Generalisations of Preprojective algebras

Peter Harry Clifford Shaw

Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

The University of Leeds

School of Mathematics

September 2005

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Acknowledgements

Firstly, I would like to thank my supervisor, Bill Crawley-Boevey, for his guidance and encouragement during my PhD studies. I am extremely grateful for receiving the benefit of his knowledge and experience. I would also like to thank the other members of the Algebra group at Leeds.

I would like to thank my parents for their support throughout my studies. I would also like to acknowledge the financial support provided by the Boothman, Reynolds and Smithells scholarship.

Abstract

In this thesis, we investigate two ways of generalising the preprojective algebra.

First, we introduce the multiplicative preprojective algebra, $\Lambda^q(Q)$, which is a multiplicative analogue of the deformed preprojective algebra, introduced by Crawley-Boevey and Holland. The special case $q = 1$ is the undeformed multiplicative preprojective algebra, which is an analogue of the ordinary (undeformed) preprojective algebra. We adapt the middle convolution operation of Dettweiler and Reiter to construct reflection functors, which are used to determine the possible dimension vectors of simple modules for $\Lambda^q(Q)$. We show that $\Lambda^q(Q)$ is finite dimensional if Q is Dynkin, and that $e_1\Lambda^1(Q)e_1$ is a commutative integral domain of Krull dimension 2 if Q is extended Dynkin with 1 an extending vertex. The proofs of these results depend on applying the reduction algorithm as described by Bergman, which is recalled in the appendix. We conjecture that the undeformed multiplicative preprojective algebra is a 'preprojective algebra' in the sense of Gelfand and Ponomarev, in that as a KQ-module, it is isomorphic to the direct sum of the indecomposable preprojective KQ-modules.

Second, we extend the notion of a preprojective algebra of a quiver to the notion of a preprojective algebra for a quiver with relations. Our results show that for any Nakayama algebra A, there exists an algebra $P(A)$ such that $P(A)$ is isomorphic to the direct sum of all indecomposable A-modules.

Contents

Chapter 1

Introduction

1.1 Background

In recent decades, the representation theory of quivers has played a fundamental role in the theory of finite dimensional algebras. The first major result was obtained by Gabriel [17], when he showed that the indecomposable representations of Dynkin quivers were in correspondence with the root systems of the corresponding Lie algebra (this was generalised to all quivers by Kac [21]). Thus the only quivers with finitely many indecomposable representations are the Dynkin quivers. A number of techniques for studying quiver representations were developed in the 1970's, including the reflection functors of Bernstein, Gelfand and Ponomarev, Coxeter functors and the Auslander-Reiten translation.

The notion of a preprojective algebra was introduced by Gelfand and Ponomarev [18]. Their aim was to construct an algebra which contains the path algebra as a subalgebra and is isomorphic to the direct sum of the indecomposable preprojective modules for the path algebra, and thus one easily obtains the indecomposable representations of the quiver. This work was subsequently generalised by Dlab and Ringel [14]. Preprojective algebras had many connections with the known tools of representation theory. For example they had been used implicitly in work by Riedtmann on Coxeter functors [26], and the definition of the preprojective algebra involves relations resembling the mesh relations of the Auslander-Reiten quiver. Indeed later, an alternative definition was proposed by Baer, Geigle and Lenzing [3], which defined the preprojective algebra directly in terms the Auslander-Reiten translate. It was eventually proved by Ringel [28] that the two definitions were the same, although this had always been generally accepted.

Besides being used to study representations of quivers, the preprojective algebra was found to appear naturally in a wide variety of situations. These applications include work by Lusztig on quantum groups [23], [24], Kronheimer's work on differential geometry [22], and in particular Kleinian singularities. In order to study deformations of Kleinian singularities, the deformed preprojective algebra was introduced by Crawley-Boevey and Holland [10].

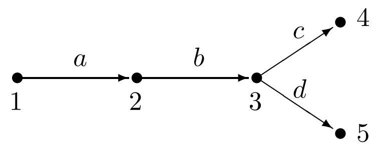
More recently, the deformed preprojective algebra was used by Crawley-Boevey [9] to solve the additive Deligne-Simpson problem, which asks for solutions to equations involving sums of matrices. This problem has connections with Fuchsian systems of ordinary differential equations.

The aim of this thesis is to investigate the question, "Is it possible to 'generalise' the notion of a preprojective algebra?". Of course, this question is rather vague, as 'generalise' has at least two different meanings in this context. One can form algebras closely related to preprojective algebras by using similar constructions (e.g. by taking a quotient of the path algebra of the double of the quiver by a similar relation). In particular, we look for a multiplicative analogue of the deformed preprojective algebra which can be applied to study the multiplicative Deligne-Simpson problem. Alternatively, one can seek a preprojective algebra for a quiver with relations in the spirit of Gelfand and Ponomarev, namely, construct an algebra which contains the path algebra modulo the relations as a subalgebra, and decomposes as a direct sum of the indecomposable 'preprojective' modules for this algebra.

1.2 Basic definitions

In this section, we give a brief overview of quivers, representations and root systems, introducing the notation which will be used, and stating some well known results. First we state some conventions. Throughout, K denotes an algebraically closed field. An algebra is a K-vector space equipped with a bilinear associative product, and is always assumed to have an identity element. Modules will typically be left modules, and are usually finite dimensional. Functions are always written on the left, so that $\theta\phi$ means 'first apply ϕ, then θ'. If r is an element of an algebra, then I_r denotes the ideal generated by r, and similarly if R is a set of elements or relations, then I_R denotes the ideal generated by R.

Quivers. A *quiver* $Q = (Q_0, Q_1, h, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows, and functions $t, h : Q_1 \to Q_0$. For each $a \in Q_1$, the vertices $t(a), h(a)$ are called the tail and head of a respectively (alternatively start/end or initial/terminal vertex etc.). We assume that the sets Q_0, Q_1 are finite. The *underlying graph* of Q is the graph obtained by 'forgetting' the orientation. If v is a vertex such that no arrow starts (ends) at v, then v is called a *sink* (*source*). A quiver is *bipartite* if every vertex is a source or a sink. Quivers are typically given as diagrams, with vertices represented by dots, and arrows pointing from the tail to the head, e.g.



A *path* of Q of length n is a word $a_n \ldots a_1$ where each $a_i \in Q_1$ and $h(a_i) = t(a_{i+1})$ for $i = 1, \ldots, n-1$. Additionally for each $v \in Q_0$ there is a *trivial path* of length zero denoted by e_v. For a path p, define $h(p), t(p)$ by $h(e_v) = t(e_v) = v$ for trivial paths and $t(a_n \ldots a_1) = t(a_1)$, $h(a_n \ldots a_1) = h(a_n)$. An *oriented cycle* of a quiver is a non trivial path with $h(p) = t(p)$. In the above quiver, the paths are $e_1, e_2, e_3, e_4, e_5, a, b, c, d, ba, cb, db, cba, dba$, and there are no oriented cycles.

Path algebras. Given a quiver Q, there is a path algebra KQ, which is the algebra whose basis is the set of paths of Q, and the multiplication of paths p_1 and p_2 is defined to be the concatenation p_1p_2 if $h(p_2) = t(p_1)$, and zero otherwise. This can easily be seen to be an associative product, and the element $\sum_{v\in Q_0} e_v$ is the identity. The $(e_v)_{v\in Q_0}$ are a complete set of primitive orthogonal idempotents. It is easy to see that the path algebra is finite dimensional if and only if there are no oriented cycles in Q. The path algebra of the example quiver is 14 dimensional, and some examples of products are $a.e_1 = a$, $e_1.a = 0$, $d.b = db$, $d.c = 0$ etc.

Representations. A *representation* X of a quiver Q is given by a vector space X_v for each $v \in Q_0$ and a linear map $X_a : X_{t(a)} \to X_{h(a)}$ for each $a \in Q_1$. The *dimension vector* of X, is $\underline{\dim} X = (\dim X_v)_{v\in Q_0}$. The *support* of X is the set $\{v \in Q_0 : X_v \neq 0\}$. The following diagram indicates a representation of the example quiver of dimension vector (1,2,2,1,1).

$$\begin{tikzcd}
 & & & K \\
K \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & K^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] & K^2 \arrow[ur, "(0\ 1)"] \arrow[dr, "(1\ 0)"'] & \\
 & & & K
\end{tikzcd}$$

There is an equivalence between the category of KQ-modules and the category of representations of Q. Given a left KQ-module M, we define a representation X by setting $X_v = e_vM$ for each $v \in Q_0$, and if $a : v_1 \to v_2$ is an arrow, $X_a : X_{v_1} \to X_{v_2}$ is the map which takes $m \in e_{v_1}M$ to $am \in e_{v_2}M$. Conversely if X is a representation of Q, there is a module $M = \oplus X_v$, where e_v acts as the projection onto X_v and $a : v_1 \to v_2$ acts as the composition $M \twoheadrightarrow X_{v_1} \xrightarrow{X_a} X_{v_2} \hookrightarrow M$.

More generally, one can speak of representations of quivers with relations. Namely, suppose $R = \{r_1, r_2, \ldots, r_k\}$ is a set of elements of KQ such that for each i there are vertices u_i, v_i with $r_i \in e_{u_i}KQe_{v_i}$, and let $A = KQ/I_R$. Given a representation of Q, one may consider the linear map $X_i : X_{v_i} \to X_{u_i}$ obtained from the expression of r_i by replacing each arrow a by X_a, and each e_v by the identity map 1_{X_v}. One can identify the category of A-modules with the category of representations of Q in which the linear maps satisfy the relations $X_i = 0$ for all i.

Roots. The *Ringel form* for a quiver Q is the bilinear form

$$\langle \alpha, \beta \rangle = \sum_{v \in Q_0} \alpha_v \beta_v - \sum_{a \in Q_1} \alpha_{h(a)} \beta_{t(a)}.$$

This gives rise to the symmetric bilinear form $(-,-)$ defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$, and a quadratic form q defined by $q(\alpha) = \langle \alpha, \alpha \rangle$.

If v is a loopfree vertex of Q, there is a reflection $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$, defined by $s_v(\alpha) = \alpha - (\alpha, \epsilon_v)\epsilon_v$, where ϵ_v is the coordinate vector at v. The *Weyl group* W is the subgroup of $\mathrm{Aut}(\mathbb{Z}^{Q_0})$ generated by the s_v, and the *fundamental region* is the set

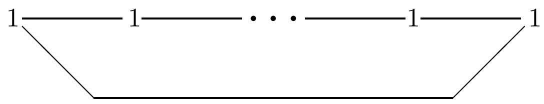
$$F = \{\alpha \in \mathbb{N}^{Q_0} : \alpha \neq 0, \alpha \text{ has connected support and } (\alpha, \epsilon_v) \leq 0 \text{ for all } v\}.$$

The *real roots* for Q are the orbits of ϵ_v under W, and the *imaginary roots* are the elements of the form $\pm w\alpha$, where $\alpha \in F$, $w \in W$ and a *root* is a real root or an imaginary root. It can be shown that if α is a root then either α is positive ($\alpha \in \mathbb{N}^{Q_0}$) or negative ($-\alpha \in \mathbb{N}^{Q_0}$).

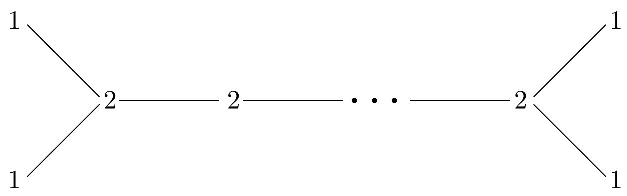
Observe that $q(s_v(\alpha)) = q(\alpha)$, and so if α is a real root, then $q(\alpha) = 1$, and if α is an imaginary root, then $q(\alpha) \leq 0$.

Dynkin quivers and extended Dynkin quivers. The *extended Dynkin quivers* are the quivers whose underlying graph is one of the following graphs, where the number of vertices is the subscript plus one. In each case, we mark a dimension vector δ by writing an integer δ_v instead of a dot to represent a vertex.

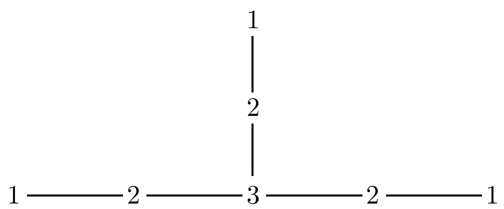
$\tilde{A}_n$, $n \geq 0$



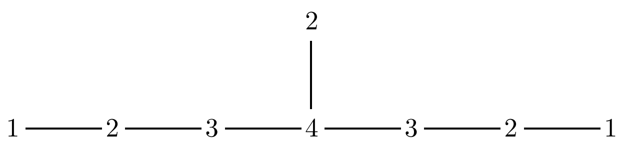
$\tilde{D}_n$, $n \geq 4$



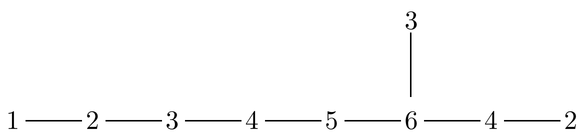
$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$



The vertices v for which $\delta_v = 1$ are called *extending vertices.* A *Dynkin quiver* is a quiver which can be obtained from an extended Dynkin quiver by removing an extending vertex.

For all Dynkin quivers, it can be shown that the quadratic form is positive definite, and thus all roots (of which there are finitely many) are real. For extended Dynkin quivers, the quadratic form is positive semi-definite, and $q(\alpha) = 0$ if and only if α is a multiple of δ. Thus the imaginary roots are the multiples of δ, and the remaining roots (of which there are infinitely many) are real.

Theorem 1.2.1. *[21] For a loopfree quiver Q, there is an indecomposable representation of Q with dimension vector α if and only if α is a positive root. If α is a positive real root, the indecomposable representation is unique up to isomorphism. If α is a positive imaginary root, there are infinitely many pairwise non-isomorphic modules of dimension vector α.*

Representation type. An algebra has *finite representation type* if there are only finitely many isomorphism classes of indecomposable modules (otherwise, it has *infinite representation type*). From the above, it is evident that a path algebra KQ has finite representation type if and only if Q is a Dynkin quiver, a theorem originally due to Gabriel.

Preprojective Modules. For a finite dimensional algebra A, there exist several competing definitions of a preprojective module. In the case where A is the path algebra KQ of a quiver, these definitions coincide with the following definition, which uses the Auslander-Reiten translate τ and its inverse τ^- (it can be shown that $\tau(M) = D\,\mathrm{Ext}^1_{KQ}(M, KQ)$ and $\tau^-(M) = \mathrm{Ext}^1_{KQ}(DM, KQ)$). An indecomposable module M for A is *preprojective* (respectively *preinjective*) if $\tau^n(M) = 0$ (respectively if $\tau^{-n}(M) = 0$) for some positive integer n. If $\tau^n(M) \neq 0$ for all $n \in \mathbb{Z}$ then M is *regular*. If KQ has finite representation type then all its modules are preprojective and preinjective. If it has infinite representation type, then the isomorphism classes of indecomposable preprojective modules and indecomposable preinjective modules are disjoint and there are an infinite number of each, as well as an infinite number of isomorphism classes of indecomposable regular modules.

Only in Chapter 6 of this thesis does the definition in general become relevant, and then only in the finite type case. We use the definition given by Auslander and Smalø, [2], which for finite type algebras means all modules are preprojective and preinjective.

1.3 Preprojective algebras

In this section, we define the preprojective algebra, and state some well known results.

Definition 1.3.1. Given a quiver Q, the *double* of Q, denoted by $\overline{Q}$, is defined to be the quiver obtained by adjoining a reverse arrow a^* for each arrow a of Q with $h(a^*) = t(a)$ and $t(a^*) = h(a)$. We extend the operation $a \mapsto a^*$ into an involution on the arrows of $\overline{Q}$, by defining $(a^*)^* = a$. We define a function $\epsilon : \overline{Q}_1 \to \{-1, 1\}$ by

$$\epsilon(a) = \begin{cases} 1 & \text{if } a \in Q_1 \\ -1 & \text{if } a^* \in Q_1. \end{cases}$$

There is a grading on $K\overline{Q}$ defined by assigning trivial paths degree 0, and the arrows degree 1, which we call the *unoriented grading*. Alternatively, one may assign the trivial paths and elements of Q_1 degree 0, and the elements of Q_1^* degree 1, and this gives rise to an *oriented grading*. We typically work with the oriented grading.

Definition 1.3.2. Given a quiver Q, the *preprojective algebra* $\Pi(Q)$ is defined to be the algebra $K\overline{Q}/I_\rho$, where

$$\rho = \sum_{a \in \overline{Q}_1} \epsilon(a) a a^*.$$

The two gradings on $K\overline{Q}$ both induce a grading on $\Pi(Q)$, since in either case ρ is a homogeneous element.

Definition 1.3.3. Let $P(A)$ be an algebra with a finite dimensional subalgebra A. $P(A)$ satisfies the *preprojective property* for A if, as a left (right) A-module,

$$P(A) \cong \bigoplus_{M \in Z} M$$

where Z is a set of indecomposable representatives for the category of preprojective left (right) A-modules (i.e. Z contains exactly one module from each isomorphism class of indecomposable preprojective modules).

Theorem 1.3.4. *If Q has no oriented cycles, then $\Pi(Q)$ has the preprojective property for KQ.*

Thus it follows quickly from Gabriel's theorem that $\Pi(Q)$ is finite dimensional if and only if Q is a Dynkin quiver. Note that $\Pi(Q)$ is not necessarily the only algebra which satisfies Theorem 1.3.4. For example, with $\rho' = \sum_{a \in \overline{Q}_1} aa^*$, one could define the algebra $K\overline{Q}/I_{\rho'}$, which would also satisfy the preprojective property, and is only known to be isomorphic to the ordinary preprojective algebra if the quiver is bipartite.

An alternative definition of the preprojective algebra was given by Baer, Geigle and Lenzing [3]. This is not important for the purposes of this thesis, but is given for completion, and it also helps understand why the preprojective algebra has the preprojective property. Given a ring A and an $A-A-$bimodule M, let $T_A(M)$ be the tensor algebra, which is defined as

$$T_A(M) = \bigoplus_{i \geq 0} M^{\otimes i},$$

where $M^{\otimes i}$ denotes the i-fold tensor power of M, with $M^{\otimes 0} = A$. The product of $x \in M^{\otimes i}$ and $y \in M^{\otimes j}$, is defined to be $x \otimes y \in M^{\otimes (i+j)}$. One can then define $\Pi(Q) = T_{KQ}(\tau^-(KQ))$. This algebra has a natural grading, where the elements of $M^{\otimes i}$ are in degree i. This definition is equivalent to Definition 1.3.2, (see [28]), and the grading coincides with the oriented grading.

If Q is extended Dynkin, then $\Pi(Q)$ has many interesting properties which are given in [3]. There is also the following nice description of the ring $e_1\Pi(Q)e_1$ where 1 is an extending vertex. This follows from work by Cassens and Slodowy, [7] and shows the connection of preprojective algebras to Kleinian singularities. The equations below are not the traditional equations associated to the Kleinian singularities (but can be shown to be equivalent after a simple change of variables), but are written in this way for later comparison.

Theorem 1.3.5. *If Q is extended Dynkin and 1 is an extending vertex, then $e_1\Pi(Q)e_1$ is a commutative algebra. More precisely,*

$$e_1\Pi(Q)e_1 \cong K[X,Y,Z]/J,$$

where J is the ideal generated by

$$\begin{aligned}
Z^{n+1} + XY &\qquad \textit{if } Q \textit{ type } \tilde{A}_n,\\
Z^2 - XY^2 - X^mY &\qquad \textit{if } Q \textit{ type } \tilde{D}_{2m},\\
Z^2 - XY^2 + X^mZ &\qquad \textit{if } Q \textit{ type } \tilde{D}_{2m+1},\\
Z^2 + X^2Z + Y^3 &\qquad \textit{if } Q \textit{ type } \tilde{E}_6,\\
Z^2 + Y^3 + X^3Y &\qquad \textit{if } Q \textit{ type } \tilde{E}_7,\\
Z^2 - Y^3 - X^5 &\qquad \textit{if } Q \textit{ type } \tilde{E}_8.
\end{aligned}$$

In order to study deformations of Kleinian singularities, Crawley-Boevey and Holland introduced the following generalisation of the preprojective algebra.

Definition 1.3.6. [10] Given a quiver Q and a *weight* $\lambda \in K^{Q_0}$, the *deformed preprojective algebra* $\Pi^\lambda(Q)$ is defined to be the algebra $K\overline{Q}/I_{\rho^\lambda}$, where ρ^λ is the element

$$\sum_{a\in\overline{Q}_1} \epsilon(a)aa^* - \sum_{v\in Q_0} \lambda_v e_v.$$

Clearly, the preprojective algebra is the special case $\lambda = 0$.

Note that the ideal I_{ρ^λ} is the same as the ideal generated by the elements

$$\rho_v^\lambda = e_v\rho^\lambda e_v = \sum_{\substack{a\in Q_1\\ h(a)=v}} aa^* - \sum_{\substack{a\in Q_1\\ t(a)=v}} a^*a - \lambda_v e_v.$$

This is helpful when considering representations of $\Pi^\lambda(Q)$, as they can be identified with representations of $\overline{Q}$ in which the linear maps satisfy the following relation at each vertex $v \in Q_0$.

$$\sum_{\substack{a\in Q_1\\ h(a)=v}} X_aX_a^* - \sum_{\substack{a\in Q_1\\ t(a)=v}} X_a^*X_a - \lambda_v 1_{X_v} = 0.$$

The following lemma is easy, but it is helpful to write out a proof, for later reference.

Lemma 1.3.7. *$\Pi^\lambda(Q)$ is independent of the orientation of Q.*

Proof. Suppose Q' is obtained from Q by removing an arrow b and replacing it with an arrow c satisfying $t(c) = h(b)$, $h(c) = t(b)$. There is an isomorphism $\theta : K\overline{Q} \to K\overline{Q'}$ which sends each e_v to e_v, b to $-c^*$, b^* to c and each remaining arrow to itself. Now

$$\begin{aligned}\theta\left(\sum_{a\in\overline{Q}_1}\epsilon(a)aa^* - \sum_{v\in Q_0}\lambda_v e_v\right) &= \sum_{\substack{a\in\overline{Q}_1\\ a\neq b,b^*}}\epsilon(a)aa^* - c^*c + cc^* - \sum_{v\in Q'_0}\lambda_v e_v\\ &= \sum_{a\in\overline{Q'_1}}\epsilon(a)aa^* - \sum_{v\in Q'_0}\lambda_v e_v,\end{aligned}$$

and so θ induces an isomorphism $\Pi^\lambda(Q) \to \Pi^\lambda(Q')$. □

Some important properties of deformed preprojective algebras are given in [10] and [8]. In the remainder of this section we recall those which are of particular interest for this thesis.

Theorem 1.3.8. *[10] If Q is a Dynkin diagram, then Π^λ is finite dimensional.*

An important tool for studying the representations of deformed preprojective algebras are *reflection functors*. Given a loopfree vertex $v \in Q_0$, define $r_v : K^{Q_0} \to K^{Q_0}$ as $(r_v(\lambda))_u = \lambda_u - (\epsilon_v, \epsilon_u)\lambda_v$. This reflection is dual to the reflection s_v, namely, $\lambda.s_v(\alpha) = r_v(\lambda).\alpha$, where $\lambda.\alpha = \sum_{v\in Q_0}\lambda_v\alpha_v$.

Theorem 1.3.9. *[10] If v is a loopfree vertex of Q with $\lambda_v \neq 0$ there is an equivalence E_v from the category of representations of Π^λ to the category of representations of $\Pi^{r_v(\lambda)}$ which acts as s_v on dimension vectors.*

Theorem 1.3.10. *[8] There is a simple representation of Π^λ of dimension vector α if and only if α is a positive root, $\lambda.\alpha = 0$, and $p(\alpha) = \sum p(\beta_i)$ for any decomposition $\alpha = \sum\beta_i$ as a sum of two or more positive roots with each $\lambda.\beta_i = 0$ (where $p(\alpha) = 1 - q(\alpha)$).*

This classification of the simple modules is used in [9] to solve the additive Deligne-Simpson problem.

1.4 Main results and thesis layout

As already stated, the aim of this thesis is to generalise the preprojective algebra, by answering the questions "Is there a multiplicative analogue of the deformed preprojective algebra?", and "Given an algebra A presented by a quiver with relations, is it possible to define an algebra $P(A)$ which satisfies Theorem 1.3.4?". Of the two questions, we were more successful with the first, and most of the thesis (Chapters 2,3,4,5) is concerned with the definition and properties of these algebras. Many of the results of these chapters can be described as the multiplicative analogue of a known result for the ordinary preprojective algebra (that is, we replace Π by Λ, λ by q, and the condition that $\lambda.\alpha = 0$ by $q^\alpha = 1$). For the second question, we were able to define a 'preprojective algebra' for a quiver with certain types of relations. We now describe our main results in more detail.

Chapter 2 is concerned with the definition of the multiplicative preprojective algebra, and its properties in the general case. This material (other than Section 2.2) is due to be published in [11], where it is used to give a partial solution to the Deligne-Simpson problem. Given a quiver Q equipped with an ordering $<$ on the arrows, and an element $q \in (K^*)^{Q_0}$, we define an algebra $\Lambda^q(Q, <)$. The first main result is the following theorem.

Theorem 2.1.3. $\Lambda^q(Q, <)$ *is independent of the orientation of* Q *and the ordering* $<$.

Thus we can write $\Lambda^q(Q)$ instead of $\Lambda^q(Q, <)$ (and in the special case of the undeformed multiplicative preprojective algebra, where $q_v = 1$ for all v, we write $\Lambda^1(Q)$). In Section 2.2 we investigate whether $\Lambda^q(Q)$ can be defined as a quotient of $K\overline{Q}$, which would be easier than using the given definition (which involves localising certain elements of $K\overline{Q}$). In [11], it was shown that this is the case for star-shaped quivers (which was the only case necessary for the purpose of solving the Deligne-Simpson problem). This is still yet to be fully understood, but we have obtained some further results. In Section 2.3, we adapt work of Dettweiler and Reiter [12] to obtain the following multiplicative analogue of Theorem 1.3.9, thus showing that reflection functors also exist for multiplicative preprojective algebras. Let $t_v : (K^*)^{Q_0} \to (K^*)^{Q_0}$ be the reflection given by $t_v(q)_u = q_u q_v^{-(\epsilon_u,\epsilon_v)}$. This is dual to the s_v, as $(t_v(q))^\alpha = q^{s_i(\alpha)}$, where $q^\alpha = \prod_{v \in Q_0} q_v^{\alpha_v}$.

Theorem 2.3.1. *If v is a loopfree vertex of Q with $q_v \neq 1$, there is an equivalence E_v from the category of representations of $\Lambda^q(Q)$ to the category of representations of $\Lambda^{t_v(q)}(Q)$ which acts as s_v on dimension vectors.*

In Section 2.4 we use reflection functors to prove the following results.

Theorem 2.4.4. *If X is a simple representation of $\Lambda^q(Q)$ of dimension vector α, then α is a positive root for Q.*

Theorem 2.4.5. *Let α be a positive real root for Q. There is a simple representation of $\Lambda^q(Q)$ of dimension vector α if and only if $q^\alpha = 1$ and there is no decomposition $\alpha = \sum \beta_i$ as a sum of two or more positive roots with each $q^{\beta_i} = 1$.*

These results give evidence towards the truth of the following conjecture, which if true would be the multiplicative analogue of Theorem 1.3.10. The truth of this conjecture would lead to a solution of the multiplicative Deligne-Simpson problem, see [11] for more details, and a proof of one implication.

Conjecture 2.4.1. *There is a simple representation of $\Lambda^q(Q)$ of dimension vector α if and only if α is a positive root, $q^\alpha = 1$ and $p(\alpha) = \sum p(\beta_i)$ for any decomposition $\alpha = \sum \beta_i$ as a sum of two or more positive roots with each $q^{\beta_i} = 1$.*

In chapters 3 and 4 we move on to considering the properties of multiplicative preprojective algebras in the Dynkin and extended Dynkin case respectively.

The main results of these chapters are following two theorems, the multiplicative analogues of Theorems 1.3.8 and 1.3.5 respectively.

Theorem 3.1.1. *If Q is a Dynkin quiver, then $\Lambda^q(Q)$ is finite dimensional.*

Theorem 4.1.1. *If Q is extended Dynkin and 1 is an extending vertex, then $e_1\Lambda^1(Q)e_1$ is a commutative algebra. More precisely,*

$$e_1\Lambda^1(Q)e_1 \cong K[X,Y,Z]/J,$$

where J is the ideal generated by

$$\begin{aligned}
Z^{n+1}+XY+XYZ &\qquad \text{if } Q \text{ type } \tilde{A}_n,\\
Z^2-p_k(X)XZ+p_{k-1}(X)X^2Y-XY^2-XYZ &\qquad \text{if } Q \text{ type } \tilde{D}_n,\\
Z^2+X^2Z+Y^3-XYZ &\qquad \text{if } Q \text{ type } \tilde{E}_6,\\
Z^2+Y^3+X^3Y-XYZ &\qquad \text{if } Q \text{ type } \tilde{E}_7,\\
Z^2-Y^3-X^5+XYZ &\qquad \text{if } Q \text{ type } \tilde{E}_8,
\end{aligned}$$

where $k=n-4$, and the p_k are polynomials defined inductively by $p_{-1}(X)=-1$, $p_0(X)=0$ and $p_{i+1}(X)=X(p_i(X)+p_{i-1}(X))$ for $i\geq 1$.

Unfortunately the proofs of these results involve a case by case analysis, and are therefore quite long. It would certainly be desirable to obtain shorter proofs. At the end of each of these chapters, we include a short section on open problems.

In Chapter 5, we investigate some further open questions regarding multiplicative preprojective algebras, in particular whether $\Lambda^1(Q)$ is a 'preprojective algebra' in the sense of satisfying Theorem 1.3.4. If this was true in general then it would perhaps lead to a better understanding (and easier proofs) of the results of Chapters 3 and 4. We show that the conjecture is true for some small examples, as well as in the easiest infinite type case.

Theorem 5.1.4. *Let Q be the quiver*

$$1 \underset{a}{\overset{b}{\rightrightarrows}} 2$$

$\Lambda^1(Q)$ has the preprojective property for KQ.

We also consider whether $\Lambda^1(Q)$ and $\Pi(Q)$ could be isomorphic as algebras, and list some other interesting questions.

In Chapter 6, we consider the second interpretation of our initial question. In Section 6.1, we define what is meant by a 'pairing', and show that if Q is a quiver equipped with a pairing Σ, then it gives rise to a quiver Q^Σ, an ideal I^Σ in the path algebra KQ^Σ, and an algebra $\Pi(Q, \Sigma)$.

Conjecture 6.2.3. *If Σ is an 'end pairing', and $A = KQ^\Sigma/I^\Sigma$ has finite representation type, then we conjecture that $\Pi(Q, \Sigma)$ satisfies the preprojective property for A.*

For a certain type of end pairing, we can prove this conjecture (which is Theorem 6.2.4). This is done in Sections 6.2-6.4. In Section 6.5 we prove the following result, which could be said to be the main result of this chapter.

Theorem 6.5.4. *If A is a Nakayama algebra, then there is a quiver Q and a pairing Σ satisfying the conditions of Theorem 6.2.4 such that $A \cong KQ^\Sigma/I^\Sigma$, and thus $\Pi(Q, \Sigma)$ is an algebra satisfying the preprojective property for A.*

It had been hoped that this chapter would lead to slightly better results. For example, it would be desirable to obtain some results in the case where A has infinite representation type, but none of our results apply to this case. It had even been hoped that one could show that preprojective algebras exist for any finite dimensional algebra, but we have a counterexample to show the conjecture is not true if we replaced 'end pairing' by 'pairing', thus suggesting this is not the case.

Finally, in the appendix, we discuss the 'reduction algorithm', which enables us to find spanning sets (or even bases) for algebras presented by generators with relations. None of this material is new (the main reference is [4]), but it is helpful for understanding the proofs in Chapters 3 and 4.

Chapter 2

The Multiplicative preprojective algebra

The material in this chapter (except for Section 2.2) appears in [11]. Our aim is to develop a multiplicative analogue of the deformed preprojective algebra. After giving a definition in Section 2.1, we explore whether we can give a simpler definition (Section 2.2). We then develop the theory of reflection functors (Section 2.3), which we use in Section 2.4 to give some conditions regarding the existence of simple modules.

The study of properties of the algebra for the Dynkin and extended Dynkin case is reserved for later chapters.

2.1 Definition

Let Q be a quiver, with vertex set Q_0, and let $q \in (K^*)^{Q_0}$. We define $\epsilon : \overline{Q}_1 \to \{-1, 1\}$ as in Definition 1.3.1. Choose an ordering $<$ on the set of arrows in $\overline{Q}$, and label the arrows as a_i so that $a_1 < a_2 < \cdots < a_n$. Given an algebra homomorphism $\theta : K\overline{Q} \to A$, we consider the properties (†) and (‡).

$$(\dagger) \qquad \theta(1 + a_i a_i^*) \text{ is invertible in } A \text{ for all } i.$$

$$(\ddagger) \qquad \prod_{i=1}^{n} (\theta(1 + a_i a_i^*))^{\epsilon(a_i)} = \sum_{v \in Q_0} \theta(q_v e_v).$$

Definition 2.1.1. [11] The *multiplicative preprojective algebra* is defined to be the algebra $\Lambda^q(Q,<)$ equipped with a homomorphism $\phi : K\overline{Q} \to \Lambda^q(Q,<)$ which is universal for homomorphisms satisfying (†) and (‡). Namely, ϕ satisfies (†) and (‡) and if $\theta : K\overline{Q} \to A$ satisfies (†) and (‡), there exists a unique map $\psi : \Lambda^q(Q,<) \to A$ such that $\psi\phi = \theta$. Since it is defined by a universal property, the multiplicative preprojective algebra is unique up to isomorphism (provided it exists). The *undeformed multiplicative preprojective algebra* is the special case where $q_v = 1$ for all v, and we write $\Lambda^1(Q,<)$.

We now prove the multiplicative preprojective algebra exists by constructing it. First an easy lemma.

Lemma 2.1.2. *If e is an idempotent in a ring A and $z \in eAe$ then $1+z$ is invertible if and only if $e+z$ is invertible in eAe. Note that we can replace 'invertible' by 'left invertible' or 'right invertible' throughout.*

Given an arrow a of $\overline{Q}$, let $r_a = e_{h(a)} + aa^*$ and $s_a = 1 + aa^*$, and let $\overline{Q_l}$ be the quiver obtained from $\overline{Q}$ by adjoining a loop l_a at $h(a)$ for each arrow a of $\overline{Q}$. Let $L_Q = K\overline{Q_l}/J$ where J is the ideal of $K\overline{Q_l}$ generated by the relations $l_a r_a - e_{h(a)}$, $r_a l_a - e_{h(a)}$ for all $a \in \overline{Q}_1$. The relations ensure each r_a has inverse l_a in $e_{h(a)} L_Q e_{h(a)}$, and by the above lemma, each s_a is invertible in L_Q, with inverse $l_a + 1 - e_{h(a)}$. We can therefore define

$$\mu_{Q,<} = \prod_{i=1}^{n} s_{a_i}^{\epsilon(a_i)} - \sum_{v \in Q_0} q_v e_v \in L_Q,$$

and form the quotient $L_Q/I_{\mu_{Q,<}}$. We claim that this is equal to the multiplicative preprojective algebra. For this we must show that the obvious homomorphism $\phi : K\overline{Q} \to L_Q/I_{\mu_{Q,<}}$ (which clearly satisfies (†) and (‡)) is universal for homomorphisms $\theta : K\overline{Q} \to A$ satisfying (†) and (‡). Given such a homomorphism, we can define $\tilde{\psi} : L_Q \to A$ to be the homomorphism which sends x to $\theta(x)$ if x is a trivial path or an arrow of $\overline{Q}$, and each l_a to $(\theta(r_a))^{-1}$, (possible by Lemma 2.1.2 since θ satisfies (†)). This is well defined since $\tilde{\psi}(l_a r_a) = (\theta(r_a))^{-1}\theta(r_a) = \theta(e_{h(a)}) = \tilde{\psi}(e_{h(a)})$, and similarly $\tilde{\psi}(r_a l_a) = \tilde{\psi}(e_{h(a)})$. Since $\tilde{\psi}(\mu_{Q,<}) = \prod_{i=1}^{n}(\theta(1 + a_i a_i^*))^{\epsilon(a_i)} - \sum_{v \in Q_0} \theta(q_v e_v)$

(which is zero since θ satisfies (‡)), $\tilde{\psi}$ induces the map $\psi : L_Q/I_{\mu_{Q,<}} \to A$ which is clearly uniquely determined.

Theorem 2.1.3. *[11] $\Lambda^q(Q,<)$ is independent of the orientation of Q and the ordering $<$.*

Proof. First prove independence of orientation. It clearly suffices to show that $\Lambda^q(Q',<') \cong \Lambda^q(Q,<)$ in the case where Q' is obtained from Q by removing an arrow c and replacing it with an arrow b with $h(b) = t(c)$, $t(b) = h(c)$, and $<'$ is the ordering on the arrows of $\overline{Q'}$ obtained from $<$ with c replaced by b^* and c^* replaced by b. There is an algebra homomorphism $\theta : L_{Q'} \to L_Q$ which sends b to c^*, b^* to $-l_c c$, l_b to r_{c^*}, l_{b^*} to r_c and sends the remaining arrows, each remaining l_a and all trivial paths to themselves. To be well defined, we must check that for all $a \in \overline{Q'_1}$, $\theta(l_a r_a - e_{h(a)}) = 0$, and $\theta(r_a l_a - e_{h(a)}) = 0$. We have

$$\theta(l_b r_b - e_{h(b)}) = r_{c^*}(e_{h(c)} - c^* l_c c) - e_{h(c^*)} = r_{a^*} - c^* r_c l_c c - e_{h(c^*)} = 0,$$

$$\theta(r_b l_b - e_{h(b)}) = (e_{h(c)} - c^* l_c c) r_{c^*} - e_{h(c^*)} = r_{a^*} - c^* l_c r_c c - e_{h(c^*)} = 0,$$

and similarly for $a = b^*$, and for the remaining arrows it is obvious. There is a map $\phi : L_Q \to L_{Q'}$ defined similarly, sending c to $-b^* l_b$, c^* to b, l_c to r_{b^*}, l_{c^*} to r_b. Since $\theta(\phi(c)) = \theta(-b^* l_b) = l_c c r_{c^*} = l_c r_c c = c$ and $\phi(\theta(b^*)) = \phi(-l_c c) = r_{b^*} b^* l_b = b r_b l_b = b$, θ and ϕ are mutual inverses, and are therefore isomorphisms. Clearly $\theta(\mu_{Q',<'}) = \mu_{Q,<}$ since $\theta(1 + bb^*) = 1 - c^* l_c c = (1 + c^* c)^{-1}$ and $\theta(1 + b^* b) = 1 - l_c c c^* = (1 + cc^*)^{-1}$, and so $\Lambda^q(Q',<') \cong \Lambda^q(Q,<)$.

We now prove independence of the ordering. First note that $I_{\mu_{Q,<}} = I_{\mu_{Q,<^\circ}}$, where $<^\circ$ is the ordering with $a_2 <^\circ a_3 <^\circ \cdots <^\circ a_n <^\circ a_1$ (this follows by conjugating $\mu_{Q,<}$ by $s_{a_1}^{\epsilon(a_1)}$). It therefore suffices to show that $\Lambda^q(Q,<) \cong \Lambda^q(Q,<'')$, where $<''$ is the ordering with $a_2 <'' a_1 <'' a_3 <'' \cdots <'' a_n$. If $h(a_1) \neq h(a_2)$, then it is trivially true since $\mu_{Q,<} = \mu_{Q,<''}$, so assume that $h(a_1) = h(a_2)$. If $a_1 = a_2^*$, then a_1 is a loop, and then $\Lambda^q(Q,<'')$ is the same as $\Lambda^q(Q',<')$ where Q' and $<'$ are obtained by reversing a_1, and the argument above shows that this is isomorphic to $\Lambda^q(Q,<)$.

So assume that $a_1 \neq a_2^*$, and by reversing arrows if necessary that $\epsilon(a_1) = \epsilon(a_2) = 1$. Define an isomorphism $\theta : L_Q \to L_Q$ which sends a_1 to $r_{a_2}a_1$, a_1^* to $a_1^* l_{a_2}$, l_{a_1} to $r_{a_2}l_{a_1}l_{a_2}$ and each remaining arrow, each remaining l_a and each trivial path to themselves. It is clear that $\theta(l_a r_a - e_{h(a)}) = 0$ and $\theta(r_a l_a - e_{h(a)}) = 0$ for each arrow a, the only non trivial case being a_1, which follows since

$$\theta(r_{a_1}) = e_{h(a_1)} + r_{a_2}a_1a_1^*l_{a_2} = r_{a_2}r_{a_1}l_{a_2}.$$

Since $\theta(1+a_1a_1^*)(1+a_2a_2^*) = (1+r_{a_2}a_1a_1^*l_{a_2})(1+a_2a_2^*) = (1+a_2a_2^*)(1+a_1a_1^*)$ and $\theta(1+a_1^*a_1) = 1+a_1^*a_1$, it is clear that $\theta(\mu_{Q,<}) = \mu_{Q,<''}$, and therefore $\Lambda^q(Q,<) \cong \Lambda^q(Q,<'')$. □

We can therefore write $\Lambda^q(Q)$ (or sometimes Λ^q) instead of $\Lambda^q(Q,<)$.

2.2 Alternative definitions

In [11, Lemma 8.1], it is shown that if Q is a star shaped quiver, the multiplicative preprojective algebra can be defined as a quotient of $K\overline{Q}$, rather than using localisation. One can ask whether this is possible for other quivers, and in this section we investigate this interesting question.

First some notation. For all arrows a of $\overline{Q}$, denote $e_{h(a)} + aa^*$ and $1 + aa^*$ by r_a and s_a respectively. Henceforth 'r_a is invertible' is taken to mean 'r_a is invertible in $e_{h(a)}\Lambda^q(Q)e_{h(a)}$'. We say an ordering on the set of arrows of $\overline{Q}$ is *admissible* if $a \in Q_1$, $b^* \in Q_1$ implies $a < b$.

Definition 2.2.1. Given an admissible ordering $<$ on the set of arrows of $\overline{Q}$, we define $\tilde{\Lambda}^q(Q,<)$ to be $K\overline{Q}/I_{\mu^q}$, where I_{μ^q} is the ideal generated by the elements

$$\mu_v^q = \prod_{i=1}^{k_v} r_{a_{vi}} - q_v \prod_{i=k_v+1}^{l_v} r_{a_{vi}},$$

where the a_{vi} are the arrows of $\overline{Q}$ with head at v, labelled so that $a_{v1} < a_{v2} < \cdots < a_{v,l_v}$, and k_v is the number of arrows of Q with head at v (so each a_{vi} with $i > k_v$ is of the form a^* for some $a \in Q_1$ with $t(a) = v$). The empty product is taken to be e_v. [Note that we can understand 'μ^q' to be the element $\sum_{v \in Q_0} \mu_v^q$, as well as the set $\{\mu_v^q : v \in Q_0\}$ since the ideal 'I_{μ^q}' is the same in both cases.]

At first sight, this is perhaps a more natural definition of a 'multiplicative preprojective algebra' (note the similarity with the definition of the of the deformed preprojective algebra). However it is almost certainly not the case that $\tilde{\Lambda}^q(Q,<)$ is independent of $<$ (which will be illustrated in Example 2.2.4), and so the original definition seems the correct one. It is more desirable to speak of 'the' multiplicative preprojective algebra for a quiver Q, rather than have one for each ordering of the arrows, which may or may not be isomorphic to each other. We are interested in whether or not $\tilde{\Lambda}^q(Q,<)$ is isomorphic to the multiplicative preprojective algebra. Let ϕ be the natural map $K\overline{Q} \to \tilde{\Lambda}^q(Q,<)$, and make the following definition.

Definition 2.2.2. If ϕ satisfies (†), then we say $<$ is a good ordering.

Lemma 2.2.3. *If $<$ is a good ordering, then $\tilde{\Lambda}^q(Q,<) \cong \Lambda^q(Q)$ (via ϕ).*

Proof. Let $<'$ be the ordering defined as follows,

$$a <' b \text{ if } \begin{cases} a < b \text{ and } b \text{ in } Q_1 \\ b < a \text{ and } b^* \text{ in } Q_1. \end{cases}$$

We show that ϕ satisfies (‡) for this ordering. Note that if we label the arrows as in Definition 2.2.1, one has $a_{v,l_v} <' a_{v,l_v-1} <' \cdots <' a_{v,k_v+1}$. In view of this, we clearly have

$$\prod_{a\in\overline{Q}_1} (\phi(s_a))^{\epsilon(a)} = \sum_{v\in Q_0} \left(\prod_{i=1}^{k_v} \phi(r_{a_{vi}})\right) \left(\prod_{i=0}^{l_v-k_v-1} \phi(r_{a_{v,l_v-i}})^{-1}\right).$$

Using the relations μ_v^q, this equals

$$\sum_{v\in Q_0} q_v \left(\prod_{i=k_v+1}^{l_v} \phi(r_{a_{vi}})\right) \left(\prod_{i=0}^{l_v-k_v-1} \phi(r_{a_{v,l_v-i}})^{-1}\right).$$

Each $\phi(r_{a_{vi}})$ cancels with a $\phi(r_{a_{vi}})^{-1}$, so it equals

$$\sum_{v\in Q_0} \phi(q_v e_v).$$

Hence ϕ satisfies both (†) and (‡), and is clearly universal. $\square$

In view of Lemma 2.1.2, it is clear that an ordering is good if and only if each r_a is invertible. The key tool for showing that the r_a are invertible is the following property.

$(*)$ $\quad r_a$ is (left/right) invertible if and only if r_{a^*} is (left/right) invertible.

This follows easily from Lemma 2.1.2 and the fact that if x, y are elements of a ring such that $1 + xy$ is invertible, then $1 + yx$ is also invertible with inverse $1 - y(1 + xy)^{-1}x$.

Example 2.2.4. Let Q be the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift left, "b"] \arrow[r, shift right, "c"'] & \underset{2}{\bullet} \end{tikzcd}$$

There are two fundamentally different orderings to consider, $<$ and $<'$ where $b < c < c^* < b^*$ and $c <' b <' c^* <' b^*$. We show that $<$ is good whereas $<'$ isn't.

In the first case we have $\tilde{\Lambda}^q(Q, <) = K\overline{Q}/I_\mu$, where I_μ is generated by $q_1 r_{c^*} r_{b^*} - e_1$, $r_b r_c - q_2 e_2$. Clearly the relations make r_c left invertible and r_{c^*} right invertible, and by $(*)$, both are invertible. Similarly r_b and r_{b^*} are both invertible and so the ordering is good and by Lemma 2.2.3, $\tilde{\Lambda}^q(Q, <)$ is isomorphic to the multiplicative preprojective algebra.

In the second case, we have $A = \tilde{\Lambda}^q(Q, <') = K\overline{Q}/I_{\mu'}$, where $I_{\mu'}$ is generated by $q_1 r_{c^*} r_{b^*} - e_1$, $r_c r_b - q_2 e_2$. If we attempt a similar argument to the one above, we can only show the invertibility of each r_a on one side. This suggests that the elements are not all invertible in A, which we now prove by constructing a representation X of A in which the corresponding linear maps (denoted by X_{r_a}) are not invertible, and so A is not the multiplicative preprojective algebra via the natural map. [Of course there may still be a universal map $K\overline{Q} \to A$ satisfying (†) and (‡), thus making A the multiplicative preprojective algebra. However, since the natural map fails, this seems highly unlikely.] Note that such a representation must be infinite dimensional since a linear map between finite dimensional vector spaces is invertible if it is invertible on one side.

Let V be the vector space with countable basis $\{v_i : i \in \mathbb{N}\}$. Let X be the representation of $\overline{Q}$ with $X_1 = X_2 = V$,

$$\begin{aligned} X_b &= I_{q_1^{-1}}(S^-_{q_1^{-1}} - 1_V), \quad X_{b^*} = I_{q_1}, \\ X_c &= (S^+_{q_2} - 1_V)I_{q_2}, \quad X_{c^*} = I_{q_2^{-1}}, \end{aligned}$$

where I_q, S^-_q, S^+_q are the linear maps defined as follows

$$\begin{aligned} I_q(v_i) &= q^i v_i \text{ for all } i, \\ S^+_q(v_i) &= qv_{i+1} \text{ for all } i, \\ S^-_q(v_i) &= \begin{cases} 0 & \text{if } i = 0 \\ qv_{i-1} & \text{if } i \geq 1. \end{cases} \end{aligned}$$

Note that $I_{q^{-1}}S^+_q I_q = S^+_1$ since

$$(I_{q^{-1}}S^+_q I_q)(v_i) = q^i(I_{q^{-1}}S^+_q)(v_i) = q^{i+1}(I_{q^{-1}})(v_{i+1}) = v_{i+1} = S^+_1(v_i),$$

and $I_q S^-_q I_{q^{-1}} = S^-_1$ since if $i > 0$ we have

$$(I_q S^-_q I_{q^{-1}})(v_i) = q^{-i}(I_q S^-_q)(v_i) = q^{1-i}(I_q)(v_{i-1}) = v_{i-1} = S^-_1(v_i),$$

and if $i = 0$, we have $(I_q S^-_q I_{q^{-1}})v_0 = (I_q S^-_q)v_0 = 0 = S^-_1(v_0)$. We therefore have

$$\begin{aligned} X_{r_{b^*}} &= 1_V + X_{b^*}X_b = S^-_{q_1^{-1}}, \\ X_{r_b} &= 1_V + X_b X_{b^*} = I_{q_1^{-1}} S^-_{q_1^{-1}} I_{q_1} = S^-_1, \\ X_{r_{c^*}} &= 1_V + X_{c^*}X_c = I_{q_2^{-1}} S_{q_2} I_{q_2} = S^+_1, \\ X_{r_c} &= 1_V + X_c X_{c^*} = S^+_{q_2}. \end{aligned}$$

Now since $S^-_q S^+_r = qr1_V$, we have $q_1 X_{r_{b^*}} X_{r_{c^*}} = 1_V$ and $X_{r_b} X_{r_c} = q_2 1_V$, so X is a representation of A, but none of the X_{r_a} are invertible.

We would like to obtain a classification of the quivers which have a good ordering, but this is a difficult problem which remains open. The previous example is a special case of the following lemma, which is the most general result we have obtained.

Lemma 2.2.5. *If Q is a bipartite quiver, then Q has a good ordering.*

Proof. Choose any ordering on the arrows of Q and label them so that $a_1 < a_2 < \cdots < a_n$. Extend this to an admissible ordering of the arrows of $\overline{Q}$:

$$a < b \text{ if and only if } \begin{cases} \text{if } a \in Q_1,\ b \in Q_1 \text{ and } a < b, \\ \text{if } a \in Q_1,\ b^* \in Q_1, \\ \text{if } a^* \in Q_1,\ b^* \in Q_1 \text{ and } b^* < a^*. \end{cases}$$

We claim that each r_a is invertible in $\tilde{\Lambda}^q(Q, <)$. Since every vertex of Q is a source or a sink, there are no arrows with tail at $h(a_1)$. Since a_1 is the minimal arrow, $\mu^q_{h(a_1)}$ has the form $r_{a_1}x - q_{h(a_1)}e_{h(a_1)}$ for some product x of some r_{a_i} with $i > 1$. This makes r_{a_1} right invertible. Similarly, the are no arrows with head at $t(a_1)$, and a_1^* is the maximal arrow, so $\mu^q_{h(a_1)}$ has the form $q_{t(a_1)}yr_{a_1{}^*} - e_{t(a)}$, where y denotes a product of some $r_{a_i{}^*}$ with $i > 1$. This ensures that $r_{a_1{}^*}$ is left invertible. Hence both r_{a_1} and $r_{a_1{}^*}$ are invertible.

Assuming that r_{a_i} and $r_{a_i{}^*}$ are invertible for all $i < k$, we show that r_{a_k} and $r_{a_k{}^*}$ are invertible. By a similar argument to the one above, $\mu^q_{h(a_k)}$ has the form $wr_{a_k}x - q_{h(a_k)}e_{h(a_k)}$ (where w, x denote a product of some r_{a_i} with $i < k$, $i > k$ respectively). Using the invertibility of w, I_{μ^q} contains $r_{a_k}xw = q_{h(a_k)}e_{h(a_k)}$ which makes r_{a_k} right invertible. Similarly $r_{a_k{}^*}$ is left invertible, so both are invertible.

Hence it follows by induction that $<$ is a good ordering. □

Along with the result for star-shaped quivers, Lemma 3.3.2, this shows that good orderings exist for most of the quivers we consider in this thesis (so that in Chapters 3,4,5, we can always assume that $\Lambda^q(Q) = K\overline{Q}/I_{\mu^q}$). However, this is far from a complete understanding, as is shown by the following example.

