contradiction.

THEOREM (D. Baer). There is a projective $P(m)$ such that any non-zero map $\tau^{-r}P(m)\longrightarrow P$ with P preprojective is mono.

REMARK. m seems to play the role of an extending vertex in Euclidean case. We call $P(m)$ a <u>mono-orbit</u> if it satisfies this condition, although Baer used this term in a more restricted context.

PROOF. Choose a non-zero regular module X.

If $Z\neq 0$ is preprojective then $\tau^{-i}Z$ sincere for $i>>0$. Thus $\tau^{-i}X$ sincere for $i>>0$ by an earlier lemma. Pick $s\leq 0$ such that $\tau^{r}X$ sincere for $r<s$.

Let M be minimum of D restricted to $\{(r,j)\mid r\geq s,\ 1\leq j\leq n\}$. Let (t,m) be minimal realizing M. Now the hypotheses of the technical lemma hold. If $f:\tau^{-r}P(m)\longrightarrow P$ is non-zero and not mono, then there is a sequence $0\longrightarrow \mathrm{Ker}(f)\longrightarrow \tau^{-r}P(m)\longrightarrow \mathrm{Im}(f)\longrightarrow 0$ which contradicts the technical lemma since $\mathrm{Im}(f)$ is preprojective.

THEOREM.

(1) If X is not preinjective then $\dim\,[\tau^{-r}X]_i\longrightarrow\infty$ as $r\longrightarrow\infty$.

(2) If X is not preprojective then $\dim\,[\tau^{r}X]_i\longrightarrow\infty$ as $r\longrightarrow\infty$.

PROOF (1) By a previous lemma we may suppose that X is indecomposable preprojective. Let $P(m)$ be a mono-orbit. We know almost all preprojectives sincere, so there is j_0 with $[\tau^{-j}X]_m\neq 0$ for $j\geq j_0$. Given a number M, by the unboundedness we have $\dim\,[\tau^{-s}P(m)]_i\geq M$ for some s. Now for $r\geq s+j_0$ we have $\mathrm{Hom}(\tau^{-s}P(m),\tau^{-r}X)\neq 0$ so $\tau^{-s}P(m)$ embeds in $\tau^{-r}X$, and so $\dim\,[\tau^{-r}X]_i\geq M$.

THEOREM (D. Baer). If $X,Y\neq 0$ are regular then $\mathrm{Hom}(\tau^{-r}X,Y)\neq 0$ for $r>>0$.

PROOF. We may assume that Y has no proper non-zero regular submodule.

Let M be minimal value of $D(r,j)$ for X. Since only finitely many $D(r,j)=M$, we can choose (t,m) minimal realizing M. Now hypotheses of technical lemma hold.

For $r>>0$ we have $\dim\,\tau^{-r}P(m)>\dim Y$ and $[\tau^{r}Y]_m\neq 0$. For such r, let $f:\tau^{-r}P(m)\longrightarrow Y$ be a non-zero map. Now f is not mono by dimensions. Kernel Z. $\mathrm{Im}(f)$ is regular by mono-orbit property. Thus f is epi by minimality of Y. By the technical lemma $\langle \underline{\dim}\,Y,\underline{\dim}\,\tau^{t-r}X\rangle<0$, so $\mathrm{Ext}^1(Y,\tau^{t-r}X)\neq 0$, so

$\mathrm{Hom}(\tau^{-(r+1-t)}X, Y) \neq 0$.

THEOREM (Kerner). If X, Y are regular then $\mathrm{Hom}(X, \tau^{-r}Y) = 0$ for $r >> 0$.

PROOF (Lukas).

There is r_0 such that $\dim \tau^r Z \geq \dim Y$ for all $r \geq r_0$ and all non-zero regular Z with $\underline{\dim}\, Z \leq \underline{\dim}\, X$. Namely, for any given Z there is r_0, and this number only depends on $\underline{\dim}\, Z$.

If $f: X \longrightarrow \tau^{-r}Y$ then $\mathrm{Im}(f)$ is regular, $\underline{\dim}\, \mathrm{Im}(f) \leq \underline{\dim}\, X$ and $\tau^r \mathrm{Im}(f) \hookrightarrow Y$ since $\mathrm{Im}(f) \hookrightarrow \tau^{-r}Y$. Impossible if $r \geq r_0$ and f is non-zero.

§4. Constructions with modules without self extensions

In this section $A = kQ$ and Q has no oriented cycles. We consider f.d. A-modules. Most of the results generalize to arbitrary f.d. algebras, provided you work with module with no self extensions and projective dimension ≤ 1. We do not pursue this generalization here.

DEFINITIONS. Let X be a module (usually without self exts).

