Rigid integral representations of quivers

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Dedicated to the memory of Maurice Auslander

We study representations of a quiver by means of finitely generated free modules for a principal ideal domain. In particular we determine all representations $X$ with $\text{Ext}^1(X, X) = 0$.

1. Introduction

Let $Q$ be a finite quiver and let $R$ be a principal ideal domain. By an $RQ$-lattice we mean a representation of $Q$ by means of finitely generated free $R$-modules; equivalently it is an $RQ$-module which is finitely generated and free over $R$. If $Q$ has vertex set $\{1, \ldots, n\}$, the rank vector $\text{rank} X \in \mathbb{N}^n$ of a lattice $X$ gives the rank of the $R$-module attached to each vertex. A lattice is rigid if $\text{Ext}^1(X, X) = 0$, and is exceptional if in addition $\text{End}(X) = R$. A lattice is absolutely indecomposable if $X^K = X \otimes_R K$ is an indecomposable $KQ$-module for each homomorphism $R \to K$ to an algebraically closed field.

In this paper we use an action of the braid group to describe the exceptional lattices. We then use exceptional lattices to compute all rigid lattices. At the end we prove some special cases of the following conjecture: for each positive real root $\alpha \in \mathbb{N}^n$ there is a unique absolutely indecomposable lattice of rank vector $\alpha$.

2. Exceptional lattices

By an $R$-field we mean a homomorphism of $R$ into a field $K$. We normally only consider algebraically closed $R$-fields, meaning that $K$ is algebraically closed. We write $M^K$ for $M \otimes_R K$. If $M$ is an $RQ$-module, then we consider $M^K$ as a $KQ$-module. Since $R$ is a principal ideal domain, a f.g. $R$-module $M$ is free if and

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only if $M^K$ has constant dimension for all algebraically closed $R$-fields $K$, and a homomorphism $\theta : M \to N$ between f.g. free $R$-modules is an isomorphism (resp. an epimorphism, resp. a split monomorphism) if and only if $\theta^K : M^K \to N^K$ is an isomorphism (resp. an epimorphism, resp. a monomorphism) for all $K$.

**Lemma 1.** If $X$ is an $RQ$-lattice, then it has projective dimension at most 1 and $\text{Ext}_{RQ}^1(X, Y)^K \cong \text{Ext}_{KQ}^1(X^K, Y^K)$ for all $RQ$-modules $Y$ and algebraically closed $R$-fields $K$. In particular $X$ is rigid if and only if $X^K$ is a rigid $KQ$-module for all algebraically closed $R$-fields $K$.

**Proof.** There is a standard exact sequence

$$0 \to RQ \otimes_S B \otimes_S RQ \to RQ \otimes_S RQ \to RQ \to 0,$$

where $S$ is the $R$-subalgebra of $RQ$ with basis the trivial paths $e_1, \ldots, e_n$, and $B$ is the free $R$-submodule of $RQ$ with basis the arrows. Applying $- \otimes_{RQ} X$ gives

$$0 = \text{Tor}_{1}^{RQ}(RQ, X) \to RQ \otimes_S B \otimes_S X \to RQ \otimes_S X \to X \to 0,$$

a projective resolution. The statement about extensions holds since if $P$ is a f.g. projective $RQ$-module then $\text{Hom}_{RQ}(P, Y)^K \cong \text{Hom}_{KQ}(P^K, Y^K)$. □

For $\alpha, \beta \in \mathbb{N}^n$ the Ringel form is defined by

$$\langle \alpha, \beta \rangle = \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{a:i \rightarrow j} \alpha_i \beta_j,$$

and $\text{ext}(\alpha, \beta) \in \mathbb{N}$ is defined inductively by $\text{ext}(\alpha, 0) = \text{ext}(0, \beta) = 0$ and

$$\text{ext}(\alpha, \beta) = \max \left\{ -\langle \alpha', \beta' \rangle \middle| \begin{array}{c} 0 \leq \alpha' \leq \alpha, \\ 0 \leq \beta' \leq \beta, \\ \text{ext}(\alpha' - \alpha, \beta' - \beta') = 0 \end{array} \right\}.$$