Example 2.2.6. Let Q be the quiver obtained by orienting the complete graph on vertices 1,2,3,4 so that if $a : u \to v$ is an arrow, $u < v$. Changing the notation slightly, for each arrow $a : u \to v$ of $\overline{Q}$, let r_{uv} denote the element $e_v + aa^*$. Let

I be the ideal of KQ generated by the relations

$$r_{12}r_{13}r_{14} = q_1e_1, \tag{2.1}$$

$$r_{24}r_{23} = q_2r_{21}, \tag{2.2}$$

$$r_{34} = q_3r_{31}r_{32}, \tag{2.3}$$

$$e_4 = q_4r_{41}r_{43}r_{42}. \tag{2.4}$$

Property $(*)$ now reads r_{uv} is invertible if and only if r_{vu} is invertible. Using this, we can work through the quiver and eventually show all the r_{uv} are invertible. From (2.4), r_{41} is right invertible and from (2.1), r_{14} is left invertible, so both are invertible. Since r_{12} is right invertible by (2.1), so is r_{21}, and then by multiplying (2.2) by this right inverse, so is r_{24}. Since r_{42} is left invertible by (2.4), both r_{42} and r_{24} are invertible. It follows that r_{43} is invertible, and so is r_{34}. Then multiplying (2.3) by the right inverse of r_{34} shows that r_{31} is right invertible. Using the invertibility of r_{14} and (2.1) shows r_{13} is left invertible, and so both r_{31} and r_{13} are invertible. It quickly follows that r_{12}, r_{21}, r_{32}, r_{23} are all invertible, and so the ordering is good and $\Lambda^q(Q) \cong K\overline{Q}/I$.

For simplicity, we now assume that $q = 1$ (in any case, it seems likely that the question of whether an ordering is good or not does not depend on q). We now attack the problem from the other direction, namely, instead of determining the quivers possessing good orderings, we give examples of quivers for which no ordering is good. We say such quivers are *bad*. Unfortunately, it is quite difficult to prove a quiver is bad, as they must be quite complicated, and checking every possible ordering is a lengthy process. For example, if we had instead chosen (2.3) to be $r_{34} = q_3r_{32}r_{31}$ in the above example, then the ordering is not good, but we have to work through most of the calculation to show this. Note that we work under the assumption that if we can't prove that the r_c are invertible by using $(*)$, then they aren't invertible - one can prove it by constructing a representation as in Example 2.2.4.

The following lemma is useful for obtaining examples of bad quivers.

Lemma 2.2.7. *If a quiver is bad, then all quivers which contain it are bad.*

Proof. Let Q be a bad quiver and suppose Q' contains Q as a subquiver. Given an ordering $<'$ on the arrows of $\overline{Q'}$, let $<$ be the induced ordering on the arrows of $\overline{Q}$. Since Q is bad, there exists a representation $X^<$ of $\tilde{\Lambda}(Q,<)$ in which $X^<_{r_a}$ is not invertible for some arrow a of $\overline{Q}$. Let $Y^<$ be the representation of Q' where $Y^<_v = X_v$ if $v \in Q_0$ and zero otherwise, and let $Y^<_a = X_a$ if $a \in \overline{Q}_1$, and zero otherwise. It is clear that $Y^<$ is a representation of $\tilde{\Lambda}(Q',<')$ in which $Y^<_{r_a}$ is not invertible. This can be done for all orderings, and so Q' is bad. $\square$

So we can attempt to find the *minimal* bad quivers, the quivers which are bad, but all subquivers of them are not. We have obtained the following list.

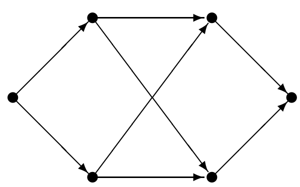
Minimal Bad Quivers.

1. A quiver of type $\tilde{A}_n$, which is oriented cyclically.

2. A quiver of type $\tilde{A}_{i,j;m,n}$, which is a quiver without an oriented cycle with a source u of outdegree 2, a sink w of indegree 2, a vertex v of indegree 2 and outdegree 2, and all other vertices being outdegree 1 and indegree 1. The numbers i, j refer to the length of the two paths between u and v, and m, n to the length of the two paths between v and w. The simplest quiver of this type is $A_{1,1;1,1}$, which looks like

$$\begin{tikzcd} \underset{u}{\bullet} \arrow[r, shift left] \arrow[r, shift right] & \underset{v}{\bullet} \arrow[r, shift left] \arrow[r, shift right] & \underset{w}{\bullet} \end{tikzcd}$$

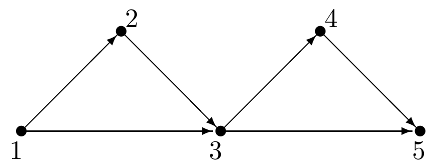
3. There are others, e.g.



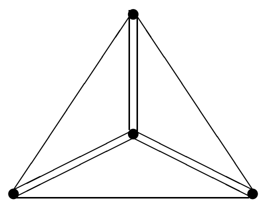
Of course, the multiplicative preprojective algebra depends only on the underlying graph of Q (see Theorem 2.1.3), and it is possible to reorient the above quivers (other than the loop) so that they have good orderings. One might conjecture that, given any graph without loops, one can find an orientation Q with a good ordering $<$, and thus one can define the multiplicative preprojective algebra as $\tilde{\Lambda}^q(Q,<)$. However this is not the case. We say a graph is *bad* if any orientation of it is bad. The following graphs are bad because any orientation must contain one of the bad quivers above as a subquiver.

Bad Graphs.

1. The graph on three vertices, with two edges between each vertex (any orientation must contain an oriented cycle or $\tilde{A}_{1,1;1,1}$).
2. The complete graph K_5. Any orientation without oriented cycles must have a source and a sink (labelled 1 and 5 say). Consider the remaining three vertices, any non cyclic orientation of the three arrows between them must determine a relative source 2 and a relative sink 4. Labelling the remaining vertex 3, we have a bad subquiver $\tilde{A}_{1,2;1,2}$:



3. An orientation of the following graph must contain either an oriented cycle, $\tilde{A}_{1,1;1,1}$, $\tilde{A}_{1,1;1,2}$ or $\tilde{A}_{1,2;1,1}$.



It is possible that these graphs have some graph-theoretic property which may give rise to a characterisation (the presence of K_5 suggests this), but we have not be able to see it.

Going back to the question raised at the start of the section, we have been interpreting the question 'Can we define $\Lambda(Q)$ as a quotient of $K\overline{Q}$?' in a restricted way, by effectively the condition 'by an ideal generated by the obvious multiplicative relation at each vertex'. This is the natural question to investigate, but it is interesting to answer question without this restriction, especially since we can obtain the nice answer that it is possible for all quivers without loops (with the drawback being that in practice it will be difficult to write down exactly what the quotient is). In the case where Q does contain a loop then little can be done, e.g., if Q consists of one vertex v and a loop a, then $K\overline{Q} \cong K\langle a, a^*\rangle$. If one takes the quotient by the relation $\mu_v^1 = aa^* - a^*a$, the algebra obtained is the commutative ring in two variables, which is already smaller than $\Lambda^1(Q) \cong K[x, y, (1 + xy)^{-1}]$.

Theorem 2.2.8. *If Q has no loops, $\Lambda^q(Q)$ is isomorphic to a quotient of $K\overline{Q}$.*

Proof. Since $\Lambda^q(Q)$ is independent of ordering and orientation, we can assume that Q has no oriented cycles (this would obviously be impossible if Q had a loop), and choose an admissible ordering $<$. We define an ideal J of $\overline{Q}$ as being generated by a set of elements $\{\mu_{vj} : v \in Q_0, 1 \leq j \leq l_v\}$, where l_v is the number of arrows of Q with head at v (except in the case that v is a source, when $l_v = 1$). These elements are defined in the course of the proof. Eventually we show that $\Lambda^q(Q, <)$ is isomorphic to $K\overline{Q}/J$.

Define a k-sink to be a vertex v where the maximal length of a path starting at v is k (so that a 0-sink is just a sink). At stage k we write down the μ_{vj} where v is a k-sink.

Stage 0.

Since Q has no oriented cycles, it has a 0-sink v. We label the arrows of Q with head at v so that $a_1 < \cdots < a_{l_v}$. Let j be in the range $1, \ldots, l_v$ and define

$$\mu_{vj} = r_{a_j} r_{a_{j+1}} \ldots r_{a_{l_v}} r_{a_1} \ldots r_{a_{j-1}} - q_v e_v.$$

Let

$$l_{a_i} = q_v^{-1} r_{a_{j+1}} \ldots r_{a_{l_v}} r_{a_1} \ldots r_{a_{j-1}},$$

and since J contains both $\mu_{vi} = q_v r_{a_i} l_{a_i} - q_v e_v$ and $\mu_{v,i+1} = q_v l_{a_i} r_{a_i} - q_v e_v$, we see l_{a_i} is the inverse of r_{a_i} in $e_v(K\overline{Q}/J)e_v$. Repeat with all other 0-sinks.

Stage k.

We assume that stage $k-1$ has been done. Namely, if u is an l-sink with $l \leq k-1$, then each μ_{uj} has been defined and if a is an arrow with head at u, r_a is invertible in $K\overline{Q}/J$. Suppose that v is a k-sink, and label the arrows of Q with head at v so that $a_1 < \cdots < a_{l_v}$, and let j be in the range $1, \ldots, l_v$. Define

$$\mu_{vj} = r_{a_j} r_{a_{j+1}} \ldots r_{a_{l_v}} t_v r_{a_1} \ldots r_{a_{j-1}} - q_v e_v,$$

where $t_v = \prod_{t(a)=v} r_{a^*}^{-1}$ with the product taken in the order given by $<$. Note that $r_{a^*}^{-1}$ makes sense as r_a is invertible by the comments above. By a similar argument to that in stage 0, each r_{a_i} is invertible. Repeat with all the other k-sinks. Note that in the case that a k-sink has no arrows with head at v, we define μ_{v1} to be $t_v - q_v e_v$.

Since all vertices of Q must be a k-sink for some k, this completes the definition of J, and shows that for each arrow of $\overline{Q}$, s_a is invertible. Since additionally

$$\mu = \prod_{a \in \overline{Q}_1} {s_a}^{\epsilon(a)} - \sum_{v \in Q_0} q_v e_v = \sum_{v \in Q_0} \mu_{v1},$$

the natural map $\phi : K\overline{Q} \to K\overline{Q}/J$ satisfies (†) and (‡). To show $K\overline{Q}/J$ is the multiplicative preprojective algebra, we must show that ϕ is universal. Clearly if $\theta : K\overline{Q} \to A$ satisfies (†) and (‡), then there is a unique induced map $\psi : K\overline{Q}/J \to A$, provided $\theta(J) = 0$. This is satisfied since $\theta(\mu_{v1}) = \theta(e_v \mu e_v) = 0$, and $\mu_{vj} = {r_{a_{j-1}}}^{-1} \ldots {r_{a_1}}^{-1} \mu_{v1} r_{a_1} \ldots r_{a_{j-1}}$. □

2.3 Reflection functors

In [10] it was shown that there exist reflection functors for deformed preprojective algebras. In this section we adapt the 'middle convolution' operation of Dettweiler and Reiter [12] to show that an analogue of these reflection functors exist for multiplicative preprojective algebras. The construction is very similar (but the calculations are more complicated).

We can identify representations of Λ^q with representations of $\overline{Q}$ which satisfy

$$1_{X_{h(a)}} + X_a X_{a^*} \text{ is an invertible endomorphism of } X_{h(a)} \text{ for all } a \in \overline{Q}_1, \quad (2.5)$$

$$\prod_{\substack{a\in\overline{Q}_1\\ h(a)=v}} (1_{X_{h(a)}} + X_a X_{a^*})^{\epsilon(a)} = q_v 1_{X_{h(a)}} \text{ for all } v \in Q_0. \quad (2.6)$$

Given $\alpha \in \mathbb{Z}^{Q_0}$, define $q^\alpha = \prod_i q_i^{\alpha_i}$. Recall that if v is a loop free vertex then there is a simple reflection $s_v : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ given by $s_v(\alpha) = \alpha - (\alpha, \epsilon_v)\epsilon_v$. There is a reflection $t_v : K^{Q_0} \to K^{Q_0}$ given by $t_v(q)_u = q_u q_v^{-(\epsilon_u,\epsilon_v)}$. This is dual to the s_v, as $(t_v q)^\alpha = q^{s_i(\alpha)}$.

Theorem 2.3.1. *[11] If v is a loopfree vertex of Q with $q_v \neq 1$, there is an equivalence E_v from the category of representations of $\Lambda^q(Q)$ to the category of representations of $\Lambda^{t_v(q)}(Q)$ which acts as s_v on dimension vectors.*

The proof of this theorem comprises the rest of this section. We assume that v is a sink and denote the arrows with head at v as $a_1, a_2, \ldots, a_n$ so that $a_i < a_{i+1}$ for all i. Let X be a representation of Λ^q. We identify X with a representation of $\overline{Q}$ satisfying in (2.5) and (2.6). In particular, the relation at the vertex v guarantees that

$$(1_{X_v} + X_{a_1} X_{a_1^*})(1_{X_v} + X_{a_2} X_{a_2^*}) \ldots (1_{X_v} + X_{a_n} X_{a_n^*}) = q_v 1_{X_v}. \quad (2.7)$$

For $1 \leq i \leq n+1$, we define

$$\xi_i = (1_{X_v} + X_{a_1} X_{a_1^*})(1_{X_v} + X_{a_2} X_{a_2^*}) \ldots (1_{X_v} + X_{a_{i-1}} X_{a_{i-1}^*}).$$

Lemma 2.3.2. *We have the following formulas.*

$$\sum_{j=1}^{i-1} \xi_j X_{a_j} X_{a_j^*} = \xi_i - 1_{X_v}, \quad (2.8)$$

$$\sum_{j=1}^{n} \xi_j X_{a_j} X_{a_j^*} = (q_v - 1) 1_{X_v}, \quad (2.9)$$

$$\sum_{j=1}^{i-1} \xi_j X_{a_j} X_{a_j^*} + \frac{1}{q_v} \sum_{j=i}^{n} \xi_j X_{a_j} X_{a_j^*} + \frac{1-q_v}{q_v} \xi_i = 0. \tag{2.10}$$

Proof. The first equation is obvious, and the second follows by putting $i = n+1$ in (2.8) and noting that (2.7) says $\xi_{n+1} = q_v 1_{X_v}$. Finally, the third equation is equivalent to

$$\frac{1}{q_v} \sum_{j=1}^{n} \xi_j X_{a_j} X_{a_j^*} + \frac{q_v - 1}{q_v} \sum_{j=1}^{i-1} \xi_j X_{a_j} X_{a_j^*} + \frac{1-q_v}{q_v} \xi_i = 0,$$

and using (2.8) and (2.9), this is equivalent to

$$\frac{1}{q_v}(q_v - 1) 1_{X_v} + \frac{q_v - 1}{q_v}(\xi_i - 1_{X_v}) + \frac{1-q_v}{q_v} \xi_i = 0,$$

which is obviously true. $\square$

Let

$$X_\oplus = \bigoplus_{j=1}^{n} X_{t(a_j)}.$$

We denote the natural inclusions and projections between $X_\oplus$ and $X_{t(a_i)}$ by ι_i and π_i respectively. Define maps $\iota : X_v \to X_\oplus$, $\pi : X_\oplus \to X_v$ by

$$\iota = \sum_{j=1}^{n} \iota_j X_{a_j^*}, \quad \pi = \frac{1}{q_v - 1} \sum_{j=1}^{n} \xi_j X_{a_j} \pi_j.$$

Using (2.9), we have

$$\pi\iota = \frac{1}{q_v - 1} \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_j X_{a_j} \pi_j \iota_k X_{a_k^*} = \frac{1}{q_v - 1} \sum_{j=1}^{n} \xi_j X_{a_j} X_{a_j^*} = 1_{X_v},$$

so $\iota\pi$ and $\epsilon = 1_{X_\oplus} - \iota\pi$ are idempotent endomorphisms of $X_\oplus$. Now define $\phi_i : X_{t(a_i)} \to X_\oplus$,

$$\phi_i = \sum_{j=1}^{i-1} \iota_j X_{a_j^*} X_{a_i} + \frac{1}{q_v} \sum_{j=i}^{n} \iota_j X_{a_j^*} X_{a_i} + \frac{1-q_v}{q_v} \iota_i.$$

Note that if $j < i$,

$$\pi_j \phi_i = X_{a_j^*} X_{a_i}. \tag{2.11}$$

Lemma 2.3.3. *[11] For all i, $\pi\phi_i = 0$.*

Proof. We have

$$\begin{aligned}
\pi\phi_i &= \frac{1}{q_v-1}\sum_{k=1}^{n}\xi_k X_{a_k}\pi_k\left(\sum_{j=1}^{i-1}\iota_j X_{a_j^*}X_{a_i}+\frac{1}{q_v}\sum_{j=i}^{n}\iota_j X_{a_j^*}X_{a_i}+\frac{1-q_v}{q_v}\iota_i\right)\\
&= \frac{1}{q_v-1}\left(\sum_{j=1}^{i-1}\xi_j X_{a_j}X_{a_j^*}X_{a_i}+\frac{1}{q_v}\sum_{j=i}^{n}\xi_j X_{a_j}X_{a_j^*}X_{a_i}+\frac{1-q_v}{q_v}\xi_i X_{a_i}\right)\\
&= \frac{1}{q_v-1}\left(\sum_{j=1}^{i-1}\xi_j X_{a_j}X_{a_j^*}+\frac{1}{q_v}\sum_{j=i}^{n}\xi_j X_{a_j}X_{a_j^*}+\frac{1-q_v}{q_v}\xi_i\right)X_{a_i}\\
&= 0
\end{aligned}$$

using (2.10). □

Lemma 2.3.4. *[11] For all $0 \le m \le n$ we have*

$$(1_{X_\oplus}+\phi_1\pi_1)(1_{X_\oplus}+\phi_2\pi_2)\dots(1_{X_\oplus}+\phi_m\pi_m)=1_{X_\oplus}+\frac{1-q_v}{q_v}\sum_{j=1}^{m}\epsilon\iota_j\pi_j$$

Proof. By induction on m. If $m=0$ there is nothing to prove. Assume that the formula is true for $l=m-1$. We want to show that it holds at m, namely that

$$\left(1_{X_\oplus}+\frac{1-q_v}{q_v}\sum_{j=1}^{m-1}\epsilon\iota_j\pi_j\right)\left(1_{X_\oplus}+\phi_m\pi_m\right)=1_{X_\oplus}+\frac{1-q_v}{q_v}\sum_{j=1}^{m}\epsilon\iota_j\pi_j.$$

Multiplying out and rearranging, this is equivalent to

$$\phi_m\pi_m=\frac{1-q_v}{q_v}\epsilon\left(\iota_m\pi_m-\frac{1}{q_v}\sum_{j=1}^{m-1}\iota_j\pi_j\phi_m\pi_m\right).$$

Using (2.11), the right hand side of this is

$$\frac{1-q_v}{q_v}(1_{X_\oplus}-\iota\pi)\left(\iota_m-\sum_{j=1}^{m-1}\iota_j X_{a_j}^* X_{a_m}\right)\pi_m.$$

Multiplying out, this becomes

$$\frac{1-q_v}{q_v}\left(\iota_m-\sum_{j=1}^{m-1}\iota_j X_{a_j}^* X_{a_m}-\iota\pi\iota_m+\iota\pi\sum_{j=1}^{m-1}\iota_j X_{a_j}^* X_{a_m}\right)\pi_m.$$

Now since $\pi\iota_j = \frac{1}{q_v - 1}\xi_j X_{a_j}$ this is

$$\left(\frac{1-q_v}{q_v}\left(\iota_m - \sum_{j=1}^{m-1}\iota_j X^*_{a_j}X_{a_m}\right) + \frac{1}{q_v}\iota\xi_m X_{a_m} - \sum_{j=1}^{m-1}\iota\xi_j X_{a_j}X^*_{a_j}X_{a_m}\right)\pi_m.$$

Using (2.8), this simplifies to

$$\left(\frac{1-q_v}{q_v}\left(\iota_m - \sum_{j=1}^{m-1}\iota_j X^*_{a_j}X_{a_m}\right) + \frac{1}{q_v}\iota X_{a_m}\right)\pi_m.$$

Substituting the expression for ι, this equals

$$\left(\frac{1-q_v}{q_v}\iota_m + \frac{1}{q_v}\sum_{j=m}^{n}\iota_j X^*_{a_j}X_{a_m} + \sum_{j=1}^{m-1}\iota_j X^*_{a_j}X_{a_m}\right)\pi_m,$$

which is $\phi_m\pi_m$. □

We define a representation X' of $\overline{Q}$. Let $X'_v = \mathrm{Im}(\epsilon) = \mathrm{Ker}(\iota\pi) = \mathrm{Ker}(\pi)$ and let $X'_u = X_u$ if $u \neq v$. Denote by ι' the inclusion of X'_v in $X_\oplus$. If a is not incident with v, let $X'_a = X_a$. Otherwise, let $X'_{a_i{}^*} = \pi_i\iota'$, and let X'_{a_i} be the unique map such that $\phi_i = \iota' X'_{a_i}$. This is possible since $\mathrm{Im}\,\phi_i \subseteq X'_v$ by Lemma 2.3.3 and is uniquely determined since ι' is injective. Now let $q' = t_v(q)$, and let α be the dimension vector of X.

Lemma 2.3.5. *[11] X' is a representation of $\Lambda^{q'}(Q)$ of dimension vector $s_v(\alpha)$.*

Proof. We must check that the X'_a satisfy the following relation for all $v \in Q_0$,

$$\prod_{\substack{a\in\overline{Q}_1\\ h(a)=v}} (1_{X'_v} + X'_a X'_{a^*})^{\epsilon(a)} = q'_v 1_{X'_v}.$$

At vertices different from v and the $t(a_i)$, this is trivial. For all i, it is clear that

$$\begin{aligned}
1_{X_{t(a_i)}} + X'_{a_i^*}X'_{a_i} &= 1_{X_{t(a_i)}} + \pi_i\phi_i \\
&= 1_{X_{t(a_i)}} + \frac{1}{q_v}X_{a_i^*}X_{a_i} + \frac{1-q_v}{q_v}1_{X_{t(a_i)}} \\
&= \frac{1}{q_v}(1_{X_{t(a_i)}} + X_{a_i^*}X_{a_i}),
\end{aligned}$$

so that the relation is satisfied if $v = t(a_i)$ (recall that $q'_{t(a_i)} = q_{t(a_i)} q_v^{-k}$, where k is the number of arrows between v and $t(a_i)$). Finally, we put $m = n$ in Lemma 2.3.4 and we have

$$(1_{X_\oplus} + \phi_1\pi_1)(1_{X_\oplus} + \phi_2\pi_2)\dots(1_{X_\oplus} + \phi_n\pi_n) = 1_{X_\oplus} + \frac{1-q_v}{q_v}\epsilon.$$

Restricting to X'_v gives

$$(1_{X_v} + X'_{a_1}X'_{a_1{}^*})(1_{X_v} + X'_{a_2}X'_{a_2{}^*})\dots(1_{X_v} + X'_{a_n}X'_{a_n{}^*}) = \frac{1}{q_v}1_{X_v},$$

which shows the relation at v holds. We have $X_\oplus = \mathrm{Im}(\iota\pi) \oplus \mathrm{Im}(1-\iota\pi) = X_v \oplus X'_v$, so $\dim X'_v = \dim X_\oplus - \dim X_v = \sum_i \alpha_{t(a_i)} - \alpha_v = s_v(\alpha)$, and hence $\underline{\dim}\, X' = (s_v(\alpha))_v$. □

One can define a functor by setting $E_v(X) = X'$ for any object X, and if $\theta : X \to Y$ is a morphism, we define $E_v(\theta)$ by $(E_v(\theta))_u = \theta_u$ if $u \neq v$ and $(E_v((\theta)_v$ to be the unique map with $\iota'_Y(E_v(\theta))_v = \sum_k \theta_{t(a_k)}\iota'_X$.

Lemma 2.3.6. *[11] E_v is an equivalence of categories.*

Proof. Clearly we can define a functor E'_v which takes a representation X' of $\Lambda^{q'}$ to a representation X'' of $\Lambda^{t_v(q')} = \Lambda^q$. We show that there is a natural isomorphism $X'' \to X$, and thus E'_v is the inverse of E_v. Note that

$$\bigoplus_{j=1}^{n} X'_{t(a_j)} = \bigoplus_{j=1}^{n} X_{t(a_j)} = X_\oplus,$$

and

$$\sum_{j=1}^{n} \iota_j X'_{a_j^*} = \sum_{j=1}^{n} \iota_j\pi_j\iota' = \iota',$$

so that $X'_\oplus = X_\oplus$ and ι' is the analogue of ι constructed from X'. Let π', ξ'_j be the analogues of π and ξ_j respectively. Note that

$$\begin{aligned}\xi'_i &= (1_{X'_v} + X'_{a_1}X'_{a_1^*})(1_{X'_v} + X'_{a_2}X'_{a_2^*})\dots(1_{X'_v} + X'_{a_{i-1}}X'_{a_{i-1}^*})\\ &= (1_{X_\oplus} + \phi_1\pi_1)(1_{X_\oplus} + \phi_2\pi_2)\dots(1_{X_\oplus} + \phi_{i-1}\pi_{i-1})\end{aligned}$$

restricted to X'_v. Thus by Lemma 2.3.4 we have

$$\xi'_i = 1_{X_\oplus} + \frac{1-q_v}{q_v}\sum_{j=1}^{i-1} \epsilon\iota_j\pi_j \tag{2.12}$$

restricted to X'_v. We claim that $\iota'\pi' = 1_{X_\oplus} - \iota\pi$. We have

$$\begin{aligned}\iota'\pi' &= \frac{1}{q'_v - 1}\sum_{i=1}^{n} \iota'\xi'_i X'_{a_i}\pi_i \\ &= \frac{q_v}{1-q_v}\sum_{i=1}^{n}\phi_i\pi_i + \sum_{i=1}^{n}\sum_{j=1}^{i-1}\epsilon\iota_j\pi_j\phi_i\pi_i,\end{aligned}$$

using (2.12). Now

$$\epsilon\iota_j = \iota_j - \iota\pi\iota_j = \iota_j - \frac{1}{q_v-1}\iota\xi_j X_{a_j} = \iota_j - \frac{1}{q_v-1}\sum_{k=1}^{n}\iota_k X_{a_k^*}\xi_j X_{a_j}.$$

Substituting this and using (2.11), the expression for $\iota'\pi'$ becomes

$$\frac{q_v}{1-q_v}\sum_{i=1}^{n}\phi_i\pi_i + \sum_{i=1}^{n}\sum_{j=1}^{i-1}\iota_j X_{a_j^*}X_{a_i}\pi_i - \frac{1}{q_v-1}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{n}\iota_k X_{a_k^*}\xi_j X_{a_j}X_{a_j^*}X_{a_i}\pi_i.$$

By (2.8), this is equal to

$$\frac{q_v}{1-q_v}\sum_{i=1}^{n}\phi_i\pi_i + \sum_{i=1}^{n}\sum_{j=1}^{i-1}\iota_j X_{a_j^*}X_{a_i}\pi_i - \frac{1}{q_v-1}\sum_{i=1}^{n}\sum_{k=1}^{n}\iota_k X_{a_k^*}(\xi_i - 1)X_{a_i}\pi_i.$$

By rearranging, we obtain

$$\begin{aligned}\iota'\pi' &= \frac{q_v}{1-q_v}\sum_{i=1}^{n}\phi_i\pi_i + \frac{q_v}{q_v-1}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\iota_j X_{a_j^*}X_{a_i}\pi_i \\ &\quad + \frac{1}{q_v-1}\sum_{i=1}^{n}\sum_{j=i}^{n}\iota_j X_{a_j^*}X_{a_i}\pi_i + \frac{1}{1-q_v}\sum_{i=1}^{n}\sum_{k=1}^{n}\iota_k X_{a_k^*}\xi_i X_{a_i}\pi_i.\end{aligned}$$

By expanding using the formula for ϕ_i, we obtain

$$\iota'\pi' = 1_{X_\oplus} + \frac{1}{1-q_v}\sum_{i=1}^{n}\sum_{k=1}^{n}\iota_k X_{a_k^*}\xi_i X_{a_i}\pi_i = 1_{X_\oplus} - \iota\pi,$$

as required.

Thus $\epsilon' = 1_{X_\oplus} - \iota'\pi' = \iota\pi$ and $X''_v = \mathrm{Im}(\epsilon') = \mathrm{Im}(\iota\pi) = \mathrm{Im}(\iota)$. The inclusion ι'' of X''_v in $X_\oplus$ can therefore be identified with ι. Clearly, for all remaining vertices u we have $X''_u = X_u$. The linear maps of X'' are given by

$X''_{a_i^*} = \pi_i \iota = X_{a_i^*}$ and

$$\begin{aligned}
\iota X''_{a_i} &= \sum_{j=1}^{i-1} \iota_j X'_{a_j^*} X'_{a_i} + \frac{1}{q'_v} \sum_{j=i}^{n} \iota_j X'_{a_j^*} X'_{a_i} + \frac{1-q'_v}{q'_v} \iota_i \\
&= \sum_{j=1}^{i-1} \iota_j \pi_j \phi_i + q_v \sum_{j=i}^{n} \iota_j \pi_j \phi_i + (q_v - 1)\iota_i \\
&= \sum_{j=1}^{i-1} \iota_j X_{a_j^*} X_{a_i} + q_v \left(\sum_{j=i}^{n} \iota_j \frac{1}{q_v} X_{a_j^*} X_{a_i} + \frac{1-q_v}{q_v} \iota_i \right) + (q_v - 1)\iota_i \\
&= \sum_{j=1}^{n} \iota_j X_{a_j^*} X_{a_i} \\
&= \iota X_{a_i}.
\end{aligned}$$

Thus $X''_{a_i} = X_{a_i}$ and $X'' = X$ as required. □

This completes the proof of Theorem 2.3.1.

2.4 Simple modules

The main goal regarding simple modules for multiplicative preprojective algebras is a proof of the following conjecture, which would (see [11] and [9]) lead to a solution of the multiplicative Deligne-Simpson problem.

Conjecture 2.4.1. *There is a simple representation of $\Lambda^q(Q)$ of dimension vector α if and only if α is a positive root, $q^\alpha = 1$ and $p(\alpha) > \sum p(\beta_i)$ for any decomposition $\alpha = \sum \beta_i$ as a sum of two or more positive roots with each $q^{\beta_i} = 1$.*

Unfortunately, this has not been accomplished. However, we can discuss some special cases, and some results related to this conjecture. We start with an easy lemma.

Lemma 2.4.2. *If X is a finite dimensional representation of $\Lambda^q(Q)$ with dimension vector α, then $q^\alpha = 1$.*

Proof. By [19, Theorem 1.3.20], if M_1 is an m by n matrix, and M_2 is an n by m matrix, then $\det(I_m + M_1 M_2) = \det(I_n + M_2 M_1)$. Thus for each arrow

$a : i \to j$, $\det(1_{X_{h(a)}} + X_a X_{a^*}) = \det(1_{X_{t(a)}} + X_{a^*} X_a)$. In particular

$$\prod_{a \in \overline{Q}_1} \det(1_{X_{h(a)}} + X_a X_{a^*})^{\epsilon(a)} = 1$$

since $\det(1_{X_{h(a)}} + X_a X_{a^*}) \det(1_{X_{t(a)}} + X_{a^*} X_a)^{-1} = 1$ for each arrow a (recall that $\det(1_{X_{h(a)}} + X_a X_{a^*})$ is always non zero due to (2.5)). Hence, using (2.6),

$$q^\alpha = \prod_{v \in Q_0} q_v^{\dim X_v} = \prod_{v \in Q_0} \det(q_v 1_{X_v}) = \prod_{a \in \overline{Q}_1} \det(1_{X_{h(a)}} + X_a X_{a^*})^{\epsilon(a)} = 1.$$

□

Observe that if $\dim X = \epsilon_v$, then this lemma tells us that $q_v = 1$. Of course this had to be the case, since otherwise we could apply Theorem 2.3.1 to obtain a representation of $\Lambda^{t_v(q)}$ of dimension vector $-\epsilon_v$, which is clearly nonsense. This is worth noting when following some of the later proofs.

Lemma 2.4.3. *[11] If X is a simple representation of $\Lambda^q(Q)$ of dimension vector α and v is a vertex, then either $\alpha = \epsilon_v$ or $q_v \neq 1$ or $(\alpha, \epsilon_v) \leq 0$.*

Proof. Suppose otherwise, i.e. that $\alpha \neq \epsilon_v$, $q_v = 1$ and $(\alpha, \epsilon_v) > 0$. The last condition ensures that v is loopfree. We assume that v is a sink and denote the arrows of Q with head at v as $a_1, a_2, \ldots, a_n$ so that $a_i < a_{i+1}$ for all i. As in the discussion after Theorem 2.3.1, let

$$\xi_i = (1_{X_v} + X_{a_1} X_{a_1^*})(1_{X_v} + X_{a_2} X_{a_2^*}) \ldots (1_{X_v} + X_{a_{i-1}} X_{a_{i-1}^*}).$$

Let

$$X_\oplus = \bigoplus_{i=1}^n X_{t(a_i)}.$$

Let $\theta : X_v \to X_\oplus$ be the linear map with components $X_{a_i^*}$ and let $\phi : X_\oplus \to X_v$ be the linear map with components $\xi_i X_{a_i}$. Using (2.8) with $q_v = 1$ we have $\phi\theta = 0$.

Suppose θ is not injective. Then X has a subrepresentation with vector space $\mathrm{Ker}(\theta)$ at vertex v and the zero subspace at all other vertices. By simplicity X is equal to this subrepresentation. Since v is loopfree, this implies that its dimension vector is ϵ_v, a contradiction, and so θ is injective.

Suppose ϕ is not surjective. We claim that X has a subrepresentation given by the vector space $U = \mathrm{Im}(\phi)$ at vertex v and the whole space X_w at all the other vertices w. Clearly, this will be the case if this subrepresentation makes sense, i.e. if $\mathrm{Im}(X_{a_i}) \subseteq U$ for all i. We prove this by induction on i. Clearly we have $\mathrm{Im}(\xi_i X_{a_i}) \subseteq U$ for all i. If $i = 1$, then $\xi_i = 1$, so this proves $\mathrm{Im}(X_{a_1}) \subseteq U$. Assuming that $\mathrm{Im}(X_{a_i}) \subseteq U$ for all $i < k$, it follows that $(1_{X_v} + X_{a_i} X_{a_i^*})(U) \subseteq U$ for all $i < k$, and since $(1_{X_v} + X_{a_i} X_{a_i^*})$ acts invertibly on X_v, this is an equality and we also have $(1_{X_v} + X_{a_i} X_{a_i^*})^{-1}(U) = U$. Thus $(\xi_k)^{-1}(U) = (1_{X_v} + X_{a_{k-1}} X_{a_{k-1}^*})^{-1} \dots (1_{X_v} + X_{a_1} X_{a_1^*})^{-1}(U) = U$. Now $\mathrm{Im}(\xi_k X_{a_k}) \subseteq U$, and hence $\mathrm{Im}(X_{a_k}) \subseteq \xi_k^{-1}(U) = U$ as required. Since X is simple, X is equal to this subrepresentation, so ϕ is surjective.

It follows that ϕ induces a surjective linear map $X_\oplus / \mathrm{Im}(\theta) \to X_v$, so $\dim X_\oplus \geq \dim X_v + \dim \mathrm{Im}\,\theta = 2 \dim X_v$ (since θ is injective), and then $(\alpha, \epsilon_v) = 2\alpha_v - \sum_i \alpha_{t(a_i)} \leq 0$, contradicting $(\alpha, \epsilon_v) > 0$. □

Theorem 2.4.4. *[11] If X is a simple representation of $\Lambda^q(Q)$ of dimension vector α, then α is a positive root for Q.*

Proof. Assume that the theorem is true for all $\beta < \alpha$. We can assume that $(\alpha, \epsilon_v) > 0$ for some vertex v (which must be loopfree) since otherwise α is in the fundamental region, and is therefore a root.

If $q_v = 1$, then by Lemma 2.4.3, $\alpha = \epsilon_v$ and is therefore a root.

If $q_v \neq 1$, then since v is loopfree, we can apply Theorem 2.3.1 at v. Namely, X corresponds to a simple representation of $\Lambda^{t_v(q)}$ of dimension vector $s_v(\alpha)$. Since $s_v(\alpha) = \alpha - (\alpha, \epsilon_v)\epsilon_v < \alpha$, the induction hypothesis applies, and so $s_v(\alpha)$ is a root, and hence so is α. □

In view of this theorem, the conjecture is equivalent to the statement that for any positive root for Q, there is a simple representation of dimension vector α if and only if $q^\alpha = 1$ and $p(\alpha) > \sum p(\beta_i)$ for any decomposition $\alpha = \sum \beta_i$ as a sum of two or more positive roots with each $q^{\beta_i} = 1$. If α is a positive real root, this can be simplified because $p(\alpha) = 0$ for all roots α, so any decomposition will automatically have $p(\alpha) \leq \sum p(\beta_i)$. We can use reflection functors to prove the conjecture is true in this case, and we can solve the rigid case of the Deligne-Simpson problem (see [11]).

Theorem 2.4.5. *[11] Let α be a positive real root for Q. There is a simple representation of $\Lambda^q(Q)$ of dimension vector α if and only if $q^\alpha = 1$ and there is no decomposition $\alpha = \sum \beta_i$ as a sum of two or more positive roots with each $q^{\beta_i} = 1$.*

Proof. Again, assume that the theorem is true for all $\beta < \alpha$. There is a vertex v with $(\alpha, \epsilon_v) > 0$ (since otherwise α is in the fundamental region, so is an imaginary root).

Suppose $q_v = 1$. By Lemma 2.4.3 the first condition holds if and only if $\alpha = \epsilon_v$. The second condition also holds if and only if $\alpha = \epsilon_v$, because if $\alpha \neq \epsilon_v$, then there is a decomposition $\alpha = s_v(\alpha) + (\alpha, \epsilon_v)\epsilon_v$ into a sum of at least two positive roots (and clearly there is no decomposition if $\alpha = \epsilon_v$). Thus the two conditions are equivalent.

If $q_v \neq 1$, then by Theorem 2.3.1, there is a simple representation of Λ^q of dimension vector α if and only if there is a simple representation of $\Lambda^{t_v(q)}$ of dimension vector $s_v(\alpha)$ (since $s_v(\alpha)$ is a real root less than α). By the induction hypothesis, this holds if and only if $t_v(q)^{s_v(\alpha)} = 1$ and there is no decomposition $s_v(\alpha) = \sum \beta_i$ as a sum of two or more positive roots with each $t_v(q)^{\beta_i} = 1$. We claim that this condition is equivalent to the same condition for α, which proves the theorem. First, it is obvious that $q^\alpha = 1$ if and only if $t_v(q)^{s_v(\alpha)} = 1$, and there is a decomposition $\alpha = \sum_i \beta_i$ of α into a sum of positive roots with $q^{\beta_i} = 1$ if and only if $s_v(\alpha) = \sum s_v(\beta_i)$ is a decomposition for $s_v(\alpha)$ into positive roots with each $t_v(q)^{s_v(\beta_i)} = 1$. This is true because the reflection at v of any positive root except ϵ_v is a positive root, and ϵ_v cannot appear in either decomposition because $(t_v(q))_v = q_v = q^{\epsilon_v} = 1$. Thus the theorem is true for α. □

Chapter 3

The Dynkin case

In this chapter we examine the properties of $\Lambda^q(Q)$ in the case of Q being a Dynkin diagram. Sections 3.1-3.5 prove the main result of this chapter, that $\Lambda^q(Q)$ is finite dimensional. In the last section we consider some further questions that can be asked.

3.1 The main theorem

Theorem 3.1.1. *If Q is a Dynkin diagram then $\Lambda^q(Q)$ is finite dimensional.*

The proof of the corresponding theorem for deformed preprojective algebras (Theorem 1.3.8) given in [10] depends on two ingredients.

1. It is known that $\Pi(Q)$ is finite dimensional for all Dynkin quivers (Theorem 1.3.4).

2. The oriented grading on $K\overline{Q}$ induces a filtration on Π^λ. One can then show that the associated graded ring $\operatorname{gr}\Pi^\lambda$ is a quotient of Π, and then it follows that Π^λ is finite dimensional.

Unfortunately, for the multiplicative case, this simple approach is not available. There does not seem to be a filtration on Λ^q which is suitable for this argument, and even if there was, we do not have the result corresponding to Theorem 1.3.4 for $\Lambda^1(Q)$ (although see Chapter 5). Instead we are forced to adopt a lengthy case by case analysis of the Dynkin diagrams.

3.2 Type A_n

There is nothing to do in this case, due to the following lemma.

Lemma 3.2.1. *If Q has type A_n then $\Lambda^q(Q)$ is isomorphic to a deformed preprojective algebra.*

Proof. We can assume (see Lemma 3.3.2, but it should be clear in any case) that $\Lambda^q = K\overline{Q}/I_{\mu^q}$ where Q is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "a_1"'] & \underset{3}{\bullet} \arrow[l, "a_2"'] & \cdot \arrow[l, "a_3"'] & \cdot & \cdot & \underset{n}{\bullet} \arrow[l, "a_{n-1}"'] \end{tikzcd}$$

and I_{μ^q} is the ideal generated by the elements

$$\mu_i^q = \begin{cases} (e_1 + a_1a_1^*) - q_1e_1 & \text{if } i = 1, \\ (e_i + a_ia_i^*) - q_i(e_i + a_{i-1}^*a_{i-1}) & \text{if } 2 \leq i \leq n-1, \\ e_n - q_n(e_n + a_{n-1}^*a_{n-1}) & \text{if } i = n. \end{cases}$$

Let $\theta : K\overline{Q} \to K\overline{Q}$ be the isomorphism which takes e_v to e_v, a_i to x_ia_i, and a_i^* to a_i^* where $x_i = (q_n \ldots q_{i+1})^{-1}$ for $1 \leq i \leq n-1$. Clearly θ induces an isomorphism $K\overline{Q}/I_{\mu^q} \to K\overline{Q}/\theta(I_{\mu^q})$. Now

$$\theta(\mu_1^q) = e_1 + x_1a_1a_1^* - q_1e_1 = x_1(a_1a_1^* - (q_1 - 1)q_2 \ldots q_ne_1),$$

$$\theta(\mu_n^q) = e_n - q_n(e_n + x_{n-1}a_{n-1}^*a_{n-1}) = -a_{n-1}^*a_{n-1} - (q_n - 1)e_n,$$

and for $i = 2, \ldots, n-1$,

$$\begin{aligned} \theta(\mu_i^q) &= (e_i + x_ia_ia_i^*) - q_i(e_i + x_{i-1}a_{i-1}^*a_{i-1}) \\ &= x_i(a_ia_i^* - a_{i-1}^*a_{i-1} - (q_i - 1)q_{i+1} \ldots q_ne_i). \end{aligned}$$

Let $\lambda = (\lambda_i)_{i \in Q_0}$, where $\lambda_i = (q_i - 1)q_{i+1} \ldots q_n$ for all i. We have

$$\begin{aligned} \theta(I_{\mu^q}) &= (\theta(\mu_1^q), \ldots, \theta(\mu_i^q), \ldots, \theta(\mu_n^q)) \\ &= (a_1a_1^* - \lambda_1e_1, \ldots, a_ia_i^* - a_{i-1}^*a_{i-1} - \lambda_ie_i, \ldots, -a_{n-1}^*a_{n-1} - \lambda_ne_n) \\ &= (\rho_1^\lambda, \rho_2^\lambda, \ldots, \rho_n^\lambda) \\ &= I_{\rho^\lambda}. \end{aligned}$$

Therefore θ induces an isomorphism $\Lambda^q = K\overline{Q}/I_{\mu^q} \to K\overline{Q}/I_{\rho^\lambda} = \Pi^\lambda$. $\square$

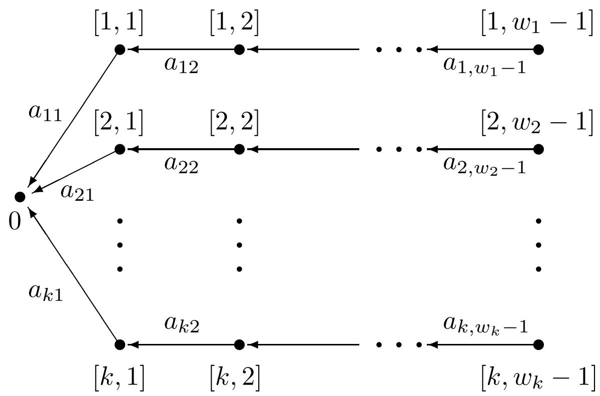
Therefore we can use the results in [10] regarding the deformed preprojective algebra. In particular by Theorem 1.3.8 we have the following corollary.

Corollary 3.2.2. *If Q has type A_n then $\Lambda^q(Q)$ is finite dimensional.*

3.3 Star-shaped quivers

In this section we prove some facts regarding the multiplicative preprojective algebra of a general star shaped quiver which will help us understand the remaining cases of Dynkin quivers.

Definition 3.3.1. A quiver Q is *star-shaped* if it has the form



That is, there are integers $k \geq 1$, $w_1, \ldots, w_k \geq 2$ such that

$$Q_0 = \{0\} \cup \{[i,j] : 1 \leq i \leq k, 1 \leq j \leq w_k - 1\},$$

$$Q_1 = \{a_{ij} : 1 \leq i \leq k, 1 \leq j \leq w_k - 1\},$$

where the arrows satisfy $t(a_{ij}) = [i,j]$ and $h(a_{ij}) = [i, j-1]$. Note that we understand that $[i, 0]$ means the vertex 0.

The Dynkin quivers are all star-shaped (provided they are given the suitable orientation), each with $k = 3$.

$$D_n : w_1 = n-2, w_2 = 2, w_3 = 2.$$

$$E_6 : w_1 = 3, w_2 = 3, w_3 = 2.$$

$$E_7 : w_1 = 4, w_2 = 3, w_3 = 2.$$

$$E_8 : w_1 = 5, w_2 = 3, w_3 = 2.$$

Throughout the rest of this section Q denotes the quiver above and we write e_{ij} instead of $e_{[i,j]}$ to denote the trivial path at vertex $[i, j]$. We work towards Lemma 3.3.7, which gives a presentation of $e_0\Lambda^q(Q)e_0$ in terms of generators and relations.

Lemma 3.3.2. *[11, Lemma 8.1]*

$$\Lambda^q(Q) \cong K\overline{Q}/I_\mu,$$

where I_μ is the ideal generated by the elements $(\mu_v^q)_{v\in Q_0}$ with

$$\begin{aligned}
\mu_0^q &= (e_0 + a_{11}a_{11}^*)\dots(e_0 + a_{k1}a_{k1}^*) - q_0e_0,\\
\mu_{ij}^q &= e_{ij} + a_{i,j+1}a_{i,j+1}^* - q_{ij}(e_{ij} + a_{ij}^*a_{ij}),\ \textit{for } j = 1,\dots,w_i - 2,\\
\mu_{i,w_i-1}^q &= e_{i,w_i-1} - q_{i,w_i-1}(e_{i,w_i-1} + a_{i,w_i-1}^*a_{i,w_i-1}).
\end{aligned}$$

Definition 3.3.3. Given integers m, n, k with $1 < k \leq m + 1, n + 1$, we say a path p of $\overline{Q}$ has type (A, m, k, n) if it has the form

$$p = a_{rm}^* \dots a_{rk}^* a_{sk} \dots a_{sn} \text{ for some } r, s.$$

Given integers m, n, l, with $m, n \geq 0$ we say a path p of $\overline{Q}$ has type (B, m, l, n) it is has the form

$$p = a_{rm}^* \dots a_{r1}^*(a_{i_1,1}a_{i_1,1}^*)\dots(a_{i_l,1}a_{i_l,1}^*)a_{s1}\dots a_{sn}$$

for some r, s and $i_1, \dots, i_l$. In either case, we say p is *normalised.* Note that in the above we are understanding that the extreme cases '$a_{rm}^* \dots a_{r,m+1}^*$' and '$a_{s,n+1} \dots a_{sn}$' mean the trivial paths e_m and e_n respectively.

Some examples of normalised paths are given below.

$$e_0 \text{ has type } (B, 0, 0, 0),$$

$$e_{ij} \text{ has type } (A, j-1, j, j-1) \text{ for all } i,$$

$$a_{12}a_{13} \text{ has type } (A, 1, 2, 3),$$

$$a_{12}^*a_{11}^*a_{11}a_{11}^*a_{21}a_{21}^*a_{11}a_{12}a_{13}a_{14} \text{ has type } (B, 2, 2, 4).$$

Lemma 3.3.4. $\Lambda^q(Q)$ *is spanned by the set of normalised paths.*

Proof. By Lemma 3.3.2, Λ^q is a quotient of $K\overline{Q}$, so is spanned by the set P of paths in $\overline{Q}$. We set up a reduction system Ω on P. For each arrow a in $\overline{Q}$ let $d(a) = j$ if $a = a_{ij}$ or $a = a_{ij}^*$. For each path $p = a_n \dots a_1 \in P$, let $d(p) = \sum_{m=1}^{n} d(a_m)$ and let $\leq$ be the partial ordering on P defined by

$$p_1 \leq p_2 \text{ if and only if } d(p_1) < d(p_2) \text{ or } p_1 = p_2.$$

This clearly satisfies (†) and (‡) of the Appendix. Let Ω be reduction system consisting of the elements

$$\{a_{ij}a_{ij}^* - q_{i,j-1}a_{i,j-1}^*a_{i,j-1} - (q_{i,j-1} - 1)e_{i,j-1} : 1 \leq i \leq k, 2 \leq j \leq w_i - 1\}.$$

The elements are obtained by monicising the elements μ_{ij}^q for $1 \leq j \leq w_i - 2$ with respect to $\leq$. Note that we ignore μ_0^q and μ_{j,w_j-1}^q so this isn't a full reduction system. By Lemma A.2.3, Λ^q is spanned by the set of irreducible paths, namely those which do not contain a subpath $a_{ij}a_{ij}^*$ with $j > 1$.

We claim that the irreducible paths are exactly all the normalised paths. No normalised path has a subpath $a_{ij}a_{ij}^*$ with $j > 1$, so all normalised paths are irreducible. We now suppose p is an irreducible path and show by induction on the length it is normalised. If p is trivial or an arrow then it normalised. We assume that the claim is true for all paths of length less than p. Suppose that $p = bp'$ where b is an arrow. Since p is irreducible then so is p', and by the induction hypothesis p' is normalised. There a number of cases to consider.

Case 1. p' has type $(A, k-1, k, n)$, i.e. $p = a_{sk} \dots a_{sn}$.
Then either (i) $b = a_{sk}^*$, and then p has type (A, k, k, n), or (ii) $b = a_{s,k-1}$, and then p has type $(A, k-2, k-1, n)$ (or type $(B, 0, 0, n)$ if $k = 2$).

Case 2. p' has type (A, m, k, n) where $m \geq k$, i.e. $p = a^*_{rm} \dots a^*_{rk} a_{sk} \dots a_{sn}$. Then either (i) $b = a_{rm}$, which contradicts the irreducibility of p since it has a subpath $a_{rm}a^*_{rm}$, or (ii) $b = a^*_{r,m+1}$, and then p has type $(A, m+1, k, n)$.

Case 3. p' has type $(B, 0, l, n)$, i.e. $p = (a_{i_1,1}a^*_{i_1,1}) \dots (a_{i_l,1}a^*_{i_l,1})a_{s1} \dots a_{sn}$. Then $b = a^*_{r1}$ for some r and p has type $(B, 1, l, m)$.

Case 4. p' has type $(B,1,l,n)$, i.e. $p = a^*_{r1}(a_{i_1,1}a^*_{i_1,1}) \dots (a_{i_l,1}a^*_{i_l,1})a_{s1} \dots a_{sn}$. Then either (i) $b = a^*_{r2}$, and then p has type $(B, 2, l, m)$, or (ii) $b = a_{r1}$, and p has type $(B, 0, l+1, n)$.