- $\mathrm{add}(X)$ = direct sums of the indecomposable summands of X.
- $\#X$ = number of non-isomorphic indecomposable summands of X. Eg $\#A = n$. Note that $\#X \le n$ if X has no self extensions.
- $\mathrm{gen}(X)$ = modules which are quotients of direct sums of copies of X, the modules <u>generated</u> by X.
- If M is a module then $g_X M = \sum_{\theta\in\mathrm{Hom}(X,M)} \mathrm{Im}\,\theta$. This is clearly the unique largest submodule of M in $\mathrm{gen}(X)$

LEMMA. If X has no self extensions then $g_X(M/g_X M)=0$. Thus g_X defines a torsion theory, with torsion class $\mathrm{gen}(X)$ and torsion-free class $\mathcal{F}_X=\{M\,|\,\mathrm{Hom}(X,M)=0\}$.

PROOF. Have $0\longrightarrow\mathrm{Hom}(X,g_X M)\xrightarrow{f}\mathrm{Hom}(X,M)\longrightarrow\mathrm{Hom}(X,M/g_X M)\longrightarrow\mathrm{Ext}^1(X,g_X M)$. Now f is epi, and the last term is zero since $X^r\twoheadrightarrow g_X M$ and $\mathrm{Ext}^1(X,X^r)=0$. Thus $\mathrm{Hom}(X,M/g_X M)=0$.

LEMMA. If X has no self exts and $\mathcal{E}_X=\{N\,|\,\mathrm{Ext}^1(X,N)=0\}$ then

$$\mathrm{add}(X)=\{M\in\mathrm{gen}(X)\,|\,\mathrm{Ext}^1(M,N)=0\ \forall N\in\mathcal{E}_X\}.$$

In particular, if $\mathrm{gen}(X)=\mathcal{E}_X$ then $\mathrm{add}(X)$ is the relative projectives of $\mathrm{gen}(X)$.

PROOF. Clearly $X \in \mathrm{gen}(X)$ and $\mathrm{Ext}^1(X,N)=0$ for $N\in\mathcal{E}$. Suppose $M\in\mathrm{gen}(X)$ and $\mathrm{Ext}^1(M,N)=0$ $\forall N\in\mathcal{E}_X$. Take a basis of $\mathrm{Hom}(X,M)$ and construct the corresponding map $X^r\longrightarrow M$. This is epi, and if N is the kernel, then $\mathrm{Ext}^1(X,N)=0$. Thus $\mathrm{Ext}^1(M,N)=0$, so $0\longrightarrow N\longrightarrow X^r\longrightarrow M\longrightarrow 0$ splits, so $M\in\mathrm{add}(X)$.

LEMMA. If X has no self extensions, M is a module and $r=\dim \mathrm{Ext}^1(X,M)$, there is <u>universal exact sequence</u> $0\longrightarrow M\longrightarrow u_X M\longrightarrow X^r\longrightarrow 0$, with $\mathrm{Ext}^1(X,u_X M)=0$.
(1) If N is a module with $\mathrm{Ext}^1(X,N)=0$ then $\mathrm{Ext}^1(u_X M,N)\cong\mathrm{Ext}^1(M,N)$.
(2) If M has no self exts and $\mathrm{Ext}^1(M,X)=0$ then $X\oplus u_X M$ has no self exts.
(3) If X is indecomposable then $\mathrm{Hom}(X,M) \cong \mathrm{Hom}(X,u_X M)$.

PROOF. Let $\xi_1,\dots,\xi_r$ be a basis of $\mathrm{Ext}^1(X,M)$. Let $\xi: 0\longrightarrow M\longrightarrow u_X M\longrightarrow X^r\longrightarrow 0$ correspond to (ξ_i) under the isomorphism $\mathrm{Ext}^1(X^r,M) \cong \mathrm{Ext}^1(X,M)^r$. In the long exact sequence

$$\longrightarrow \mathrm{Hom}(X,X^r)\xrightarrow{f}\mathrm{Ext}^1(X,M)\rightarrow\mathrm{Ext}^1(X,u_X M)\rightarrow\mathrm{Ext}^1(X,X^r)=0$$

the map f is epi. Now (1) and (2) are clear. (3) f is iso since X is brick.

THEOREM. If X has no self extensions, the following are equivalent
(1) $\#X = n$.
(2) There is an exact sequence $0\longrightarrow A\longrightarrow X'\longrightarrow X''\longrightarrow 0$ with $X',X''\in\mathrm{add}(X)$.
(3) $\mathrm{gen}(X)=\{N\mid \mathrm{Ext}^1(X,N)=0\}$.
In this case X is called a <u>tilting module</u>.

PROOF.
(1)⇒(2) $X\oplus u_X A$ has no self exts so has $\le n$ summands. By assumption each one already occurs as a summand of X, so $u_X A\in\mathrm{add}(X)$, and we can use the universal exact sequence.