By the results of [S3] and [C2] this is the general value of $\dim \text{Ext}_{KQ}^{1}(M, N)$ with $M$ and $N$ running through the varieties of $KQ$-modules of dimension vectors $\alpha$ and $\beta$, where $K$ is any algebraically closed field. In particular if $M$ and $N$ are rigid $KQ$-modules then $\dim \text{Ext}_{KQ}^{1}(M, N) = \text{ext}(\alpha, \beta)$ since they correspond to open orbits in these varieties.

**Lemma 2.** If $X$ and $Y$ are rigid lattices with rank vectors $\alpha$ and $\beta$, then $\text{Ext}_{RQ}^{1}(X, Y)$ and $\text{Hom}_{RQ}(X, Y)$ are free $R$-modules. They have ranks $\text{ext}(\alpha, \beta)$ and $\text{hom}(\alpha, \beta) = \langle \alpha, \beta \rangle + \text{ext}(\alpha, \beta)$ respectively.

**Proof.** For all algebraically closed $R$-fields $K$ the modules $X^K$ and $Y^K$ are rigid $KQ$-modules of dimension vectors $\alpha$ and $\beta$, so $\dim \text{Ext}_{KQ}^{1}(X^K, Y^K) = \text{ext}(\alpha, \beta)$ by the remarks above. By Lemma 1 it follows that $\text{Ext}_{RQ}^{1}(X, Y)$ is free over $R$ of rank $\text{ext}(\alpha, \beta)$. If $0 \to P \to P' \to X \to 0$ is a projective resolution of $X$ then the long exact sequence

$$0 \to \text{Hom}_{RQ}(X, Y) \to \text{Hom}_{RQ}(P', Y) \to \text{Hom}_{RQ}(P, Y) \to \text{Ext}_{RQ}^{1}(X, Y) \to 0$$
consists of free $R$-modules, so remains exact on tensoring with $K$. It follows that 
the natural map $\text{Hom}_{R}(X,Y)^{K} \to \text{Hom}_{K}(X^{K},Y^{K})$ is an isomorphism. Now 
$\dim \text{Hom}_{K}(X^{K},Y^{K}) = \text{hom}(\alpha, \beta)$ since the Ringel form gives the difference 
in the dimensions of Hom and Ext$^{1}$, and the assertion about $\text{Hom}_{R}(X,Y)$ follows. □

**Lemma 3.** There is at most one exceptional lattice of each rank vector.

**Proof.** Say $X$ and $Y$ are exceptional lattices of rank $\alpha$. Now $\text{hom}(\alpha, \alpha) = 1$ 
since $\text{End}_{R}(X) \cong R$, and then $\text{Hom}_{R}(X,Y) \cong R$ since $R$ is a principal ideal 
domain. Let $\theta : X \to Y$ be a generator. If $K$ is an algebraically closed $R$-field, then the map $\theta^{K} : X^{K} \to Y^{K}$ is nonzero. Since there is at most one 
rigid $KQ$-module of any given dimension vector we have $X^{K} \cong Y^{K}$. Now 
$\text{Hom}_{K}(X^{K},Y^{K})$ is one-dimensional, so $\theta^{K}$ is an isomorphism. It follows that 
$\theta$ is an isomorphism. □

By an *exceptional sequence* $(X_{1}, \ldots, X_{r})$ of length $r$ we mean a sequence of 
exceptional lattices with $\text{Hom}(X_{i},X_{j}) = \text{Ext}^{1}(X_{i},X_{j}) = 0$ for $i > j$. If $(X,Y)$
is an exceptional sequence and $X$ and $Y$ have ranks $\alpha, \beta$, then the mutations 
$L_{X}Y$ and $R_{Y}X$ are defined as follows. We write $D$ for $\text{Hom}_{R}(-,R)$, and use 
Lemma 2 to ensure that the universal constructions exist.