Case 5. p' has type (B, m, l, n) where $m > 1$, i.e. $p = a^*_{rm} \dots a^*_{r1}(a_{i_1,1}a^*_{i_1,1})$ $\dots (a_{i_l,1}a^*_{i_l,1})a_{s1} \dots a_{sn}$.
Then either (i) $b = a^*_{r,m+1}$, and then p has type $(B, m+1, l, n)$, or (ii) $b = a_{rm}$, which contradicts p being irreducible since it has a subpath $a_{rm}a^*_{rm}$. □

Lemma 3.3.5. *$e_0\Lambda^q(Q)e_0$ is generated by the paths $a_{i1}a^*_{i1}$, $1 \leq i \leq k$.*

Proof. By Lemma 3.3.4, Λ^q is spanned by the set of normalised paths. So $e_0\Lambda^q e_0$ is spanned by the set of normalised paths which start and end at 0, namely, the set of paths of type $(B, 0, l, 0)$. Clearly each path of type $(B, 0, l, 0)$ can be formed by taking a product involving the paths $a_{i1}a^*_{i1}$, $1 \leq i \leq k$. □

We now have a generating set for $e_0\Lambda^q e_0$, and we now perform some calculations which will give the relations. We define scalars s^i_{mn}, where $1 \leq i \leq k$ and $1 \leq m \leq n \leq w_i$,

$$s_i^{mn} = \prod_{l=m}^{n-1} q_{il}^{-1}$$

where the empty product is taken to be 1. Note that $s_i^{jl} = q_{ij}^{-1} s_i^{j+1,l}$ if $j < l$, and in the special case where $q_{ij} = 1$ for all i, j, each $s_i^{mn} = 1$.

Lemma 3.3.6. *Working in $\Lambda^q(Q)$, we have the following equations.*
(i) For all i, and $j < l$,

$$a^*_{ij}a_{ij} - (s_i^{jl} - 1)e_{ij} = q_{ij}^{-1}(a_{i,j+1}a^*_{i,j+1} - (s_i^{j+1,l} - 1)e_{ij}).$$

(ii) For all i, and $1 \leq t \leq w_i - 1$,

$$\prod_{j=1}^{w_i}(a_{i1}a_{i1}^* - (s_i^{1j} - 1)e_0) = \left(\prod_{r=1}^{t-1} q_{ir}^{r-w_i}\right) F_t$$

where F_t represents the expression

$$a_{i1} \dots a_{it} \left(\prod_{j=t+1}^{w_i} (a_{it}^* a_{it} - (s_i^{tj} - 1)e_{it})\right) a_{it}^* \dots a_{i1}^*.$$

(iii) For all i,

$$\prod_{j=1}^{w_i}(a_{i1}a_{i1}^* - (s_i^{1j} - 1)e_0) = 0.$$

Proof. (i) Rewriting the relation μ_{ij}^q, we have that

$$a_{ij}^* a_{ij} = q_{ij}^{-1} a_{i,j+1} a_{i,j+1}^* + (q_{ij}^{-1} - 1)e_{ij}.$$

Subtracting $(s_i^{jl} - 1)e_{ij}$ from both sides gives the required equation.

(ii) By induction on t. Since $s_i^{11} = 1$, we have

$$\prod_{j=1}^{w_i}(a_{i1}a_{i1}^* - (s_i^{1j} - 1)e_0) = a_{i1}a_{i1}^* \left(\prod_{j=2}^{w_i}(a_{i1}a_{i1}^* - (s_i^{1j} - 1)e_0)\right).$$

Rearranging the brackets, this is

$$a_{i1} \left(\prod_{j=2}^{w_i}(a_{i1}^* a_{i1} - (s_i^{1j} - 1)e_{i1})\right) a_{i1}^*,$$

which is F_1. We now show that $F_{t+1} = q_{it}^{t-w_i} F_t$ for all t. We have

$$\begin{aligned} F_t &= a_{i1} \dots a_{it} \left(\prod_{j=t+1}^{w_i} (a_{it}^* a_{it} - (s_i^{tj} - 1)e_{it})\right) a_{it}^* \dots a_{i1}^* \\ &= a_{i1} \dots a_{it} q_{it}^{t-w_i} \left(\prod_{j=t+1}^{w_i} (a_{i,t+1} a_{i,t+1}^* - (s_i^{t+1,j} - 1)e_{it})\right) a_{it}^* \dots a_{i1}^* \end{aligned}$$

by using (i) on each term of the product. We take $q_{it}^{t-w_i}$ to the front, and substitute $s_i^{tt} = 1$, to obtain

$$q_{it}^{t-w_i} a_{i1} \dots a_{it} a_{i,t+1} a_{i,t+1}^* \left(\prod_{j=t+2}^{w_i} (a_{i,t+1} a_{i,t+1}^* - (s_i^{t+1,j} - 1)e_{it})\right) a_{it}^* \dots a_{i1}^*.$$

Rearranging the brackets, this is

$$q_{it}^{t-w_i} a_{i1} \dots a_{it} a_{i,t+1} \left(\prod_{j=t+2}^{w_i} (a_{i,t+1}^* a_{i,t+1} - (s_i^{t+1,j} - 1) e_{i,t+1}) \right) a_{i,t+1}^* a_{it}^* \dots a_{i1}^*$$

which is $q_{it}^{t-w_i} F_{t+1}$.

(iii) By (ii), this is equivalent to showing that any of the expressions F_t equal zero. It is obvious that F_{w_i-1} is zero since it equals

$$a_{i1} \dots a_{i,w_i-1} (a_{i,w_i-1}^* a_{i,w_i-1} - (s_i^{w_i-1,w_i} - 1) e_{i,w_i-1}) a_{i,w_i-1}^* \dots a_{i1}^*.$$

which is

$$-q_{i,w_i-1}^{-1} a_{i1} \dots a_{i,w_i-1} \mu_{i,w_i-1} a_{i,w_i-1}^* \dots a_{i1}^* = 0.$$

□

Lemma 3.3.7. *Let $S = K\langle A_1, A_2, \dots, A_k\rangle / I_R$, where R is the set of relations*

$$(A_1 + 1)(A_2 + 1) \dots (A_k + 1) = q_0,$$
$$\prod_{j=1}^{w_k} (A_i - (s_i^{1j} - 1)) = 0, \quad \textit{for } i = 1, \dots, k.$$

Let $S^e = K\langle \alpha_1, \alpha_2, \dots, \alpha_k\rangle / I_{R^e}$, where R^e is the set of relations

$$\alpha_1 \alpha_2 \dots \alpha_k = q_0,$$
$$\prod_{j=1}^{w_k} (\alpha_j - s_i^{1j}), \quad \textit{for } i = 1, \dots, k.$$

Then $e_0 \Lambda^q(Q) e_0$ is isomorphic to both S and S^e.

Proof. By Lemma 3.3.5 there is a surjective homomorphism

$$\theta : K\langle A_1, \dots, A_k\rangle \to e_0 \Lambda^q e_0,$$

in which A_i is sent to $a_{i1} a_{i1}^*$. Clearly $(A_1 + 1)(A_2 + 1) \dots (A_k + 1) - q_0$ is sent to μ_0, which is zero, and by the previous lemma, $\prod_{j=1}^{w_k} (A_i - (s_i^{1j} - 1))$ is also sent to 0. Therefore θ induces a surjective homomorphism $\overline{\theta} : S \to e_0 \Lambda^q e_0$. Let $I = \operatorname{Ker} \overline{\theta}$. We claim that $I = 0$ and so $\overline{\theta}$ is an isomorphism.

To prove this we show that any S-module M is the restriction by $\overline{\theta}$ of an $e_0\Lambda^q e_0$-module. So, given M, we construct a representation X of Λ^q. Let $X_0 = M$ and

$$X_{ij} = (A_i - (s_i^{1j} - 1)) \dots (A_i - (s_i^{12} - 1))(A_i - (s_i^{11} - 1))M$$

Let $X_{a_{ij}}$ be the inclusion of X_{ij} in $X_{i,j-1}$ and let $X_{a_{ij}^*}$ be multiplication by $(s_i^{1j})^{-1}(A_i - (s_i^{1j} - 1))$. This clearly defines a representation of $\overline{Q}$, so for X to be a representation of Λ^q, the X_a must satisfy the appropriate relations. Since each $s_i^{11} = 1$,

$$(1 + X_{a_{11}}X_{a_{11}^*}) \dots (1 + X_{a_{k1}}X_{a_{k1}^*}) = (1 + A_1) \dots (1 + A_k) = q_0.$$

and so the X_a satisfy the relation at 0.

$$\begin{aligned} q_{ij}(1 + X_{a_{ij}}X_{a_{ij}^*}) &= q_{ij}(1 + (s_i^{1j})^{-1}(A_i - (s_i^{1j} - 1))) \\ &= (s_i^{1,j+1})^{-1}A_i + (s_i^{1,j+1})^{-1} \\ &= 1 + (s_i^{1,j+1})^{-1}(A_i - (s_i^{1,j+1} - 1)) \\ &= 1 + X_{a_{i,j+1}}X_{a_{i,j+1}^*}, \end{aligned}$$

and so the X_a satisfy the relations at $[i, j]$ where $1 \leq j \leq w_1 - 2$.

$$\begin{aligned} q_{i,w_i-1}(1 + X_{a_{i,w_i-1}^*}X_{a_{i,w_i-1}}) &= q_{i,w_i-1}(1 + (s_i^{1,w_i-1})^{-1}(A_i - (s_i^{1,w_i-1} - 1)) \\ &= (s_i^{1,w_i})^{-1}(A_i - (s_i^{1,w_i} - 1)) + 1, \end{aligned}$$

and the X_a satisfy the relations at $[i, w_i - 1]$ since $(A_i - (s_i^{i,w_i} - 1))|_{X_{i,w_i-1}} = 0$ because $(A_i - (s_i^{i,w_i} - 1)) \dots (A_i - (s_i^{12} - 1))(A_i - (s_i^{11} - 1)) = 0$.

Therefore X can be regarded as a Λ^q-module, and so $M = e_0X$ can be regarded as an $e_0\Lambda^q e_0$ module. For each $m \in M$, and each i, $1 \leq i \leq k$, the $e_0\Lambda^q e_0$ product $a_{i1}a_{i1}^*m$ is equal to $X_{a_{i1}}X_{a_{i1}}^*(m)$, which is the same as the S-module product $(s_i^{11})^{-1}(A_i - (s_i^{11} - 1))m$. Since $s_i^{11} = 1$, this is simply A_im. Since $\overline{\theta}(A_i) = a_{i1}a_{i1}^*$, we have shown that if $r \in S$ is a generator, the S-module product rm is the same as the $e_0\Lambda^q e_0$-module product $\overline{\theta}(r)m$. Since it holds for all generators, it holds for any element and we have shown that any S-module can be obtained by the restriction of an $e_0\Lambda^q e_0$ module.

Consider in particular the case $M = S$, as a module over itself. Then S is an $e_0\Lambda^q e_0 \cong S/I$ module. For each $i \in I$, we have $i = i.1_S = \overline{\theta}(i).1_S = 0$, and so $I = 0$.

The algebras S and S^e are clearly isomorphic since the map $S \to S^e$ which takes A_i to $\alpha_i - 1$ and the map $S^e \to S$ which takes α_i to $A_i + 1$ are mutual inverses. □

Lemma 3.3.8. *$\Lambda^q(Q)$ is finite dimensional if and only if $e_0\Lambda^q(Q)e_0$ is finite dimensional.*

Proof. Suppose $e_0\Lambda^q e_0$ is finite dimensional. As in the proof of Lemma 3.3.5 $e_0\Lambda^q e_0$ is spanned by the set P of paths $\{p : p \text{ is normalised of type } (B, 0, l, 0)\}$. Since $e_0\Lambda^q e_0$ is finite dimensional we can choose a finite subset P' of P so that P' spans $e_0\Lambda^q e_0$. Choose the maximal t such that there is a path of type $(B, 0, t, 0)$ in P'. We claim Λ^q is spanned by the set

$$U = \{p : p \text{ has type } (A, m, k, n)\} \cup \{p : p \text{ has type } (B, m, l, n) \text{ where } l \leq t\}.$$

To prove this we need to show that any path of type (B, m, l, n) with $l > t$ can be written as a linear combination of paths in P. Let $p = a^*_{rm} \dots a^*_{r1}(a_{i_1,1}a^*_{i_1,1}) \dots (a_{i_l,1}a^*_{i_l,1})a_{s1} \dots a_{sn}$ be such a path. Now $p_0 = (a_{i_1,1}a^*_{i_1,1}) \dots (a_{i_l,1}a^*_{i_l,1}) \in P$ so $p_0 = \sum_i \lambda_i p_i$ for some $\lambda_i \in K$ and $p_i \in P'$. Then

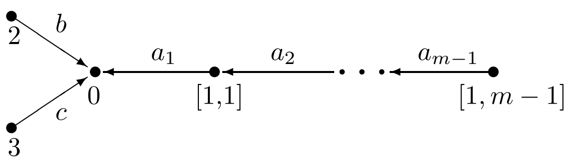
$$\begin{aligned} p &= a^*_{rm} \dots a^*_{r1} p_0 a_{s1} \dots a_{sn} \\ &= \sum_i \lambda_i a^*_{rm} \dots a^*_{r1} p_i a_{s1} \dots a_{sn} \end{aligned}$$

expresses p as required. Finally we need to show that U is a finite set. There are only finitely many paths of a given type, and since the number of possible types is bounded ($m, n, k < \max\{w_1, \dots, w_k\}$, $l \leq t$) this is clear. □

We are now ready to prove Theorem 3.1.1 for the remaining Dynkin diagrams. We have a presentation for $e_0\Lambda^q e_0$ so we apply the method described in Section A.5 to obtain a reduction system which gives a finite spanning set. Note that it is impractical to try to find a basis for $e_0\Lambda^q e_0$, since that would depend on q. We use the second presentation S^e as this allows us some shortcuts when resolving ambiguities, due to the fact that we know that the generators are invertible (they correspond with elements of $e_0\Lambda^q e_0$ of the form $e_0 + aa^*$, which are invertible). In S^e the inverse of α_i is a polynomial in α_i of degree $w_i - 1$ which can be calculated from the relation involving α_i. We can always reduce the expression '$\alpha_i\alpha_i^{-1}$' to 1. Note that we are always allowed to divide by any q_v since they are always nonzero.

3.4 Type D_n

We assume that Q is the star shaped quiver



By Lemma 3.3.7 we know that $e_0\Lambda^q e_0 \cong S^e = K\langle\alpha, \beta, \gamma\rangle/I$, where I is the ideal generated by the set of elements

$$\begin{aligned}
r_0 &= \alpha\beta\gamma - q_0, \\
r_\alpha &= (\alpha - 1)(\alpha - q_{11}^{-1})\dots(\alpha - q_{11}^{-1}\dots q_{1,m-1}^{-1}), \\
r_\beta &= (\beta - 1)(\beta - q_2^{-1}), \\
r_\gamma &= (\gamma - 1)(\gamma - q_3^{-1}).
\end{aligned}$$

Note that

$$\gamma^{-1} = (1 + q_3) - q_3\gamma, \tag{3.1}$$

$$\beta^{-1} = (1 + q_2) - q_2\beta, \tag{3.2}$$

Lemma 3.4.1. *(i) The following elements r_2, r_3, r_4 all lie in I.*

$$\begin{aligned}
r_2 &= \alpha\beta - q_0(1 + q_3) + q_0q_3\gamma, \\
r_3 &= \gamma\beta - q_0^{-1}q_2^{-1}q_3^{-1}\alpha - (1 + q_2^{-1})\gamma - (1 + q_3^{-1})\beta + (1 + q_2^{-1})(1 + q_3^{-1}), \\
r_4 &= \gamma\alpha - q_0(1 + q_2) + q_0q_2\beta.
\end{aligned}$$

(ii) The set $\Omega = \{r_\alpha, r_\beta, r_\gamma, r_2, r_3, r_4\}$ *is a (full) reduction system for* S^e *and therefore the set of irreducible words is a finite spanning set.*

Proof. (i) This could be done by considering the set $R^e = \{r_0, r_\alpha, r_\beta, r_\gamma\}$ as a reduction system and resolving the ambiguities, but the following method is equivalent and quicker.

I contains $r_0\gamma^{-1}$, so contains $\alpha\beta - q_0\gamma^{-1} = r_2$ by substituting (3.1). I contains $r_2\beta^{-1}$, so contains $\alpha - q_0(1 + q_3)\beta^{-1} + q_0q_3\gamma\beta^{-1} = -q_0q_2q_3r_3$ by substituting (3.2), and therefore contains r_3. I contains $\alpha^{-1}r_0\alpha$, so contains $\beta\gamma\alpha - q_0$. Set this equal to r_1. Then I contains $\beta^{-1}r_1$, so contains $\gamma\alpha - q_0\beta^{-1} = r_4$ by substituting (3.2).

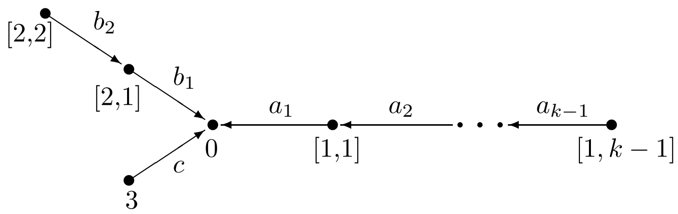
(ii) The elements of Ω are monic, so is Ω is a reduction system. [In fact Ω is full since the ideals I and I_Ω are equal - $I_\Omega \subseteq I$ was proved in part (i), and $r_0 = r_2\gamma - q_3r_\gamma \in I_\Omega$, so $I \subseteq I_\Omega$. However, this is not necessary for the rest of the proof]. The illegal words are $\{\alpha^m, \beta^2, \gamma^2, \alpha\beta, \gamma\beta, \gamma\alpha\}$, so if w is irreducible it must have the form $\beta^i\alpha^j\gamma^k$ where $i, k \in \{0, 1\}$, $j \in \{0, \ldots, m-1\}$, and so there are only finitely many possibilities. By Lemma A.2.3 this is a spanning set for S^e. $\square$

Hence $e_0\Lambda^q e_0$ is finite dimensional and by Lemma 3.3.8 we have the following corollary.

Corollary 3.4.2. *If Q has type D_n then $\Lambda^q(Q)$ is finite dimensional.*

3.5 Type E_6, E_7, E_8

We consider a general 'type E' star shaped quiver



By Lemma 3.3.7 we know that $e_0\Lambda^q e_0 \cong S^e = K\langle\alpha,\beta,\gamma\rangle/I$, where I is the ideal generated by the elements

$$\begin{aligned}
r_0 &= \alpha\beta\gamma - q_0, \\
r_\alpha &= (\alpha-1)(\alpha - q_{11}^{-1})\dots(\alpha - q_{11}^{-1}q_{12}^{-1}\dots q_{1k}^{-1}), \\
r_\beta &= (\beta-1)(\beta - q_{21}^{-1})(\beta - q_{21}^{-1}q_{22}^{-1}), \\
r_\gamma &= (\gamma-1)(\gamma - q_3^{-1}).
\end{aligned}$$

Note that

$$\beta^{-1} = q_{21}^2 q_{22}\beta^2 - (q_{21}^2 q_{22} + q_{21}q_{22} + q_{21})\beta + (1 + q_{21} + q_{21}q_{22}), \quad (3.3)$$

$$\gamma^{-1} = (1+q_3) - q_3\gamma, \quad (3.4)$$

and that α^{-1} is a polynomial in α of degree $k-1$, where the coefficient of α^{k-1} is $\prod_{i=1}^{k-1} q_{1i}^{k-i}$ which is nonzero.

Lemma 3.5.1. *The following elements all lie in I.*

$$\begin{aligned}
r_1 &= \gamma - (1 + q_3^{-1}) + q_0^{-1} q_3^{-1} \alpha\beta, \\
r_2 &= \beta^2 + q_0^{-2} q_3^{-1} q_{21}^{-2} q_{22}^{-1} \alpha\beta\alpha - q_0^{-1}(1 + q_3^{-1}) q_{21}^{-2} q_{22}^{-1} \alpha \\
&\quad -(1 + q_{21}^{-1} + q_{21}^{-1} q_{22}^{-1})\beta + (q_{21}^{-2} q_{22}^{-1} + q_{21}^{-1} q_{22}^{-1} + q_{21}^{-1}), \\
r_3 &= \beta\alpha\beta - q_0(1 + q_3)\beta + q_0^2 q_3 \alpha^{-1}, \\
r_4 &= \beta\alpha^2\beta - q_0^4 q_3^2 q_{21}^2 q_{22} \alpha^{-1}\beta\alpha^{-1} + q_0^4 q_3^2 (q_{21}^2 q_{22} - q_{21} q_{22} + q_{21})\alpha^{-2} \\
&\quad -q_0(1 + q_3)\alpha\beta + q_0(1 + q_3)\beta\alpha + q_0^2 q_3 (1 + q_{21} - q_{21} q_{22})\beta \\
&\quad -q_0^3 q_3 (1 + q_3)(1 + q_{21} - q_{21} q_{22})\alpha^{-1} + q_0^2 (1 + q_3)^2, \\
r_5 &= q_0^{-2} q_3^{-1} q_{21}^{-2} q_{22}^{-1} \beta\alpha^3\beta\alpha + q_0^4 q_3^2 q_{21}^2 q_{22} \alpha^{-1}\beta\alpha^{-1}\beta + q_0(1 + q_3)\beta\alpha\beta \\
&\quad -(1 + q_{21}^{-1} + q_{21}^{-1} q_{22}^{-1})\beta\alpha^2\beta - q_0 - 1(1 + q_3^{-1}) q_{21}^{-2} q_{22}^{-1} \beta\alpha^3 \\
&\quad +(q_{21}^{-2} q_{22}^{-1} + q_{21}^{-1} q_{22}^{-1} + q_{21}^{-1})\beta\alpha^2 + q_0(1 + q_3)\alpha\beta^2 \\
&\quad -q_0^2 q_3 (1 + q_{21} + q_{21} q_{22})\beta^2 - q_0^4 q_3^2 (q_{21}^2 q_{22} + q_{21} q_{22} + q_{21})\alpha^{-2}\beta \\
&\quad -q_0^2 (1 + q_3)^2 \beta + q_0^3 q_3 (1 + q_3)(1 + q_{21} + q_{21} q_{22})\alpha^{-1}\beta, \\
r_6 &= q_0^{-2} q_3^{-1} q_{21}^{-2} q_{22}^{-1} \alpha\beta\alpha^3\beta + q_0^4 q_3^2 q_{21}^2 q_{22} \beta\alpha^{-1}\beta\alpha^{-1} + q_0(1 + q_3)\beta\alpha\beta \\
&\quad -(1 + q_{21}^{-1} + q_{21}^{-1} q_{22}^{-1})\beta\alpha^2\beta - q_0 - 1(1 + q_3^{-1}) q_{21}^{-2} q_{22}^{-1} \alpha^3\beta \\
&\quad +(q_{21}^{-2} q_{22}^{-1} + q_{21}^{-1} q_{22}^{-1} + q_{21}^{-1})\alpha^2\beta + q_0(1 + q_3)\beta^2\alpha \\
&\quad -q_0^2 q_3 (1 + q_{21} + q_{21} q_{22})\beta^2 - q_0^4 q_3^2 (q_{21}^2 q_{22} + q_{21} q_{22} + q_{21})\beta\alpha^{-2} \\
&\quad -q_0^2 (1 + q_3)^2 \beta + q_0^3 q_3 (1 + q_3)(1 + q_{21} + q_{21} q_{22})\beta\alpha^{-1}.
\end{aligned}$$

Proof. Let W be the set of words formed from α, β, γ and let $\leq$ be the ordering $\leq_{\gamma,\beta}$ on W (see Section A.6). Observe that $r_1, \ldots, r_4$ are monic with respect to the ordering (the leading word being the first word as written out above), but r_5, r_6 do not have a leading word because the ordering is not sufficiently refined. Note that we keep expressions in terms of α^{-1}, for the purposes of the ordering they are regarded as a polynomial in α.

We show the elements lie in I by resolving some of the ambiguities created by the reduction system $\Omega_0 = \{r_0, r_\alpha, r_\beta, r_\gamma\}$. The illegal words are

$\{\alpha\beta\gamma, \alpha^k, \beta^3, \gamma^2\}$. We resolve $\alpha\beta\gamma^2$:

$$\begin{aligned}
\alpha\beta(\gamma^2) &\mapsto \alpha\beta((q_3^{-1}+1)\gamma - q_3^{-1}) \rightsquigarrow q_0(q_3^{-1}+1) - q_3^{-1}\alpha\beta, \\
(\alpha\beta\gamma)\gamma &\mapsto q_0\gamma.
\end{aligned}$$

and so $q_0(q_3^{-1}+1) - q_3^{-1}\alpha\beta - q_0\gamma = -q_0 r_1 \in I$. Monicising, we see that $r_1 \in I$. Note that this was basically the same calculation as was done in the type D_n case, and this time the leading word is γ because the ordering has changed. Now let $\Omega_1 = \Omega_0 \cup \{r_1\}$, the illegal words are $\{\alpha\beta\gamma, \alpha^k, \beta^3, \gamma^2, \gamma\}$. We resolve the inclusion ambiguity $\alpha\beta\gamma$.

$$\begin{aligned}
\alpha\beta\gamma &\mapsto q_0, \\
\alpha\beta(\gamma) &\mapsto \alpha\beta(-q_0^{-1}q_3^{-1}\alpha\beta + (1+q_3^{-1})).
\end{aligned}$$

So by monicising, we have that $r_7 = \alpha\beta\alpha\beta - q_0(1+q_3)\alpha\beta + q_0^2 q_3 \in I$. We could add this element to the reduction system and resolve the ambiguities $\alpha\beta\alpha\beta^3$ and $\alpha^k\beta\alpha\beta$ (quite a complicated process). However, we can use the fact that α and β are invertible to reach the same result. That is, since $r_7\beta^{-1} \in I$, so is $\alpha\beta\alpha - q_0(1+q_3)\alpha + q_0^2 q_3 \beta^{-1} = q_0^2 q_3 q_{21}^2 q_{22} r_2$, by using (3.3). Thus $r_2 \in I$ by monicising. Similarly, $\alpha^{-1} r_7 \in I$, and hence so is $\beta\alpha\beta - q_0(1+q_3)\beta + q_0^2 q_3 \alpha^{-1} = r_3$.

We set $\Omega_2 = \Omega_1 \cup \{r_2, r_3\}$. We can take out r_0, r_β, r_γ from Ω_2 because they are redundant and we are only interested in finding a spanning set, so it doesn't matter if Ω_2 is no longer a full reduction system (in fact $\alpha\beta\gamma, \beta^3, \gamma^2$ are reduction unique with respect to Ω_2, so Ω_2 is full). This leaves us with $\Omega_3 = \{r_\alpha, r_1, r_2, r_3\}$ with illegal words $\{\alpha^k, \gamma, \beta^2, \beta\alpha\beta\}$. We now resolve $\beta\alpha\beta^2$.

$$\begin{aligned}
(\beta\alpha\beta)\beta &\mapsto q_0(1+q_3)\beta^2 - q_0^2 q_3 \alpha^{-1}\beta, \\
&\rightsquigarrow -q_0^{-1}(1+q_3^{-1})q_{21}^{-2}q_{22}^{-1}\alpha\beta\alpha + (1+q_3)(1+q_3^{-1})q_{21}^{-2}q_{22}^{-1}\alpha \\
&\quad -q_0(1+q_3)(q_{21}^{-1} + q_{21}^{-1}q_{22}^{-1} + q_{21}^{-2}q_{22}^{-1}) \\
&\quad +q_0(1+q_3)(1+q_{21}^{-1} + q_{21}^{-1}q_{22}^{-1})\beta - q_0^2 q_3 \alpha^{-1}\beta,
\end{aligned}$$

$$\begin{aligned}
\beta\alpha(\beta^2) \quad &\mapsto \quad \beta\alpha\Big(-q_0^{-2}q_3^{-1}q_{21}^{-2}q_{22}^{-1}\alpha\beta\alpha + q_0^{-1}(1+q_3^{-1})q_{21}^{-2}q_{22}^{-1}\alpha \\
&\qquad +(1+q_{21}^{-1}+q_{21}^{-1}q_{22}^{-1})\beta - (q_{21}^{-2}q_{22}^{-1}+q_{21}^{-1}q_{22}^{-1}+q_{21}^{-1})\Big), \\
&\rightsquigarrow \quad -q_0^{-2}q_3^{-1}q_{21}^{-2}q_{22}^{-1}\beta\alpha^2\beta\alpha + q_0(1+q_3^{-1})q_{21}^{-2}q_{22}^{-1}\beta\alpha^2 \\
&\qquad +q_0(1+q_3)(1+q_{21}^{-1}+q_{21}^{-1}q_{22}^{-1})\beta - q_0^2q_3(1+q_{21}^{-1}+q_{21}^{-1}q_{22}^{-1})\alpha^{-1} \\
&\qquad -(q_{21}^{-2}q_{22}^{-1}+q_{21}^{-1}q_{22}^{-1}+q_{21}^{-1})\beta\alpha.
\end{aligned}$$

Equating the two reductions gives a new element of I, which we monicise (the leading term being $\beta\alpha^2\beta\alpha$, as its coefficient is nonzero). That is, we obtain $r_8 \in I$, where

$$\begin{aligned}
r_8 \quad = \quad & \beta\alpha^2\beta\alpha - q_0^4q_3^2q_{21}^2q_{22}\alpha^{-1}\beta + q_0^4q_3^2(q_{21}^2q_{22} - q_{21}q_{22} + q_{21})\alpha^{-1} \\
& -q_0(1+q_3)\alpha\beta\alpha + q_0(1+q_3)\beta\alpha^2 + q_0^2q_3(1+q_{21}-q_{21}q_{22})\beta\alpha \\
& -q_0^3q_3(1+q_3)(1+q_{21}-q_{21}q_{22}) + q_0^2(1+q_3)^2\alpha.
\end{aligned}$$

Adjoining r_8 to Ω_3 creates an ambiguity $\beta\alpha^2\beta\alpha^k$. To resolve it, it is equivalent to multiply r_8 by α^{-1} and we obtain $r_4 \in I$. Setting $\Omega_4 = \Omega_3 \cup \{r_4\}$ we have illegal words $\{\alpha^k, \gamma, \beta^2, \beta\alpha\beta, \beta\alpha^2\beta\}$. This leads to ambiguities $\beta\alpha^2\beta^2$ and $\beta^2\alpha^2\beta$. We resolve $\beta\alpha^2\beta^2$,

$$\begin{aligned}
(\beta\alpha^2\beta)\beta \quad &\mapsto \quad q_0^4q_3^2q_{21}^2q_{22}\alpha^{-1}\beta\alpha^{-1}\beta - q_0^4q_3^2(q_{21}^2q_{22}+q_{21}q_{22}+q_{21})\alpha^{-2}\beta \\
&\qquad +q_0(1+q_3)\alpha\beta^2 + q_0(1+q_3)\beta\alpha\beta - q_0^2q_3(1+q_{21}+q_{21}q_{22})\beta^2 \\
&\qquad +q_0^3q_3(1+q_3)(1+q_{21}+q_{21}q_{22})\alpha^{-1}\beta - q_0^2(1+q_3)^2\beta, \\
\beta\alpha^2(\beta^2) \quad &\mapsto \quad -q_0^{-2}q_3^{-1}q_{21}^{-2}q_{22}^{-1}\beta\alpha^3\beta\alpha + q_0^{-1}(1+q_3^{-1})q_{21}^{-2}q_{22}^{-1}\beta\alpha^3 \\
&\qquad +(1+q_{21}^{-1}+q_{21}^{-1}q_{22}^{-1}))\beta\alpha^2\beta - (q_{21}^{-2}q_{22}^{-1}+q_{21}^{-1}q_{22}^{-1}+q_{21}^{-1})\beta\alpha^2.
\end{aligned}$$

Equating the two single step reductions shows $r_5 \in I$. Similarly we can resolve $\beta^2\alpha^2\beta$, which is the same calculation as for $\beta\alpha^2\beta^2$, except for reversing the words. This shows $r_6 \in I$. □

Lemma 3.5.2. S^e *is finite dimensional for* $k = 3, 4, 5$.

Proof. Let Ω be the reduction system $\{r_\alpha, r_1, r_2, r_3, r_4\}$. The corresponding set of illegal words is $\{\alpha^k, \gamma, \beta^2, \beta\alpha\beta, \beta\alpha^2\beta\}$. With $k = 3$, the irreducible words are exactly all the subwords of $\alpha^2\beta\alpha^2$ (a finite set), and by Lemma A.2.3 this is a finite spanning set for S^e.

The cases $k = 4, 5$ require us to refine the ordering so that we use r_5 and r_6. Let r_5' be the complete reduction of r_5 with respect to Ω, and let $\leq$ be the ordering $\leq_{\gamma,\beta,(\alpha,\beta;m)}$ (see Section A.6), with $m = 3$. We claim that the leading word of r_5' is $\beta\alpha^3\beta\alpha$. First, observe that its coefficient in r_5 is nonzero. It therefore suffices to prove that $\beta\alpha^3\beta\alpha > u$ for all other words u appearing in r_5'. Clearly the only term which concerns us is $\alpha^{-1}\beta\alpha^{-1}\beta$ as the remaining terms involving two occurrences of β have been reduced. We expand $\alpha^{-1}\beta\alpha^{-1}\beta$ using the expression for α^{-1} as a polynomial in α, so we need to show that if $r, s \leq k - 1 \leq 4$ then $g^m_{a,b}(\beta\alpha^3\beta\alpha) > g^m_{a,b}(\alpha^r\beta\alpha^s\beta)$. This is true because $g^m_{a,b}(\beta\alpha^3\beta\alpha) = 3m + m^2$ and $g^m_{a,b}(\alpha^r\beta\alpha^s\beta) = r + sm$ (and clearly $3m + m^2 > r + sm$ for $m = 3$, $r, s \leq 4$).

This gives enough information to settle the case $k = 4$. Let Ω_5 be the reduction system $\{r_\alpha, r_1, r_2, r_3, r_4, r_5'\}$. By the above claim, the corresponding set of illegal words is $\{\alpha^4, \gamma, \beta^2, \beta\alpha\beta, \beta\alpha^2\beta, \beta\alpha^3\beta\alpha\}$. Clearly the irreducible words are exactly all the subwords of $\alpha^3\beta\alpha^3\beta$ (a finite set), and by Lemma A.2.3 this is a finite spanning set for S^e.

Now assume that $k = 5$, and let r_6' be the reduction of r_6 with respect to Ω. We claim that the leading word of r_6' is $\beta\alpha^4\beta\alpha^4$. Its coefficient in r_6 is $t^2q_0^4q_3^2q_{21}^2q_{22}$ (where t^2 is the coefficient of α^4 in α^{-1}). It therefore suffices to prove that $\beta\alpha^4\beta\alpha^4 > u$ for all other words u appearing in r_6'. We can forget about words other than $\alpha\beta\alpha^3\beta$ and $\beta\alpha^{-1}\beta\alpha^{-1}$. We expand the latter term as a linear combination of words $\beta\alpha^r\beta\alpha^s$, $r, s \leq 4$. We wish to calculate for which word $g^m_{a,b}$ takes its maximal value. $g^m_{a,b}(\beta\alpha^r\beta\alpha^s) = rm + sm^2$, which is maximised when r and s are maximised, i.e. when $r = s = 4$. This is clearly greater than $1 + 3m = g^m_{a,b}(\alpha\beta\alpha^3\beta)$.

We are now ready to prove the result for $k = 5$. Let Ω_6 be the reduction system $\{r_\alpha, r_1, r_2, r_3, r_4, {r_5}', {r_6}'\}$. By the above claim, the corresponding set of illegal words is $\{\alpha^5, \gamma, \beta^2, \beta\alpha\beta, \beta\alpha^2\beta, \beta\alpha^3\beta\alpha, \beta\alpha^4\beta\alpha^4\}$. Clearly the irreducible words are exactly all the subwords of $\alpha^4\beta\alpha^4\beta\alpha^3\beta$ (a finite set), and by Lemma

A.2.3 this is a finite spanning set for S^e. $\square$

Now $k = 3, 4, 5$ corresponds to Q being type E_6, E_7, E_8 respectively, so we obtain the following corollary.

Corollary 3.5.3. *If Q has type E_6, E_7, E_8 then $\Lambda^q(Q)$ is finite dimensional.*

Now we combine Corollaries 3.2.2, 3.4.2, 3.5.3 to complete the proof of Theorem 3.1.1.

3.6 Open problems

Now that is has been shown that $\Lambda^q(Q)$ is finite dimensional for Q Dynkin, the next obvious problem is to determine its dimension. This will almost certainly depend on q, and in particular the positive roots for which $q^\alpha = 1$. In fact, we immediately have the following result.

Corollary 3.6.1. *If $q^\alpha \neq 1$ for all positive roots for Q, then $\Lambda^q(Q)$ is zero.*

Proof. By combining Lemma 2.4.3 and Theorem 2.4.4, we have that the dimension vector α of a finite dimensional simple representation of Λ^q must be a positive root and must satisfy $q^\alpha = 1$. Hence, by the hypothesis, Λ^q has no finite dimensional simple representations. The only finite dimensional algebra without any finite dimensional simple representations is zero. $\square$

It should be possible to obtain further results by using the methods of [10, Section 7], though it is unclear whether this would lead to a proof of the following conjecture.

Conjecture 3.6.2. *Let $q \in (K^*)^{Q_0}$ and $\lambda \in K^{Q_0}$. If we can partition the set of roots for Q into a pair of subsets R_1, R_2 so that $q^\alpha = 1$ and $\lambda.\alpha = 0$ for $\alpha \in R_1$ and $q^\alpha \neq 1$ and $\lambda.\alpha \neq 0$ for $\alpha \in R_2$, then* $\dim \Lambda^q(Q) = \dim \Pi^\lambda(Q)$.

In the undeformed case, this conjecture can be concisely stated.

Conjecture 3.6.3. $\dim \Lambda^1(Q) = \dim \Pi(Q)$.

Whereas the first conjecture is made with little evidence, the second one is made with more confidence. It is obviously true in type A_n, and it is also true in type D_4 (see Lemma 5.1.3). Additionally one can calculate the dimension of $e_0\Lambda^1 e_0$ by using the reduction algorithm (the calculation is analogous to the previous sections), and this has been found to be the same as $e_0\Pi e_0$. We could (in theory) do the same with Λ^1, but this is rather impractical. It seems 'unnatural' that the dimensions of Λ^1 and Π could be different, whilst the dimensions of $e_0\Lambda^1 e_0$ and $e_0\Pi e_0$ be the same, since the algebras have the same presentation except for taking a different relation at 0. This suggests that the conjecture is true, but we have been unable to work out a specific reason why 'unnatural' should imply 'impossible'.

We end this section by considering how the undeformed multiplicative preprojective algebra relates to the 'deformed preprojective algebra of generalised Dynkin type', $P^f(Q)$, introduced by Białkowski, Erdmann and Skowroński, [5]. Note that these algebras are not the same as the deformed preprojective algebra of Crawley-Boevey and Holland. Let Q be a star-shaped quiver of type D_n, E_6, E_7 or E_8 with central vertex 0 (other situations are considered in [5], but they are not relevant to this discussion). Let R_Q be the algebra

$$R_Q = K\langle x, y\rangle/(x^{w_3}, y^{w_2}, (x+y)^{w_1}),$$

where the w_i are the integers given in Section 3.3. It can be easily checked that $R_Q \cong e_0\Pi e_0$. Let $f(x,y) \in \operatorname{rad}^2 R_Q$ and define $P^f(Q) = K\overline{Q}/I$, where I is the ideal generated by the elements $(\rho_v^f)_{v\in Q_0}$ and $\tilde{\rho}$ where

$$\begin{aligned}
\rho_0^f &= a_{11}a_{11}^* + a_{21}a_{21}^* + a_{31}a_{31}^* + f(a_{21}a_{21}^*, a_{31}a_{31}^*), \\
\rho_v^f &= \sum_{\substack{a\in\overline{Q}_1 \\ h(a)=v}} \epsilon(a)aa^*, \text{ if } v \neq 0, \\
\tilde{\rho} &= (a_{21}a_{21}^* + a_{31}a_{31}^*)^{w_1}.
\end{aligned}$$

In the following, we write A for $a_{11}a_{11}^*$, B for $a_{21}a_{21}^*$, C for $a_{31}a_{31}^*$ and e for e_0 and Λ for $\Lambda^1(Q)$. It can be shown that Λ is equal to $K\overline{Q}/\tilde{I}$, where $\tilde{I}$ is the ideal generated by the relations $(\rho_v^f)_{v\in Q_0}$, with $f = -xy$ in type D_n and $f = -xy - y^2 + xy^2$ otherwise. Since $\rho_v^f = \mu_v^1$ for all $v \neq 0$, it suffices to show that $\mu_0^1 \in \tilde{I}$ and $\rho_0^f \in I_\mu$. The first is true since we can cancel the terms involving B^{w_2} and C^{w_3} in $\rho_0^f(e+B)(e+C)$ to obtain μ_0^1. The second is true because in type D_n we can do a similar process on $\mu_0^1(e-C)(e-B)$ to obtain ρ_0^f, and in type E_n we do the same with $\mu_0^1(e-C)(e-B+B^2)$.

This shows that $P^f(Q)$ is the quotient of $\Lambda^1(Q)$ by the relation $\tilde{\rho} = (B+C)^{w_1}$. It is natural to ask whether $\tilde{\rho} \in I_\mu$, in which case this is a trivial quotient, and so $\Lambda^1(Q) = P^f(Q)$ (this is one example of the last question posed at the end of Chapter 5). The question is connected to an assertion made in the proof of Lemma 3.2 of [5], where it is stated that for any f and any Dynkin quiver Q,

$$eP^f(Q)e \cong R_Q. \tag{3.5}$$

If Q has type D_n, it can be shown that this claim is correct. Since we have already stated in the discussion after Conjecture 3.6.3 that $\dim e\Lambda e = \dim e\Pi e = \dim R_Q$, it must be the case that $P^f(Q) = \Lambda$, and in particular $(B+C)^{w_1} \in I_\mu$. We verify this - Since I_μ contains $(B+C-CB)^{w_1}$, it contains $L = (B+C-CB)^{w_1}((e+B)(e+C))^{\frac{w_1}{2}}$ (assuming w_1 is even, if w_1 is odd then multiplying by $((e+B)(e+C))^{\frac{w_1-1}{2}}(e+B)$ will work the same way). We can manipulate L (and remain in I_μ) using the following rules: $(B+C-CB)(1+B)$ can be replaced by $(B+C)$, $(B+C)(1+C)$ can be replaced by $(1+B)(B+C)$, and $(B+C)^2(1+B)$ can be replaced by $(1+B)(B+C)^2$ as each pair of expressions are the same if one cancels terms involving B^2 or C^2. It is easy to see that the expression eventually obtained is $(B+C)^{w_1}$.

However, if Q has type E_6, then (3.5) is not true for all f, and in particular for $f = -xy - y^2 + xy^2$. [It can also be shown that $\tilde{\rho} \notin I_\mu$, and so $\Lambda^1(Q) \ncong P^f(Q)$.] There is a surjective map θ from R_Q to $eP^f(Q)e$ which takes x to C and y to B. Clearly $(x+y+f(x,y))^3$ is sent to zero, so if θ were an isomorphism, then

$(x + y + f(x, y))^3$ must equal zero in R_Q. Consider the following matrices

$$x = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&-1&-1&-1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0&0&-1&-1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, \quad y = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

It is easily checked that these matrices satisfy $x^2 = 0$, $y^3 = 0$, $(x + y)^3 = 0$, and therefore define a representation of R_Q (in fact the regular representation). However, we have

$$(x + y + f(x, y))^3 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&-1&-1&2 \\ 0&0&0&0&0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \neq 0,$$

and thus $(x + y + f(x, y))^3 \neq 0$ in R_Q. Thus (3.5), and consequently the statement that $\dim P^f(Q) = \dim \Pi(Q)$ (which is the final part of [5, Lemma 3.2]) are incorrect. Possibly (depending on the other results of [5]) one should define $P^f(Q)$ without the extra relation $\tilde{\rho}$, as then $\dim P^f(Q) = \dim \Pi(Q)$ would very likely be true (Conjecture 3.6.3 is a special case of this).

Chapter 4

The extended Dynkin case

In this chapter we are concerned with the properties of $\Lambda^q(Q)$ where Q is an extended Dynkin diagram. The main result is a nice description of the ring $e_1\Lambda^1(Q)e_1$ (the analogue of Theorem 1.3.5), where 1 is an extending vertex. As in the previous chapter, the proof is a long case by case analysis, which comprises the majority of the chapter. In the final section, we consider the implications of this theorem with regard to the properties of Λ, and list some further open questions.

4.1 The main theorem

Theorem 4.1.1. *If Q is extended Dynkin and 1 is an extending vertex, then $e_1\Lambda(Q)e_1$ is a commutative algebra. More precisely,*

$$e_1\Lambda(Q)e_1 \cong K[X,Y,Z]/J,$$

where J is the ideal generated by

$$\begin{aligned} Z^{n+1} + XY + XYZ \qquad & \textit{if } Q \textit{ type } \tilde{A}_n, \\ Z^2 - p_k(X)XZ + p_{k-1}(X)X^2Y - XY^2 - XYZ \qquad & \textit{if } Q \textit{ type } \tilde{D}_n, \\ Z^2 + X^2Z + Y^3 - XYZ \qquad & \textit{if } Q \textit{ type } \tilde{E}_6, \\ Z^2 + Y^3 + X^3Y - XYZ \qquad & \textit{if } Q \textit{ type } \tilde{E}_7, \\ Z^2 - Y^3 - X^5 + XYZ \qquad & \textit{if } Q \textit{ type } \tilde{E}_8, \end{aligned}$$

where $k = n-4$, and the p_k are polynomials defined inductively by $p_{-1}(X) = -1$, $p_0(X) = 0$ and $p_{i+1}(X) = X(p_i(X) + p_{i-1}(X))$ for $i \geq 1$.

It would be interesting to determine the significance of these polynomials, given their similarity to the polynomials of the Kleinian singularities. It can be verified that they are irreducible, and have a unique singular point at zero. One can ask if each ring is isomorphic to the coordinate ring at the corresponding Kleinian singularity, (i.e. is $e_1\Lambda(Q)e_1$ is isomorphic $e_1\Pi(Q)e_1$?) as well as some other questions.

The proof of the theorem is done by a case by case analysis, starting with the star shaped quivers $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, then $\tilde{A}_n$ (omitting $\tilde{A}_0$ - this will be discussed in the final section) and finally $\tilde{D}_n$ with $n > 4$. The proof in each case splits into four parts.

1. We can assume that $\Lambda = K\overline{Q}/I_\mu$ (using Lemma 3.3.2 in the star shaped cases, and being careful to those the correct ordering in the $\tilde{A}_n$ case). The object of this part is to obtain some 'useful' elements of I_μ to be used in later stages (in the $\tilde{A}_n$ case we can move directly to stage 2). Assuming the quiver is star shaped (the $\tilde{D}_n$ with $n > 4$ are more complicated) and that 0 is the central vertex, then we can use the presentation $S = K\langle A_1, A_2, \ldots, A_k\rangle/I_R$ of $e_0\Lambda e_0$ given in Lemma 3.3.7, and then find elements of I_R (which can be considered to be members of I_μ) in a similar fashion to the previous chapter. We consider the rings Λ_0 and S_0 defined to be the analogues of Λ and S obtained by ignoring the complicated relation at the central vertex. There is a natural map $K\overline{Q} \to \Lambda_0$ (which induces a natural map $K\langle A_1, A_2, \ldots, A_k\rangle \to S_0$) which is denoted by an underline. This enables us to easily make the 'obvious' reductions by the simpler arm relations. Clearly, if $\underline{x} = \underline{y}$, then $x - y \in I_\mu$. Observe that during this part we are operating in the path algebra or the free algebra $K\langle A_1, A_2, \ldots, A_k\rangle$ (or in the rings Λ_0 and S_0 in the case where elements are underlined).

2. The object of this part is to show that each element of $e_1\Lambda e_1$ can be written as a linear combinations of products of X, Y and Z (where X, Y and

Z are certain well chosen paths). For the star shaped cases, there is an easy lemma which is helpful. We denote the shortest path of $\overline{Q}$ from 0 to 1 by [, and the shortest path from 1 to 0 by] (this might look rather ugly, but it seems the most efficient notation).

Lemma 4.1.2. *If H is a spanning set for $e_0\Lambda e_0$, then $[H] = \{[h] : h \in H\}$ is a spanning set for $e_1\Lambda e_1$.*

Proof. By Lemma 3.3.4, $e_1\Lambda e_1$ is spanned by the set of all normalised paths which start and end at 1. Let p be such a path. If p doesn't visit 0, the we use the reduction system $\{a^*_{ij}a_{ij} - a_{i,j+1}a^*_{i,j+1} : 1 \leq i \leq k, 1 \leq j \leq w_i - 2\} \cup \{a^*_{i,w_i-1}a_{i,w_i-1} : 1 \leq i \leq k\}$ to prove that $p = 0$. We can therefore assume that p does visit 0, which means p has the form $[p']$ for some path p' which starts and ends at 0. Clearly $p' \in KH$, so $p \in K[H]$. □

The method used is a 'reduction algorithm' which uses the elements obtained in part 1. For the $\tilde{D}_n$, cases, the standard method described in the appendix works nicely, but the remaining cases are slightly complicated, because the standard method will only get only part of the way towards the desired result. At that point we employ a modified reduction algorithm, which makes substitutions based on the position of a particular subword in a word. Of course care must be taken with this approach, but in each case it should be clear this is a valid argument. Some final comments about this stage - In the type $\tilde{E}_n$ cases, we define a sequence notation to better describe words. Although these sequences are just an alternative way of describing elements of the path algebra, we always use the convention that these sequences are elements of Λ (so that a sequence really represents the image under the map $K\overline{Q} \to \Lambda$).

3. The next stage is to show that there is a surjective map $\overline{\theta} : L \to e_1\Lambda e_1$ (where L is the appropriate $K[X,Y,Z]/J$). In part 2 we have shown there is a map $K\langle X,Y,Z\rangle \to e_1\Lambda e_1$, so this is simply a set of calculations which show X,Y and Z commute, and that J is sent to zero. In these calculations, we explain each step by stating which substitution is being used. At all times during this stage the convention is that we are working in Λ, so when a word or sequence is

written down, it is understood this means the image under the map $K\overline{Q} \to \Lambda$ (or $\Lambda_0 \to \Lambda$).

4. Once we have the surjective map $\overline{\theta}$, the final stage is to prove that it is an isomorphism. We write down a family $\{M^{st} : (s,t) \in V\}$ of Λ modules, where V is a 2 dimensional affine variety in K^2. The M^{st} are given as representations in $\overline{Q}$ satisfying the appropriate relations, where the matrices are rational functions in s and t defined on V, and are therefore algebraic. In each case $e_1 M^{st}$ is a one dimensional module for $e_1 \Lambda e_1$, and is therefore simple.

There is a morphism of varieties $\tilde{\phi} : V \to \operatorname{Spec} e_1 \Lambda e_1$ which takes (s,t) to $\operatorname{Ann}(e_1 M^{st})$. Since $e_1 \Lambda e_1$ is a quotient of L via $\overline{\theta}$, we can identify $\operatorname{Spec} e_1 \Lambda e_1$ with a closed subset of K^3, and this gives rise to a morphism of varieties $\phi : V \to K^3$ with takes (s,t) to (x_{st}, y_{st}, z_{st}), where x_{st} is the entry of the 1 by 1 matrix obtained by substituting the matrices in M^{st} for the arrows in $\overline{\theta}(X)$ (and similarly y_{st} and z_{st} are defined using $\overline{\theta}(Y)$ and $\overline{\theta}(Z)$ respectively).

Lemma 4.1.3. *If ϕ is injective, then $\overline{\theta} : L \to e_1 \Lambda e_1$ is an isomorphism.*

Proof. First note that in each case the ideal J is a prime ideal of height 1 since it is generated by a single irreducible polynomial, and so $K[X,Y,Z]/J$ is a domain of Krull dimension 2. To prove $\overline{\theta}$ is an isomorphism, it suffices to prove that $e_1 \Lambda e_1$ has Krull dimension 2 since in this case there exists a chain $P_0 \subset P_1 \subset P_2$ of prime ideals of $e_1 \Lambda e_1$, and hence a corresponding chain $P_0' \subset P_1' \subset P_2'$ of prime ideals of $K[X,Y,Z]/J$ containing $\operatorname{Ker} \overline{\theta}$. If $\operatorname{Ker} \overline{\theta} \neq 0$, this can be extended to a chain $\{0\} \subset P_0' \subset P_1' \subset P_2'$, contradicting $K[X,Y,Z]/J$ being a domain of Krull dimension 2. Since ϕ is injective, $\dim \operatorname{Spec} e_1 \Lambda e_1 \geq \dim \overline{\operatorname{Im} \phi} \geq \dim V = 2$, and so the Krull dimension of $e_1 \Lambda e_1$ is 2 (since it cannot be greater than 2). $\square$

Note that V is always chosen so that the x_{st}, y_{st}, z_{st} are non zero, which is essential when proving that ϕ is injective.