(2)⇒(3) Clearly $\mathrm{gen}(X) \subseteq \{M\mid\mathrm{Ext}^1(X,M)=0\}$. If $\mathrm{Ext}^1(X,M)=0$ then $\mathrm{Hom}(X,M/g_X M)=0$ and $\mathrm{Ext}^1(X,M/g_X M)=0$. Apply $\mathrm{Hom}(-,M/g_X M)$ to get $\mathrm{Hom}(A,M/g_X M)=0$. Thus $M = g_X M \in \mathrm{gen}(X)$.

(3)⇒(1) Let $\mathcal{X} \subseteq \mathbb{Z}^n$ be the $\mathbb{Z}$-linear combinations of the dimension vectors of the summands of X. We have exact sequences $0\longrightarrow P(i)\longrightarrow u_X P(i)\longrightarrow X^r\longrightarrow 0$. Now $\mathrm{Ext}^1(X,u_X P(i))=0$ so $u_X P(i)\in\mathrm{gen}(X)$. Also $u_X P(i)$ is a relative projective of $\mathrm{gen}(X)$, so $u_X P(i)\in\mathrm{add}(X)$ by the lemma. Thus $\underline{\dim}\, P(i)\in\mathcal{X}$, so $\mathcal{X}=\mathbb{Z}^n$, and hence $\#X=n$.

COROLLARY. Any module X without self exts can be enlarged to a tilting module $X\oplus u_X A$, called the <u>Bongartz completion</u>.

REMARK. We are not doing any tilting theory. That subject is too big.

DEFINITION. $X^{\perp}=\{M \mid \mathrm{Hom}(X,M)=\mathrm{Ext}^1(X,M)=0\}$ is the <u>perpendicular</u> <u>category</u> to X.

EXAMPLES.

- $P(i)^{\perp}$ consists of the modules which are zero at i.
- An arrow $\rho: i \longrightarrow j$ gives mono $\theta: P(j) \longrightarrow P(i)$, and $\mathrm{Coker}(\theta)^{\perp}$ consists of representations M with M_{ρ} an iso. Equivalent to reps of quiver in which ρ is shrunk.

LEMMA. $X^{\perp}$ is closed under extensions, images, kernels, cokernels, so is an abelian category. If $N \subseteq M$ are in $X^{\perp}$ and M is relative projective, so is N.

PROOF. Easy.

LEMMA. Suppose X has no self exts. The assignment sending M to $p_X M = u_X M / g_X(u_X M)$ induces a left adjoint to the inclusion of $X^{\perp}$ in A-mod.

PROOF. It is easy to check that $p_X M \in X^{\perp}$. There is a map $f_M: M \hookrightarrow u_X M \twoheadrightarrow p_X M$, and if $N \in X^{\perp}$ it is easy to check that f_M gives an iso $\mathrm{Hom}(p_X M,N) \longrightarrow \mathrm{Hom}(M,N)$. Now we can make p_X into a functor by sending $\theta: M \longrightarrow M'$ to the map in $\mathrm{Hom}(p_X M, p_X M')$ corresponding to $f_{M'} \circ \theta$ in $\mathrm{Hom}(M, p_X M')$.

LEMMA. If X has no self exts and P is projective then $\underline{\dim}\, p_X P - \underline{\dim}\, P$ is a $\mathbb{Z}$-linear combination of the dimension vectors of the indecomposable summands of X. If X is indecomposable, it is $-\langle \underline{\dim}\, X, \underline{\dim}\, P \rangle\, \underline{\dim}\, X$.

PROOF. We have $0 \longrightarrow P \longrightarrow u_X P \longrightarrow X^r \longrightarrow 0$ and $0 \longrightarrow g_X u_X P \longrightarrow u_X P \longrightarrow p_X P \longrightarrow 0$. If $N \in \mathcal{E}_X$ then $\mathrm{Ext}^1(u_X P,N)=0$, so $\mathrm{Ext}^1(g_X u_X P,N)=0$, and hence $g_X u_X P \in \mathrm{add}(X)$ by the lemma. For the last statement apply $\langle \underline{\dim}\, X, - \rangle$ and use the fact that $\langle \underline{\dim}\, X, \underline{\dim}\, X \rangle = 1$.

THEOREM. If X has no self exts there is an equivalence $F_X: kQ_X\text{-mod} \longrightarrow X^{\perp}$ with Q_X a quiver with no oriented cycles and $n-\#X$ vertices. Moreover F_X induces isos $\mathrm{Ext}^i_{kQ_X}(M,N) \cong \mathrm{Ext}^i_A(F_X M, F_X N)$.