1. If $\text{Hom}(X,Y) = 0$ then $L_{X}Y$ and $R_{Y}X$ are defined by the universal exact 
sequences

\[0 \to Y \to L_{X}Y \to X \otimes_{R} \text{Ext}^{1}(X,Y) \to 0\]

\[0 \to Y \otimes_{R} D\text{Ext}^{1}(X,Y) \to R_{Y}X \to X \to 0.\]

2. Suppose that $\text{Hom}(X,Y) \neq 0$. If $\beta \geq \langle \alpha, \beta \rangle \alpha$ then $L_{X}Y$ is the cokernel of 
the universal map

\[X \otimes_{R} \text{Hom}(X,Y) \to Y,\]

and otherwise it is the kernel. If $\alpha \geq \langle \alpha, \beta \rangle \beta$ then $R_{Y}X$ is the kernel of 
the universal map

\[X \to Y \otimes_{R} D\text{Hom}(X,Y),\]

and otherwise it is the cokernel.

**Lemma 4.** $(L_{X}Y,X)$ and $(Y,R_{Y}X)$ are exceptional sequences, and 

\[\text{rank } L_{X}Y = \pm(\beta - \langle \alpha, \beta \rangle \alpha), \quad \text{rank } R_{Y}X = \pm(\alpha - \langle \alpha, \beta \rangle \beta).\]

**Proof.** Passing to an algebraically closed $R$-field $K$, these are the standard 
constructions of the mutations in [C1], cf. [Ru]. The assertion follows. For 
example if $\text{Hom}(X,Y) \neq 0$ and $\alpha \geq \langle \alpha, \beta \rangle \beta$ we have an exact sequence

\[0 \to R_{Y}X \to X \to Y \otimes_{R} D\text{Hom}(X,Y) \to C \to 0,\]

and hence

\[X^{K} \to [Y \otimes_{R} D\text{Hom}(X,Y)]^{K} \to C^{K} \to 0.\]
Now the standard mutation is

\[ 0 \rightarrow R_Y K X^K \rightarrow X^K \rightarrow Y^K \otimes D_K \text{Hom}(X^K, Y^K) \rightarrow 0 \]

where \( D_K \) is the duality with \( K \). Thus \( C^K = 0 \), so \( C \) is zero. It follows that

\[ (R_Y X)^K \cong R_Y K X^K, \]

so \( R_Y X \) is an exceptional lattice of rank \( \alpha - \langle \alpha, \beta \rangle \beta \).

Recall that the braid group on \( r \) strings is generated by \( \sigma_1, \ldots, \sigma_{r-1} \) subject to the relations \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( i \neq j \pm 1 \) and \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \). By Lemma 3 and \([C1]\) we obtain

Lemma 5. The braid group on \( r \) strings acts naturally on the set of exceptional sequences of length \( r \) via the assignments

\[ \sigma_i(X_1, \ldots, X_r) = (X_1, \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, \ldots, X_r) \]

\[ \sigma_i^{-1}(X_1, \ldots, X_r) = (X_1, \ldots, X_{i-1}, L_X, X_{i+1}, X_i, X_{i+2}, \ldots, X_r). \]

Recall that \( \alpha \in \mathbb{N}^n \) is a real Schur root if over an algebraically closed field there is an exceptional representation of dimension vector \( \alpha \). By Kac's canonical decomposition the real Schur roots can be characterized as the \( \alpha \in \mathbb{N}^n \) with \( \langle \alpha, \alpha \rangle = 1 \) and with no non-trivial decomposition \( \alpha = \beta + \gamma \) with \( \beta, \gamma \in \mathbb{N}^n \) and \( \text{ext}(\beta, \gamma) = \text{ext}(\gamma, \beta) = 0 \), see \([K1, \S4]\). This characterization shows that the real Schur roots do not depend on the field.