It is worth describing the method used to find the M^{st}. We consider the one parameter family F of regular simple representations of Q of dimension vector

δ (these can be obtained in [13] for example). They can be extended to representations of $\overline{Q}$ by using arbitrary matrices of the correct size to the represent the M_{a^*}, for each $M \in F$. We calculate which of these are representations of Λ. Namely, we determine which M_{a^*} satisfy the equations

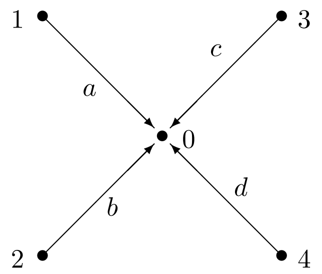
$$\prod_{\substack{a \in \overline{Q}_1 \\ h(a)=v}} (1_{M_{h(a)}} + M_a M_{a^*})^{\epsilon(a)} = 1_{M_{h(a)}} \text{ for all } v \in Q_0.$$

In each case, this leads to set of m equations in m unknowns (regarding the original parameter t as a constant). They are nonlinear, but by regarding some of the variables to be constants, can be assumed to be linear in the remaining variables. Solving for these variables and substituting, we obtain another set of equations with fewer variables, and can repeat the equations are all solved. This can be done easily on a computer. The solution set is always one dimensional (depending on s say) and we therefore obtain a family depending on two parameters, s and t.

A final comment, which is worth noting when comparing this theorem with Theorem 1.3.5. It is possible to repeat this entire calculation with $\Pi(Q)$ (where it is much easier). The paths we obtained in part 2 for $\Lambda(Q)$ will also generate $e_1\Pi(Q)e_1$, and the relation obtained in part 3 is exactly the relation given in the statement of Theorem 1.3.5 (which is the reason for the slightly strange looking polynomials). These relations only differ by XYZ in most cases.

4.2 Type $\tilde{D}_4$

We can assume that $\Lambda(Q) = K\overline{Q}/I_\mu$, where Q is the quiver



and I_μ is the ideal generated by the elements a^*a, b^*b, c^*c, d^*d and $\mu_0 = (e_0 + aa^*)(e_0+bb^*)(e_0+cc^*)(e_0+dd^*)-e_0$. Let $A = aa^*$, $B = bb^*$, $C = cc^*$, $D = dd^*$. Note that $e_0\Lambda e_0 \cong S = K\langle A, B, C, D\rangle/I_R$ where $R = \{A^2, B^2, C^2, D^2, s_0\}$ (with $s_0 = (1+A)(1+B)(1+C)(1+D)-1$).

Lemma 4.2.1. *The following elements lie in I_R (and hence in I_μ).*

$$\begin{aligned} s_1 &= D + A + B + C + AB + AC + BC + ABC, \\ s_2 &= CB + AB + AC + BA + BC + CA + ABC + ABA + ACA \\ &\quad +BCA + ABCA, \\ s_3 &= CAB - BAC - ABC - ABA - ACA - BCA - ABCA. \end{aligned}$$

Proof. Since $s_0 \in I_R$, so is $s_0(1+D)^{-1} = (1+A)(1+B)(1+C) - (1-D) = s_1$. Now we 'resolve' D^2: Since $D^2 \in R$, so is $s_4 = (1-(1+A)(1+B)(1+C))^2$. Multiplying s_4 by $(1+C)^{-1}(1+B)^{-1} = (1-C)(1-B)$ we get

$$s_2' = (1+A)(1+B)(1+C)(1+A) - 2(1+A) + (1-C)(1-B) \in I_R,$$

and then $\underline{s_2}' = \underline{s_2}$, so $s_2 \in I_R$. Multiplying s_4 by $(1+B)^{-1}(1+A)^{-1}$ on the left and $(1+C)^{-1}$ on the right we get

$$s_3' = (1+C)(1+A)(1+B) - 2 + (1-B)(1-A)(1-C) \in I_R,$$

and then $s_3 = s_3' - s_2 \in I_R$. $\square$

Lemma 4.2.2. *$e_1\Lambda e_1$ is generated by $X = [B]$, $Y = [C]$, $Z = [BC]$.*

Proof. Let Ω be the reduction system $\{A^2, B^2, C^2, s_1, s_2\}$, with respect to the ordering $\leq_{D,C,B,(B,C;3)}$. The leading words of s_1 and s_2 are D and CB respectively. By Lemma A.2.3, $e_0\Lambda e_0$ is spanned by the set H of irreducible words, namely, all words which do not involve D, no letter occurs two or more times consecutively, and C never occurs immediately to the left of B, and so by Lemma 4.1.2, $[H]$ is a spanning set for $e_1\Lambda e_1$. Now let G be the subset of H containing the empty word and all words which start or end with A. Since $a^*A, Aa, a^*a \in I_\mu$, if $z_0 \in G$, $[z_0]$ is zero in Λ, and therefore $H' = [(H \setminus G)]$ is a spanning set for $e_1\Lambda e_1$.

Elements of H' have the form $[x_1Ax_2A\ldots Ax_k]$ where $x_i \in \{B, C, BC\}$. Since $A = a^*a =][$, we can bracket this as $[x_1][x_2]\ldots[x_k]$ which completes the proof. $\square$

Lemma 4.2.3. *There is a surjective map $\overline{\theta} : K[X, Y, Z]/(Z^2 - X^2Y - XY^2 - XYZ) \to e_1\Lambda e_1$.*

Proof. By the previous lemma, there is a surjective map $\theta : K\langle X, Y, Z\rangle \to e_1\Lambda e_1$, which maps X to a^*Ba, Y to a^*Ca and Z to a^*BCa. Observe that

$$\begin{aligned}
\theta(YX - XY) &= a^*BACa - a^*CABa = \underline{a^*s_3a} = 0,\\
\theta(ZX - XZ) &= a^*BCABa - a^*BABCa = \underline{a^*Bs_3a} = 0,\\
\theta(YZ - ZY) &= a^*CABCa - a^*BCACa = \underline{a^*s_3Ca} = 0,
\end{aligned}$$

which shows X, Y, Z commute, and

$$\begin{aligned}
\theta(Z^2 - X^2Y - XY^2 - XYZ) &= a^*BCABCa - a^*BABACa - a^*BACACa\\
&\quad - a^*BABCACa\\
&= \underline{a^*Bs_3Ca} = 0,
\end{aligned}$$

which shows that θ induces $\overline{\theta}$. $\square$

Lemma 4.2.4. *$\overline{\theta}$ is an isomorphism.*

Proof. For all $(s,t) \in K^2$ such that $t^2s - ts + 1 \neq 0$, $s \neq 0$, $t \neq 0, 1$, we consider the matrices

$$\alpha = \begin{pmatrix} 0 & \frac{s(t-1)}{t^2s-ts+1} \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 \\ ts(t-1) & 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \frac{ts}{t^2s-ts+1} & \frac{-ts}{t^2s-ts+1} \\ \frac{ts}{t^2s-ts+1} & \frac{-ts}{t^2s-ts+1} \end{pmatrix}, \quad \delta = \begin{pmatrix} -ts & s \\ -t^2s & ts \end{pmatrix}.$$

One can check that $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 0$, and $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) = 1$. This implies (see the proof of Lemma 3.3.7) that one gets a representation M^{st} (see the diagram below) of Λ in which $M_0^{st} = K^2$, $M_1^{st} = \operatorname{Im}\alpha$, $M_2^{st} = \operatorname{Im}\beta$, $M_3^{st} = \operatorname{Im}\gamma$, $M_4^{st} = \operatorname{Im}\delta$, M_a^{st} is the inclusion of $\operatorname{Im}\alpha$ in K^2, and $M_{a^*}^{st}$ is α (and similarly for b, c, d).

$$\begin{tikzcd}
K \arrow[dr, shift left, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & & K \arrow[dl, shift left, "\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)"] \\
& K^2 \arrow[ul, shift left, "\left(\begin{smallmatrix} 0 & \frac{s(t-1)}{t^2s-ts+1} \end{smallmatrix}\right)"] \arrow[ur, shift left, "\left(\begin{smallmatrix} \frac{ts}{t^2s-ts+1} & \frac{-ts}{t^2s-ts+1} \end{smallmatrix}\right)"] \arrow[dl, shift left, "\left(\begin{smallmatrix} ts(t-1) & 0 \end{smallmatrix}\right)"] \arrow[dr, shift left, "\left(\begin{smallmatrix} -ts & s \end{smallmatrix}\right)"] & \\
K \arrow[ur, shift left, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & & K \arrow[ul, shift left, "\left(\begin{smallmatrix} 1 \\ t \end{smallmatrix}\right)"]
\end{tikzcd}$$

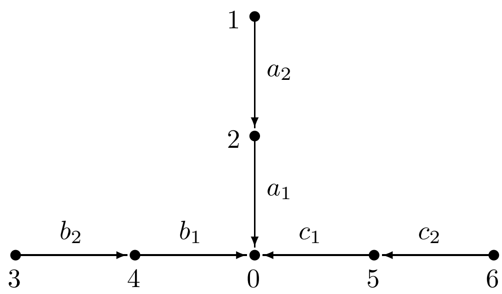
If we calculate x_{st}, y_{st}, z_{st} as described before Lemma 4.1.3, we find

$$x_{st} = \frac{(t-1)^2ts^2}{t^2s - ts + 1}, \quad y_{st} = \frac{(t-1)ts^2}{(t^2s - ts + 1)^2}, \quad z_{st} = \frac{(t-1)^2t^2s^3}{(t^2s - ts + 1)^2}.$$

If we assume that $(x_{st}, y_{st}, z_{st}) = (x_{s't'}, y_{s't'}, z_{s't'})$, then in particular $t = z^{st}/x_{st}y_{st}(y_{st} + z_{st}) = z_{s't'}/x_{s't'}y_{s't'}(y_{s't'} + z_{s't'}) = t'$ which shows $t = t'$ and then $ts(t-1) = z_{st}/y_{st} = z_{s't'}/y_{s't'} = t's'(t'-1)$ which shows $s = s'$. Thus $\overline{\theta}$ is an isomorphism by Lemma 4.1.3. $\square$

4.3 Type $\tilde{E}_6$.

We can assume that $\Lambda(Q) = K\overline{Q}/I_\mu$, where Q is the quiver



and I_μ is the ideal generated by the elements $a_2^*a_2, b_2^*b_2, c_2^*c_2, a_2a_2^* - a_1^*a_1, b_2b_2^* - b_1^*b_1, c_2c_2^* - c_1^*c_1$ and $\mu_0 = (e_0 + a_1a_1^*)(e_0 + b_1b_1^*)(e_0 + c_1c_1^*) - e_0$. Let $A = a_1a_1^*$, $B = b_1b_1^*$, $C = c_1c_1^*$. Note that $e_0\Lambda e_0 \cong K\langle A, B, C\rangle/I_R$ where $R = \{A^3, B^3, C^3, s_0\}$ with $s_0 = (1 + A)(1 + B)(1 + C) - 1$.

Lemma 4.3.1. *The following elements lie in I_R (and hence in I_μ).*

$$\begin{aligned}
s_1 &= C + A + B - BA - A^2 - B^2 + BA^2 + B^2A - B^2A^2, \\
s_2 &= BA^2 + A^2B + AB^2 + B^2A + BAB + ABA + ABAB - B^2A^2, \\
s_3 &= BA^2 + A^2B + AB^2 + B^2A + BAB + ABA + BABA - A^2B^2, \\
s_4 &= B^2A^2 + BABA - A^2B^2 - ABAB.
\end{aligned}$$

Proof. Since $s_0 \in I_R$, so is $(1+B)^{-1}(1+A)^{-1}s_0 = (1+C) - (1+B)^{-1}(1+A)^{-1} = s_1$. Now we 'resolve' C^3: Since $C^3 \in I_R$, so is $s_5 = ((1 + B)^{-1}(1 + A)^{-1} - 1)^3$. Multiplying s_5 on the right by $(1 + A)(1 + B)(1 + A)(1 + B)$ we get

$$(1+B)^{-1}(1+A)^{-1} - 3 + 3(1+A)(1+B) - (1+A)(1+B)(1+A)(1+B) = s_2' \in I_R,$$

and then $\underline{s_2}' = \underline{s_2}$, so $s_2 \in I_R$. Multiplying s_5 on the left by $(1+B)(1+A)(1+B)$ and the right by $(1 + A)$ we get

$$(1+A)^{-1}(1+B)^{-1} - 3 + 3(1+B)(1+A) - (1+B)(1+A)(1+B)(1+A) = s_3' \in I_R,$$

and then $\underline{s_3}' = \underline{s_3}$, so $s_3 \in I_R$. Finally $s_4 = s_3 - s_2 \in I_R$. $\square$

Lemma 4.3.2. *$e_1 \Lambda e_1$ is generated by $X = [B]$, $Y = [B^2]$, $Z = [BAB^2]$.*

Proof. As in the $\tilde{D}_4$ case, we find a suitable spanning set for $e_0 \Lambda e_0$. Using the reduction system $\{A^3, B^3, s_1\}$ with respect to the ordering $\leq_C$ shows that the set H of all words which do not involve C, B^3, A^3 is a spanning set for $e_0 \Lambda e_0$, and by Lemma 4.1.2, $[H]$ is a spanning set for $e_1 \Lambda e_1$. Let H' be the set of all elements of H which start and end with B. Since $[A,\ A]$, $[] \in I_\mu$, $[H']$ is a spanning set for $e_1 \Lambda e_1$. Attempts to reduce H' further by using s_2 in some reduction system do not give the required answer, so we have to use other methods.

We can denote an element of $[H']$ of the form $[B^{i_1} A B^{i_2} A \dots A B^{i_k}]$ as a sequence of integers, $[i_1, i_2, \dots, i_k]$, where each $i_l = 1, 2$. Since $A^2 = a_1^* a_2^* a_2 a_1$ can replaced by][, for all $h \in H'$ we can write $[h]$ as a product of sequences, e.g. $[B^2 ABABA^2 B^2 A^2 B^2] = [2,1,1][2][1]$.

We claim that all sequences $[i_1, i_2, \dots, i_k]$ can be written (as elements of Λ) as a linear combination of elements which are products of the sequences [1],[2],[1,2], and this completes the proof since these sequences are equal to X,Y,Z respectively.

Proof of claim: By induction on the length k of the sequence. We first check the small cases. If $k = 1$ then there is nothing to prove. Suppose $k = 2$. Then $[1,2]$ is trivial, $[1,1] = \underline{[s_2]} = 0$. The claim follows for the sequences [2,1] and [2,2] because $\underline{[Bs_3]} = [1][1] + [2,1] + [1,2] - [1][2] = 0$ and $\underline{[B^2 s_3]} = [2][1] + [2,2] - [2][2] = 0$. Now suppose that $k \geq 3$ and assume that the claim is true for all sequences of length less than k.

(1) Since $\underline{[s_2 A} = \underline{[B^2 A^2 + [BABA}$, we have that

$$[1, 1, i_3, \dots, i_k] = -[2][i_3, \dots, i_k].$$

Using the induction hypothesis, the claim is verified for sequences of the form $[1, 1, \dots]$.

(2) Since

$$\begin{aligned}\underline{[BAs_2} &= \underline{[BABA^2 + [BA^2B^2 + [BAB^2A + [BABAB + [BA^2BA} \\ &\quad \underline{[BA^2BAB - [BAB^2A^2},\end{aligned}$$

we have that

$$\begin{aligned}[1, 2, i_3, \ldots, i_k] &= -[1,1][i_3, \ldots, i_k] - [1][i_3 + 2, \ldots, i_k] - [1, 1, i_3 + 1, \ldots, i_k] \\ &\quad -[1][1, i_3, \ldots, i_k] - [1][1, i_3 + 1, \ldots, i_k] + [1, 2][i_3, \ldots, i_k].\end{aligned}$$

By the induction hypothesis, and the result for sequences of the form $[1, 1, \ldots]$, the claim is true for sequences of the form $[1, 2, \ldots]$ (note that sequences involving integers greater than 2 can be ignored because this corresponds to having a subword B^3, which is zero).

(3) Since $\underline{[s_2} = \underline{[BA^2 + [B^2A + [BAB - [B^2A^2}$, we have that

$$[2, i_2, \ldots, i_k] = -[1][i_2, \ldots, i_k] - [1, i_2 + 1, \ldots, i_k] + [2][i_2, \ldots, i_k],$$

and the claim is true for sequences of the form $[2, \ldots]$, and therefore for all sequences. □

Lemma 4.3.3. *There is a surjective map* $\theta : K[X, Y, Z]/(Z^2 - X^2Y - XY^2 - XYZ) \to e_1\Lambda e_1$.

Proof. By the previous lemma, there is a surjective map $\theta : K\langle X, Y, Z\rangle \to e_1\Lambda e_1$, which maps X to $[B]$, Y to $[B^2]$ and Z to $[BAB^2]$. Observe that

$$\begin{aligned}\theta(XY) &= [(BA^2)B^2] = -[BABAB] - [B^2AB^2], \\ \theta(YX) &= [B^2(A^2B)] = -[BABAB] - [B^2AB^2].\end{aligned}$$

This is true by using s_3 in line 1 and s_2 in line 2 to substitute the bracketed term.

$$\begin{aligned}\theta(XZ) &= [BA(ABAB)B] \\ &= [BAB^2A^2B] + [(BAB)ABAB] \\ &= [BAB^2A^2B] - [B^2][BAB] \\ &= \theta(ZX)).\end{aligned}$$

Lines 2,3 are true by using s_4 and s_2 respectively to substitute the bracketed words, and line 4 since $[BAB]$ is zero.

$$\begin{aligned}\theta(YZ) &= [(B^2A)ABAB^2] = -[BABABAB^2],\\ \theta(ZY) &= [(BAB(BA^2)B^2] = -[BABABAB^2].\end{aligned}$$

We have used s_2 to substitute the bracketed term on each line. This shows that X, Y, Z commute. Finally,

$$\begin{aligned}\theta(Z^2) &= [BAB^2(A^2B)AB^2]\\ &= -[BA(B^2A)B^2AB^2] - [BAB^2ABA^2B^2] + [BAB^2A^2B^2AB^2]\\ &= [BAB][B^2AB^2] + [B][BAB^2AB^2] - [BAB^2AB][B^2]\\ &= \theta(X)[BAB^2AB^2] - [BAB^2AB^2]\theta(Y)\\ &= \theta(X)\theta(Z)\theta(Y) - \theta(X)\theta(X)\theta(Z) - \theta(Y)\theta(Y)\theta(Y)\\ &= \theta(XYZ - X^2Z - Y^3).\end{aligned}$$

In line 2 and 3, we used s_3 and s_2 respectively to substitute the bracketed word, and line 4 we cancelled the term involving $[BAB]$. The next line uses the facts that $[1, 2, 1] = -[1, 1, 2] = [2][2]$ and $[1, 2, 2] = -[1][1, 2] + [1, 2][2]$, which can be easily verified by following the proof of the previous lemma. □

Lemma 4.3.4. *$\overline{\theta}$ is an isomorphism.*

Proof. For all $(s,t) \in K^2$ such that $t^2s - ts + 1 \neq 0$, $s \neq 0$, $t \neq 0, 1$, we consider the matrices

$$\alpha = \begin{pmatrix} 0 & \frac{ts(t-1)}{t^2s-ts+1} & -ts(t-1)\\ 0 & 0 & -ts(t-1)\\ 0 & 0 & 0\end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 0 & 0\\ s(t-1) & 0 & 0\\ \frac{-s}{t^2s-ts+1} & \frac{ts}{t^2s-ts+1} & 0\end{pmatrix}$$

$$\gamma = \begin{pmatrix} 0 & \frac{-ts(t-1)}{t^2s-ts+1} & \frac{ts(t-1)}{t^2s-ts+1}\\ -s(t-1) & \frac{s^2t(t-1)^2}{t^2s-ts+1} & \frac{ts(-3t^2s+3ts-s+t^3s+t-1)}{t^2s-ts+1}\\ s & \frac{-ts(ts-s+1)}{t^2s-ts+1} & \frac{-s^2t(t-1)^2}{t^2s-ts+1}\end{pmatrix}.$$

One can check that $\alpha^3 = \beta^3 = \gamma^3 = 0$, and $(\alpha+1)(\beta+1)(\gamma+1) = 1$. This implies (see the proof of Lemma 3.3.7) that one gets a representation M^{st} of Λ,

$$\begin{tikzcd}
 & & \operatorname{Im}\gamma^2 \arrow[d, shift left] & & \\
 & & \operatorname{Im}\gamma \arrow[u, shift left] \arrow[d, shift left] & & \\
\operatorname{Im}\alpha^2 \arrow[r, shift left] & \operatorname{Im}\alpha \arrow[l, shift left] \arrow[r, shift left] & K^3 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift right] & \operatorname{Im}\beta \arrow[l, shift right] \arrow[r, shift right] & \operatorname{Im}\beta^2 \arrow[l, shift right]
\end{tikzcd}$$

where the linear maps are $M^{st}_{a_i{}^*} = \alpha|_{\operatorname{Im}\alpha^{i-1}}$ and $M^{st}_{a_i}$ is the inclusion of $\operatorname{Im}\alpha^i$ in $\operatorname{Im}\alpha^{i-1}$ (and similarly for the b_i and c_i). This is easily seen to have dimension vector δ. Calculating x_{st}, y_{st}, z_{st} as described before Lemma 4.1.3, we find

$$x_{st} = \frac{(t-1)^2 t^2 s^3}{(t^2 s - ts + 1)^2}, \quad y_{st} = \frac{-(t-1)^3 t^3 s^4}{(t^2 s - ts + 1)^2}, \quad z_{st} = \frac{(t-1)^5 t^4 s^6}{(t^2 s - ts + 1)^3}.$$

If we assume that $(x_{st}, y_{st}, z_{st}) = (x_{s't'}, y_{s't'}, z_{s't'})$, then in particular $(t-1)/t = -z_{st}^2/y_{st}^3 = -z_{s't'}^2/y_{s't'}^3 = (t'-1)/t'$ which shows $t = t'$ and then $ts(t-1) = y_{st}/x_{st} = y_{s't'}/x_{s't'} = t's'(t'-1)$ which shows $s = s'$. Thus $\overline{\theta}$ is an isomorphism by Lemma 4.1.3. $\square$

4.4 Type $\tilde{E}_7$.

We can assume that $\Lambda(Q) = K\overline{Q}/I_\mu$, where Q is the quiver

$$\begin{tikzcd}
 & & & \overset{4}{\bullet} \arrow[d, "c"] & & & \\
\underset{1}{\bullet} \arrow[r, "a_3"] & \underset{2}{\bullet} \arrow[r, "a_2"] & \underset{3}{\bullet} \arrow[r, "a_1"] & \underset{0}{\bullet} & \underset{5}{\bullet} \arrow[l, "b_1"'] & \underset{6}{\bullet} \arrow[l, "b_2"'] & \underset{7}{\bullet} \arrow[l, "b_3"']
\end{tikzcd}$$

and I_μ is the ideal generated by the elements $a_3^* a_3, b_3^* b_3, c^* c, a_3 a_3^* - a_2^* a_2, b_3 b_3^* - b_2^* b_2, a_2 a_2^* - a_1^* a_1, b_2 b_2^* - b_1^* b_1$ and $\mu_0 = (e_0 + a_1 a_1^*)(e_0 + b_1 b_1^*)(e_0 + cc^*) - e_0$. We set $A = a_1 a_1^*$, $B = b_1 b_1^*$, $C = cc^*$. Note that $e_0 \Lambda e_0 \cong K\langle A, B, C\rangle / I_R$ where R is the set of elements $\{A^4, B^4, C^2, s_0\}$ with $s_0 = (1+A)(1+B)(1+C) - 1$.

Lemma 4.4.1. *The following elements lie in I_R (and hence in I_μ).*

$$\begin{aligned} s_1 &= B + A + C - A^2 - AC + A^3 + A^2C - A^3C, \\ s_2 &= CA^3 + A^3C + CA^2C - CA^3C + ACA^2 + A^2CA \\ &\quad +ACAC + CACA + ACACA. \end{aligned}$$

Proof. Since $s_0 \in I_R$, so is $(1+A)^{-1}s_0(1+C)^{-1} = (1+B)-(1+A)^{-1}(1+C)^{-1} = s_1$. Now we 'resolve' B^4: Since $B^4 \in I_R$, so is $s_3 = ((1+A)^{-1}(1+C)^{-1} - 1)^4$. Let $s_4 = (1+A)(1+C)(1+A)(1+C)(1+A)s_3$. One finds that $\underline{s_4} = \underline{s_2}$, so $s_2 \in I_R$. $\square$

Lemma 4.4.2. *$e_1\Lambda e_1$ is generated by $X = [C]$, $Y = [CAC]$, $Z = [CACA^2C]$.*

Proof. Once again, we show that each element of $e_1\Lambda e_1$ can be written as linear combinations of products of X,Y and Z. Using the reduction system $\{A^4, C^2, s_1\}$ with respect to the ordering $\leq_B$, we see that $e_0\Lambda e_0$ is spanned by the set H of all words not containing B, C^2, A^4 as a subword, and hence $e_1\Lambda e_1$ is spanned by $[H]$. Since $[A,\ A]$, $[] \in I_\mu$, we can replace $[H]$ by $[H']$, where H' is the subset of H containing all words which start and end with C. We can express an element $[CA^{i_1}CA^{i_2}C\ldots CA^{i_n}C]$ of $[H']$ as a sequence $[i_1, i_2, \ldots, i_k]$, (note that we can assume that each $i_l = 1, 2, 3$ and we use [.] to denote $[C]$). We claim that each sequence can be written as a linear combination of products of the sequences [.],[1],[1,2], which completes the proof of the lemma.

Proof of claim - By a 'reduction algorithm' on sequences. Consider the ordering $\leq_{3,2,1,lex}$ on the set of all sequences, where the lexographic ordering has $2 > 1$. We write down a list of substitutions which writes a sequence as a linear combination of products of lesser sequences. Note that this is what was effectively being done for $\tilde{E}_6$, only there the ordering was just $\leq_{len}$.

(1) Since $[i_1, \ldots, i_{j-1}, 3, i_{j+1}, \ldots, i_k] = [i_1, \ldots, i_{j-1}][i_{j+1}, \ldots, i_k]$, we can assume that each $i_l = 1, 2$.

(2) Since $\underline{Cs_2C} = \underline{CACA^2C + CA^2CAC + CACACAC}$, we have that

$$\begin{aligned} [i_1, \ldots, i_{j-1}, 2, 1, i_{j+2}, \ldots, i_k] &= -[i_1, \ldots, i_{j-1}, 1, 2, i_{j+2}, \ldots, i_k] \\ &\quad -[i_1, \ldots, i_{j-1}, 1, 1, 1, i_{j+2}, \ldots, i_k]. \end{aligned}$$

By applying this substitution repeatedly, we can write any sequence as a linear combination of sequences of the form $[1, 1, \ldots, 1, 2, 2, \ldots, 2]$, and thus it suffices to prove the claim only for sequences of this form.

(3) Since $\underline{[s_2C} = \underline{[CA^3C + [CACAC}$, we have that

$$[1, 1, i_3, \ldots, i_k] = -[3][i_3, \ldots, i_k] = -[.][.][i_3, \ldots, i_k],$$

and it follows that it suffices to prove the claim for sequences of the form $[1, 2, 2, \ldots, 2]$.

(4) Since $\underline{CA^2s_2]} = \underline{CA^2CA^2C] - CA^2CA^3C] + CA^3CAC]}$, we have that

$$\begin{aligned} [i_1, \ldots, i_{k-2}, 2, 2] &= [i_1, \ldots, i_{k-2}, 2, 3] - [i_1, \ldots, i_{k-2}, 3, 1] \\ &= [i_1, \ldots, i_{k-2}, 2][.] - [i_1, \ldots, i_{k-2}][1]. \end{aligned}$$

and it follows that we can write any sequence as a linear combination of products of the sequences $[.], [1], [2], [1, 2]$. Since $\underline{[s_2]} = \underline{[CA^2C] - [CA^3C]}$, then $[2] = [3] = [.][.]$, which completes proof of the claim. $\square$

Lemma 4.4.3. *There is a surjective map* $\theta : K[X, Y, Z]/(Z^2 + X^3Y + Y^3 - XYZ) \to e_1\Lambda e_1$.

Proof. By the previous lemma, there is a surjective map $\theta : K\langle X, Y, Z\rangle \to e_1\Lambda e_1$, which maps X to $[C]$, Y to $[CAC]$ and Z to $[CACA^2C]$. Observe that by expanding using s_2 we have

$$\begin{aligned} \theta(XY) &= [(CA^3C)AC] = [CA^2CAC] + [CACA^2C], \\ \theta(YX) &= [(CA(CA^3C)] = [CACA^2C] + [CA^2CAC]. \end{aligned}$$

which shows X and Y commute. In the following calculations, we make substitutions of the bracketed part using s_2, convert into the sequence notation and follow the method described in the above proof.

$$\begin{aligned}
\theta(ZX) &= [CA(CA^2C)A^3C] = [CACA^3CA^3C] - [CA^2(CACA)A^2C] \\
&= [1,3,3] + [2,2,2] - [2,3,2] + [3,1,2] + [3,1,3] \\
&= [1][.][.] + [2,2][.] - [2][1] - [.][.][.][.] + [.][1,2] + [.][1][.] \\
&= [1][.][.] + [.][.][.][.] - [.][1][.] - [.][.][1] - [.][.][.][.] + [.][1,2] + [.][1][.] \\
&= [.][1,2] \\
&= \theta(XZ)
\end{aligned}$$

$$\begin{aligned}
\theta(YZ) &= [CACA^3(CACA)AC] \\
&= -[CAC(A^3C)A^2CAC] + [CACA^3CA^3CAC] \\
&= [1,2,3,1] + [1,1,1,2,1] + [1,1,1,3,1] + [1,3,3,1] \\
&= [1,2][1] - [1,1,1,1,2] - [1,1,1,1,1,1] + [1,1,1][1] + [1][.][1] \\
&= [1,2][1] + [.][1,1,2] + [.][1,1,1,1] - [.][1][1] + [1][.][1] \\
&= [1,2][1] - [.][.][2] - [.][.][1,1] \\
&= [1,2][1] - [.][.][.][.] + [.][.][.][.] \\
&= \theta(ZY).
\end{aligned}$$

$$\begin{aligned}
\theta(Z^2) &= [CACA^2(CA^3)CACA^2C] \\
&= -[1,3,2,1,2] - [1,2,1,1,1,2] - [1,3,1,1,1,2] \\
&= -[1][2,1,2] - [1,2,1,1,1,2] - [1][1,1,1,2] \\
&= [1][1,2,2] + [1,1,2,1,1,2] + [1,1,1,1,1,1,2] \\
&= [1][1,2,2] - [1,1,1,2,1,2] \\
&= [1][1,2,2] + [1,1,1,1,2,2] + [1,1,1,1,1,1,2] \\
&= [1][1,2,2] + [.][.][2,2] - [.][.][.][2] \\
&= [1][1,2][.] - [1][1][1] - [.][.][.][1] \\
&= \theta(XYZ - Y^3 - X^3Y).
\end{aligned}$$

□

Lemma 4.4.4. *$\overline{\theta}$ is an isomorphism.*

Proof. For all $(s,t) \in K^2$ such that $s \neq 0, -1$, $t \neq 0, 1$, we consider the matrices

$$\alpha = \begin{pmatrix} 0 & \frac{s(t-1)}{s+1} & -s(t-1) & \frac{s(t-1)}{s+1} \\ 0 & 0 & -s & \frac{s(t-1)}{t(s+1)} \\ 0 & 0 & 0 & \frac{s(t-1)}{t(s+1)} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{s}{t} & 0 & 0 & 0 \\ \frac{-s(t-1)}{t(s+1)} & \frac{s(t-1)}{s+1} & 0 & 0 \\ -s & s(t-1) & s & 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 0 & \frac{-s(t-1)}{s+1} & \frac{s(t-1)}{s+1} & \frac{-s(t-1)}{s+1} \\ \frac{-s}{t} & \frac{s^2(t-1)}{t(s+1)} & \frac{s(t+s)}{t(s+1)} & \frac{-s(t-1)}{t(s+1)} \\ \frac{s(t-1)}{t} & \frac{-s(-t-2ts+s+t^2s+t^2)}{t(s+1)} & \frac{-s^2(t-1)}{t(s+1)} & \frac{-s(t-1)}{t(s+1)} \\ s & -s(t-1) & -s & 0 \end{pmatrix}.$$

One can check that $\alpha^4 = \beta^4 = \gamma^2 = 0$, and $(\alpha+1)(\beta+1)(\gamma+1) = 1$. This implies (see the proof of Lemma 3.3.7) that one gets a representation M^{st} of Λ,

$$\begin{tikzcd}
 & & & \operatorname{Im}\gamma \arrow[d, shift left] & & & \\
\operatorname{Im}\alpha^3 \arrow[r, shift left] & \operatorname{Im}\alpha^2 \arrow[l, shift left] \arrow[r, shift left] & \operatorname{Im}\alpha \arrow[l, shift left] \arrow[r, shift left] & K^4 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \operatorname{Im}\beta \arrow[l, shift left] \arrow[r, shift left] & \operatorname{Im}\beta^2 \arrow[l, shift left] \arrow[r, shift left] & \operatorname{Im}\beta^3 \arrow[l, shift left]
\end{tikzcd}$$

where the linear maps are $M^{st}_{a_i{}^*} = \alpha|_{\operatorname{Im}\alpha^{i-1}}$ and $M^{st}_{a_i}$ is the inclusion of $\operatorname{Im}\alpha^i$ in $\operatorname{Im}\alpha^{i-1}$ (and similarly for the b_i and c_i). This is easily seen to have dimension vector δ. Now

$$x_{st} = \frac{-(t-1)^2 s^4}{t(s+1)^2}, \quad y_{st} = \frac{(t-1)^3 s^6}{t^2(s+1)^3}, \quad z_{st} = \frac{-(t-1)^5 s^9}{t^3(s+1)^4}.$$

If we assume that $(x_{st}, y_{st}, z_{st}) = (x_{s't'}, y_{s't'}, z_{s't'})$, then in particular $t = -x_{st}^3/y_{st}^2 = -x_{s't'}^3/y_{s't'}^2 = t'$ and $s/(s+1) = x_{st}y_{st}/z_{st} = x_{s't'}y_{s't'}/z_{s't'} = s'/(s'+1)$ which shows $s = s'$. Thus $\overline{\theta}$ is an isomorphism by Lemma 4.1.3. $\square$

4.5 Type $\tilde{E}_8$.

We can assume that $\Lambda(Q) = K\overline{Q}/I_\mu$, where Q is the quiver

$$\begin{tikzcd}
 & & & & & 7 \arrow[d, "c"] & & \\
1 \arrow[r, "a_5"] & 2 \arrow[r, "a_4"] & 3 \arrow[r, "a_3"] & 4 \arrow[r, "a_2"] & 5 \arrow[r, "a_1"] & 0 & 8 \arrow[l, "b_1"'] & 9 \arrow[l, "b_2"']
\end{tikzcd}$$

and I_μ is the ideal generated by the elements $a_5^*a_5, b_2^*b_2, c^*c, a_5a_5^* - a_4^*a_4, a_4a_4^* - a_3^*a_3, a_3a_3^* - a_2^*a_2, a_2a_2^* - a_1^*a_1, b_2b_2^* - b_1^*b_1$ and $\mu_0 = (e_0 + a_1a_1^*)(e_0 + b_1b_1^*)(e_0 + cc^*) - e_0$. We set $A = a_1a_1^*$, $B = b_1b_1^*$, $C = cc^*$. Note that $e_0\Lambda e_0 \cong K\langle A, B, C\rangle/I_R$ where $R = \{A^6, B^3, C^2, s_0\}$ with $s_0 = (1+A)(1+B)(1+C) - 1$.

Lemma 4.5.1. *The following elements lie in I_R (and hence in I_μ).*

$$\begin{aligned}
s_1 &= B + A + C - A^2 - AC + A^3 + A^2C - A^4 - A^3C + A^5 + A^4C - A^5C, \\
s &= CACAC + CA^2C + ACAC + CACA + ACA \\
&\quad +CAC + A^2C + CA^2 + A^5 - A^4 + A^3.
\end{aligned}$$

The following elements of $K\overline{Q}$ lie in I_μ.

$$\begin{aligned}
t_1 &= CACAC + CA^5C - CA^4C + CA^3C, \\
t_2 &= CACACAC + CACA^2C + CA^2CAC - CA^5C + CA^4C, \\
t_3 &= CACACA^2C + CACA^3C + CA^2CA^2C + CA^5C, \\
t_4 &= CACACA^3C + CACA^4C + CA^2CA^3C, \\
t_5 &= CACACA^4C + CACA^5C + CA^2CA^4C, \\
t_6 &= [CACAC] + [CA^2C] + [CAC], \\
t_7 &= [CACAC + [CA^2C, \\
t_8 &= CACAC] + CA^2C], \\
t_9 &= [CACACA^4C + [CA^2CA^4C + [CACA^5C + [CACA^4C, \\
t_{10} &= [CACACA^3C + [CA^2CA^3C + [CACA^4C + [CACA^3C + [CA^5C,
\end{aligned}$$

Proof. Since $s_0 \in I_R$, so is $(1+A)^{-1}s_0(1+C)^{-1} = (1+B)-(1+A)^{-1}(1+C)^{-1} = s_1$. Now we 'resolve' B^3: Since $B^4 \in R$, so is $s_3 = ((1+A)^{-1}(1+C)^{-1}-1)^3$. Let $s_4 = (1+C)(1+A)(1+C)(1+A)(1+C)s_3$. Observe that $\underline{s_4} = \underline{s}$, so $s \in I_R$. The $t_i \in I_\mu$ since $\underline{t_1} = \underline{CsC}$, $\underline{t_2} = \underline{CsAC}$, $\underline{t_3} = \underline{CsA^2C}$, $\underline{t_4} = \underline{CsA^3C}$, $\underline{t_5} = \underline{CsA^4C}$, $\underline{t_6} = \underline{[s]}$, $\underline{t_7} = \underline{[sC}$, $\underline{t_8} = \underline{C]}$, $\underline{t_9} = \underline{[sA^4C}$, $\underline{t_{10}} = \underline{[sA^3C}$. $\square$

Lemma 4.5.2. *$e_1\Lambda e_1$ is generated by the elements $X = [C]$, $Y = [CACAC]$, $Z = [CACACA^2CAC]$.*

Proof. Using the reduction system $\{A^6, C^2, s_1\}$ with respect to the ordering $\leq_B$, we see that $e_0\Lambda e_0$ is spanned by the set H of all words not containing B, C^2, A^6 as a subword, and hence $e_1\Lambda e_1$ is spanned by $[H]$. Since $[A,\ A]$, $[] \in I_\mu$, we can replace $[H]$ by $[H']$, where H' is the subset of H containing all words which start and end with C. We can express an element $[CA^{i_1}CA^{i_2}C\ldots CA^{i_n}C]$ of $[H']$ as a sequence $[i_1, i_2, \ldots, i_k]$, (note that each $i_l = 1,2,3,4,5$ and we use [.] to denote $[C]$). We claim that each sequence can be written as a linear combination of products of the sequences [.],[1,1],[1,1,2,1], which completes the proof of the lemma.

Proof of claim - By a 'reduction algorithm' on sequences. In order to simplify things, we drop the commas from sequences (since we only deal with single digit numbers this shouldn't cause confusion). The following set of equations show that it suffices to prove the claim for all sequences consisting of 1's and 2's.

$$[...3...] = -[...12...]-[...21...]-[...11...]-[...111...], \tag{4.1}$$

$$[...5...] = [...111...]+[...1111...]+[...121...]-[...22...], \tag{4.2}$$

$$[...4...] = [...1111...]+[...121...]-[...22...]-[...12...]-[...21...]. \tag{4.3}$$

(4.1) is obtained by adding t_1 to t_2, (4.2) is obtained by combining (4.1) with t_3, and (4.3) is obtained by combining (4.2) with t_2. We now show that the following property $(*)$ holds.

$(*)$ Any sequence consisting of 1's and 2's of length k can be written as a linear combination of products of $[.], [11]$ and sequences of the form $[112...]$ of length at most $k+1$.

We have the following equations

$$\begin{aligned} [2...] &= -[11...], && (4.4)\\ [1] &= 0, && (4.5)\\ [15...] &= 0, && (4.6)\\ [14...] &= 0, && (4.7)\\ [13...] &= -[.][...]. && (4.8) \end{aligned}$$

(4.4) is obtained from t_7, and then (4.5) can be obtained by combining this with t_6. Since we can replace 5 with][, this gives us (4.6). Combining (4.4), (4.6) and t_9 gives us (4.7), and combining (4.4), (4.7) with t_{10} gives us (4.8). We write down some further equations (to be verified later).

$$\begin{aligned} [111...] &= [.][...] = [5...], && (4.9)\\ [121...] &= -[.][1...] - [112...], && (4.10)\\ [122...] &= [.][1...] + [.][11...] + [1121...], && (4.11)\\ [12] &= -[.][.], && (4.12)\\ [112] &= 0. && (4.13) \end{aligned}$$

It should be easy to see that equations (4.4), (4.5) and (4.9)-(4.13) are sufficient to prove $(*)$. We now verify (4.9)-(4.13).

$$\begin{aligned} [1\underline{11}...] &= -[15...] + [14...] - [13...]\\ &= [.][...]. \end{aligned}$$

Line 1 follows by using t_1 to substitute the underlined segment, and line 2 by using (4.6), (4.7) and (4.8).

$$\begin{aligned} [1\underline{21}...] &= -[1111...] - [112...] + [15...] - [14...]\\ &= -[.][1...] - [112...]. \end{aligned}$$

Line 1 follows by using t_2, and line 2 by using (4.9), (4.7) and (4.6).

$$\begin{aligned} [1\underline{22}...] &= [1111...] + [11111...] + [1121...] - [15...]\\ &= [.][1...] + [.][11...] + [1121...]. \end{aligned}$$

Line 1 follows from using (4.2), and line 2 by using by using (4.9) and (4.6). The final two equations follow easily from t_8, (4.9) and (4.5).

We are now ready to prove the claim. By ($*$), it suffices to show that all sequences of the form [112...] can be written as a linear combination of products of [.],[11],[1121] and shorter sequences of the form [112...]. We have the following equations (verified below).

$$\begin{aligned} [1122...] &= [.][11..] - [11][...], & (4.14)\\ [11211...] &= -[.][12...] - [.][.][...], & (4.15)\\ [11212...] &= [11][1...] - [.][112...]. & (4.16) \end{aligned}$$

This completes the proof, since by ($*$), the arbitrary sequences appearing in these equations can be replaced by sequences of the form starting [112...] increasing the length by at most one. The only sequence not considered is [1121], which is of course trivial. The verification of equations (4.14)-(4.16) follows.

$$\begin{aligned} [11\underline{22}...] &= [11111...] + [111111...] + [11121...] - [115...]\\ &= [.][11..] - [11][...]. \end{aligned}$$

Line 1 follows from using (4.2) to substitute the underlined segment, and line 2 follows by simplifying using (4.4) and (4.9).

$$\begin{aligned} [112\underline{11}...] &= [11\underline{24}...] - [112\underline{5}...] - [11\underline{23}...]\\ &= -[11114...] - [1115...] - [112][...] + [1114...] + [11113...]\\ &= -[.][14...] - [.][.][...] + [.][4...] + [.][13...]\\ &= [.]\Big(- [.][11...] - [1121...] + [122...] - [22...] - [12...]\\ &\quad -[21...] - [111...] - [1111...]\Big)\\ &= -[.][12...] - [.][.][...]. \end{aligned}$$

Line 1 follows from using t_1 to substitute the underlined segment, line 2 follows by using t_4 and t_5 in the same way, line 3 uses (4.9) and (4.13), line 4 uses (4.3) and (4.2), and finally line 5 uses (4.4) and (4.9)-(4.11) to simplify.

$$\begin{aligned}
[112\underline{12}...] &= -[112111...] - [11221...] + [112\underline{5}...] - [11\underline{24}...] \\
&= [.][121...] + [.][.][1...] - [.][111...] + [11][1...] + [112][...] \\
&\quad +[11114...] + [1115...] \\
&= -[.][112...] + [11][1...] + [.][14...] \\
&= -[.][112...] + [11][1...] + [.][11111...] + [.][1121...] - [.][122...] \\
&\quad -[.][112...] - [.][121...] \\
&= [11][1...] - [.][112...].
\end{aligned}$$

Line 1 follows from using t_2 to substitute the underlined segment, line 2 follows from using t_5 in the same way, and also using (4.15) and (4.14). Line 3 follows by simplifying using (4.9), (4.10), line 4 follows by using (4.3), and line 5 by simplifying using (4.9)-(4.11). This completes the proof of the lemma. □

Lemma 4.5.3. *There is a surjective map* $\overline{\theta} : K[X,Y,Z]/(Z^2 - X^5 - Y^3 + XYZ) \to e_1\Lambda e_1$.

Proof. By the previous lemma, there is a surjective map $\theta : K\langle X,Y,Z\rangle \to e_1\Lambda e_1$, which maps X to $[C]$, Y to $[CACAC]$ and Z to $[CACACA^2CAC]$. We perform the usual calculations to show the commutativity relations (note that in the following $(4.9)'$ denotes the reversed form of (4.9), namely $[...111] = [...][.] = [...5]$. We have

$$\begin{aligned}
[11111] &= [511] = [.][11] = \theta(XY), \\
&= [115] = [11][.] = \theta(YX),
\end{aligned}$$

by using (4.9) and $(4.9)'$.

$$\begin{aligned}
\theta(ZX) &= [1121][.] = [11215] \\
&= [1121111] \\
&= -[.][1211] - [.][.][11] \\
&= [.][1121] \\
&= \theta(XZ).
\end{aligned}$$

Line 2 follows from $(4.9)'$, line 3 from (4.15), and line 4 from (4.10).

$$
\begin{aligned}
\theta(ZY) &= [1121][11] = [1121511] \\
&= [112111111] + [1121111111] + [112112111] - [11212211] \\
&= [112][.][.] + [1121111][.] + [12112][.] - [11212211] \\
&= -[11212211] \\
&= -[11][1211] + [.][112211] \\
&= [11][.][11] + [11][1121] + [.][.][1111] - [.][11][11] \\
&= [11][1121] \\
&= \theta(YZ).
\end{aligned}
$$

Line 2 follows by substituting using (4.2), line 3 by using $(4.9)'$, and line 4 by using t_8 and (4.13) to cancel. Line 5 uses (4.16), line 6 uses (4.10) and (4.14), and line 7 uses (4.9), (4.5) and the fact that X and Y commute to cancel.

$$
\begin{aligned}
\theta(Z^2) &= [1121][1121] = [112151121] \\
&= [11211111121] + [112111111121] + [11211211121] - [1121221121] \\
&= -[.][12111121] - [.][.][111121] - [.][121111121] - [.][12211121] \\
&\quad -[11][121121] + [.][11221121] \\
&= [.][.][111121] + [.][11211121] - [.][.][.][121] + [.][.][1111121] \\
&\quad +[.][112111121] - [.][.][111121] - [.][.][1111121] - [.][112111121] \\
&\quad +[11][.][1121] + [11][112121] + [.][.][111121] - [.][11][1121] \\
&= [.][11211121] + [11][112121] \\
&= -[.][.][12121] - [.][.][.][121] + [11][11][11] - [11][.][1121] \\
&= [.][.][11221] + [11][11][11] - [11][.][1121] \\
&= [.][.][.][.][.] + [11][11][11] - [11][.][1121] \\
&= \theta(X^5 - XYZ + Y^3).
\end{aligned}
$$

Line 2 follows by using (4.2), and line 3 by using (4.15),(4.16) and then using (4.4) to cancel two terms. Line 4 uses (4.9), (4.10), (4.11) and (4.14), and then we cancel many terms to obtain line 5. Line 6 follows by using (4.15) and (4.16), and then we apply (4.10) to obtain line 7. Finally we use (4.14), (4.9) and (4.5) to obtain line 8, and thus θ induces $\overline{\theta}$. $\square$

Lemma 4.5.4. *$\overline{\theta}$ is an isomorphism.*

Proof. For all $(s,t) \in K^2$ such that $st \neq 1$, $s \neq 0$, $t \neq 0, 1$, we consider the matrices

$$\alpha = \begin{pmatrix} 0 & \frac{-ts(t-1)}{ts-1} & t^2s - ts & \frac{ts(t^3s-2t^2s+t-1+ts)}{ts-1} & 0 & \frac{ts(t-1)}{ts-1} \\ 0 & 0 & -ts & \frac{-t^2s^2(t-1)}{ts-1} & ts & 0 \\ 0 & 0 & 0 & \frac{ts(t-1)}{ts-1} & 0 & \frac{(t-1)s}{ts-1} \\ 0 & 0 & 0 & 0 & ts & \frac{(t-1)s}{ts-1} \\ 0 & 0 & 0 & 0 & 0 & \frac{(t-1)s}{ts-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (t-1)s & \frac{-ts(t-1)}{ts-1} & 0 & 0 & 0 & 0 \\ -s & 0 & 0 & 0 & 0 & 0 \\ \frac{(t-1)s}{ts-1} & \frac{-ts(t-1)}{ts-1} & 0 & \frac{ts(t-1)}{ts-1} & 0 & 0 \\ ts & ts(t-1) & -ts & -ts(t-1) & 0 & 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 0 & \frac{ts(t-1)}{ts-1} & \frac{ts(t-1)}{ts-1} & \frac{-ts(t-1)}{ts-1} & 0 & \frac{-ts(t-1)}{ts-1} \\ 0 & 0 & ts & 0 & -ts & 0 \\ -ts+s & \frac{-tsf(s,t)}{ts-1} & \frac{ts^2(t-1)}{ts-1} & \frac{tsf(s,t)}{ts-1} & 0 & \frac{-s(t-1)}{ts-1} \\ s & \frac{ts^2(t-1)}{ts-1} & \frac{ts^2(t-1)}{ts-1} & \frac{-ts^2(t-1)}{ts-1} & -ts & \frac{-s(t-1)}{ts-1} \\ -ts+s & \frac{-tsf(s,t)}{ts-1} & \frac{ts^2(t-1)}{ts-1} & \frac{tsf(s,t)}{ts-1} & 0 & \frac{-s(t-1)}{ts-1} \\ -ts & -t^2s+ts & ts & t^2s-ts & 0 & 0 \end{pmatrix},$$

where $f(s,t) = -2ts + 1 + s + t^2s - t$. One can check that $\alpha^6 = \beta^3 = \gamma^2 = 0$, and $(\alpha+1)(\beta+1)(\gamma+1) = 1$. This implies (see the proof of Lemma 3.3.7) that one gets a representation M^{st} of Λ.

$$\begin{tikzcd}
 & & & & & \operatorname{Im}\gamma \arrow[d, shift left] & & \\
\operatorname{Im}\alpha^5 \arrow[r, shift left] & \operatorname{Im}\alpha^4 \arrow[l, shift left] \arrow[r, shift left] & \operatorname{Im}\alpha^3 \arrow[l, shift left] \arrow[r, shift left] & \operatorname{Im}\alpha^2 \arrow[l, shift left] \arrow[r, shift left] & \operatorname{Im}\alpha \arrow[l, shift left] \arrow[r, shift left] & K^6 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift right] & \operatorname{Im}\beta \arrow[l, shift right] \arrow[r, shift right] & \operatorname{Im}\beta^2 \arrow[l, shift right]
\end{tikzcd}$$

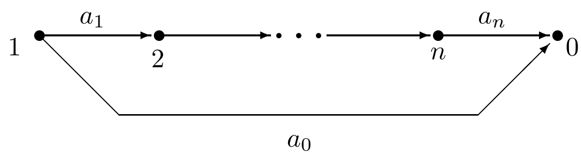
where the linear maps are $M^{st}_{a_i^*} = \alpha|_{\operatorname{Im}\alpha^{i-1}}$ and $M^{st}_{a_i}$ is the inclusion of $\operatorname{Im}\alpha^i$ in $\operatorname{Im}\alpha^{i-1}$ (and similarly for the b_i and c_i). This is easily seen to have dimension vector δ. Now

$$x_{st} = \frac{-t^5 s^6 (t-1)^3}{(ts-1)^3}, \quad y_{st} = \frac{t^8 s^{10}(t-1)^5}{(ts-1)^5}, \quad z_{st} = \frac{t^{12}s^{15}(t-1)^8}{(ts-1)^7}.$$

If we assume that $(x_{st}, y_{st}, z_{st}) = (x_{s't'}, y_{s't'}, z_{s't'})$, then in particular $t = -x_{st}^5/y_{st}^3 = -x_{s't'}^5/y_{s't'}^3 = t'$ and $ts(t-1) = x_{st}z_{st}/y_{st} = x_{s't'}z_{s't'}/t_{s't'} = t's'(t'-1)$ which shows $s = s'$. Thus $\overline{\theta}$ is an isomorphism by Lemma 4.1.3. $\square$

4.6 Type $\tilde{A}_n$, $n > 0$

We can assume that $\Lambda(Q) = K\overline{Q}/I_\mu$, where Q is the quiver



and I_μ is the ideal generated by the elements $a_0^*a_0 + a_1^*a_1 + a_0^*a_0a_1^*a_1$, $a_0a_0^* + a_na_n^* + a_na_n^*a_0a_0^*$ and $a_ia_i^* - a_{i+1}^*a_{i+1}$ for $i = 1, \dots, n-1$. [To see this, it suffices to prove that each $e_{h(a)} + aa^*$ is invertible by Lemma 2.2.3. The relations $(e_0 + a_na_n^*)(e_0 + a_0a_0^*) - e_0$ and $(e_1 + a_0^*a_0)(e_1 + a_1^*a_1) - e_1$ make $e_0 + a_0a_0^*$ and $e_1 + a_0^*a_0$ invertible (using the key fact of Section 2.2), the latter having inverse $(e_1 + a_1^*a_1)$. Since $(e_{i+1} + a_ia_i^*) = (e_{i+1} + a_{i+1}^*a_{i+1})$ for $1 \le 1 \le n-1$, it follows from the key fact that each $e_{h(a)} + aa^*$ is invertible.] Note that $a_na_n^* = (e_0 + a_0a_0^*)^{-1} - e_0 = -a_0(e_1 + a_0^*a_0)^{-1}a_0^* = -a_0a_0^* - a_0a_1^*a_1a_0^*$, and it can be easily checked that $a_0^*a_na_n^* = a_1^*a_1a_0^*$.