PROOF. $p_X A$ is a relative projective generator for $X^{\perp}$, so $X^{\perp}$ is equivalent to a module category $\mathrm{End}(p_X A)$-mod. Submodules of relative projectives are relative projective, so $\mathrm{End}(p_X A)$ is hereditary.

Since k is algebraically closed we have an equivalence $F_X: kQ_X\text{-mod} \longrightarrow X^{\perp}$ for some quiver Q_X without oriented cycles (proof omitted). This functor induces isos on Ext^1 since $X^{\perp}$ is closed under extensions.

Let $\mathcal{X} \subseteq \mathbb{Z}^n$ be the $\mathbb{Z}$-linear combinations of the dimension vectors of indecomposable summands of X, so rk $\mathcal{X} = \#X$. Let $\mathcal{Y}$ be the $\mathbb{Z}$-linear combinations of dimension vectors of modules in $X^{\perp}$. Any such module has a resolution by relative projectives of $X^{\perp}$, so rk $\mathcal{Y}$ is the number of vertices of Q_X. If P is projective then $\underline{\dim}\, p_X P - \underline{\dim}\, P \in \mathcal{X}$ by the lemma, so $\underline{\dim}\, P \in \mathcal{X}+\mathcal{Y}$, so $\mathcal{X}+\mathcal{Y}=\mathbb{Z}^n$. Thus rk $\mathcal{X}$+rk $\mathcal{Y} \geq n$. Also $\langle \mathcal{X},\mathcal{Y} \rangle = 0$ and $\langle -,- \rangle$ induces a non-degenerate bilinear form so rk $\mathcal{X}$+rk $\mathcal{Y} \leq n$ (extend to $\mathbb{Q}$ if it makes you happier).

LEMMA. Let X, Y be indecomposables without self exts. Suppose that
$\mathrm{Hom}(X,Y)=\mathrm{Hom}(Y,X)=\mathrm{Ext}^1(Y,X)=0$, and $r = \dim \mathrm{Ext}^1(X,Y)$.
There is an equivalence $F: kQ_r\text{-mod} \longrightarrow \mathcal{E}(X,Y)$ where

$$Q_r = 1 \overset{r \text{ arrows}}{\rightrightarrows} 2.$$

and $\mathcal{E}(X,Y)$ is the category of modules with a filtration in which the quotients are X and Y. This equivalence sends S(1) to X and S(2) to Y. Moreover F induces isos $\mathrm{Ext}^*_{kQ_r}(M,N) \cong \mathrm{Ext}^*_A(FM,FN)$.

IDEA OF PROOF. Any kQ_r-module M fits in an exact sequence $0 \longrightarrow S(2)^a \longrightarrow M \longrightarrow S(1)^b \longrightarrow 0$, while any module $N \in \mathcal{E}(X,Y)$ fits in an exact sequence $0 \longrightarrow Y^a \longrightarrow N \longrightarrow X^b \longrightarrow 0$. These exact sequences are classified by the same data. etc.

§5. Quasi-period of regular modules

In this section $A = kQ$ with Q wild connected and without oriented cycles.

DEFINITION. If X is a quasi-simple module, then <u>quasi-period</u> X is

$$qpX = \min \{m \in \mathbb{Z} \mid \mathrm{rad}(X, \tau^m X) \neq 0\}$$

Note that $\mathrm{rad}(X,\tau^m X) = \mathrm{Hom}(X,\tau^m X)$ if $m \neq 0$. The theorems of Baer and Kerner say that $\mathrm{Hom}(X,\tau^m X)$ is non-zero for $m>>0$ and is zero for $m<<0$, so the quasi-period is an integer. Also $qp\, X = qp\, \tau^r X$, so we can talk about the quasi-period of a regular AR component by choosing any quasi-simple.

LEMMA. If X is regular and $r>0$ there is no mono or epi $f: X \longrightarrow \tau^{-r}X$.

PROOF. We have used this before. If f is epi then $\tau^{-r}X \twoheadrightarrow \tau^{-2r}X$, etc, a contradiction. If f is mono then $\tau^r X \hookrightarrow X$, $\tau^{2r}X \hookrightarrow \tau^r X$, etc.

LEMMA. If X is a quasi-simple brick, then $\mathrm{Hom}(X,\tau^- X)=0$.

