Degenerations of finite dimensional modules are given by extensions

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Abstract. Let A be a finite dimensional k-algebra over algebraically closed field k and $\operatorname{mod} A$ be the category of finite dimensional left A-modules. We show that a module M in $\operatorname{mod} A$ degenerates to another module N in $\operatorname{mod} A$ if and only if there is an exact sequence $0 \to N \to M \oplus Z \to Z \to 0$ in $\operatorname{mod} A$ for some A-module Z. Moreover, we give a description of minimal degenerations of finite dimensional A-modules.

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1. Introduction and main results

Let A be a finite dimensional associative k-algebra with an identity over an algebraically closed field k and $\operatorname{mod} A$ be the category of finite dimensional left A-modules. If $a_1 = 1, \ldots, a_\alpha$ is a basis of A over k, we have the structure constants a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\operatorname{mod}_A^d(k)$ of d-dimensional unital left A-modules consists of α-tuples $m = (m_1, \ldots, m_\alpha)$ of $d \times d$-matrices with coefficients in k such that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices i and j. Any such α-tuple m corresponds to a d-dimensional module $M \in \operatorname{mod} A$ in the obvious way. The general linear group $\mathrm{Gl}_d(k)$ acts on $\operatorname{mod}_A^d(k)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules in $\operatorname{mod} A$ (see [7]). We denote by $\mathcal{O}(m)$ the $\mathrm{Gl}_d(k)$-orbit of a point m in $\operatorname{mod}_A^d(k)$. By abuse of notation N is a degeneration of M if n belongs to the Zariski closure $\overline{\mathcal{O}(m)}$ of $\mathcal{O}(m)$ in $\operatorname{mod}_A^d(k)$, and we denote this fact by $M \leq_{\deg} N$. Thus $\leq_{\deg}$ is a partial order on the set of isomorphism classes of A-modules of a given (finite) dimension. It was not clear how to characterize $\leq_{\deg}$ in terms of representation theory.

Let M, N, Z be modules in $\operatorname{mod} A$ such that there is an exact sequence in $\operatorname{mod} A$ of one of the forms

$$0 \to N \to M \oplus Z \to Z \to 0 \qquad \text{or} \qquad 0 \to Z \to Z \oplus M \to N \to 0.$$

In [11] Riedtmann proved that then $M \leq_{\deg} N$. We shall show that the reverse implication is also true.

THEOREM 1. *Let m, n be points in* $\operatorname{mod}_A^d(k)$, $d \geq 1$. *Then the following conditions are equivalent:*

(1) $M \leq_{\deg} N$.

(2) There is a short exact sequence $0 \to N \to M \oplus Z \to Z \to 0$ *in* $\operatorname{mod} A$ *for some module* Z *in* $\operatorname{mod} A$.

(3) There is a short exact sequence $0 \to Z \to Z \oplus M \to N \to 0$ *in* $\operatorname{mod} A$ *for some module* Z *in* $\operatorname{mod} A$.

As a direct consequence of Theorem 1 and the proof of Proposition 3.4 in [11] we get

COROLLARY 2. *Let m, n be points in* $\operatorname{mod}_A^d(k)$ *such that* $M \leq_{\deg} N$. *Then there is a nonempty open subset* $\mathcal{C}$ *of* k, *a morphism* $\mu : \mathcal{C} \to \overline{\mathcal{O}(m)}$ *and a point* c_0 *in* $\mathcal{C}$, *such that* $\mu(c_0) = n$ *and* $\mu(c) \in \mathcal{O}(m)$ *for all* $c \neq c_0$.

Following Abeasis and del Fra [1] we may consider another partial order $\leq_{\text{ext}}$ defined as follows:

- $M \leq_{\text{ext}} N$: $\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\operatorname{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.

Then for modules M and N in $\operatorname{mod} A$ the following implication holds:

$$M \leq_{\text{ext}} N \quad \Longrightarrow \quad M \leq_{\deg} N$$

(see [3], [11]). Observe that for any modules M, N in $\operatorname{mod} A$ with $M <_{\text{ext}} N$, the module N is decomposable. Since there exist proper degenerations to indecomposable modules even for very simple representation-finite algebras (see [11]), the reverse implication is not true in general. Our next result concerns degenerations $M <_{\deg} N$ which are not given by a sequence of the form $0 \to N' \to M \to N'' \to 0$ with $N = N' \oplus N''$.

THEOREM 3. *Let* M, N, N', N'' *be modules in* $\operatorname{mod} A$ *such that* $M \leq_{\deg} N$ *and* $N \simeq N' \oplus N''$. *If every exact sequence in* $\operatorname{mod} A$ *of the form*

$$0 \to N' \to W \to N'' \to 0 \quad \text{or} \quad 0 \to N'' \to W \to N' \to 0$$

with $M \leq_{\deg} W$ *is splittable* ($W \simeq N' \oplus N''$), *then there are modules* M', M'' *in* $\operatorname{mod} A$ *such that* $M' \leq_{\deg} N'$, $M'' \leq_{\deg} N''$ *and* $M \simeq M' \oplus M''$.

It will give us the following theorem about minimal degenerations.

THEOREM 4. *Let M, N be modules in* $\operatorname{mod} A$ *such that N is a minimal degeneration of M. Then $M <_{\text{ext}} N$ or there are modules W, $\overline{M}$, $\overline{N}$ in* $\operatorname{mod} A$ *such that $M \simeq W \oplus \overline{M}$, $N \simeq W \oplus \overline{N}$, $\overline{M} <_{\text{deg}} \overline{N}$ and the module $\overline{N}$ is indecomposable.*

As a direct consequence of Theorem 4 we get the following fact.

COROLLARY 5. *The orders $\leq_{\text{ext}}$ and $\leq_{\text{deg}}$ are equivalent for all modules in* $\operatorname{mod} A$ *if and only if for any modules M, N in* $\operatorname{mod} A$ *with $M <_{\text{deg}} N$, the module N is decomposable.*

It is known (see Corollary 2 in [15]) that, if A is an algebra and for any proper degeneration $M <_{\text{deg}} N$ of A-modules the module N is decomposable, then A is tame, that is, the indecomposable A-modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. Recall also that an algebra A is called quasi-tilted if A is of global dimension at most 2 and each indecomposable finite dimensional A-module has projective dimension at most one or injective dimension at most one. The structure of tame quasi-tilted algebras and their module categories has been described by Skowroński in [13]. Then, applying results of [14], we proved in our joint paper ([15], Theorem 3) that for any proper degeneration $M <_{\text{deg}} N$ of modules over a tame quasi-tilted algebra A, the module N is decomposable. Applying Corollary 5 we may reformulate it now as follows.