Lemma 4.6.1. *$e_1\Lambda e_1$ is generated by $X = a_0^*a_n \dots a_2a_1$, $Y = a_1^*a_2^* \dots a_n^*a_0$, $Z = a_1^*a_1$.*

Proof. We show that if p is a path of $\overline{Q}$ which starts and ends at 1 then $p \in \Lambda$ can be written as a linear combination of products of X, Y, Z. We can assume that p doesn't visit vertex 1 other than at the start and end, since all paths which start and end at 1 can be written as the product of paths of this form. We split into three cases.

Case 1 - p does not involve a_0^*.
We use the reduction system $\{a_{i+1}^* a_{i+1} - a_i a_i^* : i = 1, \ldots, n-1\}$. We claim that any complete reduction p' of p is a product involving Z and Y.

Namely, we assume that all reduced paths of length less than p' are such a product, and use induction to show that this is the case for p'. First, note that $p' = a_1^* p_1$, since a_1^* is the only arrow other than a_0^* which ends at 1. Now if $p_1 = a_1 p_1'$ then $p' = Z p_1'$, and by the induction hypothesis, p' is a product involving Z and Y. On the other hand, if $p_1 = a_2^* p_2$, then p' must have the form $a_1^* a_2^* \ldots a_n^* a_0 p_1'$, since it cannot have a subpath $a_i^* a_i$ where $i \geq 2$. Therefore $p' = Y p_1'$ and the result follows.

Case 2 - p does not involve a_0.
A similar argument shows that p can be written as product involving Z and X.

Case 3 - p involves a_0 and a_0^*.
By the comment given at the start of the proof, we can assume that $p = a_0^* p_0 a_0$, where p_0 moves between vertices $2, \ldots, n, 0$. We consider the reduction system $\{a_i a_i^* - a_{i+1}^* a_{i+1} : i = 1, \ldots, n-1\}$ (changing the ordering so that $a_i a_i^*$ is the leading word). Suppose that p' is a complete reduction of p. It is clear that p' has the form $a_0^* (a_n a_n^*)^i a_0$. Now, since $a_n a_n^* - a_0 a_0^* - a_0 a_1^* a_1 a_0^* \in I_\mu$, we can add it to the reduction system (it can be assumed that the leading word is $a_n a_n^*$), and compute the complete reduction of p'. It is clear this is a linear combination of products involving Z and $W = a_0^* a_0$. To complete the proof, we show that W can be expressed as a product involving X, Y and Z. First, note that

$$\begin{aligned}
Z^{n+1} &= (a_1^* a_1)^{n+1} \\
&= a_1^* a_2^* \ldots a_n^* (a_n a_n^*) a_n a_{n-1} \ldots a_2 a_1 \\
&= a_1^* a_2^* \ldots a_n^* (-a_0 a_0^* - a_n a_n^* a_0 a_0^*) a_n a_{n-1} \ldots a_2 a_1 \\
&= -YX - a_1^* a_1 a_1^* a_2^* \ldots a_n^* a_n a_n^* X \\
&= -YX - ZYX.
\end{aligned}$$

Now $(e_1 + Z)(e_1 - Z + Z^2 - \cdots + (-1)^n Z^n + (-1)^n YX) = e_1 + (-1)^n Z^{n+1} + (-1)^n (YX + ZYX) = e_1$. Therefore $W = (e_1 + Z)^{-1} - e_1 = -Z + Z^2 - \cdots + (-1)^n (Z^n + YX)$, which completes the proof. $\square$

Lemma 4.6.2. *There is a surjective map* $\overline{\theta} : K[X, Y, Z]/(Z^{n+1} + XY + XYZ) \to e_1 \Lambda e_1$.

Proof. By the previous lemma, there is a surjective map $\theta : K\langle X, Y, Z\rangle \to e_1 \Lambda e_1$, which maps X to $a_0^* a_n \dots a_2 a_1$, Y to $a_1^* a_2^* \dots a_n^* a_0$ and Z to $a_1^* a_1$. Observe that

$$\begin{aligned}
\theta(XZ) &= a_0^* a_n \dots a_2 a_1 a_1^* a_1 \\
&= a_0^* a_n a_n^* a_n \dots a_2 a_1 \\
&= -a_0^* a_0 a_1^* a_1 a_0^* a_n \dots a_2 a_1 - a_0^* a_0 a_0^* a_n \dots a_2 a_1 \\
&= a_1^* a_1 a_0^* a_n \dots a_2 a_1 \\
&= \theta(ZX). \\
\theta(ZY) &= a_1^* a_1 a_1^* a_2^* \dots a_n^* a_0 \\
&= a_1^* a_2^* \dots a_n^* a_n a_n^* a_0 \\
&= -a_1^* a_2^* \dots a_n^* a_0 a_0^* a_0 - a_1^* a_2^* \dots a_n^* a_0 a_1^* a_1 a_0^* a_0 \\
&= a_1^* a_2^* \dots a_n^* a_0 a_1^* a_1 \\
&= \theta(YZ). \\
\theta(XY) &= a_0^* a_n \dots a_2 a_1 a_1^* a_2^* \dots a_n^* a_0 \\
&= a_0^* (a_n a_n^*)^n a_0 \\
&= (a_1^* a_1)^n a_0^* a_0 \\
&= (a_1^* a_1)^{n+1} (e_1 + a_0^* a_0) \\
&= \theta(Z^{n+1}(1 + W)) \\
&= \theta(YX + YXZ)(1 + Z)^{-1} \\
&= \theta(YX).
\end{aligned}$$

which shows X, Y, Z commute, and $\theta(Z^{n+1} + XY + XYZ) = 0$ was shown in the proof of the previous lemma, and so θ induces $\overline{\theta}$. $\square$

Lemma 4.6.3. $\overline{\theta}$ *is an isomorphism.*

Proof. For $s, t \in V$, where $V = \{(s,t) \in K^2 : s \neq 0, -1, \, t \neq 0, 1\}$, consider the following representation M^{st} of $\overline{Q}$, which is easily seen to be a representation of Λ.

$$\begin{tikzcd}
K \arrow[r, shift right, "1"'] \arrow[rrrr, bend right=40, "t"'] & K \arrow[l, shift right, "s"'] \arrow[r, shift right, "1"'] & \cdots \arrow[l, shift right, "s"'] \arrow[r, shift right, "1"'] & K \arrow[l, shift right, "s"'] \arrow[r, shift right, "1"'] & K \arrow[l, shift right, "s"'] \arrow[llll, bend right=30, "\frac{-s}{t(s+1)}"']
\end{tikzcd}$$

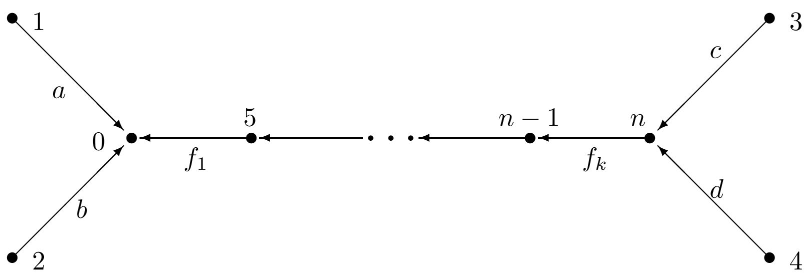
Now

$$x_{st} = t, \quad y_{st} = \frac{-s^{n+1}}{t(s+1)}, \quad z_{st} = s,$$

If we assume that $(x_{st}, y_{st}, z_{st}) = (x_{s't'}, y_{s't'}, z_{s't'})$, then clearly $s = s'$ and $t = t'$. Thus $\overline{\theta}$ is an isomorphism by Lemma 4.1.3. $\square$

4.7 Type $\tilde{D}_n$, $n > 4$

Let $k = n - 4$. We can assume that $\Lambda(Q) = K\overline{Q}/I_\mu$, where Q is the quiver



and I_μ is the ideal generated by the elements $a^*a, b^*b, c^*c, d^*d, s_0, s_1$ and m_i for $1 \leq i \leq k-1$ where $s_0 = (e_0 + aa^*)(e_0 + bb^*)(e_0 + f_1 f_1^*) - e_0$, $s_1 = (e_n + f_k^* f_k) - (e_n + cc^*)(e_n + dd^*)$, and each $m_i = f_i^* f_i - f_{i+1} f_{i+1}^*$. Let $A = aa^*$, $B = bb^*$, $C = cc^*$, $D = dd^*$, $F = f_1 f_1^*$, $G = f_k^* f_k$, $M = f_1 f_2 \dots f_k$, $N = f_k^* \dots f_2^* f_1^*$, so $s_0 = (e_0 + A)(e_0 + B)(e_0 + F) - e_0$, and $s_1 = (e_n + G) - (e_n + C)(e_n + D)$.

We follow a similar path to the one we took when dealing with $\tilde{D}_4$, except that more calculations are necessary. We split them into several lemmas so that it is easier to compare with the $\tilde{D}_4$ case. First we list some elements of I_μ which are relatively easy to obtain.

Lemma 4.7.1. *The following elements all lie in I_μ.*

$$\begin{aligned}
s_A &= A + B + F + BF, \\
s_B &= B + F + FA + A, \\
s_F &= F + A + B - BA, \\
s_D &= D + C - G + CG, \\
u_1 &= F^k - NM, \\
u_2 &= G^k - MN, \\
u_3 &= GM - MF, \\
u_4 &= NG - FM.
\end{aligned}$$

Proof. Since $(e_0 + A)^{-1} s_0 \in I_\mu$, so is s_A, Similarly, the fact that s_B, s_F, s_D are in I_μ follows from considering the expressions $(e_0 + B)^{-1}(e_0 + A)^{-1} s_0 (e_0 + A)$, $(e_0 + B)^{-1} s_A$ and $s_D = -(e_n + C)^{-1} s_1$ respectively. The u_i are easily obtained by using the m_i. □

Observe that from u_3 and u_4 we have $(1 + G)M - M(1 + F), N(1 + G) - (1 + F)N \in I_\mu$, and consequently so are $(1 + G)^{-1}M - M(1 + F)^{-1}$ and $N(1 + G)^{-1} - (1 + F)^{-1}N$. Compare the following lemma with Lemma 4.2.1, and observe that if k is assumed to be zero (and thus N and M can be ignored), the formulas will coincide.

Lemma 4.7.2. *The following elements lie in I_μ.*

$$\begin{aligned}
s_2 &= ND - N + (e_0 + A)(e_0 + B)N(e_n + C),\\
s_2' &= DM + M - (e_n - C)M(e_0 - B)(e_0 - A),\\
s_3 &= NCMB + ABNM + ANCM + BNCM + ABNCM + BNMA\\
&\quad +NCMA + ABNMA + ANCMA + BNCMA + ABNCMA\\
&\quad +ANMA + ANM + BNM - NMB - NMA,\\
s_4 &= NCMAB - BANCM - ABNCM - ABNMA - ANCMA\\
&\quad -BNCMA - ABNCMA + BAF^{k-1}BA.
\end{aligned}$$

Proof. We have $-N(e_n+G)^{-1}s_1(e_n+D)^{-1} = -N(e_n+G)^{-1}((e_n+G)-(e_n+C)(e_n+D))(e_n+D)^{-1} \in I_\mu$, and therefore so is $t_2 = ND-N+N(e_n+G)^{-1}(e_n+C)$, and by the comments above, so is $s_2 = ND - N + (e_0 + F)^{-1}N(e_n + C)$. Similarly, $-(e_n+C)^{-1}s_1M = -(e_n+C)^{-1}((e_n+G)-(e_n+C)(e_n+D))M \in I_\mu$, and therefore so is $(e_n - C)(e_n + G)M + (e_n + D)M$, and also $t_2' = -(e_n - C)M(e_0 + F) + M + DM$. Finally, by substituting the F in t_2' using s_F, we have $s_2' \in I_\mu$.

Now assuming ND, DM to be the leading words of s_2 and s_2', we resolve NDM:

$$\begin{aligned}
N(DM) &\mapsto -NM + N(e_n - C)M(e_0 - B)(e_0 - A),\\
(ND)M &\mapsto NM - (e_0 + A)(e_0 + B)N(e_n + C)M.
\end{aligned}$$

Thus $s_5 = N(e_n-C)M(e_0-B)(e_0-A)+(e_0+A)(e_0+B)N(e_n+C)M-2NM \in I_\mu$. Then since $s_5(e_0 + A) \in I_\mu$, so is $s_3 = N(e_n - C)M(e_0 - B) + (e_0 + A)(e_0 + B)N(e_n + C)M(e_0 + A) - 2NM(e_0 + A)$.

Since $(e_0-B)(e_0-A)s_5(e_0+A)(e_0+B) \in I_\mu$, so is $s_6 = (e_0-B)(e_0-A)N(e_n-C)M + N(e_n+C)M(e_0+A)(e_0+B) - 2(e_0-B)(e_0-A)NM(e_0+A)(e_0+B)$.

Multiply this out, we have

$$\begin{aligned} s_6 &= BNCM + ANCM - BANCM + NCMA + NCMB + NCMAB \\ &\quad +BNM + ANM - NMA - NMB - BANM + 2BNMA \\ &\quad +2BNMB + 2ANMA + 2ANMB - NMAB - 2BANMA \\ &\quad -2BANMB + 2BNMAB + 2ANMAB - 2BANMAB. \end{aligned}$$

Substituting the term $NCMB$ using s_3, we obtain $s_7 \in I_\mu$, where

$$\begin{aligned} s_7 &= NCMAB - BANCM - ABNCM - ABNMA - ANCMA \\ &\quad -BNCMA - ABNCMA + (-BANM - NMAB + BNMA \\ &\quad +2BNMB + ANMA + 2ANMB - ABNM - 2BANMA \\ &\quad -2BANMB + 2BNMAB + 2ANMAB - 2BANMAB). \end{aligned}$$

Observe that this is equal to s_4, except that the terms in the bracket are replaced by $BAF^{k-1}BA$. To prove this substitution can be made, we must perform another tricky calculation to show that the term inside the bracket can be reduced to $BAF^{k-1}BA$ by adding elements of I_μ. First, replace each NM by F^k, and then use s_F can be used to substitute F by $-A-B-BA$, so that each term starts and ends with A or B. Set s_8 equal to this element. We have

$$\begin{aligned} s_8 &= -BAF^{k-1}BA + BAF^{k-1}B + BAF^{k-1}A - BAF^{k-1}AB + BF^{k-1}AB \\ &\quad +AF^{k-1}AB + BF^kA + 2BF^kB + AF^kA + 2AF^kB - ABF^{k-1}BA \\ &\quad +ABF^{k-1}B + ABF^{k-1}A - 2BAF^kA - 2BAF^kB + 2BF^kAB \\ &\quad +2AF^kAB - 2BAF^kAB. \end{aligned}$$

We consider the terms starting and ending with B. This is equal to

$$\begin{aligned} &BAF^{k-1}B - BAF^{k-1}AB + BF^{k-1}AB + 2BF^kB \\ &\qquad\qquad - 2BAF^kB + 2BF^kAB - 2BAF^kAB. \end{aligned}$$

The final three terms cancel, since $BAF^kB = -BAF^{k-1}BAB - BAF^{k-1}AB = BF^kAB - BAF^kAB$ by using s_F. The first four terms also cancel, since BF^kB

is equal to both $-BAF^{k-1}$ and $BF^{k-1}BAB - BF^{k-1}AB$ (using s_F). Now consider the terms starting and ending with A:

$$AF^kA - ABF^{k-1}BA + ABF^{k-1}A.$$

The can easily be reduced to zero, since $AF^kA = -ABF^kA - ABF^{k-1}A = ABF^{k-1}BA - ABF^{-1}A$, using s_A and then s_F. Continue with terms starting with A and ending with B:

$$AF^{k-1}AB + 2AF^kB + ABF^{k-1}B + 2AF^kAB.$$

Subtracting $2AF^{k-1}s_BB$, we obtain $ABF^{k-1}B - AF^{k-1}AB$. We can assume that $k > 1$, (otherwise it is trivially zero) and substitute an F in both terms, leaving us with $AB(F^{k-2}B - AF^{k-2})AB$. Now substitute the A and B inside the bracket to obtain zero. Finally we do the same with the terms starting with B and ending with A (and s_8 must be equal to this, since the remaining terms have cancelled):

$$-BAF^{k-1}BA + BAF^{k-1}A + BF^kA - 2BAF^kA.$$

Using s_F, the middle two terms cancel, and $BAF^kA = -BAF^{k-1}BA$, and so the whole expression is equal to $BAF^{k-1}BA$, as required. $\square$

Before we find the generators of $e_1\Lambda e_1$, we need one further calculation. At the same time, we will give a couple of formulas which will help us show that the generators commute.

Lemma 4.7.3. *The following are all elements of I_μ.*

$$\begin{aligned} s &= NCG - (AB + BA)N - (AB + A + B)NC, \\ t_i &= a^*NCG^iCMa + a^*F^iANCMa, \text{ for all } i, \\ r_i &= a^*BNCG^iCMa + a^*BF^iBNCMa, \text{ for all } i. \end{aligned}$$

Proof. Going back to the start of the proof of the previous lemma, we have two different substitutions for D in terms of C and G, namely $D = (e_n - C)(e_n +$

$G) - e_n$ and $D = e_n - (e_n + G)^{-1}(e_n + C)$. Hence $e_n - C + G - CG + (e_n + G)^{-1}(e_n + C) \in I_\mu$. Multiply by $-N$ on the left, to obtain $NCG - NG + NC - N - (e_0 + F)^{-1}N(e_n + C) \in I_\mu$. Now observe that we can substitute $(e_0 + F)^{-1}$ by $e_0 + A + B + AB$, and therefore by $e_0 - F + AB + BA$, obtaining $NCG - NG + NC - N - (N + NC - FN - FNC + ABN + ABNC + BAN + BANC) = NCG - (AB + BA)N - (AB + BA - F)NC \in I_\mu$. Finally use the substitution s_F to obtain $s \in I_\mu$.

We prove $t_i, r_i \in I_\mu$ by induction. For $i = 0$, both are trivial. So we assume that $t_i, r_i \in I_\mu$ for all $i \leq j$, and consider t_{j+1}. As elements of Λ, we have

$$\begin{aligned}
a^*NCG^{j+1}CMa &= a^*((AB + BA)N + (AB + A + B)NC)G^jCMa \\
&= a^*BANG^jCMa + a^*BNCG^jCMa \\
&= a^*BAF^jNCMa - a^*BF^jBNCMa \\
&= a^*BAF^{j-1}\underline{B}ANCMa - a^*B\underline{A}F^{j-1}BNCMa \\
&\quad -a^*B\underline{A}F^{j-1}ANCMa - a^*BF^jBNCMa \\
&= -a^*\underline{BA}F^jANCMa + a^*BF^jBNCMa \\
&\quad +a^*BF^jANCMa - a^*BF^jBNCMa \\
&= -a^*F^{j+1}ANCMa.
\end{aligned}$$

Line 1 follows by using s, and then we make easy cancellations to obtain line 2. Line 3 follows by substituting using r_j (possible by the induction hypothesis), and line 4 by substituting using s_F. Line 5 follows by using s_A and s_B to substitute the underlined letters, and finally line 6 follows using s_F to substitute BA and the remaining terms cancel. This shows $t_{j+1} \in I_\mu$.

We now do the same to prove $r_{j+1} \in I_\mu$.

$$\begin{aligned}
a^*BNCG^{j+1}CMa &= a^*B((AB+BA)N+(AB+A+B)NC)G^jCMa \\
&= a^*BABNG^jCMa + a^*BABNCG^jCMa \\
&\quad +a^*BANCG^jCMa \\
&= a^*BABF^jNCMa - a^*BABF^jBNCMa \\
&\quad -a^*BAF^jANCMa \\
&= a^*BA(BF^{j-1}BA - BF^{j-1}B - BF^{j-1}A - BF^jB \\
&\quad -BAF^{j-1}A + BF^{j-1}A)NCMa \\
&= a^*BA(-BF^{j-1}B - BF^jB)NCMa \\
&= a^*BA(-BF^{j-1}B + BAF^{j-1}B)NCMa \\
&= a^*BAF^jBNCMa \\
&= -a^*BF^{j+1}BNCMa.
\end{aligned}$$

Line 1 follows by substituting using s, and then we make easy cancellations to obtain line 2. By the induction hypothesis, we can use r_j and t_j to obtain line 3. Applying s_F gives us line 4. Line 5 follows from the easily verified fact that $BAF^{j-1}A = BF^{j-1}BA$ for all j. We then use s_F to obtain line 6, and again to obtain line 7. Finally, line 8 follows by using s_A, and this completes the proof. $\square$

Lemma 4.7.4. *$e_1\Lambda e_1$ is generated by the paths $X = a^*Ba$, $Y = a^*NCMa$, $Z = a^*BNCMa$.*

Proof. We have that $e_1\Lambda e_1$ is spanned by the set of all paths which start and end at 1. Using s_B and s_D in a reduction system, we see that $e_1\Lambda e_1$ is spanned by the set H of paths which start and end at 1 and do not visit 2 or 4. We show that any path p which doesn't visit 2 and 4 can be written as a linear combination of products of X, Y, Z. We can assume that p doesn't visit 1 except at its start and end. We split into two cases.

(1) If p does not visit 3 (i.e. does not involve C), then we use the reduction system $\{m_i : i = 1, \ldots, k-1\}$ where the ordering is chosen so that each m_i

has leading word $f_{i+1}f_{i+1}^*$. This shows that p can be assumed to have the form a^*F^ia, and then by using s_F we can express p as a polynomial in X.

(2) If p does visit 3 then we use the reduction system $\{m_i : i = 1, \ldots, k-1\} \cup \{s\}$ where the ordering is chosen so that each m_i has leading word $f_i^*f_i$ and s has leading word NCG. Then it follows that p can be written as a linear combination of elements of the form a^*NG^iCMa, which in turn can be written as a linear combination of elements of the form a^*F^iNCMa. Use s_F to eliminate each F and the resulting expression is a linear combination of elements of the form $a^*(BA)^jNCMa$ and $a^*(BA)^jBNCMa$, which are X^jY, X^jZ respectively. $\square$

Note that $a^*F^ia = p_i(X)$ for all $i \geq 0$. For $i = 0$ it is clear and for $i = 1$, $a^*Fa = -a^*Ba = -X = p_1(X)$. For $i > 1$, $a^*F^ia = a^*BAF^{i-1}a - a^*BF^{i-1}a = a^*BAF^{i-1}a + a^*BAF^{i-2}a = X(p_{i-1}(X) + p_{i-2}(X)) = p_i(X)$. To make the final calculations easier to follow, we define Λ_0 to be the algebra $K\overline{Q}/I$, where I is the ideal generated by the relations a^*a, b^*b, c^*c, d^*d, and denote the map $K\overline{Q} \to \Lambda_0$ by an underline. The calculation is similar to Lemma 4.2.3.

Lemma 4.7.5. *There is a surjective map* $\overline{\theta} : K[X,Y,Z]/(Z^2 - p_k(X)XZ + p_{k-1}(X)X^2Y - XY^2 - XYZ) \to e_1\Lambda e_1$.

Proof. By the previous lemma, there is a surjective map $\theta : K\langle X,Y,Z\rangle \to e_1\Lambda e_1$, which maps X to a^*Ba, Y to a^*NCMa and Z to a^*BNCMa. Observe that

$$\begin{aligned}
\theta(XY - YX) &= a^*BANCMa - a^*NCMABa = \underline{a^*s_4a} = 0, \\
\theta(ZX - XZ) &= a^*BNCMABa - a^*BABNCMa = \underline{a^*Bs_4a} = 0, \\
\theta(YZ - ZY) &= a^*NCMABNCMa - a^*BNCMANCMa \\
&= a^*NCMABNCMa - a^*BNCMANCMa - Xt_k \\
&= a^*NCMABNCMa - a^*BNCMANCMa \\
&\quad -aBANCMNCMa + aBAF^{k-1}BANCMa \\
&= \underline{a^*s_4NCMa} = 0,
\end{aligned}$$

which shows X, Y, Z commute. Let $T = \theta(-Xp_k(X)Z + X^2p_{k-1}(X)Y)$. We have

$$\begin{aligned}
T &= -a^*BA\underline{F}F^{k-1}ABNCMa + a^*BAB\underline{A}F^{k-1}ANCMa \\
&= -a^*BABAF^{k-1}ABNCMa + a^*BABF^{k-1}\underline{A}BNCMa \\
&\quad -a^*BABF^kANCMa \\
&= a^*BABF^kBNCMa - a^*BABF^kANCMa \\
&= -a^*BABNCG^kCMa - a^*BABF^kANCMa \\
&= -a^*BABNCMNCMa - a^*BABNMANCMa
\end{aligned}$$

Line 2 follows by using s_A and s_F to substitute the underlined letters, line 3 by using s_A, and the fact that $BAF^kA = BF^kBA$. Line 4 follows by using r_k. Now it is clear that $\theta(Z^2 - Xp_k(X)Z + X^2p_{k-1}(X)Y - XY^2 - XZY)$ is equal to

$$\begin{aligned}
&a^*BNCMABNCMa - a^*BANCMANCMa - a^*BABNCMANCMa \\
&\qquad - a^*BABNCMNCMa - a^*BABNMANCMa,
\end{aligned}$$

which is $\underline{a^*Bs_4NCMa} = 0$. Thus θ induces $\overline{\theta}$. $\square$

Lemma 4.7.6. *$\overline{\theta}$ is an isomorphism.*

Proof. For $s, t \in V$, where $V = \{(s,t) \in K^2 : t^2s - ts + 1 \neq 0,\ s \neq 0,\ t \neq 0, 1\}$, we consider matrices

$$\alpha = \begin{pmatrix} 0 & \frac{s(t-1)}{t^2s-ts+1} \\ 0 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 0 \\ ts(t-1) & 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \frac{ts}{t^2s-ts+1} & \frac{-ts}{t^2s-ts+1} \\ \frac{ts}{t^2s-ts+1} & \frac{-ts}{t^2s-ts+1} \end{pmatrix}$$

$$\delta = \begin{pmatrix} -ts & s \\ -t^2s & ts \end{pmatrix}$$

$$\zeta = \begin{pmatrix} 0 & \frac{-s(t-1)}{t^2s-ts+1} \\ -ts(t-1) & \frac{ts^2(t-1)^2}{t^2s-ts+1} \end{pmatrix}$$

One can check that $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 0$, and that $(1+\alpha)(1+\beta)(1+\zeta) = 1$ and $(1+\zeta) = (1+\gamma)(1+\delta)$, and thus one gets a representation M^{st} of Λ of dimension vector δ where $M^{st}_v = K^2$ for $v \neq 1,2,3,4$, $M^{st}_1 = \operatorname{Im}\alpha$, $M^{st}_2 = \operatorname{Im}\beta$, $M^{st}_3 = \operatorname{Im}\gamma$, $M^{st}_4 = \operatorname{Im}\delta$, and the linear maps are $M^{st}_{a^*} = \alpha$, M^{st}_a is the inclusion of $\operatorname{Im}\alpha$ in K^2 (and similarly for b, c and d), $M^{st}_{f_i^*} = \zeta$ for all i and each $M^{st}_{f_i}$ is the identity map. For each k, let $V_k = \{(s,t) \in V :$ Each component of cf^k is non zero$\}$. Since V_k is the complement in K^2 of the set of zeroes of a finite list of polynomials, it is a 2 dimensional variety. Now

$$x_{st} = \frac{ts^2(t-1)^2}{t^2s - ts + 1}, \quad z_{st} = ts(t-1)y_{st},$$

and y_{st} is a complicated expression which is guaranteed to be nonzero by the assumption on V_k. Additionally note that $z_{st}+y_{st} = y_{st}(ts(t-1)+1) = y_{st}(t^2s - ts+1)$ is also non zero. If we assume that $(x_{st}, y_{st}, z_{st}) = (x_{s't'}, y_{s't'}, z_{s't'})$, then in particular, $t = z_{st}^2/(x_{st}y_{st}(y_{st}+z_{st}) = z_{s't'}^2/(x_{s't'}y_{s't'}(y_{s't'}+z_{s't'}) = t'$, and then it follows that $s = s'$. Thus $\overline{\theta}$ is an isomorphism by Lemma 4.1.3. □

4.8 Open problems

The main theorem gives rise to the following corollary.

Corollary 4.8.1. *If Q is extended Dynkin, and 1 is an extending vertex, then $e_1\Lambda^1(Q)e_1$ is a commutative Noetherian domain of Krull dimension 2.*

Of course, it would be desirable to obtain the properties for general q, and in particular a multiplicative analogue of [10, Theorem 0.4]. It may be the case here that simply substituting '$\lambda.\alpha = 0$' with '$q^\alpha = 1$' throughout is not correct, and one should instead use 'q^α is a root of 1'. The best way to attack this problem is probably to look at the simplest extended Dynkin quiver, which has one vertex and one loop. If $q = 1$, this is isomorphic to a localised polynomial ring $K[x, y, (1+xy)^{-1}]$ (which is isomorphic to the algebra $K[X, Y, Z]/(Z + XY + XYZ)$ via the isomorphism which sends x to X, y to Y and $(1+xy)^{-1}$ to $1+Z$, thus verifying Theorem 4.1.1 in the $\tilde{A}_0$ case). If $q \neq 1$, then $\Lambda^q(Q) =$

$K\langle a, a^*, (1+aa^*)^{-1}\rangle/((1+aa^*) = q(1+a^*a))$, which is isomorphic to a localised first quantised Weyl algebra, which is discussed in [20].

The proof of many results for Π^λ in the extended Dynkin case rely on the following construction, given in [10]. Any extended Dynkin quiver Q corresponds to a subgroup $\Gamma \in \mathrm{SL}_2(K)$, (shown in [25]). Using the natural action of Γ on $K\langle x, y\rangle$, one can form the skew group ring $K\langle x, y\rangle * \Gamma$. Given $\lambda \in Z(K\Gamma)$, let $\tilde{\Pi}^\lambda$ be the ring $(K\langle x, y\rangle * \Gamma)/(xy - yx - \lambda)$. One can identify λ with an element of K^{Q_0}, and then it can be shown that $\Pi^\lambda(Q)$ is Morita equivalent to $\tilde{\Pi}^\lambda(Q)$, and $e_1\Pi^\lambda(Q)e_1 \cong e\tilde{\Pi}^\lambda(Q)e$, where e is the average of the group elements. Our attempts to find a multiplicative analogue of this construction have been unsuccessful.

We would also like to obtain properties of Λ^q. We make the following conjecture (based on [10, Corollary 3.6]).

Conjecture 4.8.2. *If Q is extended Dynkin, then $\Lambda^q(Q)$ is a prime Noetherian ring of* GK *dimension 2.*

Whereas it is difficult to know where to start on a proof for general q, it ought to be possible to use Corollary 4.8.1 to make progress in the case $q = 1$. It may be possible to derive these (and other) results from results on 'generalized double affine Hecke algebras', which are defined in [16]. This paper includes an appendix which shows that these algebras are isomorphic to $e_0\Lambda^q(Q)e_0$, where Q is a star shaped extended Dynkin quiver, and 0 is the central vertex.

Chapter 5

Further Investigations

In this chapter we examine some miscellaneous questions regarding multiplicative preprojective algebras. In Section 5.1.1, we conjecture that $\Lambda^1(Q)$ is a 'preprojective algebra' in the sense of satisfying the preprojective property, and give some examples where this is true. In Section 5.2 we consider whether $\Lambda^1(Q)$ could be isomorphic to $\Pi(Q)$ as an algebra, and in Section 5.3 we list some other questions.

5.1 Are Λ and Π isomorphic as KQ-modules?

In this section we assume that $q = 1$, and write $\Lambda(Q)$ (or simply Λ) for $\Lambda^1(Q)$. We propose the following conjecture (clearly equivalent to the question in the title of this section having the answer yes).

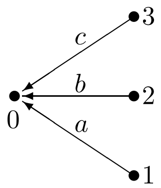
Conjecture 5.1.1. *For any quiver Q, $\Lambda(Q)$ satisfies the preprojective property for KQ.*

If true, this conjecture would have some interesting implications. One immediate consequence would be the truth of Conjecture 3.6.3. It would also perhaps lead to some easier proofs of the results in the previous two chapters. Unfortunately, the proof in the general case is likely to be very difficult. Instead, we can show the result is true in some special cases (note that we only show the preprojective property holds for left modules, but each proof can be easily adapted to show it for right modules). First the 'trivial' case.

Lemma 5.1.2. *If Q has type A_n then Conjecture 5.1.1 is true.*

Proof. This is not quite as trivial as might first appear [It is not always the case that two isomorphic algebras with a common subalgebra A are isomorphic as A-modules, one also requires that the isomorphism between them is the identity map when restricted to A, which is what is shown here]. Let Q' be the quiver given in Lemma 3.2.1. The conjecture is clearly true in this case, because the algebras $\Lambda(Q')$ and $\Pi(Q')$ are the same, as they are given by the same presentation. Clearly Q can be obtained from Q' by reversing some arrows. That is, partition the integers $1, \ldots, n-1$ into two disjoint sets R and S so that $Q_1 = \{b_i : i \in R\} \cup \{c_i : i \in S\}$ where $t(b_i) = i+1$, $h(b_i) = i$ and $t(c_i) = i$, $h(c_i) = i+1$. By Lemma 2.1.3 there is an isomorphism $\theta : \Lambda(Q) \to \Lambda(Q')$ which satisfies $\theta(b_i) = a_i$ and $\theta(c_i) = a_i^*$. Now (see Lemma 1.3.7) there is an isomorphism $\phi : \Pi(Q') \to \Pi(Q)$ which in particular maps a_i to b_i if $i \in R$, and a_i^* to c_i if $i \in S$. The composition $\psi = \phi\theta$ is an algebra isomorphism $\Lambda(Q) \to \Pi(Q)$ which acts as the identity map on the subalgebras KQ. Therefore if $x \in KQ$ and $y \in \Lambda(Q)$, $\psi(xy) = \psi(x)\psi(y) = x\psi(y)$, so ψ is also a KQ-module isomorphism. $\square$

We can also verify it in the smallest non trivial case, where Q is the following quiver of type D_4.



Lemma 5.1.3. *If Q is the quiver given above, the conjecture is true.*

Proof. We show the set $\{e_0, e_1, e_2, e_3, a, a^*, b, b^*, c, c^*, b^*a, c^*a, aa^*, a^*b, c^*b, bb^*, a^*c, b^*c, bb^*a, b^*aa^*, c^*aa^*, aa^*b, a^*bb^*, aa^*c, a^*bb^*a, bb^*aa^*, b^*aa^*b, c^*aa^*c\}$ is a basis of both $\Pi(Q)$ and $\Lambda(Q)$, by using the reduction algorithm. Let $\Omega_1 = \{a^*a, b^*b, c^*c, aa^* + bb^* + cc^*\}$. This is clearly a full reduction system for $\Pi = K\overline{Q}/I_\rho$ (we are working with the $\leq_{c,b,len,lex}$ ordering, where the lexographic ordering is chosen so that $bb^*aa^* < aa^*bb^*$). Resolving the ambiguities cc^*c and c^*cc^* shows that $\omega_1 = bb^*c + aa^*c$ and $\omega_2 = c^*bb^* + c^*aa^*$ are in I_ρ. Let $\Omega_2 = \Omega_1 \cup \{\omega_1, \omega_2\}$. Resolving the ambiguity cc^*bb^* shows that $\omega_3 = aa^*bb^* + bb^*aa^*$ is in I_ρ. Let $\Omega_3 = \Omega_2 \cup \{\omega_3\}$. Resolving the ambiguities $a^*aa^*bb^*, b^*bb^*c, c^*bb^*b, aa^*bb^*b$ shows that $a^*bb^*aa^*, b^*aa^*c, c^*aa^*b, bb^*aa^*b \in I_\rho$. Let Ω be the union of these elements and Ω_3. One can check that all ambiguities are reduction unique, and thus the set of irreducible words above is a basis of Π.

We do a similar process for $\Lambda = K\overline{Q}/I_\mu$, this time starting with $\Omega_0 = \{a^*a, b^*b, c^*c, \omega_0\}$ where $\omega_0 = aa^* + bb^* + cc^* + aa^*bb^* + aa^*cc^* + bb^*cc^* + aa^*bb^*cc^*$. We first verify that $I_{\Omega_0} = I_{\Omega_1}$, where Ω_1 is the same as Ω_0, except that we replace ω_0 by $\omega_0' = aa^* + bb^* + cc^* + aa^*bb^*$. This is clear, since $\omega_0 = \omega_0'(e_0 + cc^*) - cc^*cc^*$ shows the $\subseteq$ inclusion, and then $\omega_0' = \omega_0(e_0 - cc^*) + (e_0 + aa^* + bb^* + aa^*bb^*)cc^*cc^*$ shows the other inclusion. Resolving the ambiguity cc^*c in Ω_1 shows that $aa^*c + (e_0 + aa^*)bb^*c \in I_\mu$, and thus so is $(e_0 - aa^*)aa^*c + bb^*c$. It follows that ω_1 (as given above) is in I_μ. Similarly $\omega_2 \in I_\mu$, by resolving c^*cc^*. By following the rest of the calculation that was done for Π in a virtually identical fashion, we find that Ω (as given above except that $aa^* + bb^* + cc^*$ replaced by ω_0') is a full reduction system for Λ in which all ambiguities are reduction unique. The set of illegal words are the same as for Π, and therefore Λ has the same basis as Π.

We have KQ-module decompositions

$$\Pi = \bigoplus_{v \in Q_0} \Pi e_v,$$

$$\Lambda = \bigoplus_{v \in Q_0} \Lambda e_v,$$

where each Πe_v, Λe_v is spanned by the paths starting at v. We want to show that $\Pi e_v \cong \Lambda e_v$ for all v. Let us consider $v = 0$, which is the only non trivial case. Using the information above we calculate the representation of Q corresponding to Πe_0.

$$\begin{tikzcd}
 & (a^*, a^*cc^*) \arrow[dl, "\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}"'] \\
(e_0, aa^*, bb^*, bb^*aa^*) & & (b^*, b^*aa^*) \arrow[ll, "\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}"'] \\
 & (c^*, c^*aa^*) \arrow[ul, "\begin{pmatrix}0&0\\-1&0\\-1&0\\0&-1\end{pmatrix}"]
\end{tikzcd}$$

Note that $(x_1, x_2, \ldots, x_n)$ denotes the vector space with basis $\{x_1, x_2, \ldots, x_n\}$ and the matrices define the linear maps with respect to this basis. This representation clearly decomposes as

$$\begin{tikzcd}
 & 0 \arrow[dl] & & (a^*) \arrow[dl, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & & (c^*aa^*) \arrow[dl] \\
(e_0) & 0 \arrow[l] & \bigoplus (aa^*, bb^*) & (b^*) \arrow[l, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] \; \bigoplus & (bb^*aa^*) & (b^*aa^*) \arrow[l] \\
 & 0 \arrow[ul] & & (c^*) \arrow[ul, "\left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right)"] & & (a^*bb^*) \arrow[ul]
\end{tikzcd}$$

Similarly for Λe_0, we obtain

$$\begin{tikzcd}
 & & (a^*, a^*cc^*) \arrow[dll, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \\
(e_0, aa^*, bb^*, bb^*aa^*) & & & (b^*, b^*aa^*) \arrow[lll, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \\
 & & (c^*, c^*aa^*) \arrow[ull, "\left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ 1 & -1 \end{smallmatrix}\right)"]
\end{tikzcd}$$

which decomposes as

$$\begin{tikzcd}
 & 0 \arrow[dl] & & (a^*) \arrow[dl, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & & (c^*aa^*) \arrow[dl] \\
(e_0) & 0 \arrow[l] \;\bigoplus & (aa^*, bb^*) & (b^*) \arrow[l, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] \;\bigoplus & (bb^*aa^*) & (b^*aa^*) \arrow[l] \\
 & 0 \arrow[ul] & & (c^* - c^*aa^*) \arrow[ul, "\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)"] & & (a^*bb^*) \arrow[ul]
\end{tikzcd}$$

It clear from this that $\Pi e_0 \cong \Lambda e_0$. For the remaining vertices, it is obvious that $\Pi e_v \cong \Lambda e_v$, since their representations (before decomposing) are identical. Thus the conjecture is true in this case. $\square$

This example illustrates the difficulty in proving the conjecture. Since the ordinary preprojective algebra is graded, there is an automatic decomposition

$$\Pi \cong \bigoplus_{\substack{v \in Q_0 \\ k \geq 0}} \Pi_k e_v,$$

where Π_k denotes the span of the paths of degree k (using the oriented grading). This decomposition is in fact the decomposition into indecomposable

KQ-modules. The multiplicative preprojective algebra does not inherit the oriented grading from $K\overline{Q}$, and one has to consider the inhomogeneous element $c^* - c^*aa^*$ in order to obtain the decomposition. It is not at all obvious why this particular element is needed. One possible strategy of proving the conjecture (in the Dynkin case) that we investigated was to use a descending filtration on Λ, and show that the associated graded ring is isomorphic to Π. This would be true if Conjecture 3.6.3 was true, but even if $\operatorname{gr}\Lambda \cong \Pi$ as algebras, it does not seem to follow easily that $\Lambda \cong \Pi$ as modules.

The best evidence we have for the truth of the conjecture is the following result in the simplest infinite type case. It would seem unlikely that the conjecture could be true in this case by accident.

Theorem 5.1.4. *Let Q be the quiver*

$$\begin{tikzcd} 1 \arrow[r, shift left, "b"] \arrow[r, shift right, "a"'] & 2 \end{tikzcd}$$

$\Lambda(Q)$ is isomorphic to the direct sum of a set of representatives of the indecomposable preprojective KQ modules.

The (lengthy) proof of this theorem comprises the rest of this section. Firstly, we use the reduction algorithm to obtain a basis P of Λ (Corollary 5.1.7). We then find (Lemma 5.1.13) an alternative basis L which is suitable for decomposing of Λ into a direct sum of indecomposable modules. The long part of the proof is taken up by proving that L is a basis. It would be desirable to obtain a simpler proof, by determining the significance of the elements of L. This may also lead to a proof of the conjecture in general.

We can assume that $\Lambda \cong K\overline{Q}/I$, where I is the ideal generated by $r_1 = aa^*bb^* + aa^* + bb^*$ and $r_2 = b^*ba^*a + a^*a + b^*b$. We define some elements of $K\overline{Q}$. Set $c_1 = b^*aa^* - a^*ab^*$, $c_2 = ba^*a - aa^*b$, $c_3 = bb^*a - ab^*b$, $c_4 = b^*ba^* - a^*bb^*$.

Given an integer $i \geq 0$, let

$$\begin{aligned} f_i &= aa^*(ba^*)^i bb^* + a(a^*b)^i a^* + b(a^*b)^i b^*, \\ g_i &= a^*a(b^*a)^i b^*b + a^*(ab^*)^i a + b^*(ab^*)^i b. \end{aligned}$$

Lemma 5.1.5. *The elements $c_1, \ldots, c_4$, and each f_i, g_i lie in I.*

Proof. Consider the reduction system $\Omega_0 = \{r_1, r_2\}$ with respect to the ordering $\leq_{len,lex}$ with $a < b$, $a^* < b^*$. Resolve $aa^*bb^*ba^*a$:

$$\begin{aligned} (aa^*bb^*)ba^*a &\mapsto -aa^*ba^*a - b(b^*ba^*a) \mapsto -aa^*ba^*a + bb^*b + ba^*a, \\ aa^*b(b^*ba^*a) &\mapsto -aa^*ba^*a - (aa^*bb^*)b \mapsto -aa^*ba^*a + bb^*b + aa^*b. \end{aligned}$$

Thus $c_2 \in I$. Resolve $b^*ba^*aa^*bb^*$:

$$\begin{aligned} b^*ba^*(aa^*bb^*) &\mapsto -(b^*ba^*a)a^* - b^*ba^*bb^* \mapsto a^*aa^* + b^*ba^* - b^*ba^*bb^*, \\ (b^*ba^*a)a^*bb^* &\mapsto -a^*(aa^*bb^*) - b^*ba^*bb^* \mapsto a^*aa^* + a^*bb^* - b^*ba^*bb^*. \end{aligned}$$

Thus $c_3 \in I$. Let $\Omega_1 = \Omega_0 \cup \{c_2, c_3\}$. Resolve b^*ba^*a:

$$\begin{aligned} b^*ba^*a &\mapsto -b^*b - a^*a, \\ b^*(ba^*a) &\mapsto b^*aa^*b, \\ (b^*ba^*)a &\mapsto a^*bb^*a. \end{aligned}$$

This shows that both $s_1 = b^*aa^*b + b^*b + a^*a$ and $s_2 = a^*bb^*a + b^*b + a^*a$ lie in I. Let $\Omega_2 = \Omega_1 \cup \{s_1, s_2\}$. Resolve $b^*aa^*bb^*$:

$$\begin{aligned} b^*(aa^*bb^*) &\mapsto -b^*aa^* - b^*bb^*, \\ (b^*aa^*b)b^* &\mapsto -b^*bb^* - a^*ab^*. \end{aligned}$$

This shows that $c_1 \in I$. Resolve aa^*bb^*a:

$$\begin{aligned} (aa^*bb^*)a &\mapsto -aa^*a - bb^*a, \\ a(a^*bb^*a) &\mapsto -aa^*a - ab^*b. \end{aligned}$$

This shows that $c_4 \in I$. Now let $\Omega_3 = \{r_1, r_2, c_1, c_2, c_3, c_4\}$ (s_1, s_2 are now redundant). Note that the combination of $c_1, \ldots, c_4$ imply that given a path p

we can always reduce it to the unique path which contains the same number of occurrences of each arrow as p, and which starts/ends in at the same vertices as p, and a (respectively a^*) never occurs to the right of b (respectively b^*). For example, if $p = bb^*aa^*ab^*bb^*ab^*$, then we can reduce it to $aa^*ab^*ab^*bb^*bb^*$.

We now show by induction that each $f_i, g_i \in I$. First, $f_0 = r_1 \in I$ and $r_2 \to g_0$ so $g_0 \in I$. Now assume that $f_j, g_j \in I$ for all $j \leq i$. Since they are monic, we can therefore include them in a reduction system. We resolve $aa^*(ba^*)^i bb^*ba^*$:

$$\begin{aligned}
(aa^*(ba^*)^i bb^*)ba^* &\mapsto -a(a^*b)^i a^*ba^* - b(a^*b)^i a^*bb^*, \\
&\to -a(a^*b)^{i+1}a^* - b(a^*b)^{i+1}b^*, \\
aa^*(ba^*)^i b(b^*ba^*) &\mapsto aa^*(ba^*)^i ba^*bb^*, \\
&\to aa^*(ba^*)^{i+1}bb^*.
\end{aligned}$$

This shows $f_{i+1} \in I$.

$$\begin{aligned}
(a^*a(b^*a)^i b^*b)b^*a &\mapsto -a^*(ab^*)^i ab^*a - b^*(ab^*)^i ab^*b, \\
&\to -a^*(ab^*)^{i+1}a - b^*(ab^*)^{i+1}b, \\
a^*a(b^*a)^i b^*(bb^*a) &\mapsto a^*a(b^*a)^i b^*ab^*b, \\
&\to a^*a(b^*a)^{i+1}b^*b.
\end{aligned}$$

This shows $g_{i+1} \in I$, which completes the proof. □

Lemma 5.1.6. *The set $\Omega = \{c_1, c_2, c_3, c_4\} \cup \{f_i : i \geq 0\} \cup \{g_i : i \geq 0\}$ is a full reduction system for Λ in which all minimally ambiguous words are reduction unique.*

Proof. The elements of Ω are monic, and we have shown that $I_\Omega \subseteq I$, so Ω is a reduction system. To show it is full, we only need show that $r_1, r_2 \in I_\Omega$, since then $I = I_\Omega$. This is clear, since $r_1 = f_0$, and $r_2 = b^*ba^*a + b^*b + a^*a = g_0 + b^*c_2 + c_1b \in I_\Omega$. We must now show that all minimally ambiguous words are reduction unique. There are no inclusion ambiguities, so we must look for overlaps. The only overlaps between the c_i are bb^*aa^*, b^*ba^*a, which are very easy to check. There are no overlaps which involve only the f_i and g_i, which leaves us with the overlap of a c_k and an f_i or g_i. First we check the overlaps with f_0, namely, $aa^*bb^*aa^*$, $aa^*bb^*ba^*$, $ba^*aa^*bb^*$, $bb^*aa^*bb^*$ (there is no need to check aa^*bb^*a, $b^*aa^*bb^*$ since if for example aa^*bb^*a is not reduction unique, then neither is $aa^*bb^*aa^*$). We compute all single step reductions, and show each of them has a reduction to a common value.

$$\begin{aligned}
aa^*b(b^*aa^*) &\mapsto aa^*(ba^*a)b^* \mapsto aa^*(aa^*bb^*) \mapsto -aa^*aa^* - aa^*bb^*,\\
aa^*(bb^*a)a^* &\mapsto a(a^*ab^*b)a^* \mapsto -aa^*aa^* - a(b^*ba^*) \mapsto -aa^*aa^* - aa^*bb^*,\\
(aa^*bb^*)aa^* &\mapsto -aa^*aa^* - bb^*aa^* \to -aa^*aa^* - aa^*bb^*.
\end{aligned}$$

$$\begin{aligned}
aa^*b(b^*ba^*) &\mapsto aa^*ba^*bb^* \mapsto -aa^*ba^* - ba^*bb^*,\\
(aa^*bb^*)ba^* &\mapsto -aa^*ba^* - b(b^*ba^*) \mapsto -aa^*ba^* - ba^*bb^*.
\end{aligned}$$

$$\begin{aligned}
(ba^*a)a^*bb^* &\mapsto aa^*ba^*bb^* \mapsto -aa^*ba^* - ba^*bb^*,\\
ba^*(aa^*bb^*) &\mapsto (ba^*a)a^* - ba^*bb^* \mapsto -aa^*ba^* - ba^*bb^*.
\end{aligned}$$

$$\begin{aligned}
(bb^*a)a^*bb^* &\mapsto a(bb^*a^*)bb^* \mapsto (aa^*bb^*)bb^* \mapsto -aa^*bb^* - bb^*bb^*,\\
b(b^*aa^*)bb^* &\mapsto b(a^*ab^*b)b^* \mapsto -(ba^*a)b^* - bb^*bb^* \mapsto -aa^*bb^* - bb^*bb^*,\\
bb^*(aa^*bb^*) &\mapsto -bb^*aa^* - bb^*bb^* \to -aa^*bb^* - bb^*bb^*.
\end{aligned}$$

The overlaps with g_0 are also easily seen to be reduction unique (the calculations are the same, apply * to each arrow). We now show that for each k, the overlaps involving f_k and g_k are reduction unique. It is only necessary to check f_k, since the calculations for g_k are the same except for the stars. First the two shorter overlaps.

$$\begin{aligned}
aa^*(ba^*)^k(bb^*a) &\mapsto aa^*(ba^*)^kab^*b \to aa^*(aa^*(ba^*)^{k-1}bb^*)b,\\
&\mapsto -aa^*a(a^*b)^k - a(a^*b)^kb^*b,\\
(aa^*(ba^*)^kbb^*)a &\mapsto -a(a^*b)^ka^*a - b(a^*b)^kb^*a,\\
&\to -aa^*a(a^*b)^k - a(a^*b)^kb^*b.
\end{aligned}$$

$$\begin{aligned}
(b^*aa^*)(ba^*)^k bb^* &\mapsto a^*ab^*(ba^*)^k bb^* \to a^*(aa^*(ba^*)^{k-1}bb^*)bb^*,\\
&\mapsto -a^*a(a^*b)^{k-1}a^*bb^* - (a^*b)^k b^*bb^*,\\
b^*(aa^*(ba^*)^k bb^*) &\mapsto -b^*a(a^*b)^k a^* - b^*b(a^*b)^k b^*,\\
&\to -a^*a(a^*b)^{k-1}a^*bb^* - (a^*b)^k b^*bb^*.
\end{aligned}$$

Now we check the four longer overlaps.

$$\begin{aligned}
aa^*(ba^*)^k b(b^*aa^*) &\mapsto aa^*(ba^*)^k ba^*ab^* \to aa^*(aa^*(ba^*)^k bb^*),\\
&\mapsto -aa^*a(a^*b)^k a^* - a(a^*b)^{k+1}b^*,\\
aa^*(ba^*)^k(bb^*a)a^* &\mapsto aa^*(ba^*)^k ab^*ba^* \to aa^*aa^*(ba^*)^k bb^*,\\
(aa^*(ba^*)^k bb^*)aa^* &\mapsto -a(a^*b)^k a^*aa^* - b(a^*b)^k b^*aa^*,\\
&\to -aa^*a(a^*b)^k a^* - a(a^*b)^{k+1}b^*.
\end{aligned}$$

$$\begin{aligned}
aa^*(ba^*)^k b(b^*ba^*) &\mapsto aa^*(ba^*)^k ba^*bb^* \mapsto -a(a^*b)^{k+1}a^* - b(a^*b)^{k+1}b^*,\\
(aa^*(ba^*)^k bb^*)ba^* &\mapsto -a(a^*b)^k a^*ba^* - b(a^*b)^k b^*ba^*,\\
&\to -a(a^*b)^{k+1}a^* - b(a^*b)^{k+1}b^*.
\end{aligned}$$

$$\begin{aligned}
(ba^*a)a^*(ba^*)^k bb^* &\mapsto aa^*ba^*(ba^*)^k bb^* \mapsto -a(a^*b)^{k+1}a^* - b(a^*b)^{k+1}b^*,\\
ba^*(aa^*(ba^*)^k bb^*) &\mapsto -ba^*a(a^*b)^k a^* - ba^*b(a^*b)^k b^*,\\
&\to -a(a^*b)^{k+1}a^* - b(a^*b)^{k+1}b^*.
\end{aligned}$$

$$\begin{aligned}
(bb^*a)a^*(ba^*)^k bb^* &\mapsto abb^*a^*(ba^*)^k bb^* \to (aa^*(ba^*)^k bb^*)bb^*,\\
&\mapsto -a(a^*b)^{k+1}b^* - b(a^*b)^k b^*bb^*,\\
b(b^*aa^*)(ba^*)^k bb^* &\mapsto ba^*ab^*(ba^*)^k bb^* \to (aa^*(ba^*)^k bb^*)bb^*,\\
bb^*(aa^*(ba^*)^k bb^*) &\mapsto -bb^*a(a^*b)^k a^* - bb^*b(a^*b)^k b^*,\\
&\to -a(a^*b)^{k+1}b^* - b(a^*b)^k b^*bb^*.
\end{aligned}$$

□

Putting this together with Lemma A.4.3 gives the following corollary.