PROOF. Let $0 \neq f: X \longrightarrow \tau^- X$. This is not mono or epi by the lemma. The algebra is hereditary so the map $\mathrm{Ext}^1(\mathrm{Cok}\, f, X) \longrightarrow \mathrm{Ext}^1(\mathrm{Cok}\, f, \mathrm{Im}\, f)$ is epi, and hence we can fill in the module L in the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & X \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \mathrm{Cok}\, f \arrow[r] \arrow[d, equal] & 0 \\
\xi : & 0 \arrow[r] & \mathrm{Im}\, f \arrow[r] & \tau^- X \arrow[r] & \mathrm{Cok}\, f \arrow[r] & 0.
\end{tikzcd}$$

This gives a sequence $0 \longrightarrow X \longrightarrow \mathrm{Im}\, f \oplus L \xrightarrow{p} \tau^- X \longrightarrow 0$, which is not split since $\mathrm{Im}\, f$ is not iso to X or $\tau^- X$. Thus p factors through the sink map $E \longrightarrow \tau^- X$. Say

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "h"] & \mathrm{Im}\, f \oplus L \arrow[r] \arrow[d] & \tau^- X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & E \arrow[r] & \tau^- X \arrow[r] & 0
\end{tikzcd}$$

Now X is a brick and $h \neq 0$ since $E \longrightarrow \tau^- X$ is not se. Thus h is an automorphism, so $E \cong \mathrm{Im}\, f \oplus L$. But E is indecomposable.

LEMMA. Suppose X is quasi-simple and $[r]X$ is a brick.

(1) $\mathrm{Ext}^1([i]X,[j]X)=0$ for $1\le i<r$ and $1\le j\le r$.

(2) $\mathrm{Hom}([i]X,[j]X)=0$ for $1\le i<j\le r$.

PROOF. (1) If $r>1$ then $[r-1]X$ has no self exts, for if $f:[r-1]X\longrightarrow\tau[r-1]X$ is non-zero, then so is the composition $[r]X\twoheadrightarrow[r-1]X\longrightarrow\tau[r-1]X\hookrightarrow[r]X$. Thus $[r-1]X$ is a brick. By induction $[i]X$ has no self extensions (and is brick) for $i<r$. Thus $\mathrm{Ext}^1([i]X,[j]X)=0$ for $j\le i<r$ since $[i]X\twoheadrightarrow[j]X$.

We now show $\mathrm{Ext}^1([i]X,[j]X)=0$ for $i<j\le r$ by induction on $j-i$.

Any non-zero map $[j]X\longrightarrow[i+1]X$ is epi. Namely,

. if $i+1=j$ then $[j]X$ is a brick,

. if $i+1<j$ then $\mathrm{Ext}^1([i+1]X,[j]X)=0$ by induction, so any map $[j]X\longrightarrow[i+1]X$ is mono or epi, so is epi by dimensions.

Now any non-zero map $f:[j]X\longrightarrow\tau[i]X$ composes with the mono $\tau[i]X\longrightarrow[i+1]X$ to give a non-zero map $[j]X\longrightarrow[i+1]X$ which is not an epi. Contradiction. Thus $\mathrm{Ext}^1([i]X,[j]X)=0$.

(2) There is an epi $[j]X\twoheadrightarrow[i]X$ whose composition with a non-zero map $[i]X\longrightarrow[j]X$ is a non-zero non-iso $[j]X\longrightarrow[j]X$. But $[j]X$ is a brick.

THEOREM (Kerner). Let X be quasi-simple and $r\ge1$. Then

$[r]X$ is a brick $\Leftrightarrow$ $\mathrm{qp}\ X\ge r$.

$[r]X$ no self exts $\Leftrightarrow$ $\mathrm{qp}\ X>r$.

PROOF. We show first that if X is quasi-simple then X is a brick $\Leftrightarrow$ $\mathrm{qp}\ X\ge1$. It suffices to show that if X a brick then $\mathrm{Hom}(X,\tau^{-r}X)=0$ for $r>0$. We use induction on r. The case $r=1$ is a lemma, so suppose $r>1$. By hypothesis

$$\mathrm{Ext}^1(\tau^{-r}X,X)\cong D\mathrm{Hom}(X,\tau^{-(r-1)}X)=0$$

so a non-zero map $f:X\longrightarrow\tau^{-r}X$ is mono or epi. Both are impossible.

Now for $r\ge1$ we show the following are equivalent

(1) $[r+1]X$ is a brick.

(2) $[r]X$ has no self extensions.

(3) $\mathrm{qp}\ X>r$.

$(1)\Rightarrow(2)$ is lemma.

$(2)\Rightarrow(3)$ By the lemma $[i]X$ has no self exts for $1\le i\le r$. Now $[i]X \twoheadrightarrow X$ and $\tau^{i-1}X \hookrightarrow [i]X$ so $\tau^{i}X \hookrightarrow \tau[i]X$. Since $\mathrm{Hom}([i]X,\tau[i]X)=0$ we must have $\mathrm{Hom}(X,\tau^{i}X)=0$. Also $\mathrm{rad}(X,X)=0$ since no self exts, and $\mathrm{Hom}(X,\tau^{-j}X)=0$ by the first part of the proof.