COROLLARY 6. *Let A be a tame quasi-tilted algebra. Then the orders $\leq_{\text{ext}}$ and $\leq_{\text{deg}}$ are equivalent for all modules in* $\operatorname{mod} A$.

The paper is organized as follows. In Section 2 we give characterisations of splittable exact sequences and introduce the notion of an affine scheme mod_A^d, playing a fundamental role in our proofs of Theorems 1 and 3. Sections 3, 4 and 5 are devoted to the proofs of Theorems 1, 3 and 4, respectively.

For basic background on the topics considered here we refer to [3], [4], [7] and [12]. The author would like to thank C. M. Ringel and A. Skowroński for helpful suggestions and comments during the preparation of this paper. The author also gratefully acknowledges support from the Polish Scientific Grant KBN No. 2 PO3A 012 14 and Sonderforschungsbereich 343 (Universität Bielefeld).

2. Preliminary results

2.1. Throughout the paper A denotes a fixed finite dimensional associative k-algebra with an identity over an algebraically closed field k. We denote by $\operatorname{mod} A$ the category of finite dimensional left A-modules.

Let R be a ring and d', d'' be two natural numbers. We denote by $\mathcal{M}^{\alpha}_{d'\times d''}(R)$ the set of all α-tuple of $d' \times d''$ matrices with coefficients in R (so $\operatorname{mod}^{d}_{A}(k) \subseteq \mathcal{M}^{\alpha}_{d\times d}(k)$). Clearly, $\mathcal{M}^{\alpha}_{d'\times d''}$ is a functor from the category of rings to the category of sets. Let $m = (m_1, \ldots, m_\alpha)$ belongs to $\mathcal{M}^{\alpha}_{d'\times d''}(R)$ and h' (resp. h'') be any $c' \times d'$ (resp. $d'' \times c''$) matrix with coefficients in R, for some natural numbers c', c''. Then we define

$$\begin{aligned} h' \star m &= (h'm_1, \ldots, h'm_\alpha) \in \mathcal{M}^{\alpha}_{c'\times d''}(R), \\ m \star h'' &= (m_1 h'', \ldots, m_\alpha h'') \in \mathcal{M}^{\alpha}_{d'\times c''}(R). \end{aligned}$$

In particular, if $g \in \mathrm{Gl}_d(k)$ and $m \in \operatorname{mod}^{d}_{A}(k)$, then $g \cdot m = g \star m \star g^{-1}$ defines the action of $\mathrm{Gl}_d(k)$ on $\operatorname{mod}^{d}_{A}(k)$ that was mentioned in the introduction.

2.2. For an exact sequence $\Sigma : 0 \to U \to W \to V \to 0$ in $\operatorname{mod} A$ we define an additive function δ_Σ from $\operatorname{mod} A$ to the set of integers as follows:

$$\delta_\Sigma(X) = \dim_k \operatorname{Hom}_A(U \oplus V, X) - \dim_k \operatorname{Hom}_A(W, X),$$

for any module X in $\operatorname{mod} A$. Then the following fact holds:

LEMMA. Let $\Sigma : 0 \to U \to W \to V \to 0$ be an exact sequence in $\operatorname{mod} A$. Then $\delta_\Sigma(X) \geq 0$ for any module X in $\operatorname{mod} A$. Moreover, the following conditions are equivalent:

(1) the sequence Σ is splittable,

(2) $W \simeq U \oplus V$,

(3) $\delta_\Sigma(X) = 0$ for any module X in $\operatorname{mod} A$.

Proof. The exact sequence $\Sigma : 0 \to U \xrightarrow{f} W \xrightarrow{g} V \to 0$ induces the following exact sequence

$$0 \to \operatorname{Hom}_A(V, X) \xrightarrow{g^\star} \operatorname{Hom}_A(W, X) \xrightarrow{f^\star} \operatorname{Hom}_A(U, X),$$

what leads to $\delta_\Sigma(X) \geq 0$, for any module X in $\operatorname{mod} A$. Clearly, the condition (1) implies (2) and the condition (2) implies (3).

Assume that $\delta_\Sigma(X) = 0$ for any module X in $\operatorname{mod} A$. In particular, $\delta_\Sigma(U) = 0$ and hence the sequence

$$0 \to \operatorname{Hom}_A(V, U) \xrightarrow{g^\star} \operatorname{Hom}_A(W, U) \xrightarrow{f^\star} \operatorname{Hom}_A(U, U) \to 0$$

is exact. This implies that $1_U = f^\star(\gamma) = \gamma \circ f$, for some homomorphism γ in $\mathrm{Hom}_A(W, U)$. Then the sequence Σ is splittable and the condition (1) holds.

2.3. Now we give another characterisation of splittable sequences. Let d, d', d'' be natural numbers with $d = d' + d''$ and let u, w, v be points in $\mathrm{mod}_A^{d'}(k)$, $\mathrm{mod}_A^{d}(k)$, $\mathrm{mod}_A^{d''}(k)$, respectively. Then the following lemma holds (see II.2.7 in [8]):

LEMMA. Assume that $w = \begin{bmatrix} u & z \\ 0 & v \end{bmatrix}$ for some $z \in \mathcal{M}^{\alpha}_{d' \times d''}(k)$. Then there is an exact sequence Σ in mod A of the form $0 \to U \to W \to V \to 0$. Moreover, the sequence Σ is splittable if and only if $z = u \star h - h \star v$, for some $d' \times d''$ matrix h with coefficients in k.

2.4. The affine variety $\mathrm{mod}_A^d(k)$ extends to an affine algebraic k-scheme mod_A^d (see, for example [2], [4], [10]). The scheme mod_A^d may be described in the functorial point of view as follows: for any commutative k-algebra R with an identity, the set $\mathrm{mod}_A^d(R)$ consists of α-tuples $m = (m_1, m_2, \ldots, m_\alpha)$ in $\mathcal{M}^{\alpha}_{d \times d}(R)$, such that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices i and j. Clearly, any α-tuple $m \in \mathrm{mod}_A^d(R)$ gives the A-R-bimodule structure on R^d with the natural action of R. The affine algebraic group scheme Gl_d acts on mod_A^d by conjugation.

Let u be a point in $\mathrm{mod}_A^d(R)$ and U be the corresponding A-R-bimodule. Observe that, for any homomorphism $\varphi : R \to S$ of k-algebras, the A-S-bimodule corresponding to the point $\mathrm{mod}_A^d(\varphi)(u)$ in $\mathrm{mod}_A^d(S)$ is isomorphic to $U \otimes_R S$.