Corollary 5.1.7. *A basis for Λ is given by the set of P paths of $\overline{Q}$ in which*

(i) a never occurs to the right of b,

(ii) a^* never occurs to the right of b^*,

(iii) At least one arrow does not occur (e.g. b does not occur in $aa^*ab^*ab^*$).

Proof. Such paths are irreducible, as they cannot contain an element of Ω as a subpath. Conversely, if a path is reducible by some c_i, it cannot satisfy both (i) and (ii), and if it is reducible by some f_i or g_i, it cannot satisfy (iii). □

We can calculate the representation of Q corresponding to Λ with respect to this basis, but unfortunately it is not convenient for obtaining the decomposition of Λ into indecomposable modules. Instead we have to calculate a new basis so that Λ is easily seen to decompose. We define some notation to easily write down elements P. Suppose u, v are vertices of the quiver and $i, j \geq 0$ be integers (exclude the cases ${}_1(a,0,0)_2$, ${}_1(b,0,0)_2$, ${}_2(a^*,0,0)_1$, ${}_2(b^*,0,0)_1$ which don't make sense).

1. Let ${}_u(a,i,j)_v$ be the unique path from v to u in P which contains i occurrences of a^*, j occurrences of b^*, and does not involve b.

2. Let ${}_u(b,i,j)_v$ be the unique path from v to u in P which contains i occurrences of a^*, j occurrences of b^*, and does not involve a.

3. Let ${}_u(a^*,i,j)_v$ be the unique path from v to u in P which contains i occurrences of a, j occurrences of b, and does not involve b^*.

4. Let ${}_u(b^*,i,j)_v$ be the unique path from v to u in P which contains i occurrences of a, j occurrences of b, and does not involve a^*.

For example, ${}_1(a^*,1,3)_1$ is equal to the basis element $a^*aa^*ba^*ba^*b$. It cannot be equal to any other basis element, as once we know the number of occurrences of a and b, the order in which they appear is determined by (i). We know each remaining arrow is that appears is a^* so the path is determined by the starting/ending vertices. Conversely, every basis element can be represented in this way, as by (iii) it must be included at at least one of the categories. Note that some basis elements are not uniquely represented, specifically those in which at least two arrows do not occur, e.g. $a^*ba^*ba^*$ is equal to both ${}_1(a^*,0,2)_2$ and

${}_1(b,3,0)_2$. Clearly the set P_{uv} of all elements of the form ${}_u(c,i,j)_v$, where c can be any arrow in $\overline{Q}$, is a basis of $e_u\Lambda e_v$ (provided the appropriate identifications are made).

Let $A = -(e_2 + aa^*)$, $A^* = -(e_1 + a^*a)$, $B = -(e_2 + bb^*)$, $B^* = -(e_1 + b^*b)$. As elements of Λ, $bb^* = aa^*B = Baa^*$, $aa^* = Abb^* = bb^*B$, $b^*b = a^*aB^* = B^*a^*a$, $a^*a = A^*b^*b = b^*bB^*$, $Ab = bA^*$, $Ba = aB^*$, $Aa = aA^*$, $Bb = bB^*$.

Definition 5.1.8. We define a set of elements $\{x_l^i : l \in \mathbb{N}, 1 \leq i \leq l\} \in e_1\Lambda$. Let $x_1^1 = e_1$, and for all $j \geq 1$, let $x_{2j}^1 = (a^*b)^{j-1}a^*$, $x_{2j+1}^1 = (a^*b)^j$, $x_{2j}^{2j} = A^{*j}(b^*a)^{j-1}b^*$, $x_{2j+1}^{2j+1} = A^{*j}(b^*a)^j$. The remaining x_l^i are defined by induction, $x_{l+2}^{i+1} = a^*ax_l^i$, (it should be easy to see that this gives a valid definition for all i, l in the given range).

Lemma 5.1.9. *For all $i = 1, \ldots, l-1$, $ax_l^i = bx_l^{i+1}$.*

Proof. We split into the cases of l even or odd. Suppose $l = 2j$. We proceed by induction on j. We have

$$ax_2^1 = aa^* = Abb^* = bA^*b^* = bx_2^2,$$

so the formula is true for $j = 1$. Assume that $j > 1$, and that the formula holds for all values less than j. For the cases $i = 1, 2j-1$, we have

$$\begin{aligned} ax_{2j}^1 &= a(a^*b)^{j-1}a^* = ba^*a(a^*b)^{j-2}a^* = ba^*ax_{2j-2}^1 = bx_{2j}^2, \\ ax_{2j}^{2j-1} &= aa^*ax_{2j-2}^{2j-2} = aa^*aA^{*j-1}(b^*a)^{j-2}b^* = aa^*A^{j-1}(ab^*)^{j-1} \\ &= bb^*AA^{j-1}(ab^*)^{j-1} = bA^{*j}(b^*a)^{j-1}b^* = bx_{2j}^{2j}. \end{aligned}$$

For the cases $i \neq 1, 2j-1$, we can use the induction hypothesis,

$$ax_{2j}^i = aa^*ax_{2j-2}^{i-1} = aa^*bx_{2j-2}^i = ba^*ax_{2j-2}^i = bx_{2j}^{i+1},$$

which completes the proof for even l. Now suppose $l = 2j+1$, and again proceed by induction on j. We have

$$\begin{aligned} ax_3^1 &= aa^*b = ba^*a = ba^*ax_1^1 = bx_3^2, \\ ax_3^2 &= aa^*a = Abb^*a = bA^*b^*a = bx_3^3, \end{aligned}$$

so the formula is true for $j = 1$. Assume that $j > 1$, and that the formula holds for all values less than j. For the cases $i = 1, 2j$, we have

$$\begin{aligned} ax^1_{2j+1} &= a(a^*b)^j = aa^*b(a^*b)^{j-1} = ba^*a(a^*b)^{j-1} = ba^*ax^1_{2j-1} = bx^2_{2j+1}, \\ ax^{2j}_{2j+1} &= aa^*ax^{2j-1}_{2j-1} = aa^*a(A^*)^{j-1}(b^*a)^{j-1} = aa^*A^{j-1}(ab^*)^{j-1}a \\ &= bb^*A^j(a^*b)^{j-1} = b(A^*)^j(b^*a)^j = bx^{2j+1}_{2j+1}. \end{aligned}$$

For the cases $i \neq 1, 2j$, we can use the induction hypothesis,

$$ax^i_{2j+1} = aa^*ax^{i-1}_{2j-1} = aa^*bx^i_{2j-1} = ba^*ax^i_{2j-1} = bx^{i+1}_{2j+1},$$

which completes the proof. □

Definition 5.1.10. We can now define a set of elements $\{y^i_l : l \in \mathbb{N}, 0 \leq i \leq l\} \in e_2\Lambda$ by setting $y^0_0 = e_2$, and otherwise let $y^i_l = ax^i_l$ and $y^{i-1}_l = bx^i_l$. The previous lemma shows that this is well defined.

Definition 5.1.11. Let k be an integer. Set $x^i_{2j}[k] = x^i_{2j}B^k$, $x^i_{2j+1}[k] = x^i_{2j+1}B^{*k}$, $y^i_{2j}[k] = y^i_{2j}B^k$, $y^i_{2j+1}[k] = y^i_{2j+1}B^{*k}$.

Lemma 5.1.12. *The elements of P can be written in the form $x^i_l[k]$ or $y^i_l[k]$ for some i, l, k. Specifically, for all valid r, s,*

$$\begin{aligned} {}_1(a,r,s)_2 &= x^{r+2s}_{2r+2s}[s], & {}_2(a,r,s)_2 &= y^{r+2s}_{2r+2s}[s], \\ {}_1(b,r,s)_2 &= x^{s+1}_{2r+2s}[s], & {}_2(b,r,s)_2 &= y^{s}_{2r+2s}[s], \\ {}_1(a^*,r,s)_2 &= x^{r+1}_{2r+2s+2}[0], & {}_2(a^*,r,s)_2 &= y^{r}_{2r+2s}[0], \\ {}_1(b^*,r,s)_2 &= x^{2r+s+2}_{2r+2s+2}[r+s+1], & {}_2(b^*,r,s)_2 &= y^{2r+s}_{2r+2s}[r+s]. \end{aligned}$$

$$\begin{aligned} {}_1(a,r,s)_1 &= x^{r+2s+1}_{2r+2s+1}[s], & {}_2(a,r,s)_1 &= y^{r+2s+1}_{2r+2s+1}[s], \\ {}_1(b,r,s)_1 &= x^{s+1}_{2r+2s+1}[s], & {}_2(b,r,s)_1 &= y^{s}_{2r+2s+1}[s], \\ {}_1(a^*,r,s)_1 &= x^{r+1}_{2r+2s+1}[0], & {}_2(a^*,r,s)_1 &= y^{r}_{2r+2s-1}[0], \\ {}_1(b^*,r,s)_1 &= x^{2r+s+1}_{2r+2s+1}[r+s]. & {}_2(b^*,r,s)_1 &= y^{2r+s-1}_{2r+2s-1}[r+s-1]. \end{aligned}$$

Proof. First, we verify the top four equations on the left. If $s > 0$, then

$$\begin{aligned} x^{r+2s}_{2r+2s}[s] &= (a^*a)^r x^{2s}_{2s}[s] = (a^*a)^r A^{*s}(b^*a)^{s-1}b^*B^s \\ &= (a^*a)^r A^{*s}B^{*s}(b^*a)^{s-1}b^* = (a^*a)^r(b^*a)^{s-1}b^*, \end{aligned}$$

since $A^*B^* = e_1$, and the resulting expression is is equal to ${}_1(a, r, s)_2$. If $s = 0$, then the equation is true since for $r > 0$

$$x^r_{2r}[0] = (a^*a)^{r-1}x^1_2[0] = (a^*a)^{r-1}a^* = {}_1(a, r, 0)_2,$$

and ${}_1(a, 0, 0)_2$ is not defined. If $r > 0$, then

$$x^{s+1}_{2r+2s}[s] = (a^*a)^s x^1_{2r}[s] = (a^*a)^s(a^*b)^{r-1}a^*B^s = (a^*b)^{r-1}a^*(bb^*)^s,$$

which is ${}_1(b, r, s)_2$ and if $r = 0$, then the equation is true since for $s > 0$,

$$\begin{aligned} x^{s+1}_{2s}[s] &= (a^*a)^{s-1}x^2_2[s] = (a^*a)^{s-1}A^*b^*B^s = A^*(a^*a)^{s-1}(B^*)^s b^* \\ &= A^*B^*(b^*b)^{s-1}b^* = (b^*b)^{s-1}b^* = {}_1(b, 0, s)_2, \end{aligned}$$

and ${}_1(b, 0, 0)_2$ is not defined. The third equation is true since

$$x^{r+1}_{2r+2s+2}[0] = (a^*a)^r x^1_{2s+2} = (a^*a)^r(a^*b)^s a^* = {}_1(a^*, r, s)_2,$$

and the fourth is true since

$$\begin{aligned} x^{2r+s+2}_{2r+2s+2}[r+s+1] &= (a^*a)^s x^{2r+2}_{2r+2}[r+s+1] = (a^*a)^s(A^*)^{r+1}(b^*a)^r b^*B^{r+s+1} \\ &= (a^*a)^s(B^*)^s(b^*a)^r b^* = (b^*b)^s(b^*a)^r b^* \\ &= {}_1(b^*, r, s)_2. \end{aligned}$$

The top four equations on the right are now easy to verify. Note first that $(c, 0, 0) = e_2 = y^0_0[0]$ for all arrows c. For the rest we use the equations just verified.

$${}_2(a, r, s)_2 = a({}_1(a, r, s)_2) = ax^{r+2s}_{2r+2s}[s] = y^{r+2s}_{2r+2s}[s],$$

$${}_2(b, r, s) = b({}_1(b, r, s)_2) = bx^{s+1}_{2r+2s}[s] = y^s_{2r+2s}[s].$$

If $s > 0$, then

$${}_2(a^*, 0, s)_2 = b({}_1(a^*, 0, s-1)_2) = bx^1_{2s}[0] = y^0_{2s}[0].$$

If $r > 0$, then

$$_2(a^*, r, s)_2 = a(_1(a^*, r-1, s)_2) = ax^r_{2r+2s}[0] = y^r_{2r+2s}[0].$$

If $s > 0$, then

$$_2(b^*, 0, s)_2 = b(_1(b^*, 0, s-1)_2) = bx^{s+1}_{2s}[s] = y^s_{2s}[s].$$

If $r > 0$, then

$$_2(b^*, r, s)_2 = a(_1(b^*, r-1, s)_2) = ax^{2r+s}_{2r+2s}[r+s] = y^{2r+s}_{2r+2s}[r+s].$$

For the equations at the bottom, we do the same thing. Note first that $(c, 0, 0) = e_1 = x^1_1[0]$ for all arrows c. If $s > 0$, then

$$x^{r+2s+1}_{2r+2s+1}[s] = (a^*a)^r x^{2s+1}_{2s+1}[s] = (a^*a)^r A^{*s}(b^*a)^s(B^*)^s = (a^*a)^r(b^*a)^s,$$

which is equal to $_1(a, r, s)_1$. If $s = 0$, then the equation is true since

$$x^{r+1}_{2r+1}[0] = (a^*a)^r x^1_1[0] = (a^*a)^r = {}_1(a, r, 0)_1.$$

If $r > 0$, then

$$x^{s+1}_{2r+2s+1}[s] = (a^*a)^s x^1_{2r+1}[s] = (a^*a)^s(a^*b)^r(B^*)^s = (a^*b)^r(b^*b)^s = {}_1(b, r, s)_1,$$

and if $r = 0$, then the equation is true since

$$x^{s+1}_{2s+1}[s] = (a^*a)^s x^1_1[s] = (a^*a)^s(B^*)^s = (b^*b)^s = {}_1(b, 0, s)_1.$$

The next two equations are satisfied since

$$x^{r+1}_{2r+2s+1}[0] = (a^*a)^r x^1_{2s+1} = (a^*a)^r(a^*b)^s = {}_1(a^*, r, s)_1.$$

$$\begin{aligned} x^{2r+s+1}_{2r+2s+1}[r+s] &= (a^*a)^s x^{2r+1}_{2r+1}[r+s] = (a^*a)^s(A^*)^r(b^*a)^r(B^*)^{r+s} \\ &= (a^*a)^s(B^*)^s(b^*a)^r = (b^*b)^s(b^*a)^r = {}_1(b^*, r, s)_1. \end{aligned}$$

Finally, we verify the four equations on the bottom right

$$_2(a, r, s)_1 = a(_1(a, r, s)_1) = ax^{r+2s+1}_{2r+2s+1}[s] = y^{r+2s+1}_{2r+2s+1}[s].$$

$$ {}_2(b,r,s)_1 = b({}_1(b,r,s)_1) = bx^{s+1}_{2r+2s+1}[s] = y^s_{2r+2s}[s]. $$

If $s > 0$, then

$$ {}_2(a^*,0,s)_1 = b({}_1(a^*,0,s-1)_1) = bx^1_{2s-1}[0] = y^0_{2s-1}[0]. $$

If $r > 0$, then

$$ {}_2(a^*,r,s)_1 = a({}_1(a^*,r-1,s)_1) = ax^r_{2r+2s-1}[0] = y^r_{2r+2s-1}[0]. $$

If $s > 0$, then

$$ {}_2(b^*,0,s)_1 = b({}_1(b^*,0,s-1)_1) = bx^s_{2s-1}[s-1] = y^{s-1}_{2s-1}[s-1]. $$

If $r > 0$, then

$$ {}_2(b^*,r,s)_1 = a({}_1(b^*,r-1,s)_1) = ax^{2r+s-1}_{2r+2s-1}[r+s-1] = y^{2r+s-1}_{2r+2s-1}[r+s-1]. $$

If r, s are both zero, then there is nothing to check, because ${}_1(a^*,0,0)_2$ and ${}_1(b^*,0,0)_2$ are not defined. $\square$

We can now state the crucial lemma.

Lemma 5.1.13. *Set $k_l = \lfloor \frac{l}{4} \rfloor$ (i.e. the integer part of $\frac{l}{4}$). The set $L = \cup L_{uv}$ is a basis for Λ, where*

$$ \begin{aligned} L_{12} &= \{x^i_{2j}[k_{2j}] : j \in \mathbb{N}, 1 \le i \le 2j\}, \\ L_{22} &= \{y^i_{2j}[k_{2j}] : j \in \mathbb{N}, 1 \le i \le 2j\}, \\ L_{11} &= \{x^i_{2j+1}[k_{2j+1}] : j \in \mathbb{N}, 0 \le i \le 2j+1\}, \\ L_{21} &= \{y^i_{2j+1}[k_{2j+1}] : j \in \mathbb{N}, 0 \le i \le 2j+1\}. \end{aligned} $$

The proof of this lemma is quite long. The basic idea is to write the original basis elements in terms of the $x^i_l[k_l]$, $y^i_l[k_l]$, in such a way that it guarantees they also form a basis of Λ. We split the proof into four separate parts, each part showing that L_{uv} is a basis of $e_u \Lambda e_v$. We define chains of sets $(P_{uv}(m))_{m\in\mathbb{N}}$, $(L_{uv}(m))_{m\in\mathbb{N}}$ with the properties $P_{uv}(m) \subseteq P_{uv}$, $\cup_m P_{uv}(m) = P_{uv}$ and the same with L replacing P. Note that although we use the (c,r,s) notation to write down the elements of $P_{uv}(m)$, they are identified if they correspond with the same path. We show that for each m the following properties hold,

$$(\dagger) \quad |L_{uv}(m)| = |P_{uv}(m)|,$$

$$(\ddagger) \quad KP_{uv}(m) \subseteq KL_{uv}(m).$$

It then follows that L_{uv} is a basis of $e_u\Lambda e_v$. Namely,

(i) It spans $e_u\Lambda e_v$. Given an element α of $e_u\Lambda e_v$, we can write α as a linear combination of elements of P_{uv} (since P_{uv} is a basis of $e_u\Lambda e_v$). We can choose m so that each of these elements is in $P_{uv}(m)$. Then by ($\ddagger$), $\alpha \in KL_{uv}(m)$, and hence certainly in KL_{uv}.

(ii) It is independent. If this is not the case, then some linear combination of elements of L_{uv} is zero. Choose m so that each of these elements lies in $L_{uv}(m)$. Thus $L_{uv}(m)$ is not independent. However ($\ddagger$) tells us that $L_{uv}(m)$ spans $KP_{uv}(m)$, and ($\dagger$) tells us that it has the same number of elements as a basis of $KP_{uv}(m)$, and is therefore a basis. This is a contradiction.

The awkward part is proving ($\ddagger$). Using Lemma 5.1.12, we write each element of P_{uv} as some $x_l^i[k]$ or $y_l^t[k]$, and then use the following fact.

Lemma 5.1.14. *Given any $x_l^i[k]$, we can write it as a linear combination of elements of L, specifically*

$$x_l^i[k] = \sum_{t=0}^{w} \lambda_t x_{l+2t}^{i+t}[k_{l+2t}],$$

where the λ_t are scalars, and

$$w = \begin{cases} \lfloor \frac{l}{2} - 2k - 1 \rfloor & \text{if } \frac{l}{4} - k \geq 1, \\ -\lfloor \frac{l}{2} - 2k \rfloor & \text{if } \frac{l}{4} - k < 0, \\ 0 & \text{otherwise.} \end{cases}$$

The same formula also holds with x replaced by y.

Proof. First, note the following formula, which holds for all valid i, l.

$$x_l^i[k] + x_{l+2}^{i+1}[k] + x_l^i[k-1] = 0. \tag{5.1}$$

This holds since $x^{i+1}_{l+2} = aa^* x^i_l$, and so $x^{i+1}_{l+2} + x^i_l = -A^* x^i_l$. Multiplying on the right by B^k or $(B^*)^k$ (depending on whether l is even or odd) gives the equation. Note that the formula also holds with x replaced by y - For $i = 1, \ldots, l$, multiply the formula for x (with the same i) on the left by a, and for $i = 0$, multiply the formula with $i = 1$ by b.

Let $z(x^i_l[k])$ be the z-*value* of $x^i_l[k]$, defined to be $\frac{l}{4} - k$. This measures how close k is to k_l. Clearly, $x^i_l[k] \in L$ if and only if $0 \leq z(x^i_l[k]) < 1$. Note that

$$\begin{aligned} z(x^i_l[k]) &= \frac{l}{4} - k, \\ z(x^{i+1}_{l+2}[k]) &= \frac{l-2}{4} - k, \\ z(x^i_l[k-1]) &= \frac{l-4}{4} - k. \end{aligned}$$

By choosing an appropriate term, we can use (5.1) as a substitution to write $x^i_l[k]$ as a linear combination of elements with greater/lesser z-value, and repeat until each z-value lies in the range [0,1), and thus each term is in L. That is, we rewrite (5.1) in two different ways:

$$x^i_l[k] = -x^{i+1}_{l+2}[k] - x^i_l[k-1]. \tag{5.2}$$

$$x^i_l[k] = -x^i_l[k+1] - x^{i+1}_{l+2}[k+1]. \tag{5.3}$$

Then, for example, suppose we wish to express $x^3_{10}[0]$ in terms of the new basis elements. Since $z(x^3_{10}[0]) = \frac{5}{2}$ we use (5.3).

$$\begin{aligned} x^3_{10}[0] &= -x^3_{10}[1] - x^4_{12}[1], \\ &= x^3_{10}[2] + 2x^4_{12}[2] + x^5_{14}[2], \\ &= x^3_{10}[2] - 2x^4_{12}[3] - 3x^5_{14}[3] - x^6_{16}[3], \\ &= x^3_{10}[2] - 2x^4_{12}[3] - 3x^5_{14}[3] + x^6_{16}[4] - x^7_{18}[4]. \end{aligned}$$

Observe that we only use (5.3) on the terms whose z-value is at least 1 and we leave the rest alone. If instead we wish to do the same with $x^3_{10}[4]$, then since $z(x^3_{10}[4]) = -\frac{3}{2}$ we must use (5.2) (in this case only substituting the terms with

z-value less than zero).

$$\begin{aligned} x_{10}^3[4] &= -x_{12}^4[4] - x_{10}^3[3], \\ &= x_{14}^5[4] + 2x_{12}^4[3] + x_{10}^3[2], \\ &= -x_{16}^6[4] - x_{14}^5[3] + 2x_{12}^4[3] + x_{10}^3[2]. \end{aligned}$$

It is obvious that we can use this method to express any given $x_l^i[k]$ in the form given in the statement of the lemma. One needs only check that the given w is correct, which is the the maximum number of iterations of (5.3) needed. If the z-value $\frac{l}{4} - k$ is at least one, then after n iterations, the maximum z-value which occurs in the expression is $\frac{l}{4} - k - \frac{n}{2}$. We require this to be less than one, i.e. we can take w to be the minimum integer n for which $\frac{l}{4} - k - \frac{n}{2} < 1$, which is $\lfloor \frac{l}{2} - 2k - 1 \rfloor$. If the z-value $\frac{l}{4} - k$ is less than 0, then after n iterations of (5.2) the minimum z-value is $\frac{l}{4} - k + \frac{n}{2}$. We require this to be at least 0, i.e. we can take w to be the minimum integer n for which $\frac{l}{4} - k - \frac{n}{2} \geq 0$, which is $-\lfloor \frac{l}{2} - 2k \rfloor$. If $0 \leq z(x_l^i[k]) < 1$, then we can take $w = 0$.

Clearly, the same argument is valid with y replacing x throughout. $\square$

Part 1: $u = 1$**,** $v = 2$**.**

Since during this part we are dealing exclusively with paths of type ${}_1(c, r, s)_2$ we can just write (c, r, s) (also recall that if $c = a, b$ then r, s cannot both be zero). Given an integer $m > 0$, let

$$P_{12}(m) = \{(c, r, s) \in P_{12} : r, s \leq m \text{ if } c = a, b \text{ , } r + s < m \text{ if } c = a^*, b^*\},$$

$$L_{12}(m) = \{x_{2j}^i[k_{2j}] : j \leq 2m, j - m < i \leq j + m\}.$$

We verify (†). We count the number of distinct elements of $P_{12}(n) \backslash P_{12}(n-1)$ (assuming $P_{12}(0)$ to be the empty set). We claim that this is equal to $6n - 2$. If $n = 1$, the four distinct basis elements are $(a, 1, 0) = (b, 1, 0) = (a^*, 0, 0)$, $(a, 0, 1) = (b, 0, 1) = (b^*, 0, 0)$, $(a, 1, 1)$, $(b, 1, 1)$. There are no more, since $(a, 0, 0)$, $(b, 0, 0)$ don't make sense. For $n > 1$, there are $2(2n + 1) + 2(n - 2) = 6n - 2$ distinct elements, namely

$$(a, n, 0), (a, n, 1), \ldots, (a, n, n - 1), (a, n, n), (a, n - 1, n), \ldots, (a, 1, n), (a, 0, n),$$

$$(b,n,0),(b,n,1),\ldots,(b,n,n-1),(b,n,n),(b,n-1,n),\ldots,(b,1,n),(b,0,n),$$

$$(a^*,1,n-2),(a^*,2,n-3),\ldots,(a^*,n-2,1),$$

$$(b^*,1,n-2),(b^*,2,n-3),\ldots,(b^*,n-2,1).$$

There are no more, since $(a^*,0,n-1)=(b,n,0)$, $(b^*,0,n-1)=(b,0,n)$, $(a^*,n-1,0)=(a,n,0)$, $(b^*,n-1,0)=(a,0,n)$. So

$$|P_{12}(m)|=\sum_{n=1}^{m}6n-2=6\sum_{n=1}^{m}n-2m=3m(m+1)-2m=3m^2+m.$$

We compute the number of elements of $L_{12}(m)$.

$$\begin{aligned}
|L_{12}(m)| &= |\{x_{2j}^{i}[k_{2j}]:j\le 2m, j-m<i\le j+m\}|\\
&= |\{(i,2j)\in\mathbb{N}^2:j\le 2m, j-m<i\le j+m, 1\le i\le 2j\}|\\
&= |\{(i,2j)\in\mathbb{N}^2:j\le m, 1\le i\le 2j\}|\\
&\quad +|\{(i,2j)\in\mathbb{N}^2:m<j\le 2m, j-m<i\le j+m\}|\\
&= \sum_{j=1}^{m}2j+2m^2\\
&= m(m+1)+2m^2\\
&= 3m^2+m.
\end{aligned}$$

Thus (†) is satisfied. We now show (‡) is satisfied. First the case $m=1$.

$$(a,1,1)=x_4^3[1],\qquad (b,1,1)=x_4^2[1],$$

$$(a^*,0,0)=x_2^1[0],\qquad (b^*,0,0)=x_2^2[1]=-x_4^3[1]-x_2^2[0],$$

each of which is in $KL_{12}(1)$. Assuming the claim has been proved for $m-1$, every element of $P_{12}(m-1)$ has been shown to be in $KL_{12}(m-1)$, and therefore in $KL_{12}(m)$. We must therefore write down each element of $P_{12}(m)\setminus P_{12}(m-1)$ as a linear combination of elements of $L_{12}(m)$, using Lemmas 5.1.12 and 5.1.14. This splits into six cases (some of which overlap).

(1) For all r with $0\le r\le m$,

$$(a,r,m)=x_{2r+2m}^{r+2m}[m]=\sum_{t=0}^{m-r}\lambda_t x_{2r+2m+2t}^{r+2m+t}[k_{2r+2m+2t}].$$

Each term is in $L_{12}(m)$, since setting $i_t = r + 2m + t$ and $j_t = r + m + t$, we have $j_t = r+m+t \leq r+m+m-r \leq 2m$, and $i_t = j_t + m$, so lies in the range $j_t - m < i_t \leq j_t + m$ (we omit this verification in all subsequent cases, as it is trivial to check).

(2) For all s with $0 \leq s < m$,

$$(a, m, s) = x_{2m+2s}^{m+2s}[s] = \sum_{t=0}^{m-s-1} \lambda_t x_{2m+2s+2t}^{m+2s+t}[k_{2m+2s+2t}].$$

(3) For all r with $0 \leq r \leq m$,

$$(b, r, m) = x_{2r+2m}^{m+1}[m] = \sum_{t=0}^{m-r} \lambda_t x_{2r+2m+2t}^{m+1+t}[k_{2r+2m+2t}].$$

(4) For all s with $0 \leq s < m$,

$$(b, m, s) = x_{2m+2s}^{s+1}[s] = \sum_{t=0}^{m-s-1} \lambda_t x_{2m+2s+2t}^{s+1+t}[k_{2m+2s+2t}].$$

(5) For all r with $0 \leq r \leq m-1$,

$$(a^*, r, m-r-1) = x_{2m}^{r+1}[0] = \sum_{t=0}^{m-1} \lambda_t x_{2m+2t}^{r+1+t}[k_{2m+2t}].$$

(6) For all r with $0 \leq r \leq m-1$,

$$(b^*, r, m-r-1) = x_{2m}^{m+r+1}[m] = \sum_{t=0}^{m} \lambda_t x_{2m+2t}^{m+r+1+t}[k_{2m+2t}].$$

Thus (‡) is satisfied.

Part 2: $u = 2$**,** $v = 2$**.**

We write (c, r, s) for ${}_2(c, r, s)_2$. Given an integer $m > 0$, let

$$P_{22}(m) = \{(c, r, s) \in P_{12} : r, s \leq m \text{ if } c = a, b \text{ , } r + s \leq m \text{ if } c = a^*, b^*\},$$

$$L_{22} = \{x_{2j}^i[k_{2j}] : j \leq 2m, j - m \leq i \leq j + m\}.$$

We verify (†). For convenience, set $P_{22}(0) = \{e_2\}$. For $n \geq 1$, we claim that the number of distinct elements of $P_{22}(n) \setminus P_{22}(n-1)$ is $6n$. If $n = 1$, they are

$(a,0,1) = (b^*,1,0)$, $(b,0,1) = (b^*,0,1)$, $(a,1,0) = (a^*,1,0)$, $(b,1,0) = (a^*,0,1)$, $(a,1,1)$, $(b,1,1)$. For $n > 1$, there are $2(2n+1)+2(n-1) = 6n$ distinct elements, namely

$$(a,n,0),(a,n,1),\ldots,(a,n,n-1),(a,n,n),(a,n-1,n),\ldots,(a,1,n),(a,0,n),$$

$$(b,n,0),(b,n,1),\ldots,(b,n,n-1),(b,n,n),(b,n-1,n),\ldots,(b,1,n),(b,0,n),$$

$$(a^*,1,n-1),(a^*,2,n-2),\ldots,(a^*,n-1,1),$$

$$(b^*,1,n-1),(b^*,2,n-2),\ldots,(b^*,n-1,1).$$

There are no more, since $(a^*,0,n) = (b,n,0)$, $(b^*,0,n) = (b,0,n)$, $(a^*,n,0) = (a,n,0)$, $(b^*,n,0) = (a,0,n)$. So

$$|P_{22}(m)| = 1 + \sum_{n=1}^{m} 6n = 1 + 6\sum_{n=1}^{m} n = 1 + 3m(m+1) = 3m^2+3m+1.$$

We compute the number of elements of $L_{22}(m)$.

$$\begin{aligned}
|L_{22}(m)| &= |\{y^i_{2j}[k_{2j}] : j \le 2m, j-m \le i \le j+m\}| \\
&= 1 + |\{(i,2j) \in \mathbb{N}^2 : 1 \le j \le 2m, j-m \le i \le j+m, 0 \le i \le 2j\}| \\
&= 1 + |\{(i,2j) \in \mathbb{N}^2 : 1 \le j \le m, 0 \le i \le 2j\}| \\
&\quad + |\{(i,2j) \in \mathbb{N}^2 : m < j \le 2m, j-m \le i \le j+m\}| \\
&= 1 + \sum_{j=1}^{m} 2j+1 + m(2m+1) \\
&= 1 + m(m+1) + m + 2m^2 + m \\
&= 3m^2+3m+1.
\end{aligned}$$

Thus (†) is satisfied. We now show (‡) is satisfied. First, the case $m = 1$. Clearly $e_2 = y^0_0[0] \in L_{22}(1)$, and so are

$$(b^*,1,0) = y^2_2[1] = -y^3_4[1] - y^2_2[0], \quad (b^*,0,1) = y^1_2[1] = -y^2_4[1] - y^1_2[0],$$

$$(a^*,1,0) = y^1_2[0], \quad (a^*,0,1) = y^0_2[0], \quad (a,1,1) = y^3_4[1], \quad (b,1,1) = y^1_4[1].$$

Assuming the claim has been proved for $m-1$, we must show that each element of $P_{22}(m) \setminus P_{22}(m-1)$ lies in $KL_{22}(m)$. Again, this splits into six cases.

$$\text{For } r \le m, \quad (a,r,m) = y^{r+2m}_{2r+2m}[m] = \sum_{t=0}^{m-r} \lambda_t y^{r+2m+t}_{2r+2m+2t}[k_{2r+2m+2t}],$$

$$\text{For } s < m, \quad (a,m,s) = y^{m+2s}_{2m+2s}[s] = \sum_{t=0}^{m-s-1} \lambda_t y^{m+2s+t}_{2m+2s+2t}[k_{2m+2s+2t}],$$

$$\text{For } r \le m, \quad (b,r,m) = y^{m}_{2r+2m}[m] = \sum_{t=0}^{m-r} \lambda_t y^{m+t}_{2r+2m+2t}[k_{2r+2m+2t}],$$

$$\text{For } s < m, \quad (b,m,s) = y^{s}_{2m+2s}[s] = \sum_{t=0}^{m-s-1} \lambda_t y^{2s+t}_{2m+2s+2t}[k_{2m+2s+2t}],$$

$$\text{For } r \le m, \quad (a^*,r,m-r) = y^{r}_{2m}[0] = \sum_{t=0}^{m-1} \lambda_t y^{r+t}_{2m+2t}[k_{2m+2t}],$$

$$\text{For } r \le m, \quad (b^*,r,m-r) = y^{m+r}_{2m}[m] = \sum_{t=0}^{m} \lambda_t y^{m+r+t}_{2m+2t}[k_{2m+2t}].$$

Thus (‡) is satisfied.

Part 3: $u = 1$**,** $v = 1$**.**

We write (c,r,s) for ${}_1(c,r,s)_1$. Given an integer $m \ge 1$, let

$$P_{11}(m) = \{(c,r,s) \in P_{12} : r,s \le m \text{ if } c = a,b \text{ , } r+s \le m \text{ if } c = a^*, b^*\},$$

$$L_{11} = \{x^i_{2j}[k_{2j}] : j \le 2m, j-m+1 \le i \le j+m+1\}.$$

We verify (†). The calculation of the number of elements of $P_{11}(m)$ is virtually identical to the one for $P_{22}(m)$, and has the same answer, $|P_{11}(m)| = 3m^2 + 3m + 1$.

We compute the number of elements of $L_{11}(m)$.

$$
\begin{aligned}
|L_{11}(m)| &= |\{y^i_{2j}[k_{2j+1}] : j \le 2m, j-m+1 \le i \le j+m+1\}| \\
&= 1 + |\{(i, 2j+1) \in \mathbb{N}^2 : 1 \le j \le 2m, \\
&\qquad j-m+1 \le i \le j+m+1, 1 \le i \le 2j+1\}| \\
&= 1 + |\{(i, 2j+1) \in \mathbb{N}^2 : 1 \le j \le m, 1 \le i \le 2j+1\}| \\
&\quad + |\{(i, 2j+1) \in \mathbb{N}^2 : m < j \le 2m, j-m+1 \le i \le j+m+1\}| \\
&= 1 + \sum_{j=1}^{m} 2j+1 + m(2m+1) \\
&= 1 + m(m+1) + m + 2m^2 + m \\
&= 3m^2 + 3m + 1.
\end{aligned}
$$

Thus (†) is satisfied. We now show (‡) is satisfied. First, the case $m = 1$. Clearly $e_1 = x^1_1[0] \in L_{11}(1)$, and so are

$$
(b^*, 1, 0) = x^3_3[1] = -x^4_5[1] - x^3_3[0], \quad (b^*, 0, 1) = y^2_3[1] = -y^3_5[1] - y^2_3[0],
$$

$$
(a^*, 1, 0) = y^2_3[0], \quad (a^*, 0, 1) = y^1_3[0], \quad (a, 1, 1) = y^4_5[1], \quad (b, 1, 1) = y^2_5[1].
$$

Assuming the claim has been proved for $m-1$, we can show it holds for m by a similar process to the previous parts by splitting into six cases, and using Lemmas 5.1.12 and 5.1.14.

$$
\begin{aligned}
&\text{For } r \le m, \quad (a, r, m) = x^{r+2m+1}_{2r+2m+1}[m] = \sum_{t=0}^{m-r} \lambda_t x^{r+2m+t+1}_{2r+2m+2t+1}[k_{2r+2m+2t+1}], \\
&\text{For } s < m, \quad (a, m, s) = x^{m+2s+1}_{2m+2s+1}[s] = \sum_{t=0}^{m-s-1} \lambda_t x^{m+2s+t+1}_{2m+2s+2t+1}[k_{2m+2s+2t+1}], \\
&\text{For } r \le m, \quad (b, r, m) = x^{m+1}_{2r+2m+1}[m] = \sum_{t=0}^{m-r} \lambda_t x^{m+t+1}_{2r+2m+2t+1}[k_{2r+2m+2t+1}], \\
&\text{For } s < m, \quad (b, m, s) = x^{s+1}_{2m+2s+1}[s] = \sum_{t=0}^{m-s-1} \lambda_t x^{2s+t+1}_{2m+2s+2t+1}[k_{2m+2s+2t+1}], \\
&\text{For } r \le m, \quad (a^*, r, m-r) = x^{r+1}_{2m+1}[0] = \sum_{t=0}^{m-1} \lambda_t x^{r+t+1}_{2m+2t+1}[k_{2m+2t+1}], \\
&\text{For } r \le m, \quad (b^*, r, m-r) = x^{m+r+1}_{2m+1}[m] = \sum_{t=0}^{m} \lambda_t x^{m+r+t+1}_{2m+2t+1}[k_{2m+2t+1}].
\end{aligned}
$$

Thus (‡) is satisfied.

Part 4: $u = 2$**,** $v = 1$**.**
We write (c, r, s) for ${}_2(c, r, s)_1$. Recall that if $c = a^*, b^*$ then r, s cannot both be zero. Given an integer $m \geq 1$, let

$$P_{21}(m) = \{(c, r, s) \in P_{12} : r, s \leq m \text{ if } c = a, b\ , r + s \leq m + 1 \text{ if } c = a^*, b^*\},$$

$$L_{21} = \{x^i_{2j}[k_{2j}] : j \leq 2m, j - m \leq i \leq j + m + 1\}.$$

We verify (†). For convenience, set $P_{21}(0) = \{a, b\}$. For $n \geq 1$, we claim that the number of distinct elements of $P_{21}(n) \setminus P_{21}(n-1)$ is $6n+2$. If $n = 1$, they are $(a, 0, 1) = (b^*, 2, 0)$, $(b, 0, 1) = (b^*, 0, 2)$, $(a, 1, 0) = (a^*, 2, 0)$, $(b, 1, 0) = (a^*, 0, 2)$, $(a, 1, 1)$, $(b, 1, 1)$, $(a^*, 1, 1)$, $(b^*, 1, 1)$. For $n > 1$ there are $2(2n+1)+2n = 6n+2$ distinct elements, namely

$$(a, n, 0), (a, n, 1), \ldots, (a, n, n-1), (a, n, n), (a, n-1, n), \ldots, (a, 1, n), (a, 0, n),$$

$$(b, n, 0), (b, n, 1), \ldots, (b, n, n-1), (b, n, n), (b, n-1, n), \ldots, (b, 1, n), (b, 0, n),$$

$$(a^*, 1, n), (a^*, 2, n-1), \ldots, (a^*, n, 1),$$

$$(b^*, 1, n), (b^*, 2, n-1), \ldots, (b^*, n, 1).$$

There are no more, since $(a^*, 0, n+1) = (b, n, 0)$, $(b^*, 0, n+1) = (b, 0, n)$, $(a^*, n+1, 0) = (a, n, 0)$, $(b^*, n+1, 0) = (a, 0, n)$. So

$$|P_{21}(m)| = 2+\sum_{n=1}^{m} 6n+2 = 1+6\sum_{n=1}^{m} n+2m = 2+3m(m+1)+2m = 3m^2+5m+2.$$

We compute the number of elements of $L_{21}(m)$.

$$\begin{aligned}
|L_{21}(m)| &= |\{y^i_{2j}[k_{2j+1}] : j \leq 2m, j-m \leq i \leq j+m+1\}| \\
&= 2 + |\{(i, 2j+1) \in \mathbb{N}^2 : 1 \leq j \leq 2m, \\
&\qquad j-m \leq i \leq j+m+1, 0 \leq i \leq 2j+1\}| \\
&= 2 + |\{(i, 2j+1) \in \mathbb{N}^2 : 1 \leq j \leq m, 0 \leq i \leq 2j+1\}| \\
&\quad + |\{(i, 2j+1) \in \mathbb{N}^2 : m < j \leq 2m, j-m \leq i \leq j+m+1\}| \\
&= 2 + \sum_{j=1}^{m} 2j + 2 + m(2m+2) \\
&= 2 + m(m+1) + 2m + 2m^2 + 2m \\
&= 3m^2 + 5m + 2.
\end{aligned}$$

Thus (†) is satisfied. We now show (‡) is satisfied. First, the case $m = 1$.

$$(a, 0, 0) = a = y^1_1[0], \quad (b, 0, 0) = b = y^0_1[0],$$

$$(a, 0, 1) = y^3_3[1] = -y^4_5[1] - y^3_3[0], \quad (b, 0, 1) = y^1_3[1] = -y^2_5[1] - y^1_3[0],$$

$$(a, 1, 0) = y^2_3[0], \quad (b, 1, 0) = y^0_3[0], \quad (a, 1, 1) = y^4_5[1], \quad (b, 1, 1) = y^1_5[1],$$

$$(a^*, 1, 1) = y^1_3[0], \quad (b^*, 1, 1) = y^2_3[1] = -y^3_5[1] - y^2_3[0],$$

each of which lie in $L_{21}(1)$. Assuming the claim has been proved for $m-1$, we can show it holds for m by a similar process to the previous parts by splitting into six cases, and using Lemmas 5.1.12 and 5.1.14.

$$\text{For } r \leq m, \quad (a,r,m) = y^{r+2m+1}_{2r+2m+1}[m] = \sum_{t=0}^{m-r} \lambda_t y^{r+2m+t}_{2r+2m+2t}[k_{2r+2m+2t}],$$

$$\text{For } s < m, \quad (a,m,s) = y^{m+2s+1}_{2m+2s+1}[s] = \sum_{t=0}^{m-s-1} \lambda_t y^{m+2s+t}_{2m+2s+2t}[k_{2m+2s+2t}],$$

$$\text{For } r \leq m, \quad (b,r,m) = y^{m}_{2r+2m+1}[m] = \sum_{t=0}^{m-r} \lambda_t y^{m+t}_{2r+2m+2t}[k_{2r+2m+2t}],$$

$$\text{For } s < m, \quad (b,m,s) = y^{s}_{2m+2s+1}[s] = \sum_{t=0}^{m-s-1} \lambda_t y^{2s+t}_{2m+2s+2t}[k_{2m+2s+2t}],$$

$$\text{For } r \leq m+1, \quad (a^*,r,m-r+1) = y^{r}_{2m+1}[0] = \sum_{t=0}^{m-1} \lambda_t y^{r+t}_{2m+2t}[k_{2m+2t}],$$

$$\text{For } r \leq m+1, \quad (b^*,r,m-r+1) = y^{m+r}_{2m+1}[m] = \sum_{t=0}^{m} \lambda_t x^{m+r+t}_{2m+2t}[k_{2m+2t}].$$

Thus (‡) is satisfied. Putting the four parts together completes the proof of Lemma 5.1.13.

We are now finally in a position to complete the proof of Theorem 5.1.4. We have the following description of the preprojective modules for Q, first obtained by Kronecker.

Lemma 5.1.15. *If M in an indecomposable preprojective module for KQ, then M has dimension vector $(n, n+1)$ for some integer $n \geq 0$ and M is isomorphic to the module corresponding to the representation*

$$\begin{tikzcd}[row sep=huge] K^n \arrow[d, shift right=2ex, swap, "\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}"] \arrow[d, shift left=2ex, "\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}"] \\ K^{n+1} \end{tikzcd}$$

Thus, if M is an indecomposable preprojective module, there are bases $\{f_1, \ldots, f_n\}$ for e_1M and $\{g_0, \ldots, g_n\}$ for e_2M so that $af_i = bf_{i+1} = g_i$ for all $i = 0, \ldots, n-1$. The following diagram illustrates this,

$$
\begin{tikzcd}
x_n \arrow[d] \arrow[dr] & x_{n-1} \arrow[d] \arrow[dr] & & \cdots & x_1 \arrow[d] \arrow[dr] & \\
y_n & y_{n-1} & y_{n-2} & & y_1 & y_0
\end{tikzcd}
$$

where the vertical arrows represent multiplication by a, and the diagonal arrows represent multiplication by b.

Write Λ as a representation of Q with respect to the basis given in Lemma 5.1.13. Since $ax_l^i[k_l] = bx_l^{i+1}[k_l] = y_l^i[k_l]$, it decomposes as

$$
\begin{tikzcd}
 & & x_1^1[0] \arrow[dd] \arrow[ddr] & & & x_2^2[0] \arrow[dd] \arrow[ddr] & x_2^1[0] \arrow[dd] \arrow[ddr] & & & \\
 & \bigoplus & & & \bigoplus & & & & \bigoplus & \cdots \\
y_0^0[0] & & y_1^1[0] & y_1^0[0] & & y_2^2[0] & y_2^1[0] & y_2^0[0] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & x_l^l[k_l] \arrow[dd] \arrow[ddr] & x_l^{l-1}[k_l] \arrow[dd] \arrow[ddr] & & & x_l^1[k_l] \arrow[dd] \arrow[ddr] & & & \\
\cdots & \bigoplus & & & & \cdots & & & \bigoplus & \cdots \\
 & & y_l^l[k_l] & y_l^{l-1}[k_l] & y_l^{l-2}[k_l] & & y_l^1[k_l] & y_l^0[k_l] & &
\end{tikzcd}
$$

Clearly this is isomorphic to the direct sum of all the indecomposable preprojectives, with one taken from each isomorphism class.

5.2 Are Λ and Π isomorphic as algebras?

Since we conjecture that Λ and Π are isomorphic as KQ-modules (and so have the same dimension in the Dynkin case), a reasonable question to ask is whether they could be isomorphic as algebras, especially since they obviously are in the case of Q being type A_n. We consider the smallest non trivial case.

Lemma 5.2.1. *If K has characteristic 2, then $\Pi(Q) \ncong \Lambda(Q)$, where Q is the quiver given in Lemma 5.1.3.*

Proof. Suppose that there is an isomorphism $\theta : \Lambda \to \Pi$. In steps (1)-(4), we show that we can modify θ to an isomorphism satisfying increasingly stronger properties, and so we only need show that there are no isomorphisms of the type given in (4). Let $S = Ke_0 + Ke_1 + Ke_2 + Ke_3$.

(1) We can assume that $\theta(S) = S$.

Let $S' = \theta(S)$. Clearly S is a semisimple subalgebra of Π, and as vector spaces $\Pi = S \oplus \operatorname{rad}\Pi$. Similarly, $\Lambda = S \oplus \operatorname{rad}\Lambda$ and therefore $\theta(\Lambda) = \theta(S) \oplus \theta(\operatorname{rad}\Lambda)$, i.e. $\Pi = S' \oplus \operatorname{rad}\Pi$. By the Wedderburn-Malcev theorem, [15, Theorem 6.2.1], there is an invertible element $x \in \Pi$ such that $S = x^{-1}S'x$. Let $\phi : \Pi \to \Pi$ be the automorphism defined by $\phi(y) = x^{-1}yx$. By composing θ with ϕ we obtain an isomorphism $\theta' : \Lambda \to \Pi$ which does satisfy $\theta'(S) = S$.

(2) We can assume that $\theta(e_i) = e_{\sigma(i)}$ for some permutation σ.

Assuming (1), we have that $\theta(e_i) = \sum t_{ij}e_j$ for some scalars t_{ij}. Now it is clear that given i, at least one t_{ij} is nonzero, since otherwise $\theta(e_i) = 0$. Additionally, given j, at most one t_{ij} is nonzero since if both t_{ij} and t_{kj} are nonzero, then

$$0 = \theta(e_ie_k) = \theta(e_i)\theta(e_k) = \left(\sum_r t_{ir}e_r\right)\left(\sum_s t_{ks}e_s\right) = \sum_l t_{il}t_{kl}e_l \neq 0,$$

since $t_{ij}t_{kj} \neq 0$. These two conditions show that there are exactly four nonzero t_{ij}, and thus given i, exactly one t_{ij} is nonzero, (say $j = \sigma(i)$), and given j, exactly one t_{ij} is nonzero, (say $i = \rho(j)$). Clearly σ and ρ are inverses of each other, and so they are permutations, and $\theta(e_i) = t_{i,\sigma(i)}e_{\sigma(i)}$. Since for all i we have $\theta(e_i) = \theta(e_i)^2$, we have $t_{i,\sigma(i)}e_{\sigma(i)} = (t_{i,\sigma(i)})^2e_{\sigma(i)}^2$, hence $t_{i,\sigma(i)} = (t_{i,\sigma(i)})^2$, and therefore $t_{i,\sigma(i)} = 1$. This completes the proof of (2).

(3) We can assume $\theta(e_0) = e_0$.

Suppose otherwise, i.e. $\theta(e_0) = e_i$, where $i = 1, 2, 3$. Then $\theta(e_0 \Lambda e_0) = e_i \Pi e_i$. However this is impossible, as $\dim e_0 \Lambda e_0 = 10$ and $\dim e_i \Pi e_i = 6$ for $i \neq 0$ (see Lemma 5.1.3).

(4) We can assume that $\theta(e_i) = e_i$ for all i.

Given any permutation ρ of $\{0, 1, 2, 3\}$ which sends 0 to 0, there is an automorphism of Π which sends e_i to $e_{\rho(i)}$. Apply this with $\rho = \sigma^{-1}$.