$(3)\Rightarrow(1)$ By induction on r we know $[r]X$ is a brick. We have a non-split exact sequence $\xi: 0\longrightarrow\tau^{r}X\longrightarrow[r+1]X\longrightarrow[r]X\longrightarrow 0$. Now $[r]X$ has a filtration with quotients $X,\tau X,\dots,\tau^{r-1}X$ so there are no non-zero maps between $\tau^{r}X$ and $[r]X$. It follows that $[r+1]X$ is brick.

THEOREM (Hoshino). Every quasi-simple has $\mathrm{qp} \le n-1$.

PROOF. Suppose $\mathrm{qp}\ X = p>1$. Now $Y=[1]X\oplus\dots\oplus[p-1]X$ has no self exts and $Y^{\perp} \cong kQ_{Y}\text{-mod}$ contains $[p]X$ which has self extensions. Thus Q_Y has ≥ 2 vertices, so $n - (p-1) \ge 2$.

THEOREM (Ringel). Suppose Q is a connected non-Dynkin quiver. There is a regular tilting module $\Leftrightarrow$ Q wild with $n\ge 3$ vertices.

PROOF (Partly D. Baer).

If Q is Euclidean then the dimension vectors α of the regular modules satisfy $\langle\alpha,\delta\rangle=0$, so we cannot find n linearly independent ones. If Q wild and $n=2$ then Hoshino's Theorem implies no regulars without self exts.

Suppose Q wild and $n\ge 3$. There is vertex i such that after deleting i you get a connected quiver which is not Dynkin. To see this, you need a case-by-case analysis.

Now there is a regular module X without self exts: delete the vertex i and look at the preprojectives for that subquiver. Only finitely many are preprojective or preinjective for Q.

Replacing X by a translate, we may assume $\tau^{i}X$ sincere for all $i\ge 0$. Now $X\oplus u_X A$ is a tilting module and $0\longrightarrow A\longrightarrow u_X A\longrightarrow X^{r}\longrightarrow 0$ so the summands are preprojective or regular. No preprojective summand since

$$\mathrm{Ext}^1(X,\tau^{-i}P(j)) \cong D\mathrm{Hom}(\tau^{-i}P(j),\tau X) \cong D\mathrm{Hom}(P(j),\tau^{i+1}X) \neq 0.$$

REMARK. Some theorems of Kerner and Lukas are as follows.

Theorem (K&L). There are quasi-simples with qp arbitrarily highly negative.

Theorem (K&L). Almost all non-sincere quasi-simples have $qp \leq 2$.

Conjecture (Unger). If X quasi-simple and $\tau^r X$ sincere $\forall r$ then qp $X \leq 2$. Proved by K&L in case eg Q has a multiple arrow.

Theorem (K). There are only finitely many AR components in which the quasi-simples X have the property that with $\mathrm{rad}(X,\tau^i X) \neq 0$ and $\mathrm{rad}(X,\tau^{i+1}X)=0$ for some i, called <u>exceptional</u> <u>components</u>.

Theorem (K). As X varies over the quasi-simples, the function

<u>quasi-rank</u> $X = \min\{m_0 \mid \mathrm{rad}(X,\tau^m X)\neq 0 \;\forall m\geq m_0\} = \max\{m\in\mathbb{Z} \mid \mathrm{rad}(X,\tau^{m-1}X)=0\}$

is bounded (since, apart from the exceptional components, the quasi-rank is the same as the quasi-period, which is bounded by Hoshino).

§6. An algorithm for computing real Schur roots

In this section $A = kQ$ with Q having no oriented cycles. A <u>real Schur root</u> is the dimension vector of an indecomposable module without self extensions. For α a real Schur root there is a unique such module, denoted by $G(\alpha)$. We describe work of Schofield in this section.

EXAMPLE. For the quiver $Q_r = 1 \overset{r \text{ arrows}}{\rightrightarrows} 2$.

If $r=1$ then the real Schur roots are $(1,0)$ $(0,1)$ and $(1,1)$.

If $r=2$ then get preprojectives and preinjectives, so $(m,m\pm1)$.

If $r>2$ then get preprojectives and preinjectives by Hoshino.

We can easily compute all real Schur roots. If $\alpha=(c,d)$ is a sincere real Schur root then the unique predecessor of $G(\alpha)$ in the AR quiver has dimension vector $(rc-d,c)$.

LEMMA. Let X have no self exts and let S be a simple object of $X^{\perp}$ which is not injective as an A-module. If P is a relative projective of $(\tau^{-}S)^{\perp}$ and $\mathrm{Hom}(P,X)=0$, then P is projective.