2.5. Since any two points on an irreducible variety can be connected by an irreducible curve (see A.I.4.5 in [8]), then we get the following characterization of orbit closure (see Theorem 1.2 in [5], also [9]).

PROPOSITION. Let m and n be any points in $\mathrm{mod}_A^d(k)$. Then $M \leq_{\deg} N$ if and only if there is a discrete valuation k-algebra R with the maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$, whose quotient field K is finitely generated over k of transcendence degree one, and an element $y \in \mathrm{mod}_A^d(R)$ such that

- $\mathrm{mod}_A^d(\tau)(y) = g \cdot (\mathrm{mod}_A^d(\tau\eta)(m)), \quad \text{for some } g \in \mathrm{Gl}_d(K)$
- $\mathrm{mod}_A^d(\pi)(y) = n,$

where η, τ and π are the canonical homomorphisms

$$\begin{array}{ccccc} k & \xrightarrow{\eta} & R & \xrightarrow{\tau} & K \\ & & \downarrow{\scriptstyle\pi} & & \\ & & R/\mathfrak{m} = k. & & \end{array}$$

There is a geometric interpretation of this characterization. Such a k-algebra R and a field K correspond to the local ring of a point c_0 of some nonsingular affine curve $\mathcal{C}$ and the field of rational function on $\mathcal{C}$, respectively (see, for example, I.6 in [6]). Hence, $M \leq_{\deg} N$ if and only if there is a nonsingular affine curve $\mathcal{C}$, a point c_0 in $\mathcal{C}$ and a morphism $\mu : \mathcal{C} \to \overline{\mathcal{O}(m)}$ such that $\mu(c_0) = n$ and $\mu(c) \in \mathcal{O}(m)$ for c in an open dense subset of $\mathcal{C}$ (compare with the proof of Theorem 1.2 in [5]).

3. The proof of Theorem 1

3.1. (2) implies (1) and (3) implies (1), by Proposition 3.4 in [11]. Let m, n be points in $\operatorname{mod}_A^d(k)$, $d \geq 1$. By transposing all matrices in m and n we get points m' and n', respectively, in $\operatorname{mod}_{A'}^d(k)$, where A' denotes an opposite algebra of A. One sees that the module M degenerates to N if and only if the same holds for dual modules M' and N' over A'. Hence, it remains to show that $M \leq_{\deg} N$ implies that there is an exact sequence in $\operatorname{mod} A$ of the form $0 \to N \to M \oplus Z \to Z \to 0$, for some A-module Z.

Thus assume that $M \leq_{\deg} N$. We apply Proposition 2.5 and use the notation introduced there. Let Y be an A-R-bimodule on R^d corresponding to y. Denote by $\operatorname{fin} R$ the category of finite dimensional (over k) R-modules. Since Y_R is a free R-module of finite rank, we have the exact functor $\mathcal{F} = {}_AY \otimes_R (-) : \operatorname{fin} R \to \operatorname{mod} A$. For $i \geq 1$, let $N_i = \mathcal{F}(R/\mathfrak{m}^i) = Y/Y\mathfrak{m}^i \in \operatorname{mod} A$. Since $Mod_A^d(\pi)(y) = n$, then

$${}_AN_1 = {}_A(Y/Y\mathfrak{m}) = {}_A(Y \otimes_R R/\mathfrak{m}) = {}_AN.$$

We fix an element $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since R is a discrete valuation ring, then $\mathfrak{m} = (f)$ and consequently, $\mathfrak{m}^i = (f^i)$, for any $i \geq 1$. Since $\mathcal{F}$ is exact, the following exact sequences:

$$0 \to R/\mathfrak{m} \xrightarrow{\beta_i} R/\mathfrak{m}^{i+1} \xrightarrow{\gamma_i} R/\mathfrak{m}^i \to 0, \quad i \geq 1,$$

where $\beta_i(r + \mathfrak{m}) = f^i \cdot r + \mathfrak{m}^{i+1}$, $\gamma_i(r + \mathfrak{m}^{i+1}) = r + \mathfrak{m}^i$, for all $r \in R$, gives the exact sequence

$$0 \to N_1 \xrightarrow{\mathcal{F}(\beta_i)} N_{i+1} \xrightarrow{\mathcal{F}(\gamma_i)} N_i \to 0, \quad i \geq 1,$$

in $\operatorname{mod} A$. Since $N_1 = N$, it remains to show that $N_{i+1} \simeq M \oplus N_i$ for some $i \geq 1$. The remaining part of this section is devoted to the proof that, for sufficiently large h, there is an A-module isomorphism $N_{h+1} \simeq M \oplus N_h$.

3.2. The assumption that $\operatorname{mod}_A^d(\tau)(y) = g \cdot (\operatorname{mod}_A^d(\tau\eta)(m))$, for some $g \in \mathrm{Gl}_d(K)$, means that the function $g : M \otimes_k K \to Y \otimes_R K$ is an isomorphism of A-K-bimodules. Let $g = [g_{ij}]_{i,j \leq d}$, with $g_{ij} \in K$. Since K is the quotient field of the discrete valuation ring R, there is a number $b \geq 0$ such that $f^b \cdot g_{ij} \in R$, for all $1 \leq i, j \leq d$. Let $\tilde{g} = f^b \cdot g$ and observe that

$$\tilde{g} \cdot ((\tau\eta)_*(m)) = \tilde{g} \star ((\tau\eta)_*(m)) \star \tilde{g}^{-1} = g \star ((\tau\eta)_*(m)) \star g^{-1} = g \cdot ((\tau\eta)_*(m)),$$

where $(\tau\eta)_* = \operatorname{mod}_A^d(\tau\eta)$. Thus we may assume that $b = 0$, $\tilde{g} = g$, and consequently $g_{ij} \in R$, for all $1 \leq i, j \leq d$. Then we get the monomorphism

$$\varphi = g|_{M \otimes_k R} : M \otimes_k R \to Y$$

of A-R-bimodules.

We note that results in the remaining part of this section and their proofs extend without any changes to an arbitrary field k.

3.3. Let Y be an A-R-bimodule which is, as an R-module, free of rank d. Assume that there is a monomorphism $\varphi : M \otimes_k R \to Y$ of A-R-bimodules, for some d-dimensional A-module M. We set $X = \operatorname{im} \varphi$.

LEMMA. There is a natural number t with $Y\mathfrak{m}^t \subseteq X$. In particular, the module Y/X is finite-dimensional.

Proof. Since φ is a monomorphism between free R-modules of rank d (and since R is a principal ideal domain), then Y/X, as an R-module, is isomorphic to

$$R/\mathfrak{m}^{t_1} \oplus R/\mathfrak{m}^{t_2} \oplus \ldots \oplus R/\mathfrak{m}^{t_d}.$$

Thus, the R-module Y/X is annihilated by $\mathfrak{m}^t$, where t is the maximum of $t_1, t_2, \ldots, t_d$. Hence, $Y\mathfrak{m}^t \subseteq X$.