It therefore suffices to prove there is no isomorphism which satisfies (4), so assume θ to be such an isomorphism. We have $\theta(a) = \theta(e_0)\theta(a)\theta(e_1) = e_0\theta(a)e_1$, that is, $\theta(a)$ is a linear combination of paths from 1 to 0. By Lemma 5.1.3, we see that $e_1 \Pi e_0$ is a 2 dimensional space with basis $\{a, bb^*a\}$, and so $\theta(a) = \lambda_a a + \mu_a bb^*a$ for some scalars λ_a, μ_a. Similarly, $\theta(a^*) = \lambda_{a^*} a^* + \mu_{a^*} a^*bb^*$, $\theta(b) = \lambda_b b + \mu_b aa^*b$, $\theta(b^*) = \lambda_{b^*} b^* + \mu_{b^*} b^*aa^*$, $\theta(c) = \lambda_c c + \mu_c aa^*c$, $\theta(c^*) = \lambda_{c^*} c^* + \mu_{c^*} c^*aa^*$. Note that the λ scalars are all non zero, since otherwise θ is not surjective.

Now $0 = \theta(a^*a) = \theta(a^*)\theta(a) = \lambda_{a^*}\lambda_a a^*a + \mu_{a^*}\lambda_a a^*bb^*a + \lambda_{a^*}\mu_a a^*bb^*a + \mu_{a^*}\mu_a a^*bb^*bb^*a = (\mu_{a^*}\lambda_a + \lambda_{a^*}\mu_a)a^*bb^*a$ (since the other terms are equal to 0 in Π). So $\mu_{a^*}\lambda_a + \lambda_{a^*}\mu_a = 0$ (since a^*bb^*a is not zero in Π). Similarly $\mu_{b^*}\lambda_b + \lambda_{b^*}\mu_b = 0$ and $\mu_{c^*}\lambda_c + \lambda_{c^*}\mu_c = 0$.

Finally, $0 = \theta(aa^* + bb^* + cc^* + aa^*bb^*) = \lambda_a\lambda_{a^*}aa^* + \lambda_a\mu_{a^*}aa^*bb^* + \mu_a\lambda_{a^*}bb^*aa^* + \lambda_b\lambda_{b^*}bb^* + \lambda_b\mu_{b^*}bb^*aa^* + \mu_b\lambda_{b^*}aa^*bb^* + \lambda_c\lambda_{c^*}cc^* + \lambda_c\mu_{c^*}cc^*aa^* + \mu_c\lambda_{c^*}aa^*cc^* + \lambda_a\lambda_{a^*}\lambda_b\lambda_{b^*}aa^*bb^* = (\lambda_a\lambda_{a^*} - \lambda_c\lambda_{c^*})aa^* + (\lambda_b\lambda_{b^*} - \lambda_c\lambda_{c^*})bb^* + (2(\lambda_a\mu_{a^*} - \lambda_b\mu_{b^*} + \lambda_c\mu_{c^*}) + \lambda_a\lambda_{a^*}\lambda_b\lambda_{b^*})aa^*bb^*$ (using the formulas obtained in the previous paragraph, and the reduction formulas obtained in Lemma 5.1.3). Since aa^*, bb^*, aa^*bb^* are independent in Π, the coefficients must be zero. In particular, since K has characteristic 2, $\lambda_a\lambda_{a^*}\lambda_b\lambda_{b^*} = 0$, which is impossible. $\square$

Of course, if K does not have characteristic 2, then one can use this proof to construct an isomorphism, e.g. define $\theta : K\overline{Q} \to K\overline{Q}$ to be the map which sends a to $a + \frac{1}{2}bb^*a$, a^* to $a^* - \frac{1}{2}a^*bb^*$ and each remaining arrow (and each trivial path) to itself. Then one can check that $\theta(\mu^1) = \rho^0$ and thus θ induces a map $\Lambda \to \Pi$. It is easy to construct an inverse, e.g. the map which sends a to $a - \frac{1}{2}bb^*a$ and a^* to $a^* + \frac{1}{2}a^*bb^*$, and each remaining arrow to itself. However, this example doesn't really suggest how this question can be answered in general, because it is not practical to attempt this analysis for the larger Dynkin quivers.

The question is quite interesting to ask for the quivers of type $\tilde{A}_n$. If we orient the quiver cyclically, then it is easy to see that Π is a subalgebra of Λ. Namely, construct Λ as described in Section 2.1, and observe that $\rho = \sum_{a \in \overline{Q}_1} \epsilon(a)aa^*$ is zero. Therefore the natural map $K\overline{Q} \to \Lambda$ induces an inclusion $\theta : \Pi \to \Lambda$ because ρ is sent to zero. This is not an isomorphism, because the representation X of $\overline{Q}$ with $X_v = K$ for all v, $X_a = 1$, $X_{a^*} = -1$ for all $a \in Q_1$ is a representation of Π, but not of Λ because each $1+X_aX_{a^*}$ is zero and is therefore not invertible. Thus the image of θ cannot contain l_a. Strictly, this does not show that Π is not isomorphic to Λ as algebras, since we have not shown that no isomorphism exists, only that θ is not an isomorphism.

5.3 Other questions

We list some other questions, whose answers may turn out to be of interest.

1. When is $\Lambda^q(Q) \cong \Lambda^{q'}(Q)$? We do have the following theorem, which may suggest an answer to this question. It could have been placed in Chapter 2, as it follows immediately from Theorem 2.3.1.

Theorem 5.3.1. *$\Lambda^q(Q)$ is Morita equivalent to $\Lambda^{q'}$ for all $q' \in Wq$.*

2. Is there any significance to the numbers $\dim(\Lambda^1(Q)_{\leq i}/\Lambda^1(Q)_{\leq i-1})$? Note that $\Lambda^1(Q)_{\leq i} = \sum_i \Lambda^1(Q)_i$ where $\Lambda^1(Q)_i$ is the span of the paths of degree i using the oriented grading on $K\overline{Q}$. The reason why these numbers may be of interest is that the corresponding numbers for Π are equal to the dimension of the module $\tau^{-i}(KQ)$. The numbers for Λ will be different (provided Q does not have type A_n), e.g. if Q is the quiver of type D_4 given in 5.1.3, then $\dim(e_0\Lambda_{\leq 1}e_0/e_0\Lambda_{\leq 0}e_0)$ is equal to 3 (because aa^*, bb^* cc^* are independent), but $\dim(e_0\Pi_{\leq 1}e_0/e_0\Pi_{\leq 0}e_0)$ is 2.

3. One can define a class of algebras $K\overline{Q}/I_{\rho^x}$, where $\rho^x = \sum_{a\in\overline{Q}_1} \epsilon(a)aa^* + x$ and x is a linear combination of paths formed by composing paths two or more paths of the form aa^*. One can ask whether such algebras are isomorphic to Π, or to Λ (Λ being one special case). This has been explored in the Dynkin case in [5].

4. It may be interesting to consider the intersection between the ideals I_ρ and I_μ. We were surprised during our calculations how often the elements of I_μ were homogeneous, and that these homogeneous elements were also elements of I_ρ (e.g. the c_i in obtained in Lemma 5.1.5).

Chapter 6

Preprojective algebras for quivers with relations

The purpose of this chapter is to determine whether the construction of the preprojective algebra (that is, an algebra satisfying the preprojective property) can be generalised to algebras given by quivers with relations. The algebras investigated are those arising from 'pairings'. Such algebras can have both finite and infinite representation type, but our results only apply in the finite type case. We use the convention that all modules are preprojective (e.g. we can use the definition of a preprojective module given by Auslander and Smalø, [2]), so that a 'preprojective algebra' $P(A)$ of an algebra A should have A as a subalgebra, and should decompose as the direct sum of all indecomposable modules for A, one from each isomorphism class. It would be desirable to obtain some results for algebras of infinite representation type, but this appears to be difficult.

In Section 6.1, we introduce the notion of a 'pairing' for a quiver, and show that a quiver Q equipped with a pairing Σ gives rise to a new quiver Q^{Σ} with relations (and hence an algebra A). We then can use the preprojective algebra for Q to construct an algebra $\Pi(Q, \Sigma)$. In Section 6.2, we conjecture that if Σ is a certain type of pairing (an 'end pairing'), then $\Pi(Q, \Sigma)$ satisfies the preprojective property for A (provided A has finite type). After giving a counterexample to show that this is not true for all pairings, we prove that the conjecture holds in a special case (Sections 6.3 and 6.4). Finally, in Section 6.5, we show that this result is sufficient to show that for any Nakayama algebra A, there exists an algebra satisfying the preprojective property for A.

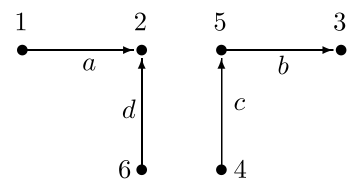
6.1 Pairings

Definition 6.1.1. A *pairing* Σ of a quiver Q is a triple (Q', Q'', σ) where Q', Q'' are full subquivers of Q, and $\sigma : Q'' \to Q'$ is an isomorphism.

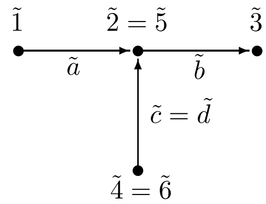
We write $v_1 \sim v_2$ if $v_1 = \sigma(v_2)$ and extend $\sim$ to an equivalence relation on Q_0 (i.e. $u \sim v$ if and only if there is a sequence $u = v_1, v_2, \ldots, v_k = w$ with either $v_i = \sigma(v_{i+1})$ for all i or $v_i = \sigma(v_{i-1})$ for all i). We define an equivalence relation on Q_1 in the same way. Clearly if $a_1 \sim a_2$, then $h(a_1) \sim h(a_2)$ and $t(a_1) \sim t(a_2)$. This fact enables us to make the following definition.

Definition 6.1.2. We define the *glued quiver* Q^{Σ} to be the quiver with vertex set $Q_0/\sim$ and arrows $Q_1/\sim$, with $h(\tilde{a}) = \widetilde{h(a)}$, and $t(\tilde{a}) = \widetilde{t(a)}$.

Example 6.1.3. Let Q be the quiver



and set $Q'_0 = \{2, 6\}$ and $Q''_0 = \{5, 4\}$. The corresponding glued quiver is



Given a quiver Q and a pairing Σ, there is an induced pairing $\overline{\Sigma} = (\overline{Q'}, \overline{Q''}, \overline{\sigma})$ on $\overline{Q}$, where $\overline{\sigma}$ is the extension of σ obtained by defining $\overline{\sigma}(a^*)$ to be $(\sigma(a))^*$.

It is clear that the quiver $\overline{Q}^{\overline{\Sigma}}$ may be identified with $\overline{Q^\Sigma}$. If we denote the set of paths of $\overline{Q}$ by P, and the set of paths of $\overline{Q^\Sigma}$ by P^Σ, then there is a map $\eta : P \to P^\Sigma$ which takes a path $a_n \dots a_1$ to $\tilde{a}_n \dots \tilde{a}_1$ and a trivial path e_v to $e_{\tilde{v}}$. For $\mu \in P^\Sigma$ we define

$$\overline{\theta}(\mu) = \sum_{q \in \eta^{-1}(\mu)} q,$$

and extend to a vector space homomorphism $K\overline{Q^\Sigma} \to K\overline{Q}$ (note that if $\mu \notin \operatorname{Im}\eta$, then the sum is taken to be zero).

Lemma 6.1.4. *$\overline{\theta}$ is an algebra homomorphism.*

Proof. We need to check that $\overline{\theta}(\mu_1\mu_2) = \overline{\theta}(\mu_1)\overline{\theta}(\mu_2)$ for all $\mu_1, \mu_2 \in P^\Sigma$. This follows from the fact that if $\mu_1, \mu_2 \in P^\Sigma$ with $h(\mu_2) = t(\mu_1)$, then

$$\eta^{-1}(\mu_1\mu_2) = \{q_1q_2 \in P : q_1 \in \eta^{-1}(\mu_1), q_2 \in \eta^{-1}(\mu_2)\}.$$

If $h(\mu_2) \neq t(\mu_1)$, then $h(q_2) \not\sim t(q_1)$ for all $q_1 \in \eta^{-1}(\mu_1)$ and $q_2 \in \eta^{-1}(\mu_2)$, and so $\overline{\theta}(\mu_1)\overline{\theta}(\mu_2) = 0 = \overline{\theta}(\mu_1\mu_2)$. Finally it is clear that $\overline{\theta}(1_{K\overline{Q^\Sigma}}) = \overline{\theta}(\sum_{\tilde{v} \in Q_0^\Sigma} e_{\tilde{v}}) = \sum_{v \in Q_0} e_v = 1_{K\overline{Q}}$. ∎

Clearly one can restrict $\overline{\theta}$ to an algebra homomorphism $\theta : KQ^\Sigma \to KQ$. Let $I^\Sigma = \operatorname{Ker}\theta$. It is the ideal of KQ^Σ generated by the paths of Q^Σ not in $\operatorname{Im}\eta$ (so in the example I^Σ is generated by $\tilde{b}\tilde{a}$). The algebra KQ^Σ/I^Σ can be embedded in KQ, namely, there is a map $\phi : KQ^\Sigma/I^\Sigma \hookrightarrow KQ$ induced from θ.

Definition 6.1.5. Given a quiver Q with a pairing Σ, we denote the algebra $\operatorname{Im}\pi\overline{\theta}$ as $\Pi(Q, \Sigma)$ (recall that π denotes the natural surjection $K\overline{Q} \to \Pi(Q)$).

Clearly $\Pi(Q, \Sigma)$ inherits an oriented grading from the oriented grading on $\Pi(Q)$. It is clear that

$$\pi\overline{\theta}\left(\sum_{a \in \overline{Q^\Sigma}_1} \epsilon(a)aa^*\right) = \pi\left(\sum_{a \in \overline{Q}_1} \epsilon(a)aa^*\right) = 0,$$

so $\Pi(Q, \Sigma)$ is a quotient of $\Pi(Q^\Sigma)$.

Lemma 6.1.6. *KQ^Σ/I^Σ is a subalgebra of $\Pi(Q, \Sigma)$.*

Proof. The composition $\xi : KQ^{\Sigma} \xrightarrow{\theta} KQ \hookrightarrow K\overline{Q} \twoheadrightarrow \Pi(Q)$ maps into $\Pi(Q,\Sigma)$. Clearly $\operatorname{Ker}\theta \subseteq \operatorname{Ker}\xi$, and if $x \in \operatorname{Ker}\xi$, then $\theta(x) \in \operatorname{Ker}\pi \cap KQ$, so $\theta(x) = 0$. So $\operatorname{Ker}\xi = \operatorname{Ker}\theta = I^{\Sigma}$, and there is an induced injective algebra homomorphism $KQ^{\Sigma}/I^{\Sigma} \to \Pi(Q,\Sigma)$. □

Thus $\Pi(Q,\Sigma)$ has a natural KQ^{Σ}/I^{Σ}-module structure.

6.2 The main theorem

Given vertices u, v of a quiver, if there is a path from u to v, then u is said to be a predecessor of v, and v a successor of u.

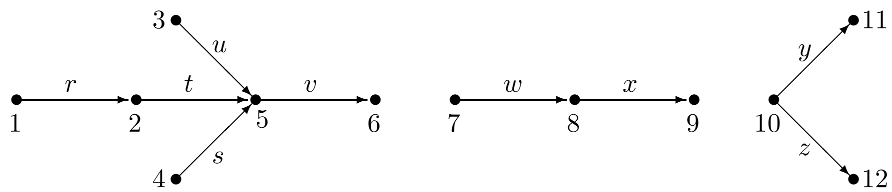
Definition 6.2.1. A pairing of a quiver Q is an *end pairing* if and only if

(1) There are non source vertices $u_1, u_2, \ldots, u_l \in Q_0$ such that Q' is the full subquiver of Q with vertex set consisting of all successors of the u_i, and if $a : u \to v$ is an arrow of Q which is incident with a vertex of Q', then either $a \in Q'_1$ or $v = u_i$ for some i.

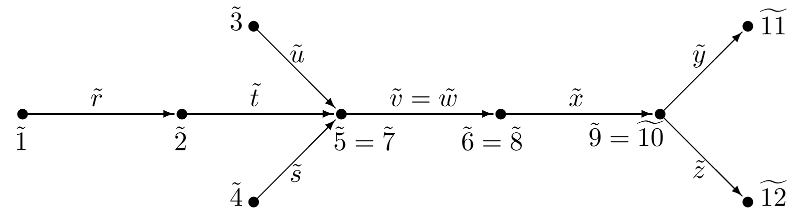
(2) There are non sink vertices $w_1, w_2, \ldots, w_m \in Q_0$ such that Q'' is the full subquiver of Q with vertex set consisting of all predecessors of the w_i, and if $a : u \to v$ is an arrow of Q which is incident with a vertex of Q'', then either $a \in Q''_1$ or $u = w_i$ for some i.

In particular, we see that a vertex with no successors in Q' (i.e. a sink in Q') cannot have any successors in Q (so is a sink in Q), whereas a source in Q' cannot be a source in Q because the only sources in Q' are the u_i. The same is true for Q'', but the other way round.

Example 6.2.2. (1) Let Q be the quiver



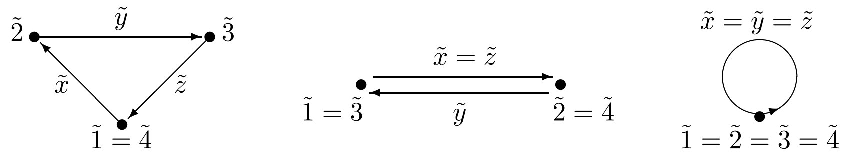
One possible end pairing is determined by setting $u_1 = 5$, $u_2 = 9$, $w_1 = 8$, $w_2 = 10$ and then $Q_0' = \{5, 6, 9\}$ and $Q_0'' = \{7, 8, 10\}$. The corresponding glued quiver is



(2) Let Q be the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "x"] & \underset{2}{\bullet} \arrow[r, "y"] & \underset{3}{\bullet} \arrow[r, "z"] & \underset{4}{\bullet} \end{tikzcd}$$

Since Q' cannot contain the source vertex 1, and is closed under successors, we must have either $Q_0' = \{4\}$ or $\{3, 4\}$ or $\{2, 3, 4\}$. The corresponding Q_0'' are $\{1\}$ or $\{1, 2\}$ or $\{1, 2, 3\}$. Note that Q' and Q'' may intersect, so the third case is allowed. The corresponding glued quivers are



(3) We can see by inspection that the pairing given in Example 6.1.3 is not an end pairing, since there is a source 6 in Q' which is also a source in Q, which is impossible.

Conjecture 6.2.3. *Let Q be a quiver with an end pairing Σ, and denote KQ^{Σ}/I^{Σ} by A. If A has finite representation type then $\Pi(Q, \Sigma)$ has the preprojective property for A.*

It is almost certainly the case that the conjecture can be extended in some way to the case of A being infinite representation type. Clearly one would have to instead use the category of 'preprojective' A-modules, but then the problem arises that there are several different definitions of what a preprojective module for a non-hereditary algebra should be, and it is unclear which we should be using. The definition given by Auslander and Smalø works well enough in the finite type case (in the sense that all modules are preprojective), but if this were the correct definition to use, one might expect that $(\Pi(Q,\Sigma))_i$ would be the direct sum of the modules in the i-th component of the preprojective partition, but there are many examples to show this is not the case (e.g., one can take a quiver of type D_4 with the empty pairing, and then $\Pi(Q,\Sigma)$ is the ordinary preprojective algebra, which fails to satisfy this condition).

We are able to prove the conjecture in the following special case.

Theorem 6.2.4. *Let Q be a quiver with an end pairing Σ, such that the connected components Q'_i of Q' have Dynkin type A_{n_i}, oriented to have exactly one source and one sink. If $A = KQ^\Sigma/I^\Sigma$ has finite representation type then $\Pi(Q,\Sigma)$ has the preprojective property for A.*

We give an outline of the proof of the theorem, which relies on proving two lemmas.

Label the vertices of each connected component Q'_i of Q' as $u^i_1,\dots,u^i_{n_i}$, and arrows $b^i_1,\dots,b^i_{n_i-1}$ so that $t(b^i_j)=u^i_j$ and $h(b^i_j)=u^i_{j+1}$. We define a function $d: Q_0\to\mathbb{N}$ by

$$d(v)=\begin{cases} l & \text{if } v=u^i_l \text{ for some } i\\ 0 & \text{otherwise}\end{cases}$$

We label the vertices of Q'' in the same way as Q', but use 'w' and 'c' instead of 'u' and 'b'. Note that some vertices/arrows of Q may have two labels, although the situations in which this may occur are quite restricted. That is, we claim that if $v=w^i_m=u^j_n$, then v must be contained in a component $\dot{Q}$ of Q of type A_k for some k, and $m>n$. To see this, first note that $\dot{Q}$ clearly contains the predecessors of v in Q (which are the predecessors of v in Q'', the w^i_l for $l<m$) and the successors of v in Q (which are the successors of v in Q', the u^j_l for $l>n$). Therefore $\dot{Q}$ contains the following subquiver, and there can be no arrow with head at w_l^i with $l \leq m$ or with tail at u_l^j with $l \geq n$ other than those included in the diagram.

$$\begin{tikzcd}
w_1^i \arrow[r] & w_2^i & \cdots & w_{m-1}^i \arrow[r] & \overset{w_m^i}{\underset{u_n^j}{\bullet}} \arrow[r] & u_{n+1}^j & \cdots & u_{n_j-1}^j \arrow[r] & u_{n_j}^j
\end{tikzcd}$$

In fact Q is must be equal to this subquiver. Any arrow of Q with tail at w_l^i with $l < m$ would, using the last part of property (2) of end pairings, have to be in Q'' (i.e. must be the arrow already in the diagram, c_l^i). Similarly there can be no arrows with head at u_l^j for $l > n$. One must have $b_{n+k}^j = c_{m+k}^i$, $u_{n+k}^j = w_{m+k}^i$ for all $k \in \mathbb{Z}$ which make sense, and if $n \leq m$, then Q' contains $u_{m-n+1}^j = w_1^i$, which is a source in Q, a contradiction. Note that in this situation, there is exactly one arrow of Q (namely c_{m-n}^i) with head at u_1^i, and conversely if more than one arrow of Q ends at u_1^i, then Q_i' is disjoint from Q''.

Lemma 6.2.5. *(Main Lemma 1.) Let $k \geq 0$, $\nu \in Q_0^\Sigma$. As A-modules,*

$$\Pi(Q,\Sigma)_k e_\nu \cong \bigoplus_{\substack{v \in \nu \\ d(v) \leq k}} \Pi(Q)_k e_v,$$

where the module structure on the right hand side is restriction via ϕ of the natural KQ-module structure.

The proof of this lemma is done in Section 6.3.

Now we relate the category of A-modules with the category of KQ-modules, using the process described in [27]. Via the embedding $\phi : A \hookrightarrow KQ$, any KQ-module becomes an A-module by restriction. We can describe this in terms of a functor $F : \operatorname{Rep} Q \to \operatorname{Rep} A$. Given a representation X of Q, let $F(X)$ be the representation Y of Q^Σ, where $Y_{\tilde{v}} = \oplus_{u \in \tilde{v}} X_u$ and if $\tilde{a} : \tilde{v} \to \tilde{w} \in Q_1^\Sigma$ then for $x \in X_u$ where $u \in \tilde{v}$, define

$$Y_{\tilde{a}}(x) = \begin{cases} X_b(x) & \text{if } b \in \tilde{a} \text{ with } h(b) = u. \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is well defined as if b exists, it is unique.

Lemma 6.2.6. *(Main Lemma 2.) F induces a bijection from the indecomposable representations X of Q with $X_v \neq 0$ for some $v \notin Q'_0$ to the indecomposable representations of A.*

The proof of this lemma is done in Section 6.4. Assuming that Theorem 6.2.4 is proved, this lemma gives rise to the formula $\dim(\Pi(Q,\Sigma)) = \dim(\Pi(Q)) - \dim(\Pi(Q'))$ (by applying Theorem 1.3.4).

Assuming the two Main Lemmas are proved, we prove Theorem 6.2.4 as follows (this only proves the left modules part of the preprojective property, but it should be easy to see that the every part of the proof can be done in the same way with right modules). Using Main Lemma 1, and the fact that $\Pi(Q,\Sigma)$ is graded, we have that as A-modules

$$\Pi(Q,\Sigma) \cong \bigoplus_{\substack{\nu\in Q_0^\Sigma \\ k\geq 0}} \Pi(Q,\Sigma)_k e_\nu \cong \bigoplus_{\substack{v\in Q_0 \\ k\geq d(v)}} \Pi(Q)_k e_v.$$

In view of Theorem 1.3.4, this is equivalent to

$$\Pi(Q,\Sigma) \cong \bigoplus_{M\in Z\setminus Z'} F(M),$$

where Z is a set of representatives for the category of representations of Q, and Z' is the subset of Z consisting of those representations of Q which correspond to $\Pi(Q)_k e_v$, with $k < d(v)$. We claim that $M \in Z'$ if and only if $M_u = 0$ for all $u \notin Q'_0$, and then Theorem 6.2.4 follows from Main Lemma 2.

Clearly, if $M \in Z'$, then M corresponds to some $\Pi(Q)_k e_v$ with $k < d(v)$. If $u \notin Q'_0$, then any path from v to u most contain at least $d(v)$ arrows in Q_1^*, and thus $e_u\Pi(Q)_k e_v = 0$, and so $M_u = 0$. Conversely, if $M \in Z\setminus Z'$, then M corresponds to some nonzero $\Pi(Q)_k e_v$ with $k \geq d(v)$. That is, there is some nonzero path p of degree k starting at some $u^i_{d(v)}$ for which $\pi(p) \neq 0$. By 'normalising', we can assume that p can be written as $p = q(b_1^i)^* b_1^i r$, where r is the shortest path from $u^i_{d(v)}$ to u^i_1, (see Lemma 6.3.5 for the details, but this type of calculation should be familiar from previous chapters). Using property

(1) of end pairings and the preprojective relation of u_1^i, we have

$$\pi((b_1^i)^* b_1^i) = \sum_{\substack{a \in Q_1 \\ h(a) = u_1^i}} \pi(aa^*).$$

Since $\pi(p) \neq 0$, there is some arrow $a \in Q_1$ with $h(a) = u_1^i$ such that $\pi(aa^*r) \neq 0$. Setting $u = t(a)$, (which not in Q') then $e_u \Pi(Q) e_v \neq 0$, and thus $M_u \neq 0$.

If Q and Σ satisfy Theorem 6.2.4, then it is reasonable to call $\Pi(Q, \Sigma)$ a preprojective algebra for $A = KQ^{\Sigma}/I^{\Sigma}$. Of course, we would like to say that $\Pi(Q, \Sigma)$ is *the* preprojective algebra for A, but it is possible that A may be obtained from more than one quiver and pairing, and would consequently have more than one preprojective algebra. However we make the following conjecture, which (if true) would eliminate this problem.

Conjecture 6.2.7. *If $\dot{Q}$ and $\ddot{Q}$ are quivers with end pairings $\dot{\Sigma}$ and $\ddot{\Sigma}$ respectively, such that $K\dot{Q}^{\dot{\Sigma}}/I^{\dot{\Sigma}} \cong K\ddot{Q}^{\ddot{\Sigma}}/I^{\ddot{\Sigma}}$, then the pairings are isomorphic (i.e. $\dot{Q} \cong \ddot{Q}$ via an isomorphism which respects $\dot{\sigma}, \ddot{\sigma}$), and so $\Pi(\dot{Q}, \dot{\Sigma}) \cong \Pi(\ddot{Q}, \ddot{\Sigma})$.*

Unfortunately, we are unable to prove this conjecture (even in the case where Theorem 6.2.4 applies).

To end this section, we show that the conjectures cannot be extended to apply to all pairings. Let $\dot{Q}$ and $\dot{\Sigma}$ be the quiver and pairing given in Example 6.1.3, and let $A \cong K\dot{Q}^{\dot{\Sigma}}/I^{\dot{\Sigma}}$. Let $\ddot{Q}$ be the quiver

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[r, "a"'] & \overset{2}{\bullet} & \overset{5}{\bullet} \arrow[r, "b"'] & \overset{3}{\bullet} \\
 & & \bullet_{4} \arrow[u, "c"] &
\end{tikzcd}$$

and $\ddot{\Sigma}$ be the pairing determined by setting $Q_0' = \{2\}$, $Q_0'' = \{5\}$ (which is not an end pairing because 5 is a source in Q'' which is not a source in Q). Then $A = K\ddot{Q}^{\Sigma}/I^{\ddot{\Sigma}}$, which shows that Conjecture 6.2.7 does not hold if 'end pairing' is replaced by 'pairing'.

One can calculate using the Auslander-Reiten quiver the indecomposable modules for A, and adding their dimensions we find that $\Pi(Q, \Sigma)$ would have to be 16 dimensional in order for it to satisfy the preprojective property. Since $\Pi(\ddot{Q})$ is 14 dimensional, $\Pi(\ddot{Q}, \ddot{\Sigma})$ can be at most 14 dimensional (in fact it is 13 dimensional), and thus $\Pi(\ddot{Q}, \ddot{\Sigma})$ does not satisfy the preprojective property.

For the other pairing it is less obvious. We can calculate that $\{e_1, a, a^*, c^*a, a^*c, aa^*, e_3, b, b^*, bc, c^*b^*, b^*, e_2, e_5, e_4, e_6, c, d, c^*, d^*\}$ is a basis for $\Pi(\dot{Q})$, and that $\Pi(\dot{Q}, \dot{\Sigma})$ is the subspace spanned by the first twelve elements and the elements $e_2 + e_5, e_4 + e_6, c + d, c^* + d^*$. Thus the dimension is correct, and one has to investigate further. One can decompose $\Pi(\dot{Q}, \dot{\Sigma})$ as

$$\Pi(\dot{Q}, \dot{\Sigma}) \cong \bigoplus_{\substack{\nu \in Q_0^\Sigma \\ k \geq 0}} \Pi(\dot{Q}, \dot{\Sigma})_k e_\nu.$$

In particular, $\Pi(\dot{Q}, \dot{\Sigma})_1 e_{\tilde{1}} = Kc^*a \cong S_4$ and $\Pi(\dot{Q}, \dot{\Sigma})_2 e_{\tilde{3}} = Kc^*b^* \cong S_4$. Thus $\Pi(\dot{Q}, \dot{\Sigma})$ has two isomorphic indecomposable summands, and so $\Pi(\dot{Q}, \dot{\Sigma})$ does not satisfy the preprojective property.

Note that one can relate the category of A-modules and the category of $K\dot{Q}$-modules as in Main Lemma 2 (see [27]), and in fact in can be checked that $\Pi(\dot{Q}, \dot{\Sigma})$ does satisfy the preprojective property for right A-modules.

6.3 Proof of Main Lemma 1

We prove Main Lemma 1 by constructing an A-module isomorphism. Before we can do this, it is necessary to prove several preliminary lemmas.

Lemma 6.3.1. *If $v \in Q_0''$, then $d(\sigma(v)) > d(v)$.*

Proof. We have that $v = w_m^j$ for some j, m, and $\sigma(v) = u_m^j$. If $v \notin Q_0'$, then $d(\sigma(v)) = m > 0 = d(v)$, so the result is true in this case. So suppose that $v = u_n^i \in Q_0'$. By the discussion after the statement of the theorem, we have $m > n$, i.e. $d(\sigma(v)) > d(v)$. □

Thus if $v_1 = \sigma^n(v_2)$, then n is uniquely determined, since if $\sigma^n(v) = \sigma^m(v)$

for some $n > m$, then $v = \sigma^{n-m}(v)$ which is impossible as $d(\sigma^{n-m}(v)) > d(v)$. Given a path p of $\overline{Q''}$ let $\omega(p)$ be the corresponding path of $\overline{Q'}$ induced by $\overline{\sigma}$.

Lemma 6.3.2. *$\eta(p) = \eta(q) \iff p = \omega^n(q)$ for some $n \in \mathbb{Z}$ which is uniquely determined.*

Proof. The $\Leftarrow$ implication is obvious. Assume that $\eta(p) = \eta(q)$. Clearly $t(p) = \sigma^n(t(q))$ for some uniquely determined n. We prove that $p = \omega^n(q)$ by induction on the length of the path. If p and q are trivial paths then we are done. Suppose $p = a_1a_2\dots a_k$ and $q = b_1b_2\dots b_k$. We must show $a_i = \sigma^n(b_i)$ for all i. Since $\eta(p) = \eta(q)$, $a_k \sim b_k$, and so $a_k = \sigma^m(b_k)$. Clearly $m = n$ since $t(p) = t(a_k) = \sigma^m(t(b_k)) = \sigma^m(t(q))$, and thus $h(a_k) = \sigma^n(h(b_k))$. If $k = 1$, then we are done, and if $k > 1$ then let $p' = a_1a_2\dots a_{k-1}$ and $q' = b_1b_2\dots b_{k-1}$, and one clearly has $\eta(p') = \eta(q')$. Since $t(a_{k-1}) = \sigma^n(t(b_{k-1}))$, we can use the induction hypothesis to show that $p' = \omega^n(q')$, and then $a_i = \sigma^n(b_i)$ for $i = 1, \dots, k-1$. We already have this for $i = k$, and so $p = \omega^n(q)$. $\square$

Thus if $\mu \in \operatorname{Im}\eta$, there is a total ordering on $\eta^{-1}(\mu)$, for $p, q \in \eta^{-1}(\mu)$ define $p \leq q \iff q = \omega^n(p)$ for some $n \geq 0$ (equivalently $p \leq q \iff d(t(p)) \leq d(t(q))$). Henceforth we write $d(p)$ instead of $d(t(p))$. We define $\hat{P}$ to be the set $\{p \in P : d(p) \leq \deg(p)\}$, and $\hat{P}^{max}$ to be the set $\{p \in \hat{P} : \text{If } \omega(p) \text{ exists, then } \omega(p) \notin \hat{P}\}$.

Lemma 6.3.3. *If $\mu \in \operatorname{Im}\eta$, there is $\hat{p} \in \hat{P}^{max}$ with $\eta(\hat{p}) = \mu$.*

Proof. We need to show that $\eta^{-1}(\mu)$ contains some element $p \in \hat{P}$. We can then take $\hat{p}$ to be the maximal such element. Suppose p is the minimal member of $\eta^{-1}(\mu)$, and assume for a contradiction that $p \notin \hat{P}$, i.e. that $m > \deg(p)$, where $m = d(p)$.

Clearly p must involve an arrow not in $\overline{Q'}$ since otherwise $\omega^{-1}(p) < p$ is a member of $\eta^{-1}(\mu)$. We write $p = qr$ where r involves arrows in $\overline{Q'}$ and is chosen to be as long as possible. Since q must be non trivial, we can write $q = q'a$ where a is an arrow. We have $t(p) = u^i_m$ for some i. We must have $t(q) = u^i_1$ since this is the only vertex which is connected to u^i_m and is incident with arrows not in $\overline{Q'}$. Now $\deg(r) \geq m-1$, since r must use each of the arrows $(b_l^i)^*$ for $1 \leq l \leq m-1$, and $\deg(q) \geq 1$ because $a \in Q_1^*$ as the only arrow of Q starting at u_1^i is b_1^i, and $a = b_1^i$ would contradict the choice of r. So $\deg(p) = \deg(q) + \deg(r) \geq m$, a contradiction. □

Lemma 6.3.4. *Given $q \in \hat{P}^{max}$ and $p \in P$ such that p only involves arrows in Q and $t(p) = h(q)$, then $pq \in \hat{P}^{max}$.*

Proof. Clearly in this case $d(pq) = d(q) \leq \deg(q) = \deg(pq)$, so $pq \in \hat{P}$. If $pq \notin \hat{P}^{max}$, then $\omega(pq) \in \hat{P}$, and hence $\omega(q) \in \hat{P}$, which is impossible because $q \in \hat{P}^{max}$. □

We now derive some properties of $\hat{P}^{max}$ relating to preprojective algebras. We can extend the operation ω to an algebra homomorphism $\omega^+ : K\overline{Q} \to K\overline{Q}$, which sends p to $\omega(p)$ if p only visits vertices in Q'' and zero otherwise, and similarly $\omega^- : K\overline{Q} \to K\overline{Q}$ using ω^{-1}. We can define ω^n for all $n \in \mathbb{Z}$ by applying ω^+ (respectively ω^-) n times if n is positive (respectively negative). Of course, it is necessary to take care because ω^+ and ω^- are not mutual inverses. It is clear that $\overline{\theta}\eta(x) = \sum_{n\in\mathbb{Z}}(\omega^n(x))$.

Lemma 6.3.5. *If $p \in \hat{P}$, then $\pi(\omega^-(p)) = 0$. Thus if $p \in \hat{P} \setminus \hat{P}^{max}$, then $\pi(p) = 0$ (since $p = \omega^-(p')$ for some $p' \in \hat{P}$).*

Proof. We can assume that p is a path of $\overline{Q'}$ since otherwise $\omega^-(p) = 0$ anyway. Thus $t(p) = u_m^i$ for some i, where $m = d(p)$, and thus $t(\omega^{-1}(p)) = w_m^i$. We construct a sequence of paths $(p_j)_{0\leq j\leq n}$ of $\overline{Q''}$ starting at w_m^i, such that $p_0 = \omega^{-1}(p)$, $\pi(p_{j+1} - p_j) = 0$ and $\pi(p_n) = 0$, which proves the result.

To construct p_{j+1} from p_j, write $p_j = q_j r_j$, where r_j does not involve an arrow in Q'' and is chosen to be as long as possible. Let $d_j = \deg r_j$. Since r_j can only involve the arrows $(c_l^i)^*$ for $1 \leq l \leq m-1$ (each at most once), $d_j \leq m-1$, so $\deg(q_j) = \deg(p_j) - \deg(r_j) \geq 1$, and hence q_j is not trivial. Write $q_j = s_j t_j$, where t_j uses only arrows in Q'' and is chosen to be as long as possible. Let $l_j = \text{length}(t_j)$, and set $f_j = m - 1 + l_j - d_j$. Since $\deg(s_j) = \deg(q_j) \geq 1$, s_j is not trivial, and can be written $s_j = s'_j(c^i_{f_j})^*$, and similarly $t_j = c^i_{f_j} t'_j$. If $f_j \neq 1$, let $p_{j+1} = s'_j(c^i_{f_j-1})^* c^i_{f_j-1} t'_j r_j$. If $f_j = 1$, then let $n = j$ and stop.

The sequence has the desired property as $\pi(p_{j+1} - p_j) = s'_j((c^i_{f_j-1})^* c^i_{f_j-1} - (c^i_{f_j})^* c^i_{f_j}) t'_j r_j = 0$, and p_n has the form $s(c^i_1)^* c^i_1 t$ for some paths s, t, and so $\pi(p_n) = 0$. Observe that stage n occurs when d_j has its maximum value $m-1$, and l_j has its minimum value 1. It must eventually be reached as the sequence of ordered pairs (d_j, l_j) goes $(d_0, l_0), (d_0, l_0 - 1), \ldots, (d_0, 1), (d_0 + 1, l_{j_1}), (d_0 + 1, l_{j_1} - 1), \ldots, (d_0 + 1, 1), (d_0 + 2, l_{j_2}), \ldots, \ldots, (m-1, l_{j_k}), \ldots, (m-1, 1)$. $\square$

Let $\pi' : K\overline{Q'} \to \Pi(Q')$ and $\pi'' : K\overline{Q''} \to \Pi(Q'')$ denote the natural maps. Clearly $\operatorname{Ker}\pi \cap K\overline{Q'} \subseteq \operatorname{Ker}\pi'$ and $\operatorname{Ker}\pi \cap K\overline{Q''} \subseteq \operatorname{Ker}\pi''$.

Lemma 6.3.6. *If $\pi(x) = 0$ and $\omega^+(x) \in K(P \setminus \hat{P})$, then $\pi(\omega^+(x)) = 0$.*

Proof. Let $x \in \operatorname{Ker}\pi$. We can assume that $x \in K\overline{Q''}$ since otherwise $\omega^+(x) = 0$ anyway. Hence $x \in \operatorname{Ker}\pi''$, and so $y = \omega^+(x) \in \operatorname{Ker}\pi'$ since $\omega^+(\operatorname{Ker}\pi'') = \operatorname{Ker}\pi'$. We want to show that $y \in \operatorname{Ker}\pi$. We can write

$$y = \sum_{\substack{k \geq 0 \\ v \in Q'_0}} y_{kv}$$

where y_{kv} is a linear combination of paths of degree k starting at v. Now since $y \in K(P \setminus \hat{P})$, $y_{kv} = 0$ if $d(v) \leq k$, so it suffices to prove $y_{kv} \in \operatorname{Ker}\pi$ for all k, v with $d(v) > k$. Since $y \in \operatorname{Ker}\pi' \iff y_{kv} \in \operatorname{Ker}\pi'$ for all k, v, we have $y_{kv} \in \operatorname{Ker}\pi'$, i.e.

$$y_{kv} = \sum_j r_j \left(\sum_{a \in Q'_1} aa^* - a^*a \right) s_j$$

for some paths r_j, s_j with (in particular) $\deg(s_j) \leq \deg(y_{kv}) - 1 = k - 1$ and $t(s_j) = v$. Now

$$\sum_j r_j \left(\sum_{a \in Q'_1} aa^* - a^*a \right) s_j = \sum_j r_j \left(\sum_{a \in Q_1} aa^* - a^*a \right) s_j$$

since if $a \in \overline{Q}_1 \setminus \overline{Q'_1}$, $h(a) \neq t(s_j)$ because $d(h(s_j)) \geq d(t(s_j)) - \deg(s_j) \geq$

$d(v) - k + 1 > 1$ and $d(h(a)) \leq 1$. Thus

$$y_{kv} = \sum_j r_j \left(\sum_{a \in Q_1} aa^* - a^*a \right) s_j \in \operatorname{Ker} \pi$$

as required. ☐

Lemma 6.3.7. *If $x \in K\hat{P}^{max}$, then $\pi(x) = 0 \iff \pi\overline{\theta}\eta(x) = 0$.*

Proof. Clearly $\pi\overline{\theta}\eta(x) = 0 \iff \pi(\omega^n(x)) = 0$ for all n, so the $\Leftarrow$ implication is obvious. For $\Rightarrow$, assume that $\pi(x) = 0$. Then by the previous lemma, $\pi(\omega^n(x)) = 0$ for all $n > 0$. By Lemma 6.3.5, $\pi(\omega^n(x)) = 0$ for all $n < 0$, and thus $\pi\overline{\theta}\eta(x) = \sum_{n\in\mathbb{Z}} \pi(\omega^n(x)) = 0$. ☐

We now construct a map

$$\xi : \bigoplus_{\substack{v\in\nu \\ d(v)\leq k}} \Pi(Q)_k e_v \to \Pi(Q, \Sigma)_k e_\nu,$$

which will be shown to be an A-module isomorphism. Given $p \in \hat{P}^{max}$, with $t(p) = v \in \nu$, $\deg(p) = k$, let

$$\xi(\pi(p)) = \pi\overline{\theta}\eta(p) \in \Pi(Q, \Sigma)_k e_\nu.$$

We set $P_{kv} = \{p \in P, t(p) = v, \deg(p) = k\}$, and similarly with $\hat{P}$ and $\hat{P}^{max}_{kv}$.

Lemma 6.3.8.

$$\bigoplus_{\substack{v\in\nu \\ d(v)\leq k}} \Pi(Q)_k e_v = \sum_{v\in\nu} \pi\left(K\hat{P}^{max}_{kv}\right).$$

Proof. Clearly, one has

$$\bigoplus_{\substack{v\in\nu \\ d(v)\leq k}} \Pi(Q)_k e_v = \sum_{\substack{v\in\nu \\ d(v)\leq k}} \pi\left(KP_{kv}\right).$$

By definition of $\hat{P}$, this is equivalent to

$$\bigoplus_{\substack{v\in\nu \\ d(v)\leq k}} \Pi(Q)_k e_v = \sum_{v\in\nu} \pi\left(K\hat{P}_{kv}\right).$$

The result follows by Lemma 6.3.5. ☐

In view of this lemma and Lemma 6.3.7, ξ extends to a well defined injective vector space homomorphism.

Lemma 6.3.9. *ξ is surjective.*

Proof. $\Pi(Q,\Sigma)_k e_\nu$ is spanned by the elements $\{\pi\overline{\theta}(\mu) : \mu \in S\}$, where S is the set of paths of $\overline{Q^\Sigma}$ starting at ν of degree k. Since $\overline{\theta}(\mu) = 0$ for $\mu \notin \operatorname{Im}\eta$, we can replace S by the set S' of paths in $\operatorname{Im}\eta$ starting at ν of degree k. By Lemma 6.3.3, each element of S' has the form $\eta(p)$ for some $p \in \hat{P}^{max}$. Thus $\Pi(Q,\Sigma)_k e_\nu$ is spanned by the elements $\{\pi\overline{\theta}(p) : p \in P^{max}_{kv}\}$, as required. ☐

Lemma 6.3.10. *ξ is an A-module map.*

Proof. It suffices to check that if $\rho \in P^\Sigma$ and $q \in \hat{P}^{max}$, then $\xi(\pi(\phi(\rho)q))$ is the same as the A-module product $\rho\xi(\pi(q))$ which is by definition $\rho\pi\overline{\theta}\eta(q) = \pi\overline{\theta}(\rho\eta(q))$. We assume that there is a unique $p \in P$ with $\eta(p) = \rho$ and $t(p) = h(q)$, since otherwise $\phi(\rho)q$ and $\rho\eta(q)$ are both zero. Then $\xi(\pi(\phi(\rho)q)) = \xi\pi(pq)$. Since $pq \in \hat{P}^{max}$ by Lemma 6.3.4, this is $\pi\overline{\theta}\eta(pq) = \pi(\overline{\theta}(\eta(p)\eta(q)) = \pi\overline{\theta}(\rho\eta(q))$ as required. ☐

This completes the proof of Main Lemma 1.

6.4 Proof of Main Lemma 2

We prove Main Lemma 2 by constructing a 'inverse' G of F. Note that G will not be a functor. A representation Y of A can be identified with a representation of Q^Σ which satisfies the relations

$$Y_\gamma Y_{\tilde{b}^i_{n_i-1}} \dots Y_{\tilde{b}^i_1} Y_\beta = 0,$$

for all arrows $\beta, \gamma \in Q_1^\Sigma$ with $t(\gamma) = w^i_{n_i}$, $h(\beta) = u^i_1$. Given such a representation, define vector spaces M_v and N_v for all $v \in Q_0$. If $v = u^i_m \in Q'_0$, then let

$$N_v = \sum_{\substack{\alpha \in Q_1^\Sigma \\ h(\alpha) = \tilde{u}^i_1}} \operatorname{Im} Y_{\tilde{b}^i_{m-1}} \dots Y_{\tilde{b}^i_1} Y_\alpha,$$

and otherwise let $N_v = Y_{\tilde{v}}$. If $v = w^i_m \in Q''_0$, then let

$$M_v = \sum_{\substack{\alpha \in Q^\Sigma_1 \\ h(\alpha) = \tilde{w}^i_1}} \operatorname{Im} Y_{\tilde{c}^i_{m-1}} \dots Y_{\tilde{c}^i_1} Y_\alpha,$$

and otherwise let $M_v = 0$.

Lemma 6.4.1. *(i) For all $v \in Q_0$, we have $M_v \subseteq N_v \subseteq Y_{\tilde{v}}$.*
(ii) For all $a : v \to v' \in Q_1$, we have $Y_{\tilde{a}}(M_v) \subseteq M_{v'}$ and $Y_{\tilde{a}}(N_v) \subseteq N_{v'}$.
(iii) $N_{u^i_m} = M_{w^i_m}$. Thus, if we label the members of each $\tilde{v}$ as $v_1, \dots, v_k$ so that $\sigma(v_l) = v_{l+1}$, one has $N_{v_{l+1}} = M_{v_l}$.

Proof. (i) The only non trivial case is where $v \in Q'_0 \cap Q''_0$, i.e. $v = u^j_n = w^i_m$. We know that $m > n$, and that $\tilde{b}^j_{n+k} = \tilde{c}^i_{m+k}$ for all $k \in \mathbb{Z}$ which make sense. Thus clearly

$$\begin{aligned} M_v &= \sum_{\substack{\alpha \in Q^\Sigma_1 \\ h(\alpha) = \tilde{w}^i_1}} \operatorname{Im} Y_{\tilde{c}^i_{m-1}} \dots Y_{\tilde{c}^i_1} Y_\alpha \\ &\subseteq \operatorname{Im} Y_{\tilde{c}^i_{m-1}} \dots Y_{\tilde{c}^i_{m-n+1}} \\ &= \operatorname{Im} Y_{\tilde{b}^j_{n-1}} \dots Y_{\tilde{b}^j_1} Y_{\tilde{a}} \\ &= N_v, \end{aligned}$$

where $a = c^i_{n-m}$ is the unique arrow of Q with head at $u^j_1 = w^i_{m-n}$.
(ii) If $v \notin Q''_0$, then it is obvious, so assume that $v = w^i_m \in Q''_0$. If $m < n_i$, then $a = c^i_m$, and $v' = w^i_{m+1}$ and so $Y_{\tilde{a}}(M_v) = M_{v'}$. If $m = n_i$, then $Y_{\tilde{a}}(M_v) = 0$ because of the relations. The proof for N is similar.
(iii) is clear. ∎

We define $G(Y)$ to be the representation X of Q given by $X_v = N_v/M_v$, and $X_a : X_v \to X_{v'}$ to be the map induced by $Y_{\tilde{a}}$, i.e. for $x \in N_v$ define $X_a(x + M_v)$ to be $Y_{\tilde{a}}(x) + M_{v'}$. Parts (i) and (ii) of the previous lemma show that this is well defined, and part (iii) shows that

$$FG(Y_{\tilde{v}}) = \bigoplus_{v \in \tilde{v}} N_v/M_v = Y_{\tilde{v}}/M_{v_1} \oplus M_{v_1}/M_{v_2} \oplus \dots \oplus M_{v_{k-1}},$$

which can be identified with $Y_{\tilde{v}}$, and hence $FG(Y)$ can be identified with Y. We now need to show that if X is an indecomposable representation of Q with $X_v \neq 0$ for some $v \notin Q_0'$, then $GF(X)$ can be identified with X, which will complete the proof.

Lemma 6.4.2. *Let X be an indecomposable representation of Q with $X_v \neq 0$ for some $v \notin Q_0'$. For all i,*

$$\sum_{\substack{a \in Q_1 \\ h(a)=u_1^i}} X_a = X_{u_1^i},$$

and each $X_{b_m^i}$ is surjective.

Proof. Suppose otherwise. Let W be the representation of Q where $W_{u_1^i}$ is the complement of the sum in $X_{u_1^i}$, and the remaining W_v are zero. There is a homomorphism f from X to W (take $f_{u_1^i}$ to be the projection of $X_{u_1^i}$ onto $W_{u_1^i}$). However, this is impossible because the only indecomposable representations of Q which have non zero maps to W have support contained in Q'. The second part follows easily because we can restrict X to a representation X_i of each connected component Q_i' of Q'. Each X_i decomposes as a direct sum $\oplus_j X_{ij}$, with each $(X_{ij})_{u_1^i} \neq 0$ (because otherwise X_{ij} is a proper summand of X), and one therefore has that each $(X_{ij})_{b_m^i}$ is surjective, and therefore so is each $X_{b_m^i}$. □

Lemma 6.4.3. *Let X be an indecomposable representation Q with $X_v \neq 0$ for some $v \notin Q_0'$, let $Y = F(X)$, and define the spaces M_v and N_v. One has $N_v = \oplus_{w \geq v} X_v$ and $M_v = \oplus_{w > v} X_v$.*

Proof. If $v \notin Q_0'$, then v is the maximal element of $\tilde{v}$, and so $N_v = Y_{\tilde{v}} = \oplus_{w \geq v} X_v$ is clear. Suppose then that $v = u_m^i \in Q_0'$. If there is more than one arrow of Q with head at u_1^i, then $v \notin Q_0''$ and so v is the maximal element of $\tilde{v}$, and one has

$$\begin{aligned}
N_v &= \sum_{\substack{\alpha \in Q_1^\Sigma \\ h(\alpha)=\tilde{u}_1^i}} \operatorname{Im} Y_{\tilde{b}^i_{m-1}} \dots Y_{\tilde{b}^i_1} Y_\alpha \\
&= \sum_{\substack{a \in Q_1 \\ h(a)=u_1^i}} \operatorname{Im} X_{b^i_{m-1}} \dots X_{b^i_1} X_a \\
&= X_v,
\end{aligned}$$

using the previous lemma. If there is exactly one arrow a of Q ending at u_1^i, then

$$\begin{aligned}
N_v &= \operatorname{Im} Y_{\tilde{b}^i_{m-1}} \dots Y_{\tilde{b}^i_1} Y_{\tilde{a}} \\
&= \bigoplus \operatorname{Im} X_{\sigma^l(b^i_{m-1})} \dots X_{\sigma^l(b^i_1)} X_{\sigma^i(a)},
\end{aligned}$$

where the sum is taken over all $l \in \mathbb{Z}$ which make sense. Clearly $a \notin Q_0'$, so we need only consider $l \geq 0$. By the previous lemma, we have

$$\operatorname{Im} X_{\sigma^l(b^i_{m-1})} \dots X_{\sigma^l(b^i_1)} X_{\sigma^l(a)} = X_{\sigma^l(u^i_m)},$$

for all l, which gives the result. The assertion for M_v follows from (iii) of Lemma 6.4.1. □

6.5 Nakayama algebras

Although we are only able to prove the conjecture in a special case, the special case is wide enough to show that a 'preprojective algebra' exists for all Nakayama algebras.