Theorem 1. The assignment \( X \mapsto \text{rank} X \) induces a 1-1 correspondence between exceptional lattices and real Schur roots.

Proof. Thanks to lemma 3 we only need to show that the rank vectors of the exceptional lattices are the real Schur roots. Let \( K \) be an algebraically closed \( R \)-field. If \( X \) is an exceptional lattice of rank \( \alpha \) then \( X^K \) is an exceptional \( KQ \)-module, so \( \alpha \) is a real Schur root. On the other hand, if \( \alpha \) is a real Schur root then there is an exceptional \( KQ \)-module of dimension \( \alpha \). This implies that the full subquiver of \( Q \) on the support of \( \alpha \) has no oriented cycles (for otherwise one can construct representations of dimension \( \alpha \) on which the trace of the oriented cycle is arbitrary, but the variety of representations of dimension \( \alpha \) has a dense orbit, so the trace must be constant). Thus, passing to the support of \( \alpha \) we may suppose that \( Q \) has no oriented cycles. Suppose the vertices are ordered so that the sequence of projectives \( (KQe_1, \ldots, KQe_n) \) is an exceptional sequence. By the result of \([C1]\) there is an element \( g \) of the braid group on \( n \) strings such that \( g(KQe_1, \ldots, KQe_n) \) includes the exceptional \( KQ \)-module of dimension \( \alpha \). Then \( g(RQe_1, \ldots, RQe_n) \) includes an exceptional lattice of rank \( \alpha \). \( \square \)
3. Rigid modules

An $RQ$-module $X$ is rigid if $\text{Ext}^1(X, X) = 0$. We can now give a complete description of the rigid $RQ$-modules which are f.g. over $R$.

**Theorem 2.** (1) Any rigid $RQ$-module, f.g. over $R$, is a lattice.

(2) Any rigid lattice is a direct sum of exceptional lattices, and the terms in this direct sum are unique up to isomorphism and reordering.

(3) There is at most one rigid lattice of each rank vector, and the rank vectors which arise in this way do not depend on $R$.

**Proof.** (1) Let $X$ be the module, and consider it as a representation of $Q$, with a f.g. $R$-module $X_i$ for each vertex $i$, and a homomorphism $X_a : X_i \to X_j$ for each arrow $a : i \to j$. Since $R$ is a principal ideal domain we can choose a decomposition of each $X_i$ as a direct sum of copies of $R$ and $R/\pi^r$ with $\pi$ prime and $r \geq 1$. Let $0 \to X_i \to E_i \to X_i \to 0$ be the direct sum of a split exact sequence $0 \to R \to R^2 \to R \to 0$ for each occurrence of $R$ as a summand of $X_i$, and a non-split exact sequence

$$0 \to R/\pi^r \xrightarrow{(\frac{1}{\pi})} R/\pi^{r-1} \oplus R/\pi^{r+1} \xrightarrow{(-\pi^{r-1})} R/\pi^r \to 0$$

for each occurrence of $R/\pi^r$ as a summand of $X_i$. If $a : i \to j$ is an arrow then the map $X_a : X_i \to X_j$ can be completed to a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & X_i & \to & E_i & \to & X_i & \to & 0 \\
& & \downarrow{x_a} & & \downarrow{E_a} & & \downarrow{x_a} & & \\
0 & \to & X_j & \to & E_j & \to & X_j & \to & 0.
\end{array}
$$

To prove this we may suppose that $X_i$ and $X_j$ are indecomposable.

If $X_i = R$ then the top sequence splits, $X_a$ factors through the epi $E_j \to X_j$, and it is easy to construct $E_a$. On the other hand, if $X_i = R/\pi^r$ and the map $X_a$ is nonzero then $X_j$ must be of the form $R/\pi^s$. Now $X_a$ is multiplication by $x \in R$ with $\pi^r x \in \pi^s R$, and one can take $E_a$ with matrix $\left( \begin{array}{cc} \pi^r & 0 \\ 0 & x \end{array} \right)$.