LEMMA 3.4. There exists a natural number s such that $X\mathfrak{m}^s$ is a direct summand of the A-module Y.

Proof. Given a subset $\mathcal{C}$ of R, we denote by $\langle \mathcal{C} \rangle$ the k-subspace generated by $\mathcal{C}$. Let $\mathcal{B}$ be a k-basis of R. For $b \in \mathcal{B}$, let $M_b = \varphi(M \otimes_k \langle b \rangle)$. This is an A-submodule of X and is isomorphic to ${}_AM$. Of course, $X = \bigoplus_{b \in \mathcal{B}} M_b$, and this is a direct decomposition of A-modules.

We devide the proof into several steps.

(1) There exists an A-submodule Z of Y which satisfies $Z + X = Y$ and $Z \cap X = \bigoplus_{b \in \mathcal{V}} M_b$ for some finite subset $\mathcal{V}$ of $\mathcal{B}$.

Take a projective cover $\xi : P \to Y/X$ of the A-module Y/X and lift this map to Y. We obtain an A-module homomorphism $\xi' : P \to Y$, say with image Z_0, and such that $Z_0 + X = Y$. By construction, Z_0 is a finite dimensional A-submodule of Y. Consider $Z_0 \cap X = Z_0 \cap \bigoplus_{b \in \mathcal{B}} M_b$.

Since Z_0 is finite dimensional, there is a finite subset $\mathcal{V}$ of $\mathcal{B}$ such that $Z_0 \cap \bigoplus_{b\in\mathcal{B}} M_b = Z_0 \cap \bigoplus_{b\in\mathcal{V}} M_b$. Let $Z = Z_0 + \bigoplus_{b\in\mathcal{V}} M_b$. Then Z is finite dimensional and $Z \cap X = \bigoplus_{b\in\mathcal{V}} M_b$.

(1') As a consequence: $Y = Z \oplus C$, where $C = \bigoplus_{b\in\mathcal{B}\setminus\mathcal{V}} M_b$.

We are going to replace the direct summand C by an A-submodule C' of Y which contains $X\mathfrak{m}^s$ as a submodule. Actually, $X\mathfrak{m}^s$ will be required to be even a direct summand of C'. This exchange will be done inside X: both C and C' will be A-submodules of X, they will be direct complements of $(Z \cap X)$ in X. First, we deal with the ring R itself.

(2) Let $\mathcal{V}$ be a finite subset of $\mathcal{B}$. Then there is a natural number s and a finite subset $\mathcal{W}$ of $\mathcal{B}$ such that

$$\mathfrak{m}^s \oplus \langle\mathcal{W}\rangle \oplus \langle\mathcal{V}\rangle = R.$$

Since $\langle\mathcal{V}\rangle$ is finite dimensional and $\bigcap_{i\geq 1} \mathfrak{m}^i = 0$, then $\langle\mathcal{V}\rangle \cap \mathfrak{m}^s = 0$, for some natural number s. Consider the subspace $\mathfrak{m}^s \oplus \langle\mathcal{V}\rangle$ of R. Since $\mathcal{B}$ is a K-basis of R, we can find a subset $\mathcal{W}$ of $\mathcal{B}$ as required.

(3) Of course, we may tensor the above decomposition of R with M over k and apply the monomorphism φ. Note first that $\varphi(M \otimes_k \mathfrak{m}^s) = X\mathfrak{m}^s$. Second, denote $\varphi(M \otimes_k \langle\mathcal{W}\rangle) = \bigoplus_{b\in\mathcal{W}} M_b$ by W. Third, recall that $\varphi(M \otimes_k \langle\mathcal{V}\rangle) = \bigoplus_{b\in\mathcal{V}} M_b = Z \cap X$. Altogether we get

$$X\mathfrak{m}^s \oplus W \oplus (Z \cap X) = X.$$

As mentioned above we denote $\bigoplus_{b\in\mathcal{B}\setminus\mathcal{V}} M_b = C$. Let $C' = X\mathfrak{m}^s \oplus W$. Then we see that

$$C \oplus (Z \cap X) = X = C' \oplus (Z \cap X).$$

Note that $Y = C \oplus Z$, according to (1').

(4) It follows that

$$Y = C' \oplus Z = X\mathfrak{m}^s \oplus W \oplus Z,$$

thus $W \oplus Z$ is a direct complement to $X\mathfrak{m}^s$.

Namely, we have $C' \cap Z = C' \cap Z \cap X = 0$, since $C' \subseteq X$. Also, $C' + Z = C' + Z + (Z \cap X) = Z + X = Y$, since $Z \cap X \subseteq Z$. This completes the proof.

PROPOSITION 3.5. Let Y be an A-R-bimodule which is, as an R-module, free of rank d. Assume that there is a monomorphism $\varphi : M \otimes_k R \to Y$ of A-R-bimodules, for some d-dimensional A-module M. Then, for a sufficiently large h, there is an A-module isomorphism

$$Y/Y\mathfrak{m}^{h+1} \simeq Y/Y\mathfrak{m}^h \oplus M.$$

Proof. Recall that $X = \operatorname{im}\varphi$. It follows from Lemma 3.4 that there exists a natural number s such that $X\mathfrak{m}^s$ is a direct summand of Y. Fix a direct complement Z' of $X\mathfrak{m}^s$ in Y. Next observe that, for each $i \geq 0$, $X\mathfrak{m}^{i+1}$ is a direct summand of $X\mathfrak{m}^i$, with a direct complement isomorphic to M. Further, by Lemma 3.3 there is a natural number t with $Y\mathfrak{m}^t \subseteq X$. Write $Z'' = X/Y\mathfrak{m}^t$. For any natural number j, we have $Y\mathfrak{m}^{t+j} \subseteq X\mathfrak{m}^j$. The multiplication by f^j (recall that $\mathfrak{m}^j = (f^j)$) induces an isomorphism $Z'' \xrightarrow{\sim} X\mathfrak{m}^j/Y\mathfrak{m}^{t+j}$.