PROOF. Set $Y=\tau^{-}S\neq0$ since S is not injective. We have $X\in Y^{\perp}$ since $S\in X^{\perp}$. We may assume that P is indecomposable, so a summand of $p_Y A$. As Y is indecomposable and non-projective, $\mathrm{Hom}(Y,u_Y A)\cong\mathrm{Hom}(Y,A)=0$, and hence $p_Y A\cong u_Y A$. Applying τ to the universal exact sequence we get $0\longrightarrow\tau A\longrightarrow\tau(u_Y A)\longrightarrow S^{r}$, and so $\tau P\hookrightarrow S^{r}$. Now $\tau P\in X^{\perp}$, since

. $\mathrm{Hom}(X,\tau P)\cong D\mathrm{Ext}^1(P,X) = 0$ since $P,X\in Y^{\perp}$ and P is relative projective.

. $\mathrm{Ext}^1(X,\tau P)\cong D\mathrm{Hom}(\tau P,\tau X)\cong D\mathrm{Hom}(\tau^{-}\tau P,X) = 0$ since $\mathrm{Hom}(P,X)=0$.

Now τP embeds in the semisimple object S^{r} of $X^{\perp}$. If $\tau P\neq0$ then $\tau P\cong S$, so $P\cong Y\notin Y^{\perp}$, nonsense. Thus P is projective.

THEOREM. If X is indecomposable, without self exts, and not simple then there are indecomposables C,D without self exts, with $\mathrm{Hom}(C,D)=\mathrm{Hom}(D,C)=\mathrm{Ext}^1(D,C)=0$, $r = \dim\mathrm{Ext}^1(C,D) > 0$, and an exact sequence $0\longrightarrow D^{d}\longrightarrow X\longrightarrow C^{c}\longrightarrow0$ with (c,d) a sincere real Schur root for Q_r.

PROOF. Suppose X is supported at s vertices, so $s\geq2$. If $s=2$ we can use the corresponding simples for C and D. The number of arrows connecting the two vertices is r>0, and <u>dim</u> X is a real Schur root for Q_r.

Suppose s>2. Now n-s of the indecomposable injectives lie in $X^{\perp}$. Therefore at most n-s out of n-1 relative simples of $X^{\perp}$ can be injective. Thus at least s-1 relative simples are not injective. Pick one, say S.

Now X belongs to $(\tau^{-}S)^{\perp}$. There are n-s indecomposable projectives with no non-zero map to X. By the lemma, at most n-s indecomposable relative projectives have no non-zero map to X. Thus at least s-1 indecomposable relative projectives have a non-zero map to X. Thus X is supported at ≥2 points as a module for $kQ_{\tau^{-}S}$. Now use induction on n.

DEFINITION. An r-<u>decomposition</u> of $\alpha \in \mathbb{N}^n$ is an expression $\alpha = c\gamma + d\delta$ with

. γ and δ real Schur roots.

. $G(\gamma) \in G(\delta)^{\perp}$, $\mathrm{Hom}(G(\gamma),G(\delta))=0$, $r = \dim \mathrm{Ext}^1(G(\gamma),G(\delta)) > 0$.

. (c,d) a sincere real Schur root for Q_r.

COROLLARY. If $\alpha \in \mathbb{N}^r$ then α is a real Schur root $\Leftrightarrow$ $\alpha = \varepsilon_i$ or α has an r-decomposition (some r).

PROOF. If $\alpha \neq \varepsilon_i$ is a real Schur root then $G(\alpha)$ is not simple so the theorem gives a decomposition.

Conversely suppose α has an r-decomposition $\alpha = c\gamma + d\delta$. Now $\mathcal{E}(G(\gamma),G(\delta))$ is equivalent to kQ_r-mod. The indecomposable kQ_r-module without self exts and dimension vector (c,d) gives an indecomposable A-module without self exts of dimension α.

NEXT we need a numerical criterion for the existence of a decomposition.

LEMMA. If M is module and $\alpha \in \mathbb{N}^n$ then $\{x \in \mathrm{Rep}(\alpha) \mid \mathrm{Hom}(M,R_x)=0\}$ is open.

PROOF. A homomorphism $M \longrightarrow R_x$ is given by linear maps θ_i satisfying certain commutativity conditions. Thus, given any set of linear maps θ_i, the set V_θ of points $x \in \mathrm{Rep}(\alpha)$ such that (θ_i) is a homomorphism is closed. Our set is $\bigcup_{(\theta_i)\neq 0} (\mathrm{Rep}(\alpha) \setminus V_\theta)$ so is open.

REMARK. More generally the function

$$f : \mathrm{Rep}(\alpha)\times\mathrm{Rep}(\beta)\longrightarrow\mathbb{N},\ (x,y) \longmapsto \dim \mathrm{Hom}(R_x,R_y)$$

is upper semicontinuous, ie the set $\{(x,y)\mid f(x,y)<r\}$ is open $\forall r$.