We have constructed an exact sequence of representations $0 \to X \to E \to X \to 0$, and by assumption this splits. Thus each short exact sequence $0 \to X_i \to E_i \to X_i \to 0$ splits, so each $X_i$ must be free. This proves the result.

(2) First uniqueness. If $X$ can be written as a direct sum of exceptional lattices $Y_1 \oplus \cdots \oplus Y_r$ and $K$ is an algebraically closed $R$-field then $X^K$ is the direct sum of indecomposables $Y_1^K \oplus \cdots \oplus Y_r^K$. By the Krull-Schmidt theorem the $Y_i^K$ are uniquely determined, so by Theorem 1 the $Y_i$ are uniquely determined.

For the existence of a decomposition, fix an algebraically closed $R$-field $F$ and let $M_1, \ldots, M_r$ be the non-isomorphic indecomposable summands of $X^F$, say

$$X^F \cong M_1^{m_1} \oplus \cdots \oplus M_r^{m_r}.$$ 

The $M_i$ are exceptional and by [HR, Corollary 4.2] they can be ordered so that $(M_1, \ldots, M_r)$ is an exceptional sequence. Let $(X_1, \ldots, X_r)$ be the sequence
of exceptional lattices with the same ranks. It is an exceptional sequence by
Lemma 2. For any algebraically closed $R$-field $K$ we have

$$X^K \cong (X^K_1)^{m_1} \oplus \cdots \oplus (X^K_r)^{m_r}$$

since both sides are rigid of the same dimension vector. Also

$$\text{rank } \text{Hom}(X_r, X) = \dim \text{Hom}(X^K_r, X^K) = m_r > 0.$$ 

Let $\theta : X_r \otimes_R \text{Hom}(X_r, X) \to X$ be the universal map and let $C$ be its cokernel.
Now the map $\theta^K$ can be identified with the universal map

$$X^K_r \otimes_K \text{Hom}(X^K_r, X^K) \to X^K$$

which is the inclusion of $(X^K_r)^{m_r}$ as a direct summand of $X^K$. Since this holds
for all $K$, the map $\theta$ is mono and $C$ is free over $R$. Also

$$C^K \cong \text{Coker}(\theta^K) \cong (X^K_1)^{m_1} \oplus \cdots \oplus (X^K_{r-1})^{m_{r-1}}$$

so $\text{Ext}^1(C^K, C^K) = 0$ and $\text{Ext}^1(C^K, X^K_r) = 0$, and hence $C$ is rigid and
$\text{Ext}^1(C, X_r) = 0$. Thus $X \cong X_r \otimes_R \text{Hom}(X_r, X) \oplus C$, and by induction
$X$ is a direct sum of exceptional lattices.

(3) If $X$ and $Y$ are rigid lattices, both of rank $\alpha$, and if $K$ is an algebraically
closed $R$-field, then $X^K$ and $Y^K$ are rigid $KQ$-modules of dimension $\alpha$, so isomorphic.
As in the uniqueness part of (2), the assertion $X \cong Y$ follows on
considering the decompositions of $X$ and $Y$ into exceptional lattices. The rank
vectors which arise can be characterized as the sums $\alpha_1 + \cdots + \alpha_r$ of real Schur
roots with $\text{ext}(\alpha_i, \alpha_j) = 0$ for all $i, j$. □

Remark 1. We can reformulate some of the previous theorem. Fix an $R$-field
$R \to K$ and let $M$ be an $KQ$-module. By an $R$-form of $M$ we mean an $RQ$-
lattice $X$ and an isomorphism $X^K \to M$. We say that an $R$-form is rigid if it is
so as an $RQ$-lattice. We say that two $R$-forms $X, Y$ are conjugate if there is
an isomorphism $\theta : X \to Y$. In this case $\theta^K$ is an isomorphism $X^K \to Y^K$, and
identifying both sides with $M$ we obtain $\phi \in \text{Aut}_{KQ}(M)$ making the diagram

$$\begin{array}{ccc}
X^K & \longrightarrow & M \\
\phi \downarrow & & \downarrow \phi \\
Y^K & \longrightarrow & M
\end{array}$$

commute.