For any natural number i, consider now the following chain of inclusions:

$$Y\mathfrak{m}^{s+t+i} \subseteq X\mathfrak{m}^{s+i} \subseteq X\mathfrak{m}^s \subseteq Y.$$

It follows from the above remarks that the last two inclusions have direct complements, namely M^i and Z', while the first factor $X\mathfrak{m}^{s+i}/Y\mathfrak{m}^{s+t+i}$ is isomorphic to Z''. Thus we get

$$Y/Y\mathfrak{m}^{s+t+i} \simeq Z'' \oplus M^i \oplus Z'.$$

Therefore,

$$Y/Y\mathfrak{m}^{s+t+i} \simeq Y/Y\mathfrak{m}^{s+t} \oplus M^i.$$

In particular, for $h = s+t$ and $i = 1$, we obtain $Y/Y\mathfrak{m}^{h+1} \simeq Y/Y\mathfrak{m}^h \oplus M$. This completes the proof of Proposition 3.5 and also the proof of Theorem 1.

4. The proof of Theorem 3

4.1. Let d, d', d'' be natural numbers with $d = d' + d''$. Let m, n be points in $\operatorname{mod}_A^d(k)$ such that $M \leq_{\deg} N$, and n', n'' be points in $\operatorname{mod}_A^{d'}(k)$, $\operatorname{mod}_A^{d''}(k)$, respectively, such that $N \simeq N' \oplus N''$. Assume that every exact sequence in $\operatorname{mod} A$ of the form

$$0 \to N' \to W \to N'' \to 0 \quad \text{or} \quad 0 \to N'' \to W \to N' \to 0$$

with $M \leq_{\deg} W$, is splittable. We apply Proposition 2.5 for points m, n, and we use the notation introduced there. Denote by $\pi_i : R \to R/\mathfrak{m}^i$ and $\epsilon_i : R/\mathfrak{m}^{i+1} \to R/\mathfrak{m}^i$, $i \geq 1$, the natural epimorphisms of k-algebras. Clearly, then $\pi_i = \epsilon_i\pi_{i+1}$, for any $i \geq 1$. As in (3.1), let Y be the A-R-bimodule on R^d corresponding to y, let $\mathcal{F}$ be the exact functor ${}_AY \otimes_R (-) : \operatorname{fin} R \to \operatorname{mod} A$, and finally let $N_i = \mathcal{F}(R/\mathfrak{m}^i) \in \operatorname{mod} A$, for any $i \geq 1$. Then $N_1 = N$ and the module N_i corresponds to the point $\operatorname{mod}_A^d(\pi_i)(y)$. Then the following fact holds.

LEMMA 4.2. For any $i \geq 1$, there is a point $y_i \in \operatorname{mod}_A^d(R)$ such that

(1) $y_i = g_i \cdot y$, for some $g_i \in \mathrm{Gl}_d(R)$,

(2) $\mathrm{mod}\,^d_A(\pi_i)(y_i) = \begin{pmatrix} n'_i & 0 \\ 0 & n''_i \end{pmatrix}$, for some elements n'_i in $\mathrm{mod}\,^{d'}_A(R/\mathfrak{m}^i)$ and n''_i in $\mathrm{mod}\,^{d''}_A(R/\mathfrak{m}^i)$,

(3) if $i > 1$, then $\mathrm{mod}\,^{d'}_A(\epsilon_{i-1})(n'_i) = n'_{i-1}$ and $\mathrm{mod}\,^{d''}_A(\epsilon_{i-1})(n''_i) = n''_{i-1}$.

Moreover, $n'_1 = n'$ and $n''_1 = n''$.

Proof. Since $R/\mathfrak{m} = k$, we may assume that $\pi\eta = 1_k$. The isomorphism $N \simeq N' \oplus N''$ means that there is an element $g \in \mathrm{Gl}_d(k)$ such that $g \cdot n = \begin{pmatrix} n' & 0 \\ 0 & n'' \end{pmatrix}$. Let $g_1 = \mathrm{Gl}_d(\eta)(g)$ and $y_1 = g_1 \cdot y$. Then

$$\begin{aligned} \mathrm{mod}\,^d_A(\pi_1)(y_1) &= \mathrm{mod}\,^d_A(\pi)(\mathrm{Gl}_d(\eta)(g) \cdot y) \\ &= \mathrm{Gl}_d(\pi)(\mathrm{Gl}_d(\eta)(g)) \cdot \mathrm{mod}\,^d_A(\pi)(y) \\ &= \mathrm{Gl}_d(\pi\eta)(g) \cdot n = g \cdot n = \begin{pmatrix} n' & 0 \\ 0 & n'' \end{pmatrix}. \end{aligned}$$

Assume now that there is a point $y_i \in \mathrm{mod}\,^d_A(R)$ satisfying the conditions (1), (2) and (3), for some $i \geq 1$. Then $y_i = \begin{bmatrix} y'_i & \tilde{u} \\ \tilde{v} & y''_i \end{bmatrix}$, for some points y'_i, $\tilde{u}$, $\tilde{v}$, y''_i in $\mathcal{M}^\alpha_{d'\times d'}(R)$, $\mathcal{M}^\alpha_{d'\times d''}(R)$, $\mathcal{M}^\alpha_{d''\times d'}(R)$, $\mathcal{M}^\alpha_{d''\times d''}(R)$, respectively. Applying (2) we get

$$\mathcal{M}^\alpha_{d'\times d'}(\pi_i)(y'_i) = n'_i \qquad \text{and} \qquad \mathcal{M}^\alpha_{d''\times d''}(\pi_i)(y''_i) = n''_i.$$

Moreover, since $\ker \pi_i = \mathfrak{m}^i = (f^i)$, then $\tilde{u} = f^i \cdot u$ and $\tilde{v} = f^i \cdot v$, for some $u \in \mathcal{M}^\alpha_{d'\times d''}(R)$ and $v \in \mathcal{M}^\alpha_{d''\times d'}(R)$. Take $g' = \begin{bmatrix} f^i \cdot 1_{d'} & 0 \\ 0 & 1_{d''} \end{bmatrix} \in \mathrm{Gl}_d(K)$ and $l = \begin{bmatrix} y'_i & u \\ f^{2i} \cdot v & y''_i \end{bmatrix} \in \mathrm{mod}\,^d_A(R)$. Then $\mathrm{mod}\,^d_A(\tau)(l)$ equals $g' \cdot \mathrm{mod}\,^d_A(\tau)(y_i)$. We set $w = \mathrm{mod}\,^d_A(\pi)(l)$. Then $w = \begin{bmatrix} n' & \hat{u} \\ 0 & n'' \end{bmatrix} \in \mathrm{mod}\,^d_A(k)$, where $\hat{u} = \mathcal{M}^\alpha_{d'\times d''}(\pi)(u)$. Applying Lemma 2.3 we obtain an exact sequence Σ in $\mathrm{mod}\, A$ of the form $0 \to N' \to W \to N'' \to 0$. Observe that $\mathrm{mod}\,^d_A(\tau)(l) = (g'\mathrm{Gl}_d(\tau)(g_i)) \cdot \mathrm{mod}\,^d_A(\tau)(y) = (g'\mathrm{Gl}_d(\tau)(g_i)g) \cdot \mathrm{mod}\,^d_A(\tau\eta)(m)$, by (1). Then $M \leq_{\mathrm{deg}} W$, by Proposition 2.5. Hence, by our assumptions, the sequence Σ is splittable. Consequently, by Lemma 2.3, there is a $d' \times d''$ matrix $\hat{h}'$ with coefficients in k, such that $\hat{u} = n' \star \hat{h}' - \hat{h}' \star n''$. This implies that $u - y'_i \star h' + h' \star y''_i = f \cdot z'$, for some $z' \in \mathcal{M}^\alpha_{d'\times d''}(R)$, where h' is a $d' \times d''$ matrix with coefficients in R, and these coefficients are the images of the corresponding coefficients of $\hat{h}'$ via the homomorphism η. Dually, we conclude that there is a $d'' \times d'$ matrix h'' with coefficients in R such that $v - y_i'' \star h'' + h'' \star y_i' = f \cdot z''$, for some $z'' \in \mathcal{M}^{\alpha}_{d'' \times d'}(R)$. Consider the $d \times d$ matrix