Definition 6.5.1. A module is *uniserial* if its submodules are totally ordered by inclusion.

Definition 6.5.2. A finite dimensional algebra is a *Nakayama algebra* if both its indecomposable projective and indecomposable injective modules are uniserial.

If A is an indecomposable Nakayama algebra, then one can label its indecomposable projective modules $P_0, P_1, \ldots, P_{n-1}$, so that P_{i+1} is the projective cover of $\operatorname{rad} P_i$ for $i = 0, 1, \ldots, n-2$, and P_0 is the projective cover of $\operatorname{rad} P_{n-1}$ if P_{n-1} is not simple. The sequence $(f_0, f_1, \ldots, f_{n-1})$, where $f_i = \operatorname{length}(P_i)$ is called the *admissible sequence of* A. It has the property that $f_{i+1} \geq f_i - 1 \geq 1$ for $i = 0, 1, \ldots, n-2$ and $f_0 \geq f_{n-1} - 1$. Any sequence with this property is called *admissible*.

Theorem 6.5.3. *[1] Given an admissible sequence* $(f_0, f_1, \ldots, f_{n-1})$*, there is a Nakayama algebra* A *such that* $(f_0, f_1, \ldots, f_{n-1})$ *is the admissible sequence of* A.

Proof. If $f_{n-1} = 1$, then let Q be the quiver

$$\begin{tikzcd} \underset{0}{\bullet} \arrow[r, "a_1"] & \underset{1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n-2}{\bullet} \arrow[r, "a_{n-1}"] & \underset{n-1}{\bullet} \end{tikzcd}$$

If $f_{n-1} > 1$, then let Q be the quiver

$$\begin{tikzcd} 0\,\bullet \arrow[r, "a_1"] & \underset{1}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n-2}{\bullet} \arrow[r, "a_{n-1}"] & \bullet\, n-1 \arrow[llll, bend left, "a_0"] \end{tikzcd}$$

In either case let I be the ideal of KQ generated by the set of paths $\{p_i : 0 \leq i \leq n-1\}$, where p_i is the unique path of length f_i starting at vertex i. It can be easily checked that $A = KQ/I$ is a Nakayama algebra, and $(f_0, f_1, \ldots, f_{n-1})$ is its admissible sequence. $\square$

Theorem 6.5.4. *If* A *is a Nakayama algebra, then there is a quiver* Q *and a pairing* Σ *satisfying the conditions of Theorem 6.2.4 such that* $A \cong KQ^{\Sigma}/I^{\Sigma}$*, and thus* $\Pi(Q, \Sigma)$ *is an algebra satisfying the preprojective property for* A.

Let $(f_0, \ldots, f_{n-1})$ be the admissible sequence corresponding to A. Let L be the set $\{m : f_m > f_{m-1} - 1\}$ (assuming 'f_{-1}' is equal to f_{n-1}), and list its members as $l_1, l_2, \ldots, l_k$ so that $l_i < l_{i+1}$. Additionally, for later convenience, set $l_0 = l_k - n$ and $l_{k+1} = l_1 + n$ (they are not members of L), and put $f_{l_0} = f_{l_k}$ and $f_{l_{k+1}} = f_{l_1}$. We write down some easy properties of these numbers which are required for later.

Lemma 6.5.5. *We have the following properties.*

(1) Given $l \leq m$, if there exits $i \in L$ with $l < i \leq m$ then $f_l + l < f_m + m$, otherwise $f_l + l = f_m + m$.

(2) For $i = 1, 2, \ldots, k-1$, $l_{i+1} \leq l_i + f_{l_i} - 1$.

(3) $l_1 \leq f_{n-1} - 1$ (with equality only when $f_{n-1} = 1$).

(4) $l_k + f_{l_k} = f_{n-1} + n - 1$.

Proof. (1) Provided $l+1, l+2, \ldots, m$ are not in L, one has $f_m + m = f_{m-1} + m - 1 = \cdots = f_l + l$. Otherwise at least one equality must be replaced by $>$.

(2) Assuming (1), we have $2 \leq f_{l_{i+1}-1} = f_{l_i} - l_{i+1} + l_i + 1$, which is equivalent to (2).

(3) Observe first that if $f_{n-1} = 1$, then $l_1 = 0$ as required. We can therefore suppose that $f_{n-1} \neq 1$, and assume that $l_1 > 0$ (since otherwise the claim is obviously true). Then by (1), $f_{l_1-1} + l_1 - 1 = f_0$, and since $l_1 > 0$, we have $f_0 = f_{n-1} - 1$, and thus $l_1 = f_{n-1} - f_{l_1-1} < f_{n-1} - 1$.

(4) Follows immediately from (1). $\square$

We define Q to be the quiver with vertex set $\{(m,i) \in \mathbb{N}^2 : 1 \leq m \leq k, l_m \leq i \leq l_m + f_{l_m} - 1\}$ and arrows $\{(m,i) \to (m,i+1) : 1 \leq m \leq k, l_m \leq i \leq l_m + f_{l_m} - 2\}$. Clearly Q has k connected components, a typical one being

$$\underset{(m,l_m)}{\bullet} \longrightarrow \underset{(m,l_m+1)}{\bullet} \longrightarrow \underset{(m,l_m+2)}{\bullet} \longrightarrow \cdots \longrightarrow \underset{(m,l_m+f_{l_m}-1)}{\bullet}$$

Set Q', Q'' to be the full subquivers of Q with vertex sets $\{(m,i) : l_{m+1} \leq i \leq l_m + f_{l_m} - 1\}$ and $\{(m,i) : l_m \leq i \leq l_{m-1} + f_{l_{m-1}} - 1\}$ respectively. Given $(m,i) \in Q''_0$, define

$$\sigma((m,i)) = \begin{cases} (m-1,i) \text{ if } m > 1 \\ (k, i+n) \text{ if } m = 1 \end{cases}$$

Lemma 6.5.6. *$\Sigma = (Q', Q'', \sigma)$ is an end pairing satisfying the hypothesis of Theorem 6.2.4.*

Proof. We have $(1,i) \in Q_0'' \iff l_1 \leq i \leq l_1 + f_{l_0} - 1 \iff l_1 + n \leq i + n \leq l_0 + n + f_{l_0} - 1 \iff l_{k+1} \leq i + n \leq l_k + f_{l_k} - 1 \iff (k, i+n) \in Q_0'$, and if $m > 1$ it is clear that $(m,i) \in Q_0'' \iff (m-1,i) \in Q_0'$. Thus σ is a well defined bijective map, and can be extended to a quiver isomorphism $\sigma : Q'' \to Q'$. Thus $\Sigma = (Q', Q'', \sigma)$ is a pairing. Since clearly $l_{m+1} > l_m$ and, using (1), $l_{m-1} + f_{l_{m-1}} - 1 < l_m + f_{l_m} - 1$ for all m, it is and end pairing, which clearly satisfies the hypothesis of Theorem 6.2.4. $\square$

Lemma 6.5.7. *Q^Σ is the quiver given in Theorem 6.5.3, and I^Σ is the corresponding ideal.*

Proof. For $j = 0, 1, \ldots, n-1$, set $v_j = \{(m,i) \in Q_0 : i \equiv j \mod n\}$. We claim that each v_j is an equivalence class. It is clear that if $v \in v_j$ and $v' \in v_{j'}$ with $j \neq j'$, then $v \not\sim v'$, so it remains to check that all members of each v_j are equivalent. This is done in several stages.

First we show that if $(m_1,i),(m_2,i) \in Q_0$ with $m_1 \leq m_2$, then $(m_1,i) \sim (m_2,i)$. Given m with $m_1 \leq m \leq m_2$, we have $(m,i) \in Q_0$ because $l_m \leq l_{m_2} \leq i$ and $i \leq l_{m_1} + f_{l_{m_1}} - 1 \leq l_m + f_{l_m} - 1$ using (1). Now if $(m,i),(m+1,i) \in Q_0$, then $(m+1,i) \in Q_0''$ since $i \geq l_{m+1}$. Thus $(m,i) \sim (m+1,i)$ for all $m_1 \leq m < m_2$, and so $(m_1,i) \sim (m_1+1,i) \sim \cdots \sim (m_2,i)$.

Now we show that if $(m_1,i),(m_2,i+n) \in Q_0$, then $(m_1,i) \sim (m_2,i+n)$. We have $l_1 \leq l_{m_1} \leq i$ and $i + n \leq l_{m_2} + f_{l_{m_2}} - 1 \leq l_{n-1} + f_{l_{n-1}} - 1$ using the fact that $(m_2, i+n) \in Q_0$ and (1). Thus $i \leq l_{n-1} - n + f_{n-1} - 1 = l_0 + f_{l_0} - 1$, and so $(1,i) \in Q_0''$ and then $\sigma(1,i) = (k,i+n) \in Q_0$. Since $(m_1,i) \sim (1,i) \sim (k,i+n) \sim (m_2,i+n)$, we have the result.

Finally, we check that $\{i : (m,i) \in Q_0 \text{ for some } m\} = \{i : l_1 \leq i \leq l_k + f_{l_k} - 1\}$. The $\subseteq$ inclusion is obvious. Suppose i lies in the range $l_1 \leq i \leq l_k + f_{l_k} - 1$. If $(m,i) \notin Q_0$ for all m, then for each m either $i < l_m$ or $i > l_m + f_{l_m} - 1$. Since $l_1 \leq i$, the set $\{m : l_m \leq i\}$ is non empty. Choose its maximal member j, we must have $i > l_j + f_{l_j} - 1$. Since $i \leq l_k + f_{l_k} - 1$, $i < l_k$, so $j \neq k$. Thus we have $i < l_{j+1}$ (since j was maximal) and $i > l_j + f_{l_j} - 1$ which is impossible by (2).

This completes the proof that each v_j is an equivalence class, since if (m, i), $(m', i') \in Q_0$ with $i' = i + kn$, there is $m_0, m_1, \ldots, m_k$ such that $m_0 = m$ and $m_k = m'$ such that $(m_j, i + jn) \in Q_0$ for all j. We then have $(m, i) = (m_0, i) \sim (m_1, i+n) \sim \cdots \sim (m_k, i+kn) = (m', i')$. Finally since $l_k + f_{l_k} - l_1 = n - 1 + f_{n-1} - l_1 \geq n$ (using (1) and (3)), each equivalence class has at least one member.

We now check the arrows are correct. First suppose $f_{n-1} = 1$, (and thus $l_1 = 0$). We claim there is an arrow $v_j \to v_{j+1}$ for $j = 0, 1, \ldots, n-2$. For each j, there is some $(m, j) \in Q_0$. If additionally $(m, j+1) \in Q_0$, there is clearly an arrow as required. Supposing $(m, j+1) \notin Q_0$, we must have $j = l_m + f_{l_m} - 1$, and thus $m < k$ (as using (1), $l_k + f_{l_k} = f_{n-1} + n - 1 = n > j + 1$). Then we have $l_{m+1} \leq j$ using (2), and $j + 1 \leq l_{m+1} + f_{l_{m+1}} - 1$ using (1), and thus $(m+1, j), (m+1, j+1) \in Q_0$, and there is an arrow $v_j \to v_{j+1}$. It is clear there are no other arrows (there is no arrow starting at v_{n-1} because $(m, n) \notin Q$ for all m since $l_k + f_{l_k} - 1 = n - 1$).

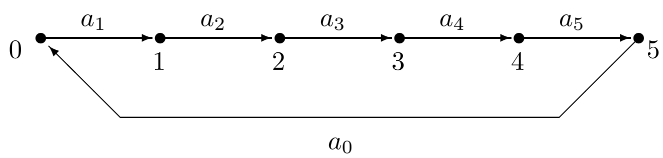
Now suppose that $f_{n-1} > 1$. We show that there is an arrow $v_j \to v_{j+1}$ for all j. We know that $(m, j) \in Q_0$ for some m, and as above we can assume that $j = l_m + f_{l_m} - 1$. Suppose $m = k$. Using (3) and (4), we have $l_1 \leq f_{n-1} - 2 = f_{l_k} + l_k - n - 1 = j - n$, and using (1) and (4), $j - n + 1 = l_k + f_{l_k} - n = f_{n-1} - 1 \leq f_0 \leq l_1 + f_{l_1} - 1$, and thus $(1, j-n), (1, j-n+1) \in Q_0$. If $m < k$, then $(m+1, j), (m+1, j+1) \in Q_0$ as in the $f_{n-1} = 1$ case. Either way, there is an arrow $v_j \to v_{j+1}$.

The final thing to check is that I^{Σ} is the correct ideal. Given i, let p_i be the path starting at i of shortest length which is not in $\operatorname{Im} \eta$. We claim length$(p_i) = f_i$, and thus I^{Σ} coincides with the ideal given in Theorem 6.5.3. Clearly there is a path of length d starting at v_i in $\operatorname{Im} \eta$ if and only if there exists $(m, j) \in Q_0$ with $j \equiv i \mod n$ and $(m, j+d) \in Q_0$.

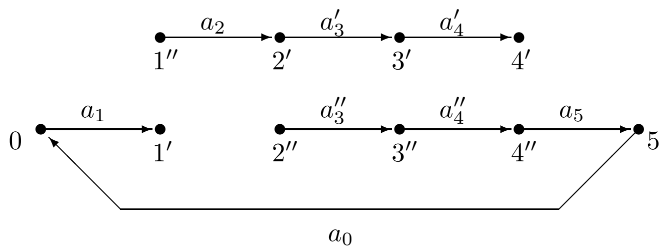
If there exists m with $l_m \leq i$, then let $\overline{m}$ be the maximal such m. Then one has $(\overline{m}, i)$ and $(\overline{m}, i + f_i - 1)$ in Q_0 because $l_{\overline{m}} \leq i$ and $i + f_i - 1 = l_m + f_{l_m} - 1$ by

(1). Thus there is a path of length $f_i - 1$ starting at v_i in $\operatorname{Im}\eta$, which is clearly the longest possible. Otherwise, if there is no such m, then one has $(k, i+n)$ and $(k, i+n+f_i-1)$ in Q_0 because $l_k \leq i+n$ is clear and $i+f_i+n-1 = f_0+n-1 = f_{n-1}+n-2 = l_k+f_{l_k}-1$, using (1) and (4). Again we have a path of length $f_i - 1$ starting at v_i, which is clearly the longest possible. $\square$

This completes the proof of Theorem 6.5.4. It is possible to give the preprojective algebra of a Nakayama algebra as a quiver with relations. That is, suppose $A = K\dot{Q}/I$, where $\dot{Q}$ is a quiver of type A_n or $\tilde{A}_n$, and I is the ideal generated by paths $p_j = a_{j1}a_{j2}\dots a_{jn_j}$. It has been stated in Section 6.1 that $\Pi(Q,\Sigma)$ is a quotient of $\Pi(\dot{Q})$ and it is reasonably straightforward to see that it is the quotient generated by the paths μ such that $\pi\overline{\theta}(\mu) = 0$, i.e. the paths μ such that $\pi(p) = 0$ for all paths p such that $\eta(p) = \mu$. It can be shown that (but the proof is long and is omitted) that this quotient is the same as the quotient by the paths p_{jk}, where $p_{jk} = a^*_{jk}\dots a^*_{j1}a_{j1}\dots a_{jn_j-k}$. We can illustrate it with an example. Suppose $\dot{Q}$ is the quiver



and let I be the ideal generated by the paths a_2a_1 and $a_5a_4a_3a_2$. Then if Q is the quiver



and Σ is the pairing with Q' (respectively Q'') being the full subquiver of Q consisting of the vertices marked with $'$ (respectively $''$), we have $Q^\Sigma = \dot{Q}$ and $I^\Sigma = I$. It is easy to see that if μ is any of the paths a_2a_1, $a_2^*a_2$, $a_1^*a_2^*$, $a_5a_4a_3a_2$,

$a_5^* a_5 a_4 a_3$, $a_4^* a_5^* a_5 a_4$, $a_3^* a_4^* a_5^* a_5$, $a_2^* a_3^* a_4^* a_5^*$, then $\pi\overline{\theta}(\mu) = 0$. In each case there is at most one path p of $\overline{Q}$ such that $\eta(p) = \mu$, and then one has $\pi(p) = 0$. To see that these are the only relations, one can in theory calculate the dimension $\Pi(A)$ (by constructing a basis), and show that it is the same as the dimension of $\Pi(Q, \Sigma)$ (which is known to be the direct sum of the indecomposable modules for A), which in this case is 65. [Another way to calculate the dimension of $\Pi(Q, \Sigma)$ is to use the formula given after the statement of Main Lemma 2, $\dim(\Pi(Q, \Sigma)) = \dim(\Pi(Q)) - \dim(\Pi(Q'))$. The dimension of a preprojective algebra of type A_n is $\frac{1}{6}n(n+1)(n+2)$, so $\dim(\Pi(Q)) = 56 + 20 = 76$ and $\dim(\Pi(Q')) = 10 + 1 = 11$, so the formula is satisfied].

Appendix A

The Reduction Algorithm

Given a K-algebra A which is presented by generators and relations, it is desirable to obtain a standard form for the elements of A which is unique, i.e., two elements of A are equal if and only if they have the same standard form. The most obvious example of this is a basis for A, since then we can express the elements of A uniquely as a linear combination of the basis elements. For some algebras, however, it is not obvious how to construct a basis. This problem has been considered many times before, e.g. in [4], [6], in settings far more general than is necessary for our purposes. In this Appendix we simplify this material so that we can more easily apply it to the algebras studied in this thesis.

A.1 Introduction

The following example illustrates the purpose of this Appendix.

Example A.1.1. Let A be the algebra

$$K\langle b, c\rangle/(b^3, c^2, cb^2c - cbcb - bcbc).$$

Can we find a basis for A? We write down a naive argument, which produces an incorrect answer, then analyse what is wrong with it.

(1) A is spanned by the set of all words formed from b and c.

(2) Since $b^3 = c^2 = 0$, A is spanned by the set of all words formed from b, c which do not include bbb or cc as a subword.

(3) If w is a word containing $cbbc$ as a subword (say $w = ucbbcv$), then $w = ucbcbv + ubcbcv$ since $cb^2c = cbcb + bcbc$. We can repeat until we have expressed w as a linear combination of words not including $cbbc$ as a subword. Hence A is spanned by the set of words not including bbb, cc or $cbbc$ as a subword.

(4) Since we have used all relations, this is a basis for A.

Now whilst statements (1) and (2) are correct, the logic of (3) is flawed because if $w = cbbcbc$ then it reduces to $cbcbbc + bcbcbc$ and then if we attempt to reduce $cbcbbc$ we only end up where we started. This mistake might be considered obvious but in a complicated situation it may not be so easy to see whether such a statement is valid. There may be several statements involved in a circular argument. Statement (4) is also clearly wrong since $cbcbc = (cb^2c - bcbc)c = (cb^2 - bcb)c^2 = 0$ so the set of irreducible words is not independent.

The idea of starting with a spanning set and obtaining equations which enable us to reduce it is correct, but we need to formulate some rules which will prevent such errors in logic from occurring. Explicitly we need to ensure that

1. There is no possibility of circular arguments such as that in (3).

2. There is a condition which can be used to guarantee that a set of irreducible words is a basis.

In the following sections we describe a suitable algorithm which consists of forming a sequence of improving 'reduction systems' from which we can find spanning sets for A. Section A.2 defines reduction systems and shows why they produce a spanning set for A. Section A.3 explains how a reduction system can be modified into a 'better' one. Section A.4 tells us how we can determine when we have have arrived at the best possible reduction system, one which leads to a basis for A.

We describe the setup which is used throughout this appendix. Let A be an algebra generated by $X = \{x_1, \ldots, x_n\}$ with relations $R = \{r_1, \ldots, r_m\}$ (we assume that X and R are finite for simplicity). We write $K\langle X\rangle$ for the algebra $K\langle x_1, x_2, \ldots, x_n\rangle$. Let W be the set of words formed from $x_1, \ldots, x_n$, and denote the length of a word u by $|u|$. Let $\leq$ be a partial ordering on W with the following properties:

(†) If $u < v$ then $rus < rvs$ for all $r, s, u, v \in W$.

(‡) $\leq$ has descending chain condition.

Note that in [4], a partial ordering satisfying (†) is called a semigroup partial ordering. We are assuming (‡) so that all words will be 'reduction finite'. Section A.6 discusses some possible orderings of this type.

A.2 Reduction Systems

Definition A.2.1. [6] Let $v \in K\langle X\rangle$. If there is a word $\overline{v}$ such that $v = \lambda\overline{v} - z$, where z is a linear combination of words strictly less than $\overline{v}$ (with respect to $\leq$) and λ is a nonzero scalar, then $\overline{v}$ is the *leading word* of v. If additionally $\lambda = 1$, then v is said to be *monic.*

If $\leq$ is a total ordering then every nonzero element of $K\langle X\rangle$ must have a leading word, but otherwise there may be incomparable words.

Definition A.2.2. A *reduction system* for $A = K\langle X\rangle/I_R$ is a set $\Omega \subseteq K\langle X\rangle$ of monic elements such that $I_\Omega \subseteq I_R$. A reduction system Ω is *full* if $I_\Omega = I_R$. If $\overline{v}$ is the leading word of some $v \in \Omega$, then we say $\overline{v}$ is an *illegal word.* Let W_Ω be the set of illegal words. We say a word is *irreducible* (with respect to Ω) if it has no subword in W_Ω, otherwise we say w is *reducible.* Let W_Ω^{irr} be the set of irreducible words.

If we choose an ordering which is sufficient to ensure the elements of R all have leading words (e.g. a total ordering), then we can form a reduction system by *monicising* the elements of R, that is, we divide an element by the coefficient of its leading word. Clearly if we take Ω to be the set of all monicised elements of R, then Ω is a full reduction system. If we only take a subset, then the reduction system may not be full. We allow the possibility that a reduction system may not be full because sometimes (e.g. Lemma 3.3.4) we are only interested in finding a spanning set for A, and it is convenient to ignore some of the relations. Clearly A is a quotient of $K\langle X\rangle/I_\Omega$, so a spanning set for $K\langle X\rangle/I_\Omega$ is a spanning set for A. If Ω is a full reduction system then $A = K\langle X\rangle/I_\Omega$, so any basis of $K\langle X\rangle/I_\Omega$ is a basis of A.

Lemma A.2.3. *If Ω is a reduction system then $K\langle X\rangle/I_\Omega$ (and hence A) is spanned by the set of irreducible words.*

Proof. Clearly $K\langle X\rangle/I_\Omega$ is spanned by the set of of words. We show that each word in $W \setminus W_\Omega^{irr}$ can be expressed as a linear combination of words in W_Ω^{irr}.

Supposing otherwise, we can choose such a word $w_1 \in W \setminus W_\Omega^{irr}$ which cannot be so expressed. Since $w_1 \notin W_\Omega^{irr}$, w_1 has an illegal subword $\overline{v} \in W_\Omega$, say $w_1 = r\overline{v}s$. Now there is an element $v = \overline{v} - z \in \Omega$, where $z = \sum_j \lambda_j u_j$ for some scalars λ_j and words $u_j < \overline{v}$. Now

$$w_1 - rzs = r(\overline{v} - z)s = rvs \in I_\Omega,$$

and therefore

$$w_1 = rzs = \sum_j \lambda_j r u_j s,$$

as elements of $K\langle X\rangle/I_\Omega$. We assumed that w_1 cannot be expressed as a linear combination of irreducible words, so some $ru_l s \notin W_\Omega^{irr}$. Let $w_2 = ru_l s$. Now $u_l < \overline{v}$ so by (†), $ru_l s < r\overline{v}s$, that is, $w_2 < w_1$. We repeat the same process with w_2, obtaining $w_3 < w_2$. Continuing, we obtain an infinite non stabilising chain

$$w_1 \geq w_2 \geq w_3 \geq \dots$$

contradicting (‡), and completing the proof. □

The proof of the lemma leads us to the following definitions.

Definition A.2.4. Suppose $w = r\overline{v}s$ for some $v \in \Omega$. Let $z = \overline{v} - v$. We say rzs is a *single step reduction* of w and write $w \mapsto rzs$. Clearly we can extend this definition in the obvious way to apply to any element of $K\langle X\rangle$, not just words. Namely, if $y = \lambda w + y'$ for some $y' \in K\langle X\rangle$, then $y \mapsto \lambda rzs + y'$. If

$y_1 \mapsto y_2 \mapsto \ldots \mapsto y_k$ is a sequence of one or more single step reductions then we say y_k is a *reduction* of y_1 and write $y_1 \to y_k$. If additionally y_k is a linear combination of irreducible words (so cannot be reduced further), then we say y_k is a *complete reduction* of y_1 and it is customary to write $y_1 \rightsquigarrow y_k$.

Note that a single step reduction is just an addition of the element $-rvs \in I_\Omega$, so if $w \to y$, $w = y$ as elements of $K\langle X\rangle/I_\Omega$ (and as elements of A).

We can now see what went wrong with Example A.1.1. In statement (2) we were effectively working with the reduction system $\{b^3, c^2\}$ which is fine, but in order for statement (3) to make sense, we require the leading word of $cb^2c - bcbc - cbcb$ to be cb^2c. Namely, we would have to find some partial order $\leq$ on the set of words satisfying (†) and (‡) with $cbbc > bcbc, cbcb$. This is impossible, as by (†) we must have both

$$(cbbc)bc > (cbcb)bc, \quad cb(cbbc) > cb(bcbc).$$

So in order to get a full reduction system, we must make either $bcbc$ or $cbcb$ the leading term (in both cases we can define an ordering suitable for this purpose).

A.3 Modifying reduction systems

Given a reduction system Ω, the above describes how we write a reducible word w as a linear combination of irreducible words. However it may be the case that forming a reduction system by monicising the relations does not give a suitable set of irreducible words. In this case we wish to add some more elements to the reduction system which will result in a smaller set of irreducible words. Sometimes one may be able to see a suitable element by inspection, but sometimes it may not be obvious. In this case we can use the 'resolving' method. It turns out that we can use the fact that the reduction system is not suitable to improve it. We refer back to Example A.1.1 to illustrate this.

We gave the equation $cbcbc = 0$ to show that the set of irreducible words was not independent (we ignore the fact that this isn't a valid reduction system because that is not relevant here). This equation was obtained by looking at the reductions of cb^2c^2. This has both cb^2c and c^2 as a subword, and so could be reduced in two different ways,

$$(cb^2c)c \mapsto (cbcb - bcbc)c \mapsto cbcbc,$$

and

$$cb^2(c^2) \mapsto 0.$$

and so $cbcbc = 0$ as an element of $K\langle X\rangle/I_\Omega$. This leads to the following definition.

Definition A.3.1. [4] Given a reduction system Ω, and a word w we can *resolve* w, which means we compute all possible complete reductions of w. If all complete reductions are equal then we say w is *reduction unique*.

If w is not reduction unique, then we have unequal elements $w_1, w_2 \in K\langle X\rangle$ such that $w \rightsquigarrow w_1$ and $w \rightsquigarrow w_2$, and hence $w_1 - w_2 \in I_\Omega$. By multiplying by a scalar (and refining the partial ordering if necessary) we can assume that $w_1 - w_2$ is a monic element v, and let $\Omega' = \Omega \cup \{v\}$. The ideals $I_{\Omega'}$ and I_Ω are equal (since $v \in I_\Omega$) and so Ω' is a reduction system. Clearly $W_{\Omega'}^{irr} \subseteq W_\Omega^{irr}$, so Ω' is an 'improved' reduction system.

Lemma A.3.2. *If all words are reduction unique then* $KW_\Omega^{irr} \cap I_\Omega = \{0\}$.

Proof. We claim that $KW_\Omega^{irr} \cap I_\Omega = \{y : w \rightsquigarrow y \text{ for some } w \in I_\Omega\}$.

$\subseteq$ is trivial since any element of I_Ω which is a linear combination of irreducible words is already reduced.

For $\supseteq$ we require that a reduction of an element of an element of I_Ω is in I_Ω. This is clear since each single step reduction is just an addition of an element of I_Ω.

We also claim that each element of I_Ω has a reduction to 0. This is true because if $x \in I_\Omega$ then $x = \sum_i x_i$, where $x_i = r_i v_i s_i = r_i(\overline{v}_i - z_i)s_i$ for some $v_i \in \Omega$. Now each $r_i\overline{v}_i s_i \mapsto r_i z_i s_i$, so each $x_i \rightsquigarrow 0$, and so $x \rightsquigarrow 0$. Now since all words are reduction unique, this is the only possible complete reduction, so by the first claim we have the result. ∎

Lemma A.3.3. *[4, Theorem 1.2, (b) $\Rightarrow$ (c)]. If Ω is full reduction system in which all words are reduction unique, then W_Ω^{irr} is a basis for A.*

Proof. Suppose that some linear combination of irreducible words y is equal to 0 in A, i.e., $y \in I_R$. Then $y \in I_\Omega$ since Ω is a full reduction system. Now by the previous lemma, $y = 0$, and so W_Ω^{irr} is a linearly independent set. Since we know it spans A, it is therefore a basis. $\square$

A.4 The Diamond Lemma

Definition A.4.1. [4] A word w is said to be *minimally ambiguous* if $w = rst$ for some words r, s, t which satisfy the following conditions.

(*i*) $rs, st \in W_\Omega$, with r, t having length at least 1,

(*ii*) $s, rst \in W_\Omega$.

In the first case we say w is an *overlap ambiguity*, in the second we say that w is an *inclusion ambiguity.*

Lemma A.4.2. *[4, Theorem 1.2, $(a') \Rightarrow (b)$)]. If Ω is a reduction system such that all minimally ambiguous words are reduction unique, then all words are reduction unique.*

Proof. Assuming the conditions of the theorem are satisfied, we prove by induction that any word w is reduction unique. Assume that all words less than w are reduction unique. Let $w = r_1\overline{v}_1s_1$ where $\overline{v}_1 \in W_\Omega$ is chosen so that if $w = r_2\overline{v}_2s_2$ with $\overline{v}_2$ in W_Ω then either $|r_1| > |r_2|$ or $|r_1| = |r_2|$ and $|v_2| \geq |v_1|$.

Since $\overline{v}_1 \in W_\Omega$, then there is $v_1 \in \Omega$ such that $v_1 = \overline{v}_1 - z_1$. Let $y_1 = r_1z_1s_1$. Clearly $w \mapsto y_1$. By the induction hypothesis, $y_1 \rightsquigarrow y$ for some unique y. We want to show that if $w \rightsquigarrow y'$, then $y' = y$. A reduction of w must start with some single step reduction $w \mapsto y_2$. Suppose $w = r_2\overline{v}_2s_2$ and $y_2 = r_2z_2s_2$ where $v_2 = \overline{v}_2 - z_2 \in \Omega$ and $y_2 \rightsquigarrow y'$.

We claim that $y_2 \rightsquigarrow y$, and this will complete the proof, since y_2 is reduction unique by the induction hypothesis, so $y' = y$. By the choice of v_1, we must have one of the following cases.

(1) $|r_1| \geq |r_2| + |v_2|$, so that v_1 and v_2 do not intersect.

In this situation we can write $w = r_2\overline{v}_2t\overline{v}_1s_1$ for some word t, and so we have $y_1 = r_2\overline{v}_2tz_1s_1$ and $y_2 = r_2z_2t\overline{v}_1s_1$. We have $y_1 \mapsto r_2z_2tz_1s_1$ (using the single step reduction of $\overline{v}_2$), and since $y_1 \rightsquigarrow y$ is unique, $r_2z_2tz_1s_1 \rightsquigarrow y$. Now since $y_2 \mapsto r_2z_2tz_1s_1$ (using the single step reduction of $\overline{v}_1$), we see $y_2 \rightsquigarrow y$, as claimed.

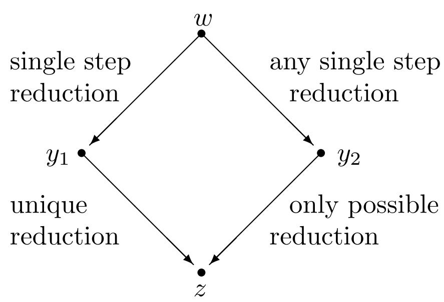
(2) $|r_1| < |r_2| + |v_2|$ and $|r_1| + |v_1| > |r_2| + |v_2|$ so that v_1 and v_2 overlap.

We can write $w = r_2rsts_1$ where $rs = \overline{v}_2$ and $st = \overline{v}_1$, so $y_1 = r_2rz_1s_1$ and $y_2 = r_2z_2ts_1$. We know rst is reduction unique, since it is minimally ambiguous, say $rst \rightsquigarrow u$. In particular, $rst \mapsto rz_1$, so $rz_1 \rightsquigarrow u$ and similarly $rst \mapsto z_2t$, so $z_2t \rightsquigarrow u$. We have $y_1 = r_2rz_1s_1 \rightarrow r_2us_1$, so $r_2us_1 \rightsquigarrow y$ (since $y_1 \rightsquigarrow y$ is unique). Now $y_2 = r_2z_2ts_1 \rightarrow r_2us_1$, and so $y_2 \rightsquigarrow y$, as claimed.

(3) $|r_1| < |r_2| + |v_2|$ and $|r_1| + |v_1| \leq |r_2| + |v_2|$ so that v_2 includes v_1.

This follows in the same way as the previous case, again using the fact that rst is reduction unique since it is minimally ambiguous. □

The proof can be illustrated with the following diagram.



For this reason this result is known as 'The Diamond Lemma'.

Corollary A.4.3. *If Ω is a full reduction system in which all minimally ambiguous words are reduction unique, then W_{Ω}^{irr} is a basis for A.*

A.5 How this works in practice.

It is possible use the Diamond Lemma to formulate an algorithm which would allow a computer to find bases in this way. However, there are problems with this, e.g.

1. It is not always obvious which ordering we should use without doing some initial calculations. If an unsuitable ordering is chosen, then one may end up with an unfeasibly large reduction system.

2. One may be able to spot suitable elements of a reduction system 'by inspection', and not by resolving some ambiguity. This happens frequently in the case where the algebra has invertible elements. It is sensible to use whatever tools we have available, rather than restrict ourselves to one set of rules.

3. In some cases, we are only interested in showing an algebra is finite dimensional, and so, instead of trying to find the best possible reduction system, we can stop once the set of irreducible words becomes finite.

4. When dealing with an infinite class of algebras (e.g. when showing $\Lambda^q(Q)$ is finite dimensional for all quivers of type D_n), one needs an argument which deals with all the cases.

So instead, we write out our proofs like this:

(1) Prove a set Ω is a reduction system.

This may be done by showing the elements of Ω lie in I_R by a direct calculation, or by the following step by step process.

(i) Set $\Omega_0 = R$ with respect to some partial ordering $\leq$.

(ii) Given Ω_i, we form Ω_{i+1} by finding some elements $s_{i1}, \ldots, s_{ik} \in I_{\Omega_i}$ (usually resolving some ambiguities in Ω_i). Set $\Omega_{i+1} = \Omega_i \cup \{s_{i1}, s_{i2}, \ldots, s_{ik}\}$ and repeat (note that sometimes we may remove elements from Ω_i before continuing).

If we are only interested in finding a spanning set for the algebra, then this is sufficient. In the cases where we are trying to find a basis, we must prove (2) and (3) which follow.

(2) Prove Ω is a full reduction system.

For this, we need only show that $I_R \subseteq I_\Omega$, or equivalently each $r \in I_\Omega$ for all $r \in R$.

(3) Prove all minimally ambiguous words are reduction unique.

To do this, we list all minimally ambiguous words in order $\{w_1, w_2, \ldots, w_n\}$.

Note that if $w_1, \ldots, w_{k-1}$ have been shown to be reduction unique, then all words below w_k are reduction unique. So to show w_k is reduction unique, we need to consider all its single step reductions (usually only two), say $x_{k1}, \ldots, x_{km}$. Then we find some y so that each x_{kj} has a reduction to y. Since each x_{kj} is reduction unique, y is the unique reduction, and so w_k is reduction unique.

Example A.5.1. We return to the example given at the start of this Appendix, and show how the reduction algorithm works. Let A be the algebra

$$K\langle b, c\rangle / I_R,$$

where R is the set of elements $\{b^3, c^2, cb^2c - cbcb - bcbc\}$. Can we find a basis for A?

We must choose a suitable partial ordering on the set of words. This ordering must make either $cbcb$ or $bcbc$ the leading word as we have already seen that we cannot make cb^2c the leading word. The obvious ordering to use (see the next section) is the length-lexographic ordering with $b < c$. We claim that $\Omega = \{b^3, c^2, cbcb - cb^2c + bcbc, cb^2cb^2 - bcb^2cb + b^2cb^2c\}$ is a full reduction system in which all minimally ambiguous words are reduction unique and therefore the set of irreducible words $\{1, b, c, b^2, bc, cb, b^2c, bcb, cb^2, cbc, b^2cb, bcb^2, bcbc, cb^2c, b^2cb^2, b^2cbc, bcb^2c, cb^2cb, b^2cb^2c, bcb^2cb, cb^2cbc, b^2cb^2cb, bcb^2cbc, b^2cb^2cbc\}$ is a basis for A.

We first prove that Ω is a reduction system. We only need show that $cb^2cb^2 - bcb^2cb + b^2cb^2c \in I_R$. This could be done in several ways, but the simplest is to consider R as a reduction system. This has an ambiguity $cbcb^3$. We resolve it:

$$\begin{aligned} cbc(b^3) &\mapsto 0. \\ (cbcb)b^2 &\mapsto cb^2cb^2 - b(cbcb)b \\ &\mapsto cb^2cb^2 - bcb^2cb + b^2(cbcb) \\ &\mapsto cb^2cb^2 - bcb^2cb + b^2cb^2c - (b^3)cbc \\ &\mapsto cb^2cb^2 - bcb^2cb + b^2cb^2c. \end{aligned}$$

Equating the two reductions shows that $cb^2cb^2 - bcb^2cb + b^2cb^2c \in I_R$.

It is obvious that Ω is a full reduction system. We now show that all minimally ambiguous words for Ω are reduction unique. Arranging them in order, they are $\{c^3, b^4, b^5, c^2bcb, cbcb^3, cbcbcb, cb^2cb^3, c^2b^2cb^2, cbcb^2cb^2, cb^2cb^2cb^2\}$. The first three are clearly reduction unique.

$$\begin{aligned}
(c^2)bcb &\mapsto 0.\\
c(cbcb) &\mapsto (c^2)b^2c - (cbcb)c \mapsto 0 - cb^2(c^2) + bcb(c^2) \mapsto 0.
\end{aligned}$$

$$\begin{aligned}
cbc(b^3) &\mapsto 0.\\
(cbcb)b^2 &\mapsto (cb^2cb^2) - b(cbcb)b \mapsto bcb^2cb - b^2cb^2c - bcb^2cb + b^2(cbcb)\\
&\mapsto -b^2cb^2c + b^2cb^2c - (b^3)cbc \mapsto 0.
\end{aligned}$$

$$\begin{aligned}
(cbcb)cb &\mapsto cb^2(c^2)b - bcb(c^2)b \mapsto 0.\\
cb(cbcb) &\mapsto (cbcb)bc - cb^2cbc \mapsto -b(cbcb)c \mapsto -bcb^2(c^2) + b^2cb(c^2) \mapsto 0.
\end{aligned}$$

$$\begin{aligned}
cb^2c(b^3) &\mapsto 0.\\
(cb^2cb^2)b &\mapsto b(cb^2cb^2) - b^2cb^2cb \mapsto (b^3)cb^2c \mapsto 0.
\end{aligned}$$

$$\begin{aligned}
(c^2)b^2cb^2 &\mapsto 0.\\
c(cb^2cb^2) &\mapsto (cbcb)bcb - cb^2cb^2c \mapsto cb^2(cbcb) - b(cbcbcb) - cb^2cb^2c\\
&\mapsto cb^2cb^2c - c(b^3)cbc - cb^2cb^2c \mapsto 0.
\end{aligned}$$

$$
\begin{aligned}
cb(cb^2cb^2) &\mapsto cb^2cb^2cb - c(b^3)cb^2c \mapsto cb^2cb^2cb. \\
(cbcb)bcb^2 &\mapsto cb^2(cbcb)b - b(cbcbcb)b \mapsto cb^2cb^2cb - c(b^3)cbcb \\
&\mapsto cb^2cb^2cb.
\end{aligned}
$$

$$
\begin{aligned}
cb^2(cb^2cb^2) &\mapsto cb^3cb^2cb - cb^4cb^2c \mapsto 0. \\
(cb^2cb^2)cb^2 &\mapsto bcb^2(cbcb)b - b^2cb^2(c^2)b^2 \mapsto b(cb^2cb^2)cb - bc(b^3)cbcb \\
&\mapsto b^2cb^2(cbcb) - (b^3)cb^2c^2b \mapsto b^2(cb^2cb^2)c - b^2c(b^3)cbc \\
&\mapsto (b^3)cb^2cbc - b^4cb^2(c^2) \mapsto 0.
\end{aligned}
$$

A.6 Orderings for reduction systems

In this section we discuss some orderings which satisfy (†) and (‡), so are suitable for reduction systems. We start with the two most obvious examples.

Definition A.6.1. The *length ordering*, $\leq_{len}$ is defined by

$$u \leq_{len} v \text{ if and only if } |u| < |v| \text{ or } u = v.$$

Given a letter a, define $|w|_a$ to be total number of occurrences of the letter a in w. The *a-degree ordering* $\leq_a$ is defined by

$$u \leq_a v \text{ if and only if } |u|_a < |v|_a \text{ or } u = v.$$

It is clear that these orderings satisfy (†) and (‡). In fact they satisfy a stronger version of (†), namely, $u < v$ if and only if $rus < rvs$. From now on, when we refer to (†), we mean this stronger property. We can combine orderings which satisfy (†) and (‡) in the following way to produce some more refined orderings satisfying (†) and (‡).

Definition A.6.2. Given two partial orderings $\leq_*$, $\leq_{**}$, we define the following *combination ordering*, $\leq_{*,**}$ by

$$u \leq_{*,**} v \text{ if and only if } u \leq_* v \text{ or } u \not\leq_* v \text{ and } v \not\leq_* u \text{ and } u \leq_{**} v$$

In other words, to compare u and v we first examine them under the $\leq_*$ ordering. Only if they are incomparable do we then try the $\leq_{**}$ ordering. For example, let $u = aaca$, $v = bbca$, $w = cabbab$. Some relations are $v \leq_{len,a} u$, $w \leq_{a,len} u$, $u \leq_{len,a} w$. The last two show that $\leq_{*,**}$ is in general not the same as $\leq_{**,*}$.

Lemma A.6.3. *If $\leq_*$, $\leq_{**}$ satisfy (†) and (‡), then so does $\leq_{*,**}$.*

Proof. Denote $\leq_{*,**}$ by $\leq$. We need to show that for all r, s, u, v,

$$u < v \text{ if and only if } rus < rvs.$$

There are three ways u and v can be related with respect to $\leq_*$. We check this statement for each case. If $u <_* v$ then $rus <_* rvs$ by (†) for $\leq_*$ and the statement follows since both sides are true. If $v <_* u$, then $rvs <_* rus$ and the statement follows since both sides are false. So suppose that u and v are incomparable with respect to $\leq_*$. The left hand side is true if and only if $u <_{**} v$ which is true if and only if $rus <_{**} rvs$ (using (†) for $\leq_{**}$), which is true if and only if $rus < rvs$. The last part is true because rus and rvs are incomparable (using (†) for $\leq_*$).

To check (‡) is satisfied, we let

$$u_1 \geq u_2 \geq u_3 \geq \dots$$

be an infinite non stabilising descending chain for $\leq$. Define a sequence of integers i_n inductively by $i_0 = 1$ and i_{n+1} to be the least integer satisfying $u_{i_{n+1}} < u_{i_n}$. The sequence must be infinite for if i_n is the last member then

$$u_{i_n} \geq_{**} u_{i_n+1} \geq_{**} u_{i_n+2} \geq_{**} \dots$$

is an infinite non stabilising descending chain for $\leq_{**}$. However we now have an infinite descending chain for $\leq_*$, namely

$$u_{i_0} \geq_* u_{i_1} \geq_* u_{i_2} \geq_* \dots$$

which is a contradiction. ∎

Sometimes it is possible that no such combination of the length and degree orderings will give a sufficiently refined ordering. If two words have the same number of occurrences of each letter (and so are the same length), then they are incomparable under the length and degree ordering (and so are incomparable under any combination of them). So if a relation involves two or more such words (as in Example A.1.1), then in order to write the relation into standard reduction form, it is necessary to introduce another way to compare words.

One example is the (left) lexographic ordering. First one chooses a total ordering $\leq$ on the letters and then define

$$u \leq_{lex} v \text{ if and only if } u = v \text{ or } u = rbs \text{ and } v = rct \text{ where } b < c.$$

This does not satisfy (‡), since $c \geq bc \geq b^2c \geq \dots$ is an infinite non stabilising descending chain. Instead one will generally use the length-lexographic ordering $\leq_{len,lex}$. This is a total ordering, so this would always be sufficient to write all equations in standard reduction form.

In some cases, the length-lexographic ordering is not suitable, and instead we use more complicated *base ordering*, defined below.

Let u be a word, and let a, b be letters, and let i be an integer between 0 and $|u|_b$. Let $f_i^{a,b}(u)$ be the number of occurrences of a between the i-th b and the $i+1$-th b (counting from the left). For example, if $u = acbaaccabbcaa$, then $f_0^{a,b}(u) = 1$, $f_1^{a,b}(u) = 3$, $f_2^{a,b}(u) = 0$, $f_3^{a,b}(u) = 2$. Let $m > 0$ be an integer. Define

$$g_m^{a,b}(u) = \sum_{i=0}^{|u|_b} f_i^{a,b} m^i$$

So for the word above, $g_m^{a,b}(u) = 1 + 3m + 2m^3$.

Lemma A.6.4. *For all words u, v*

$$g_m^{a,b}(uv) = g_m^{a,b}(u) + m^{|u|_b} g_m^{a,b}(v).$$

Proof. For simplicity write f_i and g instead of $f_i^{a,b}$, $g_m^{a,b}$.

$$\begin{aligned}
g(uv) &= \sum_{i=0}^{|uv|_b} f_i(uv)m^i \\
&= \sum_{i=0}^{|u|_b-1} f_i(uv)m^i + f_{|u|_b}(uv)m^{|u|_b} + \sum_{i=|u|_b+1}^{|uv|_b} f_i(uv)m^i \\
&= g(u) - f_{|u|_b}(u)m^{|u|_b} + f_{|u|_b}(uv)m^{|u|_b} + m^{|u|_b}\sum_{j=1}^{|u|_b} f_j(v)m^j \\
&= g(u) + f_0(v)m^{|u|_b} + m^{|u|_b}\sum_{j=1}^{|v|_b} f_j(v)m^j \\
&= g(u) + m^{|u|_b}g(v).
\end{aligned}$$

□

Definition A.6.5. Define the *base ordering*, $\leq_{(a,b;m)}$ by

$$u \leq_{(a,b;m)} v \text{ if and only if } g_{a,b}^m(u) < g_{a,b}^m(v) \text{ or } u = v.$$

Lemma A.6.6. *The ordering $\leq_{b,(a,b;m)}$ satisfies (†) and (‡).*

Proof. Write $\leq$ for $\leq_{b,(a,b;m)}$. It is clearly a partial order with satisfying (‡). Clearly (†) is satisfied in the case that $|u|_b \neq |v|_b$. So suppose $|u|_b = |v|_b$. Using the previous lemma,

$$g_{a,b}^m(rus) = g_{a,b}^m(r) + m^{|r|_b}g_{a,b}^m(u) + m^{|ru|_b}g_{a,b}^m(s),$$

$$g_{a,b}^m(rvs) = g_{a,b}^m(r) + m^{|r|_b}g_{a,b}^m(v) + m^{|rv|_b}g_{a,b}^m(s).$$

Since $|ru|_b = |rv|_b$, $g_{a,b}^m(rvs) - g_{a,b}^m(rus) = m^{|r|_b}(g_{a,b}^m(v) - g_{a,b}^m(u))$. Now

$$\begin{aligned}
rus < rvs &\Leftrightarrow g_{a,b}^m(rvs) - g_{a,b}^m(rus) > 0, \\
&\Leftrightarrow m^{|r|_b}(g_{a,b}^m(v) - g_{a,b}^m(u)) > 0, \\
&\Leftrightarrow g_{a,b}^m(v) - g_{a,b}^m(u) > 0, \\
&\Leftrightarrow u < v.
\end{aligned}$$

So $\leq_{(a,b;m)}$ satisfies (†), and therefore so does $\leq_{b,(a,b;m)}$. □

Bibliography

[1] M.Auslander, I.Reiten and S.O.Smalø, *Representation theory of Artin algebras*, Cambridge studies in Adv. Math., Vol., **36** Cambridge university press, Cambridge, 1995.

[2] M.Auslander and S.O.Smalø, *Preprojective modules over Artin algebras*, J. Algebra **66** (1980), 61–122.

[3] D.Baer, W.Geigle, H.Lenzing, *The preprojective algebra of a tame hereditary Artin algebra*, Comm. Algebra **15** (1987), 425–457.

[4] G.M.Bergman, *The diamond lemma for ring theory*, Adv. Math. **29** (1978), 178–218.

[5] J.Białkowski, K.Erdmann and A.Skowroński *Deformed preprojective algebras of generalized dynkin type*, to appear in Trans. Math.

[6] L.A.Bokut' and G.P.Kukin *Algorithmic and combinatorial algebra*, in Mathematics and its applications, Vol. **255**, Kluwer academic press, Dordrecht, 1994.

[7] H.Cassens and P.Slodowy, *On Kleinian Singularities and quivers*, in Singularities, (Oberwolfach 1996) Prog. Math., Vol. **162**, Birkhäuser, Basel, 1998 pp. 263–288.

[8] W.W.Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. **126** (2001), 257–293.

[9] W.W.Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero*, Duke Math. J. **118** (2003), 339–352.

[10] W.W.Crawley-Boevey and M.P.Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), 605–635.

[11] W.W.Crawley-Boevey and P.H.C.Shaw, *Multiplicative Preprojective algebras, middle convolution and the Deligne-Simpson problem*, to appear in Adv. Math. (www.arXiv.org/math.RA/0404186).

[12] M.Dettweiler and S.Reiter, *An algorithm of Katz and its application to the Inverse Galois problem*, J. Symbolic Comput. **30** (2000), 761–798.

[13] V.Dlab and C.M.Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. **6** (1976).

[14] V.Dlab and C.M.Ringel, *The preprojective algebra of a modulated graph*, in Representation theory II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., Vol. **832**, Springer, Berlin-New York 1980 pp. 216–231.

[15] Y.A.Drozd and V.V.Kirichenko, *Finite-dimensional algebras*, Springer, Berlin 1994.

[16] P. Etingof, A. Oblomkov and E.Rains, *Generalized double affine Hecke algebras of rank 1 and quantised Del Pezzo surfaces*, (www.arXiv.org/math.QA/0406480).

[17] P.Gabriel, *Unzerlegbare Darstellungen. I.*, Manuscripta Math. **6** (1972), 71–103.

[18] I.M.Gel'fand and V.A.Ponomarev *Model algebras and representations of graphs*, Funktsional. Anal. i Prilozhen. **13** (1979) 1–12.

[19] R.A.Horn and C.R.Johnson *Matrix Analysis*, Cambridge University Press, Cambridge, 1990.

[20] D.A.Jordan, *A simple localization of the quantized Weyl algebra*, J. Algebra **174** (1995), 267–281.

[21] V.G.Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), 57–92.

[22] P.Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Diff. Geom. **29** (1989), 665–683.

[23] G.Lusztig, *Quivers, perverse sheaves and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421.

[24] G.Lusztig, *Affine quivers and canonical bases*, Publ. Math. IHES, **76** (1992), 111–163.

[25] J. McKay, *Graphs, singularities, and finite groups*, in The Santa Cruz Conference on Finite Groups (Univ. of California, Santa Cruz, Calif., 1979) Proc. Sympos. Pure Math. **37**, Amer. Math. Soc., Providence, 1980 pp. 183–186.

[26] C.Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. **55** (1980), 199–224.

[27] C.M.Ringel, *On algorithms for solving vector space problems II. Tame algebras.*, in Representation theory I, (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Lecture Notes Math., Vol. **831**, Springer, Berlin-New York 1980 pp. 137–287.

[28] C.M.Ringel, *The preprojective algebra of a quiver*, in Algebras and Modules II, CMS Conference Proceedings Vol. **24** Amer. Math. Soc., Providence, RI, 1998 pp. 467–480.