Observe that in case the homomorphism $R \to K$ is mono the $R$-forms of $M$
can be identified with $RQ$-submodules of $M$, and two $R$-forms $X, Y$ are conjugate
if and only if there is $\phi \in \text{Aut}_{KQ}(M)$ with $\phi(X) = Y$.

Now Theorem 2 implies that any rigid $KQ$-module has a rigid $R$-form, unique
up to conjugacy.
Remark 2. The theory of perpendicular categories extends to integral representations. If \( Q \) has no oriented cycles, \( X \) is an exceptional lattice of rank \( \alpha \), and \( X^\perp \) is the category of \( RQ \)-modules \( M \) with \( \text{Hom}(X, M) = \text{Ext}^1(X, M) = 0 \), then there is an equivalence \( RQ' \text{-Mod} \rightarrow X^\perp \) for some quiver \( Q' \) with \( n-1 \) vertices and no oriented cycles. To prove this, recall that if \( K \) is an algebraically closed \( R \)-field then the category of \( KQ \)-modules perpendicular to \( X^K \) is equivalent to \( KQ' \text{-Mod} \) for some quiver \( Q' \) with \( n-1 \) vertices [S1]. Now the dimension vectors \( p_1, \ldots, p_{n-1} \) of the indecomposable relative projectives in \((X^K)^\perp \) do not depend on \( K \) since they can be characterized as the real Schur roots \( p \) with \( \text{hom}(\alpha, p) = \text{ext}(\alpha, p) = 0 \) and \( \text{ext}(p, \beta) = 0 \) for all real Schur roots \( \beta \) with \( \text{hom}(\alpha, \beta) = \text{ext}(\alpha, \beta) = 0 \). Also the dimension vectors \( s_1, \ldots, s_{n-1} \) of the corresponding simple objects in \((X^K)^\perp \) are independent of \( K \), since they can be deduced from the Cartan matrix \((p_i, p_j)_{ij}\). Let \( P_i \) and \( S_i \) be the exceptional lattices of rank vectors \( p_i \) and \( s_i \). Now \( P = P_1 \oplus \cdots \oplus P_{n-1} \) is rigid, belongs to \( X^\perp \), and is a generator for \( X^\perp \) (for if \( M \in X^\perp \) then \( M^K \in (X^K)^\perp \), so \( P^K \) generates \( M^K \)). Thus \( P \) is a f.g. projective generator for \( X^\perp \), so \( X^\perp \) is equivalent to \( \text{End}(P)_{\text{op}} \text{-Mod} \). Now \( \text{Hom}(P_i, S_i) \cong R \), so we can choose an \( R \)-module generator \( \theta_i : P_i \rightarrow S_i \). Since \( \theta_i^K \) is an epimorphism for all \( K \), with relative projective kernel, it follows that \( \theta_i \) is an epimorphism with kernel isomorphic to a direct sum of copies of the \( P_j \). Moreover the number of copies of \( P_j \) in a direct sum decomposition of \( \text{Ker}(\theta_i) \) is the number of arrows \( i \rightarrow j \) in \( Q' \). Thus there is a natural map \( RQ' \rightarrow \text{End}(P)_{\text{op}} \), which is an isomorphism, since it is so on inducing to any \( R \)-field.