$$h = \begin{bmatrix} 1_{d'} & f^i \cdot h' \\ 0 & 1_{d''} \end{bmatrix} \cdot \begin{bmatrix} 1_{d'} & 0 \\ f^i \cdot h'' & 1_{d''} \end{bmatrix}$$

with coefficients in R. Then $h \in \mathrm{Gl}_d(R)$, since $\det h = 1_R \in R \setminus \mathfrak{m}$. We set $y_{i+1} = h \cdot y_i$ and $g_{i+1} = h \cdot g_i$. Then $y_{i+1} = g_{i+1} \cdot y$ and the condition (1) holds for $i+1$. Observe that y_{i+1} equals $\begin{bmatrix} 1_{d'} & f^i \cdot h' \\ 0 & 1_{d''} \end{bmatrix} \cdot \begin{bmatrix} 1_{d'} & 0 \\ f^i \cdot h'' & 1_{d''} \end{bmatrix} \star \begin{bmatrix} y_i' & f^i \cdot u \\ f^i \cdot v & y_i'' \end{bmatrix} \star \begin{bmatrix} 1_{d'} & 0 \\ -f^i \cdot h'' & 1_{d''} \end{bmatrix} \cdot \begin{bmatrix} 1_{d'} & -f^i \cdot h' \\ 0 & 1_{d''} \end{bmatrix}$. Multiplying these matrices we get

$$y_{i+1} = \begin{bmatrix} y_i' & f^i(u - y_i' \star h' + h' \star y_i'') \\ f^i(v - y_i'' \star h'' + h'' \star y_i') & y_i'' \end{bmatrix} + f^{2i} \cdot z,$$

for some $z \in \mathcal{M}^{\alpha}_{d \times d}(R)$. Invoking the above equalities we obtain

$$y_{i+1} = \begin{bmatrix} y_i' & 0 \\ 0 & y_i'' \end{bmatrix} + f^{i+1} \cdot \begin{bmatrix} 0 & z' \\ z'' & 0 \end{bmatrix} + f^{2i} \cdot z.$$

Consequently,

$$\operatorname{mod}^d_A(\pi_{i+1})(y_{i+1}) = \begin{bmatrix} n'_{i+1} & 0 \\ 0 & n''_{i+1} \end{bmatrix},$$

where $n'_{i+1} = \mathcal{M}^{\alpha}_{d' \times d'}(\pi_{i+1})(y_i')$ and $n''_{i+1} = \mathcal{M}^{\alpha}_{d'' \times d''}(\pi_{i+1})(y_i'')$. Clearly, n'_{i+1} and n''_{i+1} are elements in $\operatorname{mod}^{d'}_A(R/\mathfrak{m}^{i+1})$ and $\operatorname{mod}^{d''}_A(R/\mathfrak{m}^{i+1})$, respectively, and hence the condition (2) for $i+1$ holds. Moreover,

$$\begin{aligned} \operatorname{mod}^{d'}_A(\epsilon_i)(n'_{i+1}) &= \operatorname{mod}^{d'}_A(\epsilon_i)\mathcal{M}^{\alpha}_{d' \times d'}(\pi_{i+1})(y_i') = \mathcal{M}^{\alpha}_{d' \times d'}(\epsilon_i \pi_{i+1})(y_i') \\ &= \mathcal{M}^{\alpha}_{d' \times d'}(\pi_i)(y_i') = n_i'. \end{aligned}$$

Similarly, $\operatorname{mod}^{d''}_A(\epsilon_i)(n''_{i+1}) = n''_i$. Hence, the condition (3) holds for $i+1$. This finishes the proof.

LEMMA 4.3. For any $i \geq 1$, the A-modules N_i and $N_i' \oplus N_i''$ are isomorphic.

Proof. Applying the conditions (1) and (2) of Lemma 4.2, we get

$$\begin{bmatrix} n_i' & 0 \\ 0 & n_i'' \end{bmatrix} = \operatorname{mod}^d_A(\pi_i)(g_i \cdot y) = \mathrm{Gl}_d(\pi_i)(g_i) \cdot \operatorname{mod}^d_A(\pi_i)(y)$$

$$= \mathrm{Gl}_d(\pi_i)(g_i) \cdot n_i,$$

for some $g_i \in \mathrm{Gl}_d(R)$. But this implies that even as A-$R/\mathfrak{m}^i$-bimodules N_i and $N_i' \oplus N_i''$ are isomorphic.

LEMMA 4.4. For any $i \geq 1$, there are in $\mathrm{mod}\, A$ exact sequences

$$\begin{array}{l} \Sigma_i' : \; 0 \to N_{i+1}' \to N_i' \oplus N_{i+2}' \to N_{i+1}' \to 0, \\ \Sigma_i'' : \; 0 \to N_{i+1}'' \to N_i'' \oplus N_{i+2}'' \to N_{i+1}'' \to 0. \end{array}$$