THEOREM. If $\alpha\in\mathbb{N}^n$ then $\alpha = c\gamma+d\delta$ is an r-decomposition $\Leftrightarrow$

. γ and δ are real Schur roots

. $\langle\gamma,\delta\rangle = -r < 0$

. γ is a sum of dimension vectors of simple objects of $G(\delta)^{\perp}$.

. (c,d) is a sincere real Schur root for Q_r.

PROOF. Suppose these conditions hold. There is a semisimple object M of $G(\delta)^{\perp}$ with dimension vector γ. Thus the set of $x\in\mathrm{Rep}(\gamma)$ with $\mathrm{Hom}(G(\delta),R_x)=0$ is non-empty. Now this set and $\mathcal{O}_{G(\gamma)}$ are non-empty open, so they intersect. Thus $\mathrm{Hom}(G(\delta),G(\gamma))=0$. Also

$$\langle\delta,\gamma\rangle = \dim \mathrm{Hom}(G(\delta),M) - \dim \mathrm{Ext}^1(G(\delta),M) = 0,$$

so $\mathrm{Ext}^1(G(\delta),G(\gamma))=0$, so hence $G(\gamma)\in G(\delta)^{\perp}$. Since $\langle\gamma,\delta\rangle < 0$ we have $\mathrm{Ext}^1(G(\gamma),G(\delta))\neq0$. Now any nonzero map $G(\gamma)\longrightarrow G(\delta)$ is mono or epi, and

. if $\exists$ epi there is a non-zero composition $G(\gamma)\twoheadrightarrow G(\delta)\longrightarrow\tau G(\gamma)$.

. if $\exists$ mono there is a non-zero composition $\tau^{-}G(\delta)\longrightarrow G(\gamma)\hookrightarrow G(\delta)$.

Both are impossible, so $\mathrm{Hom}(G(\gamma),G(\delta))=0$.

NEXT we need to know the dimension vectors of simples of $G(\alpha)^{\perp}$.

It suffices to know the relative projectives.

By induction we know all decompositions of α.

THEOREM. If α is a real Schur root, the indecomposable relative projectives in $G(\alpha)^{\perp}$ have dimension vectors

(1) $\underline{\dim}\, P(i) - \langle\alpha,\underline{\dim}\, P(i)\rangle\, \alpha$ with i a vertex with $\alpha_i=0$.

(2) $(rc-d)\gamma + c\delta$ with $\alpha = c\gamma+d\delta$ an r-decomposition.

PROOF.

(1) are dimension vectors of indecomposable relative projectives in $G(\alpha)^{\perp}$: this is the dimension vector of $p_{G(\alpha)}P(i)$. Now $\langle\underline{\dim}\, P(i),\alpha\rangle = 0$, and it follows that $\langle\underline{\dim}\, p_{G(\alpha)}P(i),\underline{\dim}\, p_{G(\alpha)}P(i)\rangle = 1$, so $p_{G(\alpha)}P(i)$ is indecomposable.

(2) are dimension vectors of indecomposable relative projectives in $G(\alpha)^{\perp}$: there is an equivalence $F: kQ_r\text{-mod} \longrightarrow \mathscr{E}(G(\gamma), G(\delta))$. Now $G(\alpha) \cong F(U)$ with $\underline{\dim}\, U = (c,d)$, and U has predecessor V with dimension $(rc-d, c)$. Now $V \in U^{\perp}$ and $V \hookrightarrow U^{r}$. Thus $F(V) \in G(\alpha)^{\perp}$ and $F(V) \hookrightarrow G(\alpha)^{r}$. This implies that F(V) is a relative projective of $G(\alpha)^{\perp}$.

(3) These give enough relative projectives. Suppose α has support s vertices. If s=1 then construction (1) gives enough. If s=2 then (1) and (2) applied to the decomposition by simples give enough. Suppose s>2 and Z is an indecomposable relative projective of $G(\alpha)^{\perp}$. Now $G(\alpha)^{\perp}$ contains at least two relative simples which are not injective. One of these, say S, has Hom(Z,S)=0.

Now Z and $G(\alpha)$ belong to $(\tau^{-}S)^{\perp}$ and Z is relative projective of $G(\alpha)^{\perp}$ computed in $(\tau^{-}S)^{\perp}$. By induction $\underline{\dim}$ Z comes from a decomposition or from an indecomposable relative projective P of $(\tau^{-}S)^{\perp}$ with $\mathrm{Hom}(P, G(\alpha))=0$. But then P is actually projective. Thus $\underline{\dim}$ Z arises from (1) or (2).

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