Remark 3. If \( Q \) is Dynkin then the lattices \( X \) which are direct sums of exceptionals can be characterized as those with \( \dim \tau_K^n X^K \) and \( \dim \tau_K^{-n} X^K \) independent of \( K \) for all \( n \in \mathbb{N} \), where \( \tau_K \) and \( \tau_K^{-n} \) are the Auslander-Reiten translates for \( KQ \). For a proof one uses the functors \( \tau(-) = D \text{Ext}^1(-, RQ) \) and \( \tau^(-) = \text{Ext}^1(D(-), RQ) \).

4. A conjecture

One can conjecture that for each positive real root \( \alpha \in \mathbb{R}^n \) there is a unique absolutely indecomposable lattice of rank vector \( \alpha \).

(1) In case \( R \) is an algebraically closed field the conjecture is part of Kac's Theorem, see [K2], to which we also refer for the definition of real roots in case the quiver has loops. For \( R \) an arbitrary field the conjecture holds by a Galois-theoretic argument of Schofield [S2].

(2) In case \( \alpha \) is a real Schur root the conjecture follows from Theorem 1. If \( X \) is a lattice of rank \( \alpha \) then by definition \( X \) is absolutely indecomposable if and only if \( X^K \) is indecomposable for each algebraically closed \( R \)-field \( K \). Since \( \alpha \) is a real root, by Kac's theorem the exceptional \( KQ \)-module of dimension \( \alpha \) is the unique indecomposable \( KQ \)-module of dimension \( \alpha \). Thus \( X \) is absolutely
indecomposable if and only if each $X^K$ is exceptional, so if and only if $X$ is exceptional.

(3) In case $Q$ has only two vertices the conjecture holds: if either vertex has
a loop there is nothing to do, since there is at most one positive real root, while
if there are no oriented cycles then all positive real roots are real Schur roots.
It only remains to deal with the quiver $\Gamma_{ab}$ with $a > 0$ arrows from 1 to 2 and
$b > 0$ arrows from 2 to 1. In this case the claim follows from Ringel's work [Ri].
All we need is the following version of Ringel’s reflection functor $\sigma_S$.

Let $S$ be an exceptional lattice. Let $\mathcal{M}^S_S$ be the category of lattices $X$ such
that for all algebraically closed $R$-fields $K$, $\text{Ext}^1(S^K, X^K) = \text{Ext}^1(X^K, S^K) = 0$
and $X^K$ has no summand which can be embedded in, or is a quotient of, a
direct sum of copies of $S^K$. Write $\mathcal{M}^S_S/S$ for the category obtained by fatoring
out those maps which factor through a direct sum of copies of $S$. Let $\mathcal{M}^{-S}_S$
be the category of lattices $X$ with $\text{Hom}(S^K, X^K) = \text{Hom}(X^K, S^K) = 0$ for all
$K$. If $X \in \mathcal{M}^S_S$, it is easy to see that $\text{Hom}(S, X)$ and $\text{Hom}(X, S)$ are both free,
$\text{Hom}(S, X)^K \cong \text{Hom}(S^K, X^K)$, and $\text{Hom}(X, S)^K \cong \text{Hom}(X^K, S^K)$. Now the
universal maps $\theta : S \otimes_R \text{Hom}(S, X) \to X$ and $\phi : X \to S \otimes_R D\text{Hom}(X, S)$
are a mono- and an epimorphism, since they are all inducing to any $R$-field [Ri, Lemma 2],
and the composition $\phi \theta$ is zero, since it is zero over any $R$-field,
[Ri, p470]. The assertion is that the functor $\sigma_S(X) = \text{Ker}(\phi)/\text{Im}(\theta)$ defines an
equivalence $\mathcal{M}^S_S/S \to \mathcal{M}^{-S}_S$, cf. [Ri, Proposition 2]. The proof is straightforward.

(4) In case $Q$ is Dynkin or Euclidean the conjecture holds: in order to deal
with positive real roots which are not real Schur roots one can use perpendicular
categories to reduce to $\tilde{A}_n$ with cyclic orientation, and thence to the quiver $\Gamma_{11}$.

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