Proof. Take a natural number $i \geq 1$ and set $j = i + 2$. Since $n_j' \in \mathrm{mod}_A^{d'}(R/\mathfrak{m}^j)$, then N_j' is an A-$R/\mathfrak{m}^j$-bimodule, which as an $R/\mathfrak{m}^j$-module is free of finite rank. Then the functor $\mathcal{F}' = N_j' \otimes_{R/\mathfrak{m}^j} (-) : \mathrm{fin}\,(R/\mathfrak{m}^j) \to \mathrm{mod}\, A$ is exact. Hence the exact sequence

$$0 \to R/\mathfrak{m}^{i+1} \xrightarrow{\beta_i} R/\mathfrak{m}^i \oplus R/\mathfrak{m}^{i+2} \xrightarrow{\gamma_i} R/\mathfrak{m}^{i+1} \to 0$$

in $\mathrm{fin}\,(R/\mathfrak{m}^j)$, where $\beta_i(r + \mathfrak{m}^{i+1}) = (r + \mathfrak{m}^i, f \cdot r + \mathfrak{m}^{i+2})$, $\gamma_i(r + \mathfrak{m}^i, r' + \mathfrak{m}^{i+2}) = f \cdot r - r' + \mathfrak{m}^{i+1}$, for any $r, r' \in R$, gives the exact sequence:

$$0 \to \mathcal{F}'(R/\mathfrak{m}^{i+1}) \xrightarrow{\mathcal{F}'(\beta_i)} \mathcal{F}'(R/\mathfrak{m}^i) \oplus \mathcal{F}'(R/\mathfrak{m}^{i+2}) \xrightarrow{\mathcal{F}'(\gamma_i)} \mathcal{F}'(R/\mathfrak{m}^{i+1}) \to 0$$

in $\mathrm{mod}\, A$. Applying the condition (3) of Lemma 4.2, we obtain

$$\mathrm{mod}_A^{d'}(\epsilon_{i+1})(n_j') = n_{i+1}' \qquad \text{and} \qquad \mathrm{mod}_A^{d'}(\epsilon_i \epsilon_{i+1})(n_j') = n_i'.$$

This implies that

$$\begin{array}{rcl} \mathcal{F}'(R/\mathfrak{m}^i) & = & N_j' \otimes_{R/\mathfrak{m}^j} R/\mathfrak{m}^i \simeq N_i', \\ \mathcal{F}'(R/\mathfrak{m}^{i+1}) & = & N_j' \otimes_{R/\mathfrak{m}^j} R/\mathfrak{m}^{i+1} \simeq N_{i+1}', \\ \mathcal{F}'(R/\mathfrak{m}^{i+2}) & = & N_j' \otimes_{R/\mathfrak{m}^j} R/\mathfrak{m}^j \simeq N_j' = N_{i+2}'. \end{array}$$

Thus there exists an exact sequence Σ_i' in $\mathrm{mod}\, A$ of the form $0 \to N_{i+1}' \to N_i' \oplus N_{i+2}' \to N_{i+1}' \to 0$, and, by symmetry, also the exact sequence $\Sigma_i'' : 0 \to N_{i+1}'' \to N_i'' \oplus N_{i+2}'' \to N_{i+1}'' \to 0$.

4.5. Applying Proposition 3.5 we get a number $h \geq 1$ such that $N_{i+2} \oplus N_i \simeq N_{i+1} \oplus N_{i+1}$ for any $i \geq h$. Moreover, we have the following lemma.

LEMMA. $N_{i+2}' \oplus N_i' \simeq N_{i+1}' \oplus N_{i+1}'$ and $N_{i+2}'' \oplus N_i'' \simeq N_{i+1}'' \oplus N_{i+1}''$, for any $i \geq h$.

Proof. From Lemma 4.4, there are in $\mathrm{mod}\, A$ exact sequences

$$\begin{array}{l} \Sigma_i' \; : 0 \to N_{i+1}' \to N_i' \oplus N_{i+2}' \to N_{i+1}' \to 0, \\ \Sigma_i'' \; : 0 \to N_{i+1}'' \to N_i'' \oplus N_{i+2}'' \to N_{i+1}'' \to 0. \end{array}$$

Then $\delta_{\Sigma'_i}(X) \geq 0$ and $\delta_{\Sigma''_i}(X) \geq 0$, for any module X in $\operatorname{mod} A$, by Lemma 2.2. Applying Lemma 4.3 we get

$$\begin{aligned}\delta_{\Sigma'_i}(X) + \delta_{\Sigma''_i}(X) &= \dim_k \operatorname{Hom}_A(N_{i+1} \oplus N_{i+1}, X) \\ &\quad - \dim_k \operatorname{Hom}_A(N_{i+2} \oplus N_i, X) = 0,\end{aligned}$$

and consequently $\delta_{\Sigma'_i}(X) = 0 = \delta_{\Sigma''_i}(X)$, for any module X in $\operatorname{mod} A$. Then the claim follows from Lemma 2.2.

LEMMA 4.6. There are modules M' and M'' in $\operatorname{mod} A$ such that $N'_{h+1} \simeq M' \oplus N'_h$ and $N''_{h+1} \simeq M'' \oplus N''_h$.

Proof. Applying Lemma 4.5 several times we obtain that

$$\bigoplus_{0 \leq i < l} (N'_{h+i+2} \oplus N'_{h+i}) \simeq \bigoplus_{0 \leq i < l} (N'_{h+i+1} \oplus N'_{h+i+1}),$$

for any $l \geq 1$. If we cancel the common direct summand $\bigoplus_{1 \leq i < l}(N'_{h+i+1} \oplus N'_{h+i})$, then we get an isomorphism of A-modules $N'_{h+l+1} \oplus N'_h \simeq N'_{h+l} \oplus N'_{h+1}$, for any $l \geq 1$. Hence, there is an isomorphism

$$\bigoplus_{1 \leq l < j} (N'_{h+l+1} \oplus N'_h) \simeq \bigoplus_{1 \leq l < j} (N'_{h+l} \oplus N'_{h+1}),$$

for any $j \geq 2$. After the cancellation of the common direct summand $\bigoplus_{2 \leq l < j} N'_{h+l}$, we get an isomorphism $N'_{h+j} \oplus (N'_h)^{j-1} \simeq (N'_{h+1})^j$ of A-modules, for any $j \geq 2$. This implies that

$$\mu(N'_{h+j}, X) + (j-1) \cdot \mu(N'_h, X) = j \cdot \mu(N'_{h+1}, X),$$

for any indecomposable module X in $\operatorname{mod} A$ and $j \geq 2$, where $\mu(Y, X)$ denotes the multiplicity of an indecomposable module $X \in \operatorname{mod} A$ as a direct summand of a module $Y \in \operatorname{mod} A$. Hence,

$$\mu(N'_h, X) \leq \lim_{j \to \infty} \frac{j}{j-1} \cdot \mu(N'_{h+1}, X) = \mu(N'_{h+1}, X),$$

for any indecomposable module X in $\operatorname{mod} A$. This implies that the module N'_h is a direct summand of N'_{h+1}, so $N'_{h+1} \simeq M' \oplus N'_h$ for some $M' \in \operatorname{mod} A$. Similarly, we conclude that there is a module M'' in $\operatorname{mod} A$ such that $N''_{h+1} \simeq M'' \oplus N''_h$.

4.7. The following lemma completes the proof of Theorem 3.
LEMMA. $M' \leq_{\deg} N'$, $M'' \leq_{\deg} N''$ and $M \simeq M' \oplus M''$.

Proof. Applying Proposition 3.5 and Lemmas 4.3, 4.6 we get

$$M\oplus N_h \simeq N_{h+1} \simeq N'_{h+1}\oplus N''_{h+1} \simeq M'\oplus N'_h\oplus M''\oplus N''_h \simeq M'\oplus M''\oplus N_h,$$

and consequently $M \simeq M' \oplus M''$. Up to symmetry, it remains to show that $M' \leq_{\mathrm{deg}} N'$. We set $j = h+1$. As in (4.4) we may consider the exact functor $\mathcal{F}' = N'_j \otimes_{R/\mathfrak{m}^j} (-) : \mathrm{fin}\,(R/\mathfrak{m}^j) \to \mathrm{mod}\,A$. Then the exact sequence

$$0 \to R/\mathfrak{m} \xrightarrow{\beta_i} R/\mathfrak{m}^{i+1} \xrightarrow{\gamma_i} R/\mathfrak{m}^i \to 0$$

(see (3.1)) induces the following exact sequence

$$0 \to \mathcal{F}'(R/\mathfrak{m}) \xrightarrow{\mathcal{F}'(\beta_i)} \mathcal{F}'(R/\mathfrak{m}^{h+1}) \xrightarrow{\mathcal{F}'(\gamma_i)} \mathcal{F}'(R/\mathfrak{m}^h) \to 0.$$

Applying the condition (3) of Lemma 4.2 several times, we obtain

$$\begin{aligned}&\mathrm{mod}\,^{d'}_A(\epsilon_h)(n'_j) = n'_i,\\ &\mathrm{mod}\,^{d'}_A(\epsilon_1 \ldots \epsilon_h)(n'_j) = n'_1 = n'.\end{aligned}$$

Together with Lemma 4.6 this implies that $\mathcal{F}'(R/\mathfrak{m}^h) \simeq N'_h$, $\mathcal{F}'(R/\mathfrak{m}) \simeq N'$ and $\mathcal{F}'(R/\mathfrak{m}^{h+1}) \simeq N'_{h+1} \simeq M' \oplus N'_h$. Then we have an exact sequence

$$0 \to N' \to M' \oplus N'_h \to N'_h \to 0$$

in $\mathrm{mod}\,A$, and so $M' \leq_{\mathrm{deg}} N'$, by Proposition 3.4 in [11].

5. The proof of Theorem 4

Let M, N be modules in $\mathrm{mod}\,A$ such that N is a minimal degeneration of M and $M \not<_{\mathrm{ext}} N$. We proceed by induction on the number of direct summands of a decomposition of N into the direct sum of indecomposable modules. If N is indecomposable, then the claim follows for $\overline{M} = M$, $\overline{N} = N$ and $W = 0$. Assume now that $N \simeq N' \oplus N''$ for some nonzero modules N', N'' in $\mathrm{mod}\,A$. Suppose that there is a nonsplittable exact sequence in $\mathrm{mod}\,A$ of the form

$$0 \to N' \to W \to N'' \to 0 \quad \text{or} \quad 0 \to N'' \to W \to N' \to 0$$

with $M \leq_{\mathrm{deg}} W$. Then $M \leq_{\mathrm{deg}} W <_{\mathrm{ext}} N$ and $M \simeq W$, because N is a minimal degeneration of M. Therefore $M <_{\mathrm{ext}} N$ what contradicts our assumptions. Hence, by Theorem 3, there are modules M', M'' in $\mathrm{mod}\,A$ such that $M' \leq_{\mathrm{deg}} N'$, $M'' \leq_{\mathrm{deg}} N''$ and $M \simeq M' \oplus M''$. Observe that

$$M \simeq M' \oplus M'' \leq_{\mathrm{deg}} N' \oplus M'' \leq_{\mathrm{deg}} N' \oplus N'' \simeq N.$$

Then either $M' \simeq N'$ or $M'' \simeq N''$, since the degeneration $M <_{\mathrm{deg}} N$ is minimal. We may assume that $M'' \simeq N''$. Then we have the minimal degeneration $M' <_{\mathrm{deg}} N'$ and $M' \not<_{\mathrm{ext}} N'$. Hence, the claim follows from our inductive assumption, applied to the minimal degeneration $M' <_{\mathrm{deg}} N'$. This finishes the proof.

References

1. Abeasis,S. and del Fra, A., 1985, Degenerations for the representations of a quiver of type $\mathring{A}_m$, *J. Algebra* 93, 376–412.
2. Bongartz,K., 1991, A geometric version of the Morita equivalence, *J. Algebra* 139, 159–171.
3. Bongartz, K., 1996, On degenerations and extensions of finite dimensional modules, *Advances Math.* 121, 245–287.
4. Gabriel,P., Finite representation type is open, in *Representations of Algebras*, Springer Lecture Notes in Math. 488, 132–155.
5. Grunewald,F. and O'Halloran,J., 1988, A characterization of orbit closure and applications *J. Algebra* 116, 163–175.
6. Hartshorn,R., 1977, *Algebraic geometry*, Springer.
7. Kraft,H., 1982, Geometric methods in representation theory, in *Representations of Algebras*, Springer Lecture Notes in Math. 944, 180–258.
8. Kraft,H., 1984, *Geometrische Methoden in der Invariantentheorie*, Vieweg.
9. Lubotsky,A. and Magid,A., 1985, *Memoirs of the AMS, Vol.336, Varieties of Representations of Finitely Generated Groups.*
10. Procesi,C., 1974, Finite dimensional representations of algebras, *Israel J. Math.* 19, 169–182.
11. Riedtmann,C., 1986, Degenerations for representations of quivers with relations, *Ann. Sci. École Normale Sup.* 4, 275–301.
12. Ringel,C.M., 1984, *Springer Lecture Notes in Math., Vol.1099, Tame Algebras and Integral Quadratic Forms.*
13. Skowroński,A., 1998, Tame quasi-tilted algebras, *J. Algebra* 203, 470–490.
14. Skowroński,A. and Zwara,G., 1996, On degenerations of modules with nondirecting indecomposable summands, *Canad. J. Math.* 48, 1091–1120.
15. Skowroński,A. and Zwara,G., 1998, Degenerations for indecomposable modules and tame algebras, *Ann. Sci. École Normale Sup.* 31, 153–180.